TATE'S THESIS IN THE DE RHAM SETTING

JUSTIN HILBURN AND SAM RASKIN

ABSTRACT. We calculate the category of D-modules on the loop space of the affine line in coherent terms. Specifically, we find that this category is derived equivalent to the category of ind-coherent sheaves on the moduli space of rank one de Rham local systems with a flat section. Our result establishes a conjecture coming out of the $3d$ mirror symmetry program, which obtains new compatibilities for the geometric Langlands program from rich dualities of QFTs that are themselves obtained from string theory conjectures.

CONTENTS

1. INTRODUCTION

1.1. **Statement of the main results.** We work over a field k of characteristic zero.

Let $\mathcal{Y}$ be the moduli space of rank 1 de Rham local systems on the punctured disc equipped with a flat section, and let $\mathfrak{L}\mathbb{A}^1$ denote the algebraic loop space of $\mathbb{A}^1$.

Our main result asserts:

Theorem 1.1.0.1 (Thm. 8.4.0.1)**.** *There is a canonical equivalence of DG categories:*

$$\Delta : D^!(\mathfrak{L}\mathbb{A}^1) \simeq \mathsf{IndCoh}^*(\mathcal{Y}).$$

Moreover, this equivalence is compatible with the local geometric class field theory of Beilinson-Drinfeld.

Here the left hand side is the category of D-modules on $\mathfrak{L}\mathbb{A}^1$, as defined in [Ber], [Ras3]. The right hand side is the category of ind-coherent sheaves on $\mathcal{Y}$, which we construct in §4 (following [Ras7]). In both cases, there are infinite-dimensional subtleties in the definitions; these are far more severe on the right hand side.

Date: July 15, 2021. Last updated: September 7, 2021.

Remark 1.1.0.2. The assertions of the theorem are made more precise in the body of the paper. For now, we content ourselves with the following remark on local class field theory.

Beilinson-Drinfeld construct an equivalence:

$$D^*(\mathfrak{L}\mathbb{G}_m) \simeq \mathsf{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}) \tag{1.1.1}$$

for $\mathrm{LocSys}_{\mathbb{G}_m}$ the moduli space of rank 1 de Rham local systems on the punctured disc. The left hand side here naturally acts on the left hand side of Theorem 1.1.0.1, while the right hand side naturally acts on the right hand side of Theorem 1.1.0.1. The compatibility asserts that the equivalence of Theorem 1.1.0.1 is compatible with these actions.

Remark 1.1.0.3. Both sides of the above equivalence have natural t-structures. However, this equivalence is *not* t-exact; it is necessarily an equivalence of derived categories. This is in contrast to (1.1.1), which largely amounts to an equivalence of abelian categories.

Remark 1.1.0.4. For K a local field, Tate's thesis [Tat] (see also [Wei]) considers the decomposition of the space $\mathcal{D}(K)$ of tempered distributions on K as a representation for $K^\times$. We consider Theorem 1.1.0.1 as an analogue of the arithmetic situation. We observe that our result applies not only at the level of eigenspaces, but describes a categorical analogue of the whole space $\mathcal{D}(K)$.

We plan to return to global aspects of the subject in future work.

Remark 1.1.0.5. As far as we are aware, our work is the first one in geometric representation theory to prove a theorem about coherent sheaves on a space of the form $\mathcal{M}aps(\overset{\circ}{\mathcal{D}}_{dR}, Y)$ for Y an Artin stack that is not a scheme and is not a classifying stack (for us: $Y = \mathbb{A}^1/\mathbb{G}_m$). As we will see in §2 and 4, there are substantial technical difficulties that arise in this setting. Roughly speaking, the singularities of the space $\mathcal{Y}$ are genuinely of infinite type. We can only overcome these difficulties in our specific (abelian) setting.

Remark 1.1.0.6. As discussed later in §1.3.2, the main piece of our construction is a realization of D-modules on $\mathbb{A}^n$ in spectral terms; this is the content of §5. Already in the $n = 1$ case, our realization of D-modules on $\mathbb{A}^1$ as coherent sheaves on some space is novel; in this case, we highlight that the expression is as coherent sheaves on the subspace $\mathcal{Y}^{\leqslant 1} \subseteq \mathcal{Y}$ discussed in detail in §2.11.

Remark 1.1.0.7. In the physicist's language, we are comparing line operators for the A-twist of a pure hypermultiplet with the B-twist of a $U(1)$-gauged hypermultiplet. Moreover, we consider these $3d$ $\mathcal{N} = 4$ theories as boundary theories for (suitably twisted) pure $U(1)$-gauge theory (with electro-magnetic duality exchanging its A and B-twists).

1.2. **Connections to $3d$ mirror symmetry.**

The equivalence of Theorem 1.1.0.1 is a first[1] instance of what is expected to be a broad family of equivalences, which goes under the heading $3d$ *mirror symmetry*. This subject is closely tied to the geometric Langlands program.

As context for our results, we provide an informal introduction to these ideas below. Our objective is to connect our equivalence as closely as possible with the mathematical physics literature, and to promote those ideas to mathematicians interested in the circle of ideas around geometric Langlands.

We hope the reader will forgive the redundancy of our discussion given the numerous other great sources in the literature.

We emphasize that we do not claim originality for any of the ideas appearing below. We withhold attributions and leave much contact with the existing literature until §1.2.24.

Remark 1.2.0.1. For another exposition of this subject also targeted at mathematicians, see [BF]. For a recent physics-oriented exposition, we refer to [DGGH] §2.

[1]Specifically, as an equivalence of categories of line operators, with both matter and a non-trivial gauge group on the B-side.

1.2.1. The physics of the last thirty years has highlighted the role of *dualities*: quantum field theories that are equivalent by non-obvious means. We emphasize that physicists regard these dualities as conjectural: they do not claim to know how to match the QFTs, only (at best) parts of them.

Examples abound. For instance, Montonen–Olive's (conjectural) S-duality for 4-dimensional Yang-Mills theory with $\mathcal{N} = 4$ supersymmetry for two Langlands dual gauge groups G and $\check{G}$ is of keen interest.

The dualities physicists consider fit into a sophisticated logical hierarchy. But briefly, most are subsumed in the existence of Witten's (conjectural) $\mathcal{M}$-theory, which is supposed to recover other (known) QFTs by various constructions.

1.2.2. A general problem is to extract mathematical structures from quantum field theories.[2] In this case, physical dualities relate to mathematical conjectures.

For instance, in [KW], Kapustin and Witten found such a relationship between Montonen–Olive's (conjectural) duality for 4-dimensional Yang-Mills with $\mathcal{N} = 4$ supersymmetry and geometric Langlands conjectures in mathematics. Their perspective was developed in [Cos1] and [EY], which exactly clarified some means of extracting algebro-geometric invariants from Lagrangian field theories.

1.2.3. *3d σ-models.* In one setting, so-called[3] $3d$ $\mathcal{N} = 4$ quantum field theories, there are rich relations with algebraic geometry.

First, for every algebraic stack $\mathcal{X}$, we suppose there is (in some algebraic sense) a $3d$ (Lagrangian) quantum field theory $T_{\mathcal{X}}$ with $\mathcal{N} = 4$ supersymmetry; we spell out some more precise expectations below.

Remark 1.2.3.1. At the quantum level, physicists agree our supposition is problematic: cf. the discussion at the end of [RW] §2.3. (The classical field theory is fine, but may be unrenormalizable, and even if it is renormalizable, there may not be a distinguished scheme.) So it is better to regard $T_{\mathcal{X}}$ as a heuristic: its twists (see below) are all that is defined (at the quantum level).

Remark 1.2.3.2. In fact, physicists say that $T_{\mathcal{X}}$ depends only on the symplectic stack $T^*\mathcal{X}$. At the classical level, this theory is the $3d$ σ-model with target $T^*\mathcal{X}$. The $\mathcal{N} = 4$ supersymmetry comes from the symplectic structure on $T^*\mathcal{X}$.

I.e., more generally, there is a *classical* $3d$ $\mathcal{N} = 4$ theory corresponding to a σ-model with target any symplectic stack $\mathcal{S}$. One hopes for a quantization associated with a Lagrangian foliation $\lambda : \mathcal{S} \to \mathcal{X}$; at the moment, this quantization (or rather, its A/B twists) is only understood for $\mathcal{S}$ a twisted cotangent bundle over some $\mathcal{X}$ (and λ the canonical projection).

This perspective is important, but sometimes inconvenient (especially at the quantum level), so we emphasize the role of $\mathcal{X}$ e.g. in our notation.

1.2.4. *Algebraic QFTs.* Next, we describe how we think about a $3d$ QFT T in algebraic geometry. We include this discussion to make precise the connection between the mathematical physics we wish to discuss and the algebraic geometry we wish to study.

[2]Often, this is by a sort of analogy. Physicists are drawn to differential geometry: their bread and butter are moduli spaces of solutions to non-linear PDEs (namely, Euler-Lagrange equations).

But for a mathematician, it may be preferable to consider algebraic varieties as analogous to Kähler manifolds, or symplectic varieties as analogous to hyper-Kähler manifolds. Often, rich algebraic geometry results.

[3]We remind what these parameters encode in the subsequent discussion.

1.2.5. Recall from [Lur2] that an *oriented* 3*-dimensional fully-extended TFT* T would attach a number $T(M)$ to a closed 3-manifold M, a vector space $T(M)$ to a closed 2-manifold M, a DG category $T(M)$ to a closed 1-manifold M, and a DG 2-category (i.e., DGCat_{cont}-module category) to a 0-manifold M. Cobordisms define morphisms. Disjoint unions go to tensor products.

A variant: given an ambient 3-manifold N, a fully-extended TFT *on* N should assign such data to manifolds M equipped with embeddings into N. (We are not aware of a reference.)

1.2.6. The algebraic situation is similar, but only some data is defined. We heuristically describe the main structures, without giving formal definitions.

Fix a smooth, projective, geometrically connected algebraic curve X, which we regard as analogous to a 2-manifold. (The curve X should not be confused with the $\mathcal{X}$ that sometimes occurs when considering a σ-model with target $T^*\mathcal{X}$.)

A[4] 3d *field theory* T on[5] X includes the data of a vector space $T(X) \in \mathsf{Vect}$ and a category $T(\mathring{\mathcal{D}}_x) \in \mathsf{DGCat}_{cont}$ for every point $x \in X$, regarding $\mathring{\mathcal{D}}_x$ as analogous to a circle.

We regard the formal disc $\mathcal{D}_x$ as a cobordism $\varnothing \to \mathring{\mathcal{D}}_x$. There is an associated functor:

$$\mathsf{Vect} = T(\varnothing) \to T(\mathring{\mathcal{D}}_x) \in \mathsf{DGCat}_{cont}$$

i.e., there is a preferred object of $T(\mathring{\mathcal{D}}_x)$, the so-called *vacuum* (or *unit*) object.

We regard $X\backslash x$ as a cobordism $\mathring{\mathcal{D}}_x \to \varnothing$. There is an associated functor:

$$T(\mathring{\mathcal{D}}_x) \to T(\varnothing) = \mathsf{Vect} \in \mathsf{DGCat}_{cont}.$$

This functional evaluated on the vacuum is $T(X) \in \mathsf{Vect}$.

Remark 1.2.6.1. Unlike in the topological situation, the above is not symmetric. I.e., we only allow cobordisms in the directions specified above.

Remark 1.2.6.2. In this 3d setting, the category $T(\mathring{\mathcal{D}}_x)$ is often called the *category of line operators* for the theory. (Physicists would draw the line $x \times \mathbb{R} \subseteq X \times \mathbb{R}$ passing through the interior of our circle $\mathring{\mathcal{D}}_x \times 0 \subseteq X \times \mathbb{R}$.)

Remark 1.2.6.3. The assignment $x \rightsquigarrow T(\mathring{\mathcal{D}}_x)$ should extend to a *factorization category on* X in the sense of [Ras2]. The vacuum objects should correspond to a unital structure on this factorization category. The functionals above should extend to a well-defined functional on the *independent category* of this unital factorization category; see [Ras1] for the definition.

Example 1.2.6.4. In [BD], Beilinson and Drinfeld defined such a structure for any chiral algebra $\mathcal{A}$ on X, i.e., they (in effect) defined a 3d field theory $T_{\mathcal{A}}$. The vector space $T_{\mathcal{A}}(X)$ is the space of *conformal blocks* of $\mathcal{A}$. The category $T_{\mathcal{A}}(\mathring{\mathcal{D}}_x)$ is $\mathcal{A}\text{–}\mathsf{mod}_x$, the category of (unital) chiral modules for $\mathcal{A}$ supported at x. The vacuum object is the vacuum representation of $\mathcal{A}$, and the functional $\mathcal{A}\text{–}\mathsf{mod}_x \to \mathsf{Vect}$ is the functor $C^{ch}(X, \mathcal{A}, -)$ from *loc. cit.* §4.2.19.

[4]As the input to the discussion is an algebraic curve X, we sometimes refer to this formalism as *algebraic QFT*. We emphasize that we are *not* referring to some kind of "algebraic twist" (analogous to a "holomorphic twist") of a supersymmetric Lagrangian field theory — the formalism is insensitive to such considerations. Rather, this terminology refers to the fact that this formalism is defined without analysis, and makes sense in the general setting of algebraic geometry (of curves here, with evident adaptations in the terminology otherwise) over fields (perhaps of characteristic zero).

[5]It might be better to say the theory lives on $X \times \mathbb{R}$.

(This theory is analogous to the $3d$ TFT associated with a $\mathcal{E}_2$-algebra $\mathcal{A}$ in [Lur2] Theorem 4.1.24.[6])

Remark 1.2.6.5. Informally, it is good to think of $3d$ theories on X as the natural home for the Morita theory of chiral algebras, just as DGCat_{cont} is the natural home for Morita theory of associative algebras.

Remark 1.2.6.6. The relationship between chiral algebras and $3d$ $\mathcal{N} = 4$ (and sometimes $\mathcal{N} = 2$) theories is the starting point of the series of works [CG1], [CCG], and [CDG], which aim to use chiral algebras to study mirror dual theories in this way. To describe this work in more detail, we use ideas reviewed later in this introduction.

Specifically, a $3d$ $\mathcal{N} = 4$ theory T with a suitably deformable (cf. [CG1] §2.3; there is an implicit choice of A or B twists fixed in our discussion here) $\mathcal{N} = (0, 4)$ boundary condition $\mathcal{B}$ should yield an algebraic $3d$ field theory T^{alg} and a boundary condition $\mathcal{B}^{alg}$ for it; here we regard $\mathcal{B}^{alg}$ as an interface $T^{alg} \to \mathrm{triv}$ *to* the trivial theory. This data yields a factorization algebra $\mathcal{A}_{T,\mathcal{B}}$ on X: its fiber at x is by evaluating $\mathcal{B}^{alg}(\mathring{\mathcal{D}}_x) : T(\mathring{\mathcal{D}}_x) \to \mathrm{triv}(\mathring{\mathcal{D}}_x) = \mathsf{Vect}$ at the vacuum object for T. In this case, there is a canonical interface $T^{alg} \to T_{\mathcal{A}_{T,\mathcal{B}}}$. Sometimes, it is even an equivalence, meaning that one can study the full theory T^{alg} via the factorization algebra $\mathcal{A}_{T,\mathcal{B}}$.

Remark 1.2.6.7. As David Ben-Zvi emphasized to us, unital factorization categories with a functional on their independent categories do seem to provide a robust definition of $[1, 2]$-extended $3d$ algebraic QFT (while allowing for some quite degenerate theories). He also informs us that these ideas were developed in collaboration with Sakellaridis and Venkatesh, and will be developed in their forthcoming work.

Question 1.2.6.8. Is there some sense in which the theories of Example 1.2.6.4 are $[0, 2]$-extended, as for their E_2-counterparts? And what general constructions of morphisms (alias: *interfaces*) between such theories exist?[7]

1.2.7. *Supersymmetry.* We now recall the main practical application of supersymmetry: supersymmetric QFTs (typically) come with canonical deformations,[8] called *twists*.[9] Twisting preserves dimensions but lowers the amount of supersymmetry; in our examples, there is no residual supersymmetry.

---

[6] We note that *loc. cit.* regards this theory as a fully-extended $2d$ theory with more categorical complexity than we outlined before. We instead regard the theory as a not-fully-extended $3d$ theory of the type outlined above. This compares to Remark 4.1.27 from [Lur2].

[7] This question is of practical importance to us. Our main theorem amounts to a construction of a non-trivial interface between two $3d$ theories. It is desirable to understand this construction on better conceptual grounds.

[8] Sometimes, we think of deformations *as* quantizations. For us, the difference is as follows.

Briefly, in 0-dimensions, $\mathcal{P}_0$-algebras can be quantized by $\mathcal{E}_0$-algebras, i.e., pointed vector spaces. On the other hand, pointed vector spaces V may be further *deformed* by finding a filtered vector space V^{def} with $\mathrm{gr}_\bullet V^{def} = V$. (In homological settings, such a structure is equivalent to equipping V with a differential in a suitable sense, or one might say, deforming the differential on V.)

In higher dimensions, one can say essentially the same words, following [CG2]. Lagrangian densities give rise to factorization $\mathcal{P}_0$-algebras, which govern the corresponding classical field theory. These may be quantized by factorization $\mathcal{E}_0$-algebras, i.e., factorization algebras; this amounts to quantizing the classical field theory (or more specifically, constructing the OPE for local operators). These factorization algebras may be further deformed to other factorization algebras.

[9] For instance, for a manifold X, its de Rham complex deforms its complex of global differential forms (considered as equipped with zero differential). Similarly, for X a smooth algebraic variety, its de Rham complex deforms its Hodge cohomology (or rather, the cochain analogue). These examples appear in supersymmetric quantum mechanics.

Remark 1.2.7.1. See e.g. [Cos1] and [CS] for an introduction to the twisting procedure. See [ES] and [ESW] for a detailed classification of the twists of supersymmetric QFTs. For the specific twists considered in $3d$ mirror symmetry, see [DGGH] §2.2.

1.2.8. In the setting of $3d$ QFTs T with $\mathcal{N} = 4$ supersymmetry, there are two twists of interest for us: the A-twist T_A and the B-twist T_B. In practice, these are *algebraic* theories.

For our theories $T_{\mathcal{X}}$ of interest, we let $T_{\mathcal{X},A}$ and $T_{\mathcal{X},B}$ denote the corresponding $3d$ QFTs.

Example 1.2.8.1. The category $T_{\mathcal{X},A}(\mathring{\mathcal{D}}_x)$ of line operators for the A-twisted theory should be the category[10] $D(\mathfrak{L}\mathcal{X})$ of D-modules on the algebraic loop space:

$$\mathfrak{L}\mathcal{X} = \mathcal{M}aps(\mathring{\mathcal{D}}_x, \mathcal{X})$$

of $\mathcal{X}$.

Example 1.2.8.2. The category $T_{\mathcal{X},B}(\mathring{\mathcal{D}}_x)$ of line operators for the B-twisted theory should be:

$$\mathsf{IndCoh}\big(\mathcal{M}aps(\mathring{\mathcal{D}}_{x,dR}, \mathcal{X})\big).$$

Remark 1.2.8.3. If $\mathcal{X}$ is smooth affine, each of the categories above essentially come from chiral algebras, as in Example 1.2.6.4. Indeed, up to mild corrections, the A-twist is essentially governed by a CDO for $\mathcal{X}$, while the B-twist is governed by the commutative chiral algebra of functions on[11] $\mathcal{X}$.

Remark 1.2.8.4. The algebraic QFTs appearing above are of a special nature. First, they are defined functorially on every smooth curve X. Moreover, for $X = \mathbb{A}^1$, the resulting sheaves of categories are strongly $\mathbb{G}_a$-equivariant (cf. [But] Chapter 2). These observations are a sort of de Rham analogue of the fact that the A and B-twists are *topological* twists.

1.2.9. Below, we speak of both algebraic and non-algebraic (or *analytic*) theories. Typically, we speak of supersymmetry for the non-algebraic theories, and obtain algebraic theories by twisting.

1.2.10. *$3d$ mirror symmetry (first pass).* Given a $3d$ $\mathcal{N} = 4$ theory T, there is another $3d$ $\mathcal{N} = 4$ theory $T^\star$; this theory has the same underlying QFT as T, but the *embedding* of the supersymmetry algebra is modified by an automorphism of that algebra. The salient property for us is that the A-twist $T^\star_A$ of $T^\star$ is the B-twist T_B of the original theory. Moreover, $(T^\star)^\star = T$. We refer to $T^\star$ as the *abstract mirror dual* to the theory T.

1.2.11. We now describe a simplified version of $3d$ mirror symmetry conjectures.

These conjectures state that for certain *$3d$ mirror dual pairs* $(\mathcal{X}, \mathcal{X}^\star)$ of algebraic stacks, there is an equivalence:

$$T^\star_{\mathcal{X}} \simeq T_{\mathcal{X}^\star}$$

of $3d$ field theories.

For instance, passing to B-twists and categories of line operators on both sides, such a conjecture predicts an equivalence:

$$D(\mathfrak{L}\mathcal{X}) \simeq \mathsf{IndCoh}\big(\mathcal{M}aps(\mathring{\mathcal{D}}_{x,dR}, \mathcal{X}^\star)\big).$$

[10] At this point, we are operating heuristically and do not want to be overly prescriptive, so we do not consider finer points such as $D^!$ vs. D^*. Similarly in Example 1.2.8.2.

With that said, in our example, $D^!$ is what appears.

[11] As in [BD], it is convenient to think of $\mathcal{X}$ as $\mathcal{M}aps(\mathcal{D}_{dR}, \mathcal{X})$ here.

Example 1.2.11.1. The pair $(\mathcal{X}, \mathcal{X}^\star) = (\mathbb{A}^1, \mathbb{A}^1/\mathbb{G}_m)$ is supposed to be a $3d$ mirror dual pair of stacks. In this case, Theorem 1.1.0.1 amounts to an equivalence of line operators for the corresponding twisted[12] theories.

1.2.12. *S-duality.* The theory of $3d$ mirror symmetry as presented above admits a generalization in which there is an auxiliary pair $(G, \check{G})$ of Langlands dual reductive groups; the previous setting amounts to a pair of trivial groups.

The theory becomes much richer in this setting. There are connections to S-duality in $4d$ gauge theory and the geometric Langlands correspondence. Moreover, many of the examples in conventional (i.e., trivial group) $3d$ mirror symmetry can be better understood as being built from more primitive examples in the general setting.

The major cost is that even the formulation of the conjectures becomes conditional on forms of S-duality conjectures.

We discuss these ideas in more detail below.

1.2.13. First, we need to discuss $4d$ algebraic QFTs, which we denote T.

This is easy: the formalism is the same as in §1.2.6, but one categorical level higher. For instance, there should be a DG category attached to X, and a DG 2-category $\mathsf{T}(X)$ (of *surface defects*) attached to $\mathring{\mathcal{D}}_x$.

We can speak of *interfaces* $\mathsf{T}_1 \to \mathsf{T}_2$ between theories, which are morphisms in the natural sense. An interface $\mathsf{triv} \to \mathsf{T}$ from the trivial[13] theory is a *boundary condition* for T. This amounts to objects:

$$\mathcal{B}_X \in \mathsf{T}(X)(\in \mathsf{DGCat}_{cont})$$
$$\mathcal{B}_x \in \mathsf{T}(\mathring{\mathcal{D}}_x)(\in \mathsf{2DGCat}).$$

Remark 1.2.13.1. A boundary condition $\mathsf{triv} \to \mathsf{triv}$ is the same as a $3d$ theory.

Remark 1.2.13.2. Note that theories can be tensored. The $4d$ theories we consider are naturally self-dual. So we can (and do) consider interfaces and boundary conditions as operating in both directions. In such a situation, given boundary conditions $\mathcal{B}_1, \mathcal{B}_2$ for T, we let $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ denote the resulting boundary condition $\mathsf{triv} \to \mathsf{triv}$ obtained by composing $\mathsf{triv} \xrightarrow{\mathcal{B}_1} \mathsf{T} \xrightarrow{\mathcal{B}_2} \mathsf{triv}$ and applying Remark 1.2.13.1.

1.2.14. We now discuss twists.

For $4d$ $\mathcal{N} = 4$ (non-algebraic) theory T, there are supposed to be A and B twists T_A and T_B. Again, there is an involutive operation of *abstract mirror dual* $\mathsf{T}^\star$, exchanging A and B twists.

1.2.15. We now recall that in $4d$, for a reductive group G, there is Yang-Mills YM_G theory with $\mathcal{N} = 4$ supersymmetry. Again, there are A and B-twists with algebraic meaning.

Example 1.2.15.1. The A-twisted theory $\mathsf{YM}_{G,A}$ attaches to X the DG category $D(\mathrm{Bun}_G(X))$ of D-modules on $\mathrm{Bun}_G(X)$, and to $\mathring{\mathcal{D}}_x$ the DG 2-category $\mathfrak{L}G$–mod of DG categories with a (strong) action of the loop group $\mathfrak{L}G$ to $\mathring{\mathcal{D}}_x$. For the latter, the vacuum object is $D(\mathrm{Gr}_G)$ and the ("chiral homology") functional $\mathfrak{L}G$–$\mathsf{mod} \to \mathsf{Vect}$ is given by tensoring over $D^*(\mathfrak{L}G)$ with $D^*(\mathrm{Bun}_G^{level,x})$.

Example 1.2.15.2. The B-twisted theory $\mathsf{YM}_{G,B}$ attaches to X the DG category $\mathsf{QCoh}(\mathrm{LocSys}_G(X))$ of quasi-coherent sheaves on $\mathrm{LocSys}_G(X)$, and to $\mathring{\mathcal{D}}_x$ attaches the DG 2-category $\mathsf{ShvCat}_{/\mathrm{LocSys}_G(\mathring{\mathcal{D}}_x)}$.

[12]This may seem incomplete, given that $3d$ mirror symmetry is stated more symmetrically in terms of untwisted theories. However, as in Remark 1.2.3.1, the untwisted QFTs are on unsteady ground.

[13]This theory attaches Vect to X and DGCat_{cont} to $\mathring{\mathcal{D}}_x$.

1.2.16. Given a stack $\mathcal{X}$ with a G-action, there is a corresponding boundary condition $\mathcal{B}_{\mathcal{X}}$ for YM_G.

For $\mathcal{X} = G$, this boundary condition is called the *Dirichlet* boundary condition; we denote it by $\mathcal{D}ir_G$.

For $\mathcal{X} = \mathrm{Spec}(k)$, this boundary condition is called the *Neumann* boundary condition; we denote it by $\mathcal{N}eu_G$.

For any boundary condition $\mathcal{B}$ of YM_G, we let $T_{\mathcal{B}}$ denote the $3d$ theory $\langle \mathcal{B}, \mathcal{D}ir_G \rangle$. By standard compatibility of $3d$ and $4d$ supersymmetry algebras, any such $T_{\mathcal{B}}$ has $\mathcal{N} = 4$ supersymmetry.

For a $3d$ $\mathcal{N} = 4$ theory T, the data of *G-flavor symmetry* is the data of a boundary condition $\mathcal{B}$ for YM_G and an identification $T \simeq T_{\mathcal{B}}$. Similarly, *G-gauge symmetry* for T is the data of a boundary condition $\mathcal{B}$ and an identification $T \simeq \langle \mathcal{B}, \mathcal{N}eu_G \rangle$.

The above is compatible with twists; $\mathcal{B}$ gives a boundary condition $\mathcal{B}_A$ for the A-twist $\mathsf{YM}_{G,A}$, and similarly for B-twists. The resulting $3d$ (algebraic) theories are $T_{\mathcal{B},A}$ and $T_{\mathcal{B},B}$.

Remark 1.2.16.1. For $\mathcal{X}$ as above, the boundary condition $\mathcal{B}_{\mathcal{X},A}$ should define the object $D(\mathfrak{L}\mathfrak{X}) \in \mathfrak{L}G\text{–}\mathsf{mod}$, and the boundary condition $\mathcal{B}_{\mathcal{X},B}$ should define $\mathsf{IndCoh}(\mathcal{M}aps(\mathring{\mathcal{D}}_{x,dR}, \mathcal{X}/G)) \in \mathsf{ShvCat}_{/\mathrm{LocSys}_G(\mathring{\mathcal{D}})}$. Globally, there should be objects of $D(\mathrm{Bun}_G)$ and $\mathsf{QCoh}(\mathrm{LocSys}_G)$ as well. These objects are considered by Ben-Zvi, Sakellaridis, and Venkatesh, who term them *period/L-sheaves*, interpreting them as sheaf-theoretic analogues of period integrals/L-function values from harmonic analysis and number theory. They were also considered (away from singular loci) in [Gai1] §9-10.

Remark 1.2.16.2. Parallel to Remark 1.2.3.2, at least at the classical (non-quantum) level, the boundary condition $\mathcal{B}_{\mathcal{X}}$ can be defined more generally for Hamiltonian G-spaces, with $\mu : T^*\mathcal{X} \to \mathfrak{g}^\vee$ inducing $\mathcal{B}_{\mathcal{X}}$ (and again, something weaker than a suitable Lagrangian foliation should be needed to quantize).

1.2.17. In the above setting, *S-duality* is supposed to be an equivalence:

$$\mathsf{YM}_{\check{G}}^{\star} \simeq \mathsf{YM}_G$$

of $4d$ theories. Passing to B-twists, we obtain:

$$\mathsf{YM}_{G,A} \simeq \mathsf{YM}_{\check{G},B}.$$

Up to smearing[14] away homological algebra subtleties (cf. [AG]), this is a form of the geometric Langlands conjectures, encoding local and global theories and compatibilities between them (cf. [Gai2]).

The subsequent discussion is predicated on the existence of S-duality, so we assume it in what follows.

1.2.18. Given a boundary condition $\mathcal{B}$ for YM_G, we let $\mathcal{B}^\star$ denote the resulting *abstract mirror dual* boundary condition for $\mathsf{YM}_G^\star$, and then let $\check{\mathcal{B}}$ denote the resulting S-dual boundary condition for $\mathsf{YM}_{\check{G}}$.

1.2.19. *$3d$ mirror symmetry redux.* Now fix a reductive group G.

In general, a[15] $3d$ *mirror dual pair (for $(G, \check{G})$)* consists of a pair $(\mathcal{X}, \mathcal{X}^\star)$ where $G \curvearrowright \mathcal{X}$, $\check{G} \curvearrowright \mathcal{X}^\star$, and there is an equivalence:

$$\check{\mathcal{B}}_{\mathcal{X}} \simeq \mathcal{B}_{\mathcal{X}^\star}$$

[14]Throughout this exposition of the general ideas, we give ourselves the freedom of ignoring these subtleties. In the body of this paper (specifically, §2 and §4), we treat these technical points in detail in our context. The non-abelian future of the subject will of course similarly need to confront these points more seriously.

[15]Traditionally, *3d mirror symmetry* refers to the case where $G = \check{G} = \mathrm{triv}$. We find the present terminology convenient, although it departs somewhat from established conventions.

of boundary conditions for $\mathsf{YM}_{\check{G}}$. I.e., the boundary condition S-dual to $\mathcal{B}_{\mathcal{X}}$ should be $\mathcal{B}_{\mathcal{X}^\star}$. For G trivial, this recovers our earlier notion of $3d$ mirror dual pairs.

1.2.20. Suppose $(\mathcal{X}, \mathcal{X}^\star)$ as above. Passing to B-twists, we find that the objects:

$$D(\mathfrak{L}\mathcal{X}) \in \mathfrak{L}G\text{–}\mathsf{mod} \simeq \mathsf{ShvCat}_{/\operatorname{LocSys}_{\check{G}}(\mathring{\mathcal{D}})} \ni \mathsf{IndCoh}(\mathcal{M}aps(\mathring{\mathcal{D}}_{x,dR}, \mathcal{X}^\star/\check{G}))$$

are supposed to match, where the middle isomorphism is local geometric Langlands (considered modulo homological subtleties).

We remark that this does not amount to an equivalence of categories between the categories on the left and on the right here. Rather, this statement is inherently conditional on local geometric Langlands. (And we intend it to be a little fuzzy regarding homological subtleties.)

With that said, in the case that G is *abelian*, the above does amount to an equivalence of categories compatible with certain symmetries. (In general, the Whittaker category for the LHS above should be close to the RHS above.)

1.2.21. *Some examples of mirror dual pairs.* To illustrate the above, we briefly include some examples. For many other examples, we refer to [Wan], which is a table of examples maintained by Jonathan Wang. (The terminology in the first two examples is taken from Ben-Zvi, Sakellaridis, and Venkatesh, who are appealing to the analogy with harmonic analysis.)

Example 1.2.21.1 (Godement-Jacquet). For $G = \check{G} = GL(V) \times GL(V)$, the pair $(GL(V) \times V, \operatorname{End}(V))$ should be $3d$ mirror dual.

In particular, the objects:

$$D(\mathfrak{L}GL(V) \times \mathfrak{L}V) \in \mathfrak{L}GL(V) \times \mathfrak{L}GL(V)\text{–}\mathsf{mod}$$

should correspond to:

$$\mathsf{IndCoh}(\mathcal{M}aps(\mathring{\mathcal{D}}_{dR}, GL(V)\backslash \operatorname{End}(V)/GL(V)) \in \mathsf{ShvCat}_{/\operatorname{LocSys}_{GL(V)\times GL(V)}(\mathring{\mathcal{D}})}$$

under local geometric Langlands.

Example 1.2.21.2 (Tate). For $V = \mathbb{A}^1$, the above asserts that:

$$D(\mathfrak{L}\mathbb{G}_m \times \mathfrak{L}\mathbb{A}^1) \in \mathfrak{L}\mathbb{G}_m \times \mathfrak{L}\mathbb{G}_m\text{–}\mathsf{mod}$$

corresponds to:

$$\mathsf{IndCoh}(\mathcal{Y} \times \operatorname{LocSys}_{\mathbb{G}_m}).$$

As our group is abelian, we expect an honest equivalence of categories. Passing to $\mathfrak{L}\mathbb{G}_m$-invariants $\leftrightarrow$ {fiber at triv $\in \operatorname{LocSys}_{\mathbb{G}_m}$}, we obtain a conjecture:

$$D(\mathfrak{L}\mathbb{A}^1) \simeq \mathsf{IndCoh}(\mathcal{Y}).$$

This is our main result.

Example 1.2.21.3 (Gaiotto-Witten [GW], §3.3.1). Let $G = \check{G} = GL(V)$ for $\dim(V) = n$. Let $\mathcal{L}au$ be the (Laumon) moduli space of data $(V_1, V_2, \ldots, V_{n-1}, f_1, \ldots, f_{n-1})$ where V_i is a vector space of dimension i and $f_i : V_i \to V_{i+1}$ is a morphism, where by definition, $V_n = V$.

Then the pair:

$$(\mathcal{X}, \mathcal{X}^\star) = (GL(V), \mathcal{L}au)$$

is expected to be $3d$ mirror dual (with respect to our $(G, \check{G})$).

Unwinding, this in effect gives a sort of formula (or kernel) for geometric Langlands for GL_n; unfortunately, the necessary symmetries on the resulting object are not apparent.

1.2.22. *Where do mirror dual pairs come from?* We briefly address the stated question. There are two styles of answer.

First, one can try to reverse engineer examples. If one believes a mirror dual $\mathcal{X}^\star$ to some $\mathcal{X}$ should exist, one can sometimes calculate the ring of functions on $T^*\mathcal{X}^\star$ as the Coulomb branch[16] of $\mathcal{X}^\star$, which must correspond to the Higgs branch[17] of $\mathcal{X}$.

Alternatively, as in the work of Ben-Zvi, Sakellaridis, and Venkatesh, one can reverse engineer examples by fitting phenomena from number theory into the above framework via the analogy between automorphic forms and automorphic D-modules.

But sometimes (always?) there is a better answer than either. Many $3d$ mirror symmetry examples can be derived from (super)string theory dualities and the existence of $\mathcal{M}$-theory. (For example, see [GW] §3.1.2 for the derivation in our example.) We are not aware of a mathematical counterpart to this idea, i.e., a simple conjectural framework that subsumes all examples in $3d$ mirror symmetry. Clearly this would be highly desirable.

Remark 1.2.22.1. Forthcoming work of the first author and Philsang Yoo will develop in detail the mathematical derivation of mirror pairs from S-duality in string theory.

1.2.23. *The role of this paper.* Our objective in writing this paper was to test the above ideas in the simplest[18] case of interest, which is the Tate case discussed above. Here $(G, \check{G}) = (\mathbb{G}_m, \mathbb{G}_m)$ and[19] $(\mathcal{X}, \mathcal{X}^\star) = (\mathbb{A}^1, \mathbb{A}^1)$.

As we consider abelian gauge groups, for which geometric Langlands is unconditional, this example can be studied in complete detail. We perform this study at the level of categories of line operators. We will return to global considerations (compatibility with chiral homology) in future work.

Given that we obtain complete positive results in this case, and because many of the technical difficulties inherent in the $3d$ mirror symmetry project occur in our example, we believe that our results provide strong support for the general conjectures.

1.2.24. *Some references.* Many of the ideas discussed above were developed in collaborative work, often unpublished, of a number of mathematical physicists. We are grateful to many people who have shared these ideas with us over the years, and find their intellectual generosity inspiring. The downside of this situation is that we find some difficulty in accurately attributing priority for the ideas. We do our best here, but apologize in advance for any omissions or inaccuracies, which are not intended as slights.

The first instances of $3d$ mirror symmetry were considered in [IS]. The connection with string theory dualities was made in [HW]. These ideas were developed further in [GW], which considered interactions between the [HW] constructions and S-duality for super Yang-Mills. In turn, [Gai1] translated those ideas into conjectures regarding geometric Langlands via the Kapustin-Witten dictionary [KW] between Langlands and S-duality. The physics literature on BPS line operators is too vast to survey here, but we refer the reader to [DGGH] for a review of the literature in the $3d$ $\mathcal{N} = 4$ context.

The algebro-geometric interpretation of the A-side discussed above is suggested by work of Braverman, Finkelberg and Nakajima in [Nak], [BFN1], [BFN2], which emphasized the role of

[16] I.e., total cohomology of endomorphisms of the unit object in $T_{\mathcal{X}^\star,B} = \mathsf{IndCoh}(\mathcal{M}aps(\mathring{\mathcal{D}}_{dR}, \mathcal{X}))$.

[17] I.e., total cohomology of endomorphisms of the unit object in $T_{\mathcal{X},A} = D(\mathfrak{L}\mathcal{X})$.

[18] Not covered by geometric Langlands/S-duality for YM, and in which the full weight of the infinite dimensional geometry appears.

[19] The discrepancy here with the ungauged version considered in Example 1.2.11.1 is explained by the fact that $\mathcal{D}ir$ and $\mathcal{N}eu$ are S-dual for tori.

Coulomb branches. The description of categories of line operators in twisted $3d$ $\mathcal{N} = 4$ theories was given in unpublished work of Kevin Costello, Tudor Dimofte, Davide Gaiotto, Philsang Yoo, and the first author.[20] The earliest derivation was based on applying the method of Elliott-Yoo [EY] in the $3d$ $\mathcal{N} = 4$ context, i.e., calculating (derived) moduli spaces of solutions to Euler-Lagrange equations and quantizing (in the shifted symplectic sense).[21] By applying the yoga of $3d$ mirror symmetry, this description of categories of line operators also led to a conjectural form of Theorem 1.1.0.1 (and other related examples).[22] Philsang Yoo and the first author considered these line operator categories in the framework of local geometric Langlands as a mathematical interpretation of [GW] (inspired in part by [BFN3]).[23] On the A-side, an alternative derivation of the category of line operators is suggested by [CG1] §4; here the method is as described in Remark 1.2.6.6.

Connections between $3d$ mirror symmetry and number theory (e.g., duality for spherical varieties and period integrals [SV]) were developed by Ben-Zvi, Sakellaridis, and Venkatesh, in forthcoming[24] work, which has led to profound new insights and many new examples of dual pairs.

Finally, in addition to the above works, we have drawn inspiration particularly from [BF], [BLPW1], [BLPW2] [BPW], [BZN], [BZGN], [KRS], [KSV], and [Tel] in our thinking about mirror symmetry.

1.2.25. *Some related works.* We also highlight some works that are closely related to ours.

First, [CCG] studies a version of our problem near the formal completion of $0 \in \mathcal{Y}$ (i.e., Theorem 1.1.0.1 *in perturbation theory*) using VOAs; in their picture (based on the method of Remark 1.2.6.6), the Lie algebra controlling the deformation theory is a $\mathfrak{psu}(1|1)$ Kac-Moody algebra. We understand that these ideas are currently being developed further by Andrew Ballin, Thomas Creutzig, Tudor Dimofte, and Wenjun Niu.

Second, the works [BFGT] and [BFT] of Braverman-Finkelberg-Ginzburg-Travkin also obtained geometric results from mirror symmetry conjectures, but in the non-abelian setting. There it is not possible at present to prove that boundary conditions match under S-duality, since local geometric Langlands is only conjectural for non-abelian G. Therefore, the cited authors instead consider the categories obtained by pairing the vacuum boundary condition(s) for $\mathsf{YM}_G/\mathsf{YM}_{\check{G}}$, and establish the resulting conjectures for certain examples of $3d$ mirror dual pairs $(G, \check{G})$ and $(\mathcal{X}, \mathcal{X}^\star)$.

1.3. **Outline of the paper.**

1.3.1. We now describe the contents of the present work.

1.3.2. First, we describe the general strategy.

By definition, there are fully faithful functors:

$$D(\mathfrak{L}_n^+\mathbb{A}^1) \hookrightarrow D^!(\mathfrak{L}^+\mathbb{A}^1) \hookrightarrow D^!(\mathfrak{L}\mathbb{A}^1).$$

Here $\mathfrak{L}_n^+\mathbb{A}^1$ is the (scheme corresponding to) the vector space $k[[t]]/t^n k[[t]]$; $\mathfrak{L}^+\mathbb{A}^1$ is the (scheme corresponding to the pro-finite dimensional) vector space $k[[t]]$, and $\mathfrak{L}\mathbb{A}^1$ is the (indscheme corresponding to the Tate) vector space $k((t))$. The first functor is a !-pullback and the second functor is a (suitably normalized) $*$-pushforward functor.

[20] These ideas were recorded in §1.1 of [DGGH] and in [BF] §7.

[21] Because we are not aware of a publicly available account of this derivation, we direct the reader to [Cos2], which is a series of talks Kevin Costello gave in 2014 that discuss some of the main ingredients. It is the earliest talk we know of that contains these components.

[22] For a physics-oriented discussion of this work, see [Dim].

[23] See [Yoo] and [Hil] for discussions of this work.

[24] The work is also highly publicized – there are recorded lectures about the work readily available online (see e.g. [BZ]).

The main step in our construction is the construction of corresponding functors:

$$\Delta_n : D(\mathfrak{L}_n^+\mathbb{A}^1) \to \mathsf{IndCoh}^*(\mathcal{Y}).$$

As the left hand side is modules over a Weyl algebra, this amounts to constructing an object of the right hand side with an action of a Weyl algebra. We construct this object explicitly, via generators and relations.

From there, we bootstrap up to construct a functor Δ as in Theorem 1.1.0.1. We show it is an equivalence using the geometry of $\mathcal{Y}$.

Remark 1.3.2.1. There is an adage in geometric Langlands that most of the work occurs on the geometric (i.e., automorphic) side. In our work it is the opposite: the geometric side is easy, and most of our work is on the spectral side.

1.3.3. We introduce $\mathcal{Y}$ (and related spaces) in §2. Here we give the main geometric tools in our study of $\mathcal{Y}$. We study ind-coherent sheaves on $\mathcal{Y}$ in §4, after some preliminary remarks about Abel-Jacobi maps in §3. We construct the functors Δ_n in §5; as indicated above, this is our main construction. We check the corresponding compatibility of this construction with class field theory in §6. We then show fully faithfulness of Δ_n in §7. We then prove Theorem 1.1.0.1 in §8.

1.4. **Categorical conventions.** At various points, we use homotopical algebra and derived algebraic geometry in Lurie's higher categorical form, cf. [Lur1], [Lur3], [Lur4], though our notation more closely follows the conventions of [GR2].

In general, our terminology should be assumed to be derived. We refer to $(\infty, 1)$-categories as *categories* and ∞-groupoids as *groupoids*. We let Gpd denote the category of groupoids.

We let DGCat_{cont} denote the category of cocomplete[25] DG categories (over k) and continuous DG functors, cf. [GR2] §I.1.10. We let $\otimes$ denote its standard (*Lurie*) tensor product. We let $\mathsf{Vect} \in \mathsf{DGCat}_{cont}$ denote the DG category of (chain complexes of) vector spaces. For a DG algebra A/k, we let A–mod denote the corresponding DG category. For a DG category $\mathcal{C}$ and $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, we let $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \in \mathsf{Vect}$ denote the complex of maps from $\mathcal{F}$ to $\mathcal{G}$.

For a DG category $\mathcal{C}$ with a t-structure, we use cohomological indexing conventions and let $\mathcal{C}^{\leqslant 0}$ denote the connective objects, $\mathcal{C}^{\geqslant 0}$ denote the coconnective objects, and let $\mathcal{C}^{\heartsuit} := \mathcal{C}^{\leqslant 0} \cap \mathcal{C}^{\geqslant 0}$ denote the heart of the t-structure. Similarly, we let $\mathcal{C}^+$ denote the eventually coconnective objects.

By default, *schemes* are DG[26] schemes over k. We let AffSch denote the category of affine schemes (over k), and Sch the category of schemes (over k).

When we wish to refer to consider various non-derived objects as derived objects, we use the term *classical*. So we may speak of classical schemes, classical rings, classical vector spaces, classical A-modules, etc., all of which are full subcategories of the corresponding (suitably) derived categories.

Finally, we sometimes use standard ideas and notation from the theory of categories with G-actions. We refer to [Ber] for an introduction to these ideas.

1.5. **Acknowledgements.** We thank Sasha Beilinson, David Ben-Zvi, Dario Beraldo, Mat Bullimore, Dylan Butson, Sasha Braverman, Justin Campbell, Kevin Costello, Gurbir Dhillon, Tudor Dimofte, Chris Elliott, Tony Feng, Davide Gaiotto, Nik Garner, Dennis Gaitsgory, Sam Gunningham, Joel Kamnitzer, Ivan Mirkovic, Tom Nevins, Wenjun Niu, Pavel Safronov, Yiannis Sakellaridis, Ben Webster, Jonathan Wang, Alex Weekes, and Philsang Yoo for many helpful and inspiring

[25]Rather, presentable.

[26]It will turn out a posteriori that the primary objects we consider are classical schemes (or stacks). But because we are studying derived categories of coherent sheaves, the machinery and perspective of derived algebraic geometry is quite convenient.

conversations related to this work. The first author particularly wishes to acknowledge and thank Philsang Yoo for joint work conjecturing Theorem 1.1.0.1.

J.H. is part of the Simons Collaboration on Homological Mirror Symmetry supported by Simons Grant 390287. S.R. was supported by NSF grant DMS-2101984. This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Research, Innovation and Science.

2. Geometry of the spectral side

2.1. **Overview.** In this section, we define $\mathcal{Y}$ and establish our main algebro-geometric tools for studying its coherent sheaves.

Here is a more detailed overview of this section.

In §2.2, we give background on jet and loop spaces, essentially setting up notation for later use. In §2.3, we give background on $\mathrm{LocSys}_{\mathbb{G}_m}$. In §2.4, we define $\mathcal{Y}$. In §2.5, we introduce variants of $\mathcal{Y}$ that are more traditional objects of algebraic geometry. In §2.6, we prove a series of flatness results; these are the key tools for studying $\mathcal{Y}$ and its coherent sheaves.

The remainder of the section consists of a series of codas, essentially developing material and notation for later reference. These may be skipped at first pass and referred to as needed. In §2.7, we introduce additional notation for later reference. In §2.8, we deduce some exact sequences from our flatness results; these will be used later in some inductive arguments. In §2.9, we describe certain well-behaved opens in $\mathcal{Z}^{\leqslant n}$. In §2.10, we explicitly describe $\mathcal{Y}$ using generators and relations. Finally, in §2.11, we draw pictures explicitly describing the geometry in the regular singular situation.

2.2. Notation for jet and loop spaces.

2.2.1. For a commutative DG algebra $A \in \mathsf{ComAlg}$, $A[[t]] \in \mathsf{ComAlg}$ is defined as $(\lim_n A \otimes k[[t]]/t^n)$ and $A((t))$ is defined as $A[[t]] \otimes_{k[[t]]} k((t))$. We also let $A[[t]]/t^n := A \otimes_k k[[t]]/t^n$.

2.2.2. For a prestack Z, $\mathfrak{L}Z \in \mathsf{PreStk} := \mathsf{Hom}(\mathsf{AffSch}^{op}, \mathsf{Gpd})$ denote the functor:

$$\mathrm{Spec}(A) \mapsto \mathrm{Hom}_{\mathsf{AffSch}}(\mathrm{Spec}(A((t))), Z).$$

Similarly, we let $\mathfrak{L}^+Z$ denote the functor:

$$\mathrm{Spec}(A) \mapsto \mathrm{Hom}_{\mathsf{AffSch}}(\mathrm{Spec}(A[[t]]), Z).$$

For $n \geqslant 0$, we let $\mathfrak{L}_n^+$ denote the functor:

$$\mathrm{Spec}(A) \mapsto \mathrm{Hom}_{\mathsf{AffSch}}(\mathrm{Spec}(A[[t]]/t^n), Z).$$

There is an evident comparison map:

$$\varepsilon_Z : \mathfrak{L}^+Z \to \lim_n \mathfrak{L}_n^+ Z.$$

We often use the following lemma without mention.

Lemma 2.2.2.1. *For Z an algebraic stack of the form Y/G with Y affine and G an affine algebraic group, the map κ_Z is an isomorphism.*

Proof. The result is obvious when $Z = Y$ is affine. For $Z = \mathbb{B}G$, we refer e.g. to [Ras5] Lemma 2.12.1. The argument from *loc. cit.* immediately extends to the general form of Y considered here. □

Remark 2.2.2.2. In fact, this result is true much more generally for Noetherian algebraic stacks with affine diagonals: see [BHL] Corollary 1.5.

Notation 2.2.2.3. We often let $\mathrm{ev} : \mathfrak{L}^+Z \to \mathfrak{L}^+_1 Z = Z$ denote the evaluation map.

2.2.3. We have the following basic representability result.

Proposition 2.2.3.1. *Suppose Z is an affine scheme almost of finite type. Then $\mathfrak{L}^+Z$ is an affine scheme and $\mathfrak{L}Z$ is an indscheme.*

2.2.4. We have the following variant of the above.

Suppose Z is equipped with a $\mathbb{G}_m$-action. We let $\mathfrak{L}Zdt \in \mathsf{PreStk}$ denote the functor:

$$\mathfrak{L}Zdt := \mathfrak{L}(Z/\mathbb{G}_m) \underset{\mathfrak{L}\mathbb{B}\mathbb{G}_m}{\times} \mathrm{Spec}(k)$$

where $\mathrm{Spec}(k) \to \mathfrak{L}\mathbb{B}\mathbb{G}_m$ corresponds to the (continuous) tangent sheaf $T_{\mathring{\mathcal{D}}}$ on $\mathring{\mathcal{D}}$. We define $\mathfrak{L}^+Zdt$ similarly. A choice of trivialization of $T_{\mathring{\mathcal{D}}}$ defines an isomorphism $\mathfrak{L}Zdt \simeq \mathfrak{L}Z$ identifying $\mathfrak{L}^+Zdt$ and $\mathfrak{L}^+Z$; in particular, Proposition 2.2.3.1 applies in this setting.

Informally, $\mathfrak{L}Zdt$ parametrizes sections of the fiber bundle $\mathring{\Theta}(\Omega^1_{\mathring{\mathcal{D}}}) \overset{\mathbb{G}_m}{\times} Z \to \mathring{\mathcal{D}}$, where $\Omega^1_{\mathring{\mathcal{D}}}$ is the line bundle of (continuous) 1-forms on $\mathring{\mathcal{D}}$ and $\Theta(-)$ (resp. $\mathring{\Theta}$) denotes the (resp. punctured) total space of a line bundle.

Example 2.2.4.1. $\mathfrak{L}\mathbb{A}^1$ is the algebro-geometric version of the space of Laurent series, while $\mathfrak{L}\mathbb{A}^1dt$ is the algebro-geometric verison of the space of 1-forms on the punctured disc (for the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$ by homotheties). Similarly, $\mathfrak{L}\mathbb{G}_m$ is the algebro-geometric version of the space of invertible Laurent series. One easily finds that these indschemes are formally smooth and classical.

2.3. **Rank 1 local systems.** We now give a detailed study of $\mathrm{LocSys}_{\mathbb{G}_m}$, the moduli space of rank 1 local systems on the punctured disc. This material is well-known, but it is convenient to review to introduce notation and constructions we will need in the more complicated setting of our space $\mathcal{Y}$.

2.3.1. We define $\mathrm{LocSys}_{\mathbb{G}_m}$ as follows.

We have a map $\mathfrak{L}\mathbb{G}_m \to \mathfrak{L}\mathbb{A}^1dt$ sending $f \in \mathfrak{L}\mathbb{G}_m$ to $d\log(f)$, cf. [Ras4] §1.12. This map is a map of (classical) group indschemes, where $\mathfrak{L}\mathbb{A}^1dt$ is given its natural additive structure.

We define $\mathrm{LocSys}_{\mathbb{G}_m}$ as the stack[27] quotient $\mathfrak{L}\mathbb{A}^1dt/\mathfrak{L}\mathbb{G}_m$.

Remark 2.3.1.1. For normalization purposes, we remark that for a point $\omega \in \mathfrak{L}\mathbb{A}^1dt$, we consider $\nabla = d - \omega$ as the corresponding rank 1 connection (on the trivial line bundle).

2.3.2. We need variants of $\mathrm{LocSys}_{\mathbb{G}_m}$ as well.

Define:

$$\mathrm{LocSys}_{\mathbb{G}_m,\log} := \mathrm{LocSys}_{\mathbb{G}_m} \underset{\mathfrak{L}\mathbb{B}\mathbb{G}_m}{\times} \mathfrak{L}^+\mathbb{B}\mathbb{G}_m = \mathfrak{L}\mathbb{A}^1dt/\mathfrak{L}^+\mathbb{G}_m.$$

This is the moduli space of line bundles on the disc with a connection on the punctured disc.

There is an evident action of $\mathrm{Gr}_{\mathbb{G}_m} := \mathfrak{L}\mathbb{G}_m/\mathfrak{L}^+\mathbb{G}_m$ on $\mathrm{LocSys}_{\mathbb{G}_m,\log}$ such that $\mathrm{LocSys}_{\mathbb{G}_m} = \mathrm{LocSys}_{\mathbb{G}_m,\log}/\mathrm{Gr}_{\mathbb{G}_m}$.

[27] I.e., we sheafify for the fppf topology. It is equivalent to sheafify for the Zariski topology as $\mathrm{Ker}(\mathfrak{L}^+\mathbb{G}_m \to \mathbb{G}_m)$ is pro-unipotent. For the same reason, the resulting prestack is a sheaf for the fpqc topology. In other words, there is no room for ambiguity in sheafifying.

2.3.3. Define $\mathfrak{L}^{pol}\mathbb{A}^1 dt$ as the quotient $\mathfrak{L}\mathbb{A}^1 dt/\mathfrak{L}^+\mathbb{A}^1 dt$, the quotient being with respect to the additive structure. In other words, $\mathfrak{L}^{pol}\mathbb{A}^1 dt$ parametrizes polar parts of differential forms on the disc. Note that $\mathfrak{L}^{pol}\mathbb{A}^1 dt$ is an indscheme of ind-finite type.

As $d\log$ maps $\mathfrak{L}^+\mathbb{G}_m$ into $\mathfrak{L}^+\mathbb{A}^1 dt$, the projection map:

$$\mathfrak{L}\mathbb{A}^1 dt \to \mathfrak{L}^{pol}\mathbb{A}^1 dt$$

intertwines the gauge action of $\mathfrak{L}^+\mathbb{G}_m$ on the left hand side with its trivial action on the right hand side. Therefore, we obtain a canonical map:

$$\mathrm{Pol} : \mathrm{LocSys}_{\mathbb{G}_m,\log} \to \mathfrak{L}^{pol}\mathbb{A}^1 dt.$$

This map takes the polar part of a connection.

2.3.4. By construction, there is a map:

$$\mathrm{LocSys}_{\mathbb{G}_m,\log} \to \mathbb{B}\mathfrak{L}^+\mathbb{G}_m \xrightarrow{\mathrm{ev}} \mathbb{B}\mathbb{G}_m.$$

This map sends a pair $(\mathcal{L}, \nabla)$ (with $\mathcal{L}$ a line bundle on the disc and ∇ its connection on the punctured disc) to the fiber of $\mathcal{L}$ at the origin.

The following result is well-known.

Proposition 2.3.4.1. *The map:*

$$\mathrm{LocSys}_{\mathbb{G}_m,\log} \to \mathfrak{L}^{pol}\mathbb{A}^1 dt \times \mathbb{B}\mathbb{G}_m$$

is an isomorphism.

Proof. The map $\mathfrak{L}^+\mathbb{G}_m \xrightarrow{d\log,\mathrm{ev}} \mathfrak{L}^+\mathbb{A}^1 dt \times \mathbb{G}_m$ is easily seen to be an isomorphism. The result is immediate from here.

□

2.3.5. We now deduce a similar description of $\mathrm{LocSys}_{\mathbb{G}_m}$ from Proposition 2.3.4.1.

There is a canonical residue map $\mathrm{Res} : \mathfrak{L}^{pol}\mathbb{A}^1 dt \to \mathbb{A}^1$. There is a canonical projection $\mathfrak{L}^{pol}\mathbb{A}^1 dt \to \mathrm{Ker}(\mathrm{Res})$ as the latter identifies with $\mathfrak{L}\mathbb{A}^1 dt/t^{-1} \cdot \mathfrak{L}^+\mathbb{A}^1 dt$, so we obtain a product decomposition:

$$\mathfrak{L}^{pol}\mathbb{A}^1 dt = \mathbb{A}^1 \times \mathrm{Ker}(\mathrm{Res}).$$

Proposition 2.3.5.1. *There is an isomorphism* $\mathrm{LocSys}_{\mathbb{G}_m} \simeq \mathbb{A}^1/\mathbb{Z} \times \mathrm{Ker}(\mathrm{Res})_{dR} \times \mathbb{B}\mathbb{G}_m$ *fitting into a commutative diagram:*

$$\begin{array}{ccc} \mathrm{LocSys}_{\mathbb{G}_m,\log} & \xrightarrow{\text{Prop. 2.3.4.1}} & \mathfrak{L}^{pol}\mathbb{A}^1 dt \times \mathbb{B}\mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathrm{LocSys}_{\mathbb{G}_m} & \xrightarrow{\simeq} & \mathbb{A}^1/\mathbb{Z} \times \mathrm{Ker}(\mathrm{Res})_{dR} \times \mathbb{B}\mathbb{G}_m \end{array}$$

Proof. There is a canonical homomorphism $\mathrm{Gr}_{\mathbb{G}_m} \to \mathbb{Z}$ given by minus[28] the valuation map. This map splits canonically as well: $\mathbb{Z} = \mathrm{Gr}_{\mathbb{G}_m}^{red}$. Therefore, we obtain a canonical product decomposition:

$$\mathrm{Gr}_{\mathbb{G}_m}^{red} = \mathbb{Z} \times \mathrm{Gr}_{\mathbb{G}_m}^{\circ}$$

for $\mathrm{Gr}_{\mathbb{G}_m}^{\circ}$ the connected component of the identity. Here we consider $\mathbb{Z}$ as a discrete indscheme over k in the natural way, i.e., as $\coprod_{n\in\mathbb{Z}} \mathrm{Spec}(k)$.

[28] This sign is included to match with standard normalizations.

We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Gr}_{\mathbb{G}_m} & \xrightarrow{d\log} & \mathfrak{L}^{pol}\mathbb{A}^1 dt \\ \| & & \| \\ \mathbb{Z} \times \mathrm{Gr}^{\circ}_{\mathbb{G}_m} & \longrightarrow & \mathbb{A}^1 \times \mathrm{Ker}(\mathrm{Res}) \end{array}$$

where the bottom map is obtained as the product of the homomorphisms $\mathbb{Z} \hookrightarrow \mathbb{A}^1$ and $d\log : \mathrm{Gr}^{\circ}_{\mathbb{G}_m} \to \mathrm{Ker}(\mathrm{Res})$. The latter is easily seen to induce an isomorphism between $\mathrm{Gr}^{\circ}_{\mathbb{G}_m}$ and the formal group $\mathrm{Ker}(\mathrm{Res})^{\wedge}_0$ of Ker(Res). This gives the claim.

□

Remark 2.3.5.2. This isomorphism actually depends mildly on the choice of uniformizer t; this is needed to trivialize the action of $\mathrm{Gr}_{\mathbb{G}_m}$ on the $\mathbb{B}\mathbb{G}_m$-factor.

2.3.6. We will also use *truncated* versions of the above spaces in which we bound the irregularity of our local systems.

We fix $n \in \mathbb{Z}^{\geqslant 0}$ in what follows.

2.3.7. Define $\mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt \subseteq \mathfrak{L}\mathbb{A}^1 dt$ as the classical closed subscheme whose points are differential forms with poles of order at most n. Therefore, $\mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt$ is the image of $\mathfrak{L}^+\mathbb{A}^1 dt$ under the map $t^{-n} \cdot - : \mathfrak{L}\mathbb{A}^1 dt \to \mathfrak{L}\mathbb{A}^1 dt$.

We similarly define $\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt$ as $\mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt/\mathfrak{L}^+\mathbb{A}^1 dt$. For $n > 0$, we define $\mathrm{Ker}(\mathrm{Res})^{\leqslant n}$ as $\mathrm{Ker}(\mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt \to \mathbb{A}^1)$.

2.3.8. For $n > 0$, define $\mathfrak{L}^{\leqslant n}\mathbb{G}_m$ as the fiber product:

$$\mathfrak{L}^{\leqslant n}\mathbb{G}_m \underset{\mathfrak{L}\mathbb{A}^1 dt}{\times} \mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt$$

where we are using $d\log : \mathfrak{L}\mathbb{G}_m \to \mathfrak{L}\mathbb{A}^1 dt$.

For[29] $n = 0$, define $\mathfrak{L}^{\leqslant 0}\mathbb{G}_m$ as $\mathfrak{L}^+\mathbb{G}_m$.

Finally, define $\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m}$ as $\mathfrak{L}^{\leqslant n}\mathbb{G}_m/\mathfrak{L}^+\mathbb{G}_m$.

Lemma 2.3.8.1. *$\mathfrak{L}^{\leqslant n}\mathbb{G}_m$ is a formally smooth classical indscheme.*

Proof. It suffices to show the same for $\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m}$.

As in the proof of Proposition 2.3.5.1, the map $\mathrm{Gr}_{\mathbb{G}_m} \xrightarrow{d\log} \mathfrak{L}\mathbb{A}^1 dt \to \mathrm{Ker}(\mathrm{Res})$ induces an isomorphism:

$$\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m} \xrightarrow{\simeq} \mathbb{Z} \times (\mathrm{Ker}(\mathrm{Res})^{\leqslant n})^{\wedge}_0.$$

Here $\mathrm{Ker}(\mathrm{Res})^{\leqslant n}$ is defined as the kernel of the residue on polar forms with poles of order $\leqslant n$. As the latter is an affine space, its formal completion at the origin is formally smooth and classical. This gives the claim.

□

[29] We separate the cases to be derivedly correct.

2.3.9. Next, define $\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m}$ as the quotient $\mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt/\mathfrak{L}^{\leqslant n}\mathbb{G}_m$ under the gauge action. We similarly define $\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}$ as $\mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt/\mathfrak{L}^{+}\mathbb{G}_m$. By definition, we have a Cartesian diagram:

Proposition 2.3.9.1. *The isomorphisms from Propositions 2.3.4.1 and 2.3.5.1 induce further isomorphisms:*

$$\begin{array}{ccc} \mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log} & \xrightarrow{\simeq} & \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt \times \mathbb{B}\mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathrm{LocSys}_{\mathbb{G}_m,\log} & \xrightarrow{\simeq} & \mathfrak{L}^{pol}\mathbb{A}^1 dt \times \mathbb{B}\mathbb{G}_m \end{array}$$

and:

$$\begin{array}{ccc} \mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m} & \xrightarrow{\simeq} & \mathbb{A}^1/\mathbb{Z} \times \mathrm{Ker}(\mathrm{Res})^{\leqslant n}_{dR} \times \mathbb{B}\mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathrm{LocSys}_{\mathbb{G}_m} & \xrightarrow{\simeq} & \mathbb{A}^1/\mathbb{Z} \times \mathrm{Ker}(\mathrm{Res})_{dR} \times \mathbb{B}\mathbb{G}_m. \end{array}$$

This result is clear from our earlier analysis.

2.4. Definition of $\mathcal{Y}$.

2.4.1. We define $\mathcal{Y}$, the moduli of rank 1 de Rham local systems on $\mathring{\mathcal{D}}$ with a flat section, as follows.

First, observe that we have a map $d : \mathfrak{L}\mathbb{A}^1 \to \mathfrak{L}\mathbb{A}^1 dt$ defined by the exterior derivative on $\mathring{\mathcal{D}}$. We also have a product map $\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt \to \mathfrak{L}\mathbb{A}^1 dt$ coming from the product $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$.

Let $\mathcal{Y}'$ denote the equalizer:

$$\mathcal{Y}' := \mathrm{Eq}\left(\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt \underset{(g,\omega)\mapsto g\omega}{\overset{d\circ\pi_1}{\rightrightarrows}} \mathfrak{L}\mathbb{A}^1 dt\right) \in \mathsf{PreStk}$$

We emphasize that this is a *derived* equalizer; although the terms appearing are classical indschemes, this equalizer is a priori a DG indscheme.

There is a canonical $\mathfrak{L}\mathbb{G}_m$-action on $\mathcal{Y}'$, heuristically given by the formula:

$$(f \in \mathfrak{L}\mathbb{G}_m, (g,\omega) \in \mathcal{Y}') \mapsto (fg, \omega + d\log(f)).$$

We will define this action more rigorously below, but in the meantime, we define:

$$\mathcal{Y} := \mathcal{Y}'/\mathfrak{L}\mathbb{G}_m.$$

Remark 2.4.1.1. Let us explain the above formulae.

Note that $\mathcal{Y}'$ by definition parametrizes pairs $(g \in \mathfrak{L}\mathbb{A}^1, \omega \in \mathfrak{L}\mathbb{A}^1 dt)$ with $dg = g\omega$, which we can rewrite as $\nabla g = 0$ for $\nabla := d - \omega$. This data amounts to a connection on the trivial bundle on $\mathring{\mathcal{D}}$ (corresponding to ω) and a flat section (corresponding to g).

Quotienting by $\mathfrak{L}\mathbb{G}_m$ amounts to modding out by gauge transformation, i.e., not fixing a trivialization on our line bundle on $\mathring{\mathcal{D}}$.

2.4.2. Above, we did not define the action of $\mathfrak{L}\mathbb{G}_m$ on $\mathcal{Y}'$ completely rigorously: as $\mathcal{Y}'$ is a priori DG, such formulae are not sufficient. (In fact, using Theorem 2.4.3.1, $\mathcal{Y}'$ is classical, and the implicit anxiety here is not needed.)

Here are two approaches.

First, in [Gai4], Gaitsgory defines a prestack $\mathcal{M}aps(\mathring{\mathcal{D}}_{dR}, Y)$ for any prestack Y. Taking $Y = \mathbb{A}^1/\mathbb{G}_m$ then immediately gives the definition of $\mathcal{Y}$ as $\mathcal{M}aps(\mathring{\mathcal{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m)$. (One readily finds

$\mathcal{M}aps(\mathring{\mathcal{D}}_{dR}, \mathbb{B}\mathbb{G}_m) = \mathrm{LocSys}_{\mathbb{G}_m}$ and $\mathcal{Y}' = \mathcal{M}aps(\mathring{\mathcal{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m) \times_{\mathcal{M}aps(\mathring{\mathcal{D}}_{dR},\mathbb{B}\mathbb{G}_m)} \Omega^1_{\mathbb{G}_m}$, consistent with our constructions.)

We prefer an alternative approach that we find more explicit, and that we describe in full detail here. The reader who is not concerned about homotopical details (which are not serious anyway by Theorem 2.4.3.1) may skip this material.

Below, we make various constructions with $\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt$. Here we may just work with formulae as this indscheme is classical.

We consider two monoid structures on $\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt$. The first has product:

$$(g, \omega) \bullet (\tilde{g}, \tilde{\omega}) := (g\tilde{g}, \omega + \tilde{\omega})$$

while the second has product:

$$(g, \omega) \star (\tilde{g}, \tilde{\omega}) := (g\tilde{g}, g\tilde{\omega} + \tilde{g}\omega).$$

Moreover, the map:

$$\mu : \mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt \to \mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt$$
$$(g, \omega) \mapsto (g, dg - g\omega)$$

is a map of monoids $(\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt, \bullet) \to (\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt, \star)$.

We obtain a commutative diagram of maps of monoids:

$$\begin{array}{ccc} \mathfrak{L}\mathbb{G}_m & \xrightarrow{f \mapsto (f, d\log(f))} & (\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt, \bullet) \\ \downarrow & & \downarrow \mu \\ * & \longrightarrow & (\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt, \star) \end{array}$$

Therefore, we obtain a canonical map of monoids:

$$\mathfrak{L}\mathbb{G}_m \to (\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt, \bullet) \underset{(\mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt, \star)}{\times} \mathfrak{L}\mathbb{A}^1$$

where the previously unconsidered map is this fiber product is $\mathfrak{L}\mathbb{A}^1 \xrightarrow{(\mathrm{id},0)} \mathfrak{L}\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt$; the source $\mathfrak{L}\mathbb{A}^1$ is given its natural product structure.

The right hand side above evidently identifies with $\mathcal{Y}'$, so we obtain a monoid structure on $\mathcal{Y}'$ and a map of monoids $\mathfrak{L}\mathbb{G}_m \to \mathcal{Y}'$. In particular, we obtain an action of $\mathfrak{L}\mathbb{G}_m$ on $\mathcal{Y}'$; this is our desired action.

2.4.3. We now formulate the following result, whose proof will be given in §2.6.6.

Theorem 2.4.3.1. *$\mathcal{Y}$ is a classical prestack.*

In particular, $\mathcal{Y}$ is completely determined by its values on usual commutative rings, i.e., not commutative DG rings. (We formally deduce the same for $\mathcal{Y}'$.)

2.5. **Intermediate spaces.** As for $\mathrm{LocSys}_{\mathbb{G}_m}$, there are several variants of $\mathcal{Y}$ that will be crucial to our study.

2.5.1. First, we define:

$$\mathcal{Y}_{\log} = \mathcal{Y} \times_{\mathfrak{L}\mathbb{B}\mathbb{G}_m} \mathfrak{L}^+\mathbb{B}\mathbb{G}_m = \mathcal{Y}'/\mathfrak{L}^+\mathbb{G}_m.$$

Note that we have a canonical action of $\mathrm{Gr}_{\mathbb{G}_m}$ on $\mathcal{Y}_{\log}$ with $\mathcal{Y}_{\log}/\mathrm{Gr}_{\mathbb{G}_m} = \mathcal{Y}$.

2.5.2. By construction, $\mathcal{Y}_{\log}$ parametrizes the data $(\mathcal{L}, \nabla, s)$ where $\mathcal{L}$ is a line bundle on the disc $\mathcal{D}$, a connection ∇ on $\mathcal{L}|_{\mathring{\mathcal{D}}}$, and $s \in \Gamma(\mathring{\mathcal{D}}, \mathcal{L})$ a flat section.

We have a natural ind-closed $\mathcal{Z} \subseteq \mathcal{Y}_{\log}$ parametrizing similar data, but with $s \in \Gamma(\mathcal{D}, \mathcal{L})$ flat as a section on the punctured disc.

Formally, we define:

$$\mathcal{Z} := \operatorname{Eq}\left(\mathfrak{L}^+\mathbb{A}^1 \times \mathfrak{L}\mathbb{A}^1 dt \underset{(g,\omega)\mapsto g\omega}{\overset{d\circ\pi_1}{\rightrightarrows}} \mathfrak{L}\mathbb{A}^1 dt\right)/\mathfrak{L}^+\mathbb{G}_m \in \mathsf{PreStk}.$$

Here the $\mathfrak{L}^+\mathbb{G}_m$-action is constructed as in §2.4.2.

Remark 2.5.2.1. The subspace $\mathcal{Z} \subseteq \mathcal{Y}_{\log}$ plays a key role in our main construction in §5.

Remark 2.5.2.2. In formulae, we have:

$$\mathcal{Z} = \mathcal{M}aps(\mathring{\mathcal{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m) \underset{\mathcal{M}aps(\mathring{\mathcal{D}}, \mathbb{A}^1/\mathbb{G}_m)}{\times} \mathcal{M}aps(\mathcal{D}, \mathbb{A}^1/\mathbb{G}_m)$$

for suitable meaning of $\mathring{\mathcal{D}}_{dR}$ (cf. [Gai4]).

Remark 2.5.2.3. A little informally, $\mathcal{Z}$ is the moduli of $(\mathcal{L}, \nabla, s)$ with $\mathcal{L}$ a line bundle on $\mathcal{D}$, ∇ a connection on the punctured disc, and $s \in \Gamma(\mathcal{D}, \mathcal{L})$ with $\nabla(s) = 0 \in \Gamma(\mathring{\mathcal{D}}, \mathcal{L} \otimes \Omega^1)$.

2.5.3. Next, we define truncated versions of the above spaces. Fix $n \geqslant 0$.

We then define:

$$\mathcal{Y}_{\log}^{\leqslant n} := \mathcal{Y}_{\log} \underset{\mathrm{LocSys}_{\mathbb{G}_m,\log}}{\times} \mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n}.$$

We define:

$$\mathcal{Z}^{\leqslant n} := \operatorname{Eq}\left(\mathfrak{L}^+\mathbb{A}^1 \times \mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt \underset{(g,\omega)\mapsto g\omega}{\overset{d\circ\pi_1}{\rightrightarrows}} \mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt\right)/\mathfrak{L}^+\mathbb{G}_m$$

similarly to $\mathcal{Z}$; again, the construction of §2.4.2 applies and provides rigorous meaning to the $\mathfrak{L}^+\mathbb{G}_m$-action in this formula.

Proposition 2.5.3.1. *The prestack $\mathcal{Z}^{\leqslant n}$ is a (DG) algebraic*[30] *stack.*

Proof. By definition, we have:

$$\operatorname{Eq}\left(\mathfrak{L}^+\mathbb{A}^1 \times \mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt \underset{(g,\omega)\mapsto g\omega}{\overset{d\circ\pi_1}{\rightrightarrows}} \mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt\right) = \mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt \underset{\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n}}{\times} \mathcal{Z}^{\leqslant n}.$$

Therefore, $\mathcal{Z}^{\leqslant n} \to \mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n}$ is an affine morphism. As $\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n}$ is an algebraic stack by Proposition 2.3.9.1, we obtain the result. □

Remark 2.5.3.2. We remind that $\mathbb{B}\mathfrak{L}^+\mathbb{G}_m$ is not an algebraic stack while $\mathbb{B}\mathbb{G}_m$ is: by definition, algebraic stacks are required to admit fppf covers, not merely fpqc covers.

2.6. **Flatness results.** We now establish some technical results, showing that certain morphisms are flat. Ultimately, these results are the technical backbone of our study of $\mathcal{Y}$ and its coherent sheaves.

[30] We refer to [Gai3] for the definition.

2.6.1. We begin with the following result.

Lemma 2.6.1.1. *The map:*

$$\mu_n : \mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n} \to \mathbb{A}^n$$
$$(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \mapsto (a_0 b_0, a_0 b_1 + a_1 b_0, \ldots, \sum_{i=0}^{n-1} a_i b_{n-1-i})$$

is flat.

Proof. This map is defined by a family of homogeneous polynomials (of degree 2). Therefore,[31] it suffices to show that the fiber $Z := \mu_n^{-1}(0)$ over 0 is equidimensional with the expected dimension $2n - n = n$.

First, for $0 \leqslant m \leqslant n$, let $Z_m \subseteq Z$ be the locally closed subscheme where $a_0 = a_1 = \ldots = a_{m-1} = 0$ and $a_m \neq 0$, the last condition being considered as vacuous for $m = n$. We claim that $\dim(Z_m) = n$ for all m. (We will also see that Z_m is smooth and connected, so we deduce that Z has the $n+1$ irreducible components $\overline{Z_m}$.)

Clearly $Z_n = \mathbb{A}^n$, so suppose $m \neq n$. Then:

$$Z_m \subseteq (\mathbb{A}^1 \backslash 0) \times \mathbb{A}^{n-m-1} \times \mathbb{A}^n$$

is closed, where the coordinates on the latter affine space are $a_m, a_{m+1}, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$; the equations defining Z_m here are:

$$\sum_{i=0}^{j} a_{m+i} b_{j-i} = 0, \ldots j = 0, \ldots, n-m-1.$$

In particular, $b_j = -\frac{\sum_{i=1}^{j} a_{m+i} b_{j-i}}{a_0}$ for $j = 0, \ldots, n-m-1$. It follows that the morphism:

$$Z_m \subseteq (\mathbb{A}^1 \backslash 0) \times \mathbb{A}^{n-m-1} \times \mathbb{A}^m$$
$$(a_m, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \mapsto (a_m, \ldots, a_{n-1}, b_{n-m}, b_{n-m+1}, \ldots, b_{n-1})$$

is an isomorphism, giving the claim. ☐

Remark 2.6.1.2. The above lemma is standard. See for example [GJS] Theorem 2.2, especially the remarks following its proof.

We now have the following variant.

Corollary 2.6.1.3. *Fix a linear map $T : \mathbb{A}^{2n} \to \mathbb{A}^n$. Then $\mu_n + T : \mathbb{A}^{2n} \to \mathbb{A}^n$ is flat.*

Proof. Introduce an auxiliary parameter λ and consider the map $\mathbb{A}^{2n} \times \mathbb{A}^1_\lambda \to \mathbb{A}^n \times \mathbb{A}^1_\lambda$ given by $(\mu_n + \lambda T, \lambda)$. This map is defined by homogeneous polynomials (all but one are degree 2). Its fiber over 0 coincides with $\mu_n^{-1}(0)$, so this map is flat. Restricting to $\mathbb{A}^n \overset{x \mapsto (x,1)}{\subseteq} \mathbb{A}^n \times \mathbb{A}^1_\lambda$ gives the claim. ☐

[31] As is standard: there exists an open $U \subseteq \mathbb{A}^n$ such that geometric fibers over points in U are equidimensional with the expected dimension. By homogeneity, U is closed under the $\mathbb{G}_m$-action. As U contains 0, it must be all of $\mathbb{A}^n$. Then recall that a morphism of smooth schemes is flat if and only if its geometric fibers are equidimensional of the expected dimension.

2.6.2. We now consider variants of the above in which we pass to a limit.

Let $\mathbb{A}^\infty \in \mathsf{AffSch}$ denote the affine scheme $\mathrm{Spec}(\mathrm{Sym}(k^{\oplus \mathbb{Z}^{\geq 0}}))$.

Corollary 2.6.2.1. *The map:*

$$\mu_\infty : \mathbb{A}^\infty \times \mathbb{A}^\infty \to \mathbb{A}^\infty$$

$$\big((a_0, a_1, \ldots), (b_0, b_1, \ldots)\big) \mapsto (a_0 b_0, a_0 b_1 + a_1 b_0, \ldots, \sum_{i=0}^{j} a_i b_{j-i}, \ldots)$$

is flat.

More generally, suppose we are given linear maps $T_n : \mathbb{A}^{2n} \to \mathbb{A}^n$ fitting into commutative diagrams:

$$\begin{array}{ccc} \mathbb{A}^{2n+2} = \mathbb{A}^{n+1} \times \mathbb{A}^{n+1} & \xrightarrow{T_{n+1}} & \mathbb{A}^{n+1} \\ \downarrow & & \downarrow \\ \mathbb{A}^{2n} & \xrightarrow{T_n} & \mathbb{A}^n \end{array}$$

with vertical maps induced by the projection $\mathbb{A}^{n+1} \xrightarrow{(a_0,\ldots,a_n) \mapsto (a_0,\ldots,a_{n-1})} \mathbb{A}^n$. Let T denote the induced map $\mathbb{A}^\infty \times \mathbb{A}^\infty \to \mathbb{A}^\infty$.

Then $\mu_\infty + T$ is flat.

Proof. As $\mu_\infty + T$ is obtained from the morphisms $\mu_n + T_n$ by passing to the inverse limit in n, the result follows from Lemma 2.6.1.1.

□

2.6.3. We also need a refinement of the above. The reader may skip this material and return to it as needed.

Let $C = \mathrm{Spec}(k[x, y]/xy)$. Geometrically, C is a the union of the x and y-axes in the plane. For a scheme S and a morphism $\varphi : S \to C$, we say that φ is *flat along the x-axis* if the derived fiber product $S \times_C \mathbb{A}^1_x$ is a classical scheme. I.e., for S classical, this means no Tors are formed in forming this fiber product.

We say φ is *flat along the y-axis* if the parallel condition holds for the y-axis. We say φ is *flat along the axes* if it is flat along both the x and y-axis.

Fix $n > 0$ and let $Z = \mu_n^{-1}(0) \subseteq \mathbb{A}^{2n}$ as in the proof of Lemma 2.6.1.1. There is an evident map:

$$Z \xrightarrow{(a_0,\ldots,a_{n-1},b_0,\ldots,b_{n-1}) \mapsto (a_0,b_0)} C \subseteq \mathbb{A}^2.$$

Proposition 2.6.3.1. *The above map is flat along the axes. Moreover, there is a canonical isomorphism:*

$$\mu_{n-1}^{-1}(0) \times \mathbb{A}^1 \simeq Z \underset{C}{\times} \mathbb{A}^1_x$$

given by:

$$\big((a_0, a_1, \ldots, a_{n-2}, b_0, \ldots, b_{n-2}), \lambda\big) \in \mu_{n-1}^{-1}(0) \times \mathbb{A}^1 \mapsto (a_0, a_1, \ldots, a_{n-2}, \lambda, 0, b_0, b_1, \ldots, b_{n-2}) \in Z.$$

Remark 2.6.3.2. Recall that in the proof Lemma 2.6.1.1 that we calculated the irreducible components of Z. In [Yue], the multiplicities of these components were also calculated.[32] This latter result follows from Proposition 2.6.3.1: given $S \xrightarrow{(f,g)} C$ flat along axes, for an irreducible component S_m of S, $\mathrm{mult}_S(S_m) = \mathrm{mult}_{\{f=0\}}(S_m \cap \{f = 0\}) + \mathrm{mult}_{\{g=0\}}(S_m \cap \{g = 0\})$; one can then calculate the

[32] In the notation from the proof of Lemma 2.6.1.1, the multiplicity of Z_m is $\binom{n}{m}$.

multiplicities by induction on n. We remark that this argument is quite similar to the one given in *loc. cit.*

We will deduce the above result from the following general lemma.

Lemma 2.6.3.3. *Suppose T is a scheme equipped with a map $(f,g) : T \to \mathbb{A}^2$. Let $S = T \times_{\mathbb{A}^2} C = \{fg = 0\} \subseteq T$, where this fiber product is understood as a derived fiber product.*

If $g : T \to \mathbb{A}^1$ is flat, then $S \to C$ is flat along the x-axis.

Proof. This is tautological:

$$S \underset{C}{\times} \mathbb{A}^1_x = T \underset{\mathbb{A}^2}{\times} C \underset{C}{\times} \mathbb{A}^1_x = T \underset{\mathbb{A}^2}{\times} \mathbb{A}^1_x = T \underset{\mathbb{A}^1_y}{\times} 0.$$

As g is assumed flat, the latter scheme is classical by definition.

□

Lemma 2.6.3.4. *The morphism:*

$$\mu'_n : \mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n} \to \mathbb{A}^n$$

$$(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \mapsto (b_0, a_0 b_1 + a_1 b_0, \ldots, \sum_{i=0}^{n-1} a_i b_{n-1-i})$$

is flat.

Remark 2.6.3.5. We highlight the (only) difference between μ'_n and μ_n: in the former, the first coordinate entry is b_0, not $a_0 b_0$.

Proof of Lemma 2.6.3.4. As in the proof of Lemma 2.6.1.1, as each coordinate of μ'_n is homogeneous, we reduce to showing that $(\mu'_n)^{-1}(0)$ has the expected dimension n. But at the classical level, this fiber clearly identifies with $\mu_{n-1}^{-1}(0) \times \mathbb{A}^1_{b_{n-1}}$ (via the map in the statement of Proposition 2.6.3.1), which has dimension n by Lemma 2.6.1.1.

□

Proof of Proposition 2.6.3.1. Consider the map $\mathbb{A}^{2n} \xrightarrow{\mu_n} \mathbb{A}^n \to \mathbb{A}^{n-1}$ where the last map projects onto the last $n-1$-coordinates. Let T denote the inverse image of 0 under this map.

By Lemma 2.6.3.4, the map $b_0 : T \to \mathbb{A}^1$ is flat. Therefore, Lemma 2.6.3.3 gives the flatness along the x-axis. Flatness along the y-axis obviously follows by symmetry. The resulting description of the fiber product is evident.

□

2.6.4. We now have the following generalizations.

Corollary 2.6.4.1. *Suppose:*

$$T : \mathbb{A}^{2n} = \mathbb{A}^n \times \mathbb{A}^n \to 0 \times \mathbb{A}^{n-1} \subseteq \mathbb{A}^n$$

is a linear map.

Let $Z^q = (\mu_n + T)^{-1}(0)$. Then the natural map $Z^q \xrightarrow{(a_0,b_0)} C$ is flat along the axes.

Moreover, in the notation of Corollary 2.6.2.1, we may take $n = \infty$ in this result.

Proof. For $1 \leqslant n < \infty$, the map:

$$\mu'_n + T : \mathbb{A}^{2n} \to \mathbb{A}^n$$

is flat by the argument for Lemma 2.6.3.4, using Corollary 2.6.1.3 instead of Lemma 2.6.1.1. The proof of Proposition 2.6.3.1 then applies to see that $Z^q \to C$ is flat along the x-axis.

The proof of Corollary 2.6.2.1 allows us to deduce the $n = \infty$ case.

□

2.6.5. We now apply the above results to $\mathcal{Z}$ and $\mathcal{Y}$.

Proposition 2.6.5.1. *For every $n \geqslant 0$, $\mathcal{Z}^{\leqslant n}$ is a classical algebraic stack.*

Proof. In coordinates, the morphism:

$$\mu : \mathfrak{L}^+\mathbb{A}^1 \times \mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt \xrightarrow{(g,\omega)\mapsto g\omega - dg} \mathfrak{L}^{\leqslant n}\mathbb{A}^1 dt$$

is given by:[33]

$$(g = \sum_{i=0}^{\infty} a_i t^i, \omega = \sum_{i=0}^{\infty} b_i t^{i-n} dt) \mapsto g\omega - dg = \sum_{i\geqslant 0} \big(\sum_{j=0}^{i} a_j b_{i-j}\big) t^{-n+i} dt + \sum_{i \geqslant 0} i b_i t^{i-1} dt.$$

By Corollary 2.6.1.3, this is a flat morphism of affine schemes. Therefore, the inverse image $\mu^{-1}(0)$ is a classical affine scheme. As this inverse image is the universal $\mathfrak{L}^+\mathbb{G}_m$-torsor over $\mathcal{Z}^{\leqslant n}$, it follows that $\mathcal{Z}^{\leqslant n}$ is classical as well. We now obtain the result from Proposition 2.5.3.1.

□

2.6.6. Let $C = \operatorname{Spec}(k[x,y]/xy)$ as in §2.6.3. Consider the $\mathbb{G}_m$-action on C of horizontal homothety. I.e., for the corresponding grading on $k[x,y]/xy$, $\deg(x) = 1$ and $\deg(y) = 0$.

Fix a coordinate t on the formal disc. We have a corresponding map $\mathcal{Z}^{\leqslant n} \to C/\mathbb{G}_m$ defined as follows.

First, we we have a natural map $\mathcal{Z} \to \mathfrak{L}^+\mathbb{A}^1/\mathbb{G}_m \xrightarrow{\text{ev}} \mathbb{A}^1/\mathbb{G}_m$. This map takes $(\mathcal{L}, \nabla, s) \in \mathcal{Z}$ to $s|_0 \in \mathcal{L}|_0$ for $0 \in \mathcal{D}$ the base-point.

Next, we have a map:

$$\mathcal{Z}^{\leqslant n} \to \operatorname{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log} \xrightarrow{\text{Pol}} \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt \simeq \prod_{i=1}^{n} \mathbb{A}^1 \frac{dt}{t^i} \to \mathbb{A}^1 \frac{dt}{t^n} = \mathbb{A}^1.$$

This map records the leading (i.e., degree $-n$) coefficient of the connection.

The corresponding map $\mathcal{Z}^{\leqslant n} \to \mathbb{A}^1 \times \mathbb{A}^1/\mathbb{G}_m$ evidently maps into $C/\mathbb{G}_m \subseteq \mathbb{A}^1 \times \mathbb{A}^1/\mathbb{G}_m$.

Remark 2.6.6.1. The above is somewhat non-canonical as the second map above depends on the choice of coordinate t. The more canonical statement would be to replace the $\mathbb{A}^1$ by the (scheme corresponding to the) line $t^{-n}k[[t]]dt/t^{-n+1}k[[t]]dt$.

The following result plays a key technical role in our work.

Proposition 2.6.6.2. *Suppose $n > 0$.*

(1) The map $\mathcal{Z}^{\leqslant n} \to C/\mathbb{G}_m$ is flat along the axes.[34]

(2) We have a canonical isomorphism:

$$\mathcal{Z}^{\leqslant n} \underset{C/\mathbb{G}_m}{\times} \mathbb{A}^1/\mathbb{G}_m \simeq \mathcal{Z}^{\leqslant n-1}.$$

(3) Let $\iota : \mathcal{Z}^{\leqslant n} \to \mathcal{Z}^{\leqslant n}$ denote the map:

$$(\mathcal{L}, \nabla, s) \mapsto (\mathcal{L}(1), \nabla, s)$$

where $\mathcal{L}(1)$ is the line bundle on the disc whose sections on the disc are allowed to have a pole of order 1 at the base-point $0 \in \mathcal{D}$.[35]

[33] As always, the meaning of this formula is on A-points for any commutative ring A. That is, the a_i and b_i's are regarded as elements of A.

[34] By this, we mean that the corresponding map $\mathcal{Z}^{\leqslant n} \underset{\mathbb{B}\mathbb{G}_m}{\times} \operatorname{Spec}(k) \to C$ is flat along the axes in the sense of §2.6.3.

[35] Another way to say this: $\operatorname{Gr}_{\mathbb{G}_m}$ acts on $\mathcal{Y}_{\log}$, and the corresponding action of $1 \in \mathbb{Z} = \operatorname{Gr}_{\mathbb{G}_m}(k)$ preserves $\mathcal{Z}^{\leqslant n}$ for all n; the induced map $\mathcal{Z}^{\leqslant n} \to \mathcal{Z}^{\leqslant n}$ is our ι.

Then ι fits into a (derived) Cartesian diagram:

$$\begin{array}{ccc} \mathcal{Z}^{\leqslant n} & \xrightarrow{\iota} & \mathcal{Z}^{\leqslant n} \\ \downarrow & & \downarrow \\ \mathbb{A}^1 \times \mathbb{B}\mathbb{G}_m & \xrightarrow{(\mathrm{id},0)} & C/\mathbb{G}_m. \end{array}$$

Proof. For (1), it suffices to check flatness along axes after passing to the fpqc cover $\mathcal{Z}^{\leqslant n} \times_{\mathbb{B}\mathfrak{L}^+\mathbb{G}_m} \mathrm{Spec}(k)$. The corresponding map to $C/\mathbb{G}_m$ evidently lifts to C, and it suffices to check that the corresponding map to C is flat along axes. Using coordinates as in the proof of Proposition 2.6.5.1, we deduce the claim from Corollary 2.6.4.1.

The other assertions are immediate from the constructions. For instance, the diagram in (3) is obviously *classically* Cartesian, so derived Cartesian by (1).

□

Corollary 2.6.6.3. *The above morphism $\iota : \mathcal{Z}^{\leqslant n} \hookrightarrow \mathcal{Z}^{\leqslant n}$ is an almost finitely presented*[36] *closed embedding.*

Proof. Almost finite presentation is preserved under (derived) base-change, and any morphism of schemes almost of finite presentation is itself almost of finite presentation. So the claim follows from Proposition 2.6.6.2 (3).

□

Corollary 2.6.6.4. *$\mathcal{Y}^{\leqslant n}_{\log}$ is a classical ind-algebraic stack. More precisely, the total space of the canonical $\mathbb{G}_m$- torsor on $\mathcal{Y}^{\leqslant n}_{\log}$ is classical and a reasonable indscheme in the sense of* [Ras7] *§6.8.*

Proof. We clearly have:

$$\operatorname{colim}\left(\mathcal{Z}^{\leqslant n} \xrightarrow{\iota} \mathcal{Z}^{\leqslant n} \xrightarrow{\iota} \ldots\right) = \mathcal{Y}^{\leqslant n}_{\log}. \tag{2.6.1}$$

Each of these morphisms is an almost finitely presented closed embedding. Therefore, $\mathcal{Y}^{\leqslant n} \times_{\mathbb{B}\mathbb{G}_m} \mathrm{Spec}(k)$ is a filtered colimit of the classical affine schemes $\mathcal{Z}^{\leqslant n} \times_{\mathbb{B}\mathbb{G}_m} \mathrm{Spec}(k)$ under almost finitely presented closed embeddings.

We now remind the general definition of reasonable indscheme from [Ras7]: it is a (DG) indscheme that can be written as a colimit of eventually coconnective quasi-compact quasi-separated schemes under closed embeddings almost of finite presentation. So clearly this property is verified here.

□

Remark 2.6.6.5. One clearly obtains similar results for the Higgs analogue of $\mathcal{Y}$. A posteriori, one sees that $\mathcal{M}aps(\overset{\circ}{\mathcal{D}}, C)$ is a reasonable ind-affine indscheme. For a weaker (classical) notion of reasonable indscheme, a similar result with C replaced by any finite type affine scheme is well-known: see [BD] Lemma 2.4.8. It is natural to ask: in what generality is such a result true for the stronger (derived) notion used here?

[36] See [Lur3] Definition 7.2.4.26 for the definition in the affine case. In the following remark in *loc. cit.*, it is shown that this notion is preserved under base-change. By [Lur4] Proposition 4.1.4.3, this condition can be checked flat locally. Therefore, there is an evident notion of a representable morphism of prestacks being locally almost of finite presentation, and we are using the term in this sense.

We emphasize that in this setting, passing to an affine cover of $\mathcal{Z}^{\leqslant n}$, the corresponding map of classical affine schemes is a finitely presented morphism in the sense of classical algebraic geometry, but being almost finitely presented is a *stronger* notion (in spite of the terminology): in the more classical reference [BGI$^+$] Exposé III Définition 1.2, this property of a morphism is called *pseudo-coherence*.

Proof of Theorem 2.4.3.1. By Corollary 2.6.6.4, $\mathcal{Y}^{\leqslant n} = \mathcal{Y}^{\leqslant n}_{\log}/\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m}$ is a classical (non-algebraic) prestack. We deduce the same for $\mathcal{Y} = \mathrm{colim}_n\, \mathcal{Y}^{\leqslant n}$.

□

2.6.7. We need one mild improvement of Proposition 2.6.6.2. Roughly, the statement says that the use of the base-point $0 \in \widehat{\mathcal{D}}$ is not essential in *loc. cit.*

First, we have a map:

$$\widehat{\mathcal{D}} \times \mathcal{Z} \to \widehat{\mathcal{D}} \times \mathfrak{L}^+\mathbb{A}^1/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m \tag{2.6.2}$$

where the latter map is evaluation. Explicitly, this map sends $(\tau, (\mathcal{L}, \nabla, s)) \in \widehat{\mathcal{D}} \times \mathcal{Z}$ to $s|_\tau \in \mathcal{L}|_\tau$.

Now fix n. We let $\mathcal{D}^{\leqslant n} := \mathrm{Spec}(k[[t]]/t^n)$.

We have a second map:

$$\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \to \mathbb{A}^1/\mathbb{G}_m \tag{2.6.3}$$

defined as follows. By definition, this map will factor as:

$$\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \to \mathcal{D}^{\leqslant n} \times \mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log} \xrightarrow{\mathrm{id}\times\mathrm{Pol}} \mathcal{D}^{\leqslant n} \times \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt \to \mathbb{A}^1/\mathbb{G}_m$$

where the last map remains to be defined. First, note that the dualizing complex $\omega_{\mathcal{D}^{\leqslant n}} \in \mathsf{IndCoh}(\mathcal{D}^{\leqslant n})$ lies in degree 0 and is a line bundle; it corresponds to the module $(k[[t]]/t^n)^\vee = t^{-n}k[[t]]dt/k[[t]]dt$. Therefore, given $\tau \in \mathcal{D}^{\leqslant n}$, we obtain a line $\tau^*(\omega_{\mathcal{D}^{\leqslant n}})$. Moreover, a point $\omega \in \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt$ defines an evident section of the above line bundle $\omega_{\mathcal{D}^{\leqslant n}}$. Restricting to the point τ, we obtain the desired construction.

Remark 2.6.7.1. A choice of coordinate t trivializes the above line bundle on $\mathcal{D}^{\leqslant n}$: the basis element is $\frac{dt}{t^n}$. Such a trivialization lifts the map (2.6.3) to a map to $\mathbb{A}^1$. This map is explicitly given at $(\tau, (\mathcal{L}, \nabla, s))$ by evaluating the function $\mathrm{Pol}(\nabla) \cdot \frac{t^n}{dt}$ on $\mathcal{D}^{\leqslant n}$ at τ.

Proposition 2.6.7.2. *The map $\mathcal{Z}^{\leqslant n} \to \mathbb{A}^1/\mathbb{G}_m \times \mathbb{A}^1/\mathbb{G}_m$ factors through $C/(\mathbb{G}_m \times \mathbb{G}_m)$. For $n > 0$, the resulting map $\mathcal{Z}^{\leqslant n} \to C/(\mathbb{G}_m \times \mathbb{G}_m)$ is flat along axes.*

Proof. For the first assertion, observe that for $(\mathcal{L}, \nabla, s) \in \mathcal{Z}$, $s\,\mathrm{Pol}(\nabla) = 0$ as a polar section of the line bundle $\mathcal{L}dt$ on the disc. This implies the claim.

For flatness, note that we have a canonical evaluation morphism:

$$\widetilde{\mathrm{ev}} : \mathcal{D}^{\leqslant n} \times \mathfrak{L}^+_n C \to C.$$

It is immediate from Proposition 2.6.3.1 that this map is flat along axes. Indeed, we need to show that $\widetilde{\mathrm{ev}}^*(i_*(\mathcal{O}_{\mathbb{A}^1_x}))$ is concentrated in cohomological degree 0 (where $i : \mathbb{A}^1_x \to C$ is the embedding). It suffices to check this after further restriction to $\mathfrak{L}^+_n C$, where the assertion is exactly *loc. cit.*

From here, the claim proceeds as in the proof of Proposition 2.6.6.2.

□

2.7. **Some notation.** We now collect a bit of notation related to $\mathcal{Y}$.

2.7.1. We begin by introducing notation for various structural maps.

For $n > 0$, recall that we have the morphism:

$$\iota : \mathcal{Z}^{\leqslant n} \to \mathcal{Z}^{\leqslant n}.$$

For $r \geqslant 0$, we sometimes let ι^r denote the r-fold composition of ι.

We let:

$$\delta_n : \mathcal{Z}^{\leqslant n-1} \to \mathcal{Z}^{\leqslant n}$$
$$i_n : \mathcal{Z}^{\leqslant n} \to \mathcal{Y}^{\leqslant n}_{\log}$$
$$\pi_n : \mathcal{Z}^{\leqslant n} \to \mathcal{Y}^{\leqslant n}$$
$$\lambda_n : \mathcal{Y}_n \to \mathcal{Y}$$

denote the canonical maps.

We also have the maps ζ and $\widetilde{\zeta}$ fitting into a diagram:

$$\begin{array}{ccc} \mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log} & \xrightarrow{\widetilde{\zeta}} & \mathcal{Z}^{\leqslant n} \\ \downarrow \widetilde{\pi}_n & & \downarrow \pi_n \\ \mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m} & \xrightarrow{\zeta} & \mathcal{Y}^{\leqslant n}. \end{array}$$

Here the horizontal arrows send $(\mathcal{L}, \nabla)$ to $(\mathcal{L}, \nabla, 0)$; i.e., we take 0 as the flat section of our local system.

2.7.2. We introduce the following line bundle on $\mathcal{Z}^{\leqslant n}$.

Using our base-point $0 \in \mathcal{D}$, we obtain a canonical line bundle $\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(-1)$ on $\mathcal{Z}^{\leqslant n}$; its fiber at $(\mathcal{L}, \nabla, s)$ is the fiber $\mathcal{L}|_0$ of $\mathcal{L}$ at 0.

For $r \in \mathbb{Z}$, we let:

$$\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(r) := \mathcal{O}_{\mathcal{Z}^{\leqslant n}}(-1)^{\otimes -r}.$$

denote its (suitably normalized) tensor powers. For $\mathcal{F} \in \mathsf{QCoh}(\mathcal{Z}^{\leqslant n})$, we let $\mathcal{F}(r)$ denote its tensor with $\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(r)$.

We let $\widetilde{\mathcal{Z}}^{\leqslant n}$ denote the total space of the $\mathbb{G}_m$-bundle defined by $\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(-1)$, i.e., $\widetilde{\mathcal{Z}}^{\leqslant n} := \mathcal{Z}^{\leqslant n} \times_{\mathbb{B}\mathbb{G}_m} \operatorname{Spec}(k)$. By Proposition 2.5.3.1 (or its proof), we remark that $\widetilde{\mathcal{Z}}^{\leqslant n}$ is an affine scheme, and by Theorem 2.4.3.1, it is a classical affine scheme. We use notation ι and δ_n sometimes for the corresponding maps for $\widetilde{\mathcal{Z}}^{\leqslant n}$.

Remark 2.7.2.1. Clearly the above line bundle extends canonically to $\mathcal{Y}_{\log}$ and $\mathcal{Y}^{\leqslant n}_{\log}$; we use similar notation in those settings.

2.8. **Fundamental exact sequences.**

2.8.1. We now record two exact sequences that we will use for some inductive arguments. The results here amount to restatements of Proposition 2.6.6.2. Suppose $n > 0$ in what follows.

2.8.2. We will presently construct a short exact sequence:

$$0 \to \delta_{n,*}\mathcal{O}_{\mathcal{Z}^{\leqslant n-1}}(1) \to \mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to \iota_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to 0 \tag{2.8.1}$$

in $\mathsf{QCoh}(\mathcal{Z}^{\leqslant n})^{\heartsuit}$.

The map $\mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to \iota_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}}$ is just the canonical adjunction morphism.

Now observe that there is a canonical map:

$$\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(1) \to \mathcal{O}_{\mathcal{Z}^{\leqslant n}}; \tag{2.8.2}$$

its fiber at $(\mathcal{L}, \nabla, s)$ is the map:

$$\mathcal{L}^\vee|_0 \to \mathcal{O}|_0$$

given by pairing with the section $s|_0 \in \mathcal{L}|_0$. We claim that this map factors uniquely as:

$$\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(1) \twoheadrightarrow \delta_{n,*}\mathcal{O}_{\mathcal{Z}^{\leqslant n-1}}(1) \dashrightarrow \mathcal{O}_{\mathcal{Z}^{\leqslant n}}$$

and that the resulting map defines a short exact sequence as in (2.8.1).

In fact, this is implicit in work we have done already: these claims are obtained by pullback from the short exact sequence:

$$0 \to k[x,y]/(xy,x) \xrightarrow{x\cdot -} k[x,y]/xy \to k[x,y]/(xy,y) \to 0$$

of bi-graded $k[x,y]/xy$ modules along the morphism from Proposition 2.6.6.2. We emphasize that we are using the flatness asserted in Proposition 2.6.6.2 (1).

Variant 2.8.2.1. For later use, we record the following observation. Fix a coordinate t on the disc and use the notation of §2.10 below.

Then we also have a short exact sequence:

$$0 \to \iota_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to \mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to \delta_{n,*}\mathcal{O}_{\mathcal{Z}^{\leqslant n-1}} \to 0 \tag{2.8.3}$$

in $\mathsf{QCoh}(\mathcal{Z}^{\leqslant n})^{\heartsuit}$ in which the right map is the canonical morphism. The left arrow is the unique map fitting into a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Z}^{\leqslant n}} & & \\ \downarrow & \searrow^{b_{-n}\cdot -} & \\ \iota_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}} & \dashrightarrow & \mathcal{O}_{\mathcal{Z}^{\leqslant n}}. \end{array}$$

Here b_{-n} is as in §2.10 below. Again, the existence of the dotted arrow and the short exact sequence follow from Proposition 2.6.3.1.

2.9. **Nice opens.** At some points, it is convenient to refer to the following geometric observations about $\mathcal{Z}^{\leqslant n}$. Roughly speaking, the idea is that $\mathcal{Z}^{\leqslant n}$ is "nice" away from $\mathcal{Z}^{\leqslant n-1}$, which we sometimes use for inductive statements.

We proceed separately in the cases where $n = 1$ and $n > 1$.

2.9.1. $n > 1$ *case.* Define $\mathcal{U}_n \subseteq \mathcal{Z}^{\leqslant n}$ as the open substack:

$$\mathcal{U}_n := \mathcal{Z}^{\leqslant n} \backslash \mathcal{Z}^{\leqslant n-1}.$$

Lemma 2.9.1.1. *The natural map:*

$$\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n} \backslash \mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n-1} \to \mathcal{U}_n$$

(induced by $\widetilde{\zeta}$ from §2.7.1) is an isomorphism.

Remark 2.9.1.2. By Proposition 2.3.4.1, we deduce that $\mathcal{U}_n \simeq (\mathbb{A}^1\backslash 0) \times \mathbb{A}^{n-1} \times \mathbb{B}\mathbb{G}_m$. In particular, $\mathcal{U}_n$ is a smooth stack of finite type.

Proof. It is convenient to use the notation of §2.10, which is introduced below. In that notation, we have $\mathcal{U}_n = \mathrm{Spec}(A_n[b_{-n}^{-1}])/\mathbb{G}_m$.

In the (classical) commutative ring $A_n[b_{-n}^{-1}]$, we have relations:

$$0 = b_{-n}a_0 \Rightarrow a_0 = 0$$
$$0 = b_{-n}a_1 + b_{-n+1}a_0 = b_{-n}a_1 \Rightarrow a_1 = 0$$
$$\ldots \Rightarrow a_i = 0 \text{ for all i}$$

(using $n > 1$). It follows that we have an isomorphism:

$$A_n[b_{-n}^{-1}] = k[b_{-1}, \ldots, b_{-n}, b_{-n}^{-1}].$$

This amounts to the claim.

□

2.9.2. $n = 1$ *case.* This case is slightly more technical.

We define $\mathcal{U}_1$ as the pro-(Zariski open substack of $\mathcal{Z}^{\leqslant 1}$):

$$\mathcal{U}_1 := \lim_r \mathcal{Z}^{\leqslant 1} \backslash \big(\mathcal{Z}^{\leqslant 0} \cup \ldots \cup \iota^r(\mathcal{Z}^{\leqslant 0})\big).$$

Here each of the structural maps in the limit is affine, so the limit exists and is well-behaved.

Lemma 2.9.2.1. *The natural map:*

$$(\mathbb{A}^1 \backslash \mathbb{Z}^{\geqslant 0}) \times \mathbb{B}\mathbb{G}_m \to \mathrm{LocSys}^{\leqslant 1}_{\mathbb{G}_m, \log} \xrightarrow{\tilde{\zeta}} \mathcal{Z}^{\leqslant 1}$$

maps through $\mathcal{U}_1$ *and induces an isomorphism:*

$$(\mathbb{A}^1 \backslash \mathbb{Z}^{\geqslant 0}) \times \mathbb{B}\mathbb{G}_m \xrightarrow{\simeq} \mathcal{U}_1.$$

Proof. We again use the coordinates of §2.10. In this notation, $\mathcal{U}_1$ the quotient stack:

$$\mathrm{Spec}(A_1[(b_{-1})^{-1}, (b_{-1} - 1)^{-1}, (b_{-1} - 2)^{-1}, \ldots])/\mathbb{G}_m$$

obtained by inverting the elements $\{b_{-1} - i\}$ for each $i \geqslant 0$. In this ring, we have the relations:

$$(b_{-1} - i)a_i = 0$$

for each $i \geqslant 0$, which means that $a_i = 0$ in this localization for each i. The claim then follows.

□

2.9.3. In both cases above, we observe that $\widetilde{\mathcal{U}}_n := \widetilde{\mathcal{Z}}^{\leqslant n} \times_{\mathcal{Z}^{\leqslant n}} \mathcal{U}_n$ is a regular Noetherian affine scheme.

2.9.4. *A remark.* In spite of the above observations, we warn that the natural map:

$$\zeta : \mathrm{LocSys}_{\mathbb{G}_m} \setminus \mathrm{LocSys}^{\leqslant 0}_{\mathbb{G}_m} \to \mathcal{Y}|_{\mathrm{LocSys}_{\mathbb{G}_m} \setminus \mathrm{LocSys}^{\leqslant 0}_{\mathbb{G}_m}}$$

is *not* an isomorphism. However, it is an isomorphism if one truncates to irregularity of order $n \leqslant 1$, or if one only evaluates on Noetherian test rings (the left hand side is locally of finite type, but the right hand side is not).

2.10. Coordinates.

2.10.1. To make the above completely explicit, we provide explicit coordinates.

2.10.2. Define a classical commutative ring A_n by taking (infinitely many) generators $a_0, a_1, a_2, \ldots$ and $b_{-1}, \ldots, b_{-n}$ and (infinitely) relations coming from equating Taylor series coefficients of the formal Laurent series:

$$\sum_{i=0}^{\infty} i a_i t^{i-1} = \sum_{i=0}^{\infty} a_i t^i \cdot \sum_{i=1}^{n} b_{-i} t^{-i}.$$

We consider A_n as graded with $\deg(a_i) = 1$ for all i and $\deg(b_i) = 0$. This grading on A_n corresponds geometrically to a $\mathbb{G}_m$-action on $\mathrm{Spec}(A_n)$.

Define a map $\mathrm{Spec}(A_n) \to \mathcal{Z}^{\leqslant n}$ by sending a point $(a_0, a_1, a_2, \ldots, b_{-1}, \ldots, b_{-n}) \in \mathrm{Spec}(A_n)$ to the trivial line bundle $\mathcal{O}$ on the disc and equipping it with the connection $\nabla = d - \sum_{i=1}^{n} b_{-i} t^{-i} dt$ on the punctured disc and the section $s = \sum_{i \geqslant 0} a_i t^i$, which is annihilated by the connection by design.

This map actually factors through $\mathrm{Spec}(A_n)/\mathbb{G}_m$; the latter parametrizes the data of a line ℓ and points as above, except that the a_i are sections of ℓ. We then take our line bundle to be $\mathcal{O}\otimes\ell$, take ∇ given by the same formula, and similarly for our section s.

From Propositions 2.3.4.1 and 2.6.5.1, we obtain the next result.

Proposition 2.10.2.1. *The above map* $\mathrm{Spec}(A_n)/\mathbb{G}_m \to \mathcal{Z}^{\leqslant n}$ *is an isomorphism. I.e., we have a* $\mathbb{G}_m$*-equivariant isomorphism* $\mathrm{Spec}(A_n) \simeq \widetilde{\mathcal{Z}}^{\leqslant n}$.

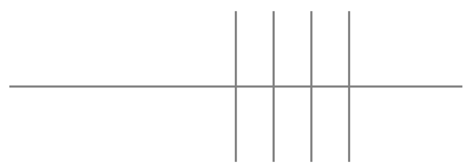
2.11. Regular singular sketches.

2.11.1. For the sake of explicitness, we draw pictures of $\mathcal{Z}^{\leqslant 1}$, $\mathcal{Y}^{\leqslant 1}_{\log}$, and $\mathcal{Y}^{\leqslant 1}$ using the presentation of §2.10.

2.11.2. By the above, A_1 is the (classical) algebra with generators $b_{-1}, a_0, a_1, \dots$ and relations:

$$(b_{-1} - i)a_i = 0.$$

For an integer $m \geqslant 0$, let $A_{1,\leqslant m}$ denote the subalgebra generated by $b_{-1}, a_0, \dots, a_m$, and let $\widetilde{\mathcal{Z}}^{\leqslant 1}_{\leqslant m} := \mathrm{Spec}(A_{1,\leqslant m})$. We obtain the following picture for $\widetilde{\mathcal{Z}}^{\leqslant 1}_{\leqslant m}$:

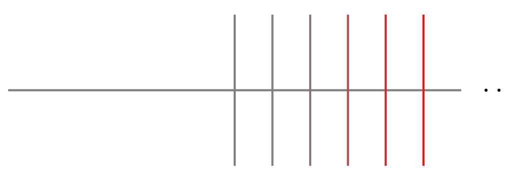
$\widetilde{\mathcal{Z}}^{\leqslant 1}_{\leqslant 3} =$ 

where the horizontal axis is the b_{-1}-axis, and the fiber over $b_{-1} = i$ is given the coordinate a_i.

The structural maps $\widetilde{\mathcal{Z}}^{\leqslant 1}_{\leqslant m+1} \to \widetilde{\mathcal{Z}}^{\leqslant 1}_{\leqslant m}$ coming from the embedding $A_{1,\leqslant m} \hookrightarrow A_{1,\leqslant m+1}$ correspond to contracting the rightmost line in the picture. Therefore, as:

$$\widetilde{\mathcal{Z}}^{\leqslant 1} = \lim_m \widetilde{\mathcal{Z}}^{\leqslant 1}_{\leqslant m}$$

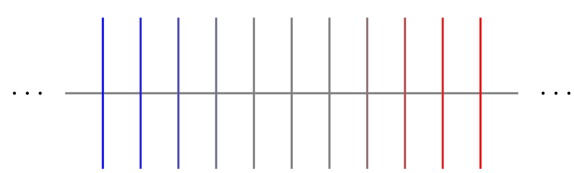
we obtain the picture:

$\widetilde{\mathcal{Z}}^{\leqslant 1} =$ 

where the reddening indicates that interpret the picture in the *pro*-sense rather than the *ind*-sense.

Remark 2.11.2.1. For $n > 1$, the stack $\mathcal{Z}^{\leqslant n}$ is non-reduced, so there is additional complexity in attempting to draw pictures. (In addition, its Krull dimension grows with n.)

2.11.3. The map $\iota : \widetilde{\mathcal{Z}}^{\leqslant 1} \to \widetilde{\mathcal{Z}}^{\leqslant 1}$ corresponds to a rightward shift in the above picture. Therefore, if we let $\widetilde{\mathcal{Y}}^{\leqslant 1}_{\log}$ similarly be the total space of the $\mathbb{G}_m$-torsor over $\mathcal{Y}^{\leqslant 1}_{\log}$, from (2.6.1) we obtain the picture:

$\widetilde{\mathcal{Y}}^{\leqslant 1}_{\log} =$ 

where the bluing in the picture indicates that we interpret the limit in the *ind*-sense (and the reddening is as before).

Informally, $\widetilde{\mathcal{Y}}_{\log}^{\leqslant 1}$ is a *semi-infinite comb*. It has infinitely many bristles, which are attached to the handle at integer points; in the positive direction, these bristles have pro-nature, and in the negative direction they have ind-nature.[37]

2.11.4. In each of these pictures, $\mathbb{G}_m$ acts by scaling in the vertical direction, i.e., scaling the bristles. So forming stacky quotients for this action, we obtain the promised sketches of $\mathcal{Z}^{\leqslant 1}$ and $\mathcal{Y}_{\log}^{\leqslant 1}$.

Finally, the action of $\mathbb{Z} = \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant 1}$ on $\mathcal{Y}_{\log}$ is generated by rightward shift in the above pictures. Quotienting by this action gives a picture[38] for $\mathcal{Y}^{\leqslant 1}$.

3. Local Abel-Jacobi morphisms

3.1. In this section, we collect some standard results about local Abel-Jacobi maps and (simplified) Contou-Carrère pairings for use in §5. This material is standard and included for the reader's convenience.

3.2. Let $\widehat{\mathcal{D}} \in \mathsf{IndSch}$ denote the formal disc $\mathrm{Spf}(k[[t]]) := \mathrm{colim}_n \mathrm{Spec}(k[[t]]/t^n)$.

Let $\widehat{\Delta} : \widehat{\mathcal{D}} \to \widehat{\mathcal{D}} \times \mathcal{D}$ denote the graph of the canonical map $\widehat{\mathcal{D}} \to \mathcal{D}$. Clearly this map is a closed embedding (in particular, schematic).

We define:

$$\mathcal{O}(-\widehat{\Delta}) := \mathrm{Ker}\left(\mathcal{O}_{\widehat{\mathcal{D}} \times \mathcal{D}} \to \widehat{\Delta}_*(\mathcal{O}_{\widehat{\mathcal{D}}})\right) \in \mathsf{QCoh}(\widehat{\mathcal{D}} \times \mathcal{D}).$$

One readily checks that $\mathcal{O}(-\widehat{\Delta})$ is a line bundle on $\widehat{\mathcal{D}} \times \mathcal{D}$: in fact, a choice of coordinate on the disc gives a trivialization of it. Moreover, as:

$$\widehat{\mathcal{D}} \underset{\widehat{\mathcal{D}} \times \mathcal{D}}{\times} (\widehat{\mathcal{D}} \times \overset{\circ}{\mathcal{D}}) = \varnothing$$

there is a canonical trivialization of $\mathcal{O}(-\widehat{\Delta})|_{\widehat{\mathcal{D}} \times \overset{\circ}{\mathcal{D}}}$.

Thus, we obtain a map:

$$\mathrm{AJ}^{-1} : \widehat{\mathcal{D}} \to \mathrm{Gr}_{\mathbb{G}_m}.$$

We define the *local Abel-Jacobi map* AJ as AJ^{-1} composed with the inversion map $\mathrm{Gr}_{\mathbb{G}_m} \to \mathrm{Gr}_{\mathbb{G}_m}$.

Explicitly, we have:

$$\begin{aligned} \mathrm{AJ} : \widehat{\mathcal{D}} &\to \mathrm{Gr}_{\mathbb{G}_m} \\ \tau &\mapsto (\mathcal{O}_{\mathcal{D}}(\tau), 1) \end{aligned}$$

where $\mathcal{O}_{\mathcal{D}}(\tau)$ is the line bundle on the disc with sections having at worst simple poles at $\tau \in \widehat{\mathcal{D}} \subseteq \mathcal{D}$ and the evident trivialization "1" on $\overset{\circ}{\mathcal{D}}$.

[37] We are unfortunately unable to produce non-synesthetic pictures adequately distinguishing between projective and inductive limits. If asked to better explain the above pictures, we would resort to the formal descriptions $\mathcal{Z}^{\leqslant 1} = \lim_m \mathcal{Z}_{\leqslant m}^{\leqslant 1}$ and $\mathcal{Y}_{\log}^{\leqslant 1} = \mathrm{colim}_\iota \mathcal{Z}^{\leqslant 1}$.

[38] Roughly speaking, this picture looks like $\mathbb{A}^1/\mathbb{Z}$ with a single bristle attached at $\mathbb{Z}/\mathbb{Z}$. But we must not forget the semi-infinite nature of that bristle, which we find difficult to visually express in this setting.

3.3. We now consider a closely related map to the Abel-Jacobi map.

We have an evident evaluation map:

$$\widehat{\mathcal{D}} \times \mathfrak{L}^+\mathbb{A}^1 \to \mathbb{A}^1 \tag{3.3.1}$$

that is linear in the second coordinate. Using the non-degeneracy of the residue pairing, we deduce that there is a unique map:

$$\widehat{\mathcal{D}} \to \mathfrak{L}^{pol}\mathbb{A}^1 dt. \tag{3.3.2}$$

such that the composition:

$$\widehat{\mathcal{D}} \times \mathfrak{L}^+\mathbb{A}^1 \xrightarrow{(3.3.2)\times \mathrm{id}} \mathfrak{L}^{pol}\mathbb{A}^1 dt \times \mathfrak{L}^+\mathbb{A}^1 \xrightarrow{(f,\omega)\mapsto \mathrm{Res}(f\omega)} \mathbb{A}^1$$

recovers the evaluation map above.

Remark 3.3.0.1. Explicitly, (3.3.2) is given in coordinates by:

$$(\tau \in \widehat{\mathcal{D}}) \mapsto \sum_{i=0}^{\infty} \frac{\tau^i}{t^{i+1}} dt.$$

We have the following elementary calculation.

Lemma 3.3.0.2. *The map* (3.3.2) *coincides with the composition:*

$$\widehat{\mathcal{D}} \xrightarrow{\mathrm{AJ}} \mathrm{Gr}_{\mathbb{G}_m} \xrightarrow{-d\log} \mathfrak{L}^{pol}\mathbb{A}^1 dt.$$

Proof. Choose coordinates as in Remark 3.3.0.1. The map $\mathrm{AJ} : \widehat{\mathcal{D}} \to \mathrm{Gr}_{\mathbb{G}_m}$ lifts to a map to $\mathfrak{L}\mathbb{G}_m$ via $\tau \mapsto \frac{1}{t-\tau}$. We then have:

$$d\log \frac{1}{t-\tau} = \frac{d\frac{1}{t-\tau}}{\frac{1}{t-\tau}} = -\frac{dt}{t-\tau} = -\frac{1}{t}\sum_{i=0}^{\infty}\frac{\tau^i}{t^i}dt$$

Comparing to Remark 3.3.0.1, we obtain the claim.

□

Remark 3.3.0.3. Following the lemma, we denote the above map by $-d\log \mathrm{AJ}$, and its additive inverse by $d\log \mathrm{AJ}$.

3.4. **Compatibility with truncations.** We use the following observation. We remind that $\mathcal{D}^{\leqslant n} := \mathrm{Spec}(k[[t]]/t^n)$.

Clearly $d\log \mathrm{AJ}$ maps $\mathcal{D}^{\leqslant n}$ maps into $\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt$. Therefore, by definition (see §2.3.8), we deduce that AJ maps $\mathcal{D}^{\leqslant n}$ into $\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}$.

3.5. **The positive Grassmannian.** Next, we define:

$$\mathfrak{L}^{pos}\mathbb{G}_m := \mathfrak{L}\mathbb{G}_m \underset{\mathfrak{L}\mathbb{A}^1}{\times} \mathfrak{L}^+\mathbb{A}^1.$$

There is an evident commutative monoid structure on $\mathfrak{L}^{pos}\mathbb{G}_m$ and homomorphism $\mathfrak{L}^+\mathbb{G}_m \to \mathfrak{L}^{pos}\mathbb{G}_m$.

We then define:

$$\mathrm{Gr}_{\mathbb{G}_m}^{pos} := \mathfrak{L}^{pos}\mathbb{G}_m/\mathfrak{L}^+\mathbb{G}_m = \mathrm{Ker}(\mathfrak{L}^+\mathbb{A}^1/\mathbb{G}_m \to \mathfrak{L}\mathbb{A}^1/\mathbb{G}_m).$$

Explicitly, $\mathrm{Gr}_{\mathbb{G}_m}^{pos}$ parametrizes the data of $\mathcal{L}$ a line bundle on the disc and $\sigma \in \Gamma(\mathcal{D}, \mathcal{L})$ a (regular) section such that $\sigma|_{\mathring{\mathcal{D}}}$ trivializes $\mathcal{L}$.

3.6. For $n \geqslant 0$, we define:

$$\operatorname{Sym}^n \widehat{\mathcal{D}} := \operatorname{Spf}(k[[t_1, \ldots, t_n]])^{S_n} \in \mathsf{IndSch}.$$

for S_n the symmetric group. We then let:

$$\operatorname{Sym} \widehat{\mathcal{D}} := \coprod_n \operatorname{Sym}^n \widehat{\mathcal{D}} \in \mathsf{IndSch}.$$

3.7. As for usual smooth curves, $\operatorname{Sym} \widehat{\mathcal{D}}$ parametrizes effective Cartier divisors on $\mathcal{D}$ supported on $\widehat{\mathcal{D}}$. Hence, we obtain an isomorphism:

$$\operatorname{Sym} \widehat{\mathcal{D}} \xrightarrow{\simeq} \operatorname{Gr}_{\mathbb{G}_m}^{pos}$$

such that the composition:

$$\widehat{\mathcal{D}} = \operatorname{Sym}^1 \widehat{\mathcal{D}} \to \operatorname{Sym} \widehat{\mathcal{D}} \simeq \operatorname{Gr}_{\mathbb{G}_m}^{pos} \subseteq \operatorname{Gr}_{\mathbb{G}_m}$$

is AJ.

This is an isomorphism of commutative monoids for the evident product on $\operatorname{Sym}(\widehat{\mathcal{D}})$.

3.8. **Log Contou-Carrére.**

3.8.1. We now observe the following.

There is a unique pairing:

$$\langle -, - \rangle : \operatorname{Gr}_{\mathbb{G}_m}^{pos} \times \mathfrak{L}^+ \mathbb{A}^1 \to \mathbb{A}^1$$

that is bilinear for multiplicative monoid structures on both sides and whose restriction to $\widehat{\mathcal{D}} \times \mathfrak{L}^+ \mathbb{A}^1$ is (3.3.1).

Explicitly, for $D \in \operatorname{Gr}_{\mathbb{G}_m}^{pos}$ an effective Cartier divisor on $\mathcal{D}$ supported on $\widehat{\mathcal{D}}$ and $f \in \mathfrak{L}^+ \mathbb{A}^1$, we have:

$$\langle D, f \rangle = f(D) := \operatorname{Nm}_D(f).$$

3.8.2. There are some variants of the pairing above.

First, note that $\langle -, - \rangle|_{\operatorname{Gr}_{\mathbb{G}_m}^{pos} \times \mathfrak{L}^+ \mathbb{G}_m}$ maps into $\mathbb{G}_m$. As $\operatorname{Gr}_{\mathbb{G}_m}$ is the group completion of $\operatorname{Gr}_{\mathbb{G}_m}^{pos}$ (in the evident sense), we obtain a pairing:

$$\langle -, - \rangle : \operatorname{Gr}_{\mathbb{G}_m} \times \mathfrak{L}^+ \mathbb{G}_m \to \mathbb{G}_m.$$

This pairing is compatible with Contou-Carrére's pairing [CC] in the evident sense.

3.8.3. Next, let $\operatorname{Sym}^m \mathcal{D}^{\leqslant n} := \operatorname{Spec}((k[[t]]/t^n)^{\otimes m, S_m})$, which is an affine scheme. We let $\operatorname{Gr}_{\mathbb{G}_m}^{pos, \leqslant n} := \coprod_m \operatorname{Sym}^m \mathcal{D}^{\leqslant n}$.

The evident map:

$$\operatorname{Gr}_{\mathbb{G}_m}^{pos, \leqslant n} \to \coprod_m \operatorname{Sym}^m \widehat{\mathcal{D}} \to \operatorname{Gr}_{\mathbb{G}_m}$$

maps into $\operatorname{Gr}_{\mathbb{G}_m}^{\leqslant n}$. Indeed, this map is a map of monoids, so the same is true of its composition with $d\log : \operatorname{Gr}_{\mathbb{G}_m} \to \mathfrak{L}^{pol} \mathbb{A}^1 dt$. As $\mathcal{D}^{\leqslant n}$ maps into $\mathfrak{L}^{pol, \leqslant n} \mathbb{A}^1 dt$ (cf. §3.4), we obtain the claim by definition of $\operatorname{Gr}_{\mathbb{G}_m}^{\leqslant n}$.

For similar reasons, the pairing:

$$\langle -, - \rangle : \operatorname{Gr}_{\mathbb{G}_m}^{pos, \leqslant n} \times \mathfrak{L}^+ \mathbb{A}^1 \to \mathbb{A}^1$$

factors through a pairing:

$$\operatorname{Gr}_{\mathbb{G}_m}^{pos, \leqslant n} \times \mathfrak{L}_n^+ \mathbb{A}^1 \to \mathbb{A}^1$$

that we also denote by $\langle -, - \rangle$.

Remark 3.8.3.1. The base-point $0 \in \widehat{\mathcal{D}}$ gives a point of $\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}$ (of degree 1), hence a translation map $T : \mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \to \mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}$. There is a natural map:

$$\operatorname{colim}\,(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \xrightarrow{T} \mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \xrightarrow{T} \ldots) \to \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}$$

that we claim is an isomorphism. Indeed, the assertion is evident at the level of k-points (both sides are $\mathbb{Z}$), so it suffices to check the assertion on tangent complexes, where it is straightforward (both sides naturally identify with $t^{-n+1}k[[t]]/k[[t]]$).

3.8.4. We will use the following constructions (in a quite simple setting).

Let V be a vector space. We consider V as a prestack[39] with functor of points $\mathrm{Spec}(A) \mapsto \Omega^\infty(A \otimes V) \in \mathsf{Gpd}$.

Now suppose X is an ind-proper indscheme equipped with a morphism $\varphi : X \to V$. We obtain a canonical "integration" map:

$$\Gamma^{\mathsf{IndCoh}}(X, \omega_X) \to V. \tag{3.8.1}$$

Indeed, our map φ is (tautologically) equivalent to a morphism $\mathcal{O}_X \to V \otimes \mathcal{O}_X \in \mathsf{QCoh}(X)$. Tensoring with the dualizing sheaf on X, we obtain a morphism $\omega_X \to V \otimes \omega_X \in \mathsf{IndCoh}(X)$, or what is the same, a map $p^!(k) \to p^!(V)$ for $p : X \to \mathrm{Spec}(k)$ the projection. By ind-properness and adjunction, we obtain a map $p_*^{\mathsf{IndCoh}} p^!(k) \to V$ as desired.

Remark 3.8.4.1. Using [GR2] §II.3 Lemma 3.3.7, one finds that this is a bijection; i.e., specifying a map (3.8.1) is equivalent to giving $X \to V$.

We also have the following variant. Suppose we are also given Y a prestack and $\pi : X \times Y \to \mathbb{A}^1$ a map.

The map π can be encoded by a morphism $X \to \Gamma(Y, \mathcal{O}_Y)$ (thinking of the target as a prestack as above). Therefore, we obtain a comparison map:

$$c_\pi : \Gamma^{\mathsf{IndCoh}}(X, \omega_X) \to \Gamma(Y, \mathcal{O}_Y) \in \mathsf{Vect}$$

by the above. (As in Remark 3.8.4.1, one can actually recover π from c_π.)

3.8.5. We now have the following non-degeneracy assertion for our pairings.

Proposition 3.8.5.1. *Each of the morphisms:*

$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}^{pos}, \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos}}) \to \Gamma(\mathfrak{L}^+\mathbb{A}^1, \mathcal{O}_{\mathfrak{L}^+\mathbb{A}^1})$$
$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}, \omega_{\mathrm{Gr}_{\mathbb{G}_m}}) \to \Gamma(\mathfrak{L}^+\mathbb{G}_m, \mathcal{O}_{\mathfrak{L}^+\mathbb{G}_m})$$
$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}, \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) \to \Gamma(\mathfrak{L}_n^+\mathbb{A}^1, \mathcal{O}_{\mathfrak{L}_n^+\mathbb{A}^1})$$
$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}, \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}}) \to \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, \mathcal{O}_{\mathfrak{L}_n^+\mathbb{A}^1})$$

coming from our pairings and §3.8.4 is an isomorphism.

Moreover, each isomorphism is a map of commutative algebras, where the left hand sides are commutative algebras using the commutative monoid structures on the relevant spaces, while the right hand sides are commutative algebras as functions on schemes.

Proof. Each of these maps is a map of commutative algebras because of the bilinearity of the pairings.

[39]If Y is coconnective, this prestack is a (DG) indscheme.

Next, we have:

$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}, \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) = \bigoplus_m \Gamma^{\mathsf{IndCoh}}(\mathrm{Sym}^m \mathcal{D}^{\leqslant n}, \omega_{\mathrm{Sym}^m \mathcal{D}^{\leqslant n}}) = \bigoplus_m ((k[[t]]/t^n)^{\otimes m, S_m})^\vee = \mathrm{Sym}((k[[t]]/t^n)^\vee)$$

where the second equality is by finiteness of $\mathrm{Sym}^m \mathcal{D}^{\leqslant n}$.

We similarly have $\mathfrak{L}_n^+\mathbb{A}^1 = \mathrm{Spec}(\mathrm{Sym}((k[[t]]/t^n)^\vee))$ by definition. Under these identifications, it suffices to show that the map:

$$\mathrm{Sym}((k[[t]]/t^n)^\vee) = \Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}, \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) \to \Gamma(\mathfrak{L}_n^+\mathbb{A}^1, \mathcal{O}_{\mathfrak{L}_n^+\mathbb{A}^1}) = \mathrm{Sym}((k[[t]]/t^n)^\vee)$$

from the proposition is the identity: it suffices to check this on the generators of the left hand side (as we have a map of commutative algebras), and there it follows by construction of the pairing.

Passing to the colimit as $n \to \infty$, we recover the first map under consideration.

The second and fourth cases follow from Remark 3.8.3.1.

□

Remark 3.8.5.2. The assertion $\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}, \omega_{\mathrm{Gr}_{\mathbb{G}_m}}) \simeq \Gamma(\mathfrak{L}^+\mathbb{G}_m, \mathcal{O}_{\mathfrak{L}^+\mathbb{G}_m})$ via this pairing is standard from Contou-Carrère [CC]. The other assertions are apparently less well-known,[40] though they easy results.

4. Coherent sheaves on $\mathcal{Y}$

4.1. **Overview.** In this section, we define IndCoh^* for $\mathcal{Y}$ and its relatives. This section follows [Ras7] §6, though we keep our exposition largely independent of *loc. cit.*

We strive to keep our exposition as explicit as possible, so assume whatever simplifying assumptions we like. We refer to *loc. cit.* for a more thorough development of the subject.

4.2. Affine case.

4.2.1. Suppose S is an eventually coconnective (e.g. classical) affine scheme.

We define the (non-cocomplete, DG) subcategory $\mathsf{Coh}(S) \subseteq \mathsf{QCoh}(S)$ to consist of objects $\mathcal{F} \in \mathsf{QCoh}(S)^+$ such that for each n, the object $\tau^{\geqslant -n}(\mathcal{F}) \in \mathsf{QCoh}(S)^{\geqslant -n}$ is compact. In other words, $\mathcal{F}$ should be eventually coconnective and $\underline{\mathrm{Hom}}_{\mathsf{QCoh}(S)}(\mathcal{F}, -)$ should commute with filtered colimits in $\mathsf{QCoh}(S)^{\geqslant -n}$ for each n. Clearly $\mathsf{Perf}(S) \subseteq \mathsf{Coh}(S)$.

Notation 4.2.1.1. For $S = \mathrm{Spec}(A)$, we also sometimes write $\mathsf{Coh}(A) \subseteq A\text{–}\mathsf{mod}$ for the subcategory $\mathsf{Coh}(S) \subseteq \mathsf{QCoh}(S) \simeq A\text{–}\mathsf{mod}$.

We define $\mathsf{IndCoh}^*(S)$ as $\mathsf{Ind}(\mathsf{Coh}(S))$. Observe that we have a natural functor $\Psi : \mathsf{IndCoh}^*(S) \to \mathsf{QCoh}(S)$.

Define $\mathsf{IndCoh}^*(S)^{\leqslant 0}$ as the subcategory generated under colimits by coherent objects that are connective (with respect to the t-structure on $\mathsf{QCoh}(S)$). There is a corresponding t-structure on $\mathsf{IndCoh}^*(S)$ with this as its connective subcategory.

Lemma 4.2.1.2 ([Ras7] Lemma 6.4.1)**.** *The above functor Ψ is t-exact and induces an equivalence $\Psi : \mathsf{IndCoh}^*(S)^+ \xrightarrow{\simeq} \mathsf{QCoh}(S)^+$.*

[40]For example, they concern Cartier duality for commutative monoids, which is less often considered than the group case.

4.2.2. *Pushforwards.* Now suppose $f : S \to T$ is a morphism of affine, eventually coconnective schemes. In this case, there is a unique continuous DG functor:

$$f_*^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(T)$$

fitting into a commutative diagram:

$$\begin{array}{ccc} \mathsf{IndCoh}^*(S) & \xrightarrow{f_*^{\mathsf{IndCoh}}} & \mathsf{IndCoh}^*(T) \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi} \\ \mathsf{QCoh}(S) & \xrightarrow{f_*} & \mathsf{QCoh}(T) \end{array}$$

and such that $f_*^{\mathsf{IndCoh}}(\mathsf{Coh}(S)) \subseteq \mathsf{QCoh}(T)^+$. (Indeed, this functor is necessarily the ind-extension of $\mathsf{Coh}(S) \xrightarrow{f_*} \mathsf{QCoh}(T)^+ \simeq \mathsf{IndCoh}^*(T)^+ \subseteq \mathsf{IndCoh}^*(T)$.

In general, the functor f_*^{IndCoh} can[41]be pathological, i.e., it may fail to map $\mathsf{IndCoh}^*(S)^+$ into $\mathsf{IndCoh}^*(T)^+$. This issue complicates the theory, compared to the finite type setting.

4.2.3. Suppose $f : S \to T$ as above is eventually coconnective (i.e., of finite Tor-amplitude). In this case, $f^* : \mathsf{QCoh}(T) \to \mathsf{QCoh}(S)$ maps $\mathsf{Coh}(T)$ to $\mathsf{Coh}(S)$. By ind-extension, we obtain a functor $f^{*,\mathsf{IndCoh}} : \mathsf{IndCoh}^*(T) \to \mathsf{IndCoh}^*(S)$. It is immediate to see that it is left adjoint to f_*^{IndCoh}.

Moreover, $f^{*,\mathsf{IndCoh}}$ is manifestly right t-exact. Therefore, f_*^{IndCoh} *is* left t-exact in this case.

4.2.4. Suppose $f : S \to T$ is proper (e.g., an almost finitely presented closed embedding).

Then $f_* : \mathsf{Coh}(S) \to \mathsf{QCoh}(T)^+$ maps into $\mathsf{Coh}(T)$. Therefore, the functor f_*^{IndCoh} admits a continuous right adjoint, that we denote $f^!$ in this case.

4.2.5. *Semi-coherence.* We now provide a general hypothesis that ensures that f_*^{IndCoh} is left t-exact, even when f is not eventually coconnective.

Definition 4.2.5.1. A connective and eventually coconnective commutative algebra A is *semi-coherent* if every $M \in A\text{–}\mathsf{mod}^+$ can be written as a filtered colimit of objects M_i such that:

- $M_i \in \mathsf{Coh}(A)$ for every i.
- There exists an integer r independent of i such that every M_i lies in $A\text{–}\mathsf{mod}^{\geqslant -r}$.

We say $S = \mathrm{Spec}(A)$ is *semi-coherent* if A is.

Remark 4.2.5.2. In the definition, if one restricts to $M \in A\text{–}\mathsf{mod}^\heartsuit$, one finds that such an r may be constructed (if it exists) independent of M (by considering direct sums of modules). It follows that it suffices to check the hypothesis just for $M \in A\text{–}\mathsf{mod}^\heartsuit$.

Example 4.2.5.3. Any Noetherian (and eventually coconnective) A is obviously semi-coherent. More generally, any coherent (and eventually coconnective) A is semi-coherent.

We now have the following simple observation.

Lemma 4.2.5.4. *Suppose S is a semi-coherent eventually connective affine scheme. Suppose $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is a DG category with a t-structure compatible with filtered colimits. Suppose $F : \mathsf{IndCoh}^*(S) \to \mathcal{C}$ is a continuous left t-exact DG functor such that $F(\mathsf{Coh}(S)) \subseteq \mathcal{C}^+$. Then $F(\mathsf{IndCoh}^*(S)^+) \subseteq \mathcal{C}^+$.*

Proof. Suppose $\mathcal{F} \in \mathsf{IndCoh}^*(S)^+$. By assumption and S, we can write $\mathcal{F}$ as a filtered colimit $\mathrm{colim}_i\, \mathcal{F}_i$ with $\mathcal{F}_i \in \mathsf{Coh}(S) \cap \mathsf{QCoh}(S)^{\geqslant -r}$ for some r; equivalently, $\mathcal{F}_i \in \mathsf{Coh}(S) \cap \mathsf{IndCoh}^*(S)^{\geqslant -r}$. Then $F(\mathcal{F}) = \mathrm{colim}_i\, F(\mathcal{F}_i) \in \mathcal{C}^{\geqslant -r}$ by assumption, giving the claim.

□

[41]Unfortunately, [Ras7], which is currently being revised, erroneously claims otherwise.

Corollary 4.2.5.5. *Suppose $f : S \to T$ is a morphism of eventually coconnective affine schemes with S semi-coherent. Then f_*^{IndCoh} is t-exact.*

Proof. By Lemma 4.2.5.4, f_*^{IndCoh} is left t-exact. Therefore, we have a commutative diagram:

$$\begin{array}{ccc} \mathsf{IndCoh}^*(S)^+ & \xrightarrow{f_*^{\mathsf{IndCoh}}} & \mathsf{IndCoh}^*(T)^+ \\ \downarrow \Psi & & \downarrow \Psi \\ \mathsf{QCoh}(S)^+ & \xrightarrow{f_*} & \mathsf{QCoh}(T)^+ \end{array}$$

in which the left arrows are t-exact equivalences and the bottom arrow is t-exact (by affineness). This gives the claim.

□

4.2.6. We have the following important technical result.

Proposition 4.2.6.1. *The (classical) commutative algebra A_n defined in §2.10 is semi-coherent.*

In other words, $\widetilde{\mathcal{Z}}^{\leqslant n}$ is semi-coherent.

We defer the proof of the proposition to §4.8, at the end of this section. Until that point, we assume it.

4.3. **Coherent sheaves with support.**

4.3.1. We now consider the following situation.

Let Z be an affine, eventually coconnective scheme. Let $f : Z \to \mathbb{A}^1$ be a function. Let $U = \{f \neq 0\} \subseteq Z$, and let $j : U \to Z$ be the embedding. Let $i : Z_0 \hookrightarrow Z$ be a closed subscheme with $Z \backslash Z_0 = U$.

The following type of result is standard in the finite type setting.

Lemma 4.3.1.1. *Suppose that:*

- *i is almost finitely presented.*
- *The map $i_*^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(Z_0) \to \mathsf{IndCoh}^*(Z)$ is t-exact.*

Then the functor:

$$(j^{*,\mathsf{IndCoh}}, i^!) : \mathsf{IndCoh}^*(Z) \to \mathsf{IndCoh}^*(U) \times \mathsf{IndCoh}^*(Z_0)$$

is conservative.

Proof. Let $\mathcal{F} \in \mathsf{IndCoh}^*(Z)$ be given. We suppose $\mathcal{F}$ is a non-zero object with $j^{*,\mathsf{IndCoh}}(\mathcal{F}) = 0$. Our goal is to construct an object $\mathcal{H} \in \mathsf{IndCoh}^*(Z_0)$ and a non-zero map $i_*^{\mathsf{IndCoh}}(\mathcal{H}) \to \mathcal{F}$.

Step 1. We have:[42]

$$j_*^{\mathsf{IndCoh}} j^{*,\mathsf{IndCoh}}(\mathcal{F}) = \operatorname{colim}\left(\mathcal{F} \xrightarrow{f} \mathcal{F} \xrightarrow{f} \dots\right) = 0.$$

Let $\mathcal{G} \in \mathsf{Coh}(Z)$ be given with a non-zero map $\alpha : \mathcal{G} \to \mathcal{F}$; such $(\mathcal{G}, \alpha)$ exists because $\mathcal{F}$ is assumed non-zero.

By compactness of $\mathcal{G}$, there is an integer n such that the composition:

$$\mathcal{G} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{f^n} \mathcal{F}$$

[42] Indeed, j_*^{IndCoh} is automatically left t-exact (hence t-exact) because j is flat; therefore, this formula follows from the quasi-coherent setting by ind-extension.

is null-homotopic. This map coincides with the composition:

$$\mathcal{G} \xrightarrow{f^n} \mathcal{G} \xrightarrow{\alpha} \mathcal{F}$$

so this map must be null-homotopic as well. It follows that α factors through a (necessarily non-zero) map:

$$\mathcal{G}/f^n := \mathrm{Coker}(f^n : \mathcal{G} \to \mathcal{G}) \to \mathcal{F}$$

for some $n \gg 0$.

We have a standard co/fiber sequence:

$$\mathcal{G}/f^{n-1} \xrightarrow{f} \mathcal{G}/f^n \to \mathcal{G}/f.$$

By descending induction, we see that there must exist a non-zero map from $\mathcal{G}/f \to \mathcal{F}$. Replacing $\mathcal{G}$ with $\mathcal{G}/f \in \mathsf{Coh}(Z)$, we can assume that f acts by zero on our original object $\mathcal{G}$.

Step 2. We now digress to make the following simple commutative algebra observation.[43] Suppose $Z = \mathrm{Spec}(A)$ below.

Let $I \subseteq H^0(A)$ be the ideal of elements vanishing on Z_0^{cl}. Let $(f) \subseteq H^0(A)$ be the ideal generated by f. We claim that there are integers r, s such that:

$$I^r \subseteq (f), (f^s) \subseteq I.$$

For any $g \in I$, we have $\{g \neq 0\} \subseteq \{f \neq 0\}$ by assumption. Therefore, f is invertible in $H^0(A)[g^{-1}]$, so $g^{r_0} \in (f)$ for some r_0 (depending on g). Because i is almost finitely presented, I is finitely generated, so we obtain $I^r \subseteq (f)$ for $r \gg 0$.

The other inclusion (which we do not need) follows similarly: $\{f \neq 0\}$ is covered by $\{g \neq 0\}$ for $g \in I$, so $1 \in I[f^{-1}] = H^0(A)[f^{-1}]$, so $f^s \in I$ for some s.

We deduce the following: any module:

$$M \in H^0(A/f)\text{–}\mathsf{mod}^{\heartsuit} \subseteq H^0(A)\text{–}\mathsf{mod}^{\heartsuit} = A\text{–}\mathsf{mod}^{\heartsuit}$$

admits a finite filtration:

$$0 = I^r M \subseteq I^{r-1} M \subseteq \ldots \subseteq IM \subseteq M$$

with subquotients lying in $H^0(A)/I\text{–}\mathsf{mod}^{\heartsuit} \subseteq A\text{–}\mathsf{mod}^{\heartsuit}$.

Step 3. We now return to earlier setting.

Recall that we have $\mathcal{G} \in \mathsf{Coh}(Z)$ on which f acts null-homotopically. Because $\mathcal{G}$ is bounded, it has a finite Postnikov filtration as an object of $\mathsf{QCoh}(Z)$ with associated graded terms $H^i(\mathcal{G})[-i]$. Each $H^i(\mathcal{G}) \in \mathsf{QCoh}(Z)^{\heartsuit}$. Because f acts by zero on them, they lie in $\mathsf{QCoh}(\{f = 0\})^{\heartsuit}$. By the previous step, each of these terms is obtained by successively extending objects in $\mathsf{QCoh}(Z_0)^{\heartsuit}$.

Now the diagram:

$$\begin{array}{ccc} \mathsf{IndCoh}(Z_0)^+ & \xrightarrow{i_*^{\mathsf{IndCoh}}} & \mathsf{IndCoh}(Z)^+ \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi} \\ \mathsf{QCoh}(Z_0)^+ & \xrightarrow{i_*} & \mathsf{QCoh}(Z)^+ \end{array}$$

commutes because we assumed i_*^{IndCoh} t-exact; moreover, the vertical arrows are equivalences. Therefore, $\mathcal{G}$ admits a finite filtration as an object of $\mathsf{IndCoh}^*(Z)$ with associated graded terms IndCoh-pushed forward from Z_0. Therefore, one of these associated graded terms must have a non-zero map to $\mathcal{F}$, giving the claim.

[43] In our eventual application, Z_0 is the classical scheme underlying $\{f = 0\}$. This step is unnecessary under that additional hypothesis.

□

4.3.2. *Application to* $\widetilde{\mathcal{Z}}^{\leqslant n}$. We now apply the above to construct generators of $\mathsf{IndCoh}^*(\widetilde{\mathcal{Z}}^{\leqslant n})$.

Proposition 4.3.2.1. *For* $n > 0$*, the DG category* $\mathsf{IndCoh}^*(\widetilde{\mathcal{Z}}^{\leqslant n})$ *is compactly generated by the objects* $\{\iota_*^{r,\mathsf{IndCoh}}(\mathcal{O}_{\widetilde{\mathcal{Z}}^{\leqslant n}})\}_{r\geqslant 0}$*. For* $n = 0$*, the DG category* $\mathsf{IndCoh}^*(\widetilde{\mathcal{Z}}^{\leqslant 0})$ *is compactly generated by* $\mathcal{O}_{\widetilde{\mathcal{Z}}^{\leqslant 0}}$*).*

Proof. We proceed by induction on n.

Step 1. The $n = 0$ case is simple: $\widetilde{\mathcal{Z}}^{\leqslant 0}$ is $\mathbb{A}^1$ (with coordinate a_0).

Step 2. We now fix $n > 0$, and that the proposition for $n-1$. We introduce the following notation.

Let $\mathcal{C} \subseteq \mathsf{IndCoh}^*(\widetilde{\mathcal{Z}}^{\leqslant n})$ be the subcategory generated by the objects $\iota_*^{r,\mathsf{IndCoh}}\mathcal{O}_{\widetilde{\mathcal{Z}}^{\leqslant n}}$. We need to show that $\mathcal{C}$ is the full IndCoh^*.

Step 3. We observe that (2.8.1) implies that $\delta_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\widetilde{\mathcal{Z}}}^{\leqslant n-1}) \in \mathcal{C}$. More generally, we find $\iota_*^{r,\mathsf{IndCoh}}\delta_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\widetilde{\mathcal{Z}}}^{\leqslant n-1}) \in \mathcal{C}$ for any $r \geqslant 0$.

Step 4. Below, we will apply Lemma 4.3.1.1 with $i : Z_0 \to Z$ corresponding to $\delta_n : \widetilde{\mathcal{Z}}^{\leqslant n-1} \to \widetilde{\mathcal{Z}}$. We remark that δ_n is almost finitely presented by Lemma 2.6.6.2 (2) (cf. the proof of Corollary 2.6.6.3), and that $\delta_{n,*}^{\mathsf{IndCoh}}$ is t-exact by Proposition 4.2.6.1.

In other words, our application of the lemma is justified.

Step 5. We now treat the $n = 1$ case.

Suppose $\mathcal{F} \in \mathsf{IndCoh}^*(\widetilde{\mathcal{Z}}^{\leqslant 1})$ lies in the right orthogonal to $\mathcal{C}$.

By Step 3 and the $n = 0$ case, $\delta_1^!(\mathcal{F}) = 0$, and $\delta_1^!(\iota^r)^!\mathcal{F} = 0$ more generally. Therefore, by Lemma 4.3.1.1, $\mathcal{F}$ is $*$-extended from:

$$\widetilde{\mathcal{Z}}^{\leqslant 1} \backslash \big(\widetilde{\mathcal{Z}}^{\leqslant 0} \cup \ldots \cup \iota^r(\widetilde{\mathcal{Z}}^{\leqslant 0})\big)$$

for each r. It follows that $\mathcal{F}$ is $*$-extended from:

$$\widetilde{\mathcal{U}}_1 := \widetilde{\mathcal{Z}}^{\leqslant 1} \times_{\mathcal{Z}^{\leqslant 1}} \mathcal{U}_1.$$

But by Lemma 2.9.2.1 (cf. §2.9.3), $\mathsf{IndCoh}^*(\widetilde{\mathcal{U}}_1) \simeq \mathsf{QCoh}(\widetilde{\mathcal{U}}_1)$ is generated by its structure sheaf. Since this is the $*$-restriction of the structure sheaf on $\mathcal{Z}^{\leqslant 1}$, we obtain the claim.

Step 6. We now assume $n > 1$ case.

In this case, we have:

$$\iota_*^{r,\mathsf{IndCoh}}\delta_{n,*}^{\mathsf{IndCoh}} \simeq \delta_{n,*}^{\mathsf{IndCoh}}\iota_*^{r,\mathsf{IndCoh}}.$$

(Here we use ι to denote the map for both $\mathcal{Z}^{\leqslant n}$ and $\mathcal{Z}^{\leqslant n-1}$.)

By induction and Step 3, $\mathcal{C}$ contains the subcategory generated under colimits by the essential image of $\delta_{n,*}^{\mathsf{IndCoh}}$. Therefore, by Lemma 4.3.1.1, any $\mathcal{F}$ in the right orthogonal to $\mathcal{C}$ is $*$-extended from $\widetilde{\mathcal{U}}_n$.

But again, $\mathsf{IndCoh}^*(\widetilde{\mathcal{U}}_n) \simeq \mathsf{QCoh}(\widetilde{\mathcal{U}}_n)$ is generated by its structure sheaf by Lemma 2.9.1.1; as $\mathcal{O}_{\widetilde{\mathcal{Z}}^{\leqslant n}} \in \mathcal{C}$, we obtain the claim.

□

4.4. **Stacky case.**

4.4.1. Now suppose S is a stack of the form T/G for G classical affine group scheme and T an eventually coconnective affine scheme.

In this case, G acts weakly on $\mathsf{IndCoh}^*(T)$, i.e., $\mathsf{QCoh}(G)$ acts canonically on $\mathsf{IndCoh}^*(T)$ (by flatness of G over a point). We set $\mathsf{IndCoh}^*(S) := \mathsf{IndCoh}^*(T)^{G,w}$.

There is a unique t-structure on $\mathsf{IndCoh}^*(S)$ such that the forgetful functor:

$$\mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(T)$$

is t-exact.

4.4.2. In the above setting, given a map $f : S_1 \to S_2$ of stacks of the above types, we obtain similar functoriality as in the non-stacky case; we omit the details.

4.4.3. We now have:

Corollary 4.4.3.1. *The category* $\mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n})$ *is compactly generated by the objects* $\iota_*^{r,\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})(m)$, *where* $r \geqslant 0$ *and* $m \in \mathbb{Z}$.

Proof. For G an affine algebraic group (in particular, of finite type) and $\mathcal{F} \in \mathcal{C}^{G,w}$, it is a general fact that $\mathcal{F}$ is compact if and only if its image $\mathrm{Oblv}(\mathcal{F}) \in \mathcal{C}$ is compact.[44] Moreover, the functor $\mathrm{Oblv} : \mathcal{C}^{G,w} \to \mathcal{C}$ always generates the target under colimits.

Therefore, writing $\mathcal{Z}^{\leqslant n} = \widetilde{\mathcal{Z}}^{\leqslant n}/\mathbb{G}_m$, the above objects are indeed compact. The generation follows from Proposition 4.3.2.1.

□

4.4.4. There is a natural functor:

$$\Psi : \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \to \mathsf{QCoh}(\mathcal{Z}^{\leqslant n})$$

obtained by passing to weak $\mathbb{G}_m$-invariants for the corresponding functor for $\widetilde{\mathcal{Z}}^{\leqslant n}$. This functor is an equivalence on coconnective (equivalently: eventually coconnective) subcategories via the corresponding assertion for $\widetilde{\mathcal{Z}}^{\leqslant n}$.

We let $\mathsf{Coh}(\mathcal{Z}^{\leqslant n}) \subseteq \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n})$ denote the subcategory of compact objects. By the above, the functor $\mathsf{Coh}(\mathcal{Z}^{\leqslant n}) \to \mathsf{QCoh}(\mathcal{Z}^{\leqslant n})$ induced by Ψ is fully faithful; its essential image is exactly the subcategory of eventually coconnective almost compact objects.

4.4.5. We now observe that the functor:

$$\iota_*^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \to \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n})$$

is t-exact.

Indeed, by definition, this assertion reduces to the analogous statement for $\widetilde{\mathcal{Z}}^{\leqslant n}$, which in turn follows from Proposition 4.2.6.1.

The same analysis applies for pushforward functors:

$$\delta_{n+1,*}^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \to \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n+1}).$$

[44] See [Ras7], proof of Lemma 5.20.2.

4.5. **Ind-algebraic stacks.** We define:

$$\mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n}) := \operatorname{colim} \left(\mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \xrightarrow{\iota_*^{\mathsf{IndCoh}}} \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \xrightarrow{\iota_*^{\mathsf{IndCoh}}} \ldots\right) \in \mathsf{DGCat}_{cont}.$$

We let:

$$i_{n,*}^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \to \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n})$$

denote the structural functor.

Similarly, we define:

$$\mathsf{IndCoh}^*(\mathcal{Y}_{\log}) := \operatorname*{colim}_n \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n}) = \operatorname*{colim}_{\substack{n,m \\ \iota_*^{\mathsf{IndCoh}}, \delta_{n,*}^{\mathsf{IndCoh}}}} \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \in \mathsf{DGCat}_{cont}.$$

By the above and [Ras6] Lemma 5.4.3, there is a unique t-structure on $\mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n})$ such that each structural functor:

$$\mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \to \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n})$$

is t-exact. Similarly, there is a unique t-structure on $\mathsf{IndCoh}^*(\mathcal{Y}_{\log})$ such that each structural functor:

$$\mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n}) \to \mathsf{IndCoh}^*(\mathcal{Y}_{\log})$$

is t-exact.

Each of these categories is compactly generated as all of our structural functors preserve compact objects.

4.6. **Grassmannian actions.** Suppose $n > 0$ in what follows.

4.6.1. Recall that $\mathrm{Gr}_{\mathbb{G}_m}$ tautologically acts on $\mathcal{Y}_{\log}$.

Observe that $\mathcal{Z} \subseteq \mathcal{Y}_{\log}$ is preserved under the action of the submonoid $\mathrm{Gr}_{\mathbb{G}_m}^{pos} \subseteq \mathrm{Gr}_{\mathbb{G}_m}$.

Indeed, suppose more generally that Y is a prestack mapping to $\mathfrak{L}\mathbb{A}^1/\mathbb{G}_m$. Then:

$$\mathrm{Gr}_{\mathbb{G}_m}^{pos} = \mathrm{Ker}(\mathfrak{L}^+\mathbb{A}^1/\mathbb{G}_m \to \mathfrak{L}\mathbb{A}^1/\mathbb{G}_m)$$

acts canonically on:

$$Z := Y \underset{\mathfrak{L}\mathbb{A}^1/\mathbb{G}_m}{\times} \mathfrak{L}^+\mathbb{A}^1/\mathbb{G}_m$$

compatibly with the canonical $\mathrm{Gr}_{\mathbb{G}_m} = \mathrm{Ker}(\mathfrak{L}^+\mathbb{B}\mathbb{G}_m \to \mathfrak{L}\mathbb{B}\mathbb{G}_m)$-action on:

$$Y_{\log} := Y \underset{\mathfrak{L}\mathbb{B}\mathbb{G}_m}{\times} \mathfrak{L}^+\mathbb{B}\mathbb{G}_m.$$

Taking $Y = \mathcal{Y}$, we recover the claim.

4.6.2. Next, observe that the action of $\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \subseteq \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}$ on $\mathcal{Y}_{\log}^{\leqslant n}$ preserves $\mathcal{Z}^{\leqslant n}$.

Indeed, this is an immediate consequence of the above.

4.6.3. By the above, there is a canonical action:

$$\mathsf{QCoh}(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}) \curvearrowright \mathsf{QCoh}(\mathcal{Z}^{\leqslant n})$$

where the left hand side is given its convolution monoidal structure.

For $\mathcal{F} \in \mathsf{Coh}(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}) \subseteq \mathsf{QCoh}(\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n})$, the corresponding action functor:

$$\mathsf{QCoh}(\mathcal{Z}^{\leqslant n}) \to \mathsf{QCoh}(\mathcal{Z}^{\leqslant n})$$

clearly preserves $\mathsf{Coh}(\mathcal{Z}^{\leqslant n})$. Therefore, we obtain a canonical action:

$$\mathsf{IndCoh}(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}) \curvearrowright \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}).$$

4.6.4. By Remark 3.8.3.1, we have:

$$\mathsf{IndCoh}(\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}) \underset{\mathsf{IndCoh}(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n})}{\otimes} \simeq \left(\mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \xrightarrow{\iota_*^{\mathsf{IndCoh}}} \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \xrightarrow{\iota_*^{\mathsf{IndCoh}}} \dots\right) \simeq \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n}).$$

Thus, we obtain a natural action of $\mathsf{IndCoh}(\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n})$ on $\mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n})$ such that the functor $i_{n,*}^{\mathsf{IndCoh}}$ is a morphism of $\mathsf{IndCoh}(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n})$-module categories.

4.6.5. The map $\delta_{n,*}^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \to \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n+1})$ is naturally a map of $\mathsf{IndCoh}(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n})$-module categories, as is evident by again considering subcategories of compact objects.

Therefore, $\mathsf{IndCoh}(\mathrm{Gr}_{\mathbb{G}_m})$ naturally acts on $\mathsf{IndCoh}(\mathcal{Y}_{\log})$, compatibly with the above actions.

4.7. Ind-coherent sheaves on $\mathcal{Y}$.

4.7.1. We now define:

$$\mathsf{IndCoh}^*(\mathcal{Y}) := \mathsf{IndCoh}^*(\mathcal{Y}_{\log})^{\mathrm{Gr}_{\mathbb{G}_m}, w}.$$

By definition, the right hand side is:

$$\mathsf{Hom}_{\mathsf{IndCoh}(\mathrm{Gr}_{\mathbb{G}_m})\text{–}\mathsf{mod}}(\mathsf{Vect}, \mathsf{IndCoh}^*(\mathcal{Y}_{\log})).$$

We let:

$$\mathrm{Oblv} : \mathsf{IndCoh}^*(\mathcal{Y}) \to \mathsf{IndCoh}^*(\mathcal{Y}_{\log})$$

denote the (conservative) forgetful functor. It admits a left adjoint:

$$\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}, w}.$$

4.7.2. The resulting monad $\mathrm{Oblv}\,\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}, w}$ on $\mathsf{IndCoh}^*(\mathcal{Y}_{\log})$ is easily seen to be t-exact.[45] Therefore, we obtain:

Proposition 4.7.2.1. *There is a unique t-structure on $\mathsf{IndCoh}^*(\mathcal{Y})$ such that the functor $\mathrm{Oblv} : \mathsf{IndCoh}^*(\mathcal{Y}) \to \mathsf{IndCoh}^*(\mathcal{Y}_{\log})$ is t-exact. Moreover, the functor $\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}, w}$ is t-exact.*

Indeed, this follows from:

Lemma 4.7.2.2. *Let $\mathcal{C} \in \mathsf{DGCat}_{cont}$ be equipped with a t-structure compatible with filtered colimits and a right t-exact monad $T : \mathcal{C} \to \mathcal{C}$. Then $T\text{–}\mathsf{mod}(\mathcal{C})$ admits a unique t-structure such that the forgetful functor $T\text{–}\mathsf{mod}(\mathcal{C}) \to \mathcal{C}$ is t-exact.*

4.7.3. In the truncated setting, we similarly define:

$$\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n}) := \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n})^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}, w}.$$

We again have adjoint functors:

$$(\mathrm{Oblv}, \mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}, w})$$

and a t-structure on $\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})$.

---

[45] This reduces to the assertion that for $\mathcal{F} \in \mathsf{IndCoh}(\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n, pos})^{\heartsuit}$, the action functor $\mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n}) \to \mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n})$ is t-exact. Filtering $\mathcal{F}$, we are reduced to the case that it is the skyscraper sheaf at a k-point. In that case, the relevant functor is a composition of functors $\iota_*^{\mathsf{IndCoh}}$, so t-exact by Proposition 4.2.6.1.

4.7.4. By functoriality of the constructions, there is a natural functor:

$$\lambda_{n,*}^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n}) \to \mathsf{IndCoh}^*(\mathcal{Y}).$$

This functor is t-exact and preserves compact objects. Moreover, we have:

Lemma 4.7.4.1. *For $n > 0$, the functor $\lambda_{n,*}^{\mathsf{IndCoh}}$ is fully faithful.*

Roughly speaking, this is true because the map $\mathcal{Y}^{\leqslant n} \to \mathcal{Y}$ is formally étale for $n > 0$, which follows from the corresponding fact for $\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n} \to \mathrm{LocSys}_{\mathbb{G}_m}$ (which follows from Proposition 2.3.4.1). Unwinding the constructions to convert this argument into a proof is straightforward.

4.7.5. For the reader's convenience, we explicitly record the following observation. Applying the definitions and Corollary 4.4.3.1, the objects:

$$\lambda_{n,*}^{\mathsf{IndCoh}} \mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}, w} i_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(m))$$

for $n \geqslant 0$ and $m \in \mathbb{Z}$ form compact generators of $\mathsf{IndCoh}^*(\mathcal{Y})$.

4.8. **Proof of Proposition 4.2.6.1.** We now give the proof of Proposition 4.2.6.1, which we deferred earlier.

4.8.1. Our argument is by induction on n. We proceed in steps.

4.8.2. *Step 1: $n = 0$ case.* The $n = 0$ case is trivial, as $A_0 = k[a_0]$ is Noetherian.

4.8.3. *Step 2: $n = 1$ case.* The inductive step we give for $n > 1$ below may be adapted to treat the $n = 1$ case, but it requires somewhat more work; we indicate how this works in §4.8.8. We prefer to give a direct argument. Actually, we will show that A_1 is coherent.

Recall the Noetherian subalgebras $A_{1,\leqslant m} \subseteq A_1$ from §2.11. It suffices to show that for any finitely presented $M \in A_1\text{–}\mathsf{mod}^{\heartsuit}$, there is an integer $m \gg 0$ and finitely generated $N \in A_{1,\leqslant m}\text{–}\mathsf{mod}^{\heartsuit}$ such that there is an isomorphism $N \otimes_{A_{1,\leqslant m}} A_1 \xrightarrow{\simeq} M$, where the tensor product is understood in the derived sense.

Choose some m_0 and $N_0 \in A_{1,\leqslant m_0}\text{–}\mathsf{mod}^{\heartsuit}$ with an isomorphism $H^0(N_0 \otimes_{A_{1,\leqslant m_0}} A_1) \xrightarrow{\simeq} M$; this may be done by choosing generators and relations for M and m_0 such that the relations all involve linear combinations with coefficients in $A_{1,\leqslant m_0}$.

Choose $m > m_0$ such that N_0 does not contain any $(b_{-1} - i)$-torsion for any integer $i \geqslant m$; this can be done because the finitely generated $A_{1,\leqslant m_0}$-module N_0 only admits finitely many associated primes. Then it is easy to see that $N := H^0(A_{1,\leqslant m} \otimes_{A_{1,\leqslant m_0}} N_0)$ satisfies the hypotheses; indeed, the localizations of N_0 and N at $b_{-1} - i$ for $i > m$ coincide, so the requisite torsion for N vanishes by assumption on m.

Remark 4.8.3.1. We see here that A_1 is actually coherent, not merely semi-coherent. We do not consider this question for general n.

4.8.4. *Step 3: filtered colimits, extensions, and effective bounds.* We now make the following observations for dévissage.

For an integer r, we say that $M \in A_n\text{–}\mathsf{mod}$ is *r-good* if M can be expressed as a filtered colimit of coherent objects concentrated in degrees $\geqslant -r$. Clearly any such M lies in $A_n\text{–}\mathsf{mod}^{\geqslant -r}$.

In other words, M is r-good if it lies in the full subcategory:

$$\mathsf{Ind}(\mathsf{Coh}(A) \cap A\text{–}\mathsf{mod}^{\geqslant -r}) \subseteq A\text{–}\mathsf{mod}^{\geqslant -r}.$$

Here the natural functor is fully faithful because $\mathsf{Coh}(A) \cap A\text{–}\mathsf{mod}^{\geqslant -r}$ consists of objects that are compact in $A\text{–}\mathsf{mod}^{\geqslant -r}$ by assumption.

It follows from the second description that r-good objects are closed under filtered colimits (for fixed r). We also observe that r-good objects are closed under extensions.

Finally, we say an object $M \in A_n\text{–}\mathsf{mod}$ is *good* if it is r-good for some r. So our task is to show that any $M \in A_n\text{–}\mathsf{mod}^+$ is good, or equivalently, any $M \in A_n\text{–}\mathsf{mod}^\heartsuit$.

4.8.5. *Step 4: reductions.* We now begin our induction. Take $n > 1$ and assume the result for $n-1$.

Suppose $M \in A_n\text{–}\mathsf{mod}^+$. It it suffices to show that:

$$M[b_{-n}^{-1}] := \operatorname*{colim}_{b_{-n}} M \text{ and } \operatorname{Coker}(M \to M[b_{-n}^{-1}])$$

are good (where Coker indicates the *homotopy cokernel*, i.e., the cone).

We check these assertions below. We remark that by Remark 4.2.5.2, we are reduced to considering $M \in A_n\text{–}\mathsf{mod}^\heartsuit$.

4.8.6. *Step 5: generic case.* Note that $M[b_{-n}^{-1}] \in A_n[b_{-n}^{-1}]\text{–}\mathsf{mod}^\heartsuit \subseteq A_n\text{–}\mathsf{mod}^\heartsuit$.

As $n > 1$, we have an isomorphism:

$$A_n[b_{-n}^{-1}] = k[b_{-1}, \dots, b_{-n}, b_{-n}^{-1}].$$

by Lemma 2.9.1.1.

As this algebra is regular Noetherian, it follows that $M[b_{-n}^{-1}]$ has bounded Tor-amplitude as an $A_n[b_{-n}^{-1}]$-module (simply because it is bounded from below), so the same is true of $M[b_{-n}^{-1}]$ as an A_n-module. By standard homological algebra, $M[b_{-n}^{-1}]$ can be represented by a bounded complex of flat (classical) A_n-modules. By Lazard, a flat A_n-module is good, so $M[b_{-n}^{-1}]$ is good.

4.8.7. *Step 6: torsion case.* We now show that $\widetilde{M} = \operatorname{Coker}(M \to M[b_{-n}^{-1}])$ is good. More generally, we show that any bounded complex $\widetilde{M} \in A_n\text{–}\mathsf{mod}^+$ such that b_{-n} acts locally nilpotently on its cohomologies is good. As the complex $\widetilde{M}$ is assumed bounded, it suffices to treat its cohomology groups one at a time, so we can assume (up to shifting) that $\widetilde{M} \in A_n\text{–}\mathsf{mod}^\heartsuit$.

By induction, there exists an integer r such that any module $N \in A_{n-1}\text{–}\mathsf{mod}^\heartsuit$ is r-good as an A_{n-1}-module. Because the projection $A_n \xrightarrow{b_{-n} \mapsto 0} A_{n-1}$ is almost finitely presented by Proposition 2.6.6.2 (2), so coherent A_{n-1}-complexes restrict to coherent A_n-complexes, N restricts to an r-good A_n-module.

Now let $\widetilde{M}_i \subseteq \widetilde{M}$ be the non-derived kernel of the map $b_{-n}^i : \widetilde{M} \to \widetilde{M}$. Clearly $\widetilde{M}_i/\widetilde{M}_{i-1} \in A_{n-1}\text{–}\mathsf{mod}^\heartsuit \subseteq A_n\text{–}\mathsf{mod}^\heartsuit$, so is r-good for the above r (which is independent of everything in sight except n). The module $\widetilde{M}_i$ is then r-good, as it is obtained by successively extending r-good modules. Finally, $\widetilde{M} = \operatorname{colim}_i \widetilde{M}_i$ is a filtered colimit of r-good modules, so is r-good. This concludes the argument.

4.8.8. *Step 7: revisiting the $n = 1$ case.* Finally, we make a remark that the above argument can be adapted to treat the $n = 1$ case, if one so desires. As in Lemma 2.9.2.1, we should consider the localization of A_1 at $\{b_{-1}, b_{-1} - 1, b_{-1} - 2, \dots\}$ (instead of simply at b_{-1}); one again then obtains a Noetherian ring of finite global dimension. The argument concludes by noting that any classical $A_1/(b_{-1} - i) = k[a_i]$-module is 0-good, so any module for A_1 that is torsion for the above elements (thus necessarily the direct sum of its torsion with respect to each) is 0-good (by an argument as above).

In other words, we need to localize at infinitely many elements before obtaining a regular ring; the saving grace is that the goodness for torsion modules at each is bounded independently of the element.

5. Spectral realization of Weyl algebras

5.1. Overview.

5.1.1. Let $i_n : \mathcal{Z}^{\leqslant n} \hookrightarrow \mathcal{Y}^{\leqslant n}_{\log}$ denote the canonical embedding, as in §2.7.

We have $i^{\mathsf{IndCoh}}_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \in \mathsf{Coh}(\mathcal{Y}^{\leqslant n}_{\log})$, which lies in $\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n}_{\log})^{\heartsuit}$ by Proposition 2.6.5.1.

Let $\mathcal{F}_n := \mathrm{Av}_!^{\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m},w}(i_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}))$, which is a compact object in $\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})$.

In this section, we construct an action of a Weyl algebra in $2n$ generators on $\mathcal{F}_n$ (considered as an object of $\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})$).

5.1.2. To formulate our construction more canonically, let W_n denote the algebra of global differential operators on the scheme $\mathfrak{L}^+_n\mathbb{A}^1 = \operatorname{Spec}\operatorname{Sym}(t^{-n}k[[t]]dt/k[[t]]dt)$.

Let W_n^{op} denote W_n with the reversed multiplication. We will construct a canonical homomorphism:

$$W_n^{op} \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \in \mathsf{Alg} = \mathsf{Alg}(\mathsf{Vect}). \tag{5.1.1}$$

Remark 5.1.2.1. In Proposition 7.1.1.1, we will show that this map is actually an isomorphism.

5.2. Reduction to generators and relations.

Note that $\mathcal{F}_n$ lies in the heart of the t-structure constructed in Proposition 4.7.2.1 (or rather, its truncated counterpart, as in §4.7.3).

Therefore, the right hand side of (5.1.1) lies in cohomological degrees $\geqslant 0$.

As the left hand side of (5.1.1) is certainly in cohomological degree 0, it suffices to construct a homomorphism:

$$W_n^{op} \to \tau^{\leqslant 0}\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) = H^0\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \in \mathsf{Alg}(\mathsf{Vect}^{\heartsuit}).$$

As both terms are now in degree 0 and the left hand side has a standard algebra presentation, such a construction may be given by constructing generators and checking relations.

We need to construct maps:

$$\begin{gathered} t^{-n}k[[t]]dt/k[[t]]dt = (k[[t]]/t^n)^\vee \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \\ (\omega \in t^{-n}k[[t]]dt/k[[t]]dt) \mapsto \varphi_\omega \end{gathered} \tag{5.2.1}$$

and:

$$\begin{gathered} k[[t]]/t^n \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \\ (f \in k[[t]]/t^n) \mapsto \xi_f. \end{gathered} \tag{5.2.2}$$

We then need to check that the φ_ω operators mutually commute, that the ξ_f operators mutually commute, and the identity:[46]

$$[\xi_f, \varphi_\omega] = -\operatorname{Res}(f\omega)\cdot \mathrm{id} \in H^0\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n). \tag{5.2.3}$$

We provide these constructions and check these identities in what follows.

5.3. Action of functions.

First, we construct the map $\omega \mapsto \varphi_\omega$ from (5.2.1).

We assume the reader is familiar with the notation from §3 below.

[46] The sign on the right hand side reflects working with W_n^{op}, not W_n.

5.3.1. We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \times \mathcal{Z}^{\leqslant n} & & \\ \downarrow \alpha & \searrow^{p_1} & \\ \mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \times \mathcal{Z}^{\leqslant n} & \xrightarrow{p_1} & \mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \end{array}$$

where $\alpha = (p_1, \mathrm{act})$ for $\mathrm{act} : \mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \times \mathcal{Z}^{\leqslant n} \to \mathcal{Z}^{\leqslant n}$ the action morphism from above.

We have an evident isomorphism:[47]

$$\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}} = p_1^*(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) \simeq \alpha^* p_1^*(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) \in \mathsf{IndCoh}^*(\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \times \mathcal{Z}^{\leqslant n})$$

giving a morphism:

$$\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to \alpha_*^{\mathsf{IndCoh}}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

Applying $p_{2,*}^{\mathsf{IndCoh}}$, we obtain a canonical morphism:

$$\Gamma^{\mathsf{IndCoh}}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) \otimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to p_{2,*}^{\mathsf{IndCoh}} \alpha_*^{\mathsf{IndCoh}}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}) = \mathrm{act}_*^{\mathsf{IndCoh}}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

We also have a canonical adjunction morphism $\gamma_*^{\mathsf{IndCoh}}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) \to \omega_{\mathrm{Gr}_{\mathbb{G}_m}}$ for $\gamma : \mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \to \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}$ the (ind-closed) embedding.

Pushing forward along $i_n : \mathcal{Z}^{\leqslant n} \to \mathcal{Y}_{\log}^{\leqslant n}$ and composing, we obtain a canonical morphism:

$$\Gamma^{\mathsf{IndCoh}}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) \otimes i_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \to \mathrm{act}_*^{\mathsf{IndCoh}}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}}^{\leqslant n} \boxtimes i_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) = \mathrm{Oblv}\,\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n},w}\, i_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

By Proposition 3.8.5.1, we have a canonical isomorphism:

$$\Gamma^{\mathsf{IndCoh}}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) \simeq \Gamma(\mathfrak{L}_n^+\mathbb{A}^1, \mathcal{O}_{\mathfrak{L}_n^+\mathbb{A}^1}).$$

Therefore, by adjunction, the above gives a morphism:

$$\Gamma(\mathfrak{L}_n^+\mathbb{A}^1, \mathcal{O}_{\mathfrak{L}_n^+\mathbb{A}^1}) \to \underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n})}\big(i_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}), \mathrm{Oblv}\,\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n},w}\, i_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})\big) =$$

$$\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}\big(\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n},w}\, i_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})\big).$$

By *loc. cit.*, this is a morphism of algebras.

For $\omega \in t^{-n}k[[t]]dt/k[[t]]dt = (k[[t]]/t^n)^\vee$, there is a corresponding linear function on $\mathfrak{L}_n^+\mathbb{A}^1$; its image under the above map is by definition φ_ω in (5.2.1). As the above map extended to the symmetric algebra on this vector space, we have verified that the operators φ_ω commute.

Remark 5.3.1.1. More evocatively, but a little less rigorously, we have monoid maps:

$$\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n} \to \mathcal{M}aps_{/\mathcal{Y}^{\leqslant n}}(\mathcal{Z}^{\leqslant n}, \mathcal{Z}^{\leqslant n}) \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n)$$

where the first map is given by the action and the second map sends a map $f : \mathcal{Z}^{\leqslant n} \to \mathcal{Z}^{\leqslant n}$ over $\mathcal{Y}^{\leqslant n}$ to the map:

$$\mathcal{F}_n = \pi_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \to \pi_{n,*}^{\mathsf{IndCoh}} f_*^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) = \pi_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) = \mathcal{F}_n$$

[47]One can work with IndCoh^* or QCoh for our purposes here. We chose IndCoh^* for the sake of definiteness.

We also have written $(-)^*$ in place of $(-)^{*,\mathsf{IndCoh}}$ to keep the notation simpler. To be clear, this notation refers to the left adjoint to the IndCoh^* pushforward functor.

where π_n is the ind-proper morphism $\mathcal{Z}^{\leqslant n} \to \mathcal{Y}^{\leqslant n}$ and the first map comes by functoriality from the canonical[48] map $\mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to f_*^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$.

We then obtain an algebra map:

$$\Gamma(\mathfrak{L}_n^+\mathbb{A}^1, \mathcal{O}_{\mathfrak{L}_n^+\mathbb{A}^1}) = \Gamma^{\mathsf{IndCoh}}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}}) \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n)$$

from (3.8.1). The detailed construction articulates this idea explicitly.

5.4. **Action of vector fields.** We now define the map $f \mapsto \xi_f$ from (5.2.2).

5.4.1. We begin with an elementary observation involving $C = \mathrm{Spec}(k[x,y]/xy)$. Let $i : \mathbb{A}^1_y \hookrightarrow C$ denote the embedding of the y-axis.

The map:

$$x \cdot - : \mathcal{O}_C \to \mathcal{O}_C$$

obviously factors through a map:

$$i_*\mathcal{O}_{\mathbb{A}^1_y} \to \mathcal{O}_C$$

that we also denote $y \cdot -$.

5.4.2. We now vary to above to incorporate $\mathbb{G}_m$-actions.

Consider $\mathbb{G}_m \times \mathbb{G}_m$ acting on C with the first $\mathbb{G}_m$ scaling the x-axis and the second $\mathbb{G}_m$ scaling the y-axis.

The map $y \cdot -$ evidently has bidegree $(0,1)$. Therefore, on the stack $C/\mathbb{G}_m \times \mathbb{G}_m$, we obtain the following.

By definition, there are tautological line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $C/\mathbb{G}_m \times \mathbb{G}_m$ equipped with sections $x \in \mathcal{L}_1$ and $y \in \mathcal{L}_2$ such that $xy = 0$ as a section of $\mathcal{L}_1 \otimes \mathcal{L}_2$.

From the above, we obtain a canonical map:

$$y \cdot - : i_*\mathcal{O}_{0/\mathbb{G}_m \times \mathbb{A}^1_y/\mathbb{G}_m} \to \mathcal{L}_2. \tag{5.4.1}$$

5.4.3. Before applying the above, we introduce some more notation.

Let $\mathrm{Gr}_{\mathbb{G}_m}^{neg} \subseteq \mathrm{Gr}_{\mathbb{G}_m}$ denote the image of $\mathrm{Gr}_{\mathbb{G}_m}^{pos}$ under the inversion map $\mathrm{Gr}_{\mathbb{G}_m} \xrightarrow{\simeq} \mathrm{Gr}_{\mathbb{G}_m}$. We use similar notation for $\mathrm{Gr}_{\mathbb{G}_m}^{neg,\leqslant n}$.

Note that $\mathrm{AJ}^{-1} : \widehat{\mathcal{D}} \to \mathrm{Gr}_{\mathbb{G}_m}$ maps into $\mathrm{Gr}_{\mathbb{G}_m}^{neg}$. Similarly, AJ^{-1} maps $\mathcal{D}^{\leqslant n}$ into $\mathrm{Gr}_{\mathbb{G}_m}^{neg,\leqslant n}$.

We let $\mathrm{act}_{neg} : \mathrm{Gr}_{\mathbb{G}_m}^{neg} \times \mathcal{Y}_{\log} \to \mathcal{Y}_{\log}$ denote the action map. We use evident variants of this as well; crucially, we also let act_{neg} denote the composition:

$$\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \xrightarrow{\mathrm{AJ}^{-1} \times i_n} \mathrm{Gr}_{\mathbb{G}_m}^{neg} \times \mathcal{Y}_{\log} \xrightarrow{\mathrm{act}_{neg}} \mathcal{Y}_{\log}.$$

5.4.4. Now recall from Proposition 2.6.7.2 that we have a canonical map:

$$\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \to C/\mathbb{G}_m \times \mathbb{G}_m.$$

In the notation of §5.4.2, the line bundle $\mathcal{L}_2$ pulls back to $\omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}$ by construction. The pullback of its section y corresponds to the canonical map:

$$\mathcal{Z}^{\leqslant n} \to \mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n} \xrightarrow{\mathrm{Pol}} \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt = \Gamma(\mathcal{D}^{\leqslant n}, \omega_{\mathcal{D}^{\leqslant n}})$$

(with the last term really meaning the scheme attached to this vector space). We sometimes denote this section as ω^{univ} in what follows.

[48]This map is tautologically equivalent via Ψ to the adjunction morphism $\mathcal{O}_{\mathcal{Z}} \to f_*^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \in \mathsf{QCoh}(\mathcal{Z}^{\leqslant n})$.

We let $\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n}) \subseteq \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$ denote the fiber product:

$$(\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}) \underset{C/\mathbb{G}_m \times \mathbb{G}_m}{\times} (0/\mathbb{G}_m \times \mathbb{A}^1_y/\mathbb{G}_m).$$

(See §5.4.5 for an explanation of the notation.) By Proposition 2.6.7.2, the derived fiber product is classical.[49] We let α denote the embedding of $\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})$ into $\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$.

Pulling back (5.4.1), we obtain a canonical map:

$$\alpha_*^{\mathsf{IndCoh}}(\mathcal{O}_{\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})}) \to \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}} \in \mathsf{IndCoh}^*(\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n})^{\heartsuit} \simeq \mathsf{QCoh}(\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n})^{\heartsuit}. \quad (5.4.2)$$

(As the notation indicates, the superscript IndCoh on α_* is a matter of perspective: it is convenient for us for later purposes to view this morphism occurring in IndCoh^* as opposed to QCoh.)

Remark 5.4.4.1. In other words, the section ω^{univ} is scheme-theoretically supported on $\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})$.

5.4.5. We now claim that $\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})$ is the *classical* (not derived!) fiber product:

$$\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n}) = \Big((\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}) \underset{\mathcal{Y}_{\log}^{\leqslant n}}{\times} \mathcal{Z}^{\leqslant n}\Big)^{cl} \subseteq \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}. \quad (5.4.3)$$

Here the morphism $\mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \to \mathcal{Y}_{\log}^{\leqslant n}$ is act_{neg}.

Indeed, we have seen that $\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})$ is a classical stack, so it suffices to check the above identity on A-points for classical commutative rings A, i.e., we are free to manipulate our A-points in the most naive way.

To calculate $\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})$, we need to recall from Proposition 2.6.7.2 that in the notation of §5.4.2, the pullback of the line bundle $\mathcal{L}$ comes from the evident universal line bundle on $\widehat{\mathcal{D}} \times \mathcal{Z}$; i.e., its fiber at a point $(\tau, (\mathcal{L}, \nabla, s)) \in \widehat{\mathcal{D}} \times \mathcal{Z}$ is $\mathcal{L}|_\tau$. Its canonical section, denoted x in §5.4.2, corresponds to $s|_\tau \in \mathcal{L}|_\tau$.

Therefore, as a classical prestack, $\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n}) \subseteq \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$ corresponds to those data $(\tau, (\mathcal{L}, \nabla, s))$ with $s \in \mathcal{L}(-\tau)$.

Clearly this description exactly matches (5.4.3).

Remark 5.4.5.1. The derived version of the fiber product (5.4.3) is easily seen to differ from $\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})$, so our notation is a bit abusive from our perspective, which emphasizes derived geometry. (However, we do not feel so bad about this as we have provided another derivedly good definition of the space.)

5.4.6. We now construct a canonical map:

$$\mathrm{can} : i_{n,*}^{\mathsf{IndCoh}} \mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to \mathrm{act}_{neg,*}^{\mathsf{IndCoh}}(\omega_{\mathcal{D}^{\leqslant n}} \boxtimes i_{n,*}\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \in \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n})$$

as follows.

By the above, we have a commutative diagram:

$$\begin{array}{ccc}
\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n}) & \xrightarrow{\alpha} & \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \\
\downarrow{\scriptstyle \mathrm{act}'_{neg}} & & \downarrow{\scriptstyle \mathrm{act}_{neg}} \\
\mathcal{Z}^{\leqslant n} & \xrightarrow{i_n} & \mathcal{Y}_{\log}^{\leqslant n}.
\end{array}$$

[49]This fact is psychologically convenient, but not literally necessary in what follows. I.e., a straightforward restructuring of the discussion that follows could avoid directly appealing to this fact.

Now applying $i_{n,*}^{\mathsf{IndCoh}}$ to the evident counit map, we obtain:

$$i_{n,*}^{\mathsf{IndCoh}}\mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to i_{n,*}^{\mathsf{IndCoh}}\,\mathrm{act}'^{,\mathsf{IndCoh}}_{neg,*}\,\mathrm{act}'^{,*,\mathsf{IndCoh}}_{neg}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) = i_{n,*}^{\mathsf{IndCoh}}\,\mathrm{act}'^{,\mathsf{IndCoh}}_{neg,*}\,\mathcal{O}_{\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})} = \mathrm{act}^{\mathsf{IndCoh}}_{neg,*}\,\alpha_*^{\mathsf{IndCoh}}\mathcal{O}_{\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})}.$$

We then applying $\mathrm{act}^{\mathsf{IndCoh}}_{neg,*}$ to (5.4.2) to obtain a map:

$$\mathrm{act}^{\mathsf{IndCoh}}_{neg,*}\,\alpha_*^{\mathsf{IndCoh}}(\mathcal{O}_{\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})}) \to \mathrm{act}^{\mathsf{IndCoh}}_{neg,*}(\omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

Composing these two maps, we obtain the desired map can.

5.4.7. More generally, suppose $f \in k[[t]]/t^n$.

There is an associated action map $f \cdot - : \omega_{\mathcal{D}^{\leqslant n}} \to \omega_{\mathcal{D}^{\leqslant n}} \in \mathsf{QCoh}(\mathcal{D}^{\leqslant n})$. By functoriality, this gives rise to a morphism:

$$f \cdot - : \mathrm{act}^{\mathsf{IndCoh}}_{neg,*}(\omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \to \mathrm{act}^{\mathsf{IndCoh}}_{neg,*}(\omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

We then define:

$$\mathrm{can}_f : i_{n,*}^{\mathsf{IndCoh}}\mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to \mathrm{act}^{\mathsf{IndCoh}}_{neg,*}(\omega_{\mathcal{D}^{\leqslant n}} \boxtimes i_{n,*}\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$$

as the composition of $\mathrm{can}(= \mathrm{can}_1)$ with $f \cdot -$.

5.4.8. We now conclude the construction. Fix f as above.

By adjunction, there is a canonical morphism $\mathrm{AJ}_*^{-1,\mathsf{IndCoh}}(\omega_{\mathcal{D}^{\leqslant n}}) \to \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}}$.

Therefore, can_f gives rise to a morphism:

$$i_{n,*}^{\mathsf{IndCoh}}\mathcal{O}_{\mathcal{Z}^{\leqslant n}} \to \mathrm{act}^{\mathsf{IndCoh}}_{neg,*}(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}} \boxtimes i_{n,*}\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) = \mathrm{Oblv}\,\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n},w}(i_{n,*}\mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

By adjunction, we obtain the desired morphism:

$$\xi_f : \mathcal{F}_n \to \mathcal{F}_n.$$

5.4.9. Next, we show that the endomorphisms ξ_f of $\mathcal{F}_n$ mutually commute.

By construction,[50] the assertion immediately reduces to the following one.

Lemma 5.4.9.1. *The section:*

$$p_{13}^*(\omega^{univ}) \otimes p_{23}^*(\omega^{univ}) \in p_1^*(\omega_{\mathcal{D}^{\leqslant n}}) \otimes p_2^*(\omega_{\mathcal{D}^{\leqslant n}}) = \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}$$

is $\mathbb{Z}/2$-equivariant (for the natural action permuting the first two factors).

Proof. There is a canonical morphism:

$$\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n} \times \mathfrak{L}_n^+\mathbb{A}^1/\mathbb{G}_m$$

given as follows. To a point of the left hand side defined by $\tau_1 \in \mathcal{D}^{\leqslant n}$, $\tau_2 \in \mathcal{D}^{\leqslant n}$, a line bundle $\mathcal{L}$ on $\mathcal{D}^{\leqslant n}$ and a section $s \in \mathcal{L}$, our map assigns the line $\mathcal{L}|_{\tau_1} \otimes \mathcal{L}|_{\tau_2}$ with its section $s|_{\tau_1} \otimes s|_{\tau_2}$.

In fact, this map factors through a map:

$$\mathrm{Sym}^2(\mathcal{D}^{\leqslant n}) \times \mathfrak{L}_n^+\mathbb{A}^1/\mathbb{G}_m.$$

Indeed, for a point $(D, (\mathcal{L}, s))$ of the left hand side, we may regard D as a finite subscheme of $\mathcal{L}$. The map then sends the datum to $(\Lambda^2(s) \in \Lambda^2(\mathcal{L}|_D)) \in \mathbb{A}^1/\mathbb{G}_m$. I.e., this is evident by the usual norm construction.

Unwinding the constructions, this implies the claim. □

[50]See §5.5.2 for a much more general setup.

Remark 5.4.9.2. We could have organized the discussion differently. Instead, we could have generalized our construction of ξ_f with $\mathrm{Gr}_{\mathbb{G}_m}^{neg,\leqslant n}$ replacing $\mathcal{D}^{\leqslant n}$ via a suitable use of norms (as in the lemma above). Then we would immediately deduce commutativity of the operators ξ_f by the same argument and for the operators φ_ω.

5.5. **Uncertainty.** We now check the remaining Weyl algebra relation (5.2.3).

5.5.1. First, let us put the above constructions a common framework to allow us to compute the relevant compositions.

Let $\mathcal{H}^{\leqslant n} := \mathcal{Z}^{\leqslant n} \times_{\mathcal{Y}^{\leqslant n}} \mathcal{Z}^{\leqslant n}$. This stack is a groupoid over $\mathcal{Z}^{\leqslant n}$ with its projections $p_1, p_2 : \mathcal{H}^{\leqslant n} \rightrightarrows \mathcal{Z}^{\leqslant n}$ ind-proper.

By adjunction and base-change, morphisms $\mathcal{F}_n \to \mathcal{F}_n$ are equivalent to sections of $p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$. For[51] $\sigma \in \Gamma^{\mathsf{IndCoh}}(\mathcal{H}^{\leqslant n}, p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}))$, we let $\psi_\sigma : \mathcal{F}_n \to \mathcal{F}_n$ denote the corresponding morphism.

In particular, suppose we are given a prestack S with an ind-proper morphism $\eta : S \to \mathcal{H}^{\leqslant n}$. Given a section of $\eta^! p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$, we obtain a section of $p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$ by functoriality, hence a morphism as above.

Example 5.5.1.1. Let $S = \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$ and let $\eta : S \to \mathcal{H}^{\leqslant n}$ be the morphism:

$$(p_2, \mathrm{act} \circ (\mathrm{AJ} \times \mathrm{id})) : (\tau, (\mathcal{L}, \nabla, s) \mapsto ((\mathcal{L}, \nabla, s), (\mathcal{L}(\tau), \nabla, s)) \in \mathcal{Z}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Z}^{\leqslant n} = \mathcal{H}^{\leqslant n}.$$

For $\omega \in (k[[t]]/t^n)^\vee \overset{Prop.3.8.5.1}{=} \Gamma^{\mathsf{IndCoh}}(\mathcal{D}^{\leqslant n}, \omega_{\mathcal{D}^{\leqslant n}})$, we obtain the evident section $p_1^*(\omega)$ of:

$$\omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}} = p_2^! \mathcal{O}_{\mathcal{Z}^{\leqslant n}} = \eta^! p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

The corresponding map $\mathcal{F}_n \to \mathcal{F}_n$ is (5.2.1).

Example 5.5.1.2. In the notation of §5.4.4, let $S = \mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n}) \subseteq \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$. Let η be the evident morphism defined using (5.4.3).[52] The construction of can (resp. can_f) amounted to showing that the section ω^{univ} (hence $p_1^*(f) \cdot \omega^{univ}$) comes from a canonical section of $\eta^! p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$ (i.e., these sections are scheme-theoretically supported on $\mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})$). By construction, the corresponding map $\mathcal{F}_n \to \mathcal{F}_n$ is (5.2.2).

Remark 5.5.1.3. Although $\mathcal{H}^{\leqslant n}$ is not classical, our constructions here are not sensitive to this. The reason is that we have a closed embedding $\iota : \mathcal{H}^{\leqslant n} \hookrightarrow \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$ (cf. §5.5.3 below). As $\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}$ is ind-finite and $\mathcal{Z}^{\leqslant n}$ is classical, $\Gamma(p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})) = \Gamma(\iota^!(\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}))$ is a coconnective complex, and its H^0 is the same for $\mathcal{H}^{\leqslant n}$ and for its underlying classical stack (both identify with sections of $\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}$ scheme-theoretically supported on $\mathcal{H}^{\leqslant n,cl}$).

5.5.2. We now spell out how to concretely compose morphisms of the above type.

We have three morphisms:

$$p_{12}, p_{23}, p_{13} : \mathcal{H}^{\leqslant n} \underset{\mathcal{Z}^{\leqslant n}}{\times} \mathcal{H}^{\leqslant n} = \mathcal{Z}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Z}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Z}^{\leqslant n} \to \mathcal{Z}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Z}^{\leqslant n} = \mathcal{H}^{\leqslant n}.$$

We remark that p_{13} corresponds to the multiplication on the groupoid.

We begin by claiming that there is a canonical isomorphism:

$$p_{13}^! p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \simeq p_{12}^* p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \otimes p_{23}^* p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}). \tag{5.5.1}$$

[51]For $V \in \mathsf{Vect}$, $v \in V$ means $v \in \Omega^\infty V$, i.e., we have a point of the underlying ∞-groupoid. If we concretely model V as a cochain complex, we obtain such data from cycles in degree 0.

[52]In the notation of §5.4.6, this morphism is given by $(p_1\alpha, \mathrm{act}'_{neg})$.

To see this, first observe that $p_1 p_{13} = p_1 p_{12}$, so:

$$p_{13}^! p_1^! (\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) = p_{12}^! p_1^! (\mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

Next, we have evident isomorphisms:

$$p_{12}^! p_1^! \mathcal{O}_{\mathcal{Z}^{\leqslant n}} = p_{12}^! (p_1^! \mathcal{O}_{\mathcal{Z}^{\leqslant n}} \underset{\mathcal{O}_{\mathcal{H}^{\leqslant n}}}{\otimes} \mathcal{O}_{\mathcal{H}^{\leqslant n}}) = p_{12}^* p_1^! \mathcal{O}_{\mathcal{Z}^{\leqslant n}} \underset{\mathcal{H}^{\leqslant n} \underset{\mathcal{Z}^{\leqslant n}}{\times} \mathcal{H}^{\leqslant n}}{\otimes} p_{12}^! \mathcal{O}_{\mathcal{H}^{\leqslant n}}.$$

Then the Cartesian diagram:

$$\begin{array}{ccccc} \mathcal{H}^{\leqslant n} \underset{\mathcal{Z}^{\leqslant n}}{\times} \mathcal{H}^{\leqslant n} & \xrightarrow{p_{12}} & \mathcal{H}^{\leqslant n} & \xrightarrow{p_1} & \mathcal{Z}^{\leqslant n} \\ \downarrow{\scriptstyle p_{23}} & & \downarrow{\scriptstyle p_2} & & \\ \mathcal{H}^{\leqslant n} & \xrightarrow{p_1} & \mathcal{Z}^{\leqslant n} & & \end{array}$$

yields an isomorphism:

$$p_{12}^! \mathcal{O}_{\mathcal{H}^{\leqslant n}} = p_{12}^! p_2^* \mathcal{O}_{\mathcal{Z}^{\leqslant n}} = p_{23}^* p_1^! \mathcal{O}_{\mathcal{Z}^{\leqslant n}}.$$

Combining the above isomorphims yields (5.5.1).

Therefore, given σ_1, σ_2 sections of $p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$, we obtain a section:

$$p_{12}^*(\sigma_1) \otimes p_{23}^*(\sigma_2)$$

of $p_{13}^! p_1^! (\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$ (using (5.5.1)). As p_{13} is ind-proper, we obtain an induced section $\sigma_1 \star \sigma_2$ of $p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$.

Then unwinding constructions, we find:

$$\psi_{\sigma_2} \psi_{\sigma_1} = \psi_{\sigma_1 \star \sigma_2}.$$

5.5.3. To analyze the composition above, it is convenient to embed our spaces as follows.

First, define a morphism:

$$\iota : \mathcal{H}^{\leqslant n} \hookrightarrow \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$$

as:

$$\mathcal{H}^{\leqslant n} = \mathcal{Z}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Z}^{\leqslant n} \hookrightarrow \mathcal{Z}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Y}_{\log}^{\leqslant n} \xleftarrow{\simeq} \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$$

where the last arrow is (p_2, act). Note that the composition of ι with the projection to $\mathcal{Z}^{\leqslant n}$ is the projection $p_1 : \mathcal{H}^{\leqslant n} \to \mathcal{Z}^{\leqslant n}$.

Next, we define a similar morphism:

$$\widetilde{\iota} : \mathcal{H}^{\leqslant n} \underset{\mathcal{Z}^{\leqslant n}}{\times} \mathcal{H}^{\leqslant n} \hookrightarrow \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$$

as:

$$\mathcal{H}^{\leqslant n} \underset{\mathcal{Z}^{\leqslant n}}{\times} \mathcal{H}^{\leqslant n} = \mathcal{Z}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Z}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Z}^{\leqslant n} \hookrightarrow \mathcal{Z}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Y}_{\log}^{\leqslant n} \underset{\mathcal{Y}^{\leqslant n}}{\times} \mathcal{Y}_{\log}^{\leqslant n} \xleftarrow{\simeq} \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$$

where this time the last arrow is $(p_3, \mathrm{act} \circ p_{23}, \mathrm{act} \circ (\mathrm{id} \times \mathrm{act}))$.

As in Remark 5.5.1.3, a section of $\Gamma^{\mathsf{IndCoh}}(\mathcal{H}^{\leqslant n}, p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}))$ is the same as a section of $\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}$ scheme-theoretically supported on $\mathcal{H}^{\leqslant n}$ (equivalently, on the underlying classical stack). The same applies for $\widetilde{\iota}$: a section of $p_{13}^! p_1^! (\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$ is the same as a section of:

$$\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}} \boxtimes \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}$$

scheme-theoretically supported on $\mathcal{H}^{\leqslant n} \times_{\mathcal{Z}^{\leqslant n}} \mathcal{H}^{\leqslant n}$ (which is equivalent to saying at the classical level).

5.5.4. Now fix $\omega \in (k[[t]]/t^n)^\vee$ and $f \in k[[t]]/t^n$. In effect, §5.5.2 constructed two sections of $p_{13}^! p_1^!(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$, which induce the compositions $\xi_f \varphi_\omega$ and $\varphi_\omega \xi_f$ (after pushforward along p_{13}).

Let us explicitly describe the resulting sections of $\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}} \boxtimes \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}$ under the above dictionary.

For $\varphi_\omega \xi_f$, the corresponding section is supported on:

$$\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \overset{\mathrm{AJ} \times \mathrm{AJ}^{-1} \times \mathrm{id}}{\hookrightarrow} \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}.$$

The corresponding section of:

$$\omega_{\mathcal{D}^{\leqslant n}} \boxtimes \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}$$

is $\omega \boxtimes p_2^*(f) \cdot \omega^{univ}$, with notation as in Examples 5.5.1.1 and 5.5.1.2.[53] It is convenient to rewrite this section as:

$$p_2^*(f) \cdot (p_1^*(\omega) \otimes p_{23}^*(\omega^{univ})) \in p_1^*(\omega_{\mathcal{D}^{\leqslant n}}) \otimes p_{23}^*(\omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

For $\xi_f \varphi_\omega$, the corresponding section is supported on:

$$\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \xrightarrow{\mathrm{AJ}^{-1} \times \mathrm{AJ} \times \mathrm{id}} \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}.$$

The corresponding section of:

$$\omega_{\mathcal{D}^{\leqslant n}} \boxtimes \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}$$

is:[54]

$$p_1^*(f) \cdot (p_2^*(\omega) \otimes (p_1 \times \mathrm{act})^*(\omega^{univ})) \in p_2^*(\omega_{\mathcal{D}^{\leqslant n}}) \otimes (p_1 \times \mathrm{act})^*(\omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}) = \\ \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leqslant n}}.$$

Now note that we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{H}^{\leqslant n} \underset{\mathcal{Z}^{\leqslant n}}{\times} \mathcal{H}^{\leqslant n} & \xrightarrow{\tilde{\iota}} & \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \\ \downarrow{\scriptstyle p_{13}} & & \downarrow{\scriptstyle m \times \mathrm{id}} \\ \mathcal{H}^{\leqslant n} & \xrightarrow{\iota} & \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \end{array}$$

for m the multiplication on $\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}$. As this multiplication is commutative, we see that $\xi_f \varphi_\omega$ could as well have been defined by the section as above obtained by transposing the first two coordinates, i.e.:

$$p_2^*(f) \cdot (p_1^*(\omega) \otimes (p_2, \mathrm{act} \circ p_{13})^*(\omega^{univ})).$$

This section has the advantage of sharing its support with our previous one for $\varphi_\omega \xi_f$.

[53]As we have shown, this section is scheme-theoretically supported on $\mathcal{D}^{\leqslant n} \boxtimes \mathrm{act}_{neg}^{-1}(\mathcal{Z}^{\leqslant n})$, hence on $(\mathcal{H}^{\leqslant n} \times_{\mathcal{Z}^{\leqslant n}} \mathcal{H}^{\leqslant n})^{cl}$.

[54]For clarity, we again highlight that we have shown this section is in fact supported on $(\mathcal{H}^{\leqslant n} \times_{\mathcal{Z}^{\leqslant n}} \mathcal{H}^{\leqslant n})^{cl}$, as it should be. Indeed, at the classical level, our previous analysis shows that it is supported on $(\mathrm{act}_{neg} \circ (\mathrm{id} \times \mathrm{act}))^{-1}(\mathcal{Z}^{\leqslant n}) \subseteq (\mathcal{H}^{\leqslant n} \times_{\mathcal{Z}^{\leqslant n}} \mathcal{H}^{\leqslant n})^{cl}$.

5.5.5. We can now conclude the argument. We use the next elementary lemmas.

Lemma 5.5.5.1. *Let:*

$$\mathrm{cotr} \in \Gamma(\mathcal{D}^{\leq n}, \mathcal{O}_{\mathcal{D}^{\leq n}}) \otimes \Gamma(\mathcal{D}^{\leq n}, \mathcal{O}_{\mathcal{D}^{\leq n}})^\vee = \Gamma^{\mathsf{IndCoh}}(\mathcal{D}^{\leq n} \times \mathcal{D}^{\leq n}, \mathcal{O}_{\mathcal{D}^{\leq n}} \boxtimes \omega_{\mathcal{D}^{\leq n}})$$

denote the canonical vector.

We then have an equality:

$$p_{23}^*(\omega^{univ}) = (p_2, \mathrm{act} \circ p_{13})^*(\omega^{univ}) + p_{12}^*(\mathrm{cotr}) \in \mathcal{O}_{\mathcal{D}^{\leq n}} \boxtimes \omega_{\mathcal{D}^{\leq n}} \boxtimes \mathcal{O}_{\mathcal{Z}^{\leq n}}.$$

Proof. For convenience, we give the first (resp. second) factor of our triple product as $\mathcal{D}_{\tau_1}^{\leq n}$ (resp. $\mathcal{D}_{\tau_2}^{\leq n}$), where τ_1, τ_2 denote the respective coordinates.

For a prestack S, note that a section of $\omega_{\mathcal{D}_{\tau_2}^{\leq n}} \boxtimes \mathcal{O}_S$ is equivalent to a map $S \to \mathfrak{L}^{pol,\leq n}\mathbb{A}^1 dt$.

For $S = \mathcal{D}_{\tau_1}^{\leq n} \times \mathcal{Z}^{\leq n}$, our sections above correspond to maps:

$$\mathcal{D}_{\tau_1}^{\leq n} \times \mathcal{Z}^{\leq n} \xrightarrow{(\tau_1,(\mathcal{L},\nabla,s)) \mapsto ???} \mathfrak{L}^{pol,\leq n}\mathbb{A}^1 dt$$

as follows:

$$\begin{aligned} p_{23}^*(\omega^{univ}) &\rightsquigarrow (\tau_1, (\mathcal{L}, \nabla, s)) \mapsto \mathrm{Pol}(\nabla) \\ (p_2, \mathrm{act} \circ p_{13})^*(\omega^{univ})) &\rightsquigarrow (\tau_1, (\mathcal{L}, \nabla, s)) \mapsto \mathrm{Pol}((\mathcal{L}(\tau_1), \nabla, s)) = \\ &\qquad \mathrm{Pol}(\nabla) + d \log \mathrm{AJ}(\tau_1) \\ p_{12}^*(\mathrm{cotr}) &\rightsquigarrow (\tau_1, (\mathcal{L}, \nabla, s)) \mapsto -d \log \mathrm{AJ}(\tau_1) \end{aligned}$$

where in the last two lines the map $d \log \mathrm{AJ}$ was considered in §3.3-3.4; the last line follows from Lemma 3.3.0.2.

We clearly obtain the assertion.

□

We also use the following observations.

Lemma 5.5.5.2. *The above section* cotr *of* $\mathcal{O}_{\mathcal{D}^{\leq n}} \boxtimes \omega_{\mathcal{D}^{\leq n}}$ *is scheme-theoretically supported on the diagonally embedded* $\mathcal{D}^{\leq n} \subseteq \mathcal{D}^{\leq n} \times \mathcal{D}^{\leq n}$.

Moreover, for $f \in k[[t]]/t^n$ *and* $\omega \in t^{-n}k[[t]]dt/k[[t]]dt$*, the section:*

$$p_2^*(f) \cdot (p_1^*(\omega) \otimes \mathrm{cotr}) \in \omega_{\mathcal{D}^{\leq n}} \boxtimes \omega_{\mathcal{D}^{\leq n}}$$

maps to $\mathrm{Res}(f\omega) \in k$ *under the adjunction morphism:*

$$\Gamma^{\mathsf{IndCoh}}(\mathcal{D}^{\leq n} \times \mathcal{D}^{\leq n}, \omega_{\mathcal{D}^{\leq n}} \boxtimes \omega_{\mathcal{D}^{\leq n}}) \to k.$$

Proof. Let A be a commutative, classical, finite k-algebra.

As $\mathrm{id}_A : A \to A$ is a morphism of A-bimodules, the corresponding element $\mathrm{cotr} \in A^\vee \otimes A \in A \otimes A\text{–}\mathsf{mod}^\heartsuit$ is scheme-theoretically supported on the diagonal $\mathrm{Spec}(A) \subseteq \mathrm{Spec}(A) \times \mathrm{Spec}(A)$.

Moreover, for $f \in A$ and $\omega \in A^\vee$, we have an element $f \otimes \omega \in A \otimes A^\vee$. Tensoring over $A \otimes A$ with $A^\vee \otimes A$, we obtain an element:

$$f \otimes \omega \cdot \mathrm{cotr} \in A^\vee \otimes A^\vee.$$

It is easy to see that when we evaluate this tensor on $1 \otimes 1 \in A \otimes A$, we obtain $\omega(f) \in k$.

Taking $A = k[[t]]/t^n$, we obtain our claim.

□

By our earlier discussion, the endomorphism $\xi_f\varphi_\omega - \varphi_\omega\xi_f$ of $\mathcal{F}_n$ corresponds to the section:

$$p_2^*(f)\cdot(p_1^*(\omega)\otimes(p_2,\mathrm{act}\circ p_{13})^*(\omega^{univ})) - p_2^*(f)\cdot(p_1^*(\omega)\otimes p_{23}^*(\omega^{univ}))$$

of $\omega_{\mathcal{D}^{\leqslant n}}\boxtimes\omega_{\mathcal{D}^{\leqslant n}}\boxtimes\mathcal{O}_{\mathcal{Z}^{\leqslant n}}$.[55]

By Lemma 5.5.5.1, the above section coincides with:

$$-p_2^*(f)\cdot p_{12}^*(p_1^*(\omega)\otimes\mathrm{cotr}).$$

By Lemma 5.5.5.2, this section is supported on the diagonally embedded:

$$\mathcal{D}^{\leqslant n}\times\mathcal{Z}^{\leqslant n}\xrightarrow{\Delta\times\mathrm{id}}\mathcal{D}^{\leqslant n}\times\mathcal{D}^{\leqslant n}\times\mathcal{Z}^{\leqslant n},$$

which evidently maps to $1\times\mathcal{Z}^{\leqslant n}\subseteq\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}\times\mathcal{Z}^{\leqslant n}$ under p_{13}. Moreover, Lemma 5.5.5.2 implies that the pushforward of the above section to $\mathcal{Z}^{\leqslant n}=1\times\mathcal{Z}^{\leqslant n}$ is:

$$-\mathrm{Res}(f\omega)\in k\subseteq\Gamma(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})\subseteq\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}\times\mathcal{Z}^{\leqslant n},\omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}}\boxtimes\mathcal{Z}^{\leqslant n}).$$

This amounts to (5.2.3), concluding the argument.

6. Compatibility with class field theory

In this section, we show that the construction of §5 is compatible with geometric class field theory in a suitable sense.

6.1. Conventions regarding D-modules.

6.1.1. Before proceeding, we take a moment to establish some notational conventions regarding D-modules. We refer to [GR1] for details.

6.1.2. Let X be a smooth variety. As in *loc. cit.*, we have the so-called *left* forgetful functor $\mathrm{Oblv}^\ell : D(X)\to\mathsf{QCoh}(X)$. It is normalized so that the functor $\mathrm{Oblv}^\ell[\dim X]$ is t-exact; for example, this functor sends the dualizing ω_X to $\mathcal{O}_X$ (with its standard left D-module structure).

6.1.3. The functor Oblv^ℓ admits the left adjoint ind^ℓ. For us, we take as a definition that the sheaf of differential operators on X is:

$$D_X := \mathrm{ind}^\ell(\mathcal{O}_X).$$

We remark that this object is concentrated in degree $-\dim X$. (On general grounds, it coincides with $\mathrm{ind}^r(\omega_X)$, where ind^r is the right D-module induction functor and $\omega_X=\Omega_X^{\dim X}[\dim X]\in\mathsf{QCoh}(X)$ is the dualizing sheaf.)

6.1.4. Now suppose X is smooth and affine. Let $\mathrm{Diff}(X)$ be the algebra of global differential operators on X.

As in [GR1], $\mathrm{End}_{D(X)}(D_X)$ canonically coincides with $\mathrm{Diff}(X)$, but with reversed multiplication. It follows that there is a canonical equivalence:

$$\mathrm{Diff}(X)\text{–}\mathsf{mod}\simeq D(X) \tag{6.1.1}$$

between left modules for $\mathrm{Diff}(X)$ and the category $D(X)$, sending $\mathrm{Diff}(X)$ to D_X. We remark that this functor is only t-exact up to shift.

6.2. Construction of functors.

[55]Of course we consider $\mathcal{D}^{\leqslant n}\times\mathcal{D}^{\leqslant n}\times\mathcal{Z}^{\leqslant n}$ mapping to $\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}\times\mathcal{Z}^{\leqslant n}$ via $(\tau_1,\tau_2,(\mathcal{L},\nabla,s))\mapsto(\mathrm{AJ}(\tau_1)\cdot\mathrm{AJ}^{-1}(\tau_2),(\mathcal{L},\nabla,s))$.

6.2.1. In §5, we constructed compact objects $\mathcal{F}_n \in \mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})^c$ and morphisms:

$$W_n^{op} \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n).$$

We obtain a corresponding functor:

$$\overline{\Delta}_n : W_n\text{–}\mathsf{mod} \overset{(6.1.1)}{\simeq} D(\mathfrak{L}_n^+\mathbb{A}^1) \to \mathsf{IndCoh}^*(\mathcal{Y}_n) \in \mathsf{DGCat}_{cont}$$

uniquely characterized by sending $D_{\mathfrak{L}_n^+\mathbb{A}^1}$ to $\mathcal{F}_n$ compatibly with the action of $W_n^{op} = \mathsf{End}_{D(\mathfrak{L}_n^+\mathbb{A}^1)}(D_{\mathfrak{L}_n^+\mathbb{A}^1})$.

Remark 6.2.1.1. Above, we carefully normalized $D_{\mathfrak{L}_n^+\mathbb{A}^1}$ to lie in cohomological degree $-n$. This is not especially relevant in our analysis until §8.

6.2.2. For later use, we also use the notation Δ_n to denote the further composition:

$$D(\mathfrak{L}_n^+\mathbb{A}^1) \xrightarrow{\overline{\Delta}_n} \mathsf{IndCoh}^*(\mathcal{Y}_n) \xrightarrow{\lambda_{n,*}^{\mathsf{IndCoh}}} \mathsf{IndCoh}^*(\mathcal{Y}).$$

6.3. **Cartier duality.** We now review some constructions from geometric class field theory.

6.3.1. Let $\mathcal{K}_n \subseteq \mathfrak{L}^+\mathbb{G}_m$ be the nth congruence subgroup, i.e., $\mathcal{K}_n := \mathrm{Ker}(\mathfrak{L}^+\mathbb{G}_m \to \mathfrak{L}_n^+\mathbb{G}_m)$.

Recall that there is a canonical bimultiplicative pairing:

$$\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n} \times (\mathfrak{L}\mathbb{G}_m/\mathcal{K}_n)_{dR} \to \mathbb{B}\mathbb{G}_m. \tag{6.3.1}$$

One basic property is that its restriction along the map:

$$\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n} \times \mathfrak{L}\mathbb{G}_m/\mathcal{K}_n \to \mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n} \times (\mathfrak{L}\mathbb{G}_m/\mathcal{K}_n)_{dR} \to \mathbb{B}\mathbb{G}_m$$

factors through the Contou-Carrère duality pairing:

$$\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n} \times \mathfrak{L}\mathbb{G}_m/\mathcal{K}_n \to \mathbb{B}\mathfrak{L}^{\leqslant n}\mathbb{G}_m \times \mathfrak{L}\mathbb{G}_m/\mathcal{K}_n \xrightarrow{CC} \mathbb{B}\mathbb{G}_m.$$

Here we normalize the sign in the Contou-Carrère pairing so that we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{B}\mathfrak{L}\mathbb{G}_m \times \mathfrak{L}^+\mathbb{G}_m & \longrightarrow & \mathbb{B}\mathfrak{L}\mathbb{G}_m \times \mathfrak{L}\mathbb{G}_m \\ \downarrow & & \downarrow CC \\ \mathbb{B}\,\mathrm{Gr}_{\mathbb{G}_m} \times \mathfrak{L}^+\mathbb{G}_m & \longrightarrow & \mathbb{B}\mathbb{G}_m \end{array}$$

where the bottom arrow is induced by the pairing of Proposition 3.8.5.1.

From (6.3.1) and functoriality, we obtain a symmetric monoidal Fourier-Mukai functor:

$$D(\mathfrak{L}\mathbb{G}_m/\mathcal{K}_n) \to \mathsf{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n}). \tag{6.3.2}$$

Theorem 6.3.1.1. *The functor* (6.3.2) *is an equivalence. In particular, there is a canonical fully faithful symmetric monoidal functor* $D(\mathfrak{L}_n^+\mathbb{G}_m) \hookrightarrow \mathsf{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n})$.

Passing to the limit over n, we also obtain the following theorem of Beilinson-Drinfeld.

Theorem 6.3.1.2. *There is a canonical symmetric monoidal equivalence:*

$$D^*(\mathfrak{L}\mathbb{G}_m) \xrightarrow{\simeq} \mathsf{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}).$$

6.4. **Formulation of the equivariance property.** Now observe that $\overline{\Delta}_n$ maps a category acted on by $D(\mathfrak{L}_n^+\mathbb{G}_m)$ to one acted on by $\mathsf{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n})$ (via the structure map $\mathcal{Y}^{\leqslant n} \to \mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n}$).

Using Theorem 6.3.1.1, we can regard both sides as acted on by $D(\mathfrak{L}_n^+\mathbb{G}_m)$.

In the remainder of this section, we show that $\overline{\Delta}_n$ is canonically a morphism of $D(\mathfrak{L}_n^+\mathbb{G}_m)$-module categories.

6.5. **Digression on Harish-Chandra data.** We will construct the equivariance structure using the theory of *Harish-Chandra data*. We review this theory below.

In what follows, G is an affine algebraic group and $A \in \mathsf{Alg}$ is a DG algebra. In our applications, A will be classical, so at some points in the discussion we will assume that.

6.5.1. First, suppose that G acts on A as an associative algebra. In other words, we assume we are given a lift of A along the forgetful functor $\mathsf{Alg}(\mathsf{Rep}(G)) \to \mathsf{Alg}(\mathsf{Vect}) = \mathsf{Alg}$.

This defines a canonical weak action of G on $A\text{–}\mathsf{mod}$. Its basic property is that the forgetful functor $A\text{–}\mathsf{mod} \to \mathsf{Vect}$ is weakly G-equivariant.

6.5.2. Now suppose that $\mathcal{C}$ is a DG category with a weak G-action.

Recall that $\mathsf{Rep}(G)$ naturally acts on $\mathcal{C}^{G,w}$. Therefore, we can consider $\mathcal{C}^{G,w}$ as enriched over $\mathsf{Rep}(G)$. For $\mathcal{F}, \mathcal{G} \in \mathcal{C}^{G,w}$, we let:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}^{enh}(\mathcal{F}, \mathcal{G}) \in \mathsf{Rep}(G)$$

denote the corresponding mapping complex.

Remark 6.5.2.1. There are natural maps:

$$\mathrm{Oblv}\, \underline{\mathrm{Hom}}_{\mathcal{C}}^{enh}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathrm{Oblv}(\mathcal{F}), \mathrm{Oblv}(\mathcal{G}))$$

that are easily seen to be isomorphisms for $\mathcal{F}$ compact.

Similarly, for $\mathcal{F} \in \mathcal{C}^{G,w}$, we let:

$$\underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F}, \mathcal{G}) \in \mathsf{Alg}(\mathsf{Rep}(G))$$

denote the inner endomorphism algebra.

6.5.3. In our earlier setting, suppose we are given a weakly G-equivariant functor:

$$F : A\text{–}\mathsf{mod} \to \mathcal{C}.$$

Note that $A \in A\text{–}\mathsf{mod}^{G,w}$, so we obtain a natural object $\mathcal{F} := F^{G,w}(A) \in \mathcal{C}^{G,w}$. Moreover, there is a canonical map:

$$\varphi : A \simeq \underline{\mathrm{End}}_{A\text{–}\mathsf{mod}}^{enh}(A) \to \underline{\mathrm{End}}_{\mathcal{C}^{G,w}}(\mathcal{F})^{op} \in \mathsf{Alg}(\mathsf{Rep}(G)).$$

(Here the superscript *op* indicates the reversed multiplication.)

Conversely, given $\mathcal{F} \in \mathcal{C}^{G,w}$ and φ as above, we obtain a functor F as above. Indeed, as is standard, a datum φ is the same as specifying a morphism

$$A\text{–}\mathsf{mod}^{G,w} = A\text{–}\mathsf{mod}(\mathsf{Rep}(G)) \to \mathcal{C}^{G,w}$$

of $\mathsf{Rep}(G)$-module categories; by de-equivariantization (i.e., tensoring over $\mathsf{Rep}(G)$ with Vect), we obtain the desired functor $F : A\text{–}\mathsf{mod} \to \mathcal{C}$.

6.5.4. Next, recall from [Ras7] §10 that $\mathsf{Alg}(\mathsf{Rep}(G))$ carries a canonical monad $B \mapsto U(\mathfrak{g})\#B$.

Here $U(\mathfrak{g})\#B$ is the usual smash (or crossed) product construction. Non-derivedly, modules for $U(\mathfrak{g})\#B$ are vector spaces equipped with an action of B and an action of $\mathfrak{g}$ that are compatible under the action of $\mathfrak{g}$ on B by derivations. For a proper derived construction (in the more complicated setting of topological algebras), we refer to [Ras7].

By definition, making $A \in \mathsf{Alg}(\mathsf{Rep}(G))$ into a module over this monad is a *Harish-Chandra* datum for A. When A is classical, this amounts to the data of a morphism:

$$i : \mathfrak{g} \to A$$

such that i is a G-equivariant morphism of Lie algebras such that $[i(\xi), -] : A \to A$ coincides with the derivation defined by $\xi \in \mathfrak{g}$ and the G-action on A.

From *loc. cit.*, it follows that specifying a Harish-Chandra datum for A is the same as extending the weak G-action on A–mod to a strong G-action.

6.5.5. Now suppose that G acts strongly on $\mathcal{C}$. Let $\mathcal{F} \in \mathcal{C}^{G,w}$ be a fixed object. We claim that $\underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F}) \in \mathsf{Alg}(\mathsf{Rep}(G))$ carries a canonical Harish-Chandra datum.

The construction is formal. Let $\mathcal{D}_0 \subseteq \mathcal{C}^{G,w}$ be the (non-cocomplete) DG category generated by $\mathcal{F}$ under finite colimits and direct summands. As in [Ber], the monoidal category:

$$\mathcal{HC}_G := \mathsf{End}_{G\text{–}\mathsf{mod}}(D(G)^{G,w})^{op} = \mathfrak{g}\text{–}\mathsf{mod}^{G,w}$$

of Harish-Chandra bimodules acts canonically on $\mathcal{C}^{G,w}$. Moreover, $\mathcal{HC}_G$ is *rigid* monoidal, so its (non-cocomplete) monoidal subcategory $\mathcal{HC}_G^c$ of compact objects preserves $\mathcal{D}_0$. Therefore, $\mathcal{HC}_G$ acts on $\mathcal{D}_1 := \mathsf{Ind}(\mathcal{D}_0)$. By de-equivariantization, $\mathcal{D}_1 = \mathcal{D}^{G,w}$ for a canonical strong G-category $\mathcal{D}$. But clearly $\mathcal{D} = \underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F})$–mod by construction, so we obtain the Harish-Chandra datum as desired (from §6.5.4).

Remark 6.5.5.1. In particular, there is a canonical map $\mathfrak{g} \to \underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F})$ of Lie algebras in $\mathsf{Rep}(G)$. This map is the standard *obstruction to strong equivariance* on $\mathcal{F}$.

Moreover, we see that if $A \in \mathsf{Alg}(\mathsf{Rep}(G))$ is equipped with a Harish-Chandra datum, giving a strongly G-equivariant functor:

$$F : A\text{–}\mathsf{mod} \to \mathcal{C}$$

is equivalent to giving $\mathcal{F} \in \mathcal{C}^{G,w}$ and a map $\varphi : A \to \underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F})^{op}$ that is *compatible with the Harish-Chandra data*.

In the case that A is classical and $\underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F})^{op}$ is coconnective, this simply amounts to saying that composition:

$$\mathfrak{g} \xrightarrow{i} A \xrightarrow{\varphi} \underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F})^{op}$$

coincides with the obstruction to strong equivariance on the object $\mathcal{F}$.

6.5.6. We now add one more mild observation before concluding our discussion. The reader may safely skip this material and return back to it as necessary.

Suppose G acts weakly on $\mathcal{C}$ and $\mathcal{F}, \mathcal{G} \in \mathcal{C}^{G,w}$. There is a canonical map:

$$\mathrm{Oblv}(\underline{\mathrm{Hom}}_{\mathcal{C}}^{enh}(\mathcal{F}, \mathcal{G})) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathrm{Oblv}(\mathcal{F}), \mathrm{Oblv}(\mathcal{G})) \in \mathsf{Vect}$$

that is an isomorphism whenever $\mathcal{F}$ is compact. (This is the reason we use the subscript $\mathcal{C}$ in $\underline{\mathrm{Hom}}_{\mathcal{C}}^{enh}$ and not $\underline{\mathrm{Hom}}_{\mathcal{C}^{G,w}}^{enh}$: the underlying vector space of the representation $\underline{\mathrm{Hom}}_{\mathcal{C}}^{enh}(\mathcal{F}, \mathcal{G})$ is comparable to, and often isomorphic to, the complex of maps from $\mathcal{F}$ to $\mathcal{G}$ in $\mathcal{C}$ itself, not in $\mathcal{C}^{G,w}$.)

Now suppose that we are in the setting of §6.5.5. Suppose $\mathcal{F} \in \mathcal{C}^{G,w}$ is compact, A is classical, and $\underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F})$ is coconnective. Then the data of a map:

$$A \to \underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F})^{op}$$

compatible with the Harish-Chandra data is equivalent to a map:

$$\varphi : A \to \underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F})^{op} \in \mathsf{Alg} = \mathsf{Alg}(\mathsf{Vect})$$

such that the composition:

$$\mathfrak{g} \xrightarrow{i} A \xrightarrow{\varphi} \underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F})^{op}$$

is the obstruction to strong equivariance. Indeed, this follows from the assumption that $\mathcal{F}$ is compact, so $\underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F})^{op} = \underline{\mathrm{End}}_{\mathcal{C}}^{enh}(\mathcal{F})^{op}$, and the connectedness of G, so that the map φ will automatically be a morphism of G-representations. (We remark that the co/connectivity assumptions allow us to replace $\underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F})^{op}$ with the classical algebra $H^0\underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F})^{op}$.)

6.6. **Reduction.** We now return to our particular setting, and apply the above material to reduce our construction to a calculation.

6.6.1. Let $G = \mathfrak{L}_n^+\mathbb{G}_m$ act strongly on $\mathcal{C} := \mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})$ via geometric class field theory. By construction, we have:

$$\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})^{\mathfrak{L}_n^+\mathbb{G}_m,w} \simeq \mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n}_{\log}).$$

Under this dictionary, the forgetful functor $\mathcal{C}^{G,w} \to \mathcal{C}$ corresponds to the IndCoh-pushforward $\mathcal{Y}^{\leqslant n}_{\log} \to \mathcal{Y}^{\leqslant n}$ (i.e., !-averaging for $\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m}$).

6.6.2. We have the object $i_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \in \mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n}_{\log})$ that maps to $\mathcal{F}_n \in \mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})$ under this forgetful functor. In §5, we constructed the map:

$$\varphi : A := W_n \to \underline{\mathrm{End}}^{enh}_{\mathcal{C}}(\mathcal{F}_n)^{op} \simeq \underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F}_n)^{op}.$$

(We remind that the displayed isomorphism is a formal consequence of compactness of $\mathcal{F}_n$.)

Now on the one hand, we have a morphism:

$$\mathrm{Lie}(\mathfrak{L}_n^+\mathbb{G}_m) \simeq k[[t]]/t^n \to \Gamma(\mathfrak{L}_n^+\mathbb{A}^1, T_{\mathfrak{L}_n^+\mathbb{A}^1}) \to W_n$$

encoding the infinitesimal action of $\mathfrak{L}_n^+\mathbb{G}_m$ on $\mathfrak{L}_n^+\mathbb{A}^1$. We denote this morphism by α.

On the other hand, we have the composition:

$$\mathrm{Lie}(\mathfrak{L}_n^+\mathbb{G}_m) \simeq k[[t]]/t^n \simeq (t^{-n}k[[t]]dt/k[[t]]dt)^\vee \subseteq \Gamma(\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt, \mathcal{O}_{\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt}) \to \underline{\mathrm{End}}^{enh}_{\mathcal{C}}(\mathcal{F}_n)^{op}$$

where the last morphism here uses the map:

$$\mathcal{Z}^{\leqslant n} \subseteq \mathcal{Y}^{\leqslant n}_{\log} \to \mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log} \xrightarrow{\mathrm{Pol}} \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt.$$

We denote this morphism by β.

In §6.7, we will prove:

Lemma 6.6.2.1. *The diagram:*

$$\begin{array}{ccc} \mathrm{Lie}(\mathfrak{L}_n^+\mathbb{G}_m) \simeq k[[t]]/t^n & & \\ \downarrow \alpha & \searrow^{\beta} & \\ W_n & \xrightarrow{\varphi} & \underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F}_n) \end{array}$$

commutes.

Observe that under Theorem 6.3.1.1, β is interpreted as the obstruction to strong equivariance. Therefore, from §6.5.6, we deduce:

Proposition 6.6.2.2. *The functor $\overline{\Delta}_n$ is canonically a morphism of $D(\mathfrak{L}_n^+\mathbb{G}_m)$-module categories.*

6.7. **Proof of Lemma 6.6.2.1.**

6.7.1. Using the additive structure on $\mathfrak{L}_n^+\mathbb{A}^1$, we compute its global vector fields as:

$$\Gamma(\mathfrak{L}_n^+\mathbb{A}^1, T_{\mathfrak{L}_n^+\mathbb{A}^1}) \simeq k[[t]]/t^n \otimes \mathrm{Sym}(t^{-n}k[[t]]dt/k[[t]]dt).$$

A straightforward calculation shows that the infinitesimal action map:

$$\mathrm{Lie}(\mathfrak{L}_n^+\mathbb{G}_m) \simeq k[[t]]/t^n \to \Gamma(\mathfrak{L}_n^+\mathbb{A}^1, T_{\mathfrak{L}_n^+\mathbb{A}^1}) \simeq k[[t]]/t^n \otimes \mathrm{Sym}(t^{-n}k[[t]]dt/k[[t]]dt)$$

which was used in the definition of α above, is the map:

$$\begin{aligned}(f \in k[[t]]/t^n) \mapsto (1 \otimes f) \cdot \mathrm{cotr} = (f \otimes 1) \cdot \mathrm{cotr} \in k[[t]]/t^n \otimes t^{-n}k[[t]]dt/k[[t]]dt \\ \subseteq k[[t]]/t^n \otimes \mathrm{Sym}(t^{-n}k[[t]]dt/k[[t]]dt).\end{aligned}$$

Here we recall the canonical vector:

$$\mathrm{cotr} \in \Gamma(\mathcal{D}^{\leqslant n}, \mathcal{O}_{\mathcal{D}^{\leqslant n}}) \otimes \Gamma(\mathcal{D}^{\leqslant n}, \mathcal{O}_{\mathcal{D}^{\leqslant n}})^{\vee} =$$
$$\Gamma^{\mathsf{IndCoh}}(\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n}, \mathcal{O}_{\mathcal{D}^{\leqslant n}} \boxtimes \omega_{\mathcal{D}^{\leqslant n}}) = k[[t]]/t^n \otimes t^{-n}k[[t]]dt/k[[t]]dt$$

from Lemma 5.5.5.1.

Therefore, we have:

$$\alpha(f) = (1 \otimes f) \cdot \mathrm{cotr} \in k[[t]]/t^n \otimes t^{-n}k[[t]]dt/k[[t]]dt \subseteq W_n.$$

6.7.2. To simplify the discussion, we take $f = 1$ for the time being. We wish to show that $\varphi\alpha(1) = \beta(1)$, i.e., we wish to show that $\varphi(\mathrm{cotr}) = \beta(1)$. We will adapt the calculation to discuss the case of general f in §6.7.7.

6.7.3. Let us unwind the construction of $\varphi(\mathrm{cotr})$ in this case.

First, we form:

$$\mathrm{cotr} \in \Gamma^{\mathsf{IndCoh}}(\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n}, \mathcal{O}_{\mathcal{D}^{\leqslant n}} \boxtimes \omega_{\mathcal{D}^{\leqslant n}}).$$

Second, we form:[56]

$$\omega^{univ} \in \Gamma^{\mathsf{IndCoh}}(\mathcal{D}^{\leqslant n} \times \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt, \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt}).$$

We then form:

$$p_{12}^*(\mathrm{cotr}) \cdot p_{23}^*(\omega^{univ}) \in \Gamma^{\mathsf{IndCoh}}(\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n} \times \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt, \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt}).$$

According to §5.5.4, $\varphi(\mathrm{cotr})$ is calculated as follows. We pull back the above section to $\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$, pushforward along:

$$\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \xrightarrow{\mathrm{AJ}^{-1} \times \mathrm{AJ} \times \mathrm{id}} \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \xrightarrow{\mathrm{mult} \times \mathrm{id}} \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathcal{Z}^{\leqslant n}$$

and apply the construction of §5.5.1.

Of course, the pullback and the pushforward commute here. Therefore, in what follows we calculate the pushforward with $\mathcal{Z}^{\leqslant n}$ replaced by $\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt$.

In what follows, let:

$$\sigma \in \Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt, \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt})$$

denote the resulting section. So our objective, in effect, is to calculate σ.

6.7.4. Recall from Proposition 3.8.5.1 that:

$$\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}, \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}}) \simeq \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, \mathcal{O}_{\mathfrak{L}_n^+\mathbb{G}_m}). \tag{6.7.1}$$

Moreover, this is an isomorphism of commutative algebras. It is convenient to reinterpret the above construction using this isomorphism.

By construction, note that the composition:

$$t^{-n}k[[t]]dt/k[[t]]dt \simeq \Gamma^{\mathsf{IndCoh}}(\mathcal{D}^{\leqslant n}, \omega_{\mathcal{D}^{\leqslant n}}) \to \Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}, \omega_{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}}) \simeq \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, \mathcal{O}_{\mathfrak{L}_n^+\mathbb{G}_m})$$

using pushforward along AJ sends:

$$\eta \in t^{-n}k[[t]]dt/k[[t]]dt$$

[56]Here we abuse notation: in §5.4.4, we used the notation ω^{univ} for a similar construction with $\mathcal{Z}^{\leqslant n}$ in place of $\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt$; that setting is pulled back from the present one.

to the function:

$$(g \in \mathfrak{L}_n^+\mathbb{G}_m) \mapsto \mathrm{Res}(g\eta).$$

Similarly, if we use AJ^{-1} instead, we obtain the function:

$$(g \in \mathfrak{L}_n^+\mathbb{G}_m) \mapsto \mathrm{Res}(g^{-1}\eta).$$

6.7.5. Our original section:

$$p_{12}^*(\mathrm{cotr}) \cdot p_{23}^*(\omega^{univ}) \in \Gamma^{\mathsf{IndCoh}}(\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n} \times \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt, \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \omega_{\mathcal{D}^{\leqslant n}} \boxtimes \mathcal{O}_{\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt})$$

can be interpreted as a map:

$$\mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt = \underline{t^{-n}k[[t]]dt/k[[t]]dt} \to \underline{t^{-n}k[[t]]dt/k[[t]]dt \otimes t^{-n}k[[t]]dt/k[[t]]dt}$$

where for a finite-dimensional vector space V, we let $\underline{V}$ denote the scheme $\mathrm{Spec}(\mathrm{Sym}(V^\vee))$. It is immediate to see that the above map is given by the formula:

$$\eta \mapsto (\eta \otimes 1) \cdot \mathrm{cotr}\,.$$

Pushing $p_{12}^*(\mathrm{cotr}) \cdot p_{23}^*(\omega^{univ})$ forward along:

$$\mathcal{D}^{\leqslant n} \times \mathcal{D}^{\leqslant n} \times \mathcal{Z}^{\leqslant n} \xrightarrow{\mathrm{AJ}^{-1} \times \mathrm{AJ} \times \mathrm{id}} \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt$$

and applying (6.7.1), we obtain a function on

$$\mathfrak{L}_n^+\mathbb{G}_m \times \mathfrak{L}_n^+\mathbb{G}_m \times \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt.$$

We deduce that it is given by the formula:

$$(g_1, g_2, \eta) \mapsto (\mathrm{Res} \otimes \mathrm{Res})((g_1^{-1} \otimes g_2) \cdot (\eta \otimes 1) \cdot \mathrm{cotr}). \tag{6.7.2}$$

6.7.6. We now wish to calculate (6.7.2) more explicitly.

First, we note the following, which is an immediate calculation.

Lemma 6.7.6.1. *For $\eta \in \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt$, we have:*

$$(\mathrm{id} \otimes \mathrm{Res})((\eta \otimes 1) \cdot \mathrm{cotr}) = \eta.$$

We then deduce:

Lemma 6.7.6.2. *For $(g, \eta) \in \mathfrak{L}_n^+\mathbb{A}^1 \times \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt$, we have:*

$$(\mathrm{Res} \otimes \mathrm{Res})((\eta \otimes g) \cdot \mathrm{cotr}) = \mathrm{Res}(g\eta).$$

Proof. We have:

$$(1 \otimes g) \cdot \mathrm{cotr} = (g \otimes 1) \cdot \mathrm{cotr}\,.$$

Indeed, this encodes the fact that the residue pairing is $k[[t]]$-equivariant.

Therefore, we have:

$$(\eta \otimes g) \cdot \mathrm{cotr} = (g\eta \otimes 1) \cdot \mathrm{cotr}$$

which gives:

$$(\mathrm{Res} \otimes \mathrm{Res})((\eta \otimes g) \cdot \mathrm{cotr}) = \mathrm{Res}(\mathrm{id} \otimes \mathrm{Res})((g\eta \otimes 1) \cdot \mathrm{cotr}).$$

Now the result follows from Lemma 6.7.6.1. ∎

In the setting of (6.7.2), we deduce:

$$(\mathrm{Res} \otimes \mathrm{Res})((g_1^{-1} \otimes g_2) \cdot (\eta \otimes 1) \cdot \mathrm{cotr}) = \mathrm{Res}(\frac{g_2}{g_1}\eta). \tag{6.7.3}$$

Now, our section σ is obtained by using a pushforward along $\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \times \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n} \to \mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}$. On the dual side, this means we restrict our function along:

$$\mathfrak{L}_n^+\mathbb{G}_m \xrightarrow{\Delta \times \mathrm{id}} \mathfrak{L}_n^+\mathbb{G}_m \times \mathfrak{L}_n^+\mathbb{G}_m \times \mathfrak{L}^{pol,\leqslant n}\mathbb{A}^1 dt.$$

By (6.7.3), we see the resulting function is:

$$(g, \eta) \mapsto \mathrm{Res}(\eta).$$

This clearly coincides with the endomorphism defined by $\beta(1)$, giving the claim (in the $f = 1$ case).

6.7.7. As promised, we now treat the case of general f.

Briefly, one replaces (6.7.2) with:

$$(g_1, g_2, \eta) \mapsto (\mathrm{Res} \otimes \mathrm{Res})((g_1^{-1} \otimes g_2) \cdot (\eta \otimes 1) \cdot (1 \otimes f) \cdot \mathrm{cotr}),$$

which Lemma 6.7.6.2 implies is:

$$(g_1, g_2, \eta) \mapsto \mathrm{Res}(\frac{fg_2}{g_1}\eta),$$

which for $g_1 = g_2 = g$ is:

$$(g, \eta) \mapsto \mathrm{Res}(f\eta).$$

Again, the induced endomorphism clearly coincides with that defined by $\beta(f)$, concluding the argument.

7. Fully faithfulness

7.1. **Overview.**

7.1.1. The main result of this section is the following.

Proposition 7.1.1.1. *For every $n \geqslant 0$, the functor Δ_n is fully faithful.*

Remark 7.1.1.2. By Lemma 4.7.4.1, the pushforward functors $\mathsf{IndCoh}^*(\mathcal{Y}_n) \to \mathsf{IndCoh}^*(\mathcal{Y})$ are fully faithful for $n > 0$. Therefore, for $n \neq 0$, the above result is equivalent to fully faithfulness of $\overline{\Delta}_n$.

Remark 7.1.1.3. As $\lambda_{n,*}^{\mathsf{IndCoh}}(\mathcal{F}_n)$ is compact in $\mathsf{IndCoh}^*(\mathcal{Y})$, Proposition 7.1.1.1 is equivalent to showing that the map:

$$W_n^{op} \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}(\lambda_{n,*}^{\mathsf{IndCoh}}\mathcal{F}_n)$$

is an isomorphism.

Our proof is essentially by induction. We settle the $n = 0$ case by explicit analysis. We then use local class field theory (or the Contou-Carrère pairing) to perform the inductive step.

7.2. **Proof for $n = 0$.** We begin with the base case of our induction.

7.2.1. *A preliminary calculation.* We begin with the following explicit calculation.

Recall the scheme $C = \mathrm{Spec}(k[x,y]/xy)$ from §2.6.3. We equip it with the $\mathbb{G}_m$-action of §2.6.6, so $\deg(x) = 1$ and $\deg(y) = 0$ for the corresponding grading.

Let $\mathcal{O}_{\mathbb{A}^1_x/\mathbb{G}_m} \in \mathsf{QCoh}(C/\mathbb{G}_m)^{\heartsuit}$ denote the structure sheaf of the x-axis, i.e., the object corresponding to the graded $k[x,y]/xy$-module $k[x] = k[x,y]/y$ (where the generator has degree 0).

Lemma 7.2.1.1. *The unit map:*

$$k \to \underline{\mathrm{End}}_{\mathsf{QCoh}(C/\mathbb{G}_m)}(\mathcal{O}_{\mathbb{A}^1_x/\mathbb{G}_m})$$

is an isomorphism.

Proof. This follows by a standard Ext-calculation that we give here.

Let $A = k[x,y]/xy$. We let $A(n)$ denote A as a graded A-module where the generator is given degree n.[57]

The complex:

$$\ldots \xrightarrow{y\cdot -} A(2) \xrightarrow{x\cdot -} A(1) \xrightarrow{y\cdot -} A(1) \xrightarrow{x\cdot -} A \xrightarrow{y\cdot -} \underset{\text{coh. deg. } 0}{A} \to 0 \to \ldots$$

gives a graded free resolution of $k[x] = A/y$.

Therefore, we may calculate $\underline{\mathrm{End}}_{A\text{–}\mathsf{mod}}(k[x])$ as a graded vector space via the complex:

$$\ldots \to 0 \to \underset{\text{coh. deg. } 0}{k[x]} \xrightarrow{0} k[x] \xrightarrow{x\cdot -} k[x](-1) \xrightarrow{0} k[x](-1) \xrightarrow{x\cdot -} k[x](-2) \xrightarrow{0} \ldots.$$

The (graded) degree 0 component of this complex, which computes:

$$\underline{\mathrm{End}}_{A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}}(k[x]) = \underline{\mathrm{End}}_{\mathsf{QCoh}(C/\mathbb{G}_m)}(\mathcal{O}_{\mathbb{A}^1_x/\mathbb{G}_m}),$$

is:

$$\ldots \to 0 \to \underset{\text{coh. deg. } 0}{k} \xrightarrow{0} k \xrightarrow{x\cdot -} k\cdot x \xrightarrow{0} k\cdot x \xrightarrow{x\cdot -} k\cdot x^2 \xrightarrow{0} \ldots.$$

Clearly this complex is acyclic outside of degree cohomological 0, and its H^0 is 1-dimensional and generated by the unit.

□

7.2.2. We have the following result, explicitly describing the geometry of our situation.

Lemma 7.2.2.1. *(1) There is a canonical isomorphism $\mathcal{Z}^{\leqslant 0} \simeq \mathbb{A}^1/\mathbb{G}_m$. Explicitly, given $(\mathcal{L}, \nabla, s) \in \mathcal{Z}^{\leqslant 0}$, we take the line $\mathcal{L}|_0$ with its section $s|_0$ (for $0 \in \mathcal{D}$ the base-point).*

(2) The canonical map $\mathcal{Z}^{\leqslant 0} \to \mathcal{Y}$ is an ind-closed embedding. Its formal completion $\mathcal{Y}^{\wedge}_{\mathcal{Z}^{\leqslant 0}}$ is isomorphic to $(C/\mathbb{G}_m)^{\wedge}_{\mathbb{A}^1/\mathbb{G}_m}$ compatibly with the above isomorphism $\mathcal{Z}^{\leqslant 0} \simeq \mathbb{A}^1/\mathbb{G}_m = \mathbb{A}^1_x/\mathbb{G}_m$.

Proof. We fix a coordinate t and then consider $\mathrm{LocSys}_{\mathbb{G}_m}$ as mapping to $\mathbb{B}\mathbb{G}_m$ via the isomorphism Proposition 2.3.5.1. For a prestack S over $\mathrm{LocSys}_{\mathbb{G}_m}$, let $\widetilde{S}$ denote the base-change $S \times_{\mathbb{B}\mathbb{G}_m} \mathrm{Spec}(k)$. (We remark that this notation is compatible with that of §2.7.2.)

By Proposition 2.6.5.1 and Theorem 2.4.3.1, $\mathcal{Z}^{\leqslant 0}$ and $\mathcal{Y}$ are classical. Moreover, the formal completion $\mathcal{Y}^{\wedge}_{\mathcal{Z}^{\leqslant 0}}$ is classical: this follows immediately from Proposition 2.6.6.2 (1).

We can explicitly calculate $\widetilde{\mathcal{Y}^{\wedge}_{\mathcal{Z}^{\leqslant 0}}}$ by classicalness and Proposition 2.3.5.1: it parametrizes $y \in \mathbb{A}^{1,\wedge}_0$ (defining the local system $(\mathcal{O}, d - y\frac{dt}{t})$) and an element $f \in (\mathfrak{L}\mathbb{A}^1)^{\wedge}_{\mathfrak{L}^+\mathbb{A}^1}$ such that $df = yf\frac{dt}{t}$. If we expand $f = \sum a_i t^i$, we find $ia_i = ya_i$. As y is nilpotent, this implies $a_i = 0$ for all $i \neq 0$ and $ya_0 = 0$.

[57] We remark that the sign conventions here are the same as in §2.7.2.

Clearly $\widetilde{\mathcal{Z}^{\leqslant 0}}$ corresponds to the locus $y = 0$. Therefore, writing x for a_0 and noting that the relevant $\mathbb{G}_m$-action scales x, we obtain the claims.

□

7.2.3. Now the $n = 0$ case of Proposition 7.1.1.1 asserts that the unit map:

$$k \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}(\mathcal{F}_0)$$

is an isomorphism. Note that $\mathcal{F}_0$ is just the pushforward of the structure sheaf of $\mathcal{Z}^{\leqslant 0}$ to $\mathcal{Y}$. The calculation of the above depends only on the formal completion, so we obtain the result from Lemmas 7.2.1.1 and 7.2.2.1.

7.3. **A fully faithfulness result.** Before proceeding to our induction, we will need the following observations.

7.3.1. Recall the maps ζ and $\widetilde{\zeta}$ from §2.7. We let:

$$\zeta_*^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n}) \to \mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})$$

denote the functor induced from the IndCoh-pushforward along $\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n} \to \mathcal{Y}_{\log}^{\leqslant n}$ on passing to weak $\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}$-invariants.

Proposition 7.3.1.1. *For any $n > 0$, the map:*

$$\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n})}(\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n},w}\mathcal{O}_{\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n}}) \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\zeta_*^{\mathsf{IndCoh}}\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n},w}\mathcal{O}_{\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n}})$$

is an isomorphism.

We prove this result in §7.3.6, after some preliminary remarks.

Remark 7.3.1.2. In the above formula and in what follows, we advise the reader to interpret $\mathrm{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n},w}$ as baroque notation for $\widetilde{\pi}_{n,*}^{\mathsf{IndCoh}}$ (or, sometimes in what follows, IndCoh-pushforward along the map $\mathcal{Y}_{\log}^{\leqslant n} \to \mathcal{Y}^{\leqslant n}$).

7.3.2. Let $A = \oplus_{i\geqslant 0}A_i$ be a $\mathbb{Z}^{\geqslant 0}$-graded (classical, say) algebra.

We let $A\text{–}\mathsf{mod}^{\mathbb{G}_m,w} \in \mathsf{DGCat}_{cont}$ denote the DG category of graded A-modules. For $M \in A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}$, we write $M = \oplus_{i\in\mathbb{Z}}M_i$ for its decomposition into weight spaces.

Lemma 7.3.2.1. *Suppose $M, N \in A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}$. Suppose that M is concentrated in positive (graded) degrees, i.e., $M = \oplus_{i>0}M_i$. Suppose N is concentrated in degree 0, i.e., $N = N_0$.*

Then:

$$\underline{\mathrm{Hom}}_{A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}}(M,N) = 0.$$

Proof. Let $A(i) \in A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}$ be as in the proof of Lemma 7.2.1.1, i.e., A graded with generator in degree i. The modules $A(i)$ for $i > 0$ generate the subcategory of $A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}$ consisting of modules concentrated in positive degrees. So we reduce to the case $M = A(i)$ for $i > 0$, for which the claim is obvious.

□

7.3.3. Suppose now that we are given a $\mathbb{Z}^{\geqslant 0}$-graded algebra $B = \oplus_{i \geqslant 0} B_i$ and a graded map $\iota^* : A \to B$ inducing an isomorphism $A_0 \xrightarrow{\simeq} B_0$. We let $\iota_* : B\text{–}\mathsf{mod}^{\mathbb{G}_m,w} \to A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}$ and $\iota^* : A\text{–}\mathsf{mod}^{\mathbb{G}_m,w} \to B\text{–}\mathsf{mod}^{\mathbb{G}_m,w}$ denote the induced adjoint functors.

Proposition 7.3.3.1. *In the above setting, suppose $M, N \in B\text{–}\mathsf{mod}^{\mathbb{G}_m,w}$ with M concentrated in non-negative graded*[58] *degrees and N concentrated in degree 0.*

Then the natural map:

$$\underline{\mathrm{Hom}}_{B\text{–}\mathsf{mod}^{\mathbb{G}_m,w}}(M,N) \to \underline{\mathrm{Hom}}_{A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}}(\iota_* M, \iota_* N)$$

is an isomorphism.

Proof.

Step 1. Let $\widetilde{M} \in A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}$ be concentrated in non-negative graded degrees. We claim that $\mathrm{Ker}(\widetilde{M} \to \iota_*\iota^*\widetilde{M})$ is concentrated in (strictly) positive graded degrees.

First, in the case $\widetilde{M} = A$, this map is $\mathrm{Ker}(A \to \iota_*(B)$, in which case the claim follows as we assumed $A \to B$ an isomorphism in graded degree 0.

In general, we have:

$$\mathrm{Ker}(\widetilde{M} \to \iota_*\iota^*\widetilde{M}) = \mathrm{Ker}(A \to \iota_*(B)) \underset{A}{\otimes} \widetilde{M}$$

which is the tensor product of a module in graded degrees $\geqslant 0$ and one in graded degrees > 0, so is in graded degrees > 0 (as A is non-negatively graded), as claimed.

Step 2. Next, we claim that $\mathrm{Ker}(\iota^*\iota_* M \to M)$ is concentrated in strictly positive graded degrees.

Clearly $\iota^*\iota_* M = B \otimes_A \iota_* M$ is in degrees $\geqslant 0$, so it suffices to see that $\iota^*\iota_* M \to M$ is an isomorphism in degree 0. We can check this after applying ι_*. Then we have a retraction:

$$\iota_* M \to \iota_*\iota^*\iota_* M \to \iota_* M,$$

so it suffices to see that the first map is an isomorphism in degree 0. Taking $\widetilde{M} = \iota_* M$, this follows from Step 1.

Step 3. Applying Lemma 7.3.2.1 to $\mathrm{Ker}(\iota^*\iota_* M \to M)$ and N, we see that the map:

$$\underline{\mathrm{Hom}}_{B\text{–}\mathsf{mod}^{\mathbb{G}_m,w}}(M,N) \to \underline{\mathrm{Hom}}_{B\text{–}\mathsf{mod}^{\mathbb{G}_m,w}}(\iota^*\iota_* M, N)$$

is an isomorphism. The right hand side identifies with $\underline{\mathrm{Hom}}_{A\text{–}\mathsf{mod}^{\mathbb{G}_m,w}}(\iota_* M, \iota_* N)$ by adjunction, and the induced map is given by ι_* functoriality, so we obtain the claim.

□

7.3.4. We will apply the above in the following setting.

Let A_n be the graded ring of §2.10. We write $A_n = \oplus_{i \geqslant 0} A_{n,i}$ for its decomposition into graded pieces. We remind from *loc. cit.* that $\mathrm{Spec}(A_n)/\mathbb{G}_m = \mathcal{Z}^{\leqslant n}$.

By construction, we have:

$$\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log} \underset{\mathbb{BG}_m}{\times} \mathrm{Spec}(k) = \mathrm{Spec}(A_{n,0})$$

such that the graded algebra maps $A_{n,0} \to A_n \to A_{n,0}$ correspond to the maps

$$\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log} \xrightarrow{\tilde{\zeta}} \mathcal{Z}^{\leqslant n} \to \mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log} \tag{7.3.1}$$

(the latter map being the projection).

[58] As opposed to cohomological.

7.3.5. Recall the maps ι, ι^r from §2.7.

The map ι arises[59] from a map $\iota^* : A_n \to A_n$ of graded rings. This map is an isomorphism on 0th graded components, as is evident from (7.3.1).

Corollary 7.3.5.1. *Suppose* $\mathcal{G} \in \mathsf{QCoh}(\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log})$. *Then the morphism:*

$$\underline{\mathrm{Hom}}_{\mathsf{QCoh}(\mathcal{Z}^{\leqslant n})}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}, \widetilde{\zeta}_*\mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathsf{QCoh}(\mathcal{Z}^{\leqslant n})}(\iota_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}}, \iota_*\widetilde{\zeta}_*\mathcal{G})$$

is an isomorphism. More generally, for any $r, s \geqslant 0$, *the morphism:*

$$\underline{\mathrm{Hom}}_{\mathsf{QCoh}(\mathcal{Z}^{\leqslant n})}(\iota^r_*(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}), \iota^s_*\widetilde{\zeta}_*\mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathsf{QCoh}(\mathcal{Z}^{\leqslant n})}(\iota^{r+1}_*(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}), \iota^{s+1}_*\widetilde{\zeta}_*\mathcal{G})$$

is an isomorphism.

Proof. Translating to graded modules and using (7.3.1), the hypotheses of Proposition 7.3.3.1 are clearly satisfied, giving the result.

□

7.3.6. *Conclusion.* We now prove Proposition 7.3.1.1.

Observe that $\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m} = \lim_n \mathcal{Z}^{\leqslant n}$, where the limit is formed under the (affine) morphism ι. Therefore, we have:

$$\widetilde{\zeta}_*\mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}} = \operatorname*{colim}_r \iota^r_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}} \in \mathsf{QCoh}(\mathcal{Z}^{\leqslant n}).$$

As all of these terms are in the heart of the t-structure, we obtain a similar identity in $\mathsf{IndCoh}(\mathcal{Z}^{\leqslant n})$, using IndCoh-pushforwards in the notation instead.

Define:

$$\mathcal{G} := \mathrm{Oblv}\,\mathrm{Av}_!^{\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m},w}\,\mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}} \in \mathsf{QCoh}(\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log})^{\heartsuit}.$$

Recall the map i_n from §2.7.1. By definition of $\zeta^{\mathsf{IndCoh}}_*$, we can rewrite the right hand side of Proposition 7.3.1.1 as:

$$\begin{gathered}\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\zeta^{\mathsf{IndCoh}}_*\,\mathrm{Av}_!^{\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m},w}\,\mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}) = \\ \underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathrm{Av}_!^{\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m},w}\,i^{\mathsf{IndCoh}}_{n,*}\widetilde{\zeta}^{\mathsf{IndCoh}}_*\mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}, \zeta^{\mathsf{IndCoh}}_*\,\mathrm{Av}_!^{\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m},w}\,\mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}).\end{gathered} \tag{7.3.2}$$

By adjunction, this term is:

$$\begin{gathered}\underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n}_{\log})}(i^{\mathsf{IndCoh}}_{n,*}\widetilde{\zeta}^{\mathsf{IndCoh}}_*\mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}, i^{\mathsf{IndCoh}}_{n,*}\widetilde{\zeta}^{\mathsf{IndCoh}}_*\,\mathrm{Oblv}\,\mathrm{Av}_!^{\mathbb{G}^{\leqslant n}_m,w}\,\mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}) = \\ \underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n}_{\log})}(i^{\mathsf{IndCoh}}_{n,*}\widetilde{\zeta}^{\mathsf{IndCoh}}_*\mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}, i^{\mathsf{IndCoh}}_{n,*}\widetilde{\zeta}^{\mathsf{IndCoh}}_*\mathcal{G}).\end{gathered}$$

By the above, we can further express this term as:

$$\begin{gathered}\underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n}_{\log})}(i^{\mathsf{IndCoh}}_{n,*}\operatorname*{colim}_r \iota^{r,\mathsf{IndCoh}}_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}}, i^{\mathsf{IndCoh}}_{n,*}\widetilde{\zeta}^{\mathsf{IndCoh}}_*\mathcal{G}) = \\ \lim_r \underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n}_{\log})}(i^{\mathsf{IndCoh}}_{n,*}\iota^{r,\mathsf{IndCoh}}_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}}, i^{\mathsf{IndCoh}}_{n,*}\widetilde{\zeta}^{\mathsf{IndCoh}}_*\mathcal{G}).\end{gathered}$$

As $\iota^r_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}} \in \mathsf{Coh}(\mathcal{Z}^{\leqslant n})$ for all r, by [Ras5] Corollary 6.5.3, we may calculate the above as:

$$\lim_r \operatorname*{colim}_s \underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n})}(\iota^{r+s,\mathsf{IndCoh}}_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}}, \iota^s_*\widetilde{\zeta}^{\mathsf{IndCoh}}_*\mathcal{G}).$$

Now for fixed r, Corollary 7.3.5.1 implies that all of the maps in the above colimit are isomorphisms. Therefore, the above may be calculated as:

$$\lim_r \underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n})}(\iota^{r,\mathsf{IndCoh}}_*\mathcal{O}_{\mathcal{Z}^{\leqslant n}}, \widetilde{\zeta}^{\mathsf{IndCoh}}_*\mathcal{G}).$$

[59] Non-canonically: i.e., we need a choice of coordinate on the disc to turn ι into a map of stacks over $\mathbb{B}\mathbb{G}_m$.

Moreover, the analysis from *loc. cit.* shows that $\mathcal{O}_{\mathcal{Z}} \to \iota^r_* \mathcal{O}_{\mathcal{Z}}$ corresponds to a map of non-negatively graded modules that is an isomorphism in degree 0, so Lemma 7.3.2.1 shows that the maps is the above limit are isomorphisms as well. Therefore, we may calculate this limit as:

$$\underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Z}^{\leqslant n})}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}, \tilde{\zeta}^{\mathsf{IndCoh}}_* \mathcal{G}). \tag{7.3.3}$$

This term is:

$$\Gamma^{\mathsf{IndCoh}}(\mathcal{Z}^{\leqslant n}, \tilde{\zeta}^{\mathsf{IndCoh}}_* \mathcal{G}) = \Gamma^{\mathsf{IndCoh}}(\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}, \mathcal{G}) =$$
$$\underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log})}(\mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}, \mathrm{Oblv}\,\mathrm{Av}_!^{\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m},w} \mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}) =$$
$$\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m})}(\mathrm{Av}_!^{\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m},w} \mathcal{O}_{\mathrm{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}).$$

This last expression is another way of writing the left hand side of Proposition 7.3.1.1. It is straightforward to see that the isomorphism we have just constructed is inverse to the map in *loc. cit.*, yielding the result.

7.4. **Structure of the argument.** We are now positioned to prove Proposition 7.1.1.1. We begin by outlining the argument.

7.4.1. The starting point is the following elementary observation.

Lemma 7.4.1.1. *Suppose $f : M \to N \in W_2\text{–}\mathsf{mod}$ is a morphism (in the derived category) of modules over the Weyl algebra W_2 in two variables, with generators denoted α, β, ∂_α and ∂_β. Then f is an isomorphism if and only if the morphisms:*

$$\begin{gathered}\operatorname*{colim}_{r,\alpha} M := \mathrm{colim}(M \xrightarrow{\alpha} M \xrightarrow{\alpha} \ldots) \to \operatorname*{colim}_{r,\alpha} N \\ \operatorname*{colim}_{r,\beta} M \to \operatorname*{colim}_{r,\beta} N \\ M^\alpha/\beta \to N^\alpha/\beta\end{gathered} \tag{7.4.1}$$

are isomorphisms in Vect. *In the last line, we define e.g. M^α as the (homotopy) kernel of $\alpha : M \to M$, and the quotient by β means the homotopy cokernel of the induced map $\beta : M^\alpha \to M^\alpha$ (which exists because α and β commute).*

Proof. In terms of D-modules on $\mathbb{A}^2$, our assumptions are that f induces an isomorphism on restriction to $(\mathbb{A}^1\backslash 0) \times \mathbb{A}^1$, $\mathbb{A}^1 \times (\mathbb{A}^1\backslash 0)$, and on !-restriction to 0 (up to a cohomological shift). This implies the claim by D-module excision.

□

7.4.2. We choose coordinates on the disc $\mathcal{D}$. We obtain an isomorphism $\mathfrak{L}^+_n\mathbb{A}^1 \simeq \mathbb{A}^n$; we let $\alpha_0, \ldots, \alpha_{n-1}$ denote the coordinates (so e.g. $\alpha_0 = \mathrm{ev}$ as maps to $\mathbb{A}^1$). We obtain an evident decomposition $W_n \simeq W_1 \otimes W_{n-1}$, where the first tensor factor uses the coordinate α_0.

Let f denote the morphism $W_n^{op} \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \in \mathsf{Alg}$. As we have a morphism $W_1^{op} \to W_n^{op}$ of algebras as above, we may regard f as a morphism of W_1-bimodules (via left and right multiplication).

Our argument applies Lemma 7.4.1.1 using this bimodule structure. Generally speaking, the first two conditions from the lemma amount to Proposition 7.3.1.1 and its relatives, and the last one is given by induction.

7.5. **Verifying the axioms.**

7.5.1. In the setting of §7.4.2, it remains to verify the conditions of Lemma 7.4.1.1. We do so below.

7.5.2. *Colimits.* We begin with checking the first two maps from (7.4.1) are isomorphisms.

Clearly:

$$\operatorname*{colim}_{r,\alpha_0\cdot -} W_n = \operatorname*{colim}_{r,-\cdot\alpha_0} W_n = W_n[\alpha_0^{-1}] = \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m}).$$

I.e., in our example, the left hand sides of the first two equations in (7.4.1) each identify with global differential operators on $\mathfrak{L}_n^+\mathbb{G}_m$.

Below, we verify the same for the right hand sides in our examples. Then we check that our map (f, in the notation of Lemma 7.4.1.1), is compatible with these identifications.

7.5.3. The left action of α_0 on $\underline{\operatorname{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leq n})}(\mathcal{F}_n)$ is, by construction, obtained by right composition with the map:

$$\mathcal{F}_n = \operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w}(i_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leq n}})) \to \operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w}(\iota_*^{\mathsf{IndCoh}} i_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leq n}})) = \mathcal{F}_n$$

obtained by applying $\operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w}$ to the canonical map:

$$\mathcal{O}_{\mathcal{Z}^{\leq n}} \to \iota_*^{\mathsf{IndCoh}}\mathcal{O}_{\mathcal{Z}^{\leq n}}.$$

Therefore, by compactness of $\mathcal{F}_n$, we see that:

$$\operatorname*{colim}_{r,\alpha_0\cdot -} \underline{\operatorname{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leq n})}(\mathcal{F}_n) \xrightarrow{\simeq} \underline{\operatorname{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leq n})}(\mathcal{F}_n, \operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w} \operatorname*{colim}_{r} (\iota_*^{r,\mathsf{IndCoh}}) i_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leq n}}).$$

The right hand side clearly identifies with:

$$\underline{\operatorname{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leq n})}(\mathcal{F}_n, \zeta_*^{\mathsf{IndCoh}} \operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w}(\mathcal{O}_{\operatorname{LocSys}_{\mathbb{G}_m,\log}^{\leq n}})).$$

Unwinding the definition of $\mathcal{F}_n$, we can further identify it with:

$$\underline{\operatorname{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leq n})}\big(i_{n,*}^{\mathsf{IndCoh}}\mathcal{O}_{\mathcal{Z}^{\leq n}}, i_{n,*}^{\mathsf{IndCoh}}\tilde{\zeta}_*^{\mathsf{IndCoh}} \operatorname{Oblv}\operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w}(\mathcal{O}_{\operatorname{LocSys}_{\mathbb{G}_m,\log}^{\leq n}})\big).$$

The comparison of (7.3.2) and (7.3.3) from §7.3.6 yields that the map:

$$\begin{gathered}\Gamma\big(\operatorname{LocSys}_{\mathbb{G}_m,\log}^{\leq n}, \operatorname{Oblv}\operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w}(\mathcal{O}_{\operatorname{LocSys}_{\mathbb{G}_m,\log}^{\leq n}})\big) = \\ \Gamma^{\mathsf{IndCoh}}\big(\mathcal{Z}^{\leq n}, \tilde{\zeta}_*^{\mathsf{IndCoh}} \operatorname{Oblv}\operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w}(\mathcal{O}_{\operatorname{LocSys}_{\mathbb{G}_m,\log}^{\leq n}})\big) = \\ \underline{\operatorname{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Z}^{\leq n})}\big(\mathcal{O}_{\mathcal{Z}^{\leq n}}, \tilde{\zeta}_*^{\mathsf{IndCoh}} \operatorname{Oblv}\operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w}(\mathcal{O}_{\operatorname{LocSys}_{\mathbb{G}_m,\log}^{\leq n}})\big) \to \\ \underline{\operatorname{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leq n})}\big(i_{n,*}^{\mathsf{IndCoh}}\mathcal{O}_{\mathcal{Z}^{\leq n}}, i_{n,*}^{\mathsf{IndCoh}}\tilde{\zeta}_*^{\mathsf{IndCoh}} \operatorname{Oblv}\operatorname{Av}_!^{\operatorname{Gr}_{\mathbb{G}_m}^{\leq n},w}(\mathcal{O}_{\operatorname{LocSys}_{\mathbb{G}_m,\log}^{\leq n}})\big)\end{gathered}$$

is an isomorphism.

By Theorem 6.3.1.1, the left hand side identifies canonically with:

$$\Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m}).$$

Putting this together, we obtain an isomorphism:

$$\operatorname*{colim}_{r,\alpha_0\cdot -} \underline{\operatorname{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leq n})}(\mathcal{F}_n) \simeq \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m}).$$

7.5.4. In §7.5.3, we constructed an isomorphism:

$$\operatorname*{colim}_{r,\alpha_0\cdot -} \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \simeq \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m}).$$

We claim that the resulting map:

$$\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \to \operatorname*{colim}_{r,\alpha_0\cdot -} \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \simeq \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m})$$

is in fact (canonically) a map of (DG) algebras, at least once the target is given the opposite multiplication.

For convenience, define:

$$\mathring{\mathcal{F}}_n := \zeta_*^{\mathsf{IndCoh}} \operatorname{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}, w} \mathcal{O}_{\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n}}.$$

Note that there is a canonical morphism:

$$\mathcal{F}_n \to \mathring{\mathcal{F}}_n.$$

We in effect showed that the natural maps:

$$\underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n, \mathring{\mathcal{F}}_n) \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathring{\mathcal{F}}_n) \leftarrow \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n})}(\operatorname{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}, w} \mathcal{O}_{\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n}})$$

are isomorphisms. The map:

$$\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \to \operatorname*{colim}_{r,\alpha_0\cdot -} \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n, \mathring{\mathcal{F}}_n)$$

is clearly a map of right $\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n)$-modules. But $\underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n, \mathring{\mathcal{F}}_n)$ carries a commuting left $\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathring{\mathcal{F}}_n)$-module structure, for which it is a free module of rank 1 by the above. As the class field theory isomorphism:

$$\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant n})}(\operatorname{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}, w} \mathcal{O}_{\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n}}) \simeq \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m}) \tag{7.5.1}$$

is an isomorphism of algebras, we obtain the claim.

7.5.5. The above calculations verify that there are canonical isomorphisms:

$$\operatorname*{colim}_{r,\alpha_0\cdot -} W_n \simeq \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m}) \simeq \operatorname*{colim}_{r,\alpha_0\cdot -} \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n).$$

We claim that this map is induced by our comparison map f from §7.4.2.

By §7.5.4, it suffices to check that the composition:

$$W_n^{op} \to \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \to \operatorname*{colim}_{r,\alpha_0\cdot -} \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \simeq \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m})^{op}$$

is the restriction map. We first check this on some particular elements.

For the linear functions $t^{-n}k[[t]]dt/k[[t]]dt \subseteq W_n^{op}$, this follows by construction. Indeed, the geometric incarnation of the action of these elements is the observation that $\mathcal{Z}^{\leqslant n} \subseteq \mathcal{Y}_{\log}^{\leqslant n}$ is closed under the action of $\mathrm{Gr}_{\mathbb{G}_m}^{pos,\leqslant n}$. The same is true for $\mathrm{LocSys}_{\mathbb{G}_m,\log}^{\leqslant n} \subseteq \mathcal{Y}_{\log}^{\leqslant n}$ — in fact, it is closed under $\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n}$. Tracing the constructions, these observations provide the claim.

Next, we claim that the vector fields on $\mathfrak{L}_n^+\mathbb{A}^1$ defined by the infinitesimal $\mathfrak{L}_n^+\mathbb{G}_m$-action:

$$k[[t]]/t^n \simeq \mathrm{Lie}(\mathfrak{L}_n^+\mathbb{G}_m) \subseteq W_n^{op}$$

have the correct image. Indeed, this follows by construction of the class field theory isomorphism and Lemma 6.6.2.1.

By the above, we see that the algebra map:

$$W_n^{op} \to \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m})^{op}$$

factors through the (non-commutative) localization of W_n^{op} at α_0, i.e., through a map:

$$W_n^{op} \to \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m})^{op} \xrightarrow{\gamma} \Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m})^{op}$$

for some algebra map γ. As the subspaces $t^{-n}k[[t]]dt/k[[t]]dt$ and $k[[t]]/t^n$ generate $\Gamma(\mathfrak{L}_n^+\mathbb{G}_m, D_{\mathfrak{L}_n^+\mathbb{G}_m})^{op}$ as an algebra, and as we have seen γ is the identity on these elements, γ itself must be the identity map, as desired.

7.5.6. Next, we calculate:

$$\underset{r,-\cdot\alpha_0}{\operatorname{colim}} \underline{\operatorname{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n).$$

For $r \in \mathbb{Z}$, let ${}^r\mathcal{Z}^{\leqslant n} \subseteq \mathcal{Y}^{\leqslant n}_{\log}$ denote the closed where the meromorphic section s has a pole of order $\leqslant r$; therefore, ${}^0\mathcal{Z}^{\leqslant n} = \mathcal{Z}^{\leqslant n}$, $\iota(\mathcal{Z}^{\leqslant n}) = {}^{-1}\mathcal{Z}^{\leqslant n}$, and α_0 can be considered as obtained by restriction ${}^r\mathcal{Z}^{\leqslant n} \to {}^{r-1}\mathcal{Z}^{\leqslant n}$ for any r.

We let $i_{n,r} : {}^r\mathcal{Z}^{\leqslant n} \to \mathcal{Y}^{\leqslant n}_{\log}$ denote the embedding.

For any $\mathcal{G} \in \mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})$, it follows by definition that:

$$\underset{r,-\cdot\alpha_0}{\operatorname{colim}} \underline{\operatorname{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n, \mathcal{G}) = \underset{r,-\cdot\alpha_0}{\operatorname{colim}} \underline{\operatorname{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(i_{n,r,*}^{\mathsf{IndCoh}}(\mathcal{O}_{{}^r\mathcal{Z}^{\leqslant n}}), \operatorname{Oblv}\mathcal{G}).$$

We have a canonical isomorphism:

$$\underset{r,-\cdot\alpha_0}{\operatorname{colim}} \underline{\operatorname{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(i_{n,r,*}^{\mathsf{IndCoh}}(\mathcal{O}_{{}^r\mathcal{Z}^{\leqslant n}}), \operatorname{Oblv}\mathcal{G}) \xrightarrow{\simeq} \Gamma^{\mathsf{IndCoh}}(\mathcal{Y}^{\leqslant n}_{\log}, \operatorname{Oblv}\mathcal{G}).$$

Now taking $\mathcal{G} = \mathcal{F}_n$, we see that:

$$\underset{r,-\cdot\alpha_0}{\operatorname{colim}} \underline{\operatorname{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \simeq \Gamma^{\mathsf{IndCoh}}(\mathcal{Y}^{\leqslant n}_{\log}, \operatorname{Oblv}\operatorname{Av}_!^{\operatorname{Gr}^{\leqslant n}_{\mathbb{G}_m},w} i_{n,*}^{\mathsf{IndCoh}}\mathcal{O}_{\mathcal{Z}^{\leqslant n}}).$$

By the analysis of §7.3.5,[60], the map:

$$\Gamma^{\mathsf{IndCoh}}(\mathcal{Y}^{\leqslant n}_{\log}, \operatorname{Oblv}\operatorname{Av}_!^{\operatorname{Gr}^{\leqslant n}_{\mathbb{G}_m},w} i_{n,*}^{\mathsf{IndCoh}}\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \to \Gamma^{\mathsf{IndCoh}}(\mathcal{Y}^{\leqslant n}_{\log}, \operatorname{Oblv}\operatorname{Av}_!^{\operatorname{Gr}^{\leqslant n}_{\mathbb{G}_m},w} i_{n,*}^{\mathsf{IndCoh}}\tilde{\zeta}_*^{\mathsf{IndCoh}}\mathcal{O}_{\operatorname{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}})$$

is an isomorphism. By functoriality, the target identifies with:

$$\Gamma(\operatorname{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}, \operatorname{Oblv}\operatorname{Av}_!^{\operatorname{Gr}^{\leqslant n}_{\mathbb{G}_m},w} \mathcal{O}_{\operatorname{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log}}).$$

Now passing to the colimit over α_0 acting on the left and right simultaneously, the above analysis actually shows that the maps:

$$\underset{r,-\cdot\alpha_0}{\operatorname{colim}} \underline{\operatorname{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \to \underset{(r_1,r_2),-\cdot\alpha_0,-\cdot\alpha_0}{\operatorname{colim}} \underline{\operatorname{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \leftarrow \underset{r,\alpha_0\cdot -}{\operatorname{colim}} \underline{\operatorname{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n.$$

are isomorphisms. It then follows from §7.5.5 that the comparison map:

$$\underset{r,-\cdot\alpha_0}{\operatorname{colim}} W_n \to \underset{r,-\cdot\alpha_0}{\operatorname{colim}} \underline{\operatorname{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n)$$

is an isomorphism.

7.5.7. *Hamiltonian reduction.* It remains to check that the third map in (7.4.1) is an isomorphism. We will see that this amounts to our inductive hypothesis.

[60]Specifically, the observations that the ring A_n from *loc. cit.* is positively graded with degree 0 component corresponding to $\operatorname{LocSys}^{\leqslant n}_{\mathbb{G}_m,\log} \subseteq \mathcal{Z}^{\leqslant n} = \operatorname{Spec}(A_n)/\mathbb{G}_m$.

7.5.8. We begin with some notation.

Suppose A is a (DG) algebra, and that we are given a map:

$$k[t] \to A.$$

We let $\alpha \in \Omega^\infty A$ denote the image of t.

In this case, we let:

$$\mathrm{QH}_\alpha(A) := A^{\alpha \cdot -}/- \cdot \alpha$$

with notation on the right hand side understood as in Lemma 7.4.1.1, i.e., we take the homotopy quotient of right multiplication by α and then the homotopy kernel of the left action of α (and of course, the order of these two operations is not important).

As is standard, we refer to $\mathrm{QH}_\alpha(A)$ as the *quantum Hamiltonian reduction* of A by α.

7.5.9. First, we note that $\mathrm{QH}_{\alpha_0}(W_1^{op}) = k$ (with $1 \in k$ mapping to the unit in W_1), so similarly, $\mathrm{QH}_{\alpha_0}(W_n^{op}) = W_{n-1}^{op}$. This is the left hand side of the third map in (7.4.1) in our setting.

7.5.10. Next, we calculate the right hand side of (7.4.1).

First, we have:

$$\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n) \xrightarrow{\simeq} \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}(\lambda_{n,*}^{\mathsf{IndCoh}}(\mathcal{F}_n))$$

by Remark 7.1.1.2 (and the assumption that $n > 0$).

By (2.8.1), we then have:

$$\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}\big(\lambda_{n,*}^{\mathsf{IndCoh}}(\mathcal{F}_n)\big)^{\alpha_0 \cdot -} = \underline{\mathrm{Hom}}_{\mathsf{IndCoh}^*(\mathcal{Y})}\big(\lambda_{n,*}^{\mathsf{IndCoh}}(\mathcal{F}_n), \lambda_{n-1,*}^{\mathsf{IndCoh}}(\mathcal{F}_{n-1})\big).$$

Applying (2.8.1) again, we have:

$$\begin{gathered}\mathrm{QH}_{\alpha_0}\Big(\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}\big(\lambda_{n,*}^{\mathsf{IndCoh}}(\mathcal{F}_n)\big)\Big) := \\ \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}\big(\lambda_{n,*}^{\mathsf{IndCoh}}(\mathcal{F}_n)\big)^{\alpha_0 \cdot -}/- \cdot \alpha_0 = \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}\big(\lambda_{n-1,*}^{\mathsf{IndCoh}}(\mathcal{F}_{n-1})\big).\end{gathered} \tag{7.5.2}$$

By induction, we know that the right hand side identifies with W_{n-1}^{op} as an algebra. Below, we will show that the comparison map from Lemma 7.4.1.1 coincides with this identification.

7.5.11. Let us return to the general setting of §7.5.8.

First, observe that $\mathrm{QH}_\alpha(A)$ carries a canonical algebra structure. Indeed, it obviously identifies with:

$$\underline{\mathrm{End}}_{A\text{–}\mathsf{mod}}(A/- \cdot \alpha) = \underline{\mathrm{End}}_{A\text{–}\mathsf{mod}}(A \underset{k[t]}{\otimes} k).$$

(Here $k[t]$ acts on k with t acting by 0.) We equip $A^{\alpha \cdot -}/- \cdot \alpha$ with the multiplication *opposite* to this one.[61]

Suppose now that A and B are (DG) algebras, and that we are given a map:

$$B[t] := B \otimes k[t] \to A.$$

We can further identify:

$$A \underset{k[t]}{\otimes} k \simeq A \underset{B[t]}{\otimes} B.$$

I.e., $A \underset{k[t]}{\otimes} k$ is canonically an (A, B)-bimodule. Therefore, by construction, we obtain a canonical morphism:

$$B \to \mathrm{QH}_\alpha(A).$$

[61]This convention is natural because $\underline{\mathrm{End}}_{A\text{–}\mathsf{mod}}(A) = A^{op}$.

Remark 7.5.11.1. We note that the diagram:

$$\begin{array}{ccc} B & \longrightarrow & \mathrm{QH}_\alpha(A) \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/-\cdot\alpha \end{array}$$

commutes by construction.

7.5.12. We apply the above with:

$$A = \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n})}(\mathcal{F}_n), \alpha = \alpha_0, B = W_{n-1}^{op}.$$

We first see that:

$$\mathrm{QH}_{\alpha_0}\left(\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}\big(\lambda_{n,*}^{\mathsf{IndCoh}}(\mathcal{F}_n)\big)\right)$$

carries a canonical algebra structure; by construction, (7.5.2) is an isomorphism of algebras, where the right hand side is given the usual algebra structure on endomorphisms.

Moreover, we deduce that the map:

$$W_{n-1}^{op} \to \mathrm{QH}_{\alpha_0}\left(\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}\big(\lambda_{n,*}^{\mathsf{IndCoh}}(\mathcal{F}_n)\big)\right).$$

(coming from Lemma 7.4.1.1) is an algebra morphism.

7.5.13. Finally, we claim that the resulting map:

$$W_{n-1}^{op} \to \mathrm{QH}_{\alpha_0}\left(\underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}\big(\lambda_{n,*}^{\mathsf{IndCoh}}(\mathcal{F}_n)\big)\right) = \underline{\mathrm{End}}_{\mathsf{IndCoh}^*(\mathcal{Y})}\big(\lambda_{n-1,*}^{\mathsf{IndCoh}}(\mathcal{F}_{n-1})\big)$$

coincides with the construction of §5 (for $n-1$ instead of n).

Indeed, as this is an algebra map, we can check this on generators of W_{n-1}. There it is essentially tautological from the constructions.

In particular, we see by induction that this map is an isomorphism. Therefore, the conditions of Lemma 7.4.1.1 are satisfied, so we have proved Proposition 7.1.1.1.

8. Proof of the main theorem

8.1. **Overview.** In this section we prove Theorem 1.1.0.1. The idea is simple given the work we have done so far: bootstrap from the functors Δ_n constructed earlier.

8.2. **Arcs.** We begin by constructing a strongly $\mathfrak{L}^+\mathbb{G}_m$-equivariant functor:

$$\Delta_\infty : D^!(\mathfrak{L}^+\mathbb{A}^1) \to \mathsf{IndCoh}^*(\mathcal{Y}).$$

8.2.1. *Notation.* In what follows, we let:

$$p_n : \mathfrak{L}^+\mathbb{A}^1 \to \mathfrak{L}_n^+\mathbb{A}^1$$

denote the structural map. For $m \geqslant n$, we similarly let:

$$p_{n,m} : \mathfrak{L}_m^+\mathbb{A}^1 \to \mathfrak{L}_n^+\mathbb{A}^1$$

denote the structural map.

8.2.2. *Overview.* First, let us unwind what it means to construct such a functor Δ_∞.

Recall from [Ras3] that by definition we have:

$$D^!(\mathfrak{L}^+\mathbb{A}^1) := \operatorname{colim}_{n\geqslant 0} D(\mathfrak{L}^+_n\mathbb{A}^1)$$

where the structural functors are the standard !-pullback functors of D-module theory. We let:

$$p^!_n : D(\mathfrak{L}^+_n\mathbb{A}^1) \to D^!(\mathfrak{L}^+\mathbb{A}^1)$$

denote the structural functor.

The above colimit is a colimit of $D^*(\mathfrak{L}^+\mathbb{G}_m)$-module categories. Moreover, as it is indexed by $\mathbb{Z}^{\geqslant 0}$, it suffices to construct functors:

$$D(\mathfrak{L}^+_n\mathbb{A}^1) \to \mathsf{IndCoh}^*(\mathcal{Y}) \in D^*(\mathfrak{L}^+\mathbb{G}_m)\text{–}\mathsf{mod} \tag{8.2.1}$$

and commutative diagrams:

$$\begin{array}{ccc} D(\mathfrak{L}^+_n\mathbb{A}^1) & & \\ {\scriptstyle p^!_{n,n+1}}\downarrow & \searrow & \\ D(\mathfrak{L}^+_{n+1}\mathbb{A}^1) & \longrightarrow & \mathsf{IndCoh}^*(\mathcal{Y}) \end{array} \tag{8.2.2}$$

of $D^*(\mathfrak{L}^+\mathbb{G}_m)$-module categories for each n. (Here $D^*(\mathfrak{L}^+\mathbb{G}_m)$–$\mathsf{mod}$ acts on $\mathsf{IndCoh}^*(\mathcal{Y})$ via geometric class field theory, as in §6.)

8.2.3. For us, by definition, the functor (8.2.1) is Δ_n. It remains to construct the commutative diagrams (8.2.2) as above.

In §8.2.4-8.2.5, we will formulate a compatibility for these commutative diagrams that characterizes them uniquely. We then turn to proving their existence.

8.2.4. *Preliminary constructions with differential operators.* We digress to some general constructions with D-modules, continuing the discussion from §6.1.

Suppose $\pi : X \to Y$ is a morphism between smooth varieties. In this case, one has a canonical natural isomorphism:

$$\operatorname{Oblv}^\ell \pi^! \simeq \pi^* \operatorname{Oblv}^\ell$$

of functors:

$$D(Y) \to \mathsf{QCoh}(X).$$

Here π^* is the quasi-coherent pullback functor. We refer to [GR1] for the construction.

The canonical map:

$$\mathcal{O}_Y \to \operatorname{Oblv}^\ell \operatorname{ind}^\ell(\mathcal{O}_Y) = \operatorname{Oblv}^\ell(D_Y)$$

then gives rise to a map:

$$\mathcal{O}_X \to \operatorname{Oblv}^\ell(D_Y) \simeq \operatorname{Oblv}^\ell \pi^!(D_Y)$$

so by adjunction a map:

$$\operatorname{can} : D_X \to \pi^!(D_Y). \tag{8.2.3}$$

In the special case where π is smooth, the map can is between objects concentrated in cohomological degree $-\dim X$, and is an epimorphism in that abelian category.

Finally, we note that if an algebraic group G acts on X and Y, and the map π is G-equivariant, the map can upgrades canonically to a map of G-weakly equivariant D-modules (by functoriality of the above constructions).

8.2.5. *A compatibility.* Let us now return to our setting. For each n, the above provides canonical morphisms:

$$\mathrm{can}_n : D_{\mathfrak{L}^+_{n+1}\mathbb{A}^1} \to p^!_{n,n+1} D_{\mathfrak{L}^+_n\mathbb{A}^1} \in D(\mathfrak{L}^+_{n+1}\mathbb{A}^1)^{\mathfrak{L}^+_{n+1}\mathbb{G}_m,w}.$$

Suppose we are given a commutative diagram (8.2.2). We then obtain a morphism:

$$\begin{gathered} i^{\mathsf{IndCoh}}_{n+1,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}}) =: \Delta_{n+1}(D_{\mathfrak{L}^+_{n+1}\mathbb{A}^1}) \xrightarrow{\Delta_{\mathrm{n+1}}(\mathrm{can_n})} \Delta_{n+1}(p^!_{n,n+1}D_{\mathfrak{L}^+_n\mathbb{A}^1}) \overset{(8.2.2)}{\simeq} \\ \Delta_n(D_{\mathfrak{L}^+_n\mathbb{A}^1}) := i^{\mathsf{IndCoh}}_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \in \mathsf{IndCoh}^*(\mathcal{Y})^{\mathfrak{L}^+\mathbb{G}_m,w} \simeq \mathsf{IndCoh}^*(\mathcal{Y}_{\mathrm{log}}). \end{gathered} \tag{8.2.4}$$

(We have omitted terms $\lambda^{\mathsf{IndCoh}}_{-,*}$ to simplify the notation. We are also abusing notation in letting e.g. Δ_n denote the induced functor on weakly equivariant categories.)

On the other hand, there is another such map coming from the embedding $\delta_{n+1} : \mathcal{Z}^{\leqslant n} \hookrightarrow \mathcal{Z}^{\leqslant n+1}$ and the adjunction map:

$$\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}} \to \delta_{n+1,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}). \tag{8.2.5}$$

We will show:

Proposition 8.2.5.1. *There exist unique commutative diagrams* (8.2.2) *so that the resulting map* (8.2.4) *is* (8.2.5)*.*

In this remainder of this subsection, we prove Proposition 8.2.5.1.

8.2.6. Let us make a preliminary observation in a general setting. Fix G an affine algebraic group.

Suppose that A is an associative algebra with a G-action and a Harish-Chandra datum as in §6.5, so G acts strongly on A–mod.

Suppose we are given $\mathcal{C}, \mathcal{D} \in G$–$\mathsf{mod}$ and functors:

$$\begin{gathered} F_1 : A\text{–}\mathsf{mod} \to \mathcal{C} \\ F_2 : \mathcal{C} \to \mathcal{D} \\ F_3 : A\text{–}\mathsf{mod} \to \mathcal{D}. \end{gathered}$$

We let:

$$\mathcal{F}_1 := F_1(A) \in \mathcal{C}^{G,w}, \mathcal{F}_3 := F_3(A) \in \mathcal{D}^{G,w}.$$

Moreover, to remove discussion of higher homotopical considerations, we assume:

- A is classical.
- $\underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F}_1)$ and $\underline{\mathrm{End}}_{\mathcal{D}}(\mathcal{F}_3)$ are classical.

Then, as in §6.5, the functor F_1 is completely encoded in the datum of compact object $\mathcal{F}_1 \in \mathcal{C}^{G,w}$ with its action of $A^{op} \in \mathsf{Alg}(\mathsf{Rep}(G))$ by endomorphisms (the data should satisfy the property that the action of $\mathfrak{g}$ through $i : \mathfrak{g} \to A$ coincides with the obstruction to strong equivariance). The same applies for F_3.

Therefore, we see that (under the homotopically simplifying assumptions above), the data of a commutative diagram:

$$\begin{array}{ccc} A\text{–}\mathsf{mod} & & \\ {\scriptstyle F_1}\downarrow & \overset{F_3}{\searrow} & \\ \mathcal{C} & \xrightarrow{F_2} & \mathcal{D} \end{array}$$

is equivalent to giving an isomorphism:

$$F_2(\mathcal{F}_1) \simeq \mathcal{F}_3 \in \mathcal{D}^{G,w}$$

compatible with the actions of $A^{op} \in \mathsf{Rep}(G)$ on both sides. (In particular, under these assumptions, it is equivalent to construct a commutative diagram of strongly or weakly G-equivariant categories.)

8.2.7. We apply this in our setting as follows.

First, the actions on the source categories factor through the localization $D^*(\mathfrak{L}^+\mathbb{G}_m) \to D(\mathfrak{L}^+_{n+1}\mathbb{G}_m)$, and the functors factor through the subcategory $\mathsf{IndCoh}^*(\mathcal{Y}^{\leqslant n+1}) \subseteq \mathsf{IndCoh}^*(\mathcal{Y})$, through which $D(\mathfrak{L}^+_n\mathbb{G}_m)$ acts. So the question only concerns the action of an affine algebraic group, not an affine group scheme.

We then see it suffices to show that there is a unique isomorphism:

$$\Delta_{n+1}(p^!_{n,n+1}D_{\mathfrak{L}^+_n\mathbb{A}^1}) \simeq \Delta_n(D_{\mathfrak{L}^+_n\mathbb{A}^1}) := i^{\mathsf{IndCoh}}_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \tag{8.2.6}$$

such that:

- The two resulting maps from $i^{\mathsf{IndCoh}}_{n+1,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}})$ (one being (8.2.4), the other being the canonical map) coincide.
- The two resulting actions of W_n^{op} on $\mathrm{Av}_!^{\mathrm{Gr}^{\leqslant n}_{\mathbb{G}_m}} i^{\mathsf{IndCoh}}_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}})$ (coming from the tautological action of W_n^{op} on $D_{\mathfrak{L}^+_n\mathbb{A}^1} \in D(\mathfrak{L}^+_n\mathbb{A}^1)$, and functoriality of $\Delta_{n+1}p^!_{n,n+1}$ and Δ_n respectively) coincide under the isomorphism.

As the canonical map:

$$i^{\mathsf{IndCoh}}_{n+1,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}}) \to i^{\mathsf{IndCoh}}_{n,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}) \in \mathsf{Coh}(\mathcal{Y}_{\log})^{\heartsuit}$$

is an epimorphism, we see that the first point above implies that such an isomorphism is unique if it exists.

To see existence, it is convenient to fix a coordinate t on the disc. We use the notation α notation of §7.4.2.

We have an evident short exact sequence:

$$0 \to D_{\mathfrak{L}^+_{n+1}\mathbb{A}^1}(-1) \xrightarrow{-\cdot\partial\alpha_n} D_{\mathfrak{L}^+_{n+1}\mathbb{A}^1} \xrightarrow{\mathrm{can}_n} p^!_{n,n+1}D_{\mathfrak{L}^+_n\mathbb{A}^1} \to 0$$

in the abelian category $D(\mathfrak{L}^+_{n+1}\mathbb{A}^1)^{\mathfrak{L}^+_{n+1}\mathbb{G}_m,w,\heartsuit}[n+1]$. Here in the first term, the twist (-1) indicates that we modify the weakly equivariant structure by tensoring with the representation $k(-1) \in \mathsf{Rep}(\mathfrak{L}^+\mathbb{G}_m)$ obtained by restriction along ev from the standard representation of $\mathbb{G}_m$.

By construction of Δ_{n+1}, on weakly equivariant categories, we have:

$$\Delta_{n+1}(D_{\mathfrak{L}^+_{n+1}\mathbb{A}^1}) \simeq i^{\mathsf{IndCoh}}_{n+1,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}})$$
$$\Delta_{n+1}(D_{\mathfrak{L}^+_{n+1}\mathbb{A}^1}(-1)) \simeq i^{\mathsf{IndCoh}}_{n+1,*}\iota^{\mathsf{IndCoh}}_*(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}}).$$

Moreover, it is easy to see that $\Delta_{n+1}(-\cdot\partial_{\alpha_n})$ goes to the map induced from:

$$\iota^{\mathsf{IndCoh}}_*(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}}) \xrightarrow{b_{-n}\cdot -} \mathcal{O}_{\mathcal{Z}^{\leqslant n+1}},$$

i.e., the left arrow in (2.8.3). From (2.8.3), it follows that there is a unique isomorphism (8.2.6) compatible with the projection from $i^{\mathsf{IndCoh}}_{n+1,*}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}})$. It is straightforward to check that this isomorphism is compatible with the action of W_n^{op}, concluding the argument.

8.3. **Compatibility with translation.** We now establish an additional equivariance property for the functor Δ_∞.

Specifically, recall the positive loop space $\mathfrak{L}^{pos}\mathbb{G}_m \subseteq \mathfrak{L}\mathbb{G}_m$ for $\mathbb{G}_m$, as defined in §3.5. This space is an ind-closed submonoid of $\mathfrak{L}\mathbb{G}_m$ containing $\mathfrak{L}^+\mathbb{G}_m$, so we have a fully-faithful monoidal functor:

$$D^*(\mathfrak{L}^{pos}\mathbb{G}_m) \subseteq D^*(\mathfrak{L}\mathbb{G}_m).$$

By construction, $\mathfrak{L}^{pos}\mathbb{G}_m$ acts on $\mathfrak{L}^+\mathbb{A}^1$, so $D^*(\mathfrak{L}^{pos}\mathbb{G}_m)$ acts on $D^!(\mathfrak{L}^+\mathbb{A}^1)$.

On the other hand, we have the local class field theory isomorphism:

$$D^*(\mathfrak{L}\mathbb{G}_m) \simeq \mathsf{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m})$$

defining a (fully faithful, symmetric) monoidal functor:

$$D^*(\mathfrak{L}^{pos}\mathbb{G}_m) \to \mathsf{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m})$$

and so an action of $D^*(\mathfrak{L}^{pos}\mathbb{G}_m)$ on $\mathsf{IndCoh}^*(\mathcal{Y})$.

Our goal is to canonically upgrade the strong $\mathfrak{L}^+\mathbb{G}_m$-equivariance of Δ_∞ to make Δ_∞ into a morphism of $D^*(\mathfrak{L}^{pos}\mathbb{G}_m)$-module categories.

8.3.1. Before proceeding, we wish to pin down this equivariance property uniquely.

Suppose $\mathcal{C}$ is a $D^*(\mathfrak{L}^{pos}\mathbb{G}_m)$-module category. We obtain a canonical functor $T : \mathcal{C}^{\mathfrak{L}^+\mathbb{G}_m} \to \mathcal{C}^{\mathfrak{L}^+\mathbb{G}_m}$ given by the action of (the δ D-module at) some (equivalently any) uniformizer $t \in \mathfrak{L}^{pos}\mathbb{G}_m$.

Our uniqueness property will involve the corresponding functors T on both sides. Below, we collect preliminary observations regarding how the functor T interacts with the source and target of our functor.

8.3.2. Suppose $\mathcal{C} = D^!(\mathfrak{L}^+\mathbb{A}^1)$. A choice of uniformizer t defines a closed embedding $\mu_t : \mathfrak{L}^+\mathbb{A}^1 \hookrightarrow \mathfrak{L}^+\mathbb{A}^1$ given by multiplication by t. The corresponding functor T above is given by the left adjoint to the pullback functor:

$$\mu_t^! : D^!(\mathfrak{L}^+\mathbb{A}^1) \to D^!(\mathfrak{L}^+\mathbb{A}^1).$$

I.e., in the notation of [Ras3], we have $T = \mu_{t,*,!-dR}$.

By adjunction, we observe that there is a canonical map:

$$\mathrm{can} : T(\omega_{\mathfrak{L}^+\mathbb{A}^1}) = \mu_{t,*,!-dR}(\omega_{\mathfrak{L}^+\mathbb{A}^1}) = \mu_{t,*,!-dR}\mu_t^!(\omega_{\mathfrak{L}^+\mathbb{A}^1}) \to \omega_{\mathfrak{L}^+\mathbb{A}^1}. \tag{8.3.1}$$

8.3.3. Now suppose $\mathcal{C} = \mathsf{IndCoh}^*(\mathcal{Y})$. Then we have:

$$\mathsf{IndCoh}^*(\mathcal{Y})^{\mathfrak{L}^+\mathbb{G}_m} \simeq \mathsf{IndCoh}^*(\mathcal{Y})_{\mathfrak{L}^+\mathbb{G}_m} \simeq$$
$$\mathsf{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant 0}) \underset{\mathsf{QCoh}(\mathrm{LocSys}_{\mathbb{G}_m})}{\otimes} \mathsf{IndCoh}^*(\mathcal{Y}) \subseteq \mathsf{IndCoh}^*(\mathcal{Y} \underset{\mathrm{LocSys}_{\mathbb{G}_m}}{\times} \mathrm{LocSys}_{\mathbb{G}_m}^{\leqslant 0}) = \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant 0}).$$

Under the above identifications, the functor T is given by tensoring with the *dual*[62] to the standard representation of $\mathbb{B}\mathbb{G}_m$, i.e., tensoring with the bundle $\mathcal{O}_{\mathcal{Y}^{\leqslant 0}}(1)$ considered earlier.

Now observe that there is a canonical map:

$$\mathrm{can} : T(\mathcal{O}_{\mathcal{Z}^{\leqslant 0}}) = \mathcal{O}_{\mathcal{Z}^{\leqslant 0}}(1) \to \mathcal{O}_{\mathcal{Z}^{\leqslant 0}} \tag{8.3.2}$$

from (2.8.2).

8.3.4. We can now formulate our results precisely.

Observe that by construction, we have an isomorphism:

$$\Delta_\infty(\omega_{\mathfrak{L}^+\mathbb{A}^1}) = \Delta_0(\omega_{\mathfrak{L}_0^+\mathbb{A}^1}) \simeq i_{0,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant 0}}) \in \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant 0}).$$

(Here we consider $\omega_{\mathfrak{L}^+\mathbb{A}^1}$ as a $\mathfrak{L}^+\mathbb{G}_m$-equivariant object.)

Proposition 8.3.4.1. *There is a unique structure of $D^*(\mathfrak{L}^{pos}\mathbb{G}_m)$-equivariance on the functor Δ_∞ such that:*

- *The underlying $D^*(\mathfrak{L}^+\mathbb{G}_m)$-equivariance is the one constructed in §8.2.*

[62]Because the Contou-Carrère pairing is skew symmetric, one has to fix a convention in setting up geometric class field theory. The sign compatibility was implicitly fixed in §3. It is a straightforward check to trace the constructions to see that our sign conventions force the appearance of the dual to the standard representation; essentially, we are on the "dual" side of the Contou-Carrère duality.

- *The map:*
$$T(\mathcal{O}_{\mathcal{Z}^{\leqslant 0}}) \to \mathcal{O}_{\mathcal{Z}^{\leqslant 0}} \in \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant 0})$$
obtained by functoriality from:
$$\Delta_\infty(\mathrm{can} : T(\omega_{\mathfrak{L}^+\mathbb{A}^1}) \xrightarrow{(8.3.1)} \omega_{\mathfrak{L}^+\mathbb{A}^1})$$
coincides with (8.3.2).

Proof. We begin with the existence. For this, it is convenient to fix a uniformizer t once and for all. Note that this choice induces a morphism $\mathrm{LocSys}_{\mathbb{G}_m} \to \mathbb{B}\mathbb{G}_m$ using Proposition 2.3.5.1. We let $\mathcal{O}_{\mathrm{LocSys}_{\mathbb{G}_m}}(-1)$ and $\mathcal{Y}(-1)$ denote the corresponding line bundles, and generally use notation as in §2.7.2.

We obtain a splitting:
$$D^*(\mathfrak{L}^{pos}\mathbb{G}_m) \simeq D^*(\mathfrak{L}^+\mathbb{G}_m) \otimes D(\mathbb{Z}^{\geqslant 0})$$
of (symmetric) monoidal DG categories. Therefore, we find that $D^*(\mathfrak{L}^{pos}\mathbb{G}_m)$-module categories are equivalent to $D^*(\mathfrak{L}^+\mathbb{G}_m)$-module categories with an endofunctor. Therefore, the existence amounts to showing that Δ_∞ intertwines $\mu_{t,*,!-dR}$ with tensoring with $\mathcal{O}_{\mathcal{Y}}(1)$.

Unwinding the constructions, this amounts to constructing commutative diagrams of strong $\mathfrak{L}^+\mathbb{G}_m$-module categories:
$$\begin{array}{ccc} D(\mathfrak{L}_n^+\mathbb{A}^1) & \xrightarrow{\mu_{t,*,dR}} & D(\mathfrak{L}_{n+1}^+\mathbb{A}^1) \\ \downarrow{\scriptstyle \Delta_n} & & \downarrow{\scriptstyle \Delta_{n+1}} \\ \mathsf{IndCoh}^*(\mathcal{Y}) & \xrightarrow{\mathcal{O}_{\mathcal{Y}}(1)\otimes -} & \mathsf{IndCoh}^*(\mathcal{Y}) \end{array}$$
compatibly in n (using the $p_-^!$ functors and the commutativity data of Proposition 8.2.5.1); here we have also let μ_t denote the induced map $\mathfrak{L}_n^+\mathbb{A}^1 \hookrightarrow \mathfrak{L}_{n+1}^+\mathbb{A}^1$ given by multiplication with t.

Using the logic of (8.2.6) (as in the proof of Proposition 8.2.5.1), we see that this amounts to constructing isomorphisms:
$$\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(1) \simeq \Delta_{n+1}(\mu_{t,*,dR} D_{\mathfrak{L}_n^+\mathbb{A}^1}) \in \mathsf{IndCoh}^*(\mathcal{Y}_{\log}^{\leqslant n+1})$$
satisfying some compatibilities (that do not involve higher homotopy theory).

For this, we observe that there is a standard short exact sequence:
$$0 \to D_{\mathfrak{L}_{n+1}^+\mathbb{A}^1} \xrightarrow{-\cdot\alpha_0} D_{\mathfrak{L}_{n+1}^+\mathbb{A}^1}(1) \to \mu_{t,*,dR} D_{\mathfrak{L}_n^+\mathbb{A}^1}[1] \to 0$$
in the abelian category $D(\mathfrak{L}_{n+1}^+\mathbb{A}^1)^{\mathfrak{L}_{n+1}\mathbb{G}_m, w, \heartsuit}[n+1]$ (with notation as before). The map $-\cdot\alpha_0$ maps under Δ_{n+1} to the canonical map:
$$\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}} \to \iota_*^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}}).$$

Therefore, we obtain an isomorphism:
$$\Delta_{n+1}(\mu_{t,*,dR} D_{\mathfrak{L}_n^+\mathbb{A}^1}) \simeq \mathrm{Coker}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}}[-1] \to \iota_*^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}}[-1])) \simeq \mathrm{Ker}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}} \to \iota_*^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n+1}})).$$
The latter is canonically isomorphism to $\delta_{n+1,*}^{\mathsf{IndCoh}}\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(1)$ by (2.8.1), as desired.

This isomorphism is easily seen to be compatible with the actions of W_n^{op} and with varying n, proving existence.

For uniqueness, we observe that Δ_∞ is fully faithful (cf. Lemma 8.4.0.2 below), so the observation follows from the observation that:
$$k \xrightarrow{\simeq} \mathrm{End}_{\mathsf{End}_{\mathfrak{L}^+\mathbb{G}_m-\mathsf{mod}}(D^!(\mathfrak{L}^+\mathbb{A}^1))}(\mu_{t,!,dR}) \in \mathsf{Gpd}$$

(which e.g. follows by replacing functors with kernels on the square $\mathfrak{L}^+\mathbb{A}^1 \times \mathfrak{L}^+\mathbb{A}^1$), meaning that any isomorphism we construct is unique up to scalars; clearly the second compatibility in the proposition pins down this scalar multiple uniquely.

□

8.4. **Conclusion of the argument.** We now have the following precise form of our main theorem.

Theorem 8.4.0.1. *There is a unique equivalence:*

$$\Delta : D^!(\mathfrak{L}\mathbb{A}^1) \xrightarrow{\simeq} \mathsf{IndCoh}^*(\mathcal{Y})$$

of $D^*(\mathfrak{L}\mathbb{G}_m)$-module categories such that the induced morphism:

$$D^!(\mathfrak{L}^+\mathbb{A}^1) \subseteq D^!(\mathfrak{L}\mathbb{A}^1) \to \mathsf{IndCoh}^*(\mathcal{Y})$$

of $D^*(\mathfrak{L}^{pos}\mathbb{G}_m)$-module categories coincides with the functor Δ_∞ from above.

Proof. There is a natural functor:

$$D^*(\mathfrak{L}\mathbb{G}_m) \underset{D^*(\mathfrak{L}^{pos}\mathbb{G}_m)}{\otimes} D^!(\mathfrak{L}^+\mathbb{A}^1) \to D^!(\mathfrak{L}\mathbb{A}^1)$$

that we observe is an equivalence. Indeed, this follows by choosing a coordinate t on the disc as before, splitting $\mathfrak{L}\mathbb{G}_m$ as $\mathfrak{L}^+\mathbb{G}_m \times \mathbb{Z}$, which then implies:

$$D^*(\mathfrak{L}\mathbb{G}_m) \underset{D^*(\mathfrak{L}^{pos}\mathbb{G}_m)}{\otimes} D^!(\mathfrak{L}^+\mathbb{A}^1) \simeq \operatorname{colim} \big(D^!(\mathfrak{L}^+\mathbb{A}^1) \xrightarrow{\mu_{t,*,!-dR}} D^!(\mathfrak{L}^+\mathbb{A}^1) \xrightarrow{\mu_{t,*,!-dR}} \dots\big).$$

Using t^{-n} to identify the nth term here with $t^{-n} \cdot \mathfrak{L}^+\mathbb{A}^1 \subseteq \mathfrak{L}\mathbb{A}^1$ and applying the definition from [Ras3], we see that this colimit canonically identifies with $D^!(\mathfrak{L}\mathbb{A}^1)$ compatibly with the natural functor considered above.

Therefore, there is a unique functor:

$$\Delta : D^!(\mathfrak{L}\mathbb{A}^1) \xrightarrow{\simeq} \mathsf{IndCoh}^*(\mathcal{Y})$$

of $D^*(\mathfrak{L}\mathbb{G}_m)$-module categories restricting to Δ_∞ (compatibly with the $D^*(\mathfrak{L}^{pos}\mathbb{G}_m)$-equivariance). It remains to show that Δ is an equivalence.

First, we note that Δ_∞ is fully faithful by Proposition 7.1.1.1 and Lemma 8.4.0.2 below. Next, we deduce that Δ itself is fully faithful by another application of Lemma 8.4.0.2.

Therefore, it remains to show that Δ is essentially surjective. Clearly its essential image contains each object:

$$\lambda_{n,*}^{\mathsf{IndCoh}} \operatorname{Av}_!^{\mathrm{Gr}_{\mathbb{G}_m}^{\leqslant n},w} i_{n,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathcal{Z}^{\leqslant n}}(m))$$

for all $n \geqslant 0$, $m \in \mathbb{Z}$. Therefore, it suffices to show that $\mathsf{IndCoh}^*(\mathcal{Y})$ is generated under colimits by these objects. But this follows immediately from the definitions and Corollary 4.4.3.1.

□

Above, we used the following simple categorical lemma.

Lemma 8.4.0.2. *Suppose we are given a filtered diagram $i \mapsto \mathcal{C}_i$ of compactly generated DG categories, and a functor:*

$$F : \mathcal{C} := \operatorname*{colim}_i \mathcal{C}_i \to \mathcal{D} \in \mathsf{DGCat}_{cont}$$

preserving compact objects. Suppose each structural functor:

$$\mathcal{C}_i \to \mathcal{C}_j$$

and each composition:

$$\mathcal{C}_i \to \mathcal{C} \to \mathcal{D}$$

is fully faithful. Then F is fully faithful.

References

[AG] Dima Arinkin and Dennis Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Mathematica*, 21(1):1–199, 2015.

[BD] Sasha Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[Ber] Dario Beraldo. Loop group actions on categories and whittaker invariants. *Advances in Mathematics*, 322:565–636, 2017.

[BF] Alexander Braverman and Michael Finkelberg. Coulomb branches of 3-dimensional gauge theories and related structures. In *Geometric Representation Theory and Gauge Theory*, pages 1–52. Springer, 2019.

[BFGT] Alexander Braverman, Michael Finkelberg, Victor Ginzburg, and Roman Travkin. Mirabolic Satake equivalence and supergroups. *arXiv preprint arXiv:1909.11492*, 2019.

[BFN1] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. *Advances in Theoretical and Mathematical Physics*, 22(5):1071–1147, 2018.

[BFN2] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Coulomb branches of $3d$ $\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian. *Advances in Theoretical and Mathematical Physics*, 23(1):75–166, 2019.

[BFN3] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Ring objects in the equivariant derived Satake category arising from Coulomb branches. *Advances in Theoretical and Mathematical Physics*, 23(2):253–344, 2019.

[BFT] Alexander Braverman, Michael Finkelberg, and Roman Travkin. Orthosymplectic Satake equivalence. *arXiv preprint arXiv:1912.01930*, 2019.

[BGI$^+$] P Berthelot, A Grothendieck, L Illusie, et al. Théorie des intersections et théoreme de Riemann–Roch, séminaire de géométrie algébrique du Bois-Marie 1966–1967 (SGA 6). *Lecture Notes in Math*, 225, 1971.

[BHL] Bhargav Bhatt and Daniel Halpern-Leistner. Tannaka duality revisited. *Advances in Mathematics*, 316:576–612, 2017.

[BLPW1] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster. Hypertoric category $\mathcal{O}$. *Advances in Mathematics*, 231(3-4):1487–1545, 2012.

[BLPW2] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster. Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality. *Astérisque*, 384:75–179, 2016.

[BPW] Tom Braden, Nicholas Proudfoot, and Ben Webster. Quantizations of conical symplectic resolutions I: local and global structure. *Astérisque*, 384:1–73, 2016.

[But] Dylan Butson. Equivariant localization in factorization homology and applications in mathematical physics I: Foundations. *arXiv preprint arXiv:2011.14988*, 2020.

[BZ] David Ben-Zvi. Electric-magnetic duality between periods and l-functions. Talk at Western Hemisphere Colloquium on Geometry and Physics, available at `https://www.youtube.com/watch?v=exf7wsHjPbU`, 2021.

[BZGN] David Ben-Zvi, Sam Gunningham, and David Nadler. The character field theory and homology of character varieties. *Mathematical Research Letters*, 26(5):1313–1342, 2019.

[BZN] David Ben-Zvi and David Nadler. Loop spaces and connections. *Journal of Topology*, 5(2):377–430, 2012.

[CC] Carlos Contou-Carrère. Jacobienne locale d'une courbe formelle relative. *Rendiconti del Seminario Matematico della Università di Padova*, 130:1–106, 2013.

[CCG] Kevin Costello, Thomas Creutzig, and Davide Gaiotto. Higgs and Coulomb branches from vertex operator algebras. *Journal of High Energy Physics*, 2019(3):1–50, 2019.

[CDG] Kevin Costello, Tudor Dimofte, and Davide Gaiotto. Boundary chiral algebras and holomorphic twists. *arXiv preprint arXiv:2005.00083*, 2020.

[CG1] Kevin Costello and Davide Gaiotto. Vertex Operator Algebras and $3d$ $\mathcal{N} = 4$ gauge theories. *Journal of High Energy Physics*, 2019(5):1–39, 2019.

[CG2] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory: Volume 1*, volume 31. Cambridge University Press, 2016.

[Cos1] Kevin Costello. Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4. *Pure and Applied Mathematics Quarterly*, 9(1):73–165, 2013.

[Cos2] Kevin Costello. Factorization algebras associated to the $(2,0)$ theory. Series of four talks, all of which are available on YouTube. The first two are available at `https://www.youtube.com/watch?v=cdlvhIMO-zQ` `https://www.youtube.com/watch?v=euWxJmIQs8E`, 2014.

[CS] Kevin Costello and Claudia Scheimbauer. Lectures on mathematical aspects of (twisted) supersymmetric gauge theories. In *Mathematical aspects of quantum field theories*, pages 57–87. Springer, 2015.

[DGGH] Tudor Dimofte, Niklas Garner, Michael Geracie, and Justin Hilburn. Mirror symmetry and line operators. *Journal of High Energy Physics*, 2020(2):1–150, 2020.

[Dim] Tudor Dimofte. Categories of line operators in $3d$ $\mathcal{N} = 4$ gauge theories. Talk at Perimeter Institute, available at `https://pirsa.org/index.php/18100103`, 2018.

[ES] Chris Elliott and Pavel Safronov. Topological twists of supersymmetric algebras of observables. *Communications in Mathematical Physics*, 371(2):727–786, 2019.

[ESW] Chris Elliott, Pavel Safronov, and Brian R Williams. A Taxonomy of Twists of Supersymmetric Yang–Mills Theory. *arXiv preprint arXiv:2002.10517*, 2020.

[EY] Chris Elliott and Philsang Yoo. Geometric Langlands twists of $\mathcal{N} = 4$ gauge theory from derived algebraic geometry. *Adv. Theor. Math. Phys.*, 22:615–708, 2018.

[Gai1] Davide Gaiotto. S-duality of boundary conditions and the Geometric Langlands program. In *Proc. Symp. Pure Math*, volume 98, page 139, 2018.

[Gai2] Dennis Gaitsgory. Quantum Langlands correspondence. *arXiv preprint arXiv:1601.05279*, 2007.

[Gai3] Dennis Gaitsgory. Stacks. *Available at* `http://www.math.harvard.edu/~gaitsgde/GL`, 2012.

[Gai4] Dennis Gaitsgory. Opers (Day VI, Talk 1). Available at `https://sites.google.com/site/geometriclanglands2014/notes`. Unpublished conference notes, 2014.

[GJS] Russell A Goward Jr and Karen E Smith. The jet scheme of a monomial scheme. *Communications in Algebra*, 34(5):1591–1598, 2006.

[GR1] Dennis Gaitsgory and Nick Rozenblyum. Crystals and D-modules. *Pure and Applied Mathematics Quarterly*, 10(1):57–154, 2014.

[GR2] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry*. American Mathematical Soc., 2017.

[GW] Davide Gaiotto and Edward Witten. S-duality of boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory. *Advances in Theoretical and Mathematical Physics*, 13(3):721–896, 2009.

[Hil] Justin Hilburn. Symplectic duality and geometric Langlands duality. Talk at IHÉS, available at `https://www.youtube.com/watch?v=ogDHRjegRXU`, 2019.

[HW] Amihay Hanany and Edward Witten. Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics. *Nuclear Physics B*, 492(1-2):152–190, 1997.

[IS] Kenneth Intriligator and Nathan Seiberg. Mirror symmetry in three dimensional gauge theories. *Physics Letters B*, 387(3):513–519, 1996.

[KRS] Anton Kapustin, Lev Rozansky, and Natalia Saulina. Three-dimensional topological field theory and symplectic algebraic geometry I. *Nuclear Physics B*, 816(3):295–355, 2009.

[KSV] Anton Kapustin, Kevin Setter, and Ketan Vyas. Surface operators in four-dimensional topological gauge theory and Langlands duality. *arXiv preprint arXiv:1002.0385*, 2010.

[KW] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *Communications in Number Theory and Physics*, 1(1):1–236, 2007.

[Lur1] Jacob Lurie. *Higher Topos Theory*, volume 189. Princeton University Press, 2009.

[Lur2] Jacob Lurie. On the classification of topological field theories. *Current developments in mathematics*, 2008:129–280, 2009.

[Lur3] Jacob Lurie. Higher algebra. Available at `http://math.harvard.edu/~lurie/papers/HigherAlgebra.pdf`, 2012.

[Lur4] Jacob Lurie. Spectral algebraic geometry. 2016. Preprint, available at `www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf`.

[Nak] Hiraku Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, I. *Advances in Theoretical and Mathematical Physics*, 20(3):595–669, 2016.

[Ras1] Sam Raskin. Factorization II (Day III, Talk 1). Available at `https://sites.google.com/site/geometriclanglands2014/notes`. Unpublished conference notes, 2014.

[Ras2] Sam Raskin. Chiral categories. Available at `https://web.ma.utexas.edu/users/sraskin/chiralcats.pdf`, 2015.

[Ras3] Sam Raskin. D-modules on infinite dimensional varieties. Available at `https://web.ma.utexas.edu/users/sraskin/dmod.pdf`, 2015.

[Ras4] Sam Raskin. On the notion of spectral decomposition in local geometric Langlands. Available at `math.utexas.edu/~sraskin/locsys.pdf.`, 2015.
[Ras5] Sam Raskin. Chiral principal series categories II: the factorizable Whittaker category. Available at `https://web.ma.utexas.edu/users/sraskin/cpsii.pdf`, 2016.
[Ras6] Sam Raskin. $\mathcal{W}$-algebras and Whittaker categories. Available at `https://web.ma.utexas.edu/users/sraskin/whit.pdf`, 2016.
[Ras7] Sam Raskin. Homological methods in semi-infinite contexts. Available at `https://web.ma.utexas.edu/users/sraskin/topalg.pdf`, 2019.
[RW] Lev Rozansky and Edward Witten. Hyper-Kähler geometry and invariants of three-manifolds. *Selecta Mathematica*, 3(3):401–458, 1997.
[SV] Yiannis Sakellaridis and Akshay Venkatesh. Periods and harmonic analysis on spherical varieties. *Asterisque*, 2017(396):1–370, 2017.
[Tat] John T Tate. Fourier analysis in number fields, and Hecke's zeta-functions. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 305–347, 1967.
[Tel] Constantin Teleman. Gauge theory and mirror symmetry. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1309–1332. Kyung Moon Sa, Seoul, 2014.
[Wan] Jonathan Wang. Examples of Relative Duality. `https://www.jonathanpwang.com/notes/RelativeDualitydb.html`.
[Wei] André Weil. Fonction zêta et distributions. *Séminaire Bourbaki*, 9(312):523–531, 1966.
[Yoo] Philsang Yoo. Symplectic duality and geometric Langlands duality. Talk at IHÉS, available at `https://www.youtube.com/watch?v=IoK6vtB3XJA`, 2019.
[Yue] Cornelia Yuen. Multiplicity of jet schemes of monomial schemes. *arXiv preprint math/0607638*, 2006.

Perimeter Institute, 31 Caroline St N, Waterloo, ON N2L 2Y5, Canada
Email address: `jhilburn@perimeterinstitute.ca`

The University of Texas at Austin, Department of Mathematics, RLM 8.100, 2515 Speedway Stop C1200, Austin, TX 78712
Email address: `sraskin@math.utexas.edu`