HOMOLOGICAL METHODS IN SEMI-INFINITE CONTEXTS

SAM RASKIN

Contents

Preface

In different forms and in disparate settings, loop spaces have been central objects of study in the mathematics of the last century.

In algebraic topology, their study is ubiquitous: the connected components of a based loop space are the fundamental group; modern methods emphasize the algebraic structure of loop spaces themselves, providing more information than was available by classical means. In geometry, the Morse theory of free loop spaces has played a distinguished role since the work [Flo] of Floer thirty years ago. Witten [Wit1] studied loop spaces via geometric analysis, previewing later developments in elliptic cohomology. Representations of of loop groups and affine Lie algebras have driven large parts of the representation theory of the last four decades. In mathematical physics, loop spaces are manifestly tied to string theory, and also arise when compactifying quantum field theories. A great deal of exciting mathematics has arisen at this interfaces between these different subjects.

Sometimes, one imagines that a loop space is like a manifold, only infinite dimensional. However, there are some peculiar phenomena characteristic of this setting and that do not appear in finite dimensions. We use the term *semi-infinite* from the title of this work to refer to these characteristic features. For our purposes, its meaning is somewhat vague, but it is meant to evoke splitting an infinite dimensional space into two pieces, where the difference of the two infinite pieces is finite dimensional.

For instance, in algebra, Laurent series split in such a way, as the sum of Taylor series and polynomials in t^{-1}. In algebraic topology, Atiyah's [Ati] proof of Bott periodicity using Fredholm operators uses such a splitting. Floer's Morse complex has this flavor, where indices of critical

points are infinite but their differences are finite. And loop spaces of manifolds have polarizations of this flavor, (see [Seg] §4, for example).In representation theory, semi-infinite cohomology of affine Lie algebras has this flavor. In number theory and arithmetic geometry, the tame symbol arises by such a construction [BBE].

This manuscript develops some foundational aspects of semi-infinite algebraic geometry. For instance, we develop a theory of coherent sheaves on suitable semi-infinite spaces. Much attention is given to symmetries of semi-infinite spaces and their categories of sheaves: a substantial portion of our study relates to group actions in this setting. In addition, we relate our theory to topological algebras in Beilinson's sense [Bei], which allows us to circumvent higher categorical issues when applying our theory to treat concrete problems. Finally, we apply our theory to settle a foundational issue of interest in the geometric Langlands program involving critical level Kac-Moody representations.

1. Introduction

1.1. **What is this work about?** Briefly, this manuscript develops foundational aspects of the algebraic geometry of loop spaces and the (higher) representation theory of loop groups.

At this point, the reader may naturally ask a number of other questions: what aspects? How do the present foundations compare to existing works? What are the main constructions and results of this work? Why are they useful, and what are they useful for? Why was this text written?

We ask the reader's patience as we defer these questions. The bulk of this introduction is intended to address them. However, given the technical nature of this work, we begin this introduction with a more informal discussion of loop spaces and their place in algebraic geometry.

1.1.1. *Loop spaces in topology.* By way of introduction, suppose M is a manifold. In this case, there are two possible interpretations of the *loop space* of M.

Define $\mathcal{L}M$ as the space of smooth maps $\{\gamma : S^1 \to M\}$ equipped with its standard topology.

One can consider $\mathcal{L}M$ as a sort of infinite dimensional manifold: at a point γ, its tangent space should be $\Gamma(S^1, \gamma^*(T_M))$. We can therefore expect to find interesting differential geometry associated with this space.

There are many instances of this idea in the literature. To name a few:

- Witten [Wit1] proposed to study elliptic operators on $\mathcal{L}M$ to obtain geometric invariants in the spirit of Atiyah-Singer. This idea fostered the development of elliptic cohomology (see [Lur1] or [Seg] for example).
- If $M = G$ is a compact Lie group, then there is a rich theory of (projective) Hilbert space representations of $\mathcal{L}M$, cf. [PS]. This theory mimics the representation theory of compact Lie groups in many respects.
- Bott [Bot1], [Bot2] used Morse theory on (based) loop spaces to study homology of $\Omega G = \mathcal{L}G \times_G *$ and to prove his celebrated periodicity theorem.
- Floer [Flo] used Morse theory on loop spaces to prove a special case of Arnold's conjecture.

In what follows, the specifics of the above examples are not so important. But we highlight a few key points. First, each of the constructions above are *geometric*, not homotopy theoretic. Second, there are evident functional analytic questions at every stage. Finally, there are significant difficulties (not all surmounted yet) in importing ideas from finite dimensional geometry into this infinite dimensional setting.

1.1.2. *Loop spaces in algebraic geometry.* Broad features of differential geometry often have counterparts in algebraic geometry. Loop spaces provide such an example, as we discuss below.

In what follows, we work over a field k fixed once and for all. We assume once and for all that k has characteristic 0, though this is not literally needed for every point of our discussion. All schemes are assumed to be k-schemes. We speak in absolute terms about relative properties of schemes to imply reference to $\mathrm{Spec}(k)$; e.g., a *smooth scheme* is a k-scheme that is smooth over $\mathrm{Spec}(k)$.

We let $K := k((t))$ and $O := k[[t]]$. We let $\mathring{\mathcal{D}} := \mathrm{Spec}(K)$ and $\mathcal{D} := \mathrm{Spec}(O)$ denote the formal punctured disc and the formal disc respectively.

We may heuristically think of $\mathring{\mathcal{D}}$ as an algebro-geometric version of the circle S^1. Here we understand the *geometric* circle, not merely its homotopy type.

1.1.3. For Y an affine scheme of finite type (over k), there is an *indscheme* (resp. scheme) $Y(K)$ (resp. $Y(O)$), the *loop (resp. arc) space* of Y, that parametrizes maps $\mathring{\mathcal{D}} \to Y$ (resp. $\mathcal{D} \to Y$).

Below, we first give explicit constructions in the case $Y = \mathbb{A}^1$. We then give the definitions in general.

1.1.4. First, suppose $Y = \mathbb{A}^1$.

Then $\mathbb{A}^1(O)$ is meant to parametrize maps $\mathcal{D} \to \mathbb{A}^1$, i.e., Taylor series $\sum_{i \geqslant 0} a_i t^i$. We take:

$$\mathbb{A}^1(O) = \mathrm{Spec}(k[a_0, a_1, \ldots]).$$

That is, of a polynomial algebra with generators labelled by $\mathbb{Z}^{\geqslant 0}$. At the risk of redundancy: the function $a_i : \mathbb{A}^1(O) \to \mathbb{A}^1$ takes the ith Taylor coefficient.

1.1.5. Similarly, $\mathbb{A}^1(K)$ should parametrize Laurent series $k((t)) = \mathrm{colim}_n t^{-n} k[[t]]$.

We take:

$$\mathbb{A}^1(K) = \mathop{\mathrm{colim}}_{n} \mathrm{Spec}(k[a_{-n}, a_{-n+1}, \ldots]).$$

Here the structural morphisms:

$$\mathrm{Spec}(k[a_{-n}, a_{-n+1}, \ldots]) \to \mathrm{Spec}(k[a_{-n-1}, a_{-n+1}, \ldots])$$

correspond to the evident ring maps:

$$k[a_{-n-1}, a_{-n}, \ldots]) \twoheadrightarrow k[a_{-n}, a_{-n+1}, \ldots]$$
$$a_{-n-1} \mapsto 0$$
$$a_i \mapsto a_i, \quad i \geqslant -n.$$

But how should this colimit be understood? We do not mean it in the category of schemes, affine or otherwise. Rather, it should be understood in a formal categorical sense, as an indscheme. More precisely, we may understand this colimit in the category of *prestacks*.

We refer to [GR3] for a introduction to indschemes (in the general setting of derived algebraic geometry).

1.1.6. An obvious variant of the above discussions hold for $Y = \mathbb{A}^n$ in place of $\mathbb{A}^1$: take $Y(O) = \prod_{i=1}^n \mathbb{A}^1(O)$, and $Y(K) = \prod_{i=1}^n \mathbb{A}^1(K)$.

More generally, if Y is an affine scheme of finite type, we may embed Y into $\mathbb{A}^n$ for some n and thereby embed $Y(O)$ (resp. $Y(K)$) into $\mathbb{A}^n(O)$ (resp. $\mathbb{A}^n(K)$) to deduce that the result is an affine scheme (resp. indscheme).

Remark 1.1.1. If Y is not affine, it is well-known that $Y(O)$ still behaves well, but $Y(K)$ does not. See [KV] for further discussion.

1.1.7.   As the case $Y = \mathbb{A}^1$ already makes clear, $Y(O)$ is (almost always) non-Noetherian, i.e., it is of *infinite type*, and $Y(K)$ is ind-infinite type.

Therefore, $Y(K)$ is infinite dimensional in two regards. It is an indscheme, which is *infinite dimensionality in the ind-direction*. Moreover, it is a union of subschemes like $Y(O)$, which reflects *infinite dimensionality in the pro-direction*. This parallels Laurent series $k((t))$, which are similarly infinite dimensional in two ways and in two directions.

In one interpretation, the word *semi-infinite* from the title of this work refers to this flavor of geometry. We expand the usual landscape of algebraic geometry in two respects: we wish to consider infinite type schemes (alias: pro-finite dimensional) and indschemes of ind-finite type (alias: ind-finite dimensional), and need a class that contains both.

We remark that the intersection of these two classes: schemes (possibly of infinite type) that are ind-finite type are exactly schemes of finite type. This parallels the linear algebra fact that topological vector spaces that are discrete and pro-finite dimensional are finite dimensional.

1.1.8.   In this manuscript, we develop some foundational aspects of algebraic geometry for such semi-infinite spaces.

We are particularly interested in studying such spaces in the context of geometric representation theory, and much of our emphasis reflects this. For instance, if G acts on Y, then $G(K)$ acts on $Y(K)$, and we might consider $G(K)$-equivariant sheaves on $Y(K)$, or other implications of these symmetries for sheaves on $Y(K)$. Or we might replace $G(K)$ by its Lie algebra and consider infinitesimal versions.

More broadly, we emphasize non-commutative geometry, derived categories of sheaves, group symmetries, and Lie algebra symmetries. Of course, the overall goal is to recover as closely as possible classical finite dimensional constructions from these theories in semi-infinite settings.

1.1.9.   Below, we begin discussing the contents of this manuscript in more detail, making reference to loop spaces as motivating examples.

At this point, we might have instead surveyed appearances of algebro-geometric loop spaces in the literature, in parallel with §1.1.1. We prefer to incorporate connections with recent research below in our discussion of the present work.

1.2. **Brief remarks on categorical conventions.** Before delving into the contents of this work, we comment on some of our conventions.

First, as remarked above, we always work over a field k of characteristic 0.

As the title of this work suggests, we use a great deal of homological algebra here. Our preferred foundations is the ∞-categorical approach to DG categories; we refer to [GR4] §I.1 for a detailed introduction to this perspective.

DG categories are a more robust substitute for triangulated categories. Informally, DG categories are categories enriched over chain complexes of k-vector spaces. The derived categories one typically runs into in algebraic geometry and representation theory all naturally come from DG categories, and we consider them as such.

DG categories are more readily manipulated than triangulated categories. For instance, if one wishes to form limits of derived categories, i.e., categories of *compatible systems of complexes up to quasi-isomorphism*, the homotopy limit of the corresponding DG categories provides an answer with suitable properties, while there is no answer using triangulated categories alone. This construction is quite useful for the purposes of this text: we generally define categories of sheaves on infinite dimensional spaces as compatible systems on finite dimensional ones.

As indicated above, DG categories are objects of homotopical nature. Therefore, we consider them as ∞-categories in the sense of [Lur2] with extra structure, again, following [GR4]. For us,

∞-categories provide a convenient, easily manipulated, and unified foundation for homotopical mathematics.

We often drop extra decorations in the terminology, and so we simply refer to *categories* and *co/limits* for ∞-categories and homotopy co/limits within them. We speak of 1-categories when we wish to emphasize categories enriched over sets rather than ∞-groupoids.

Remark 1.2.1. To some extent, the reader can probably ignore our choice of foundations during this introduction and prefer their own models. But to read the body of the work itself, it is necessary to first be acquainted with [GR4] §I.1. Therefore, during this introduction, we choose not to make every effort to avoid the terminology of DG categories and ∞-categories.

1.3. **Coherent sheaves.** In §6 of this work, we develop a theory of (ind-)coherent sheaves on semi-infinite spaces.

Below, we give motivation for this theory and describe some aspects of it.

1.3.1. *Finite dimensional recollections.* The role of coherent sheaves in conventional, finite dimensional algebraic geometry is well-known.

Many classical invariants and constructions from the Italian[1] et al.) found their home in Serre's theory [Ser1] of coherent sheaf cohomology and Serre's duality theorem; see [Die] §VIII for a discussion.

Work of Auslander-Buchsbaum [AB] and Serre [Ser2] highlighted the interplay between geometry and derived categories of sheaves, as we discuss further in §1.3.3.

In recent years, the above constructions have been abstracted, most notably in [GR4], via the functoriality of *ind-coherent sheaves* on Noetherian schemes. We provide a brief introduction to this circle of ideas below, and refer to *loc. cit.* for more context.

1.3.2. We now provide some more technical detail and notation to flesh out the discussion above. This discussion may be skipped at a first pass.

Suppose S is a (classical) scheme of finite type (over our characteristic 0 field k).

There is a traditional abelian (1-)category $\mathsf{QCoh}(S)^\heartsuit$ of quasi-coherent sheaves on S. Let $\mathsf{Coh}(S)^\heartsuit \subseteq \mathsf{QCoh}(S)^\heartsuit$ denote the subcategory of coherent sheaves.

We recall that every object of $\mathsf{QCoh}(S)^\heartsuit$ can be realized as a colimit of coherent sheaves. This can be strengthened with a categorical assertion: the natural functor $\mathsf{Ind}(\mathsf{Coh}(S)^\heartsuit) \to \mathsf{QCoh}(S)^\heartsuit$ is an equivalence. Here for a category $\mathcal{C}$, $\mathsf{Ind}(\mathcal{C})$ denotes its ind-category; in the higher categorical context, we refer to [Lur2] §5.3 for an introduction. In particular, there is a canonical categorical procedure that recovers $\mathsf{QCoh}(S)^\heartsuit$ from $\mathsf{Coh}(S)^\heartsuit$.

1.3.3. We now let $\mathsf{QCoh}(S)$ denote the derived[3] category of $\mathsf{QCoh}(S)^\heartsuit$, which we consider here as a DG category following our conventions.

Unlike the abelian categorical situation, there are *two* choices of "small" subcategory in $\mathsf{QCoh}(S)$.

First, let $\mathsf{Coh}(S) \subseteq \mathsf{QCoh}(S)$ denote the subcategory of cohomologically bounded objects with cohomologies lying in $\mathsf{Coh}(S)^\heartsuit$.

Next, let $\mathsf{Perf}(S) \subseteq \mathsf{QCoh}(S)$ denote the objects that locally on S can be represented by bounded complexes of vector bundles.

Clearly $\mathsf{Perf}(S) \subseteq \mathsf{Coh}(S)$. We have the following standard result in commutative algebra, referenced above.

---

[1] Castelnuovo, Cremona, Enriques, Segre, Severi, Zariski[2]

[3] Generally speaking, it is better to use the definition of [GR4] §I.3 rather than thinking in terms of derived categories.

Theorem 1.3.1 (Auslander-Buchsbaum, Serre). *The inclusion* $\mathsf{Perf}(S) \subseteq \mathsf{Coh}(S)$ *is an equivalence if and only if* S *is smooth.*

Next, we recall the following result of Thomason-Trobaugh.

Theorem 1.3.2. *The natural functor* $\mathsf{Ind}(\mathsf{Perf}(S)) \to \mathsf{QCoh}(S)$ *is an equivalence.*

Remark 1.3.3. For the sake of providing references for these statements in our homotopical setting, we refer to [Gai5] Proposition 1.6.4 and [Lur4] Proposition 9.6.1.1.

1.3.4. *Ind-coherent sheaves.* Above, we can also form $\mathsf{Ind}(\mathsf{Coh}(S))$, which we denote instead as $\mathsf{IndCoh}(S)$.

By the universal property of ind-categories, there is a canonical functor $\mathsf{IndCoh}(S) \to \mathsf{QCoh}(S)$, often denoted as $\Psi = \Psi_S$. If S is singular, this functor is *not* an equivalence. In general, if one has a (small) DG category $\mathcal{C}$ closed under direct summands (i.e., *idempotent complete*), $\mathcal{C}$ is the subcategory of $\mathsf{Ind}(\mathcal{C})$ consisting of *compact* objects. Therefore, Ψ being an equivalence would contradict Theorems 1.3.1 and 1.3.2.

The distinction between $\mathsf{IndCoh}(S)$ and $\mathsf{QCoh}(S)$ is not visible classically, i.e., working only with bounded derived categories. More precisely, $\mathsf{IndCoh}(S)$ has a canonical *t-structure*; we recall that this means we have subcategories[4] $\mathsf{IndCoh}(S)^{\geqslant n}$, $\mathsf{IndCoh}(S)^{\leqslant n}$ of objects in cohomological degrees $\geqslant n$ and $\leqslant n$, and the intersection $\mathsf{IndCoh}(S)^{\heartsuit} := \mathsf{IndCoh}(S)^{\geqslant 0} \cap \mathsf{IndCoh}(S)^{\leqslant 0}$ is an abelian category. Then Ψ is *t*-exact, and induces an equivalence $\mathsf{IndCoh}(S)^{\geqslant 0} \xrightarrow{\sim} \mathsf{QCoh}(S)^{\geqslant 0}$, hence on bounded below objects more generally. In particular, $\mathsf{IndCoh}(S)^{\heartsuit} \xrightarrow{\sim} \mathsf{QCoh}(S)^{\heartsuit}$. We remark the close parallel to the more classical equivalence $\mathsf{Ind}(\mathsf{Coh}(S)^{\heartsuit}) \xrightarrow{\sim} \mathsf{QCoh}(S)^{\heartsuit}$.

Therefore, the difference between IndCoh and QCoh is only relevant for unbounded derived categories.

Remark 1.3.4. Let $S = \mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$. The complex:

$$\ldots \xrightarrow{\varepsilon \cdot} k[\varepsilon]/\varepsilon^2 \xrightarrow{\varepsilon \cdot} k[\varepsilon]/\varepsilon^2 \xrightarrow{\varepsilon \cdot} \ldots \in k[\varepsilon]/\varepsilon^2\text{--mod} = \mathsf{QCoh}(S)$$

is obviously acyclic. But the formal colimit of its stupid truncations:

$$\operatorname*{colim}_n k[n] = \operatorname*{colim}_n \left(\ldots 0 \to 0 \to \underset{\text{degree } -n}{k[\varepsilon]/\varepsilon^2} \xrightarrow{\varepsilon \cdot} k[\varepsilon]/\varepsilon^2 \xrightarrow{\varepsilon \cdot} \ldots \right) \in \mathsf{IndCoh}(S)$$

is non-zero, e.g. as may be seen by computing Hom out of the augmentation module:

$$k \in k[\varepsilon]/\varepsilon^2\text{--mod}_{f.g.}^{\heartsuit} = \mathsf{Coh}(S)^{\heartsuit} \subseteq \mathsf{Coh}(S).$$

1.3.5. To summarize, IndCoh exists due to a somewhat natural construction, but the distinction with QCoh is somewhat subtle. What is its role in algebraic geometry?

We present several answers below.

- IndCoh is the natural setting to develop Grothendieck's functorial approach to Serre duality and upper-! functors. For instance, it is necessary to work in this setting for the upper-! functor to commute with direct sums. We refer to [Gai5] and [GR4] for a detailed development of this theory.
- IndCoh appears in Koszul duality problems, cf. [BGS], [Pos2], [Lur4] §13-14, [GR4].
- IndCoh appears in some problems in geometric representation theory. See e.g. [AG1], [Bez], [BF1], [BZN]. See also the discussion of §1.3.11 below.

In short, for questions for which QCoh is close-but-wrong, IndCoh often provides the answer.

---

[4]We use cohomological gradings and indexing conventions throughout this work.

1.3.6. There is another setting in which IndCoh behaves better than QCoh: when we consider indschemes rather than schemes.

For abelian categories, this is implicit already in [BD1] 7.11.4. For derived categories, this is developed in detail in [GR3] §7 and [GR4]. In short, for an indscheme S locally of finite type, IndCoh(S) has a nice t-structure with a nice corresponding abelian category, where there is not generally one on QCoh(S). And QCoh(S) is not generally compactly generated, while IndCoh(S) clearly is. So IndCoh(S) can be studied using classical, more finitary methods, while QCoh(S) generally is more pathological.

Remark 1.3.5. In the countable ind-affine case $S = \operatorname{colim}_{i \geqslant 0} \operatorname{Spec}(A_i)$, we can think of S via the topological algebra $A = \lim_i A_i$ (with the pro-topology). In this case, the abelian category IndCoh$(S)^\heartsuit$ is the category of *discrete* A-modules, i.e., those A-modules M for which every $v \in M$ has open annihilator in A.

1.3.7. *Non-Noetherian settings.* In the above situation, we have assumed finite type hypotheses. For instance, our indschemes above were assumed to be ind-finite type. However, for semi-infinite mathematics, this is too restrictive.

In §6, we introduce a class of DG indschemes S (without finiteness hypotheses) that we call *reasonable* indschemes. For instance, this class includes any quasi-compact quasi-separated DG scheme that is *eventually coconnective*,[5] or any indscheme of ind-finite type. For any reasonable indscheme S, we associate a corresponding DG category IndCoh*(S) with many similar properties to the finite type situation.

Remark 1.3.6. The class of reasonable indschemes, which is defined by analogy with a similar notion in [BD1], may be considered as an answer to the (implicit) call of §1.1.7, to provide for a general category of "semi-infinite spaces." Indeed, it is a class containing infinite type schemes (formally: that are eventually coconnective and quasi-compact quasi-separated), and indschemes of ind-finite type. Moreover, loop spaces into smooth affine targets are reasonable, cf. Example 6.8.4.

1.3.8. The construction IndCoh* is covariantly functorial: for a map $f : S \to T$, we have an induced functor IndCoh*$(S) \to$ IndCoh*(T). Following [Gai5], we denote this functor f_*^{IndCoh}.

Remark 1.3.7. Unlike in finite type, there is not generally a pullback functor for such a map. This is the reason for the notation: it is the version of IndCoh with $*$-pushforwards. As in [Gai4], there is a formally dual DG category IndCoh$^!(S) := \mathsf{IndCoh}^*(S)^\vee$ (in the notation of *loc. cit.*) with (contravariant) upper-! functors instead. In (ind-)finite type, there is a canonical equivalence IndCoh$^!(S) \simeq$ IndCoh*(S) given by Serre duality, cf. [GR4] §II.2.

We remark that our notation here is directly parallel to that of [Ras3].

Remark 1.3.8. The above discussion reflects a general principle in semi-infinite algebraic geometry:

Duality for DG categories is a convenient organizational tool in finite dimensional situations, but is largely inessential. That is, it provides an interpretation of many standard constructions, and it sometimes provides helpful structure to arguments.

But in semi-infinite situations, working with DG category duality becomes more essential. Moreover, many of the subtle aspects of the subject have to do with non-trivial duality statements. For example, as discussed below, we interpret semi-infinite cohomology for Lie algebras as a duality statement. Similarly, one can interpret CDOs for Y as coming from suitable equivalences IndCoh*$(Y(K)) \simeq$ IndCoh$^!(Y(K))$, i.e., self-duality for IndCoh*.

---

[5]This is a hypothesis particular to derived algebraic geometry: it means that the structure sheaf is bounded. Note that this condition is satisfied for any classical scheme.

1.3.9.   We also introduce some equivariant versions of IndCoh^*, i.e., we IndCoh^* on suitable stacks. The theory is somewhat more subtle in this regime, and we refer to §6 and §7 for further discussion. Ignoring some technical points, our theory in particular covers "most" quotients of reasonable indschemes by groups such as $G(K)$ for G reductive, or by $G(O)$ for G arbitrary.

1.3.10.   *Applications.* Below, we discuss some (anticipated) applications of this theory.

1.3.11. *3d mirror symmetry.* In the last five years, there have been significant advances in the mathematical understanding of $3d$ mirror symmetry conjectures.

   We refer to [BF2] §7 and the introduction to [HR] for detailed discussion of this area, and defer attributions to Remark 1.3.10. But briefly, and somewhat heuristically, certain fundamental conjectures in this area take the form:

$$\mathsf{IndCoh}^*(\mathcal{M}aps(\overset{\circ}{\mathcal{D}}_{dR}, Y_1)) \simeq D^*(Y_2(K)). \tag{1.3.1}$$

Some remarks on the notation are in order.

   - Here Y_1 and Y_2 are certain algebraic stacks of finite type; typically, they are quotients of smooth affine varieties by the action of a reductive group.
   - The relationship between Y_1 and Y_2 is not arbitrary; they should be *3d mirror dual pairs*. We refer to [BF2] §4 for some examples.
   - In (1.3.1), $\mathcal{M}aps(\overset{\circ}{\mathcal{D}}_{dR}, Y_1)$ is the moduli stack of *flat* maps from $\overset{\circ}{\mathcal{D}}$ to Y_1. For instance, if $Y_1 = \mathbb{B}G$, then this is the space of de Rham G-local systems on the punctured disc. This space is an alternative to the derived loop space of a stack, and it has similar properties (and the two coincide if Y_1 is an affine scheme).
   - In (1.3.1), D^* indicates a suitable category of D-modules, as defined in infinite type in [Ras3].
   - For physics purposes, the right (resp. left) hand side of (1.3.1) is the category of line operators in the A-twist (resp. B-twist) of the $3d$ $\mathcal{N} = 4$ quantum field theory defined by Y_i (namely, the sigma model of maps into its cotangent stack). Physics predicts that the A-twist of the theory defined by Y_2 is equivalent (as a QFT) to the B-twist of the theory defined by Y_1 for a mirror dual pair (Y_1, Y_2).

   The left hand side of (1.3.1) is not a priori defined, so this conjecture is not precisely formulated above (as acknowledged in [BF2]).

Example 1.3.9. In [HR], which is joint with Justin Hilburn, we consider the case $Y_1 = \mathbb{A}^1/\mathbb{G}_m$ and $Y_2 = \mathbb{A}^1$. We show that $\mathcal{M}aps(\overset{\circ}{\mathcal{D}}_{dR}, \mathbb{A}^1/\mathbb{G}_m)$ is the quotient of a reasonable indscheme by an action of $\mathbb{G}_m(K)$, so the present text makes sense of IndCoh^* on this mapping space. We then prove the equivalence (1.3.1) in this case, using the definitions provided in the present work for the left hand side.

   We expect the theory of IndCoh^* in §6 leads more generally to accurate and precise conjectures.

Remark 1.3.10. We now address some of the lineage of 3d mirror symmetry. In physics, the general idea that certain supersymmetric $3d$ theories might be non-trivially equivalent first appeared in [IS], and was further developed by [HW]. In unpublished work, Hilburn-Yoo gave the algebro-geometric description of the categories of line operators in A and B-twists of $3d$ $\mathcal{N} = 4$ sigma models of the type considered above, leading to conjectures of the above types. In addition, Costello, Dimofte, and Gaitto (at least) played important roles in these developments. Connections between $3d$ mirror symmetry with geometric Langlands began in physics with work [GW] of Gaitto-Witten, and was further developed by Hilburn-Yoo, Braverman-Finkelberg-Nakajima [BFN], Braverman-Finkelberg

[BF2], and Ben-Zvi-Sakellaridis-Venkatesh. See also [BDGH] and [DGGH] for recent discussion in the mathematical physics literature.

1.3.12. *Factorizable Satake.* In [CR], joint with Justin Campbell, we prove a factorizable (cf. [Ras5]) version of the derived geometric Satake theorem of [BF1].

This result was anticipated over a decade ago by Gaitsgory-Lurie, and is discussed in [Gai6] §4.7. As in *loc. cit.*, this result plays a key role in Gaitsgory's approach to global geometric Langlands conjectures.

One reason such a result was not proved earlier is that, unlike the non-factorizable version, the theorem involves IndCoh on stacks of infinite type, so a definition of one side was not readily available.

In [CR], we again see that the theory of IndCoh* provided here yields the "right" answer for (derived, factorizable) geometric Satake.

1.3.13. *Cautis-Williams.* In [CW], Cautis and Harold propose a definition for the category of half-BPS line operators in $4d$ $\mathcal{N} = 2$ gauge theories via coherent sheaves on spaces $\mathcal{R}_{V,G} :=$ $V(O)/G(O) \times_{V(K)/G(K)} V(O)/G(O)$ considered in [BFN]. Here G is a reductive group and V is a finite-dimensional G-representation.

A BFN space $\mathcal{R}_{V,G}$ is a quotient of a reasonable DG indscheme by an action of $G(O)$, although they are of infinite type and highly DG (i.e., non-classical). As such, Cautis-Williams use our theory of IndCoh* to study coherent sheaves on these spaces.

1.3.14. *Weak loop group actions.* The application of IndCoh* within the present work is to develop a theory of weak loop group actions on categories and to provide a categorical framework for semi-infinite cohomology. We discuss these applications at length in §1.4 below.

1.4. **Loop group actions on categories.** In §7, we develop a theory of *weak loop group actions on DG categories*, which constitutes a major part of the present work.

Below, we recall the finite dimensional theory (due to Gaitsgory), motivate and describe our semi-infinite theory, connect to more classical ideas in infinite dimensional algebra, and give applications.

In brief, group actions on categories provide a unifying framework for many constructions in geometric representation theory, and the theory for loop groups plays a foundational role in geometric Langlands.

1.4.1. *Preliminary remarks.* We begin with some attributions, references, and historical comments. (Some of the discussion may only make sense after reading subsequent of §1.4.)

The idea began, apparently, with [BD1] §7. In *loc. cit.*, Beilinson and Drinfeld developed some aspects of the theory of weak group actions on categories. As we discuss in §1.4.25, they used their constructions in the case of loop groups to construct Hecke eigensheaves via localization. That is to say, the initial development of the theory were in the setting of loop groups and geometric Langlands.

The ideas of Beilinson-Drinfeld were developed by Gaitsgory in a series of works, sometimes with co-authors, and sometimes only in informally distributed works. In [Gai2], he introduced abelian categories over stacks; specializing to $\mathbb{B}G$, one obtains weak group actions on categories. The Beilinson-Drinfeld ideas on Hecke patters were generalized in the appendices to [FG1], which developed a theory of algebraic groups acting on abelian categories, and some parts of the theory of loop groups acting on abelian categories. In the latter setting, it is inadequate to use bounded derived categories, leading to some deficiencies in the generality of the results in *loc. cit.* Finally, in finite dimensions, a robust derived theory was developed in [Gai8].

The theory of strong loop group actions on DG categories was developed by Beraldo in [Ber]. This work also develops some consequences of the results in [Gai8], and therefore is a convenient reference for the finite dimensional theory.

However, a well-developed theory of weak actions has not appeared before the present work. Even if one is only interested in strong actions, the weak theory is needed to connect with Kac-Moody representations; this is crucial for applications of Beilinson-Drinfeld type.

Finally, we refer to [ABC$^+$] for further discussion.

1.4.2. *Algebraic group actions.* Let G be an affine algebraic group (in particular, finite type). There are two flavors of G-actions on categories: *weak* and *strong*. We discuss each below.

1.4.3. Let DGCat_{cont} denote the symmetric monoidal (∞-)category of cocomplete DG categories; see §1.8 and §2.2 for more details.

Let $\mathsf{QCoh}(G) \in \mathsf{DGCat}_{cont}$ denote the DG category of quasi-coherent sheaves on G. The group structures on G induces a *convolution* monoidal structure on $\mathsf{QCoh}(G)$.

Definition 1.4.1. A *weak* G-action on $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is a $\mathsf{QCoh}(G)$-module structure. We let $G\text{–}\mathsf{mod}_{weak}$ denote the category of (DG) categories with strong G-actions, i.e., $\mathsf{QCoh}(G)\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont})$.

Remark 1.4.2. For a k-point $g : \mathrm{Spec}(k) \to G$, we can form a skyscraper sheaf $g_*(k) \in \mathsf{QCoh}(G)^{\heartsuit}$. For $\mathcal{C} \in G\text{–}\mathsf{mod}_{weak}$, acting by $g_*(k)$ defines an automorphism $g \cdot - : \mathcal{C} \to \mathcal{C}$. That is, we get a map $G(k) := \mathrm{Hom}(\mathrm{Spec}(k), G) \to \mathrm{Aut}(\mathcal{C})$. The above definition of a weak G-action can heuristically be understood as refining such a map $G(k) \to \mathrm{Aut}(\mathcal{C})$ to allow k-points to vary "continuously" in the natural sense of algebraic geometry.

For $\mathcal{C} \in G\text{–}\mathsf{mod}_{weak}$, we define the corresponding *weak invariants* and *weak coinvariants* categories as:

$$\mathcal{C}^{G,w} := \mathsf{Hom}_{G\text{–}\mathsf{mod}_{weak}}(\mathsf{Vect}, \mathcal{C}) \qquad\qquad \mathcal{C}_{G,w} := \mathsf{Vect} \underset{\mathsf{QCoh}(G)}{\otimes} \mathcal{C}.$$

As recalled in Theorem 5.10.1, a theorem of Gaitsgory constructs functorial equivalences $\mathcal{C}_{G,w} \simeq \mathcal{C}^{G,w}$.

Remark 1.4.3. Above, Vect is considered as a categorification of the trivial 1-dimensional representation of a group. The identity "invariants = coinvariants" is a categorification of Maschke's theorem, considering G as analogous to a finite group.

1.4.4. Let $D(G) \in \mathsf{DGCat}_{cont}$ denote the category of D-modules on G, which again carries a *convolution* monoidal structure.

Definition 1.4.4. A *strong* G-action on $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is a $D(G)$-module structure. We let $G\text{–}\mathsf{mod}$ denote the category of (DG) categories with strong G-actions, i.e., $D(G)\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont})$.

As in the notation $G\text{–}\mathsf{mod}$ above, we often omit the term *strong*, and refer simply to *actions* of G on DG categories.

We again have invariants and coinvariants categories $\mathcal{C}^G$ and $\mathcal{C}_G$ defined as in the weak setting. As in [Ber], there is again a canonical equivalence $\mathcal{C}_G \simeq \mathcal{C}^G$.

1.4.5. We now discuss examples.

If G acts on some scheme X, then G (canonically) acts weakly on $\mathsf{QCoh}(X)$. If X is locally of finite type, then G acts weakly on $\mathsf{IndCoh}(X)$ and strongly on $D(X)$ (the DG category of D-modules on X).

The weak invariant categories:

$$\mathsf{QCoh}(X)^{G,w}, \quad \mathsf{IndCoh}(X)^{G,w}$$

are tautologically DG categories of equivariant sheaves and coincide with suitable sheaves on the stack X/G. The category:

$$D(X)^G$$

of (strong) invariants is the category of D-modules on the stack X/G, and coincides with the classical equivariant derived category.

1.4.6. We can also mix the two settings.

First, the functor of (right) D-module induction ind : $\mathsf{QCoh}(G) \to D(G)$ is monoidal. This provides a forgetful functor:

$$\mathrm{Oblv} = \mathrm{Oblv}^{str \to w} : G\text{–}\mathsf{mod} \to G\text{–}\mathsf{mod}_{weak}. \tag{1.4.1}$$

In other words, categories with strong G-actions have underlying weak G-actions.

1.4.7. Now consider $D(G)$ with its natural strong $G \times G$-action. Taking weak invariants on the right, we obtain a strong G-action on:

$$D(G)^{G,w} \simeq \mathfrak{g}\text{–}\mathsf{mod}.$$

Clearly $\mathfrak{g}\text{–}\mathsf{mod}^G = \mathsf{Vect}^{G,w} = \mathsf{Rep}(G)$. More generally, given $H \subseteq G$, $\mathfrak{g}\text{–}\mathsf{mod}^H$ is the (DG) category of *Harish-Chandra modules* for the pair $(\mathfrak{g}, H)$; at the abelian categorical level, these are representations of $\mathfrak{g}$ and a lift[6] of the action of $\mathfrak{h} \subseteq \mathfrak{g}$ to an action of H.

1.4.8. It is often convenient to work with *Hecke actions*.

Given a weak (resp. strong) action of G on $\mathcal{C}$, there is an induced action of the monoidal category $\mathsf{QCoh}(H\backslash G/H)$ on $\mathcal{C}^{H,w}$ (resp. $D(H\backslash G/H)$ on $\mathcal{C}^H$). Indeed, $\mathcal{C}^{H,w} = \mathsf{Hom}_{G\text{–}\mathsf{mod}_{weak}}(\mathsf{QCoh}(G/H), \mathcal{C})$ and $\mathsf{QCoh}(H\backslash G/H) = \mathsf{End}_{G\text{–}\mathsf{mod}_{weak}}(\mathsf{QCoh}(G/H))$.

Example 1.4.5. Let G be semisimple and let $B \subseteq G$ be a Borel subgroup. We obtain an action of $D(B\backslash G/B)$ on $\mathfrak{g}\text{–}\mathsf{mod}^B$. Let $j_w : B\backslash BwB/B \hookrightarrow B\backslash G/B$ denote the locally closed embedding of a Bruhat cell. The actions of $j_{w,*,dR}(\mathrm{IC}), j_{w,!}(\mathrm{IC}) \in D(B\backslash G/B)$ on $\mathfrak{g}\text{–}\mathsf{mod}^B$ are Arkhipov twisting functors from [Ark2] (essentially by definition). Because these objects are well-known to be invertible in the monoidal category $D(B\backslash G/B)$, they define auto-equivalences $j_{w,*,dR}\star -, j_{w,!}\star - : \mathfrak{g}\text{–}\mathsf{mod}^B \xrightarrow{\simeq} \mathfrak{g}\text{–}\mathsf{mod}^B$.

These automorphisms play a fundamental role in some approaches to studying the BGG category $\mathcal{O}$, cf. [Hum]. In other words, the Hecke action on $\mathcal{O} := \mathfrak{g}\text{–}\mathsf{mod}^B$ (or a variant with generalized central character) is a crucial structure that is non-obvious from classical perspectives and transparent from the perspective of group actions on categories.

1.4.9. *The setting of loop groups.* Now suppose we replace G by $G(K)$.

In this case, a monoidal category[7] $D^*(G(K))$ of D-modules on $G(K)$ was defined in [Ber] (see also [Ras3]).

Therefore, we may define $G(K)\text{–}\mathsf{mod}$ as $D^*(G(K))\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont})$, just as in the finite dimensional setting.

---

[6]If H is connected, such a lift is unique if it exists.

[7]Here the notation D^* appears instead of D because of the semi-infinite nature of $G(K)$. The notation is exactly in parallel with that of §1.3.4.

Remark 1.4.6. The definition of invariants and coinvariants categories make sense in this setting. However, for group indschemes such as $G(K)$, there is not a natural equivalence between invariants and coinvariants, although there is for $G(O)$. Following the heuristic of Remark 1.4.3, this is because G behaves like a finite group, $G(O)$ behaves like a profinite group, and $G(K)$ behaves like a non-compact topological group.

However, in the special case when G is reductive, one may construct a (somewhat non-canonical) equivalence $\mathcal{C}^{G(K)} \simeq \mathcal{C}_{G(K)}$, bootstrapping from $G(O)$ using the ind-properness of the affine Grassmannian $\mathrm{Gr}_G = G(K)/G(O)$.

For a special (*Whittaker*) setting in which equivalences of this form hold for unipotent G, see (in increasing orders of generality) [Gai3], [Ber], and [Ras6].

1.4.10.   The setting of weak actions of loop groups on DG categories has proved more elusive, and is developed here in §7. We provide a brief overview of the theory here. For simplicity, we assume[8] G is reductive in the discussion that follows.

1.4.11.   We define a category $G(K)$–mod_{weak} in §7.

As in finite dimensions, there are fundamental functors:

$$\mathrm{Oblv}, \ (-)_{G(K),w}, \ (-)^{G(K),w} : G(K)\text{–}\mathsf{mod}_{weak} \to \mathsf{DGCat}_{cont}$$

that are the forgetful functor, (weak) coinvariants, and (weak) invariants functors.

But a major technical subtlety in this setting is that the functor:

$$\mathrm{Oblv} : G(K)\text{–}\mathsf{mod}_{weak} \to \mathsf{DGCat}_{cont}$$

is not conservative. This means that there are morphisms $\mathcal{C}_1 \to \mathcal{C}_2$ in $G(K)$–mod_{weak} that are not equivalences, but are at the level of their underlying DG categories. (In this regard, $G(K)$–mod_{weak} behaves somewhat like IndCoh on a singular affine scheme, but with one additional categorical level of complexity.)

Therefore, we cannot simply define $G(K)$–mod_{weak} as $\mathsf{IndCoh}^*(G(K))$–$\mathsf{mod}(\mathsf{DGCat}_{cont})$ or something along these lines. This presents technical challenges in the development of the theory and its functoriality, and even in the construction of natural objects here.

Remark 1.4.7. In §7, we refer to objects of $G(K)$–mod_{weak} as categories with *genuine* (weak) actions. Relatedly, we actually denote the above functor Oblv by Oblv_{gen}. The terminology is borrowed from stable homotopy theory; in some cosmetic regards, the non-conservativeness of Oblv_{gen} makes $G(K)$–mod_{weak} appear somewhat analogous to category of genuine G-spectra for a finite group G.

1.4.12.   With the above in mind, for the purposes of the introduction, we largely ignore the non-conservativeness of the forgetful functor, and essentially assume the constructions one naively expects to have (e.g., a symmetric monoidal structure on $G(K)$–mod_{weak} with unit the "trivial" representation $\mathsf{Vect} \in G(K)$–mod_{weak}) exist.

Of course, we refer to §7 for a careful development.

---

[8]More generally, our theory works for any *Tate group indscheme*, cf. §7.2. The loop group $G(K)$ satisfies this hypothesis only when G is reductive.

But for more general affine algebraic G, one can also allow the formal completion of $G(K)$ along $G(O)$, or any other congruence subgroup. This allows one to apply our methods to recover semi-infinite cohomology for Tate Lie algebras in full generality, cf. §1.4.19.

1.4.13.   What becomes of invariants and coinvariants for weak $G(K)$-actions?

Here one discovers the peculiar phenomenon characteristic of infinite dimensions: what we refer to as the *modular character* χ_{Tate} of $G(K)$.

By definition, this is a certain canonical object $\chi_{Tate} \in G(K)\text{–mod}_{weak}$. Non-canonically, $\mathrm{Oblv}(\chi_{Tate}) \in \mathsf{DGCat}_{cont}$ is isomorphic to Vect, so we can think of χ_{Tate} as a categorification of an action of $G(K)$ on a line. (As the name suggests, we regard this object as a categorification of the modular character of a Lie group, defined as usual via its Haar measure.)

The key property of χ_{Tate} is the existence of functorial equivalences:

$$(\mathcal{C} \otimes \chi_{Tate})_{G(K),w} \simeq \mathcal{C}^{G(K),w} \tag{1.4.2}$$

for $\mathcal{C} \in G(K)\text{–mod}_{weak}$. In words: *(weak) invariants and coinvariants for $G(K)$ coincide up to a twist by the modular character.*

Remark 1.4.8. If we replace $G(K)$ by an affine algebraic group G (or an affine group scheme such as $G(O)$), the modular character is canonically trivial: i.e., one has a canonical equivalence with Vect equipped with its trivial action.

In §1.4.19, we discuss how this modular character recovers classical constructions in semi-infinite algebra.

1.4.14.   Next, let us discuss the relationship with strong $G(K)$-actions and affine Lie algebras.

In §8, we construct a forgetful functor:

$$G(K)\text{–mod} \to G(K)\text{–mod}_{weak}$$

refining the natural forgetful functor $G(K)\text{–mod} \to \mathsf{DGCat}_{cont}$. (This is only non-trivial because of the subtleties in the definition of $G(K)\text{–mod}_{weak}$.)

1.4.15.   By definition, there is a canonical object $D^*(G(K)) \in G(K)\text{–mod}$. Let $D^!(G(K))$ be the dual DG category, cf. [Ras3] or Remark 1.3.7. This object is canonically a $D^*(G(K))$-module, so an object of $G(K)\text{–mod}$ as well.

Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}((t)) := \mathfrak{g} \otimes k((t))$.

In Lemma 9.13.1, we construct a canonical equivalence:

$$\mathfrak{g}((t))\text{–mod} \simeq D^!(G(K))^{G(K),w}. \tag{1.4.3}$$

Here the definition of the (unbounded) DG category $\mathfrak{g}((t))\text{–mod}$ of (smooth) $\mathfrak{g}((t))$-modules, due to Frenkel-Gaitsgory [FG2], is somewhat subtle due to semi-infinite issues; we review it in §4. That aside, in the above equivalence, the trivial representation $k \in \mathfrak{g}((t))\text{–mod}$ corresponds to the !-pullback of k along the structure map $G(K) \to \mathrm{Spec}(k)$.

1.4.16. *Applications.* We now relate the theory outlined above to other literature and provide some applications.

1.4.17. *Semi-infinite cohomology revisited.* First, we indicate how the theory of weak $G(K)$-actions provides a new perspective on classical semi-infinite cohomology, as introduced in [Fei]. This material is the subject of §9.

Remark 1.4.9. There have been various previous attempts to provide conceptual constructions of semi-infinite cohomology: see [Vor], [Ark1], and [Pos1] for example. Our perspective emphasizes the connection to the higher categorical representation theory of the loop group.

1.4.18.   First, we construct an equivalence:

$$\widehat{\mathfrak{g}}_{-Tate}\text{--mod} \simeq D^*(G(K))_{G(K),w} \tag{1.4.4}$$

similar to (1.4.3), where the notation deserves more explanation. Here $\widehat{\mathfrak{g}}_{-Tate}$ indicates the *canonical* Kac-Moody central extension of $\mathfrak{g}((t))$ (or its opposite); see e.g. [BD2] §2.7, §3.8, and [BD1] §7.13 for convenient reviews (including the relation to semi-infinite cohomology, cf. §1.4.19). We remind that this Lie algebra is the Kac-Moody extension defined by minus the Killing form for $\mathfrak{g}$. By $\widehat{\mathfrak{g}}_{-Tate}$--mod, we mean the Frenkel-Gaitsgory DG category of smooth modules on which the central element acts by the identity.

The above equivalence is actually non-canonical: it depends on a choice of a *compact open*[9] subgroup of $G(K)$; in the discussion that follows, we suppose we have chosen $G(O)$ for this purpose.

Here is the idea of the construction. First, the choice of compact open subgroup induces an equivalence $D^!(G(K)) \simeq D^*(G(K)) \in G(K)$--mod. We then have:

$$D^*(G(K))_{G(K),w} \simeq D^!(G(K))_{G(K),w} \overset{(1.4.2)}{\simeq} \left(D^!(G(K)) \otimes \chi_{Tate}^{\otimes -1}\right)^{G(K),w}.$$

This is similar to (1.4.3) except for the twist by $\chi_{Tate}^{\otimes -1}$. We explain in §9 that this twist amounts to considering modules over the canonical central extension of $\mathfrak{g}((t))$.

1.4.19.   Next, as $D^*(G(K))$ is dual to $D^!(G(K))$ and invariants are dual to coinvariants, it follows formally from (1.4.3) and (1.4.4) that $\widehat{\mathfrak{g}}_{-Tate}$--mod is dual as a DG category to $\mathfrak{g}((t))$--mod.

In particular, $\mathfrak{g}((t))$--mod identifies with the category of (continuous DG) functors $\widehat{\mathfrak{g}}_{-Tate}$--mod $\to$ Vect. Therefore, the trivial representation $k \in \mathfrak{g}((t))$--mod defines a functor:

$$\widehat{\mathfrak{g}}_{-Tate}\text{--mod} \to \text{Vect.} \tag{1.4.5}$$

There is a well-known such functor that plays a key role in the study of affine Lie algebras: semi-infinite cohomology. We recall this is a functor:

$$C^{\frac{\infty}{2}}\left(\mathfrak{g}((t)), \mathfrak{g}[[t]]; -\right) : \widehat{\mathfrak{g}}_{-Tate}\text{--mod} \to \text{Vect}$$

where, as the notation indicates, there is also (mild) dependence on the choice of compact open subgroup $\mathfrak{g}[[t]]$.

We show that under our duality, the functor (1.4.5) coincides with classical semi-infinite cohomology. This is done in §9. It follows more generally that the pairing:

$$\mathfrak{g}((t))\text{--mod} \otimes \widehat{\mathfrak{g}}_{-Tate}\text{--mod} \to \text{Vect}$$

underlying the duality $\mathfrak{g}((t))$--mod $= (\widehat{\mathfrak{g}}_{-Tate}$--mod$)^\vee$ is calculated by tensoring two modules and forming semi-infinite cohomology of the resulting $\widehat{\mathfrak{g}}_{-Tate}$-module.

Remark 1.4.10. The existence of a duality with this property was anticipated in [AG3] §2.2. Our treatment differs from *loc. cit.*: we construct a duality above on abstract grounds and then show it recovers classical semi-infinite cohomology, while Arkhipov-Gaitsgory use the latter to establish duality.

In addition, some details are missing in *arkhipov-gaitsgory-localization*. Moreover, our approach makes it essentially tautological that the duality is strongly $G(K)$-equivariant, which is of fundamental importance (see below) and difficult to see from the perspective of [AG3].

---

[9]See §7.2 for the definition.

Remark 1.4.11. There is a generalization of the above for general affine Kac-Moody algebras:

$$(\widehat{\mathfrak{g}}_\kappa\text{--mod})^\vee \simeq \widehat{\mathfrak{g}}_{-\kappa-Tate}\text{--mod}.$$

The ideas are the same, so we have not emphasized it in our present discussion. See §11 for more details.

1.4.20. *Construction of a $G(K)$-action on $\mathfrak{g}((t))$--mod.* The equivalence (1.4.3) induces a strong action of $G(K)$ on $\mathfrak{g}((t))$--mod. (There is a similar generalization allowing Kac-Moody twists, cf. §11.) As we will discuss, this construction plays a key role in some studies of Kac-Moody representations.

Note that this assertion do not mention weak $G(K)$-actions: $\mathfrak{g}((t))$--mod can be defined as a DG category directly, and we are discussing strong $G(K)$-actions. And indeed, the existence of a $G(K)$-action on $\mathfrak{g}((t))$--mod was previously outlined in [Gai7].

However, as we discuss below, our perspective here is more robust and fills a number of gaps in the literature.

1.4.21. The action defined by (1.4.3) implies a universal property for $\mathfrak{g}((t))$--mod as a strong $G(K)$-category: given $\mathcal{C} \in G(K)$--mod, a $G(K)$-equivariant functor $\mathcal{C} \to \mathfrak{g}((t))$--mod is equivalent to a weakly $G(K)$-equivariant functor $\mathcal{C} \to \mathsf{Vect}$, i.e., a functor $\mathcal{C}_{G(K),w} \to \mathsf{Vect}$.

In other words, although we are discussing a strong action, when thinking about $\mathfrak{g}((t))$--mod as a $G(K)$-module, it is helpful to know about $G(K)$--mod$_{weak}$.

1.4.22. In addition, our perspective on the $G(K)$-action here make various functoriality results that have tacitly been assumed in the literature.

For instance, [AG3] Theorem-Construction 4.2.2 is unproved in *loc. cit.* It asserts the existence of a (strongly) $G(K)$-equivariant (coherent) global sections functor:

$$\Gamma(\mathrm{Bun}_G^{level,x}, -) : D^*(\mathrm{Bun}_G^{level,x}(X)) \to \mathfrak{g}((t))\text{--mod} \tag{1.4.6}$$

where X is a smooth projective curve, $x \in X(k)$ is a k-point with formal completion identified[10] with $\mathrm{Spf}(k[[t]])$, and $\mathrm{Bun}_G^{level,x}$ is the (infinite type) scheme parametrizing G-bundle on X with a trivialization on the formal neighborhood of x.

Remark 1.4.12. Although the above functor is constructed in [AG3], the construction of its $G(K)$-equivariance is not shown, although it plays a key role in *loc. cit.* And it appears difficult to establish using the techniques of *loc. cit.*

From our perspective, the $G(K)$-equivariance is readily established. Because this plays a key role in other literature, we briefly outline the construction below.

Remark 1.4.13. As we discuss in §1.4.25, the above functor is closely related to the Beilinson-Drinfeld *localization* functor, which plays a key role in the de Rham geometric Langlands program and for which there are also gaps in the literature.

1.4.23. As just stated, we digress to outline the construction of a $G(K)$-equivariant functor (1.4.6). We find this example illustrates some important ideas in this text, but this material may safely be skipped by the reader. In particular, it relies on some basic familiarity with G-bundles on curves that is not assumed elsewhere. Moreover, it assumes some working knowledge of D-modules and IndCoh in infinite type, as developed in [Ras3] and §6 of this text.

Let Bun_G denote the finite type Artin stack of G-bundles on X. Recall that Bun_G is not quasi-compact: rather, we can write it as a union $\mathrm{Bun}_G = \mathrm{colim}_i U_i$ for $U_i \subseteq \mathrm{Bun}_G$ open quasi-compact

---

[10]To say it better: our ambient formal disc is assumed to be the one based around x.

substacks. Recall that $\mathrm{Bun}_G^{level,x} \to \mathrm{Bun}_G$ is a $G(O)$-torsor. We let $\widetilde{U}_i \subseteq \mathrm{Bun}_G^{level,x}$ denote the inverse image of U_i, which is an open quasi-compact subscheme of $\mathrm{Bun}_G^{level,x}$.

Because each $\widetilde{U}_i$ is a $G(O)$-torsor over a smooth stack U_i, it follows that $D^*(\widetilde{U}_i) \simeq D^!(\widetilde{U}_i)$ and $\mathsf{IndCoh}^*(\widetilde{U}_i) \simeq \mathsf{IndCoh}^!(\widetilde{U}_i)$ canonically; indeed, for the former equivalence, see [Ras3] Proposition 4.8.1, and for the latter, one notes that the natural functor $\Psi : \mathsf{IndCoh}^*(\widetilde{U}_i) \to \mathsf{QCoh}(\widetilde{U}_i)$ (cf. §6) is an equivalence for pro-smooth schemes, so $\mathsf{IndCoh}^*(\widetilde{U}_i)$ is canonically self-dual as a DG category, as this is always true for QCoh on a quasi-compact scheme. For convenience, we denote these categories simply by $D(\widetilde{U}_i)$ and $\mathsf{IndCoh}(\widetilde{U}_i)$ in what follows.

We then have:

$$D^*(\mathrm{Bun}_G^{level,x}) = \operatorname*{colim}_i D(\widetilde{U}_i)$$
$$\mathsf{IndCoh}^*(\mathrm{Bun}_G^{level,x}) = \operatorname*{colim}_i D(\widetilde{U}_i) \tag{1.4.7}$$

where each of these colimits is formed in DGCat_{cont} and the structure maps are pushforwards. Therefore, there is a canonical[11] forgetful functor:

$$\mathrm{Oblv} : D^*(\mathrm{Bun}_G^{level,x}) \to \mathsf{IndCoh}^*(\mathrm{Bun}_G^{level,x}).$$

Because $G(K)$ acts on $\mathrm{Bun}_G^{level,x}$, it acts strongly on $D^*(\mathrm{Bun}_G^{level,x})$ and weakly on $\mathsf{IndCoh}^*(\mathrm{Bun}_G^{level,x})$. One can readily[12] verify that there is a canonical weak $G(K)$-equivariant structure on Oblv.

---

[11]This forgetful functor is a little funny: naturally, one has a forgetful functor $\mathrm{Oblv} : D^!(S) \to \mathsf{IndCoh}^!(S)$ by dualizing the discussion in §6.20. This gives forgetful functors $D(\widetilde{U}_i) = D^!(\widetilde{U}_i) \to \mathsf{IndCoh}^!(\widetilde{U}_i) = \mathsf{IndCoh}(\widetilde{U}_i)$. These functors are naturally compatible under the structure functors in the above colimit, so give a forgetful functor of the desired type.

In other words, we use the identifications $D^!(\widetilde{U}_i) = D^*(\widetilde{U}_i)$ (and similarly for IndCoh) to construct the forgetful functor above. (This may be compared to [BD1] §7.14.3.)

[12]Let us outline the argument, since it is a little technical.

First, one more naturally shows that the forgetful functor $\mathrm{Oblv} : D^!(\mathrm{Bun}_G^{level,x}) \to \mathsf{IndCoh}^!(\mathrm{Bun}_G^{level,x})$ is canonically weakly $G(K)$-equivariant; this follows from general functoriality considerations. The difference here is that, as in the previous discussion, this forgetful functor exists more generally and is more natural, so its functoriality is more easily established.

Then we claim that there is a unique *naive* (in the sense of §7) weak $G(K)$-equivariant structure on $\mathrm{Oblv} : D^*(\mathrm{Bun}_G^{level,x}) \to \mathsf{IndCoh}^*(\mathrm{Bun}_G^{level,x})$ fitting into a commutative diagram:

$$\begin{array}{ccc} D^*(\mathrm{Bun}_G^{level,x}) & \longrightarrow & \mathsf{IndCoh}^*(\mathrm{Bun}_G^{level,x}) \\ \downarrow & & \downarrow \\ D^!(\mathrm{Bun}_G^{level,x}) & \longrightarrow & \mathsf{IndCoh}^!(\mathrm{Bun}_G^{level,x}) \end{array}$$

where the vertical maps are the canonical arrows resulting from (1.4.7).

For this, one uses Step (2) from the proof of Lemma 8.16.1 (taking $\mathcal{A}_{ren} = \mathcal{A} = \mathsf{IndCoh}^*(G(K))$ in *loc. cit*). This construction produces a certain monoidal DG category $\mathcal{B}$ with a quotient functor $\mathcal{B} \to \mathsf{IndCoh}^*(G(K))$. It is easy to establish $\mathcal{B}$-linearity of the above functors: one reduces to the (non-cocomplete) monoidal subcategory $\mathcal{B}^c$, and then because compact objects in $\mathsf{IndCoh}^*(G(K))$ have finite support, the claim is essentially formal from the definitions. Because $\mathcal{B} \to \mathsf{IndCoh}^*(G(K))$ is a quotient functor, this suffices.

Finally, using (a simple version of) the constructions from §8.14, one sees that this structure canonically upgrades to a *genuine* weak $G(K)$-equivariant structure.

1.4.24. *Duality for Kac-Moody algebras.* In the discussion above, we have emphasized an *aesthetic* feature of our theory of weak $G(K)$-actions: it reinterprets semi-infinite cohomology as a duality between DG categories. However, this result also has *practical* significance, especially in its (strongly) $G(K)$-equivariant form.

- In [Dhi], Dhillon uses Kac-Moody duality to construct semi-infinite cohomology for $\mathcal{W}$-algebras using group actions on categories and [Ras6].
- In [AG3], Arkhipov-Gaitsgory use this duality to show Kashiwara-Tanisaki localization for thin flags (at negative levels) implies their localization theorem for thick flags (at positive levels).
- In [Ras7] Appendix A, we use duality to provide a conceptual interpretation of the main construction of [AG2].
- Kac-Moody duality appears in studies of the positive level cases of Gaitsgory's FLE; see [ABC$^+$], [CDR], and [Liu].
- In [Gai9], Gaitsgory uses Kac-Moody duality throughout his study of the Kac-Moody side of Kazhdan-Lusztig style equivalences. (Note that in *loc. cit.* §1.3, introducing this duality, Gaitsgory writes "The material in this subsection does not admit adequate references in the published literature.")

To varying degrees, the above references make implicit and explicit use of $G(K)$-equivariance property of Kac-Moody duality, which was not known before the present work.

1.4.25. *Beilinson-Drinfeld localization revisited.* There is a $G(K)$-equivariant localization functor:

$$\text{Loc} : \widehat{\mathfrak{g}}_{-Tate} \to D^!(\text{Bun}_G^{level,x})$$

obtained by duality from (1.4.6).

More generally, one can incorporate a level[13] κ as in §11 to obtain dual level κ strongly $G(K)$-equivariant functors:

$$\Gamma(\text{Bun}_G^{level,x}, -) : D_\kappa^*(\text{Bun}_G^{level,x}(X)) \to \widehat{\mathfrak{g}}_\kappa\text{--mod}$$
$$\text{Loc} : \widehat{\mathfrak{g}}_\kappa \to D_{\kappa+Tate}^!(\text{Bun}_G^{level,x}) \simeq D_\kappa^!(\text{Bun}_G^{level,x})$$

where the last isomorphism follows from integrality of the Tate level; we refer to §11 for the relevant notions.

In particular, passing to $G(O)$-invariants, one obtains a localization functor:

$$\text{Loc}^{G(O)} : \widehat{\mathfrak{g}}_\kappa\text{--mod}^{G(O)} \to D_\kappa^!(\text{Bun}_G)$$

that is a morphism of module categories for the spherical Hecke category of bi-$G(O)$-equivariant κ-twisted D-modules on $G(K)$.

A weaker version of this latter functor was considered in [BD1] §7.14. More precisely, it was constructed at the triangulated level; higher coherence data regarding the Hecke action was not considered in *loc. cit.*

Remark 1.4.14. In Beilinson-Drinfeld's construction, semi-infinite linear algebra plays a major role: cf. [BD1] 7.14.2-5. Our construction provides a conceptual explanation for this: the more natural object is the global sections functor, whose categorical dual is the localization functor.

---

[13]At some points in this introduction, we assume the reader has some familiarity with standard notions from Kac-Moody representations. Briefly, we remind that a *level* is a G-invariant symmetric bilinear form on $\mathfrak{g}$, and defines a central extension of $\mathfrak{g}((t))$ in a standard way.

Remark 1.4.15. To indicate the importance of the Hecke equivariance of Loc, recall its role in [BD1]. For a $\check{G}$-*oper* χ on the curve X, which we recall is a $\check{G}$-local system with extra structure, Beilinson-Drinfeld form a corresponding quotient $\mathbb{V}_{crit,\chi} \in \hat{\mathfrak{g}}_{crit}$–$\mathsf{mod}^{G(O)}$ of the vacuum representation via the Feigin-Frenkel isomorphism (here the level is critical, cf. §11). Moreover, they show that this quotient satisfies the local Hecke property (see *loc. cit.* Theorem 5.4.11, or [Ras1]). By applying the localization functor, they deduce a global version of this result and thereby construct a Hecke eigensheaf.

Remark 1.4.16. For another perspective on the Beilinson-Drinfeld localization functor (without emphasis on Hecke symmetry), see [Roz].

1.4.26. *Localization at critical level.* In [FG1], Frenkel-Gaitsgory initiated an ambitious program to study *critical* level Kac-Moody representations using ideas from local geometric Langlands. Many of their conjectures resemble Beilinson-Bernstein-style localization theorems.

In [Ras7], we recently proved one of their outstanding conjectures for GL_2 using the (critical level) $G(K)$-action on $\hat{\mathfrak{g}}_{crit}$–$\mathsf{mod}$ constructed here. We refer to *loc. cit.* for more details on this application.

Remark 1.4.17. This application was the genesis of this text: in writing [Ras7], we found that there were a number of gaps in the literature that needed to be addressed.

1.4.27. *Local geometric Langlands.* Finally, we highlight that weak loop group actions play a distinguished role in the conjectural local geometric Langlands program.

Fix a level κ as above. Let $G(K)$–mod_κ denote the category of DG categories with a strong $G(K)$-action with *level* κ, cf. §11. For instance, for $\kappa = 0$, this is the cateogry we previously denoted as $G(K)$–mod.

Quantum local geometric Langlands predicts that there is something like an equivalence:

$$G(K)\text{–}\mathsf{mod}_\kappa \approx \check{G}(K)\text{–}\mathsf{mod}_{\check{\kappa}}.$$

For instance, if κ is generic, there is expected to be an honest equivalence of this type. We ignore the difference between $\approx$ and $\simeq$ below.

The basic feature of this (almost) equivalence is that the diagram:

$$
\begin{array}{ccc}
G(K)\text{–}\mathsf{mod}_\kappa & \!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= & \check{G}(K)\text{–}\mathsf{mod}_{-\check{\kappa}} \\
 & \searrow\!\!\scriptstyle{(-)^{G(K),w}} \quad \scriptstyle{\mathsf{Whit}}\!\!\swarrow & \\
 & \mathsf{DGCat}_{cont} &
\end{array}
$$

should (canonically) commute. Here $\check{G}$ is the Langlands dual group to G and $\check{\kappa}$ is a (suitably normalized) dual level.

In the diagram above, the functor on the right is the *Whittaker* functor, well studied in local and global geometric Langlands; see in [Ras6] in the local context.

More saliently, the functor on the left is defined as the composition:

$$G(K)\text{–}\mathsf{mod}_\kappa \to G(K)\text{–}\mathsf{mod}_{weak} \xrightarrow{(-)^{G(K),w}} \mathsf{DGCat}_{cont}.$$

As the various terms here were not previously[14] well-studied, this functor was previously not well-understood, even at a formal level.

---

[14]For instance, the construction of the forgetful functor $G(K)$–$\mathsf{mod} \to G(K)$–mod_{weak} is the subject of §8, and is somewhat involved.

In this way, weak loop group actions play a defining role in local geometric Langlands.

Remark 1.4.18. For an introduction to the above ideas, we refer to [ABC$^+$].

1.5. **Topological algebras and Harish-Chandra data.** In §1.4, we introduced the theory of loop group actions on categories and argued that it provides useful perspectives on Kac-Moody representations.

However, the theory is a priori quite abstract. For instance, in concrete circumstances, it can be quite difficult to construct a (strong, say) $G(K)$-action on a category $\mathcal{C}$: this was implicit in §1.4.20.

Remark 1.5.1. If $\mathcal{C}$ is a category of modules for a vertex algebra with *Kac-Moody symmetry*, we should expect $\mathcal{C}$ to have a strong $G(K)$-action (with suitable level).

Remark 1.5.2. We have seen in §1.4.27 that the (level κ) strong $G(K)$-action on $\widehat{\mathfrak{g}}_\kappa$–mod plays an important role in local geometric Langlands.

However, the construction of this $G(K)$-action was quite abstract. What if we wish to prove concrete results, say of the sort predicted by local geometric Langlands, about this $G(K)$-action? Certainly one needs to use *classical, non-derived* structures to be able to study the $G(K)$-action on $\widehat{\mathfrak{g}}_\kappa$–mod.

Our theories of topological DG algebras and Harish-Chandra data address these issues. We describe salient parts of this theory below.

1.5.1. *Classical Harish-Chandra data.* What finite-dimensional theory are we trying to imitate for loop groups?

Classically, suppose G is an affine algebraic group and A is an associative algebra. We suppose A is classical, i.e., not DG.

If G acts on A, then G acts *weakly* on the category A–mod.

To upgrade this weak action to a strong action, it is equivalent to specify a *Harish-Chandra datum*, i.e., a map:

$$i : \mathfrak{g} \to A$$

that is a G-equivariant morphism of Lie algebras, and such that the induced adjoint action of $\mathfrak{g}$ on A is the infinitesimal action defined by the G-action on A.

For instance, if $A = U(\mathfrak{g})$ with the adjoint G-action, we can take i as the structural map $\mathfrak{g} \to U(\mathfrak{g})$ to obtain the strong G-action on $\mathfrak{g}$–mod. Or, if X is a smooth affine variety with a G-action, we can take i as the composition $\mathfrak{g} \to \Gamma(X, T_X) \to \Gamma(X, D_X)$ to obtain the strong G-action on $\Gamma(X, D_X)$–mod $= D(X)$.

We refer to [Neg] and [FG1] §20.4 for discussion in the classical (intrinsically 1-categorical) context. I am not aware of a suitable reference in the DG setting, but it is not difficult to directly deduce the above assertion from standard theory.

Remark 1.5.3. In summary: in the above setting, certain[15] strong G-actions categories $\mathcal{C} = A$–mod can be completely encoded in terms of classical abstract algebra.

---

[15]More precisely, those actions for which the forgetful functor A–mod $\to$ Vect is given a weakly G-equivariant structure; this is equivalent to specifying our initial G-action on the algebra A itself.

1.5.2.   We desire a parallel theory suitable for loop groups.

There is an immediate difference: for loop groups, we need to work with *topological* algebras.

Indeed, a fundamental example should be the category $\mathfrak{g}((t))$–mod, where $A = U(\mathfrak{g}((t)))$ is the completed enveloping algebra of $\mathfrak{g}((t))$ (cf. [Bei]).

Note that at the abelian categorical level, $\mathfrak{g}((t))$–mod$^{\heartsuit}$ consists only of *discrete* modules over $U(\mathfrak{g}((t)))$, i.e., the definition of the category of modules is sensitive to the topology on $\mathfrak{g}((t))$.

Remark 1.5.4. The relevant class of topological algebras was introduced in [BD2] and [Bei]. In the latter, they were given the name *topological chiral algebras*. In this text, we prefer the name $\overset{\rightarrow}{\otimes}$-*algebras*.

1.5.3.   The theory of loop group actions on DG categories inherently involves derived categories.

However, to the author's knowledge, previous literature has avoided relating $\overset{\rightarrow}{\otimes}$-algebras and DG categories.

Instead, other works have only considered abelian categories of modules over $\overset{\rightarrow}{\otimes}$-algebras, and then have passed to the corresponding derived categories of modules.

Unfortunately, this approach is inadequate for setting up a theory of Harish-Chandra data for loop groups: one needs a more direct connection between $\overset{\rightarrow}{\otimes}$-algebras and their corresponding *derived* categories of modules.

To overcome this issue, we develop an *inherently derived* theory of $\overset{\rightarrow}{\otimes}$-algebras and their modules in §3, §2, and §4. (As in standard, we consider pro-complexes instead of topological vector spaces.)

Remark 1.5.5. Clausen-Scholze [Sch] have recently put forward a different theory of topological algebras in homotopical settings. Their theory in effect gives a derived version of the $\overset{*}{\otimes}$-algebras considered in [Bei].

We do not consider $\overset{*}{\otimes}$-algebras in this text, but one could formally define such a tensor structure on ProVect (or pro-abelian groups, or pro-spectra) by taking [Bei] Corollary 1.1 as a definition (using the constructions from our §3). It would be interesting to contrast the resulting theory with the Clausen-Scholze approach, which uses direct topological methods rather than pro-objects (and certainly produces an inequivalent theory).

1.5.4.   Our theory is largely parallel to that of [Bei], but there are some discrepancies.

The major one relates to the finer points of the Frenkel-Gaitsgory definition of $\mathfrak{g}((t))$–mod referenced in §1.4.15. The issue is that the forgetful functor $\mathfrak{g}((t))$–mod $\to$ Vect is *not* conservative: it sends some non-zero objects to zero. (It *is* conservative on the bounded below derived category.)

Therefore, it is not literally possible to think of general objects of $\mathfrak{g}((t))$–mod as vector spaces with extra structure (like an $\mathfrak{g}((t))$-action).

Following [FG2], we refer to these sorts of phenomena as *renormalization* problems. The finer points of the theory all relate to renormalization difficulties.

1.5.5.   Finally, in §10, we introduce a suitable theory of Harish-Chandra data for loop groups.

Due to the complications above, the theory is quite technical, and there are additional technical hypotheses that do not appear in the finite-dimensional setting.

Still, at the end of the day, for a *classical* (i.e., non-DG) $\overset{\rightarrow}{\otimes}$-algebra A (which we also assume comes from a topological vector space), and equipped with a suitable *renormalization datum* (as above), we can characterize suitable actions of $G(K)$ on the renormalized category A–mod$_{ren}$ of A-modules in terms of morphisms $i : \mathfrak{g}((t)) \to A$ satisfying some identities.

In other words, we characterize the action of $G(K)$ on A–mod_{ren} using classical linear algebraic data: an action of $G(K)$ on A and a map $i : \mathfrak{g}((t)) \to A$ satisfying some properties. Therefore, the theory of Harish-Chandra data in §10 provides a way to study certain categories with $G(K)$-actions using *purely 1-categorical methods*. In this way, we can relate classical infinite-dimensional representation theory and $G(K)$-actions on categories. (See Remark 1.5.7 for one explicit situation in which this principle applies.)

Remark 1.5.6. Our theory covers the following examples:

- The action of $G(K)$ of $\mathfrak{g}((t))$–mod.
- The action of $G(K)$ on $D^!(Y(K))$ for Y a smooth affine variety equipped with a $G(K)$-equivariant CDO (cf. [BD2] §3.9).
- Suitable extensions of the above including a level κ.

More heuristically, we expect that given a "nice" vertex algebra $\mathcal{V}$ with Kac-Moody symmetry, our theory applies to the $G(K)$-action on the $\overrightarrow{\otimes}$-algebra associated with $\mathcal{V}$.

1.5.6. *Applications.* We now discuss applications of this material to other sources.

1.5.7. *Critical level.* Suppose $\kappa = crit$ is the so-called *critical* level, which we remind is $-\frac{1}{2}$ times the Killing form. In this case, Feigin-Frenkel showed that there is a large center $\mathfrak{Z}$ of the corresponding (twisted, completed) enveloping algebra.

In §11, we apply our theory of Harish-Chandra data to give a categorical realization of these symmetries of the Feigin-Frenkel center.

The main result is Theorem 11.18.1. Roughly, this result says that $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}$ has an action of $\mathsf{IndCoh}^*(\mathrm{Spf}\,\mathfrak{Z})$, and that this action naturally commutes with the canonical (critical level) $G(K)$-action on $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}$.

Remark 1.5.7. Similar constructions are easy to perform in the finite dimensional setting. However, the author is unaware of a simpler approach than the one presented here in the affine setting. The basic issue is that we need to relate the center $\mathfrak{Z}$, which is defined using topological algebras and studied using representation theory, to derived category constructions (as in the definition of $G(K)$-actions). This is what our theories of $\overrightarrow{\otimes}$-algebras and Harish-Chandra data were designed to do.

Remark 1.5.8. A weaker version of the above construction was given in [FG1] §23. The construction is *loc. cit.* was given before there was a good theory of $G(K)$-actions on DG categories, so is inherently weaker than the construction we give here. For instance, our construction manifestly accounts for all higher homotopy coherence data, so is compatible with constructions on $G(K)$-categories.

Remark 1.5.9. This material is used in [Ras7], as mentioned in §1.4.26. As in Remark 1.4.17, much of the present text arose from trying to fill in gaps in the literature regarding categorical symmetries for Kac-Moody algebras. In particular, this is true for our theories of $\overrightarrow{\otimes}$-algebras and Harish-Chandra data; we needed the commuting actions of $G(K)$ and the critical center for the methods of *loc. cit.*

1.5.8. *BRST-style constructions.* One sometimes finds the following situation in the physics literature.

One is given a vertex algebra $\mathcal{V}$ with a $G(O)$-action and a morphism $\mathbb{V}_{\mathfrak{g},\kappa} \to \mathcal{V}$ defining Kac-Moody symmetry on $\mathcal{V}$ for some integral level κ. One wishes to form the BRST reduction $\mathrm{BRST}(\mathcal{V})$

of $\mathcal{V}$ with respect to this Kac-Moody action. However, this is only possible if $\kappa = -Tate = 2 \cdot crit$. What should be done at other levels?

In this case, there is a natural map $\widehat{\mathfrak{g}}_\kappa \to A(\mathcal{V})$, where $A(\mathcal{V})$ is the $\overset{\rightarrow}{\otimes}$-algebra controlling vertex modules for $\mathcal{V}$. This map satisfies the identities to be a Harish-Chandra datum. One should expect the technical conditions of §10 to be satisfied, so $G(K)$ acts strongly on the DG category $\mathcal{V}$–mod (which is assumed suitably renormalized).

As the level is integral, we can ignore it. Therefore, $\mathcal{V}$–mod$^{G(K)}$ is defined. Heuristically, it would be modules over BRST($\mathcal{V}$) if that reduction were defined. In this sense, the *derived category* of modules over BRST($\mathcal{V}$) is defined, even if there is an anomaly; it is the forgetful functor to Vect that is missing. (Moreover, one should be able to extend this to work with factorization categories in the sense of [Ras2], providing a substitute for the VOA structure on BRST($\mathcal{V}$).)

Remark 1.5.10. For one example in which the above principle BRST($\mathcal{V}$)–mod $\approx \mathcal{V}$–mod$^{G(K)}$ can be made quite precise, see [Ras6]. Note that in *loc. cit.*, every object of the relevant equivariant category lives in cohomological degree $-\infty$; i.e., it is essential to confront the finer points of the homological algebra.

Remark 1.5.11. The issue described here occurs many places in the literature: see [CG], [GR1], and [BLL$^+$] for some recent examples of different flavors.

Remark 1.5.12. We do not claim that this construction is suited for all purposes. For instance, important invariants such as conformal blocks are not a priori defined for categories such as $\mathcal{V}$–mod$^{G(K)}$, only for the VOA BRST (if it exists).

1.5.9. *Weil representations and Coulomb branches.* In [Ras8], we use the theory of Harish-Chandra data to construct an analogue of the Weil representation for symplectic loop groups. We apply this to construct Coulomb branches for symplectic representations of reductive groups,[16] generalizing the BFN construction [BFN] to the case when there is no Lagrangian subspace.

1.5.10. *Harish-Chandra bimodules.* Finally, we briefly want to draw the interested reader's attention to the fact that §10 implicitly provides tools to study the category of affine Harish-Chandra bimodules.

For a level κ of $\mathfrak{g}$, define $\mathcal{HC}^{\mathrm{aff}}_{G,\kappa}$ as $\mathsf{End}_{G(K)-\mathsf{mod}_\kappa}(\widehat{\mathfrak{g}}_\kappa-\mathsf{mod})$, i.e., as the monoidal DG category of endomorphisms of $\widehat{\mathfrak{g}}_\kappa$–mod considered as a category with a level κ $G(K)$-action (this notion is defined in §11). By the results of §8, this category may also be calculated as $\widehat{\mathfrak{g}}_\kappa$–mod$_{G(K),w}$, the category of weak $G(K)$-coinvariants.

The category $\mathcal{HC}^{\mathrm{aff}}_{G,\kappa}$ plays a central role in quantum local geometric Langlands, but is difficult to study explicitly. Theorem 11.18.1 amounts to a construction of a monoidal functor:

$$\mathsf{IndCoh}^!(\mathrm{Op}_{\check{G}}) \to \mathcal{HC}^{\mathrm{aff}}_{G,crit}$$

so the proof must provide some basic study of the right hand side.

The main technical work in this study is implicit in §10. The careful reader will find that the mosttechnical results in §10 are about categories $\widehat{\mathfrak{g}}_\kappa$–mod^{K,w} for $K \subseteq G(K)$ compact open, and understanding these categories is an essential prerequisite to understanding $\widehat{\mathfrak{g}}_\kappa$–mod$_{G(K),w} = \mathcal{HC}^{\mathrm{aff}}_{G,\kappa}$.

---

[16]In the absence of a certain anomaly involving $\pi_4(G)$; cf. [Wit2] for $G = SL_2$.

1.6. **Relation to older approaches.** It is roughly fair to say that this text is an update of the appendices to [FG1], incorporating modern homotopical techniques and working with unbounded derived categories.

We remark that the extension to unbounded derived categories is essential in applications (see already [FG2]), and the reader will observe that most of the difficulties that come up in our setting exactly have to do with the difference between bounded below derived categories and unbounded ones.

Many of our constructions are also close in spirit to [Pos1], although our perspective and emphasis are somewhat different.

1.7. **Leitfaden.** Here are the (essential) logical dependencies.
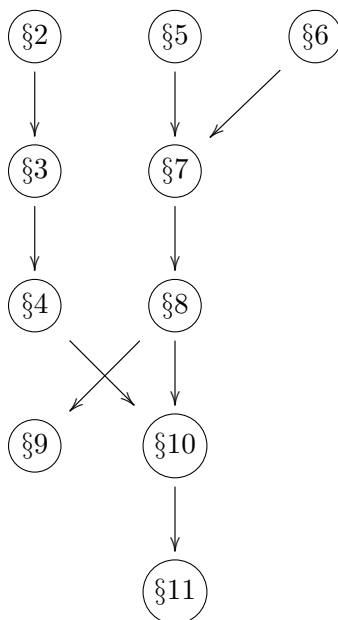
Briefly, §2-4 develops the theory of $\overrightarrow{\otimes}$-algebras and renormalization data. The theory of weak group actions on categories is developed in §5 and §7, and is related to strong group actions in §8. We apply these ideas to semi-infinite cohomology in §9. The material we need on ind-coherent sheaves is developed in §6. Finally, §10 introduces Harish-Chandra data for group indschemes acting on $\overrightarrow{\otimes}$-algebras, and §11 gives an application at the critical level.

1.8. **Conventions.** We always work over the base field k of characteristic zero.

We use higher categorical language without mention: by *category*, we mean ∞-category in the sense of [Lur2], and similarly for monoidal category and so on. We let Cat denote the category of $(\infty-)$-categories, and Gpd denote the category of $(\infty-)$groupoids. We also refer to §2.2 for some essential notation used throughout the paper.

Similarly, by *scheme* we mean *derived scheme over k* in the sense of [GR4], or *spectral scheme over k* in the sense of [Lur4]. Similarly, by *indscheme*, we mean what [GR3] calls *DG indscheme*. Algebras of all flavors are assumed to be derived unless otherwise stated. We emphasize that when we speak of *DG* objects or chain complexes of vector spaces or the like, we really understand objects of suitable ∞-categories, not explicit cochain models for them; we refer to [GR4] §1 for an introduction to this way of thinking.

For $\mathcal{C}$ a DG category, we let $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \in \mathsf{Vect}$ denote the Hom-complex between objects, and we let $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \in \mathsf{Gpd}$ denote groupoid of maps in $\mathcal{C}$ regarded as an abstract category, i.e., forgetting the DG structure. We remind that $\Omega^{\infty}\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$, where on the left hand side we are regarding Vect as the ∞-category of k-module spectra.

For a DG category $\mathcal{C}$ with a t-structure, we let $\tau^{\geqslant n}$ and $\tau^{\leqslant n}$ denote the truncation functors; we use cohomological gradings throughout (as indicated by the use of superscripts).

1.9. **Acknowledgements.** We thank Sasha Beilinson, Dylan Butson, Justin Campbell, Gurbir Dhillon, Dennis Gaitsgory, Nick Rozenblyum, and Harold Williams for a number of essential discussions related to this text.

2. Monoidal structures

2.1.   In this section, we define monoidal structures $\overset{!}{\otimes}$ and $\overset{\rightarrow}{\otimes}$ on $\mathsf{ProVect}$. This material follows [Bei], the appendices to [FG1], [GK], and [Pos1] Appendix D. The main difference with those sources is that we work in the derived setting, which requires somewhat restructuring the usual definitions.

2.2. **Notation.** Let $\mathsf{ProVect}$ denote the pro-category of (the DG category) Vect; we refer to [Lur2] §7.1.6 and [Lur4] §A.8.1 for details on pro-categories.

Remark 2.2.1. There are cardinality issues to keep in mind when working with a category such as $\mathsf{ProVect}$. Let us remind some relevant ideas from [Lur2] §5. A category is *accessible* if it satisfies a certain hypothesis involving cardinalities (and is idempotent complete); for instance, compactly generated categories are accessible, where the cardinality condition is the hypothesis that the subcategory of compact objects is *essentially small*. We also remind that *presentable* accessible and cocomplete (i.e., admitting colimits). This hypothesis is designed so the "naive" proof of the adjoint functor theorem (involving potentially large limits) goes through; in particular, presentable categories admit limits.

Now for an accessible category $\mathcal{C}$ admitting finite limits, (e.g., $\mathcal{C}$ is presentable), $\mathsf{Pro}(\mathcal{C})^{op}$ is defined to be the category of *accessible* functors $\mathcal{C} \to \mathsf{Gpd}$ preserving finite limits. Note that $\mathsf{Pro}(\mathcal{C})$ is not presentable, so the adjoint functor theorem and its relatives do not apply.

By a *DG category*, we mean a stable (∞-)category with a Vect^c-module category structure where the action is exact in each variable separately. Here $\mathsf{Vect}^c \subseteq \mathsf{Vect}$ is the subcategory of compact objects (i.e., perfect objects, i.e., bounded complexes with finite dimensional cohomologies). By a DG functor, we mean an exact functor compatible between the Vect^c-module structures. We let DGCat_{big} denote the 2-category of such.

We let $\mathsf{DGCat} \subseteq \mathsf{DGCat}_{big}$ denote the 2-category of accessible DG categories under accessible DG functors. Recall that DGCat_{cont} denotes the 2-category of cocomplete, presentable DG categories under continuous functors.

For the set up of topological algebras, it would be more natural to work with spectra and stable categories, but given our convention that we work over k, we stick to the language of DG categories.

We remind (c.f. [Lur3] §4.8.1) that DGCat_{big} has a canonical symmetric monoidal structure with unit Vect^c. We denote this monoidal structure by $\overline{\otimes}$. If we worked in the spectral setting, functors $\mathcal{C} \overline{\otimes} \mathcal{D} \to \mathcal{E}$ would be the same as functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ exact in each variable separately; in the DG setting, they should be called *bi-DG* functors.

Similarly, DGCat_{cont} has a symmetric monoidal structure $\otimes$ such that functors $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ are the same as bi-DG functors that commute with colimits in each variable separately.

For $\mathcal{F} \in \mathcal{C}$ and $\mathcal{G} \in \mathcal{D}$, we let $\mathcal{F} \boxtimes \mathcal{G}$ denote the induced object of $\mathcal{C} \overline{\otimes} \mathcal{D}$ or $\mathcal{C} \otimes \mathcal{D}$ as appropriate.

For $\mathcal{C}$ a compactly generated DG category, we let $\mathcal{C}^c$ denote its subcategory of compact objects, as in the case of Vect above.

2.3. Review of topological tensor products.

Following the above references, we seek two tensor products $\overset{!}{\otimes}$ and $\overset{\to}{\otimes}$ on ProVect. Ignoring homotopy coherences issues for the moment, we recall the basic formulae characterizing these two tensor products concretely.

Roughly, if $V = \lim_i V_i, W = \lim_j W_j \in$ ProVect are filtered limits with $V_i, W_j \in$ Vect, then:

$$V \overset{!}{\otimes} W = \lim_{i,j} V_i \otimes W_j.$$

The (non-symmetric) monoidal product $\overset{\to}{\otimes}$ is characterized by the fact that it is a bi-DG functor, and the functor:

$$V \overset{\to}{\otimes} - : \mathsf{ProVect} \to \mathsf{ProVect}$$

commutes with limits, while the functor:

$$V \overset{\to}{\otimes} - : \mathsf{Vect} \to \mathsf{ProVect}$$

commutes with colimits.

Explicitly, if $W_j = \mathrm{colim}_k W_{j,k}$ with $W_{j,k} \in \mathsf{Vect}^c$, we have:

$$V \overset{\to}{\otimes} W = \lim_j \mathrm{colim}_k V \otimes W_{j,k}.$$

(Clearly we should allow the indexing set for the terms "k" to depend on j.)

These two tensor products are connected as follows. For V_1, V_2, W_1, W_2, there is a natural map:

$$(V_1 \overset{!}{\otimes} V_2) \overset{\to}{\otimes} (W_1 \overset{!}{\otimes} W_2) \to (V_1 \overset{\to}{\otimes} W_1) \overset{!}{\otimes} (V_2 \overset{\to}{\otimes} W_2). \tag{2.3.1}$$

In particular, there is a natural map:

$$V \overset{\to}{\otimes} W \to V \overset{!}{\otimes} W.$$

2.4. Topological tensor products in the derived setting.

We now formally define the above structures and characterize their categorical properties.

The tensor product $\overset{!}{\otimes}$ on ProVect is easy: as Ind of a monoidal category has a canonical tensor product, so does Pro.

2.5.

To construct $\overset{\to}{\otimes}$, first note that $\mathsf{Pro}(\mathsf{Vect})^{op}$ is by definition the category $\mathsf{Hom}(\mathsf{Vect}, \mathsf{Gpd})$ of accessible functors $\mathsf{Vect} \to \mathsf{Gpd}$. Any such functor factors canonically as:

$$\mathsf{Vect} \xrightarrow{F} \mathsf{Vect} \xrightarrow{\mathrm{Oblv}} \mathsf{Spectra} \xrightarrow{\Omega^\infty} \mathsf{Gpd}$$

with F a DG functor, i.e., $\mathsf{ProVect} = \mathsf{Hom}_{\mathsf{DGCat}}(\mathsf{Vect}, \mathsf{Vect})^{op}$.

Notation 2.5.1. For $V \in \mathsf{ProVect}$, we let F_V denote the induced functor $\mathsf{Vect} \to \mathsf{Vect}$. Clearly $F_V = \underline{\mathrm{Hom}}_{\mathsf{ProVect}}(V, -)$. Define $V \overset{\to}{\otimes} - : \mathsf{ProVect} \to \mathsf{ProVect}$ as the "partially-defined left adjoint" to F_V, i.e., for $W \in \mathsf{Vect}$ and $U \in \mathsf{ProVect}$, we have functorial isomorphisms:

$$\underline{\mathrm{Hom}}_{\mathsf{ProVect}}(V \overset{\to}{\otimes} W, U) \simeq \underline{\mathrm{Hom}}_{\mathsf{Vect}}(W, F_V(U)). \tag{2.5.1}$$

We extend this construction to general $W \in \mathsf{ProVect}$ by right Kan extension.

We extend this construction to a monoidal structure by reinterpreting it as composition of functors in $\mathsf{Hom}_{\mathsf{DGCat}}(\mathsf{Vect}, \mathsf{Vect})$. That is, we observe:

$$F_V \circ F_W \simeq F_{V \overrightarrow{\otimes} W}.$$

The left hand side extends to the evident monoidal structure on $\mathsf{Hom}_{\mathsf{DGCat}}(\mathsf{Vect}, \mathsf{Vect})^{op}$.

2.6. **Comparison of tensor products.** We now wish to give compatibilities between $\overrightarrow{\otimes}$ and $\overset{!}{\otimes}$. Roughly, we claim that these form a "lax $\mathcal{E}_2$" structure.[17]

Let $\mathsf{Alg}(\mathsf{Cat})$ denote the category of monoidal categories and lax monoidal functors, which we consider as a symmetric monoidal category under products. We claim that $(\mathsf{ProVect}, \overrightarrow{\otimes})$ is a commutative algebra in this category with operation $\overset{!}{\otimes}$. Note that this structure encodes the natural transformations (2.3.1). (In §3.3, we give some simple consequences, and the reader may wish to skip ahead.)

To construct this compatibility, note that if we write $\mathsf{ProVect}$ as $\mathsf{End}_{\mathsf{DGCat}}(\mathsf{Vect})^{op}$, then $\overset{!}{\otimes}$ corresponds to *Day convolution*. Then this follows from formal facts about Day convolution.

With respect to the symmetric monoidal structure $\overline{\otimes}$ on DGCat, we have the internal Hom functor:

$$\mathsf{Hom}_{\mathsf{DGCat}}(\mathcal{C}, \mathcal{D}) \in \mathsf{DGCat}_{big}$$

which is the usual (DG) category of DG functors. If $\mathcal{C}$ is a monoidal DG category and $\mathcal{D} \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$, then recall that $\mathsf{Hom}_{\mathsf{DGCat}}(\mathcal{C}, \mathcal{D})$ has the usual Day convolution monoidal structure. It is characterized by the fact that:

$$\mathsf{Hom}_{\mathsf{Alg}^{lax}(\mathsf{DGCat})}(\mathcal{E}, \mathsf{Hom}_{\mathsf{DGCat}}(\mathcal{C}, \mathcal{D})) = \mathsf{Hom}_{\mathsf{DGCat}}(\mathcal{C} \overline{\otimes} \mathcal{E}, \mathcal{D})$$

where by $\mathsf{Alg}^{lax}(\mathsf{DGCat})$ we mean monoidal DG categories under *lax* monoidal functors.

It is straightforward to see that Day convolution has the property that the composition functor:

$$\mathsf{Hom}_{\mathsf{DGCat}}(\mathcal{C}, \mathcal{D}) \overline{\otimes} \mathsf{Hom}_{\mathsf{DGCat}}(\mathcal{D}, \mathcal{E}) \to \mathsf{Hom}_{\mathsf{DGCat}}(\mathcal{C}, \mathcal{E})$$

is lax monoidal (assuming $\mathcal{D}, \mathcal{E} \in \mathsf{Alg}(\mathsf{DGCat}_{cont}, \otimes)$). This immediately implies our claim about the monoidal structures on $\mathsf{ProVect}$.

2.7.   First, suppose that $\mathcal{C}, \mathcal{D} \in \mathsf{DGCat}_{cont}$ with $\mathcal{D}$ compactly generated. Then observe that there is a canonical bi-DG functor:

$$\mathsf{Pro}(\mathcal{C}) \times \mathsf{Pro}(\mathcal{D}) \xrightarrow{-\overrightarrow{\boxtimes}-} \mathsf{Pro}(\mathcal{C} \otimes \mathcal{D})$$

computed as follows.

If $\mathcal{G} \in \mathcal{D}^c$, then the induced functor $\xrightarrow{-\overrightarrow{\boxtimes}\mathcal{G}}: \mathsf{Pro}(\mathcal{C}) \to \mathsf{Pro}(\mathcal{C} \otimes \mathcal{D})$ is the right Kan extension of the functor $- \boxtimes \mathcal{G} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{D} \subseteq \mathsf{Pro}(\mathcal{C} \otimes \mathcal{D})$. In general, for $\mathcal{F} \in \mathsf{Pro}(\mathcal{C})$, the functor $\overrightarrow{\boxtimes}- : \mathsf{Pro}(\mathcal{D}) \to \mathsf{Pro}(\mathcal{C} \otimes \mathcal{D})$ is computed by first left Kan extending the above functor from $\mathcal{D}^c$ to $\mathcal{D}$, and then right Kan extending to $\mathsf{Pro}(\mathcal{D})$.

This operation is functorial in the sense that for $F : \mathcal{C}_1 \to \mathcal{C}_2 \in \mathsf{DGCat}_{cont}$, the diagram:

---

[17]For what follows, it is important to think of the $\mathcal{E}_2$ operad as $\mathcal{E}_1^{\otimes 2}$ and not as the little discs operad: the laxness evidently breaks the $SO(2)$-symmetry.

$$\begin{array}{ccc} \mathsf{Pro}(\mathcal{C}_1) \times \mathsf{Pro}(\mathcal{D}) & \longrightarrow & \mathsf{Pro}(\mathcal{C}_2) \times \mathsf{Pro}(\mathcal{D}) \\ \downarrow{\scriptstyle -\overrightarrow{\boxtimes}-} & & \downarrow{\scriptstyle -\overrightarrow{\boxtimes}-} \\ \mathsf{Pro}(\mathcal{C}_1 \otimes \mathcal{D}) & \longrightarrow & \mathsf{Pro}(\mathcal{C}_2 \otimes \mathcal{D}) \end{array}$$

canonically commutes. Indeed, this follows immediately from:

Lemma 2.7.1. *For* $F : \mathcal{C} \to \mathcal{D} \in \mathsf{DGCat}_{cont}$, $\mathsf{Pro}(F) : \mathsf{Pro}(\mathcal{C}) \to \mathsf{Pro}(\mathcal{D})$ *commutes with limits and colimits.*

Proof. $\mathsf{Pro}(F)$ tautologically commutes with limits. For the commutation with colimits, note that F admits an (accessible) right adjoint G by the adjoint functor theorem, so $\mathsf{Pro}(F)$ admits the right adjoint $\mathsf{Pro}(G)$.

$\square$

3. Modules and comodules

3.1. We let $\mathsf{Alg}^{\overrightarrow{\otimes}}$ denote the category of (associative, unital) algebras in $\mathsf{ProVect}$ with respect to $\overrightarrow{\otimes}$. We refer to objects of $\mathsf{Alg}^{\overrightarrow{\otimes}}$ as $\overrightarrow{\otimes}$-*algebras*. We remark that (lower categorical analogues of) such objects have been variously referred to as *topological algebras* or *topological chiral algebras* in the literature.

In this section, we give basic definitions about modules over $\overrightarrow{\otimes}$-algebras. Note that we are exclusively interested in *discrete* modules, i.e., modules in Vect, not in $\mathsf{ProVect}$, and our notation will always take this for granted.

Terminology 3.1.1. We generally use the term *discrete* to refer to objects of $\mathsf{Vect} \subseteq \mathsf{ProVect}$. For example, we say a $\overrightarrow{\otimes}$-algebra is *discrete* if its underlying object lies in Vect (in which case this structure is equivalent to a usual associative DG algebra structure).

This should not be confused with the usage of this phrase in homotopy theory, where it is often used for an object in the heart of a t-structure. In that setting, we prefer the term *classical*, so e.g., a *classical* $\overrightarrow{\otimes}$-algebra is one whose underlying object lies in $\mathsf{ProVect}^{\heartsuit}$.

3.2. **Comparison with comonads.** First, note that by construction, we have:

$$\mathsf{Alg}^{\overrightarrow{\otimes}} \simeq \{\text{accessible DG comonads on } \mathsf{Vect.}\}^{op}$$

Here DG indicates that we have compatible comonad and DG functor structures; in the stable setting, this would simply mean the underlying functor of our comonad is exact.

Let A be a $\overrightarrow{\otimes}$-algebra. Define A–mod_{top} as A–$\mathsf{mod}(\mathsf{ProVect})$. We define A–mod_{naive} to be the "naive" category of discrete A-modules:

$$A\text{–}\mathsf{mod}_{top} \underset{\mathsf{ProVect}}{\times} \mathsf{Vect}.$$

That is, an object of A–mod_{naive} has an underlying vector space $M \in \mathsf{Vect}$, an action map $\overrightarrow{AM} \to M \in \mathsf{ProVect}$, and the usual (higher) associativity data. (We use the notation "naive" by comparison with the renormalization setting introduced below.)

By (2.5.1), if $S := F_A$ is the comonad corresponding to A, we have a canonical equivalence:

$$A\text{–}\mathsf{mod}_{naive} \simeq S\text{–}\mathsf{comod}$$

compatible with forgetful functors to Vect.

As a consequence, A–mod_{naive} is presentable and the forgetful functor $\mathrm{Oblv} : A$–$\mathsf{mod}_{naive} \to \mathsf{Vect}$ is continuous, conservative, and, of course, comonadic.

Remark 3.2.1. To conclude: the language of $\overset{\to}{\otimes}$-algebras is equivalent to the language of (DG) comonads on Vect. Therefore, the wisdom in using this language may be reasonably questioned by the reader; we use it here to connect to older work, and because pro-vector spaces are typically nicer to describe than their corresponding comonads.

3.3. **Tensor products.** We now spell out what the material of §2.6 means for $\overset{\to}{\otimes}$-algebras and their modules. (We remind that §2.6 is a souped up version of (2.3.1), which may be more helpful to refer to.)

By §2.6, $\mathsf{Alg}^{\overset{\to}{\otimes}}$ is symmetric monoidal with tensor product:

$$A, B \mapsto A \overset{!}{\otimes} B.$$

Similarly, we have the bi-DG functor:

$$A\text{–}\mathsf{mod}_{top} \times B\text{–}\mathsf{mod}_{top} \to A \overset{!}{\otimes} B\text{–}\mathsf{mod}_{top}$$

$$(M, N) \mapsto M \overset{!}{\otimes} N.$$

Clearly this induces a bi-DG functor:

$$A\text{–}\mathsf{mod}_{naive} \times B\text{–}\mathsf{mod}_{naive} \to A \overset{!}{\otimes} B\text{–}\mathsf{mod}_{naive}$$
$$(M, N) \mapsto M \otimes N.$$

This functor commutes with colimits in each variable separately, so induces:

$$A\text{–}\mathsf{mod}_{naive} \otimes B\text{–}\mathsf{mod}_{naive} \to (A \overset{!}{\otimes} B)\text{–}\mathsf{mod}_{naive}. \tag{3.3.1}$$

To properly encode all higher categorical data, note that we have upgraded $A \mapsto A$–mod_{naive} to a contravariant lax symmetric monoidal functor from $\overset{\to}{\otimes}$-algebras to DGCat_{cont}.

3.4. **Forgetful functors.** Like every functor in DGCat_{cont}, F is pro-representable, i.e., there is a filtered projective system $i \mapsto \mathcal{F}_i \in A$–$\mathsf{mod}_{naive}$ such that:

$$\mathrm{colim}_i \underline{\mathrm{Hom}}_{A\text{–}\mathsf{mod}_{naive}}(\mathcal{F}_i, -) = \mathrm{Oblv}\,.$$

In fact, we claim that $\lim_i \mathrm{Oblv}(\mathcal{F}_i) \in \mathsf{ProVect}$ is the pro-vector space underlying A.

Indeed, let $\Phi : \mathsf{Vect} \to A$–$\mathsf{mod}_{naive}$ be the functor right adjoint to the forgetful functor. Note that $\mathrm{Oblv}\,\Phi = F_A$ ($:= \underline{\mathrm{Hom}}_{\mathsf{ProVect}}(A, -)$), as is clear in the comonadic picture.

Then for any $V \in \mathsf{Vect}$, we obtain:

$$\underline{\mathrm{Hom}}_{\mathsf{ProVect}}(\lim_i \mathrm{Oblv}(\mathcal{F}_i), V) = \mathrm{colim}_i \underline{\mathrm{Hom}}_{\mathsf{Vect}}(\mathrm{Oblv}(\mathcal{F}_i), V) =$$

$$\mathrm{colim}_i \underline{\mathrm{Hom}}_{\mathsf{Vect}}(\mathcal{F}_i, \Phi(V)) = \mathrm{Oblv}\,\Phi(V) = \underline{\mathrm{Hom}}_{\mathsf{ProVect}}(A, -)$$

as desired.

3.5. *t*-**structures.** Recall that $\mathsf{ProVect}$ has a natural *t*-structure with $(\mathsf{ProVect})^{\leqslant 0} = \mathsf{Pro}(\mathsf{Vect}^{\leqslant 0})$ and $(\mathsf{ProVect})^{\geqslant 0} = \mathsf{Pro}(\mathsf{Vect}^{\geqslant 0})$; we omit the parentheses in the sequel as there can be no confusion.

In the remainder of the section, we will be interested in *connective* $\overset{\rightarrow}{\otimes}$-algebras, i.e., such algebras A in $\mathsf{ProVect}^{\leqslant 0}$. Clearly this hypothesis is equivalent to the comonad F_A being left *t*-exact.

From this latter description, we see that $A\text{–}\mathsf{mod}_{naive}$ carries a canonical *t*-structure such that $\mathrm{Oblv} : A\text{–}\mathsf{mod}_{naive} \to \mathsf{Vect}$ is *t*-exact. Because Oblv commutes with colimits, this *t*-structure is necessarily right complete.

3.6. **Convergence.** In order to formulate Proposition 3.7.1 below, we introduce the following terminology.

For $V \in \mathsf{ProVect}$, the *convergent completion* of V is:

$$\lim_n \tau^{\geqslant -n}(V) \in \mathsf{ProVect}.$$

We say that V is *convergent* if the natural map $V \to \widehat{V}$ is an isomorphism. Note that V is convergent if and only if it lies in $\mathsf{Pro}(\mathsf{Vect}^+) \subseteq \mathsf{Pro}(\mathsf{Vect})$ (or equivalently: F_V is left Kan extended from Vect^+).

In particular, we obtain that connective convergent pro-vector spaces are (contravariantly) equivalent to left *t*-exact functors $\mathsf{Vect}^+ \to \mathsf{Vect}^+ \in \mathsf{DGCat}$. Under this dictionary, connective $\overset{\rightarrow}{\otimes}$-algebras are the same as left *t*-exact (accessible) DG comonads on Vect^+.

Remark 3.6.1. If A is a connective $\overset{\rightarrow}{\otimes}$-algebra, then its convergent completion $\widehat{A}$ is as well, and $\widehat{A}\text{–}\mathsf{mod}^+_{naive} \overset{\cong}{\to} A\text{–}\mathsf{mod}^+_{naive}$.

3.7. **Comparison with categorical data.** We have the following psychologically important result.

Proposition 3.7.1. *The functor:*

$$\{convergent, \ connective \ \overset{\rightarrow}{\otimes}\text{-}algebras\} \to \mathsf{DGCat}_{/\mathsf{Vect}^+}$$

$$A \mapsto \big(\mathrm{Oblv} : A\text{–}\mathsf{mod}^+_{naive} \to \mathsf{Vect}^+ \big)$$

is fully-faithful. A DG category $\mathcal{C}$ with structural functor $F : \mathcal{C} \to \mathsf{Vect}^+$ lies in the essential image of this map if and only if:

- *F is conservative.*
- *$\mathcal{C}$ admits a (necessarily unique) t-structure for which F is t-exact.*
- *$\mathcal{C}^{\geqslant 0}$ admits arbitrary colimits, and the functor $F : \mathcal{C}^{\geqslant 0} \to \mathsf{Vect}^{\geqslant 0}$ preserves such colimits.*

Under this equivalence, $\mathcal{C}$ is the bounded below derived category of its heart $\mathcal{C}^{\heartsuit}$ with F the derived functor of its restriction $\mathcal{C}^{\heartsuit} \to \mathsf{Vect}^{\heartsuit}$ if and only if the corresponding $\overset{\rightarrow}{\otimes}$-algebra A is classical (i.e., lies in $\mathsf{ProVect}^{\heartsuit}$).

We first recall the following standard result about simplicial objects, see e.g. [Lur3] Remark 1.2.4.3.

Lemma 3.7.2. *For a cosimplicial object $\mathcal{F}^{\bullet}$ in a stable (e.g., DG) category $\mathcal{C}$, let $\mathrm{Tot}^{\leqslant n} \mathcal{F}^{\bullet}$ be the limit over the subcategory $\boldsymbol{\Delta}_{\leqslant n} \subseteq \boldsymbol{\Delta}$ of totally ordered sets of cardinality $\leqslant n + 1$.*
  Then for $n > 0$:

$$\mathrm{Ker}(\mathrm{Tot}^{\leqslant n} \mathcal{F}^{\bullet} \to \mathrm{Tot}^{\leqslant n-1} \mathcal{F}^{\bullet})$$

is isomorphic to a direct summand of $\mathcal{F}^n[-n]$.

Proof of Proposition 3.7.1. First it is straightforward to see that $A\text{–}\mathsf{mod}^+_{naive}$ actually satisfies the above conditions.

Suppose $F : \mathcal{C} \to \mathsf{Vect}^+$ with the above properties is given. Clearly the t-structure on $\mathcal{C}$ is bounded from below (i.e., $\mathcal{C} = \mathcal{C}^+$), compatible with filtered colimits, and right complete. Clearly the equivalence follows if we can show such F is comonadic; the argument is well-known, but we reproduce it here for convenience.

First, we claim $F|_{\mathcal{C}^{\geq 0}} : \mathcal{C}^{\geq 0} \to \mathsf{Vect}^{\geq 0}$ commutes with arbitrary totalizations. By Lemma 3.7.2, if $\mathcal{F}^\bullet$ is a cosimplicial diagram in $\mathcal{C}$ with $\mathcal{F}^i \in \mathcal{C}^{\geq 0}$ for all i, then the totalization exists and is calculated by:

$$\tau^{\leq n} \operatorname{Tot} \mathcal{F}^\bullet = \tau^{\leq n} \operatorname{Tot}^{\leq n+1} \mathcal{F}^\bullet.$$

Since $\operatorname{Tot}^{\leq n+1}$ is a finite limit, t-exactness of F implies the claim.

Now observe that F admits a left t-exact (possibly non-continuous) right adjoint G, as $F|_{\mathcal{C}^{\geq 0}}$ admits a left exact right adjoint. Then for any $\mathcal{F} \in \mathcal{C}$, we have $\mathcal{F} \in \mathcal{C}^{\geq -N}$, for N large enough, so $(GF)^n(\mathcal{F}) \in \mathcal{C}^{\geq -N}$ for any n, so the totalization $\operatorname{Tot}((GF)^{\bullet+1}(\mathcal{F}))$ exists and is preserved by the conservative functor F, implying comonadicity.

It remains to show the compatibility with abelian categories. Suppose $\mathcal{A}$ is a k-linear abelian category with a k-linear functor $F^\heartsuit : \mathcal{A} \to \mathsf{Vect}^\heartsuit$ that is exact, continuous, conservative, and accessible. Then there is a pro-object $\lim \mathcal{F}_i \in \mathsf{Pro}(\mathcal{A})$ ($\mathcal{F}_i \in \mathcal{A}$) corepresenting $F^\heartsuit$. It immediately follows that this pro-object also corepresents the derived functor $F(:= RF^\heartsuit) : D^+(\mathcal{A}) \to \mathsf{Vect}^+$ (because the functor this pro-system defines maps injectives in $\mathcal{A}^\heartsuit$ into $\mathsf{Vect}^\heartsuit$). By §3.4, this implies that the corresponding $\overset{\rightarrow}{\otimes}$- algebra has underlying object $\lim F(\mathcal{F}_i) \in \mathsf{ProVect}$. Because $F^\heartsuit$ is exact, F is t-exact, so $F(\mathcal{F}_i) \in \mathsf{Vect}^\heartsuit$, implying $\lim F(\mathcal{F}_i) \in \mathsf{ProVect}^\heartsuit$.

Conversely, suppose A is classical. Let $\Phi : \mathsf{Vect} \to A\text{–}\mathsf{mod}_{naive}$ denote the (possibly discontinuous) right adjoint to the forgetful functor. For $V \in \mathsf{Vect}^\heartsuit$,[18] $\operatorname{Oblv} \Phi(V) = F_A(V) = \underline{\operatorname{Hom}}_{\mathsf{ProVect}}(A, V) \in \mathsf{Vect}^\heartsuit$, so $\Phi(V) \in A\text{–}\mathsf{mod}^\heartsuit_{naive}$. Moreover, $\Phi(V)$ is obviously injective in $A\text{–}\mathsf{mod}_{naive}$ in the sense that for any $\mathcal{F} \in A\text{–}\mathsf{mod}^{\geq 0}_{naive}$, $\underline{\operatorname{Hom}}(\mathcal{F}, \Phi(V)) = \underline{\operatorname{Hom}}_{\mathsf{Vect}}(\operatorname{Oblv}(\mathcal{F}), V) \in \mathsf{Vect}^{\leq 0}$. For $\mathcal{F} \in A\text{–}\mathsf{mod}^\heartsuit_{naive}$, the map $\mathcal{F} \to \Phi \operatorname{Oblv}(\mathcal{F})$ is a monomorphism in $A\text{–}\mathsf{mod}^\heartsuit_{naive}$ (as it splits after applying Oblv), so such there are "enough" injective objects, implying $A\text{–}\mathsf{mod}^+_{naive}$ is the bounded below derived category of its heart. Moreover, this reasoning immediately shows that the forgetful functor is the derived functor of its restriction to the hearts.

$\square$

4. RENORMALIZATION

4.1.   In our applications, the naive category $A\text{–}\mathsf{mod}_{naive}$ is typically not the one we want. For example, the forgetful functor $\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod} \to \mathsf{Vect}$ is not conservative, so the above construction *does not recover the correct category* $\widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}$, i.e., $U(\widehat{\mathfrak{g}}_\kappa)\text{–}\mathsf{mod}_{naive} \neq \widehat{\mathfrak{g}}_\kappa\text{–}\mathsf{mod}$.

Following [FG2], a key role is played by *renormalization* of derived categories. We refer to *loc. cit.*, [Gai7], and [Ras6] for introductions to this notion in the setting of Kac-Moody algebras. The basics of the theory of ind-coherent sheaves also play an instructional role: see [Gai5] for an introduction.

In this section, we give an introduction to this formalism.

4.2. **Renormalization data.**

---

[18]If we worked with a general commutative ring $k \in \mathsf{Ab}^\heartsuit$, V should be an injective k-module.

Definition 4.2.1. A *renormalization datum* for a connective $\overrightarrow{\otimes}$-algebra A is a DG category $A\text{–mod}_{ren} \in$ DGCat_{cont}, equipped with a t-structure and an equivalence $\rho : A\text{–mod}^+_{naive} \xrightarrow{\simeq} A\text{–mod}^+_{ren} \in \mathsf{DGCat}$, such that:

- ρ is t-exact.
- $A\text{–mod}_{ren}$ is compactly generated with compact generators lying in $A\text{–mod}^+_{ren}$.
- The t-structure on $A\text{–mod}_{ren}$ is compactly generated: i.e., $\mathcal{G} \in A\text{–mod}^{\geq 0}_{ren}$ if and only if $\mathrm{Hom}_{A\text{–mod}_{ren}}(\mathcal{F}, \mathcal{G}) = 0$ for every *compact* $\mathcal{F} \in A\text{–mod}^{<0}_{ren}$.

We will also say A is a *renormalized* $\overrightarrow{\otimes}$-algebra to mean A is a connective $\overrightarrow{\otimes}$-algebra equipped with a renormalization datum.

Remark 4.2.2. Once and for all, we emphasize: if A is renormalized, it is in particular connective.

Remark 4.2.3. The subcategory $A\text{–mod}^c_{ren}$ of compact objects in $A\text{–mod}_{ren}$ embeds canonically into $A\text{–mod}_{naive}$ as $A\text{–mod}^c_{ren} \subseteq A\text{–mod}^+_{ren} \simeq A\text{–mod}^+_{naive}$. It is immediate to see that a renormalization datum is equivalent to a choice of such a subcategory satisfying some conditions.

Remark 4.2.4. By Proposition 3.7.1, the category $\mathsf{Alg}^{\overrightarrow{\otimes}}_{conv,ren}$ of convergent, renormalized $\overrightarrow{\otimes}$-algebras are equivalent to some categorical data: $\mathcal{C} \in \mathsf{DGCat}_{cont}$, a continuous functor $F : \mathcal{C} \to \mathsf{Vect}$, and a t-structure on $\mathcal{C}$ such that F is t-exact and conservative on $\mathcal{C}^+$, and the t-structure on $\mathcal{C}$ is generated by eventually coconnective compact objects. (We remark that F completely determines the t-structure in this case.) As we will show in Theorem 4.6.1, this equivalence canonically upgrades to a symmetric monoidal one.

Remark 4.2.5. Suppose $\mathcal{C}$ is a compactly generated DG category with a continuous functor $F : \mathcal{C} \to \mathsf{Vect}$. Then F may be pro-represented by a pro-compact object. Comparing with §3.4, we see that this puts significant restrictions on which $\overrightarrow{\otimes}$-algebras A admit renormalization data. (For example, up to convergent completion, $A \in \mathsf{ProVect}$ must be expressible as a filtered limit of some discrete A-modules that are *almost compact*, i.e., whose truncations are compact in $A\text{–mod}^{\geq -n}_{naive}$ for all n.)

4.3. **Examples.** We now give some examples of renormalization data.

We begin with examples when A is discrete, i.e., $A \in \mathsf{Vect}^{\leq 0} \subseteq \mathsf{ProVect}^{\leq 0}$.

Example 4.3.1 (Ind-coherent sheaves). Let A be a commutative, connective k-algebra (almost) of finite type and let $S = \mathrm{Spec}(A)$. Recall that $\mathsf{IndCoh}(S)$ ($:= \mathsf{Ind}(\mathsf{Coh}(S))$) equipped with the tautological embedding $\mathsf{Coh}(S) \hookrightarrow \mathsf{QCoh}(S)^+ = A\text{–mod}^+$ defines a renormalization datum for A.

Example 4.3.2. More generally, if A is a left coherent[19] k-algebra, then define $A\text{–mod}_{coh} \subseteq A\text{–mod}$ as the subcategory of bounded complexes with finitely presented cohomologies. Then $A\text{–mod}_{ren} :=$ $\mathsf{Ind}(A\text{–mod}_{coh})$ defines a renormalization datum.

It is straightforward to show that this renormalization datum is initial among all renormalization data for A.

Example 4.3.3 (Quasi-coherent sheaves). If A is a connective associative k-algebra, then $A\text{–mod}$ itself underlies a renormalization datum if and only if A is eventually coconnective, i.e., A is also bounded below as a complex of vector spaces. Indeed, recall that for renormalization data, there is an assumption that the category be compactly generated by eventually coconnective objects, and $A\text{–mod}$ is compactly generated by perfect ones.

---

[19]Recall that an algebra is left coherent if it is connective; the category $A\text{–mod}_{coh}$ defined below is actually a DG category, i.e., it is closed under cones; and $\tau^{\geq -n} A \in A\text{–mod}_{coh}$ for all n. For example, this is the case if A is left Noetherian.

We now give some examples involving honestly topological algebras.

Example 4.3.4. Suppose $S = \operatorname{colim}_i S_i$ is an ind-affine indscheme of ind-finite type. Then $\mathsf{IndCoh}(S)$ is naturally a renormalization for the pro-algebra of functions on S. Indeed, this is a special case of Example 4.3.5.

Example 4.3.5 (Pro-algebras). Suppose $i \mapsto A_i \in \mathsf{Alg}(\mathsf{Vect}^{\leq 0})$ is a projective system of algebras. Let $A = \lim_i A_i \in \mathsf{Pro}(\mathsf{Vect})$. Then A is a $\overset{!}{\otimes}$-algebra, and a posteriori a $\overset{\to}{\otimes}$-algebra.

Suppose that:
- A_i is left coherent.
- Each structural map $\varphi_{ij} : A_i \to A_j$ is surjective on H^0 with finitely generated kernel.

Let A_i–mod_{ren} be as in Example 4.3.2. Note that our assumptions imply that restriction along φ_{ij} maps A_j–mod_{coh} to A_i–mod_{coh}. By ind-extension, we obtain t-exact functors A_j–$\mathsf{mod}_{ren} \to A_i$–$\mathsf{mod}_{ren}$.

Then define:

$$A\text{–}\mathsf{mod}_{ren} := \operatorname*{colim}_i A_i\text{–}\mathsf{mod}_{ren} \in \mathsf{DGCat}_{cont}.$$

Here the structural functors are the above functors. We claim that A–mod_{ren} naturally defines a renormalization datum.

As noted above, these functors preserve compact objects, so A–mod_{ren} is compactly generated. Moreover, by [Ras6] Lemma 5.4.3 (1), there is a canonical t-structure on A–mod_{ren} such that each functor $\operatorname{res}_i : A_i$–$\mathsf{mod}_{ren} \to A$–$\mathsf{mod}_{ren}$ is t-exact. This t-structure is tautologically compactly generated and right complete.

Moreover, there is a canonical functor $\mathsf{Oblv} : A$–$\mathsf{mod}_{ren} \to \mathsf{Vect} \in \mathsf{DGCat}_{cont}$ pro-represented by the object:

$$\lim_{i,n} \operatorname{res}_i \tau^{\geq -n} A_i \in \mathsf{Pro}(A\text{–}\mathsf{mod}_{ren}).$$

It is straightforward to show that each composition $\mathsf{Oblv} \circ \operatorname{res}_i : A_i$–$\mathsf{mod}_{ren} \to \mathsf{Vect}$ is the canonical forgetful functor on A_i–mod_{ren}. It immediately follows that Oblv is t-exact.

Moreover, we claim that Oblv is conservative on bounded below objects. Indeed, in the above pro-system, all objects are connective and all structural maps are surjective on H^0. As the objects $\operatorname{res}_i H^0(A_i)$ tautologically generate A–$\mathsf{mod}_{ren}^\heartsuit$ under colimits, this implies the claim.

Finally, it suffices to note that at the level of bounded below derived categories, this functor Oblv defines A under the dictionary of Proposition 3.7.1: indeed, Oblv of this pro-generator is manifestly the Postnikov completion of A in $\mathsf{ProVect}$.

Remark 4.3.6. In Example 4.3.5, compact objects in A–mod_{ren} are closed under truncations.

Example 4.3.7. This example appears somewhat in the wrong place: it uses some terminology from §4.4, and is really motivated by Example 4.3.8.

Suppose A is a connective $\overset{\to}{\otimes}$-algebra and that we are given a morphism $\varphi : A_0 \to A$ of connective $\overset{\to}{\otimes}$-algebras such that the forgetful functor A–$\mathsf{mod}_{naive}^+ \to A_0$–$\mathsf{mod}_{naive}^+$ is monadic.[20]

Denote this monad by T. Now suppose moreover that the composition:

$$A_0\text{–}\mathsf{mod}_{naive}^+ \xrightarrow{T} A_0\text{–}\mathsf{mod}_{naive}^+ \xhookrightarrow{\rho} A_0\text{–}\mathsf{mod}_{ren}$$

---

[20]In particular, this functor admits a left adjoint. So if A is discrete and $A_0 = k$, this forces A to be eventually coconnective.

renormalizes in the sense of §4.4. (E.g., this is automatic if A_0–mod_{ren} is given by Example 4.3.5.)

Then T clearly induces a monad on A_0–mod_{ren}, and A–$\mathsf{mod}_{ren} := T$–$\mathsf{mod}(A_0$–$\mathsf{mod}_{ren})$ obviously defines a renormalization datum for A.

Example 4.3.8 (Tate Lie algebras). Suppose $\mathfrak{h} \in \mathsf{ProVect}^\heartsuit$ is a Tate Lie algebra. By this, we mean that the dual Tate vector space $\mathfrak{h}^\vee \in \mathsf{ProVect}$ is given a coLie algebra structure with respect to the $\overset{!}{\otimes}$ symmetric monoidal structure. Recall that in this case, $\mathfrak{h}$ necessarily admits an open profinite dimensional subalgebra $\mathfrak{h}_0 \subseteq \mathfrak{h}$, where these hypotheses force $\mathfrak{h}_0 = \lim_i \mathfrak{h}_i$ for $\mathfrak{h}_i$ ranging over the finite dimensional Lie algebra quotients of $\mathfrak{h}_0$.

(For example, we might have $\mathfrak{h} = \mathfrak{g}((t))$ for finite dimensional $\mathfrak{g}$; then $\mathfrak{h}_0$ may be taken as $\mathfrak{g}[[t]]$ and $\mathfrak{h}_i = \mathfrak{g}[[t]]/t^i\mathfrak{g}[[t]]$.)

Then $A_0 = U(\mathfrak{h}_0) := \lim_i U(\mathfrak{h}_i)$ satisfies the hypotheses of Example 4.3.5 (c.f. Example 4.4.4 regarding renormalization of the monad). Note that each $U(\mathfrak{h}_i)$–$\mathsf{mod}_{ren} = U(\mathfrak{h}_i)$–$\mathsf{mod}$ here, so objects restrictions of modules $U(\mathfrak{h}_i)$ give compact generators of $U(\mathfrak{h}_0)$–mod_{ren}.

Moreover, $A = U(\mathfrak{h})$ the completed enveloping algebra of $\mathfrak{h}$, $A_0 \to A$ satisfies the hypotheses of Example 4.3.7 (say, by the PBW theorem). In particular, we obtain $U(\mathfrak{h})$–mod_{ren}.

Following Gaitsgory, we denote these DG categories by $\mathfrak{h}_0$–mod and $\mathfrak{h}$–mod, leaving renormalization out of the notation.

Note that the construction of $\mathfrak{h}$–mod recovers the format of [FG2] §23. Indeed, unwinding the constructions, we find that compact generators are given by inducing trivial modules from $\mathfrak{k}_i$ to $\mathfrak{h}$ for $\mathfrak{k}_i := \mathrm{Ker}(\mathfrak{h}_0 \to \mathfrak{h}_i)$.

Example 4.3.9. Renormalization data is given for the affine $\mathcal{W}$-algebra in [Ras6]: the compact generators are denoted $\mathcal{W}^n_\kappa$ in *loc. cit.* Outside of the Virasoro case, this example does not fit into any of the above patterns. (This is closely related to the fact that the $\mathcal{W}$-algebra chiral algebras are generally neither commutative nor chiral envelopes.)

4.4. Construction of functors. Let A be a renormalized $\overrightarrow{\otimes}$-algebra.

Suppose $\mathcal{C} \in \mathsf{DGCat}_{cont}$ and that we are given a DG functor $F : A$–$\mathsf{mod}^+_{ren} \simeq A$–$\mathsf{mod}^+_{naive} \to \mathcal{C}$.

Definition 4.4.1. F *renormalizes* if it is left Kan extended from A–mod^c_{ren}.

For F as above (not necessarily assumed to renormalize), we define $F_{ren} : A$–$\mathsf{mod}_{ren} \to \mathcal{C}$ as the ind-extension of $F|_{A\text{–}\mathsf{mod}^c_{ren}}$. Note that $F_{ren}|_{A\text{–}\mathsf{mod}^+_{ren}}$ is the left Kan extension of $F|_{A\text{–}\mathsf{mod}^c_{ren}}$; therefore, F renormalizes if and only if the natural map $F_{ren}|_{A\text{–}\mathsf{mod}^+_{ren}} \to F$ is an isomorphism.

Suppose now that $\mathcal{C}$ admits a t-structure compatible with filtered colimits, that F is t-exact, and that $F|_{A\text{–}\mathsf{mod}^{\geq 0}_{ren}}$ commutes with filtered colimits.

Warning 4.4.2. It is not true in this generality that F necessarily renormalizes: F_{ren} may fail to be (left) t-exact. (See Counterexample 4.5.4.)

However, we claim:

$$\tau^{\geq 0} F_{ren} = \tau^{\geq 0} F \tag{4.4.1}$$

when restricted to A–mod^+_{ren}.

Indeed, for $\mathcal{F} \in A$–mod^+_{ren}, write $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ with $\mathcal{F}_i$ compact. Then:

$$\tau^{\geq 0} F_{ren}(\mathcal{F}) = \tau^{\geq 0} \mathrm{colim}_i F(\mathcal{F}_i) = \mathrm{colim}_i F(\tau^{\geq 0}\mathcal{F}_i) = F(\tau^{\geq 0}\mathrm{colim}_i \mathcal{F}_i) = F(\tau^{\geq 0}\mathcal{F}).$$

There are two general settings in which F does renormalize.

Example 4.4.3. If the t-structure on $\mathcal{C}$ is left separated, then (4.4.1) clearly implies that F renormalizes.

Example 4.4.4. Suppose merely that F is left t-exact (or left t-exact up to shift) and that compact objects of A–mod_{ren} are closed under truncations. Then we claim that F renormalizes. Indeed, then every $\mathcal{F} \in A$–$\mathsf{mod}_{ren}^{\geq 0}$ can be written as a filtered colimit $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ with $\mathcal{F}_i$ compact and in A–$\mathsf{mod}_{ren}^{\geq 0}$: write $\mathcal{F}$ as a filtered colimit of arbitrary compacts and then apply $\tau^{\geq 0}$. Then we obtain:

$$F_{ren}(\mathcal{F}) = \mathrm{colim}_i F(\mathcal{F}_i) \xrightarrow{\simeq} F(\mathcal{F})$$

by assumption that F commutes with filtered colimits in A–$\mathsf{mod}_{ren}^{\geq 0}$.

Example 4.4.5 (Forgetful functors). The forgetful functor $\mathrm{Oblv} : A$–$\mathsf{mod}_{naive}^+ \to \mathsf{Vect}$ renormalizes to give a functor $\mathrm{Oblv}_{ren} : A$–$\mathsf{mod}_{ren} \to \mathsf{Vect}$ by Example 4.4.3. In what follows, we typically abbreviate the notation Oblv_{ren} to simply Oblv. (Although we call this functor *forgetful*, it is not generally conservative.)

Example 4.4.6 (Identity functor). The embedding A–$\mathsf{mod}_{naive}^+ \hookrightarrow A$–$\mathsf{mod}_{naive}$ renormalizes to give a continuous functor $\mathrm{id}_{ren} : A$–$\mathsf{mod}_{ren} \to A$–$\mathsf{mod}_{naive} \in \mathsf{DGCat}_{cont}$, again by Example 4.4.3.

4.5. **Morphisms.** We have the following notion of compatibility between algebra morphisms and renormalization data.

Definition 4.5.1. A *morphism* of renormalized $\overrightarrow{\otimes}$-algebras is a map $f : A \to B$ of $\overrightarrow{\otimes}$ such that the (t-exact) functor $\mathrm{Oblv} : B$–$\mathsf{mod}_{naive}^+ \to A$–$\mathsf{mod}_{naive}^+ \subseteq A$–$\mathsf{mod}_{ren}$ renormalizes to a functor $\mathrm{Oblv} = \mathrm{Oblv}_{ren} : B$–$\mathsf{mod}_{ren} \to A$–$\mathsf{mod}_{ren}$.

We let $\mathsf{Alg}_{ren}^{\overrightarrow{\otimes}}$ denote the category of renormalized algebras and such morphisms.

Remark 4.5.2. We emphasize that this is a property, not a structure, for the underlying map of $\overrightarrow{\otimes}$-algebras.

Example 4.5.3. Example 4.4.5 says that the unit map $k \to A$ is a morphism of renormalized $\overrightarrow{\otimes}$-algebras. More generally, this is true for any map from an eventually coconnective algebra with the "trivial" renormalization from Example 4.3.3.

Counterexample 4.5.4. Let A be a (discrete) almost finite type, eventually coconnective commutative k-algebra with $S = \mathrm{Spec}(A)$ singular.

Take A–$\mathsf{mod}_{ren_1} = \mathsf{IndCoh}(S)$ and A–$\mathsf{mod}_{ren_2} = \mathsf{QCoh}(S)$, and let us pedantically write A_{ren_1}, A_{ren_2} for the corresponding renormalized algebras. Then the identity map for A defines a morphism $A_{ren_1} \to A_{ren_2}$ of renormalized algebras, but not a morphism $A_{ren_2} \to A_{ren_1}$.

4.6. **Tensor products.** We now revisit the material of §3.3 in the presence of renormalizations. So suppose A and B are renormalized $\overrightarrow{\otimes}$-algebras.

Then we claim that A–$\mathsf{mod}_{ren} \otimes B$–$\mathsf{mod}_{ren}$ defines a renormalization datum for $A \overset{!}{\otimes} B$.

More precisely, define:

$$A \overset{!}{\otimes} B\text{–}\mathsf{mod}_{ren}^c \subseteq A \overset{!}{\otimes} B\text{–}\mathsf{mod}_{naive}$$

as the DG subcategory Karoubi generated by the essential image of the composition:

$$A\text{–}\mathsf{mod}_{ren}^c \times B\text{–}\mathsf{mod}_{ren}^c \to A\text{–}\mathsf{mod}_{naive} \otimes B\text{–}\mathsf{mod}_{naive} \xrightarrow{(3.3.1)} A \overset{!}{\otimes} B\text{–}\mathsf{mod}_{naive}.$$

Now define $A \overset{!}{\otimes} B$–mod_{ren} as $\mathsf{Ind}(A \overset{!}{\otimes} B$–$\mathsf{mod}_{ren}^c)$.

Theorem 4.6.1.   *(1) $A \overset{!}{\otimes} B$–mod_{ren} is a renormalization datum for $A \overset{!}{\otimes} B$.[21]*
   (2) The natural functor:

$$A\text{–}\mathsf{mod}_{ren} \otimes B\text{–}\mathsf{mod}_{ren} \to A \overset{!}{\otimes} B\text{–}\mathsf{mod}_{ren}$$

   is an equivalence.

Lemma 4.6.2. *Suppose $\mathcal{C}, \mathcal{D}_1, \mathcal{D}_2 \in \mathsf{DGCat}_{cont}$ have t-structures compatible with filtered colimits and $F : \mathcal{D}_1 \to \mathcal{D}_2 \in \mathsf{DGCat}_{cont}$ is a functor.*

   Recall that $\mathcal{C} \otimes \mathcal{D}_i$ admits a canonical t-structure with $(\mathcal{C} \otimes \mathcal{D}_i)^{\leq 0}$ generated under colimits by objects $\mathcal{F} \boxtimes \mathcal{G}$ for $\mathcal{F} \in \mathcal{C}^{\leq 0}$ and $\mathcal{G} \in \mathcal{D}_i^{\leq 0}$.

   (1) If F is right t-exact, then so is $\mathrm{id}_{\mathcal{C}} \otimes F : \mathcal{C} \otimes \mathcal{D}_1 \to \mathcal{C} \otimes \mathcal{D}_2$.
   (2) If the t-structure on $\mathcal{C}$ is compactly generated and F is left t-exact, then $\mathrm{id}_{\mathcal{C}} \otimes F$ is left t-exact.
   (3) Under the assumptions of (2), if the t-structure on $\mathcal{C}$ is right complete and $F|_{\mathcal{D}_1^{\geq 0}}$ is conservative, then $\mathrm{id}_{\mathcal{C}} \otimes F|_{(\mathcal{C} \otimes \mathcal{D}_1)^{\geq 0}}$ is conservative.

Proof. (1) is immediate. (2) is shown e.g. in [Ras6] Lemma B.6.2, but we recall the argument as it is used also for (3).

Let $\mathcal{F} \in \mathcal{C}^{\leq 0}$ be compact. Then $\mathcal{F}$ defines a continuous functor $\mathbb{D}\mathcal{F} := \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, -) : \mathcal{C} \to \mathsf{Vect}$. We can tensor to obtain:

$$\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_i} : \mathcal{C} \otimes \mathcal{D}_i \to \mathcal{D}_i.$$

As in the proof of [Ras6] Lemma B.6.2, if $\mathcal{F} \in \mathcal{C}^{\leq 0}$, then this functor is left t-exact, and conversely, $\mathcal{G} \in \mathcal{C} \otimes \mathcal{D}_i$ lies in cohomological degrees ≥ 0 if and only if $\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_i}(\mathcal{G}) \in \mathcal{D}_i^{\geq 0}$ for each such $\mathcal{F}$. These facts immediately imply (2).

Now for (3), suppose $\mathcal{G} \in (\mathcal{C} \otimes \mathcal{D}_1)^{\geq 0}$ with $(\mathrm{id}_{\mathcal{C}} \otimes F)(\mathcal{G}) = 0$. Then for any $\mathcal{F}$ as above, we claim:

$$(\mathbb{D}_{\mathcal{F}} \otimes \mathrm{id}_{\mathcal{D}_1})(\mathcal{G}) = 0 \in \mathcal{D}_1.$$

Indeed, this object lies in degrees ≥ 0, so it suffices to show that F applied to it is zero. Then:

$$F(\mathbb{D}\mathcal{F} \otimes \mathrm{id}_{\mathcal{D}_1})(\mathcal{G}) = (\mathbb{D}\mathcal{F} \otimes F)(\mathcal{G}) = \mathbb{D}\mathcal{F}(\mathrm{id}_{\mathcal{C}} \otimes F)(\mathcal{G}) = 0.$$

Now right completeness of the (compactly generated) t-structure on $\mathcal{C}$ is equivalent to $\mathcal{C}$ being compactly generated by objects $\mathcal{F}$ of the above type (and their shifts), so this implies that $\mathcal{G} = 0$ as desired.

$\square$

Proof of Theorem 4.6.1. Note that $A\text{–}\mathsf{mod}_{ren} \otimes B\text{–}\mathsf{mod}_{ren}$ admits a canonical t-structure, as in the statement of Lemma 4.6.2. This t-structure is obviously compactly generated by objects bounded from below, since this is true for each of the tensor factors.

By Lemma 4.6.2, the functor:

$$\mathrm{Oblv} = \mathrm{Oblv}_A \otimes \mathrm{Oblv}_B : A\text{–}\mathsf{mod}_{ren} \otimes B\text{–}\mathsf{mod}_{ren} \to \mathsf{Vect}$$

is t-exact and conservative on $(A\text{–}\mathsf{mod}_{ren} \otimes B\text{–}\mathsf{mod}_{ren})^{\geq 0}$.[22]

---

[21]C.f. Remark 4.2.3: because $A \overset{!}{\otimes} B\text{–}\mathsf{mod}_{ren}^c$ is tautologically embedded into $A \overset{!}{\otimes} B\text{–}\mathsf{mod}_{naive}^+$, this is a property, not a structure.

[22]Note that Lemma 4.6.2 as formulated should be applied to the functor:

By Remark 4.2.4, there is some convergent, connective $\overrightarrow{\otimes}$-algebra C such that A–mod_{ren} $\otimes$ B–mod_{ren} with its forgetful functor defines a renormalization datum for C.

Since the forgetful functor A–$\mathsf{mod}_{ren} \otimes B$–$\mathsf{mod}_{ren} \to \mathsf{Vect}$ lifts to $A \overset{!}{\otimes} B$–mod_{naive} (by Example 4.4.6 and §3.3), we have a canonical map $A \overset{!}{\otimes} B \to C$ of $\overrightarrow{\otimes}$-algebras. To prove the theorem, it suffices to show that this map realizes C as the convergent completion of $A \overset{!}{\otimes} B$.

For this, suppose $i \mapsto \mathcal{F}_i \in A$–$\mathsf{mod}^c_{ren}$ and $j \mapsto \mathcal{G}_j \in B$–$\mathsf{mod}^c_{ren}$ pro-represent the forgetful functors. Clearly $\lim_{i,j} \mathcal{F}_i \boxtimes \mathcal{G}_j \in \mathsf{Pro}(A$–$\mathsf{mod}_{ren} \otimes B$–$\mathsf{mod}_{ren})$ pro-represents the forgetful functor to vector spaces. As in §3.4, the object:

$$\lim_{i,j} \mathrm{Oblv}(\mathcal{F}_i \boxtimes \mathcal{G}_j)$$

is canonically isomorphic to $C \in \mathsf{ProVect}$. But we can calculate this object as:[23]

$$\lim_{i,j} \mathrm{Oblv}(\mathcal{F}_i \boxtimes \mathcal{G}_j) = \lim_{i,j} \mathrm{colim}_{k,\ell} \underline{\mathrm{Hom}}_{A\text{–}\mathsf{mod}_{ren} \otimes B\text{–}\mathsf{mod}_{ren}}(\mathcal{F}_k \boxtimes \mathcal{G}_\ell, \mathcal{F}_i \boxtimes \mathcal{G}_j) =$$

$$\lim_{i,j} \mathrm{colim}_{k,\ell} \left(\underline{\mathrm{Hom}}_{A\text{–}\mathsf{mod}_{ren}}(\mathcal{F}_k, \mathcal{F}_i) \otimes \underline{\mathrm{Hom}}_{B\text{–}\mathsf{mod}_{ren}}(\mathcal{G}_\ell, \mathcal{G}_j) \right) =$$

$$\lim_i \mathrm{Oblv}(\mathcal{F}_i) \overset{!}{\otimes} \lim_j \mathrm{Oblv}(\mathcal{G}_j).$$

This last term is the $\overset{!}{\otimes}$-tensor product of the convergent completions of A and B respectively, giving the claim.

$\square$

This construction obviously equips $\mathsf{Alg}^{\overrightarrow{\otimes}}_{ren}$ with a unique symmetric monoidal structure such that the forgetful functor to $\mathsf{Alg}^{\overrightarrow{\otimes}}$ is symmetric monoidal. For this symmetric monoidal structure, the functor:

$$\mathsf{Alg}^{\overrightarrow{\otimes}}_{ren} \to \mathsf{DGCat}_{cont}$$
$$A \mapsto A\text{–}\mathsf{mod}_{ren}$$

is symmetric monoidal by construction.

5. Weak actions of group schemes

5.1.   In this section, we begin a study of action of (suitable) group indschemes H on $\overrightarrow{\otimes}$-algebras and on categories.

We will explain, following Gaitsgory, that (under suitable hypotheses) there is a naive notion of weak H-action on a DG category, and a less naive notion, which we call *genuine weak actions*.

The bulk of this section is devoted to developing the notion of genuine actions when H is a classical affine group scheme. With that said, this section begins with a general discussion of naive actions on categories and $\overrightarrow{\otimes}$-algebras in the case of general ind-affine group indschemes.

─────────────

$$\mathrm{id} \otimes \mathrm{Oblv} : A\text{–}\mathsf{mod}_{ren} \otimes B\text{–}\mathsf{mod}_{ren} \to A\text{–}\mathsf{mod}_{ren}.$$

[23]The second equality is a general fact about maps out of external products of two compact objects.

5.2. **Topological bialgebras.** A $\overrightarrow{\otimes}$-*bialgebra* B is a coalgebra B in the symmetric monoidal category $(\mathsf{Alg}^{\overrightarrow{\otimes}}, \overset{!}{\otimes})$. In particular, such a B is equipped with an $\overrightarrow{\otimes}$-algebra structure and is equipped with a coproduct $\Delta : B \to B \overset{!}{\otimes} B$ that is a morphism of $\overrightarrow{\otimes}$-algebras.

There is a natural notion of *coaction* of such a B on a $\overrightarrow{\otimes}$-algebra A. Here we have a coaction map $\mathrm{coact} : A \to B \overset{!}{\otimes} A$, which is a map of $\overrightarrow{\otimes}$-algebras (and satisfies higher compatibilities with Δ and so on).

Variant 5.2.1. A $\overset{!}{\otimes}$-*bialgebra* is a bialgebra in the symmetric monoidal category $(\mathsf{ProVect}, \overset{!}{\otimes})$. Any such object has an underlying $\overrightarrow{\otimes}$-bialgebra structure.

Note that in the $\overset{!}{\otimes}$-setting, *commutative* and *cocommutative* $\overset{!}{\otimes}$-bialgebra structures have evident meaning, while in the $\overrightarrow{\otimes}$-setting, only *cocommutative* $\overrightarrow{\otimes}$-bialgebra structures make sense.

5.3. In the above setting, note that B–$\mathsf{mod}_{naive} \in \mathsf{DGCat}_{cont}$ inherits a canonical monoidal DG structure. For example, the monoidal operation is given by:

$$B\text{–}\mathsf{mod}_{naive} \otimes B\text{–}\mathsf{mod}_{naive} \xrightarrow{\S 3.3} B \overset{!}{\otimes} B\text{–}\mathsf{mod}_{naive} \xrightarrow{\Delta_*} B\text{–}\mathsf{mod}_{naive}$$

where Δ_* is restriction of module structures along the map Δ.

Similarly, if B coacts on A, then B–mod_{naive} acts on A–mod_{naive}.

5.4. Now suppose that B is given a renormalization datum. Recall from §4.6 that $\mathsf{Alg}^{\overrightarrow{\otimes}}_{ren}$ is a symmetric monoidal category.

Therefore, it makes sense to say that a bialgebra structure on B is *compatible* with the renormalization datum on B: this means that the counit and comultiplication maps are morphisms of renormalized $\overrightarrow{\otimes}$-algebras. Similarly, for $A \in \mathsf{Alg}^{\overrightarrow{\otimes}}_{ren}$, we may speak of a coaction of B on A being compatible with the given renormalization data: this means the coaction data makes A a comodule for B in the symmetric monoidal category $\mathsf{Alg}^{\overrightarrow{\otimes}}_{ren}$.

In such cases, B–mod_{ren} inherits a canonical monoidal structure and B–mod_{ren} acts on A–mod_{ren}.

5.5. **Group setting.** Now suppose that H is an ind-affine group indscheme. We suppose H is *reasonable* in the sense of [BD1] (or §6.8 below): that is, $H = \mathrm{colim}\, H_i$ for $H_i \subseteq H$ eventually coconnective quasi-compact quasi-separated subschemes[24] with all maps $H_i \to H_j$ almost finitely presented.

Then $B = \mathrm{Fun}(H) := \lim_i \Gamma(H_i, \mathcal{O}_{H_i}) \in \mathsf{ProVect}$ is a commutative $\overset{!}{\otimes}$-bialgebra, and in particular inherits a $\overrightarrow{\otimes}$-bialgebra structure.

We say that H *naively acts* on $A \in \mathsf{Alg}^{\overrightarrow{\otimes}}$ if B coacts on A. We let $\mathsf{Alg}^{\overrightarrow{\otimes}, H \curvearrowright}$ denote the category of $\overrightarrow{\otimes}$-algebras with naive H-actions (i.e., the category of B-comodules in $\mathsf{Alg}^{\overrightarrow{\otimes}}$).

5.6. **Naive group actions on categories.** Assume in the above notation that each of the (commutative) algebras $\Gamma(H_i, \mathcal{O}_{H_i})$ are coherent, as in Example 4.3.2. Then B admits a canonical renormalization as in *loc. cit.* We define $\mathsf{IndCoh}^*(H) := B\text{–}\mathsf{mod}_{ren}$.

Remark 5.6.1. In §6, we will define IndCoh^* in much greater generality. However, this elementary definition coincides in the present setting.

---

[24] For emphasis: the H_i may not necessarily be group subschemes.

Remark 5.6.2. The notation is taken from [Ras3] (see also [Gai7]), to which we refer for an explanation. The main purpose of this notation is to remind us that to avoid the pitfalls inherent in working with IndCoh in the infinite type setting.

Example 5.6.3. Suppose that H is the loop group $G(K)$ for G an affine algebraic group.[25] By [GR3], there is a canonical equivalence $\mathsf{QCoh}(G(K)) \simeq \mathsf{IndCoh}^*(G(K))$ defined as such. But we note that this equivalence uses the compact open subgroup $G(O)$ in an essential way: the functor $\mathrm{Oblv} : \mathsf{IndCoh}^*(G(K)) \to \mathsf{Vect}$, which tautologically exists in the above definition, corresponds to the composition:

$$\mathsf{QCoh}(G(K)) \xrightarrow{\pi_*} \mathsf{QCoh}(\mathrm{Gr}_G) \overset{-\otimes\omega_{\mathrm{Gr}_G}}{\underset{\simeq}{\longrightarrow}} \mathsf{IndCoh}(\mathrm{Gr}_G) \xrightarrow{\Gamma^{\mathsf{IndCoh}}(\mathrm{Gr}_G,-)} \mathsf{Vect}.$$

Here $\pi : G(K) \to \mathrm{Gr}_G = G(K)/G(O)$ is the projection, and IndCoh is defined in the standard sense on Gr_G because it is of ind-finite type; the rest of the notation is standard in the subject, and the functor of tensoring with the dualizing sheaf is an equivalence by a theorem of [GR3] (and formal smoothness of Gr_G).

Definition 5.6.4. A *naive weak action* of H on $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is an $\mathsf{IndCoh}^*(H)$-module[26] structure for $\mathcal{C}$.

Remark 5.6.5. The antipode for H induces a canonical equivalence between left and right modules for $\mathsf{IndCoh}^*(H)$, so we often ignore the distinction going forward.

Remark 5.6.6. We sometimes omit "weak": the distinction between naive and genuine actions in this section only occurs for weak group actions, not for strong group actions.

Example 5.6.7. H has a canonical naive action on $\mathsf{IndCoh}^*(H)$.

Example 5.6.8. H has a canonical naive action on Vect. Indeed, H naively acts on k with coaction given by the unit map, and this is compatible with renormalization.

Example 5.6.9. For any indscheme X of ind-finite type with an H action, H naively acts on $\mathsf{IndCoh}(X)$.

Example 5.6.10. If H acts on a Tate Lie algebra $\mathfrak{k}$, then H naively weakly acts on $\mathfrak{k}$–mod, (defined as in Example 4.3.8). In particular, H weakly acts on $\mathfrak{h}$–mod.

5.7. We let $\mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$ denote the category of renormalized $\vec{\otimes}$-algebras with naive H-actions compatible with the renormalization, i.e., the category of B-comodules in $\mathsf{Alg}^{\vec{\otimes}}$ for $B = \Gamma(H, \mathcal{O}_H)$ equipped with the renormalization datum $\mathsf{IndCoh}^*(H)$.

Note that for $A \in \mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$, H acts naively on A–mod_{ren} (c.f. §5.4).

5.8. We let H–$\mathsf{mod}_{weak,naive}$ denote the 2-category of categories with a naive weak action of H, i.e., $\mathsf{IndCoh}^*(H)$–$\mathsf{mod}(\mathsf{DGCat}_{cont})$.

For $\mathcal{C} \in \mathsf{IndCoh}^*(H)$–$\mathsf{mod}$, we define the *naive weak invariants* and *coinvariants* as:

$$\mathcal{C}^{H,w,naive} := \mathsf{Hom}_{H-\mathsf{mod}_{weak,naive}}(\mathsf{Vect}, \mathcal{C}), \qquad \mathcal{C}_{H,w,naive} := \mathsf{Vect} \underset{\mathsf{IndCoh}^*(H)}{\otimes} \mathcal{C}.$$

---

[25]In particular, G is classical and finite type.

[26]Here we are considering $\mathsf{IndCoh}^*(H)$ as an algebra object in DGCat_{cont}, so e.g. the action functor is $\mathsf{IndCoh}^*(H)\otimes \mathcal{C} \to \mathcal{C}$. In particular, the induced action $\mathsf{IndCoh}^*(H) \times \mathcal{C} \to \mathcal{C}$ commutes with colimits in each variable separately.

5.9. **Genuine actions.** In the remainder of this section and in §7, we study a more robust variant of the above notion, under somewhat more restrictive hypotheses. In this section, we focus on the case where H is profinite dimensional, which contains the main phenomena.

5.10. **Finite dimensional reminder.** We first remind the reader of the following foundational result, which will play a key role.

Let H be an affine algebraic group. In this case, we remove the label "naive" from the notation, e.g., $H\text{–mod}_{weak} = H\text{–mod}_{weak,naive}$ — the naiveté is only in the infinite type setting.

Theorem 5.10.1 (Gaitsgory, [Gai8]). *For H an affine algebraic group, the functor:*

$$H\text{–mod}_{weak} \to \mathsf{Rep}(H)\text{–mod} = \mathsf{Rep}(H)\text{–mod}(\mathsf{DGCat}_{cont})$$

$$\mathcal{C} \mapsto \mathcal{C}^{H,w}$$

is an equivalence. (Here the functor exists because $\mathsf{Rep}(H) := \mathsf{QCoh}(\mathbb{B}H)$ is tautologically isomorphic to $\mathsf{Hom}_{H\text{–mod}_{weak}}(\mathsf{Vect}, \mathsf{Vect})$ as a monoidal category.)

5.11. **Profinite dimensional setting.** In the remainder of this section, we suppose H is a classical affine group scheme.

5.12. Let $B = \mathsf{Fun}(H) \in \mathsf{Vect}^\heartsuit$ as before. Because H can be written as a limit of smooth schemes under smooth morphisms, the tautological functor $B\text{–mod}_{ren} = \mathsf{IndCoh}^*(H) \to \mathsf{QCoh}(H) = B\text{–mod}_{naive}$ is an equivalence.

5.13. We begin with a remark in the naive setting.

Note that the Beck-Chevalley conditions apply for the cosimplicial diagram defining $\mathcal{C}^{H,w,naive}$. Therefore, $\mathsf{Oblv} : \mathcal{C}^{H,w,naive} \to \mathcal{C}$ admits a continuous right adjoint $\mathsf{Av}_*^{w,naive}$, and Oblv is comonadic. The comonad on $\mathcal{C}$ is given by convolution with the coalgebra $\mathcal{O}_H$ in the monoidal category $\mathsf{QCoh}(H)$.

In particular, for $\mathcal{C} = \mathsf{Vect}$, we obtain that $\mathsf{Rep}(H)_{naive} := \mathsf{Vect}^{H,w,naive}(= \mathsf{QCoh}(\mathbb{B}H))$ is canonically equivalent to the category of B-comodules (with B as above).

5.14. We now define $\mathsf{Rep}(H) = \mathsf{Rep}(H)_{ren}$ as $\mathsf{Ind}(\mathsf{Rep}(H)^c))$ for $\mathsf{Rep}(H)^c \subseteq \mathsf{Rep}(H)_{naive}$ the full subcategory generated by finite dimensional representations, i.e., objects whose image in Vect is compact.[27] Since $\mathsf{Rep}(H)^c$ is closed under tensor products in $\mathsf{Rep}(H)_{naive}$, $\mathsf{Rep}(H)$ is a rigid symmetric monoidal DG category.

Note that $\mathsf{Rep}(H)$ carries a canonical t-structure for which the forgetful functor to Vect is t-exact. We have $\mathsf{Rep}(H)^+ \simeq \mathsf{Rep}(H)_{naive}^+$.

5.15. The following definition plays a key role.

Definition 5.15.1. The category $H\text{–mod}_{weak}$ of categories *with a genuine*[28] *weak H-action* is $\mathsf{Rep}(H)\text{–mod} = \mathsf{Rep}(H)\text{–mod}(\mathsf{DGCat}_{cont})$.

---

[27]Note that this example fits into the formalism of §4. Indeed, B is the union of its finite dimensional sub-coalgebras, so $B^\vee \in \mathsf{ProVect}$ is a profinite dimensional algebra, in particular, an $\overrightarrow{\otimes}$-algebra. Its modules are tautologically the same as B-comodules. This definition of $\mathsf{Rep}(H)$ is then obtained by applying Example 4.3.5.

[28]The terminology is borrowed from equivariant homotopy theory. In that context, for finite H, one extends the *naive* notion of H-action on a spectrum in such a way that the trivial representation (and more generally, any permutation representation) becomes compact. This is somewhat analogous to the present context, where we *renormalize* $H\text{–mod}_{weak,naive}$ so that the trivial representation Vect becomes (completely) compact.

Although the subtleties in our context only occur for group schemes (which are analogous to profinite groups) and group indschemes (which are analogous to locally compact totally disconnected groups), we still find this analogy to be somewhat evocative.

Construction 5.15.2. Note that $\mathsf{Rep}(H)_{naive} = \mathsf{Hom}_{H-\mathsf{mod}_{weak,naive}}(\mathsf{Vect}, \mathsf{Vect})$. In particular, Vect admits commuting actions of $\mathsf{Rep}(H)_{naive}$ and $\mathsf{QCoh}(H)$. In particular, since $\mathsf{Rep}(H)_{ren} \to \mathsf{Rep}(H)_{naive}$ is symmetric monoidal, Vect is a bimodule for $\mathsf{Rep}(H)_{ren}$ and $\mathsf{QCoh}(H)$, and therefore tensoring defines a functor:

$$H-\mathsf{mod}_{weak} \to H-\mathsf{mod}_{weak,naive}.$$

Notation 5.15.3. Following the case of finite dimensional H, we think of the underlying object of DGCat_{cont} as the weak H-invariants of a DG category acted on by H.

To accommodate this, suppose we are given an object of $H-\mathsf{mod}_{weak}$. By definition, this means that we are given an object $\mathcal{D} \in \mathsf{Rep}(H)-\mathsf{mod}$. We use the notation $\mathcal{C}^{H,w}$ in place of $\mathcal{D}$, where we let $\mathcal{C}$ denote the underlying object of $H-\mathsf{mod}_{weak,naive}$. We then abusively write $\mathcal{C} \in H-\mathsf{mod}_{weak}$ to summarize the situation.

Roughly, the reader should think $\mathcal{C} \in H-\mathsf{mod}_{weak}$ means that $\mathcal{C} \in H-\mathsf{mod}_{weak,naive}$, and we are given a "correction" $\mathcal{C}^{H,w}$ to $\mathcal{C}^{H,w,naive}$.

We emphasize that this "forgetful functor" $H-\mathsf{mod}_{weak} \to \mathsf{DGCat}_{cont}$ (factoring through $H-\mathsf{mod}_{weak,naive}$) is *not* conservative.

Remark 5.15.4. Because $\mathsf{Rep}(H)$ is rigid monoidal, Vect is dualizable over $\mathsf{Rep}(H)$. Therefore, the functor $H-\mathsf{mod}_{weak} \to H-\mathsf{mod}_{weak,naive}$ admits left and right adjoints. It is immediate to see that they are computed as strong and weak invariants respectively, with $\mathsf{Rep}(H)$ acting through $\mathsf{Rep}(H)_{naive}$.

In particular, for $\mathcal{C} \in H-\mathsf{mod}_{weak}$, there is a canonical functor:

$$\mathcal{C}^{H,w} \to \mathcal{C}^{H,w,naive}.$$

It is not difficult to see that each of these functors $H-\mathsf{mod}_{weak,naive} \to H-\mathsf{mod}_{weak}$ are fully-faithful. Indeed, it is well-known that it suffices to verify this for either functor, and for the right adjoint this is the content of [Ras4] Proposition 3.5.1 (which is proved by a standard Beck-Chevalley argument).

Example 5.15.5. We have a canonical object $\mathsf{Vect} \in H-\mathsf{mod}_{weak}$ with $\mathsf{Vect}^{H,w} = \mathsf{Rep}(H)$. Clearly $\mathsf{Hom}_{H-\mathsf{mod}_{weak}}(\mathsf{Vect}, \mathcal{C}) = \mathcal{C}^{H,w}$.

Example 5.15.6. By Theorem 5.10.1, genuine and naive actions coincide in the finite dimensional case. It is straightforward to show that if $H = \prod_{i=1}^{\infty} \mathbb{G}_a$, then $\mathsf{Rep}(H)$ is not equivalent to $\mathsf{Rep}(H)_{naive}$, so the two notions do not coincide in this case.

Remark 5.15.7. The relationship between $H-\mathsf{mod}_{weak}$ and $H-\mathsf{mod}_{weak,naive}$ is somewhat analogous to the relationship between IndCoh and QCoh, though it occurs a categorical level higher. Namely, there is a non-conservative functor $H-\mathsf{mod}_{weak} \to H-\mathsf{mod}_{weak,naive}$ analogous to the functor $\Psi : \mathsf{IndCoh}(S) \to \mathsf{QCoh}(S)$ for an eventually coconnective Noetherian scheme S, and in both cases, there are fully-faithful left and right adjoints.

5.16.   The key advantage of genuine H-actions is that the theory completely reduces to the finite dimensional setting, as we now discuss.

Indeed, recall that H is a limit $\lim_i H_i$ of affine algebraic groups under smooth[29] surjective maps.

Let $K_i \subseteq H$ denote the kernel of the map $H \to H_i$. Note that there is a canonical functor $H-\mathsf{mod}_{weak} \to H_i-\mathsf{mod}_{weak}$, sending $\mathcal{C}$ to:

―――――――――
[29]Of course we are using characteristic zero in an essential way.

$$\mathcal{C}^{K_i,w} := \mathcal{C}^{H,w} \underset{\mathsf{Rep}(H_i)}{\otimes} \mathsf{Vect}.$$

That is, we apply the restriction along the tensor functor $\mathsf{Rep}(H_i) \to \mathsf{Rep}(H)$ and the inverse to Theorem 5.10.1.

Proposition 5.16.1. *The induced functor:*

$$H\text{–}\mathsf{mod}_{weak} \to \lim_i H_i\text{–}\mathsf{mod}_{weak}$$

is an equivalence.

This follows immediately from the next lemma.

Lemma 5.16.2. *The morphism:*

$$\operatorname*{colim}_i \mathsf{Rep}(H_i) \to \mathsf{Rep}(H) \in \mathsf{ComAlg}(\mathsf{DGCat}_{cont})$$

is an equivalence.

Proof. This is a special case of Example 4.3.5.

□

Corollary 5.16.3. *For any $\mathcal{C} \in H\text{–}\mathsf{mod}_{weak}$, the functor:*

$$\operatorname*{colim}_i \mathcal{C}^{K_i,w} \to \mathcal{C} \in \mathsf{DGCat}_{cont}$$

is an equivalence. Moreover, each of the structural functors in this colimit admits a continuous right adjoint.

Corollary 5.16.4. *The functor* $\mathrm{Oblv} : \mathcal{C}^{H,w} \to \mathcal{C}$ *admits a continuous right adjoint* Av^w_*.

5.17. **Functoriality.** Suppose $f : H_1 \to H_2$ is a morphism of classical affine group schemes.
    We claim that there are induced adjoint functors:

$$\mathrm{ind}^w : H_1\text{–}\mathsf{mod}_{weak} \rightleftarrows H_2\text{–}\mathsf{mod}_{weak} : \mathrm{Res}$$

with the *weak induction* functor ind^w *also* canonically isomorphic to the right adjoint to Res.
    Indeed, unwinding the definitions, $H_i\text{–}\mathsf{mod}_{weak} \simeq \mathsf{Rep}(H_i)\text{–}\mathsf{mod}$, and we take ind^w to correspond to restriction of module categories along the symmetric monoidal functor $\mathsf{Rep}(H_2) \to \mathsf{Rep}(H_1)$. This functor obviously admits a left adjoint $\mathsf{Rep}(H_1) \otimes_{\mathsf{Rep}(H_2)} -$, which is defined to be Res. Then Res is both left and right adjoint because $\mathsf{Rep}(H_1)$ is self-dual as a $\mathsf{Rep}(H_2)$-module category by general properties of rigid monoidal DG categories, c.f. [Gai4].
    In particular, taking $H \to \mathrm{Spec}(k)$, we see $\mathrm{Res} : \mathsf{DGCat}_{cont} \to H\text{–}\mathsf{mod}_{weak}$ sends Vect to itself with the trivial H-action. The equality of left and right adjoints here should be interpreted as an "invariants = coinvariants" statement for genuine H-actions. We remark that the corresponding statement is false in the setting of naive weak actions.

5.18. **Genuine actions via canonical renormalization.** The following is a typical construction of genuine weak H-actions.

Suppose H acts naively on $\mathcal{C}$. Suppose moreover that $\mathcal{C}$ is equipped with a t-structure such that $\operatorname{Oblv} \operatorname{Av}_*^{w,naive} : \mathcal{C} \to \mathcal{C}$ is t-exact. Then $\mathcal{C}^{H,w,naive}$ has a (unique) t-structure such that $\operatorname{Oblv} : \mathcal{C}^{H,w,naive} \to \mathcal{C}$ is t-exact (c.f. the proof of Proposition 5.18.3 (1) below). Note that the functor $\operatorname{Av}_*^{w,naive}$ is also t-exact in this case.

In what follows, we let $\mathcal{C}^{H,w,c} \subseteq \mathcal{C}^{H,w,naive}$ denote the (non-cocomplete) DG subcategory of objects $\mathcal{F} \in \mathcal{C}^{H,w,naive}$ with $\operatorname{Oblv}(\mathcal{F})$ compact in $\mathcal{C}$.

Definition 5.18.1. In the above setting, we say the naive action of H on $\mathcal{C}$ *canonically renormalizes (compatibly with the t-structure)* if:

(1) $\mathcal{C}$ and its t-structure are compactly generated.
(2) Compact objects in $\mathcal{C}$ are bounded (i.e., eventually connective and coconnective).
(3) The essential image of the functor $\operatorname{Oblv} : \mathcal{C}^{H,w,c} \cap \mathcal{C}^{H,w,naive,\leqslant 0} \to \mathcal{C}^c \cap \mathcal{C}^{\leqslant 0}$ Karoubi generates. (Here $\mathcal{C}^c \subseteq \mathcal{C}$ is the subcategory of compact objects.)

Remark 5.18.2. Note that under the above hypotheses, the functor $\mathcal{C}^{H,w,c} \to \mathcal{C}^c$ Karoubi generates.

The following result summarizes the main features of this setting.

Proposition 5.18.3. *Suppose H acts naively on $\mathcal{C}$, $\mathcal{C}$ is equipped with a t-structure compatible with the H-action, and suppose the H-action canonically renormalizes.*

Define $\mathcal{C}^{H,w}$ as $\operatorname{Ind}(\mathcal{C}^{H,w,c})$; as $\mathcal{C}^{H,w,c} \subseteq \mathcal{C}^{H,w,naive}$ is a $\operatorname{Rep}(H)^c$-submodule category, $\mathcal{C}^{H,w}$ has a canonical $\operatorname{Rep}(H)$-module structure.

Let ψ denote the canonical functor:

$$\psi : \mathcal{C}^{H,w} \to \mathcal{C}^{H,w,naive} \in \operatorname{DGCat}_{cont}$$

ind-extending the embedding $\mathcal{C}^{H,w,c} \hookrightarrow \mathcal{C}^{H,w,naive}$. Note that ψ is a morphism of $\operatorname{Rep}(H)$-module categories.

We use a standard abuse of notation in letting $\operatorname{Oblv} : \mathcal{C}^{H,w} \to \mathcal{C}$ denote the composition $\mathcal{C}^{H,w} \xrightarrow{\psi} \mathcal{C}^{H,w,naive} \xrightarrow{\operatorname{Oblv}} \mathcal{C}$.

(1) *$\mathcal{C}^{H,w,naive}$ admits a unique t-structure such that such that the (conservative) forgetful functor to $\mathcal{C}$ is t-exact.*
(2) *$\mathcal{C}^{H,w}$ has a unique compactly generated t-structure such that the forgetful functor to $\mathcal{C}$ is t-exact and conservative on eventually coconnective subcategories.*
(3) *For $V \in \operatorname{Rep}(H)^{\heartsuit}$, the action functors $V \star - : \mathcal{C}^{H,w} \to \mathcal{C}^{H,w}$ and $V \star - : \mathcal{C}^{H,w,naive} \to \mathcal{C}^{H,w,naive}$ are t-exact.*
(4) *The functor ψ is t-exact and an equivalence on eventually coconnective subcategories:*

$$\psi : \mathcal{C}^{H,w,+} \xrightarrow{\simeq} \mathcal{C}^{H,w,naive,+}.$$

(5) *The composition:[30]*

$$\mathcal{C}^{H,w} \underset{\operatorname{Rep}(H)}{\otimes} \operatorname{Vect} \to \mathcal{C}^{H,w,naive} \underset{\operatorname{Rep}(H)}{\otimes} \operatorname{Vect} \to \mathcal{C}$$

is an equivalence. In particular, $\mathcal{C}^{H,w} \in \operatorname{Rep}(H)-\operatorname{mod}$ induces a canonical genuine H-action on $\mathcal{C}$.

---

[30]In fact, each functor here is an equivalence.

(6) Let $\mathrm{Av}^w_* : \mathcal{C} \to \mathcal{C}^{H,w}$ denote the right adjoint to Oblv. Then the induced natural transformation:

$$\psi \circ \mathrm{Av}^w_* \to \mathrm{Av}^{w,naive}_*$$

(of functors $\mathcal{C} \to \mathcal{C}^{H,w,naive}$) is an isomorphism.

Proof. (1) follows by noting that $\mathcal{C}^{H,w,naive}$ is the totalization $\mathrm{Tot}\left(\mathcal{C} \otimes \mathsf{QCoh}(H)^{\otimes \bullet}\right)$ and all of the structural maps in the underlying semi-cosimplicial diagram are t-exact.[31]

In (2), the uniqueness is clear: the t- structure must have $\mathcal{C}^{H,w,\leq 0}$ generated under colimits by $\mathcal{C}^{H,w,c} \cap \mathcal{C}^{H,w,naive,\leq 0}$. As is standard, this does define a t-structure, and the forgetful functor to $\mathcal{C}$ is clearly right t-exact. We will complete the proof of (2) later in the argument; but now, we will verify that (3) holds for this t-structure (without relying on any as yet unproved parts of (2)).

For $V \in \mathsf{Rep}(H)^\heartsuit$, we first show that the functor $V \star - : \mathcal{C}^{H,w,naive} \to \mathcal{C}^{H,w,naive}$ is t-exact. Here it suffices to verify this after applying Oblv. But $\mathrm{Oblv} \circ (V \star -) = V \otimes \mathrm{Oblv}(-)$, which is clearly t-exact.

To see $V \star - : \mathcal{C}^{H,w} \to \mathcal{C}^{H,w}$ is t-exact, note that we can assume V is finite dimensional (because the t-structure on $\mathcal{C}^{H,w}$, being compactly generated, is compatible with filtered colimits). By the naive case, this functor preserves $\mathcal{C}^{H,w,c} \cap \mathcal{C}^{H,w,naive,\leq 0}$, so we obtain right t-exactness in the genuine setting. Now $V \star -$ is right adjoint to the left t-exact functor $V^\vee \star -$, giving the left t-exactness.

We now make an auxiliary observation. Suppose $\mathcal{G} \in \mathcal{C}^{H,w,c} \subseteq \mathcal{C}^{H,w,naive}$. By assumption, $\mathcal{G}$ lies in cohomological degrees $\geq -N$ for $N \gg 0$. We claim that $\mathcal{G}$ is actually compact as an object of $\mathcal{C}^{H,w,naive,\geq -N}$. Indeed, because $\mathcal{G}$ is eventually connective, the functor:

$$\mathrm{Hom}_{\mathcal{C}^{H,w,naive}}(\mathcal{G}, -) : \mathcal{C}^{H,w,naive,\geq -N} \to \mathsf{Gpd}$$

factors through the subcategory of M-truncated groupoids for some $M \gg 0$ (depending on N and $\mathcal{G}$). Moreover, we have:

$$\mathrm{Hom}_{\mathcal{C}^{H,w,naive}}(\mathcal{G}, -) \xrightarrow{\simeq} \mathrm{Tot}\, \mathrm{Hom}_{\mathcal{C} \otimes \mathsf{QCoh}(H)^{\otimes \bullet}}(\mathrm{Oblv}(\mathcal{G}) \boxtimes \mathcal{O}_H^{\boxtimes \bullet}, -)$$

where each of these functors factors through M-truncated groupoids. Therefore, we have $\mathrm{Tot} \xrightarrow{\simeq} \mathrm{Tot}^{\leq M+1}$ here, so commuting finite limits with filtered colimits in Gpd gives the claim about $\mathcal{G}$.

Using this observation, we will now show (6). Note that the canonical natural transformation:

$$\mathrm{Fun}(H) \star - \to \mathrm{Av}^{w,naive}_* \mathrm{Oblv} \in \mathsf{Hom}(\mathcal{C}^{H,w,naive}, \mathcal{C}^{H,w,naive}) \tag{5.18.1}$$

is an isomorphism, where here $\star$ denotes the action of $\mathsf{Rep}(H)$ on $\mathcal{C}^{H,w,naive}$ and $\mathrm{Fun}(H)$ is the regular representation in $\mathsf{Rep}(H)^\heartsuit \subseteq \mathsf{Rep}(H)$. Indeed, the identification[32] $\mathcal{C} = \mathcal{C}^{H,w,naive} \otimes_{\mathsf{Rep}(H)} \mathsf{Vect}$ and the Beck-Chevalley formalism imply this.

We now similarly claim that there is a canonical isomorphism:

$$\mathrm{Fun}(H) \star - \xrightarrow{\simeq} \mathrm{Av}^w_* \mathrm{Oblv} : \mathcal{C}^{H,w} \to \mathcal{C}^{H,w}.$$

Both functors commute with colimits, so it suffices to verify that for every $\mathcal{F} \in \mathcal{C}^{H,w,c}$, the natural map:

$$\mathrm{Fun}(H) \star \mathcal{F} \to \mathrm{Av}^w_* \mathrm{Oblv}(\mathcal{F})$$

---

[31]Note that this argument is general for naive H-actions and compatible t-structures. I.e., it is not specific to canonical renormalization.

[32]This follows from identifying both sides with $\mathrm{Fun}(H)-\mathsf{mod}(\mathcal{C})$ using Barr-Beck and the Beck-Chevalley formalism.

is a natural isomorphism. Let $\mathcal{G} \in \mathcal{C}^{H,w,c}$ be given. Write $\mathrm{Fun}(H)$ as a filtered colimit $\mathrm{colim}_i V_i$ where $V_i \in \mathsf{Rep}(H)^{\heartsuit}$ are finite dimensional. We claim that:

$$\mathrm{colim}_i \underline{\mathrm{Hom}}_{\mathcal{C}^{H,w,c}}(\mathcal{G}, V_i \star \mathcal{F}) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathcal{C}^{H,w,naive}}(\mathcal{G}, \mathrm{colim}_i V_i \star \mathcal{F}).$$

Indeed, because $\mathcal{F}$ and $\mathcal{G}$ are eventually coconnective, by (3) (in the naive case), we have $V_i \star \mathcal{F}, \mathcal{F}, \mathcal{G} \in \mathcal{C}^{H,w,naive,\geqslant -N}$ for $N \gg 0$ (and for all i). Then the fact that $\mathcal{G}$ is compact in $\mathcal{C}^{H,w,naive,\geqslant -N-r}$ for all $r \geqslant 0$ gives the claim.

Therefore, we have:

$$\underline{\mathrm{Hom}}_{\mathcal{C}^{H,w}}(\mathcal{G}, \mathrm{Fun}(H) \star \mathcal{F}) = \underline{\mathrm{Hom}}_{\mathcal{C}^{H,w}}(\mathcal{G}, \mathrm{colim}_i V_i \star \mathcal{F}) = \mathrm{colim}_i \underline{\mathrm{Hom}}_{\mathcal{C}^{H,w}}(\mathcal{G}, V_i \star \mathcal{F}) =$$

$$\mathrm{colim}_i \underline{\mathrm{Hom}}_{\mathcal{C}^{H,w,c}}(\mathcal{G}, V_i \star \mathcal{F}) = \underline{\mathrm{Hom}}_{\mathcal{C}^{H,w,naive}}(\mathcal{G}, \mathrm{colim}_i V_i \star \mathcal{F}) = \underline{\mathrm{Hom}}_{\mathcal{C}^{H,w,naive}}(\mathcal{G}, \mathrm{Av}_*^{w,naive}(\mathcal{F})) =$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathrm{Oblv}(\mathcal{G}), \mathrm{Oblv}(\mathcal{F})).$$

To complete the argument, note that both functors $\mathrm{Av}_*^{w,naive}$ and $\psi \, \mathrm{Av}_*^w$ commute with colimits, so it suffices to show that our natural transformation is an equivalence when evaluated on compact objects in $\mathcal{C}$. Moreover, because the naive H-action on $\mathcal{C}$ canonically renormalizes, it suffices to check this on compact objects of the form $\mathrm{Oblv}(\mathcal{F})$ for $\mathcal{F} \in \mathcal{C}^{H,w,c}$. But then $\mathrm{Av}_*^{w,naive} \mathrm{Oblv}$ and $\psi \, \mathrm{Av}_*^w \mathrm{Oblv}$ are each canonically given by the action of $\mathrm{Fun}(H)$, and ψ is $\mathsf{Rep}(H)$-linear, giving the claim.

Returning to (2), we claiom that for $\mathcal{F} \in \mathcal{C}^{H,w,\geqslant 0}$ we have $\mathrm{Oblv}(\mathcal{F}) \in \mathcal{C}^{\geqslant 0}$. Note that $\mathrm{Av}_*^w \mathrm{Oblv}(\mathcal{F}) = \mathrm{Fun}(H) \star \mathcal{F}$ as before, and since $\mathrm{Fun}(H)$ is in degree 0, $\mathrm{Fun}(H) \star \mathcal{F}$ is also in degrees $\geqslant 0$. Therefore, for $\mathcal{G} \in \mathcal{C}^{H,w,c} \cap \mathcal{C}^{H,w,naive,<0}$, we have:

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{Oblv}\,\mathcal{G}, \mathrm{Oblv}\,\mathcal{F}) = \mathrm{Hom}_{\mathcal{C}^{H,w}}(\mathcal{G}, \mathrm{Av}_*^w \mathrm{Oblv}\,\mathcal{F}) = 0 \in \mathsf{Gpd}$$

By hypothesis, $\mathcal{C}^{<0}$ is generated under colimits by such objects $\mathrm{Oblv}\,\mathcal{G}$, giving the claim.

To conclude (2), we need to show that Oblv is conservative on eventually coconnective objects. Suppose $\mathcal{F} \in \mathcal{C}^{H,w,\geqslant 0}$ with $\mathrm{Oblv}(\mathcal{F}) = 0$. Then we have:

$$\mathcal{F}[1] = \mathrm{Coker}(\mathcal{F} \to \mathrm{Av}_*^w \mathrm{Oblv}(\mathcal{F})) = \mathrm{Coker}(k \xrightarrow{1 \mapsto 1} \mathrm{Fun}(H)) \star \mathcal{F}.$$

By (3), the right hand side is in $\mathcal{C}^{H,w,\geqslant 0}$, so $\mathcal{F}[1] \in \mathcal{C}^{H,w,\geqslant 0}$, so $\mathcal{F} \in \mathcal{C}^{H,w,\geqslant 1}$. Iterating this, we obtain $\mathcal{F} = 0$ as desired.

Then (4) follows from (2) and (6) by observing that these results imply that the forgetful functor $\mathcal{C}^{H,w,+} \to \mathcal{C}^+$ is comonadic with comonad given by the action of $\mathrm{Fun}(H) \in \mathsf{QCoh}(H)$.

It remains to show (5). By the Beck-Chevalley formalism, $\mathcal{C}^{H,w} \xrightarrow{\mathcal{F} \mapsto \mathcal{F} \boxtimes_{\mathsf{Rep}(H)} k} \mathcal{C}^{H,w} \otimes_{\mathsf{Rep}(H)} \mathsf{Vect}$ admits a conservative right adjoint, and the corresponding monad on $\mathcal{C}^{H,w}$ is the action of $\mathrm{Fun}(H) \in \mathsf{Rep}(H)$. Using our canonical identification of that convolution with $\mathrm{Av}_*^w \mathrm{Oblv}$, we obtain that the functor from (5) is fully-faithful. By Remark 5.18.2, it is also essentially surjective.

$\square$

5.19. Canonical renormalization for IndCoh. We also have the following variant.

Lemma 5.19.1. *Suppose that X is an indscheme locally almost of finite type acted on by H. Then the naive H-action on $\mathsf{IndCoh}(X)$ canonically renormalizes (relative to its canonical t-structure).*

Proof. We use the following construction. Suppose $\mathcal{F} \in \mathsf{IndCoh}(X)^{H,w,naive,\heartsuit}$ and $\mathcal{G} \subseteq \mathrm{Oblv}(\mathcal{F})$ is a subobject. Define a subobject $\widetilde{\mathcal{G}} \subseteq \mathcal{F} \in \mathsf{IndCoh}(X)^{H,w,naive,\heartsuit}$ as the fiber product (in the *abelian* category $\mathsf{IndCoh}(X)^{H,w,naive,\heartsuit}$):

$$
\begin{array}{ccc}
\widetilde{\mathcal{G}} = \mathcal{F} \underset{\mathrm{Av}^w_* \mathrm{Oblv}(\mathcal{F})}{\times} \mathrm{Av}^w_*(\mathcal{G}) & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow {\scriptstyle \mathrm{coact}} \\
\mathrm{Av}^w_*(\mathcal{G}) & \longrightarrow & \mathrm{Av}^w_* \mathrm{Oblv}(\mathcal{F}).
\end{array}
$$

Observe that $\mathrm{Oblv}(\widetilde{\mathcal{G}}) \subseteq \mathcal{G} \subseteq \mathrm{Oblv}(\mathcal{F})$ (using the counit of the adjunction).[33]

Now if $\mathcal{G}$ is coherent, then $\mathrm{Oblv}(\widetilde{\mathcal{G}})$ is as well (because we are in a Noetherian setup). Moreover, the map $\mathrm{SubObj}(\mathrm{Oblv}(\mathcal{F})) \xrightarrow{\mathcal{G} \mapsto \widetilde{\mathcal{G}}} \mathrm{SubObj}(\mathcal{F})$ commutes with filtered colimits (because we are taking fiber products in a Grothendieck abelian category). Therefore, we have:

$$
\mathcal{F} = \underset{\mathcal{G} \subseteq \mathrm{Oblv}(\mathcal{F}) \text{ coherent}}{\mathrm{colim}} \widetilde{\mathcal{G}}.
$$

Applying Oblv, we see that $\mathrm{Oblv}(\mathcal{F})$ is a filtered colimit of objects coming from $\mathsf{IndCoh}(X)^{H,w,c}$. Since such objects $\mathrm{Oblv}(\mathcal{F})$ generate $\mathsf{IndCoh}(X)^{\leq 0}$ under colimits (since Av^w_* is t-exact and conservative), this gives the claim. $\qquad\square$

5.20. **Varying the group.** We record the following result for later use.

Lemma 5.20.1. *Suppose $f : H_1 \to H_2$ is a morphism of classical affine group schemes. Suppose $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is equipped with a t-structure and a compatible naive action of H_2 that renormalizes. Then:*

(1) *The induced naive H_1-action also renormalizes.*
(2) *The category:*

$$
\mathsf{Rep}(H_1) \underset{\mathsf{Rep}(H_2)}{\otimes} \mathcal{C}^{H_2,w}
$$

is compactly generated, and the natural functor:

$$
\mathsf{Rep}(H_1) \underset{\mathsf{Rep}(H_2)}{\otimes} \mathcal{C}^{H_2,w} \to \mathcal{C}^{H_1,w,naive} \tag{5.20.1}
$$

maps compact objects to objects in $\mathcal{C}^{H_1,w,c}$ (with this subcategory defined as in Proposition 5.18.3).
(3) *The functor:*

$$
\left(\mathsf{Rep}(H_1) \underset{\mathsf{Rep}(H_2)}{\otimes} \mathcal{C}^{H_2,w}\right)^c \to \mathcal{C}^{H_1,w,c}
$$

induced by (5.20.1) is fully-faithful, so induces a fully-faithful functor:

$$
\mathsf{Rep}(H_1) \underset{\mathsf{Rep}(H_2)}{\otimes} \mathcal{C}^{H_2,w} \to \mathcal{C}^{H_1,w}. \tag{5.20.2}
$$

---

[33]In fact, $\widetilde{\mathcal{G}}$ is maximal among subobjects of $\mathcal{F}$ with this property.

(4) *If* $\mathcal{C} = \mathsf{IndCoh}(X)$ *for* X *locally almost of finite type and acted on by* H_2 *(as in Lemma 5.19.1), then (5.20.2) is an equivalence.*

(5) *If* H_1 *and* H_2 *are algebraic groups, then (5.20.2) is an equivalence.*

Proof. (1) is immediate from the definitions.

In (2), note that all the categories appearing in the construction computing the tensor product $\mathsf{Rep}(H_1) \otimes_{\mathsf{Rep}(H_2)} \mathcal{C}^{H_2,w}$ are compactly generated, and each of the functors in the underlying semisimplicial diagram preserve compact objects. Therefore, this tensor product is compactly generated by objects of the form $V \boxtimes_{\mathsf{Rep}(H_2)} \mathcal{F}$ for $V \in \mathsf{Rep}(H_1)^c$ and $\mathcal{F} \in \mathcal{C}^{H_2,w,c}$. Moreover, the functor (5.20.1) sends this object to $V \star \mathrm{Oblv}(\mathcal{F})$, where Oblv denotes the functor $\mathcal{C}^{H_2,w,naive} \to \mathcal{C}^{H_1,w,naive}$ and V denotes the action of $\mathsf{Rep}(H_1)$ on $\mathcal{C}^{H_1,w,naive}$. Clearly this object lies in $\mathcal{C}^{H_1,w,c}$, giving the claim.

For (3), let $\mathrm{Ind}_{H_1}^{H_2} : \mathsf{Rep}(H_1) \to \mathsf{Rep}(H_2)$ denote the (continuous) right adjoint to the restriction functor. Because $\mathrm{Ind}_{H_1}^{H_2}$ is a morphism of $\mathsf{Rep}(H_2)$-module categories (by rigid monoidality of $\mathsf{Rep}(H_2)$), we see that $\mathrm{Ind}_{H_1}^{H_2} \underset{\mathsf{Rep}(H_2)}{\otimes} \mathrm{id}_{\mathcal{C}^{H_2,w}}$ is right adjoint to the functor $(\mathcal{F} \in \mathcal{C}^{H_2,w}) \mapsto k \boxtimes_{\mathsf{Rep}(H_2)} \mathcal{F}$ (for k the trivial representation).

Now let $\mathcal{F}, \mathcal{G} \in \mathcal{C}^{H_2,w}$ and let $V, W \in \mathsf{Rep}(H_1)$ be given with $W \in \mathsf{Rep}(H_1)^c$. By the above, we have:

$$\underline{\mathrm{Hom}}_{\mathsf{Rep}(H_1) \underset{\mathsf{Rep}(H_2)}{\otimes} \mathcal{C}^{H_2,w}}\left(W \underset{\mathsf{Rep}(H_2)}{\boxtimes} \mathcal{G}, V \underset{\mathsf{Rep}(H_2)}{\boxtimes} \mathcal{F}\right) =$$

$$\underline{\mathrm{Hom}}_{\mathsf{Rep}(H_1) \underset{\mathsf{Rep}(H_2)}{\otimes} \mathcal{C}^{H_2,w}}\left(k \underset{\mathsf{Rep}(H_2)}{\boxtimes} \mathcal{G}, (W^\vee \otimes V) \underset{\mathsf{Rep}(H_2)}{\boxtimes} \mathcal{F}\right) =$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}^{H_2,w}}\left(\mathcal{G}, \mathrm{Ind}_{H_1}^{H_2}(W^\vee \otimes V) \star \mathcal{F}\right).$$

Now if we assume $\mathcal{F}, \mathcal{G} \in \mathcal{C}^{H_2,w,+}$ and $V \in \mathsf{Rep}(H_1)^+$, then by Proposition 5.18.3 the above $\underline{\mathrm{Hom}}$ maps isomorphically (via the functor ψ from *loc. cit.*) onto:

$$\underline{\mathrm{Hom}}_{\mathcal{C}^{H_2,w,naive}}\left(\mathcal{G}, \mathrm{Ind}_{H_1}^{H_2}(W^\vee \otimes V) \star \mathcal{F}\right). \tag{5.20.3}$$

Before calculating this term further, let $\mathrm{Oblv}^{H_2 \to H_1} : \mathcal{C}^{H_2,w,naive} \to \mathcal{C}^{H_1,w,naive}$ be the restriction functor and let $\mathrm{Av}_*^{w,naive,H_1 \to H_2} : \mathcal{C}^{H_1,w,naive} \to \mathcal{C}^{H_2,w,naive}$ denote its right adjoint. Note that although this latter functor may not commute with colimits,[34] its restriction to $\mathcal{C}^{H_2,w,naive,\geqslant 0}$ commutes with filtered colimits.[35] We claim that for any $U \in \mathsf{Rep}(H_1)^+$, the natural map:

$$\mathrm{Ind}_{H_1}^{H_2}(U) \star \mathcal{F} \to \mathrm{Av}_*^{w,naive,H_1 \to H_2}\left(U \star \mathrm{Oblv}^{H_2 \to H_1}(\mathcal{F})\right)$$

is an isomorphism. First, if U is the regular representation, this follows from the identification $\mathrm{Av}_*^{w,naive} \mathrm{Oblv} = \mathrm{Fun}(H) \star -$ from (5.18.1). If $U = \mathrm{Fun}(H) \otimes Q$ for $Q \in \mathsf{Vect}^+$, then the claim follows from commutation the fact that $\mathrm{Av}_*^{w,naive,H_1 \to H_2}$ commutes with colimits bounded uniformly from below. Finally, general $U \in \mathsf{Rep}(H_1)^+$ follows using the cobar resolution for U, using the t-structure to justify commuting the totalization with various functors.[36]

Applying this to $U = W^\vee \otimes V$ from above, we calculate (5.20.3) as:

---

[34] For example, if H_2 is a point and $H_1 = \prod_{i=1}^\infty \mathbb{G}_a$.

[35] Indeed, compact generation of $\mathcal{C}^{H_2,w}$ implies $\mathcal{C}^{H_2,w,\geqslant 0} \xrightarrow{\simeq} \mathcal{C}^{H_2,w,naive,\geqslant 0}$ is compactly generated by $\tau^{\geqslant 0}(\mathcal{C}^{H_2,w,c})$; these clearly map into $\tau^{\geqslant 0}(\mathcal{C}^{H_1,w,c})$ under Oblv. So Oblv : $\mathcal{C}^{H_2,w,\geqslant 0} \to \mathcal{C}^{H_1,w,\geqslant 0}$ preserves compacts, so its right adjoint preserves filtered colimits.

[36] More precisely, any truncation $\tau^{\leqslant N}$ applied to this totalization coincides with a suitable finite totalization.

$$\underline{\mathrm{Hom}}_{\mathcal{C}^{H_2,w,naive}}(\mathcal{G}, \mathrm{Ind}_{H_1}^{H_2}(W^\vee \otimes V) \star \mathcal{F}) =$$
$$\underline{\mathrm{Hom}}_{\mathcal{C}^{H_2,w,naive}}(\mathcal{G}, \mathrm{Av}_*^{w,naive,H_1 \to H_2}\left(W^\vee \otimes V \star \mathrm{Oblv}^{H_2 \to H_1}(\mathcal{F})\right)) =$$
$$\underline{\mathrm{Hom}}_{\mathcal{C}^{H_1,w,naive}}(\mathrm{Oblv}^{H_2 \to H_1}(\mathcal{G}), (W^\vee \otimes V) \star \mathrm{Oblv}^{H_2 \to H_1}(\mathcal{F})) =$$
$$\underline{\mathrm{Hom}}_{\mathcal{C}^{H_2,w,naive}}(W \star \mathrm{Oblv}^{H_2 \to H_1}(\mathcal{G}), V \star \mathrm{Oblv}^{H_2 \to H_1}(\mathcal{F})).$$

As the left hand side of (5.20.1) is compactly generated by objects of the form $V \boxtimes_{\mathsf{Rep}(H_2)} \mathcal{F}$ for $\mathcal{F} \in \mathcal{C}^{H_2,w,c}$ and $V \in \mathsf{Rep}(H_2)^c$, this gives fully-faithfulness of (5.20.1) when restricted to compact objects.

Next, (5) follows from Lemma 5.20.2 (applied to $H = H_1$ and $\mathcal{D}$ the essential image of (5.20.2)).

Finally, we show (4). It suffices to show that any object $\mathcal{F} \in \mathsf{IndCoh}(X)^{H_1,w,c}$ lies in the essential image of (5.20.2). Moreover, we can assume $\mathcal{F}$ lies in the heart of the t-structure.

In the course of reductions, we use the following (simple) observation repeatedly: if $f : Y \to X$ is an equivariant map of locally almost of finite type indschemes acted on by H_2, and $\mathcal{F}$ is of the form $f_*^{\mathsf{IndCoh}}(\mathcal{G})$ for some $\mathcal{G} \in \mathsf{IndCoh}(Y)^{H_2,w}$ such that $\mathcal{G}$ lies in the essential image of:

$$\mathsf{Rep}(H_1) \underset{\mathsf{Rep}(H_2)}{\otimes} \mathsf{IndCoh}(Y)^{H_2,w} \to \mathsf{IndCoh}(Y)^{H_1,w}$$

then $\mathcal{F}$ lies in the essential image of the functor:

$$\mathsf{Rep}(H_1) \underset{\mathsf{Rep}(H_2)}{\otimes} \mathsf{IndCoh}(X)^{H_2,w} \to \mathsf{IndCoh}(X)^{H_1,w}.$$

Because $\mathsf{IndCoh}(X)^\heartsuit = \mathsf{IndCoh}(X^{cl})^\heartsuit$ and similarly for equivariant categories, we may assume (by the above) that X is classical. Then X is a colimit $X = \mathrm{colim}\, X_i$ under closed embeddings of finite type classical schemes acted on by H_2. Therefore, applying the above reduction technique again, we may assume X is a classical scheme of finite type.

Now observe that there exists a map $H_2 \to H_2'$ of group schemes with H_2' an affine algebraic group and such that H_2 acts on X through H_2'. We claim that there is a commutative diagram:

$$\begin{array}{ccc} H_1 & \longrightarrow & H_1' \\ \downarrow & & \downarrow \\ H_2 & \longrightarrow & H_2' \end{array}$$

with H_1' again an affine algebraic group and such that the H_1-equivariant structure on $\mathcal{F}$ comes from an H_1'-equivariant structure. Indeed, we can write $H_1 = \lim_i H_{1,i}$ where each $H_{1,i}$ an affine algebraic group over H_2' (so in particular, it acts on X). Then the H_1-equivariant structure on $\mathcal{F}$ is encoded by the coaction map:

$$\mathrm{Oblv}(\mathcal{F}) \to \mathrm{Fun}(H_1) \star \mathrm{Oblv}(\mathcal{F}) \in \mathsf{IndCoh}(X)^\heartsuit.$$

The right hand side is $\mathrm{colim}_i \mathrm{Fun}(H_{1,i}) \star \mathrm{Oblv}(\mathcal{F})$, and because $\mathcal{F}$ is coherent, the above map factors through $\mathrm{Fun}(H_{1,i}) \star \mathrm{Oblv}(\mathcal{F})$ for some i. Since we are in a 1-categorical context here, it suffices to take $H_1' = H_{1,i}$ for such i.

We then have a commutative diagram:

$$\begin{array}{ccc}
\mathsf{Rep}(H_1') \underset{\mathsf{Rep}(H_2')}{\otimes} \mathsf{IndCoh}(X)^{H_2',w} & \longrightarrow & \mathsf{IndCoh}(X)^{H_1',w} \\
\downarrow & & \downarrow \\
\mathsf{Rep}(H_1) \underset{\mathsf{Rep}(H_2)}{\otimes} \mathsf{IndCoh}(X)^{H_2,w} & \longrightarrow & \mathsf{IndCoh}(X)^{H_1,w}.
\end{array}$$

We have lifted $\mathcal{F}$ along the right vertical arrow. By (5), the top arrow is an equivalence. Therefore, $\mathcal{F}$ lies in the essential image of the bottom functor, giving the result.

$\square$

In the course of the above, we appealed to the following result.

Lemma 5.20.2. *Let H be an affine algebraic group acting weakly[37] on $\mathcal{C} \in \mathsf{DGCat}_{cont}$. Suppose $\mathcal{D} \subseteq \mathcal{C}^{H,w}$ be a DG subcategory such that:*

- *$\mathcal{D}$ is closed under colimits and the $\mathsf{Rep}(H)$-action.*
- *$\mathcal{D}$ is compactly generated and the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}^{H,w}$ preserves compact objects.*
- *The composite $\mathcal{D} \hookrightarrow \mathcal{C}^{H,w} \xrightarrow{\mathrm{Oblv}} \mathcal{C}$ generates $\mathcal{C}$ under colimits.*

Then $\mathcal{D} = \mathcal{C}^{H,w}$.

Proof. We will repeatedly use the fact shown in the course of the proof of Proposition 5.18.3 that $\mathrm{Av}_*^w \mathrm{Oblv} = \mathrm{Fun}(H) \star -$ (as endofunctors on $\mathcal{C}^{H,w}$).

First, note that for $\mathcal{G} \in \mathcal{C}$, $\mathrm{Av}_*^w(\mathcal{G}) \in \mathcal{D}$. Indeed, since the functor $\mathrm{Oblv}|_{\mathcal{D}} : \mathcal{D} \to \mathcal{C}$ generates under colimits, it suffices to see that $\mathrm{Oblv}\,\mathrm{Av}_*^w : \mathcal{C}^{H,w} \to \mathcal{C}^{H,w}$ maps $\mathcal{D}$ into itself. But this functor is given by the action of the regular representation in $\mathsf{Rep}(H)$, so by assumption preserves $\mathcal{D}$.

Now let $k \in \mathsf{Rep}(H)$ denote the trivial representation. By Lemma 3.7.2, we have:

$$k \xrightarrow{\simeq} \tau^{\leqslant n} \mathrm{Tot}^{\leqslant n+1}(\mathrm{Av}_*^w \mathrm{Oblv})^{\bullet+1}(k)$$

for all $n \geqslant 0$. Because H is finite type and we are in characteristic 0, $\mathsf{Rep}(H)$ has finite cohomological dimension. Therefore, for $n \gg 0$, the boundary map:

$$\tau^{>n} \mathrm{Tot}^{\leqslant n+1}(\mathrm{Av}_*^w \mathrm{Oblv})^{\bullet+1}(k) \to k[1] \in \mathsf{Rep}(H)$$

is nullhomotopic. Therefore, k is a direct summand of $\mathrm{Tot}^{\leqslant n+1}(\mathrm{Av}_*^w \mathrm{Oblv})^{\bullet+1}(k)$ for $n \gg 0$.

Now for $\mathcal{F} \in \mathcal{C}^{H,w}$, we have $\mathcal{F} = k \star \mathcal{F}$ (for $\star$ denoting the action of $\mathsf{Rep}(H)$), which implies that $\mathcal{F}$ is a direct summand of:

$$\mathrm{Tot}^{\leqslant n+1}(\mathrm{Av}_*^w \mathrm{Oblv})^{\bullet+1}(\mathcal{F}).$$

By the above, this object lies in $\mathcal{D}$, so $\mathcal{F}$ does as well.

$\square$

6. Ind-coherent sheaves on some infinite dimensional spaces

6.1.   This section, wedged as it is between §5 and §7, is an extended digression.

To orient the reader, we provide a somewhat extended introduction.

---

[37]As H is finite type, this simply means $\mathcal{C}$ is a $\mathsf{QCoh}(H)$-module category.

Similarly, when we refer to weak invariants in this lemma, this is what we would call *naive* weak invariants in an infinite type setting. I.e., we are forming these weak invariants without any canonical renormalization or any such.

6.2.   First, let us explain the role this material plays in §7.

In *loc. cit.*, a certain monoidal category denoted $\mathsf{IndCoh}^*_{ren}(K\backslash H/K)$ plays a key role, where H a Tate group indscheme and $K \subseteq H$ is a compact open subgroup (see *loc. cit.* for the terminology).

The definition of this category is not hard: $\mathsf{IndCoh}^*_{ren}(K\backslash H/K)$ is compactly generated, and compact objects are objects of $\mathsf{IndCoh}(H/K)^{K,w,naive}$ that map to $\mathsf{Coh}(H/K) \subseteq \mathsf{IndCoh}(K\backslash H/K)$. However, this description breaks symmetry, so the monoidal structure is not so evident. This problem becomes compounded when we try to compare these categories for different compact open subgroups K, or different groups H, and so on.

Therefore, the ultimate goal of this section is to introduce a class of prestacks we call *renormalizable*, which includes prestacks of the form $K\backslash H/K$, and for which there is a robust theory of ind-coherent sheaves (denoted IndCoh^*_{ren}). As the above description indicates, the most important application of this material in §7 is to resolve some homotopy coherence issues.

6.3.   In the above example, we could avoid breaking the symmetry by regarding $\mathsf{IndCoh}^*_{ren}(K\backslash H/K)$ as $K \times K$-equivariant ind-coherent sheaves on H. However, H is of ind-infinite type, so is outside the usual framework of [Gai5] and [GR4]. Therefore, we first develop IndCoh^* on schemes (possibly of infinite type, but qcqs and eventually coconnective) and *reasonable* indschemes (see §6.8 for the definition).

6.4.   **Definition for schemes.** Let $^{>-\infty}\mathsf{Sch}_{qcqs}$ denote the category of quasi-compact quasi-separated eventually coconnective schemes.

For $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$, we define $\mathsf{Coh}(S) \subseteq \mathsf{QCoh}(S)$ as the full subcategory of objects $\mathcal{F} \in \mathsf{QCoh}(S)^+$ such that $\mathcal{F} \in \mathsf{QCoh}(S)^{\geq -N}$ implies $\mathcal{F}$ is compact in $\mathsf{QCoh}(S)^{\geq -N}$. We define $\mathsf{IndCoh}^*(S)$ as $\mathsf{Ind}(\mathsf{Coh}(S))$, and we define a t-structure on $\mathsf{IndCoh}^*(S)$ by taking connective objects to be generated under colimits by $\mathsf{Coh}(S) \cap \mathsf{QCoh}(S)^{\leq 0}$. Note that there is a canonical continuous functor $\Psi = \Psi_S : \mathsf{IndCoh}^*(S) \to \mathsf{QCoh}(S)$ ind-extending the embedding $\mathsf{Coh}(S) \hookrightarrow \mathsf{QCoh}(S)$.

Lemma 6.4.1. *Under the above hypotheses, the functor $\Psi : \mathsf{IndCoh}^*(S) \to \mathsf{QCoh}(S)$ is t-exact and an equivalence on eventually coconnective subcategories.*

Proof. Clearly Ψ is right t-exact.

Let $\mathcal{G} \in \mathsf{Perf}(S) \subseteq \mathsf{Coh}(S)$ be given. Then the functors:

$$\underline{\mathsf{Hom}}_{\mathsf{QCoh}(S)}(\mathcal{G}, \Psi(-)), \underline{\mathsf{Hom}}_{\mathsf{IndCoh}^*(S)}(\mathcal{G}, -) : \mathsf{IndCoh}^*(S) \to \mathsf{Vect}$$

are canonically isomorphism as both commute with colimits and the restriction of the two to $\mathsf{Coh}(S)$ are clearly equal.

Therefore, for $\mathcal{F} \in \mathsf{IndCoh}^*(S)^{>0}$ and $\mathcal{G} \in \mathsf{Perf}(S) \cap \mathsf{QCoh}(S)^{\leq 0}$, we have:

$$\mathsf{Hom}_{\mathsf{QCoh}(S)}(\mathcal{G}, \Psi(\mathcal{F})) = \mathsf{Hom}_{\mathsf{IndCoh}^*(S)}(\mathcal{G}, \mathcal{F}) = 0.$$

As $\mathsf{QCoh}(S)^{\leq 0}$ is generated under colimits by such $\mathcal{G}$, this implies $\Psi(\mathcal{F}) \in \mathsf{IndCoh}^*(S)^{>0}$, giving the t-exactness.

Now take $\mathcal{F} \in \mathsf{Coh}(S)$. We claim that the natural transformation:

$$\mathsf{Hom}_{\mathsf{IndCoh}^*(S)}(\mathcal{F}, \tau^{\geq 0}(-)) \to \mathsf{Hom}_{\mathsf{QCoh}(S)}(\mathcal{F}, \Psi(\tau^{\geq 0}(-)))$$

of functors $\mathsf{IndCoh}^*(S) \to \mathsf{Gpd}$ is an isomorphism. Indeed, recall that there exists $\mathcal{F}' \in \mathsf{Perf}(S)$ and a map $\mathcal{F}' \to \mathcal{F}$ inducing an isomorphism on $\tau^{\geq 0}$; therefore, we may assume $\mathcal{F} \in \mathsf{Perf}(S)$, and the result follows from the (evident) identity:

$$\mathrm{Hom}_{\mathsf{QCoh}(S)}(\mathcal{F}, \Psi(-)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{IndCoh}^*(S)}(\mathcal{F}, -)$$

for such perfect $\mathcal{F}$.

We immediately obtain that for any $\mathcal{F} \in \mathsf{IndCoh}^*(S)$, the natural transformation:

$$\mathrm{Hom}_{\mathsf{IndCoh}^*(S)}(\mathcal{F}, \tau^{\geq 0}(-)) \to \mathrm{Hom}_{\mathsf{QCoh}(S)}(\mathcal{F}, \Psi(\tau^{\geq 0}(-)))$$

is an isomorphism. Clearly this is equivalent to fully-faithfulness of $\Psi|_{\mathsf{IndCoh}^*(S)^+}$.

As $\mathsf{QCoh}(S)^{\geq 0}$ is generated under colimits by $\tau^{\geq 0}(\mathsf{Perf}(S))$ and this category is in the essential image of Ψ by t-exactness, we obtain that Ψ induces an equivalence on coconnective objects as desired.

$\square$

Remark 6.4.2. The notation IndCoh^* is parallel to similar notation from [Ras3] and is used to emphasize the differences between the Noetherian and non-Noetherian situations. Note that one can dualize to obtain a theory $\mathsf{IndCoh}^!$ parallel to $D^!$ from *loc. cit.* Because IndCoh is canonically self-dual on indschemes locally almost of finite type, we do not include superscripts when working with such objects (since our theory manifestly recovers that of [GR4] in this case).

6.5.   For $f : S \to T$ in $^{>-\infty}\mathsf{Sch}_{qcqs}$, define $f_*^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(T) \in \mathsf{DGCat}_{cont}$ to be the unique left t-exact (continuous DG) fitting into a commutative diagram:

$$
\begin{array}{ccc}
\mathsf{IndCoh}^*(S) & \xrightarrow{f_*^{\mathsf{IndCoh}}} & \mathsf{IndCoh}^*(T) \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi} \\
\mathsf{QCoh}(S) & \xrightarrow{f_*} & \mathsf{QCoh}(T).
\end{array}
$$

As in [Gai5] Proposition 3.2.4, this construction canonically upgrades to a functor $^{>-\infty}\mathsf{Sch}_{qcqs} \to \mathsf{DGCat}_{cont}$.

Remark 6.5.1. For f affine, f_*^{IndCoh} is t-exact.

Notation 6.5.2. For f the projection map $S \to \mathrm{Spec}(k)$, we use the notation $\Gamma^{\mathsf{IndCoh}}(S, -) : \mathsf{IndCoh}^*(S) \to \mathsf{Vect}$ for the corresponding pushforward functor (and similarly for the more general S considered later in this section).

6.6.   Next, we discuss behavior with respect to flat morphisms.

Lemma 6.6.1. *Suppose $f : S \to T \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ is flat. Then f_*^{IndCoh} admits a left adjoint.*

Proof. In this case, the adjoint functors $f^* : \mathsf{QCoh}(T) \rightleftarrows \mathsf{QCoh}(S) : f_*$ preserve the subcategories $\mathsf{QCoh}(-)^{\geq -n}$ for all integers n, and in particular induce an adjunction. It immediately follows that f^* maps $\mathsf{Coh}(T) \subseteq \mathsf{QCoh}(T)$ to $\mathsf{Coh}(S)$, and that the ind-extension of this functor is the sought-after left adjoint.

$\square$

For such flat f, we denote this left adjoint by $f^{*,\mathsf{IndCoh}}$.

Lemma 6.6.2. *For a Cartesian diagram:*

$$S' \xrightarrow{\ \varphi\ } T'$$
$$\downarrow{\scriptstyle \psi} \qquad \downarrow{\scriptstyle g}$$
$$S \xrightarrow{\ f\ } T$$

with g flat and S, T, T' in $^{>-\infty}\mathsf{Sch}_{qcqs}$ (so $S' \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ as well), the natural transformation:

$$g^{*,\mathsf{IndCoh}} f_*^{\mathsf{IndCoh}} \to \varphi_*^{\mathsf{IndCoh}} \psi^{*,\mathsf{IndCoh}}$$

is an isomorphism.

Proof. Clear from the corresponding statement for QCoh.

$\square$

Now by [GR4] Theorem V.1.3.2.2, the functor IndCoh^* extends canonically to a functor:

$$\mathsf{IndCoh}^* : \mathrm{Corr}(^{>-\infty}\mathsf{Sch}_{qcqs})_{all;flat} \to \mathsf{DGCat}_{cont}.$$

Here we are using the notation from *loc. cit.* We remind that the source category is the correspondence category for $^{>-\infty}\mathsf{Sch}_{qcqs}$ in which morphisms from one eventually coconnective scheme S to another T are diagrams:

$$
\begin{array}{ccc}
 & H & \\
{\scriptstyle \alpha}\swarrow & & \searrow{\scriptstyle \beta} \\
S & & T
\end{array}
$$

with α flat; the functor IndCoh^* attaches to such a correspondence the functor $\beta_*^{\mathsf{IndCoh}}\alpha^{*,\mathsf{IndCoh}}$. (We have omitted the "admissible" morphism data from *loc. cit.*; one may take only isomorphisms for our purposes, i.e., only work with a 1-category of correspondences.)

Remark 6.6.3. The above material extends if we replace flatness by finite Tor- dimension. However, we do not need this extension and therefore do not emphasize it.

6.7. We have the following basic result.

Lemma 6.7.1. IndCoh^* *satisfies Zariski descent on* $^{>-\infty}\mathsf{Sch}_{qcqs}$.

Proof. The argument from [Gai5] Proposition 4.2.1 applies in this setting.

$\square$

More generally, we have:

Proposition 6.7.2. IndCoh^* *satisfies flat descent on* $^{>-\infty}\mathsf{Sch}_{qcqs}$ *(for upper-* functors).*

Proof. Let $f : T \to S$ be a faithfully flat map in $^{>-\infty}\mathsf{Sch}_{qcqs}$. By definition, we need to show that:

$$\mathsf{IndCoh}^*(S) \to \mathrm{Tot}_{semi}(\mathsf{IndCoh}^*(T^{\times_S \bullet+1}))$$

is an isomorphism, where Tot_{semi} indicates the limit over the semisimplicial category $\mathbf{\Delta}_{inj}$ (which we use because only the semisimplicial part of the Cech nerve has flat structural maps).

Next, observe that by construction, $\mathsf{IndCoh}^*(S)$ is naturally a $\mathsf{QCoh}(S)$-module category (in DGCat_{cont}), and similarly for T. Moreover, $f^{*,\mathsf{IndCoh}} : \mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(T)$ is $\mathsf{QCoh}(S)$-linear, where $\mathsf{IndCoh}^*(T)$ is a $\mathsf{QCoh}(S)$-module category via $f^* : \mathsf{QCoh}(S) \to \mathsf{QCoh}(T)$.

Therefore, we obtain a functor:

$$f^{*,\mathsf{IndCoh},enh} : \mathsf{IndCoh}^*(S) \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(T) \to \mathsf{IndCoh}^*(T). \qquad (6.7.1)$$

The same argument as in [Gai5] Proposition 4.4.2 shows that this functor is fully-faithful. Note that the essential image of this functor is the subcategory generated under colimits and the $\mathsf{QCoh}(T)$-action by the essential image of $f^{*,\mathsf{IndCoh}}$.

Next, observe that the above constructions are suitably functorial and therefore induce a fully-faithful functor:

$$\mathrm{Tot}_{semi}(\mathsf{IndCoh}^*(S) \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(T^{\times_S \bullet+1})) \to \mathrm{Tot}_{semi}(\mathsf{IndCoh}^*(T^{\times_S \bullet+1})). \qquad (6.7.2)$$

Below, we will show that this functor is actually an equivalence.

Assuming this, let us deduce the descent claim. As $\mathsf{IndCoh}^*(S)$ is compactly generated, hence dualizable, and $\mathsf{QCoh}(S)$ is rigid monoidal, we obtain that $\mathsf{IndCoh}^*(S)$ is dualizable as a $\mathsf{QCoh}(S)$-module category. Therefore:

$$\mathsf{IndCoh}^*(S) \underset{\mathsf{QCoh}(S)}{\otimes} \mathrm{Tot}_{semi}(\mathsf{QCoh}(T^{\times_S \bullet+1})) \xrightarrow{\simeq} \mathrm{Tot}_{semi}(\mathsf{IndCoh}^*(S) \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(T^{\times_S \bullet+1})).$$

The left hand side is then $\mathsf{IndCoh}^*(S) \otimes_{\mathsf{QCoh}(S)} \mathsf{QCoh}(S)$ by flat descent for QCoh (see [Lur4] Corollary D.6.3.3).

We now show that (6.7.2) is an equivalence. Suppose we are given an object of the right hand side. In particular, we are given $\mathcal{F} \in \mathsf{IndCoh}^*(T)$ with an isomorphism $\alpha : p_1^{*,\mathsf{IndCoh}}(\mathcal{F}) \xrightarrow{\simeq} p_2^{*,\mathsf{IndCoh}}(\mathcal{F})$ for $p_i : T \times_S T \to T$ the projections.

We will show that the map $\mathcal{F} \to p_{1,*}^{\mathsf{IndCoh}} p_2^{*,\mathsf{IndCoh}}(\mathcal{F})$ adjoint to α realizes $\mathcal{F}$ as a summand. Assuming this claim, we obtain that $\mathcal{F}$ is a direct summand of $f^{*,\mathsf{IndCoh}} f_*^{\mathsf{IndCoh}}(\mathcal{F}) = p_{1,*}^{\mathsf{IndCoh}} p_2^{*,\mathsf{IndCoh}}(\mathcal{F})$, in particular, a summand of an object lying in the essential image of (6.7.1). This implies that $\mathcal{F}$ is in the essential image of (6.7.1), our original object lies in the essential image of (6.7.2), completing the argument.

Let $\Delta : T \to T \times_S T$ denote the diagonal map and let $\Delta^{*,\mathsf{IndCoh}} : \mathsf{IndCoh}^*(T \times_S T) \to \mathsf{Pro}(\mathsf{IndCoh}^*(T))$ denote the "partially-defined" left adjoint to $\Delta_*^{\mathsf{IndCoh}}$, noting that $\Delta^{*,\mathsf{IndCoh}}$ is defined on $p_i^{*,\mathsf{IndCoh}}(\mathcal{F})$. Then observe that $\Delta^{*,\mathsf{IndCoh}}(\alpha) = \mathrm{id}_{\mathcal{F}}$. Indeed, the standard argument in a simplicial setting applies in our setting: applying the partially-defined $*$-restriction along the diagonal $T \to T \times_S T \times_S T$ to the cocycle relation here gives the claim.

Therefore, the diagram:

$$
\begin{array}{ccc}
p_1^{*,\mathsf{IndCoh}}(\mathcal{F}) & \xrightarrow{\;\;\;\alpha\;\simeq\;\;\;} & p_2^{*,\mathsf{IndCoh}}(\mathcal{F}) \\
& \searrow \qquad \swarrow & \\
& \Delta_*^{\mathsf{IndCoh}}(\mathcal{F}) &
\end{array}
$$

commutes, where the diagonal arrows are the obvious ones induced by adjunction and the observation $p_i \Delta = \mathrm{id}$. By adjunction, this means that the composition map:

$$\mathcal{F} \to p_{1,*}^{\mathsf{IndCoh}}(\mathcal{F}) p_2^{*,\mathsf{IndCoh}}(\mathcal{F}) \to p_{1,*}^{\mathsf{IndCoh}} \Delta_*^{\mathsf{IndCoh}}(\mathcal{F}) = \mathcal{F}$$

is the identity for $\mathcal{F}$. But this composition is clearly the map under consideration.

$\square$

6.8. **Indschemes.** We now extend the above to the setting of indschemes.

Let PreStk_{conv} denote the category of *convergent* prestacks. Recall that these are by definition accessible functors $^{>-\infty}\mathsf{AffSch}^{op} \to \mathsf{Gpd}$; the natural functor $\mathsf{PreStk} \to \mathsf{PreStk}_{conv}$ admits fully-faithful left and right adjoints[38] given by Kan extensions. We recall that the composition $\mathsf{Sch} \hookrightarrow \mathsf{PreStk} \to \mathsf{PreStk}_{conv}$ is still fully-faithful; we regard Sch as a subcategory of convergent prestacks via this functor.

Definition 6.8.1. A *reasonable indscheme* is an object $S \in \mathsf{PreStk}_{conv}$ that can be written as a filtered colimit $\mathrm{colim}_i S_i$ in PreStk_{conv} of quasi-compact quasi-separated eventually coconnective schemes S_i under almost finitely presented closed embeddings.

Let $\mathsf{IndSch}_{reas} \subseteq \mathsf{PreStk}_{conv}$ denote the subcategory of reasonable indschemes.

Remark 6.8.2. By [Lur4] Corollary 5.2.2.2, a closed embedding $T_1 \hookrightarrow T_2 \in \mathsf{Sch}_{qcqs}$ is almost finitely presented if and only if for every n, $\tau^{\geqslant -n} i_*(\mathcal{O}_{T_1}) \in \mathsf{QCoh}(T_2)^{\geqslant -n}$ is compact (in this category). In particular, if T_1 is eventually coconnective, this is equivalent to $i_*(\mathcal{O}_{T_1})$ lying in $\mathsf{Coh}(T_2) \subseteq \mathsf{QCoh}(T_2)$.

Remark 6.8.3. Our terminology here is borrowed from [BD1] §7.11, where a similar notion was introduced for classical indschemes. However, our derived version here is much more restrictive, because almost finite presentation is much more restrictive than classical finite presentation.

Example 6.8.4. For Y a smooth affine scheme, its algebraic loop space $Y(K)$ is reasonable by [Dri] Theorem 6.3. Indeed, *loc. cit.* shows that $Y(K)$ is a filtered colimit of classical[39] affine schemes that are spectra of coherent commutative rings. Moreover, the structure maps are classically of finite presentation. For such rings, classical finite presentation is equivalent to almost finite presentation, giving the claim.

6.9. We let Sch_{reas} denote $\mathsf{Sch}_{qcqs} \cap \mathsf{IndSch}_{reas} \subseteq \mathsf{PreStk}_{conv}$. We refer to objects of this category as *reasonable* schemes. Note that any any of the following classes of quasi-compact quasi-separated schemes is reasonable:

- Eventually coconnective.
- Locally coherent.
- Locally *eventually coherent*[40] in the sense of [Gai5] §2.

---

[38]We remind that PreStk_{conv} is typically (e.g., in [GR4]) regarded as a full subcategory of PreStk via this right adjoint. This is because under this embedding, PreStk_{conv} then contains many subcategories of PreStk of interest, e.g., Sch and IndSch.

We mostly ignore this embedding in what follows and only consider the projection $\mathsf{PreStk} \to \mathsf{PreStk}_{conv}$, but it may help guide the reader to keep this in mind.

[39]There is a subtle point here: that $Y(K)$ is classical for smooth affine Y is stated as Conjecture 9.2.10 in [GR3], and shown when Y is a group. So we are implicitly assuming that conjecture, or are by fiat interpreting $Y(K)$ as the classical indscheme underlying the corresponding DG indscheme.

With that said, the conjecture from [GR3] may be proved as follows. Let S be a classical $\aleph_0$-indscheme that is *Tate-smooth* in the sense of [Dri] §6.3.7. By [Dri] Theorem 6.3, and as in [GR3] §9.1, it suffices to show that sucgh S is formally smooth when considered as a DG indscheme. This may be shown using an appropriate variant of Proposition 9.5.2 from [GR3].

As we do not use this result here, we do not supply further details.

[40]This condition for $S \in \mathsf{Sch}_{qcqs}$ means that the Postnikov maps $\tau^{\geqslant -n-1}S \to \tau^{\geqslant -n}S$ are almost finitely presented for $n \gg 0$.

6.10.   We define:

$$\mathsf{IndCoh}^* : \mathsf{IndSch}_{reas} \to \mathsf{DGCat}_{cont}$$

$$S \mapsto \mathsf{IndCoh}^*(S)$$

$$(f : S \to T) \mapsto (f_*^{\mathsf{IndCoh}} : \mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(T)).$$

by left Kan extension.

Remark 6.10.1. This definition of IndCoh^* evidently extends to all convergent prestacks (as the relevant left Kan extension). However, this definition does not recover the category we are after for the class of *(weakly) renormalizable prestacks* introduced below. Therefore, we do not consider this total left Kan extension here.

6.11.   Explicitly, for $S = \operatorname{colim}_i S_i$ with S_i eventually coconnective and quasi-compact quasi-separated schemes and structural maps being almost finitely presented closed embeddings, we have $\mathsf{IndCoh}^*(S) = \operatorname{colim}_i \mathsf{IndCoh}^*(S_i) \in \mathsf{DGCat}_{cont}$.

As each of the structure functors in this colimit is t-exact (since pushforward for affine morphisms is), $\mathsf{IndCoh}^*(S)$ inherits a canonical t-structure (see e.g. [Ras6] Lemma 5.4.3 (1)). This t-structure is characterized by the fact that each pushforward functor $\mathsf{IndCoh}^*(S_i) \to \mathsf{IndCoh}^*(S)$ is t-exact.

In addition, by Remark 6.8.2, each of these functors preserves compact objects. In particular, $\mathsf{IndCoh}^*(S)$ is compactly generated, and so is $\mathsf{IndCoh}^*(S)^{\leq 0}$.

Definition 6.11.1. $\mathsf{Coh}(S) \subseteq \mathsf{IndCoh}^*(S)$ is the subcategory of compact objects. We refer to such objects as *coherent*.

We record the following characterization of coherent sheaves for future use.

Lemma 6.11.2. *For S a reasonable indscheme, $\mathcal{F} \in \mathsf{IndCoh}^*(S)$ is coherent if and only if $\mathcal{F} \in \mathsf{IndCoh}^*(S)^+$ and for all $N \gg 0$ with $\mathcal{F} \in \mathsf{IndCoh}^*(S)^{\geq -N}$, $\mathcal{F}$ is compact in the category $\mathsf{IndCoh}^*(S)^{\geq -N}$.*

Proof.

Step 1. First, we remark that this result is immediate from the definitions and Lemma 6.4.1 when $S \in {}^{> -\infty}\mathsf{Sch}_{qcqs}$.

Step 2. For convenience, we introduce the following terminology. Suppose $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is equipped with a right separated t-structure compatible with filtered colimits. We say $\mathcal{F} \in \mathcal{C}$ is *almost compact* if $\mathcal{F} \in \mathcal{C}^+$ and for all $N \gg 0$ with $\mathcal{F} \in \mathcal{C}^{\geq -N}$, $\mathcal{F}$ is compact in $\mathcal{C}^{\geq -N}$.

In this terminology, our goal is to show that compactness is equivalent to almost compactness in $\mathsf{IndCoh}^*(S)$. One direction is evident: compactness implies almost compactness as compact objects in $\mathsf{IndCoh}^*(S)$ are eventually connective.

Step 3. Suppose $\mathcal{C}, \mathcal{D} \in \mathsf{DGCat}_{cont}$ are equipped with right separated t-structures compatible with filtered colimits. Let $F : \mathcal{C} \to \mathcal{D}$ be a t-exact functor admitting a continuous right adjoint G and such that $F|_{\mathcal{C}^+}$ conservative. We claim that for $\mathcal{F} \in \mathcal{C}^+$ with $F(\mathcal{F})$ almost compact, $\mathcal{F}$ is itself almost compact.

First, note that the t-structures are automatically right complete. Therefore, almost compactness of $F(\mathcal{F})$ implies that it is eventually connective. As $F|_{\mathcal{C}^+}$ is conservative and t-exact, this implies $\mathcal{F}$ is also eventually connective.

By Lemma 3.7.2, $F|_{\mathcal{C}^+}$ is comonadic (c.f. the proof of Proposition 3.7.1). Therefore, for an integer N and $\mathcal{G} \in \mathcal{C}^{\geq -N}$, we have:

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) = \operatorname{Tot} \operatorname{Hom}_{\mathcal{D}}(F(\mathcal{F}), F(GF)^{\bullet}(\mathcal{G})) \in \mathsf{Gpd}. \qquad (6.11.1)$$

There exists an integer M (depending on N and $\mathcal{F}$) such that each term in this totalization is an M-truncated groupoid (as $F(\mathcal{F})$ is eventually connective). Therefore, the above totalization commutes with a finite totalization. As $\mathrm{Hom}_{\mathcal{D}}(F(\mathcal{F}), -)$ and F and G all commute with filtered colimits in $\mathcal{D}^{\geqslant -N}$, and as filtered colimits commute with finite limits in Gpd, this implies that the left hand side of (6.11.1) commutes with filtered colimits in the variable $\mathcal{G} (\in \mathcal{C}^{\geqslant -N})$ as desired.

Step 4. Let $S = \mathrm{colim}_i S_i$ as in the definition of reasonable indscheme. Let $\alpha_i : S_i \to S$ denote the structural morphisms.

By definition, we have $\mathsf{IndCoh}^*(S) = \mathrm{colim}_i \mathsf{IndCoh}^*(S_i)$ (under pushforwards), with the colimit being taken in DGCat_{cont}.

We will also need the following variant. Let Cat_{pres} denote the category of presentable categories and functors commuting with colimits. For any integer n, we claim:

$$\mathsf{IndCoh}^*(S)^{\geqslant -n} = \mathrm{colim}_i \mathsf{IndCoh}^*(S_i)^{\geqslant -n} \in \mathsf{Cat}_{pres} \qquad (6.11.2)$$

with the colimit being taken in Cat_{pres}. Indeed, we have $\mathsf{IndCoh}^*(S) = \lim_i \mathsf{IndCoh}^*(S_i)$ under right adjoints, where this limit may be formed in any of DGCat_{cont}, Cat_{pres}, and Cat. As these right adjoints are left t-exact, we find that $\mathsf{IndCoh}^*(S)^{\geqslant -n} = \lim_i \mathsf{IndCoh}^*(S_i)^{\geqslant -n}$. The functors in this limit also admit left adjoints, so the limit coincides with the colimit in Cat_{pres}.

Step 5. We now conclude the argument. Suppose $\mathcal{F} \in \mathsf{IndCoh}^*(S)$ is almost compact. By assumption, $\mathcal{F} \in \mathsf{IndCoh}^*(S)^{\geqslant -n}$ for some $n \in \mathbb{Z}$.

Write $\mathcal{F} = \mathrm{colim}_j \mathcal{F}_j$ for $\mathcal{F}_j \in \mathsf{Coh}(S)$. We obtain $\mathcal{F} = \mathrm{colim}_j \tau^{\geqslant -n}(\mathcal{F}_j)$. By almost compactness of $\mathcal{F}$, there exists an index j such that $\mathcal{F}$ is a summand of $\tau^{\geqslant -n}(\mathcal{F}_j)$.

As $\mathsf{Coh}(S) = \mathrm{colim}_i \mathsf{Coh}(S_i) \in \mathsf{Cat}$ (c.f. [Lur3] Lemmas 7.3.5.10-13), there exists an index i and some $\widetilde{\mathcal{F}}_j \in \mathsf{Coh}(S_i)$ such that $\alpha_{i,*}^{\mathsf{IndCoh}}(\widetilde{\mathcal{F}}_j) = \mathcal{F}_j$. Moreover, by (6.11.2) and [Lur3] Lemma 7.3.5.10, after possibly increasing the index i there exists $\widetilde{\mathcal{F}} \in \mathsf{IndCoh}^*(S_i)^{\geqslant -n}$ (a summand of $\tau^{\geqslant -n}(\mathcal{F}_j)$) with:

$$\alpha_{i,*}^{\mathsf{IndCoh}}(\widetilde{\mathcal{F}}) = \mathcal{F}$$

(as summands of $\mathcal{F}_j$).

By Step 3, $\widetilde{\mathcal{F}}$ is almost compact in $\mathsf{IndCoh}^*(S_i)$. As in Step 1, this means $\widetilde{\mathcal{F}} \in \mathsf{Coh}(S_i)$. As $\alpha_{i,*}^{\mathsf{IndCoh}}$ admits a continuous right adjoint, we obtain the result.

$\square$

6.12.   The following technical result is convenient for comparing different possible presentations of a reasonable indscheme.

Proposition 6.12.1. *Let $S = \mathrm{colim}_{i \in \mathcal{I}} S_i^1 = \mathrm{colim}_{j \in \mathcal{J}} S_j^2$ be two expressions of S as a reasonable indscheme, i.e., these colimits are filtered, $S_i^1, S_j^2 \in {}^{> -\infty}\mathsf{Sch}_{qcqs}$ and the structure maps in each of these colimits are almost finitely presented. Let $\alpha_i : S_i^1 \to S$ and $\beta_j : S_j^2 \to S$ denote the structure maps.*

Then for any choice of indices $i \in \mathcal{I}$ and $j \in \mathcal{J}$ such that the map $\alpha_i^1 : S_i^1 \to S$ factors as $S_i^1 \xrightarrow{\iota} S_j^2 \xrightarrow{\beta_j} S$, the map ι is almost of finite presentation.

Proof. By Remark 6.8.2, it suffices to show that $\iota_*(\mathcal{O}_{S_i^1}) \in \mathsf{Coh}(S_j^2)$. Clearly this object lies in $\mathsf{IndCoh}^*(S_j^2)^+$. By Lemma 6.11.2 and Step 3 from its proof, it therefore suffices to show that $\beta_{j,*}^{\mathsf{IndCoh}} \iota_*(\mathcal{O}_{S_i^1}) \in \mathsf{Coh}(S)$, but this is clear as $\beta_{j,*}^{\mathsf{IndCoh}} \iota_* = \alpha_{i,*}^{\mathsf{IndCoh}}$.

$\square$

6.13. **Reasonable schemes.** Observe that we have a functor:

$$\mathsf{QCoh}(-) : \mathsf{Sch}_{qcqs} \to \mathsf{DGCat}_{cont}$$

encoding pushforward of quasi-coherent sheaves. By left Kan extension, there is a canonical natural transformation:

$$\Psi : \mathsf{IndCoh}^*(-)|_{\mathsf{Sch}_{reas}} \to \mathsf{QCoh}(-)|_{\mathsf{Sch}_{reas}}$$

of functors $\mathsf{Sch}_{reas} \to \mathsf{DGCat}_{cont}$.

Lemma 6.13.1. *For every* $S \in \mathsf{Sch}_{reas}$, Ψ *induces an equivalence* $\mathsf{IndCoh}^*(S)^+ \xrightarrow{\simeq} \mathsf{QCoh}(S)^+$. *Moreover,* Ψ *identifies* $\mathsf{Coh}(S) \subseteq \mathsf{IndCoh}^*(S)^+$ *(as defined in Definition 6.11.1) with the subcategory of cohomologically bounded objects in* $\mathsf{QCoh}(S)$ *that are compact in* $\mathsf{QCoh}(S)^{\geq -N}$ *for all* $N \gg 0$.

Proof. In what follows, for $\mathcal{C}$ a DG category with a t-structure and $n \geq 0$, we let $\mathcal{C}^{[-n,0]}$ denote the subcategory $\mathcal{C}$ of objects in cohomological degrees $[-n, 0]$.

Because S is reasonable, we have $S = \operatorname{colim} S_i$ a filtered colimit under almost finitely presented closed embeddings. As in Step 4 from the proof of Lemma 6.11.2, we have $\mathsf{IndCoh}^*(S)^{[-n,0]} = \operatorname{colim}_i \mathsf{IndCoh}^*(S_i)^{[-n,0]}$ with the colimit being taken in Cat_{pres} (the category of presentable categories and functors commuting with colimits).

Recall that e.g. $\mathsf{QCoh}(S)^{[-n,0]} = \mathsf{QCoh}(\tau^{\geq -n}S)^{[-n,0]}$. Therefore, we obtain:

$$\mathsf{IndCoh}^*(S)^{[-n,0]} = \operatorname*{colim}_i \mathsf{IndCoh}^*(S_i)^{[-n,0]} \xrightarrow{\Psi}_{\simeq} \operatorname*{colim}_i \mathsf{QCoh}^*(S_i)^{[-n,0]} =$$

$$\operatorname*{colim}_i \mathsf{QCoh}^*(\tau^{\geq -n}S_i)^{[-n,0]} = \operatorname*{colim}_i \mathsf{IndCoh}^*(\tau^{\geq -n}S_i)^{[-n,0]} = \mathsf{IndCoh}^*(\tau^{\geq -n}S)^{[-n,0]}$$

with all colimits being taken in Cat_{pres}, and where we have used that $\tau^{\geq -n}S = \operatorname{colim}_i \tau^{\geq -n}S_i$.

By right completeness of the t-structures on $\mathsf{IndCoh}^*(S)$ and $\mathsf{QCoh}(S)$, we obtain the claims. $\square$

6.14. **Proper morphisms.** We now discuss proper morphisms. We refer to [Lur4] Part II for an extensive discussion of such morphisms in derived algebraic geometry. However, we take a more restrictive definition than *loc. cit.* (to simplify terminology): we say $f : S \to T \in \mathsf{Sch}_{qcqs}$ is *proper* if it is proper in the sense of [Lur4] *and* almost of finite presentation (only finite type is required in *loc. cit.*).

Lemma 6.14.1. *Suppose* $f : S \to T \in \mathsf{Sch}_{reas}$ *is proper. Then* f_*^{IndCoh} *admits a continuous right adjoint* $f^!$.

Proof. This is immediate from Lemma 6.13.1 and [Lur4] Theorem 5.6.0.2. $\square$

Lemma 6.14.2. *Suppose we are given a Cartesian diagram of schemes:*

$$\begin{array}{ccc} S' & \xrightarrow{\varphi} & T' \\ \downarrow{\psi} & & \downarrow{g} \\ S & \xrightarrow{f} & T \end{array}$$

with all terms lying in Sch_{reas}[41] *and* f *proper.*

---

[41]This is *not* automatic for $S' = S \times_T T'$ even if S, T, and T' lie in Sch_{reas}.

(1) The natural map:

$$\psi_*^{\mathsf{IndCoh}} \varphi^! \to f^! g_*^{\mathsf{IndCoh}}$$

is an isomorphism.

(2) Suppose that g is flat. Then the natural[42] map:

$$\psi^{*,\mathsf{IndCoh}} f^! \to \varphi^! g^{*,\mathsf{IndCoh}}$$

is an isomorphism.

Proof. The same argument from [Gai5] Proposition 3.4.2 applies for (1). Similarly, again using Lemma 6.13.1, the same argument as in [Gai5] Proposition 7.1.6 applies[43] for (2).

$\square$

6.15. Flatness. We say a morphism $f : T_1 \to T_2 \in \mathsf{PreStk}_{conv}$ is *flat* if for any eventually coconnective affine $S \in {}^{>-\infty}\mathsf{AffSch}$ and any map $S \to T_2$, the fiber product $T_1 \times_{T_2} S$ lies in ${}^{>-\infty}\mathsf{Sch}_{qcqs}$ and its structure map to S is flat. Similarly, we say f is a *flat cover* if it is flat and $T_1 \times_{T_2} S \to S$ is faithfully flat.

Remark 6.15.1. As our definition requires flat morphisms to be schematic (in the relevant sense for convergent prestacks) and to be quasi-compact quasi-separated, it is much more stringent than usual notions of flatness. We hope the reader will forgive this abuse, which we find unburdens the terminology and notation to some degree.

Clearly flat morphisms (resp. covers) are closed under compositions and base-change.

Remark 6.15.2. If $f : S \to T$ is flat and T is a reasonable indscheme, then S is a reasonable indscheme as well.

6.16. We will now study a variety of base-change results for flat morphisms.

Lemma 6.16.1. *Let* $f : S \to T \in \mathsf{IndSch}_{reas}$ *be a flat morphism.*

(1) f_*^{IndCoh} *admits a left adjoint* $f^{*,\mathsf{IndCoh}}$.

(2) *Suppose* $T = \mathrm{colim}_i T_i$ *as in the definition of reasonable indscheme. For any index i, let* $S_i := S \times_T T_i$ *and denote the relevant structural maps as:*

$$
\begin{array}{ccc}
S_i & \xrightarrow{\ f_i\ } & T_i \\
{\scriptstyle \beta_i}\downarrow & & \downarrow{\scriptstyle \alpha_i} \\
S & \xrightarrow{\ f\ } & T.
\end{array}
$$

Let $\alpha_i^!$ *and* $\beta_i^!$ *denote the (continuous) right adjoints to* $\alpha_{i,*}^{\mathsf{IndCoh}}$ *and* $\beta_{i,*}^{\mathsf{IndCoh}}$.

---

[42]We remark that the existence of this map depends on (1).

[43]There is one small modification to make. The argument in [Gai5] uses Proposition 3.6.11 from *loc. cit.*, which in turn uses Lemma 3.6.13 from *loc. cit.* The argument from *loc. cit.* for this lemma does not literally work: [Gai5] reduces to the classical case by means which are not available to us here. The difference in the argument is not significant, but we provide the details below.

The lemma in question (in our setup) asserts that for flat $f : S \in \mathsf{Sch}_{reas}$ and $\mathcal{F} \in \mathsf{QCoh}(S)$ flat, the functor $\mathcal{F} \otimes -\mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(S)$ is t-exact. (Here we are using the natural action of $\mathsf{QCoh}(S)$ on $\mathsf{IndCoh}^*(S)$ obtained by ind-extension from the action of $\mathsf{Perf}(S)$ on $\mathsf{Coh}(S)$.)

In our setup, Zariski descent (Lemma 6.7.1 below, which is independent of Lemma 6.14.2), reduces to considering the case where S is affine. Then the result is immediate from Lazard's theorem in the derived setup: see [Lur3] Theorem 7.2.2.15.

Then the natural map:

$$f_{i,*}^{\mathsf{IndCoh}} \beta_i^! \to \alpha_i^! f_*^{\mathsf{IndCoh}}$$

is an isomorphism.

(3) *The natural map:*

$$f_i^{*,\mathsf{IndCoh}} \alpha_i^! \to \beta_i^! f^{*,\mathsf{IndCoh}}$$

is an isomorphism.

(4) *Given a Cartesian diagram in* IndSch_{reas}:

$$
\begin{array}{ccc}
S' & \xrightarrow{\;\varphi\;} & T' \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle g} \\
S & \xrightarrow{\;f\;} & T
\end{array}
$$

(with f flat), the natural morphism:

$$f^{*,\mathsf{IndCoh}} g_*^{\mathsf{IndCoh}} \to \psi_*^{\mathsf{IndCoh}} \varphi^{*,\mathsf{IndCoh}} \tag{6.16.1}$$

is an isomorphism.

Proof. By Lemma 6.14.2 (1)-(2), the adjoint functors:

$$f_i^{*,\mathsf{IndCoh}} : \mathsf{IndCoh}^*(T_i) \rightleftarrows \mathsf{IndCoh}^*(S_i) : f_{i,*}^{\mathsf{IndCoh}} \tag{6.16.2}$$

are (canonically) compatible with the structure functors in the limits $\mathsf{IndCoh}^*(S) = \lim_i \mathsf{IndCoh}^*(S_i)$ and $\mathsf{IndCoh}^*(T) = \lim_i \mathsf{IndCoh}^*(T_i)$ (the limits being under upper-! functors), and therefore induce an adjunction $(f^{*,\mathsf{IndCoh}}, f_*^{\mathsf{IndCoh}})$ satisfying (2) and (3).

For (4), we are immediately reduced to the case where $T' \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ (as $\mathsf{IndCoh}^*(T')$ is necessarily generated under colimits by objects pushed forward from eventually coconnective schemes). Then g factors as $T' \xrightarrow{\overline{g}} T_i \xrightarrow{\alpha_i} T$. By Lemma 6.6.2, we are reduced to the case where $T' = T_i$ and $g = \alpha_i$.

In this case, the claim follows from the fact that the adjoint functors (6.16.2) are (canonically) compatible with the structural functors in the colimits $\mathsf{IndCoh}^*(S) = \mathrm{colim}_i \mathsf{IndCoh}^*(S_i)$ and $\mathsf{IndCoh}^*(T) = \mathrm{colim}_i \mathsf{IndCoh}^*(T_i)$ (in DGCat_{cont}, under lower-* functors) by Lemma 6.6.2.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 6.16.2. *Let* $f : S \to T \in \mathsf{IndSch}_{reas}$ *be flat. Then* $f^{*,\mathsf{IndCoh}} : \mathsf{IndCoh}^*(T) \to \mathsf{IndCoh}^*(S)$ *is t-exact.*

Proof. We use the notation from Lemma 6.16.1 (2). Then $\mathsf{IndCoh}^*(T)^{\leqslant 0}$ is generated under colimits by objects of the form $\alpha_{i,*}^{\mathsf{IndCoh}}(\mathcal{F})$ for $\mathcal{F} \in \mathsf{IndCoh}^*(T_i)^{\leqslant 0}$. We then have:

$$f^{*,\mathsf{IndCoh}}(\alpha_{i,*}^{\mathsf{IndCoh}}(\mathcal{F})) = \beta_{i,*}^{\mathsf{IndCoh}} f_i^{*,\mathsf{IndCoh}}(\mathcal{F})$$

by Lemma 6.16.1 (4) and this is in $\mathsf{IndCoh}^*(S)^{\leqslant 0}$ because f_i is a flat map of schemes. Therefore, $f^{*,\mathsf{IndCoh}}$ is right t-exact.

For left t-exactness, suppose $\mathcal{F} \in \mathsf{IndCoh}^*(T)^{\geqslant 0}$. To see $f^{*,\mathsf{IndCoh}}(\mathcal{F}) \in \mathsf{IndCoh}^*(S)^{\geqslant 0}$, it is equivalent to show that $\beta_i^! f^{*,\mathsf{IndCoh}}(\mathcal{F}) \in \mathsf{IndCoh}^*(S_i)^{\geqslant 0}$ for all i. But by Lemma 6.14.2 (2), we have:

$$\beta_i^! f^{*,\mathsf{IndCoh}}(\mathcal{F}) = f_i^{*,\mathsf{IndCoh}} \alpha_i^!(\mathcal{F}).$$

Then $\alpha_i^!(\mathcal{F}) \in \mathsf{IndCoh}^*(T_i)^{\geq 0}$ by assumption on $\mathcal{F}$, so the same is true after applying $f_i^{*,\mathsf{IndCoh}}$ to it.

$\square$

6.17. We say that a morphism $f : S \to T$ of reasonable indschemes is *ind-proper* if for some (equivalently,[44] any) presentations $S = \mathrm{colim}_{i \in \mathcal{I}} S_i$ and $T = \mathrm{colim}_{j \in \mathcal{J}} T_j$ as in the definitions of reasonable indschemes, and any index $i \in \mathcal{I}$ there exists $j \in \mathcal{J}$ such that $S_i \to S \to T$ factors through a proper morphism $S_i \to T_j$.

Lemma 6.17.1. *Let $f : S \to T$ be an ind-proper morphism of reasonable indschemes. Then f_*^{IndCoh} admits a continuous right adjoint $f^!$.*

Proof. As $\mathsf{IndCoh}^* = \mathsf{Ind}(\mathsf{Coh})$ here, the continuity of the right adjoint $f^!$ is equivalent to f_*^{IndCoh} preserving compacts. Then we are immediately reduced to the case where $S, T \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$, which is covered by Lemma 6.14.1.

$\square$

We have the following (somewhat partial) generalization of Lemma 6.14.2.

Lemma 6.17.2. *Suppose we are given a Cartesian diagram of reasonable indschemes:*

$$
\begin{array}{ccc}
S' & \overset{\varphi}{\longrightarrow} & T' \\
\downarrow{\psi} & & \downarrow{g} \\
S & \underset{f}{\longrightarrow} & T
\end{array}
$$

with f ind-proper and g flat.

(1) The natural map:

$$\psi_*^{\mathsf{IndCoh}}\varphi^! \to f^! g_*^{\mathsf{IndCoh}}$$

*   *is an isomorphism.*
(2) The natural map:

$$\psi^{*,\mathsf{IndCoh}} f^! \to \varphi^! g^{*,\mathsf{IndCoh}}$$

*   *is an isomorphism.*

Proof. Writing $S = \mathrm{colim}_j S_j$ as in the definition of reasonable indscheme, both statements immediately reduce to the case where $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$.

Now take $T = \mathrm{colim}_i T_i$ as in the definition of reasonable indscheme. By assumption on f, the map f factors as $S \xrightarrow{\overline{f}} T_j \xrightarrow{\alpha_j} T$ for $\alpha_j : T_j \to T$ the structure map and $\overline{f}$ proper. By Lemma 6.14.2 (1) and (2), both statements reduce to the case where $S = T_j$ and $f = \alpha_j$ (Here the results follow from Lemma 6.16.1 (2)-(3).

$\square$

---

[44]C.f. Proposition 6.12.1.

6.18.   We record the following basic result for later use.

Lemma 6.18.1. *For a flat cover $f : S \to T \in \mathsf{IndSch}_{reas}$, $\mathcal{F} \in \mathsf{IndCoh}^*(T)$ lies in $\mathsf{Coh}(T)$ if and only $f^{*,\mathsf{IndCoh}}(\mathcal{F})$ lies in $\mathsf{Coh}(S)$.*

Proof. First, note that $f^{*,\mathsf{IndCoh}}$ is conservative. Indeed, if $T \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$, this follows from Proposition 6.7.2, and the general case results from this using Lemma 6.14.2 (2). (See also Theorem 6.25.1).

Therefore, if $f^{*,\mathsf{IndCoh}}(\mathcal{F}) \in \mathsf{Coh}(S)$, we find in particular that $\mathcal{F} \in \mathsf{IndCoh}^*(T)^+$. Now we obtain the result from Lemma 6.11.2 and Step 3 from the proof of *loc. cit.*

□

6.19.   We remark that because we can $*$-pullback along flat maps, the functor $\mathsf{IndCoh}^* : \mathsf{IndSch}_{reas} \to \mathsf{DGCat}_{cont}$ extends (again by [GR4] Theorem V.1.3.2.2) to a functor:

$$\mathrm{Corr}(\mathsf{IndSch}_{reas})_{all;flat} \to \mathsf{DGCat}_{cont}.$$

6.20.   **Relationship to D-modules.** Recall the functor $D^* : \mathsf{IndSch}_{reas} \to \mathsf{DGCat}_{cont}$ constructed in [Ras3]. We will construct a canonical natural transformation:

$$\mathsf{IndCoh}^* \to D^*$$

that can be thought of as inducing an ind-coherent sheaf to a D-module.

Each of these functors is by definition left Kan extended from ${}^{>-\infty}\mathsf{Sch}_{qcqs}$, so it suffices to define the natural transformation for the restrictions of these functors here. Moreover, each of these functors is a Zariski sheaf, so it suffices to define the natural transformation on ${}^{>-\infty}\mathsf{AffSch}$. This in turn is equivalent to specifying a compatible sequence of natural transformations on each ${}^{\geq-n}\mathsf{AffSch} = \mathsf{Pro}({}^{\geq-n}\mathsf{AffSch}_{ft})$.

By definition, $D^*|_{\geq-n\mathsf{AffSch}}$ is right Kan extended from ${}^{\geq-n}\mathsf{AffSch}_{ft}$. Therefore, we need to specify the natural transformation on ${}^{\geq-n}\mathsf{AffSch}_{ft}$ compatibly over all n. Here we define our natural transformation as the ("right[45]") D-module induction functor $\mathrm{ind} : \mathsf{IndCoh} \to D$ constructed in [GR4].

Remark 6.20.1. By construction, this natural transformation upgrades to a natural transformation of lax symmetric monoidal functors; see §6.30 below.

6.21.   **Weakly renormalizable prestacks.** We now introduce a convenient class of prestacks.[46]

Definition 6.21.1. A convergent prestack $S \in \mathsf{PreStk}_{conv}$ is *weakly renormalizable* if there exists a flat covering map $T \to S$ with $T \in \mathsf{IndSch}_{reas}$.

We let $\mathsf{PreStk}_{w.ren} \subseteq \mathsf{PreStk}_{conv}$ denote the subcategory of weakly renormalizable prestacks.

6.22.   **Morphisms.** We now introduce some classes of morphisms between weakly renormalizable prestacks.

Definition 6.22.1.   (1) A morphism $f : S_1 \to S_2 \in \mathsf{PreStk}_{w.ren}$ is *reasonable indschematic* if for any flat morphism $T \to S_2$ with $T \in \mathsf{IndSch}_{reas}$, the fiber product $T_1 \times_{T_2} S \in \mathsf{PreStk}_{conv}$ is a reasonable indscheme.
     (2) A morphism $S_1 \to S_2 \in \mathsf{PreStk}_{w.ren}$ is *locally flat* if for any $T \in \mathsf{IndSch}_{reas}$ and any flat morphism $T \to S_1$, the composition $T \to S_1 \to S_2$ is flat.

---

[45]As opposed to left.

[46]It is formally convenient to not have to sheafify in forming quotients such as X/H, so we prefer to work with prestacks.

Example 6.22.2. Any morphism from a reasonable indschemes to a weakly renormalizable prestack is reasonable indschematic. In particular, any morphism between reasonable indschemes is reasonable indschematic.

Remark 6.22.3. It is straightforward to show that to check $S_1 \to S_2$ is locally flat, it suffices to check the condition from the definition for some flat cover T of S_1. In particular, a morphism of reasonable indschemes is locally flat if and only if it is flat.

Example 6.22.4. If H is a classical affine group scheme, $\mathbb{B}H \to \mathrm{Spec}(k)$ is locally flat.

The following is immediate:

Lemma 6.22.5. *Reasonable indschematic and locally flat morphisms are closed under compositions. Any base-change of a reasonable indschematic morphism by a locally flat morphism is again reasonable indschematic, and any base-change of a locally flat morphism by a reasonable indschematic morphism is again locally flat.*

6.23. **Set-theoretic remarks.** In what follows, we will not explicitly address certain set-theoretic issues. More precisely, we will want to form limits e.g. over all reasonable indschemes flat over a given weakly renormalizable prestack. This indexing category is not essentially small, so there are set-theoretic issues.

To address these, fix a regular cardinal κ and replace "flat" everywhere by "flat and locally κ-presented." One should understand weakly renormalizable prestacks in this sense (i.e., these are prestacks admitting a locally κ-presented flat cover by a reasonable indscheme), and so on.

As will follow from Theorem 6.25.1, all of our constructions are invariant under extension of κ, i.e., if S is a weakly renormalizable prestack relative to κ and $\kappa' \geqslant \kappa$ is another regular cardinal, then the categories IndCoh^* defined using κ and κ' coincide.

Again, since the cutoff κ plays such a minor role, in order to simplify the exposition we do not mention it again.

6.24. IndCoh^* **on weakly renormalizable prestacks.** Define $\mathsf{PreStk}_{w.ren,loc.flat}$ as the 1-full subcategory of $\mathsf{PreStk}_{w.ren}$ where we only allow locally flat morphisms, and define $\mathsf{IndSch}_{reas,flat}$ similarly.

Definition 6.24.1. $\mathsf{IndCoh}^* : \mathsf{PreStk}^{op}_{w.ren,loc.flat} \to \mathsf{DGCat}_{cont}$ is the right Kan extension of the functor $\mathsf{IndSch}^{op}_{reas,flat} \to \mathsf{DGCat}_{cont}$ (which sends S to $\mathsf{IndCoh}^*(S)$ and sends flat $f : T_1 \to T_2$ to $f^{*,\mathsf{IndCoh}}$).

By [GR4] Theorem V.2.6.1.5,[47] the above construction upgrades canonically to a functor:

$$\mathsf{IndCoh}^* : \mathrm{Corr}(\mathsf{PreStk}_{w.ren})_{reas.indsch;loc.flat} \to \mathsf{DGCat}_{cont}.$$

Here "*reas.indsch*" is shorthand for *reasonable indschematic* and "*loc.flat*" is shorthand for *locally flat*. Therefore, the notation indicates that for a reasonable indschematic morphism $f : S \to T$ between weakly renormalizable prestacks, we have a pushforward functor $f^{\mathsf{IndCoh}}_* : \mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(T)$; for f locally flat, we have a functor $f^{*,\mathsf{IndCoh}}$; and the two satisfy base-change.

Remark 6.24.2. Using the 2-category of correspondences as in [GR4], one can further encode that $f^{*,\mathsf{IndCoh}}$ is left adjoint to f^{IndCoh}_* for flat f.

Remark 6.24.3. By [Gai5] Proposition 11.4.3, for S weakly renormalizable and locally almost of finite type, $\mathsf{IndCoh}^*(S)$ is canonically isomorphic to the usual category $\mathsf{IndCoh}((S)$.

---

[47]We remark that the hypotheses from *loc. cit.* are trivially verified in this setting, c.f. Lemma 6.22.5.

6.25.　The following result justifies the definition of IndCoh^* for weakly renormalizable prestacks.

Theorem 6.25.1. IndCoh^* *satisfies flat descent on* $\mathsf{PreStk}_{w.ren}$.

Proof. Let $S \in \mathsf{PreStk}_{w.ren}$ be given and let $f : T \to S$ be a flat cover. We need to show that:

$$\mathsf{IndCoh}^*(S) \to \mathrm{Tot}_{semi}(\mathsf{IndCoh}^*(T^{\times_S \bullet + 1}))$$

is an isomorphism.

　　We proceed in increasing generality.

Step 1. First, suppose $S \in \mathsf{IndSch}_{reas}$.

　　Let $S = \mathrm{colim}_i S_i$ as in the definition of reasonable indscheme. We then have:

$$\mathsf{IndCoh}^*(S) = \lim_{i,\text{upper-!}} \mathsf{IndCoh}^*(S_i) = \lim_{i,\text{upper-!}} \mathrm{Tot}_{semi}_{\text{upper-*}} \mathsf{IndCoh}^*(S_i \times_S T^{\times_S \bullet + 1}) \overset{Prop.6.7.2}{=}$$

$$\mathrm{Tot}_{semi}_{\text{upper-*}} \lim_{i,\text{upper-!}} \mathsf{IndCoh}^*(S_i \times_S T^{\times_S \bullet + 1}) = \mathrm{Tot}_{semi}_{\text{upper-*}} \mathsf{IndCoh}^*(T^{\times_S \bullet + 1})$$

as desired, where we have used Lemma 6.14.2 (2) to commute the limits.

Step 2. Next, suppose S is a general weakly renormalizable prestack and T is a reasonable indscheme.

　　We denote the functor under consideration by:

$$F : \mathsf{IndCoh}^*(S) := \lim_{\substack{U \in \mathsf{IndSch}_{reas} \\ U \to S \text{ flat}}} \mathsf{IndCoh}^*(U) \to \mathrm{Tot}_{semi}(\mathsf{IndCoh}^*(T^{\times_S \bullet + 1}))$$

We will show F is an equivalence by explicitly constructing an inverse functor G.

　　Namely, we have a functor (induced by $*$-pullback):

$$\mathrm{Tot}_{semi}(\mathsf{IndCoh}^*(T^{\times_S \bullet + 1})) \to \mathrm{Tot}_{semi} \lim_{\substack{U \in \mathsf{IndSch}_{reas} \\ U \to S \text{ flat}}} \mathsf{IndCoh}^*(U \times_S T^{\times_S \bullet + 1})$$

Exchanging the order of limits on the right hand side and noting that $U \times_S T^{\times_S \bullet + 1}$ is the Cech nerve of the flat cover $U \times_S T \to U \in \mathsf{IndSch}_{reas}$, the previous step implies that the right hand side is canonically isomorphic to:

$$\lim_{\substack{U \in \mathsf{IndSch}_{reas} \\ U \to S \text{ flat}}} \mathsf{IndCoh}^*(U) =: \mathsf{IndCoh}^*(S).$$

Therefore, we obtain our functor $G : \mathrm{Tot}_{semi}(\mathsf{IndCoh}^*(T^{\times_S \bullet + 1})) \to \mathsf{IndCoh}^*(S)$.

　　To verify that G and F are inverses, it suffices to show $GF \simeq \mathrm{id}$ and $FG \simeq \mathrm{id}$. We construct such isomorphisms by straightforward means below.

　　First, note that for $U \in \mathsf{IndSch}_{reas}$ equipped with a flat map to S, we have a projection morphism of augmented simplicial prestacks:

$$U \underset{S}{\times} T^{\times_S \bullet + 1} \to T^{\times_S \bullet + 1}.$$

This is functorial in U, so passing to the limits and using the augmentation to obtain the horizontal arrows, we get the commutative diagram:

$$\begin{array}{ccc}
\mathsf{IndCoh}^*(S) & \xrightarrow{\quad F \quad} & \mathrm{Tot}_{semi}\,\mathsf{IndCoh}^*(T^{\times_S \bullet+1}) \\
\| & & \downarrow \\
\lim\limits_{\substack{U \in \mathsf{IndSch}_{reas} \\ U \to S\ \mathrm{flat}}} \mathsf{IndCoh}^*(U) & \xrightarrow{\ \simeq\ } & \mathrm{Tot}_{semi} \lim\limits_{\substack{U \in \mathsf{IndSch}_{reas} \\ U \to S\ \mathrm{flat}}} \mathsf{IndCoh}^*(U \times_S T^{\times_S \bullet+1})
\end{array}$$

By definition, G is the composition of the right horizontal arrow and the inverse to the bottom arrow. The commutativity of this diagram therefore gives $GF \simeq \mathrm{id}$.

To construct an isomorphism $FG \simeq \mathrm{id}$, it suffices to do so after further composition with the functor:

$$\eta : \mathrm{Tot}_{semi}\,\mathsf{IndCoh}^*(T^{\times_S \bullet+1}) \xrightarrow{\ \simeq\ } \mathrm{Tot}_{semi}\,\mathrm{Tot}_{semi}\,\mathsf{IndCoh}^*(T^{\times_S \bullet+\bullet+2}).$$

Note that η is an isomorphism because T is a reasonable indscheme.

The target of η is the double totalization of the bi-semi-cosimplicial object obtained from $\mathsf{IndCoh}^*(T^{\times_S \bullet+1})$ by restricting along the join (alias: concatenation) map $\mathrm{join} : \mathbf{\Delta}_{inj} \times \mathbf{\Delta}_{inj} \to \mathbf{\Delta}_{inj}$. Moreover, by construction, the functor ηFG is the natural map in such a situation (from the limit of a functor to the limit of its restriction to another category).

Let $p_1 : \mathbf{\Delta}_{inj} \times \mathbf{\Delta}_{inj} \to \mathbf{\Delta}_{inj}$ be the first projection. There is an evident natural transformation $p_1 \to \mathrm{join}$ inducing a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Tot}_{semi}\,\mathsf{IndCoh}^*(T^{\times_S \bullet+1}) & & \\
\downarrow{\scriptstyle \simeq} & \searrow & \\
\mathrm{Tot}_{semi}\,\mathrm{Tot}_{semi}\,\mathsf{IndCoh}^*(T^{\times_S \bullet+1}) & \longrightarrow & \mathrm{Tot}_{semi}\,\mathrm{Tot}_{semi}\,\mathsf{IndCoh}^*(T^{\times_S \bullet+\bullet+2}).
\end{array}$$

The diagonal arrow is ηFG by the above discussion, while the left and bottom arrows compose to give η. This gives the claim.

Step 3. Finally, we treat the general case in which S and T are both weakly renormalizable prestacks.

By assumption on S, there exists $S' \in \mathsf{IndSch}_{reas}$ and $S' \to S$ a flat cover. We then obtain a commutative diagram:

$$\begin{array}{ccc}
\mathsf{IndCoh}^*(S) & \longrightarrow & \mathrm{Tot}_{semi}\,\mathsf{IndCoh}^*(T^{\times_S \bullet+1}) \\
\downarrow & & \downarrow \\
\mathrm{Tot}_{semi}\,\mathsf{IndCoh}^*(S'^{,\times_S \bullet+1}) & \longrightarrow & \mathrm{Tot}_{semi}\,\mathrm{Tot}_{semi}\,\mathsf{IndCoh}^*(S'^{,\times_S \bullet+1} \times_S T^{\times_S \bullet+1})
\end{array}$$

The left, bottom and right arrows are isomorphisms by the previous step, so the top arrow is as well.

$\square$

Corollary 6.25.2. *Let $S = T/H$ for H a classical affine group scheme acting on $T \in \mathsf{IndSch}_{reas}$. Then the functor:*

$$\mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(T)^{H,w,naive}$$

is an equivalence.

Proof. Clear from the Theorem 6.25.1 (e.g., using Proposition 6.36.4 and Proposition 6.35.2 (2) to convert IndCoh^* on the relevant products to tensor products). □

6.26. t-structures. Let $S \in \mathsf{PreStk}_{w.ren}$ be given. Then $\mathsf{IndCoh}^*(S)$ has a unique t-structure such that for every $U \in \mathsf{IndSch}_{reas}$ and flat $U \to S$, the pullback functor $\mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(U)$ is t-exact.

Indeed, by definition, we have:

$$\mathsf{IndCoh}^*(S) = \lim_{\substack{U \in \mathsf{IndSch}_{reas} \\ U \to S \text{ flat}}} \mathsf{IndCoh}^*(U)$$

and all of the structural functors are t-exact by Corollary 6.16.2.

6.27. Coherence. Next, for $S \in \mathsf{PreStk}_{w.ren}$, we define $\mathsf{Coh}(S) \subseteq \mathsf{IndCoh}^*(S)$ as:

$$\mathsf{Coh}(S) = \lim_{\substack{U \in \mathsf{IndSch}_{reas} \\ U \to S \text{ flat}}} \mathsf{Coh}(U).$$

Clearly $*$-pullback along locally flat maps preserve Coh.

6.28. Renormalizable prestacks.

Definition 6.28.1. $S \in \mathsf{PreStk}_{conv}$ is *renormalizable* if there exists a flat cover $f : T \to S$ with $T \in \mathsf{IndSch}_{laft}$ an indscheme locally almost of finite type.

For S renormalizable, we let $\mathsf{IndCoh}^*_{ren}(S)$ denote $\mathsf{Ind}(\mathsf{Coh}(S))$.

Remark 6.28.2. One might prefer a definition in greater generality (e.g., without finiteness hypotheses on T). However, this definition suffices for our applications, and this finiteness hypothesis simplifies the theory (essentially by Lemma 6.28.4 below).

Example 6.28.3. By Lemma 6.18.1 and Theorem 6.25.1, $\mathcal{F} \in \mathsf{IndCoh}^*(S)$ is coherent if and only if its $*$-pullback to some flat cover is so. In particular, $S = T/H$ for $T \in \mathsf{IndSch}_{laft}$ and H a classical affine group scheme acting on T, then S is renormalizable with $\mathsf{IndCoh}^*_{ren}(S) = \mathsf{IndCoh}^*(T)^{H,w}$.

Lemma 6.28.4. *For S a renormalizable prestack, coherent objects in $\mathsf{IndCoh}^*(S)$ are closed under truncations.*

Proof. By Theorem 6.25.1 and the definition, this reduces to the case of indschemes locally almost of finite type where it is clear. □

Proposition 6.28.5. *Let S be a renormalizable prestack. Define a t-structure on $\mathsf{IndCoh}^*_{ren}(S)$ by taking connective objects to be generated under colimits by $\mathsf{Coh}(S) \cap \mathsf{IndCoh}^*(S)^{\leq 0}$.*

Then then canonical functor $\mathsf{IndCoh}^*_{ren}(S) \to \mathsf{IndCoh}^*(S)$ is t-exact and induces an equivalence on eventually coconnective subcategories.

Proof. Lemma 6.28.4 implies that $\mathsf{Coh}(S) \hookrightarrow \mathsf{IndCoh}^*_{ren}(S)$ is closed under truncations for this t-structure. This clearly implies that $\mathsf{IndCoh}^*_{ren}(S) \to \mathsf{IndCoh}^*(S)$ is t-exact.

Next, observe that if $\mathcal{F} \in \mathsf{Coh}(S)$, then $\mathcal{F}$ is compact in $\mathsf{IndCoh}^*(S)^{\geq -N}$ for all $N \gg 0$. Indeed, this follows from Step 3 from the proof of Lemma 6.11.2. Combined with the fact that compact objects in $\mathsf{IndCoh}^*_{ren}(S)$ are closed under truncations, this implies that $\mathsf{IndCoh}^*_{ren}(S)^+ \to \mathsf{IndCoh}^*(S)^+$ is fully-faithful.

Finally, an argument as in Lemma 5.19.1 shows that the functor is essentially surjective. □

6.29. Let $f : S_1 \to S_2 \in \mathsf{PreStk}_{ren}$ be a reasonable indschematic morphism. Then f_*^{IndCoh} : $\mathsf{IndCoh}^*(S_1) \to \mathsf{IndCoh}^*(S_2)$ is left t-exact. Therefore, by Lemma 6.28.4 and Proposition 6.28.5, there exists a unique left t-exact functor $f_*^{\mathsf{IndCoh}_{ren}} : \mathsf{IndCoh}^*_{ren}(S_1) \to \mathsf{IndCoh}^*_{ren}(S_2)$ fitting into a commutative diagram:

$$
\begin{array}{ccc}
\mathsf{IndCoh}^*_{ren}(S_1) & \xrightarrow{f_*^{\mathsf{IndCoh}_{ren}}} & \mathsf{IndCoh}^*_{ren}(S_2) \\
\downarrow & & \downarrow \\
\mathsf{IndCoh}^*(S_1) & \xrightarrow{f_*^{\mathsf{IndCoh}}} & \mathsf{IndCoh}^*(S_2)
\end{array}
$$

(with vertical arrows the canonical functors).

Similarly, if f is locally flat, then $f^{*,\mathsf{IndCoh}}$ is t-exact, so there is a unique functor $f^{*,\mathsf{IndCoh}_{ren}}$: $\mathsf{IndCoh}^*_{ren}(S_2) \to \mathsf{IndCoh}^*_{ren}(S_1)$ fitting into an analogous diagram to the above.

Observe that renoralizable prestacks are closed under fiber squares with one leg locally flat and the other leg indschematic locally almost of finite type. As in [Gai5] Proposition 3.2.4, there is a unique functor:

$$
\mathsf{IndCoh}^*_{ren} : \mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat} \to \mathsf{DGCat}_{cont}
$$

equipped with a natural transformation to the functor $\mathsf{IndCoh}^* : \mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat} \to \mathsf{DGCat}_{cont}$ (obtained by restriction from the functor in §6.24) that on every $S \in \mathsf{PreStk}_{ren}$ evaluates to the canonical functor $\mathsf{IndCoh}^*_{ren}(S) \to \mathsf{IndCoh}^*(S)$. Here $indsch.lafp$ is shorthand for $indschematic$ $locally$ $almost$ of $finite$ $presentation$.[48]

6.30. **Symmetric monoidal structures.** Let $\mathcal{C} \subseteq \mathsf{PreStk}_{conv}$ denote any one of the subcategories:

$$
{}^{>-\infty}\mathsf{Sch}_{qcqs} \subseteq \mathsf{Sch}_{reas} \subseteq \mathsf{IndSch}_{reas} \subseteq \mathsf{PreStk}_{w.ren}
$$

or take $\mathcal{C} = \mathsf{PreStk}_{ren}$.

It is direct from the various definitions that $\mathcal{C}$ is closed under pairwise Cartesian products in PreStk_{conv}. In particular, the various correspondence categories we have considered admit symmetric monoidal structures with monoidal product given objectwise by Cartesian product: see [GR4] §V.3.

We will upgrade the various versions of IndCoh^* considered so far to have lax symmetric monoidal structures. This construction is straightforward following [GR4]; we indicate the logic below.

6.31. First, observe that the functor:

$$
(\Psi : \mathsf{IndCoh}^* \to \mathsf{QCoh}) : {}^{>-\infty}\mathsf{Sch}_{qcqs} \to \mathsf{Hom}(\Delta^1, \mathsf{DGCat}_{cont})
$$

admits a unique lax symmetric monoidal structure upgrading the standard symmetric monoidal structure on $\mathsf{QCoh} : {}^{>-\infty}\mathsf{Sch}_{qcqs} \to \mathsf{DGCat}_{cont}$. Indeed, this follows by the same method as in [Gai5] Proposition 3.2.4.

In particular, for $S, T \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$, there is an exterior product functor:

$$
- \boxtimes - : \mathsf{IndCoh}^*(S) \otimes \mathsf{IndCoh}^*(T) \to \mathsf{IndCoh}^*(S \times T).
$$

This functor is uniquely characterized by the fact that it maps $\mathsf{Coh}(S) \times \mathsf{Coh}(T) \to \mathsf{Coh}(S \times T)$ and is compatible with Ψ and exterior product of coherent sheaves.

Remark 6.31.1. We will discuss when the above functor is an equivalence in §6.35.

---

[48]We remark that such a morphism is in particular reasonable indschematic.

6.32.   The above lax symmetric monoidal structure canonically extends to one on the functor $\mathsf{IndCoh}^* : \mathrm{Corr}(^{>-\infty}\mathsf{Sch}_{qcqs})_{all;flat} \to \mathsf{DGCat}_{cont}$ using [GR4] Theorem V.1.3.2.2 (and the definition of the symmetric monoidal structure on correspondences from [GR4] §V.3.2.1).

6.33.   Similar logic applies for reasonable indschemes: by Kan extension, $\mathsf{IndCoh}^* : \mathsf{IndSch}_{reas} \to \mathsf{DGCat}_{cont}$ has a canonical lax symmetric monoidal structure, and this extends canonically to a lax symmetric monoidal structure on the functor $\mathrm{Corr}(\mathsf{IndSch}_{reas})_{all;flat} \to \mathsf{DGCat}_{cont}$.

6.34.   Next, we apply [GR4] Proposition V.3.3.2.4 to obtain a lax monoidal structure on $\mathsf{IndCoh}^* :$ $\mathrm{Corr}(\mathsf{PreStk}_{w.ren})_{reas.indsch;loc.flat} \to \mathsf{DGCat}_{cont}$.

Finally, by a similar argument as for eventually coconnective schemes (i.e., using t-structures), the functor $\mathsf{IndCoh}^*_{ren} : \mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat} \to \mathsf{DGCat}_{cont}$ admits a canonical lax symmetric monoidal structure (characterized by compatibility with the one on IndCoh^* and the natural transformation $\mathsf{IndCoh}^*_{ren} \to \mathsf{IndCoh}^*$).

6.35. **Strictness.** We now study when our lax symmetric monoidal functors behave as honest symmetric monoidal functors.

Definition 6.35.1.   A weakly renormalizable prestack S is *strict* if for every $T \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$, the functor:

$$- \boxtimes - : \mathsf{IndCoh}^*(S) \otimes \mathsf{IndCoh}^*(T) \to \mathsf{IndCoh}^*(S \times T) \tag{6.35.1}$$

is an equivalence.

Before giving examples, we record some basic properties about this notion.

Proposition 6.35.2.   *(1) Suppose $S \in \mathsf{PreStk}_{w.ren}$ is strict. Then for every $T \in \mathsf{IndSch}_{reas}$, the natural functor:*

$$- \boxtimes - : \mathsf{IndCoh}^*(S) \otimes \mathsf{IndCoh}^*(T) \to \mathsf{IndCoh}^*(S \times T)$$

is an equivalence.

(2) Suppose $S \in \mathsf{IndSch}_{reas}$ is strict. Then for every $T \in \mathsf{PreStk}_{w.ren}$, the natural functor:

$$- \boxtimes - : \mathsf{IndCoh}^*(S) \otimes \mathsf{IndCoh}^*(T) \to \mathsf{IndCoh}^*(S \times T)$$

is an equivalence.

(3) Suppose $S \in \mathsf{IndSch}_{reas}$ is a filtered colimit under almost finitely presented closed embeddings $S = \mathrm{colim}_i S_i$ with $S_i \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ strict. Then S is strict.

(4) Suppose $S \in \mathsf{PreStk}_{w.ren}$ admits a flat cover by a strict reasonable indscheme. Then S is strict.

Proof. (1) (resp. (3)) is immediate from the presentation of $\mathsf{IndCoh}^*(T)$ (resp. $\mathsf{IndCoh}^*(S)$) as a colimit (using that T, resp. S, is assumed to be in IndSch_{reas}). Then (2) is similarly formal, noting that we can commute the relevant tensor product and limit because $\mathsf{IndCoh}^*(S)$ is compactly generated (hence dualizable) for $S \in \mathsf{IndSch}_{reas}$. The same applies in (4): we have to commute a limit with a tensor product against $\mathsf{IndCoh}^*(T)$, which we can do because the relevant T here is assumed to be in $^{>-\infty}\mathsf{Sch}_{qcqs}$ so $\mathsf{IndCoh}^*(T)$ is compactly generated.

$\square$

Warning 6.35.3. In some contexts, people speak use a more general notion of indscheme in which transition maps are not required to be closed embeddings, and use the term *strict indscheme* to refer to the (more standard) notion of indscheme we have used. This terminology has no relationship to the above notion of strict indscheme(/prestack); we hope our use of this terminology in the present context does not create confusion.

6.36. We now give some examples of strictness. Note that Proposition 6.35.2 (3) and (4) reduce us to constructing examples of strict S with $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$.

Lemma 6.36.1. *For every* $S, T \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$, *the functor* (6.35.1) *is fully-faithful.*

Proof. The same argument as in [Gai5] Proposition 4.6.2 applies.

$\square$

Proposition 6.36.2. *If* $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ *is almost finite type (over* $\mathrm{Spec}(k)$[49]*), then* S *is strict.*

Proof. Let $T \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ be given. By Lemma 6.36.1, we need to show (6.35.1) is essentially surjective. By Zariski descent, we reduce to the case where S is separated.

We have a standard convolution functor:

$$- \star - : \mathsf{IndCoh}(S \times S) \otimes \mathsf{IndCoh}^*(S \times T) \to \mathsf{IndCoh}^*(S \times T)$$

$$\big(\mathcal{K} \in \mathsf{IndCoh}(S \times S), \mathcal{F} \in \mathsf{IndCoh}^*(S \times T)\big) \mapsto p_{23,*}^{\mathsf{IndCoh}}(\mathrm{id}_S \times \Delta_S \times \mathrm{id}_T)^!(\mathcal{K} \boxtimes \mathcal{F})$$

$$(6.36.1)$$

(where e.g., $\Delta_S : S \to S \times S$ is the diagonal and $p_{23} : S \times S \times T \to S \times T$ is the projection onto the last two coordinates).

By Lemma 6.14.2 (1), we have:

$$\Delta_{S,*}^{\mathsf{IndCoh}}(\omega_S) \star \mathcal{F} = \mathcal{F}$$

for any $\mathcal{F} \in \mathsf{IndCoh}^*(S \times T)$. In particular, (6.36.1) is essentially surjective.

Now note the composition:

$$\mathsf{IndCoh}(S) \otimes \mathsf{IndCoh}(S) \otimes \mathsf{IndCoh}^*(S \times T) \to$$

$$\mathsf{IndCoh}(S \times S) \otimes \mathsf{IndCoh}^*(S \times T) \xrightarrow{(6.36.1)} \mathsf{IndCoh}^*(S \times T)$$

$$(6.36.2)$$

factors through the subcategory $\mathsf{IndCoh}(S) \otimes \mathsf{IndCoh}^*(T)$. Now the first functor in (6.36.2) is an equivalence by [Gai5] Proposition 4.6.2, so we are done.

$\square$

Before proceeding, we need the following auxiliary result.

Lemma 6.36.3. *Suppose* $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ *is written as a filtered inverse limit* $S = \lim_i S_i$ *under flat affine structure maps with* $S_i \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$. *Then* $*$-*pullback induces an equivalence:*

$$\operatorname*{colim}_{i, upper-*} \mathsf{IndCoh}^*(S_i) \xrightarrow{\simeq} \mathsf{IndCoh}^*(S) \in \mathsf{DGCat}_{cont}.$$

Proof. First, note that the functor:

$$\operatorname*{colim}_i \mathsf{QCoh}(S_i) \to \mathsf{QCoh}(S) \in \mathsf{DGCat}_{cont} \qquad (6.36.3)$$

---

[49]As in [Gai5], it is important here that k have characteristic 0 or be a perfect field of characteristic p.

is an equivalence. Indeed, both sides are monadic over $\mathsf{QCoh}(S_{i_0})$ for any fixed index i_0, and the induced map on monads is an isomorphism (here we are not using the flatness assumption).

Therefore, for any index i_0 and any $\mathcal{F} \in \mathsf{Perf}(S_{i_0})$, $\mathcal{G} \in \mathsf{QCoh}(S_{i_0})$, the natural map:

$$\operatorname*{colim}_{i \to i_0} \mathrm{Hom}_{\mathsf{QCoh}(S_i)}(\alpha_{i_0,i}^*(\mathcal{F}), \alpha_{i_0,i}^*(\mathcal{G})) \to \mathrm{Hom}_{\mathsf{QCoh}(S)}(\alpha_{i_0}^*(\mathcal{F}), \alpha_{i_0}^*(\mathcal{G})) \in \mathsf{Gpd} \qquad (6.36.4)$$

is an isomorphism; here $\alpha_{i_0,i} : S_i \to S_{i_0}$ and $\alpha_{i_0} : S \to S_{i_0}$ denote the structural maps.

We now claim that (6.36.4) is an isomorphism $\mathcal{G} \in \mathsf{QCoh}(S_{i_0})^+$ and $\mathcal{F} \in \mathsf{Coh}(S_{i_0})$. Indeed, if $\mathcal{F}, \mathcal{G} \in \mathsf{QCoh}(S_{i_0})^{\geqslant -N}$, we choose $\mathcal{F}' \in \mathsf{Perf}(S_{i_0})$ equipped with an isomorphism $\tau^{\geqslant -N} \mathcal{F}' \xrightarrow{\simeq} \mathcal{F}$. By flatness, we have:

$$\alpha_{i_0,i}^*(\mathcal{G}) \in \mathsf{QCoh}(S_i)^{\geqslant -N}$$
$$\tau^{\geqslant -N} \alpha_{i_0,i}^*(\mathcal{F}') \xrightarrow{\simeq} \alpha_{i_0,i}^*(\mathcal{F})$$

and similarly for α_{i_0}. Therefore, the sides of (6.36.4) are unchanged under replacing $\mathcal{F}$ by $\mathcal{F}'$, so we are reduced to that case.

In particular, (6.36.4) is an isomorphism when $\mathcal{F}, \mathcal{G} \in \mathsf{Coh}(S_{i_0})$. Unwinding the above logic, it follows that the natural functor:

$$\operatorname*{colim}_{i} \mathsf{IndCoh}^*(S_i) \xrightarrow{\simeq} \mathsf{IndCoh}^*(S) \in \mathsf{DGCat}_{cont} \qquad (6.36.5)$$

is fully-faithful. To show that this functor is an equivalence, it suffices to show any $\mathcal{F} \in \mathsf{Coh}(S)$ lies in the essential image.

Suppose $\mathcal{F}$ lies in cohomological degrees $\geqslant -N$, and choose $\mathcal{F}' \in \mathsf{Perf}(S)$ with $\tau^{\geqslant -N}(\mathcal{F}') \xrightarrow{\simeq} \mathcal{F}$. Observe that we also have:

$$\tau^{\geqslant -N} \mathcal{F}' \xrightarrow{\simeq} \mathcal{F} \in \mathsf{IndCoh}^*(S)$$

where the truncation is for the t-structure on IndCoh^* (and we are using the embeddings $\mathsf{Perf}(S) \subseteq \mathsf{Coh}(S) \subseteq \mathsf{IndCoh}^*(S)$). Indeed, both sides are bounded from below, so it suffices to check this after applying the t-exact functor Ψ; then the corresponding isomorphism was a defining property of $\mathcal{F}'$.

By (6.36.3), there exists an index i_0 and some $\mathcal{F}'_{i_0} \in \mathsf{Perf}(S_{i_0})$ equipped with an isomorphism $\alpha_{i_0}^*(\mathcal{F}'_{i_0}) \xrightarrow{\simeq} \mathcal{F}'$. We then obtain:

$$\mathcal{F} = \tau^{\geqslant -N}(\mathcal{F}') = \tau^{\geqslant -N} \alpha_{i_0}^{*,\mathsf{IndCoh}}(\mathcal{F}'_{i_0}) = \alpha_{i_0}^{*,\mathsf{IndCoh}} \tau^{\geqslant -N}(\mathcal{F}'_{i_0}) \in \mathsf{IndCoh}^*(S)$$

(where all truncations are for t-structures on IndCoh^*). This shows $\mathcal{F}$ is in the essential image of (6.36.5) as desired.

$\square$

Next, we show:

Proposition 6.36.4. *Suppose* $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ *can be written*[50] *as a filtered limit* $S = \lim_i S_i$ *under flat affine structure maps with each* S_i *locally almost of finite type. Then* S *is strict.*

Proof. Let $T \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ be given. We have a commutative diagram in DGCat_{cont}:

---

[50]This is an analogue (better suited for IndCoh) of the notion of *placid* scheme introduced in [Ras3].

$$\begin{array}{ccc}
\mathsf{IndCoh}^*(S) \otimes \mathsf{IndCoh}^*(T) & \longrightarrow & \mathsf{IndCoh}^*(S \times T) \\
\uparrow & & \uparrow \\
\underset{i}{\mathrm{colim}}\, \mathsf{IndCoh}^*(S_i) \otimes \mathsf{IndCoh}^*(T) & \longrightarrow & \underset{i}{\mathrm{colim}}\, \mathsf{IndCoh}^*(S_i \times T).
\end{array}$$

By Lemma 6.36.3 (applied to S and $S \times T$), the vertical arrows are isomorphisms. By Proposition 6.36.2, the bottom arrow is an isomorphism. Therefore, the top arrow is an isomorphism, as desired.

$\square$

7. Weak actions of group indschemes

7.1. In this section, we study weak group actions on categories for *Tate group indschemes*. This is a convenient class of group indschemes containing loop groups and their relatives.

For a *compact open* subgroup $K \subseteq H$, we define a weak Hecke category $\mathcal{H}^w_{H,K}$, which is a certain monoidal DG category. We then define H–mod_{weak} in such a way that we have an equivalence:

$$(-)^{K,w} : H\text{–}\mathsf{mod}_{weak} \xrightarrow{\simeq} \mathcal{H}^w_{H,K}\text{–}\mathsf{mod}. \tag{7.1.1}$$

The main homotopical difficulty is to give a definition that is manifestly independent of K, which we do by a sort of brute force argument.

The main new feature in the setting of (*polarizable*) group indschemes, as opposed to group schemes, is the presence of the *modular character*; see §7.17. We especially draw the reader's attention to Proposition 7.18.2. The reader who is not worried about homotopical issues may essentially skip to §7.17, taking (7.1.1) as something like a definition.

7.2. **Tate group indschemes.** Let H be a group indscheme.

Definition 7.2.1. A *compact open subgroup* of H is a classical affine group scheme K with a closed embedding $K \hookrightarrow H$ that is a homomorphism, and such that H/K is an indscheme locally almost of finite type (equivalently, presentation).

A *Tate group indscheme* is a reasonable group indscheme admitting a compact open subgroup.

Example 7.2.2. For G an affine algebraic group, one can take $H = G(K)$ and $H_0 = G(O)$. More generally, a compact open subgroup is a closed group subscheme containing $\mathrm{Ker}(G(O) \to G(O/t^N))$ for $N \gg 0$.

Example 7.2.3. For G an affine algebraic group and $H_0 \subseteq G(K)$ a compact open subgroup, one can take $H = G(K)^\wedge_{H_0}$ to be the formal completion of $G(K)$ along H_0.

Remark 7.2.4. If H is an ind-affine Tate group indscheme, then H satisfies the hypotheses of §5.6, and the definition of $\mathsf{IndCoh}^*(H)$ from *loc. cit.* clearly coincides with the construction from §6 in this case. We remark that the notion of naive action of H from *loc. cit.* extends to the possibly non-ind-affine setting considered here: this means an $\mathsf{IndCoh}^*(H)$-module category.

Throughout this section, H denotes a Tate group indscheme. Our main objective in this section is to develop a theory of genuine H-actions on categories.

Remark 7.2.5. We remark that H being a Tate group indscheme implies in particular that H is reasonable (as an indscheme). Moreover, by Proposition 6.36.4 and Proposition 6.35.2 (3), H is strict (in the sense of §6.35).

7.3. **Topological conventions.** The reader may safely ignore this discussion at first pass.

Throughout this section, we impose a convention (implicit above) that all quotients are under-stood as prestack quotients (i.e., geometric realizations of the bar construction) sheafified for the Zariski topology. In particular, $\mathbb{B}(-)$ indicates the Zariski sheafified classifying space.

Therefore, in the definition of compact open subgroup above, the condition that H/K be an indscheme is a funny condition which is satisfied if e.g. the *étale* sheafification of this quotient is an indscheme and the projection from H to this quotient is a K-torsor locally trivial in the Zariski topology. This is the case when K is prounipotent or when $H = G(K)$ and $K = G(O)$ (as is well-known, see e.g. [BD1] Theorem 4.5.1).

This convention can be relaxed to the étale topology at the cost of replacing the fundamental role of quasi-compact quasi-separated schemes in §6 by quasi-compact quasi-separated algebraic spaces. That be done without serious modification (using [Lur4] Proposition 9.6.1.1 as a staring point), but we content ourselves with the above restrictions since they suffice for our applications.

(We remark that changing *Zariski* to *étale* would mean that for any $K \subseteq H$ with a prounipotent tail, whenever K "should" be regarded as a compact open subgroup, it is.)

7.4.   The following terminology will be convenient in what follows.

Definition 7.4.1. A *DG 2-category* is a category enriched over DGCat_{cont}. We let $2\text{–}\mathsf{DGCat}$ denote the category of DG 2-categories (for our purposes, it is sufficient to view $2\text{–}\mathsf{DGCat}$ as a 1-category).

7.5. **Hecke categories.** Let H be a group indscheme and let K be a group subscheme.

First, recall[51] that the (Zariski sheafified) quotient $K\backslash H/K = \mathbb{B}K \times_{\mathbb{B}H} \mathbb{B}K$ is an algebra in $\mathrm{Corr}(\mathsf{PreStk}_{conv})_{all;all}$ with unit and multiplication maps defined by the correspondences:

Now observe that by our assumptions on H and K, each of the left arrows in the above diagrams are locally flat while each of the right arrows are locally almost of finite presentation. Therefore, $K\backslash H/K$ is canonically an algebra in $\mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat}$.

Definition 7.5.1. The *(genuine) weak Hecke category* $\mathcal{H}^w_{H,K} \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$ is $\mathsf{IndCoh}(H/K)^{K,w} = \mathsf{IndCoh}^*_{ren}(K\backslash H/K)$, with monoidal structure induced from the above algebra (under correspon-dences) structure on $K\backslash H/K$ and by applying the symmetric monoidal functor:

$$\mathsf{IndCoh}^*_{ren} : \mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat} \to \mathsf{DGCat}_{cont}$$

constructed in §6.

Remark 7.5.2. We do not need the whole theory of §6 to construct the above monoidal structure on $\mathcal{H}^w_{H,K}$. Indeed, this follows from Example 8.16.3 below.

When we introduce $H\text{–}\mathsf{mod}_{weak}$ below, we will have $H\text{–}\mathsf{mod}_{weak} \simeq \mathcal{H}^w_{H,K}\text{–}\mathsf{mod}$, so this elementary construction gives a quick construction of $H\text{–}\mathsf{mod}_{weak}$.

---

[51]If H is classical, which is our main case of interest (see Remark 7.6.2), what follows is complete. If H is derived, [GR4] §V.3.4 covers the homotopy coherence issues.

However, it is not clearly independent of the choice of compact open subgroup K, and this leads to difficulties with the functoriality in H. For our purposes, the most important use of the theory of §6 is to deal with these functoriality problems.

7.6. Genuine actions. Let TateGp denote the category of Tate group indschemes.

Our first goal will be to show the following result. In what follows, we regard TateGp as a symmetric monoidal category via products, and similarly for $2\text{–}\mathsf{DGCat}$.

Proposition-Construction 7.6.1. *There is a canonical lax symmetric monoidal functor:*

$$\mathsf{TateGp}^{op} \to 2\text{–}\mathsf{DGCat}$$
$$H \mapsto H\text{–}\mathsf{mod}_{weak}$$

with the following properties.

(1) For any $H \in \mathsf{TateGp}^{op}$ and any $K \subseteq H$ compact open, there is a canonical equivalence:

$$H\text{–}\mathsf{mod}_{weak} \xrightarrow{(-)^{K,w}} \mathcal{H}^w_{H,K}\text{–}\mathsf{mod} := \mathcal{H}^w_{H,K}\text{–}\mathsf{mod}(\mathsf{DGCat}_{cont}).$$

(2) For any morphism $f : H_1 \to H_2 \in \mathsf{TateGp}$ and any $K_i \subseteq H_i$ compact open subgroups with $f(K_1) \subseteq K_2$, the functor:

$$H_2\text{–}\mathsf{mod}_{weak} \to H_1\text{–}\mathsf{mod}_{weak}$$

canonically fits into a commutative diagram:

$$
\begin{array}{ccc}
H_2\text{–}\mathsf{mod}_{weak} & \longrightarrow & H_1\text{–}\mathsf{mod}_{weak} \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
\mathcal{H}^w_{H_2,K_2}\text{–}\mathsf{mod} & \longrightarrow & \mathcal{H}^w_{H_1,K_1}\text{–}\mathsf{mod}
\end{array}
$$

where the bottom arrow is constructed using the Hecke-bimodule $\mathsf{IndCoh}^*_{ren}(K_1\backslash H_2/K_2)$.

(3) For I a finite set and $\{H_i \in \mathsf{TateGp}\}_{i\in I}$ equipped with compact open subgroups $K_i \subseteq H_i$, the functor:

$$\prod_{i\in I}(H_i\text{–}\mathsf{mod}_{weak}) \to (\prod_{i\in I} H_i\text{–}\mathsf{mod}_{weak})$$

coming from the lax symmetric monoidal structure corresponds under the equivalences from (1) to the functor:

$$\prod_{i\in I}(\mathcal{H}^w_{H_i,K_i}\text{–}\mathsf{mod}) \to \mathcal{H}^w_{\prod_{i\in I} H_i,\prod_{i\in I} K_i}\text{–}\mathsf{mod} \simeq (\bigotimes_i \mathcal{H}^w_{H_i,K_i})\text{–}\mathsf{mod}$$

$$\{\mathcal{C}_i \in \mathcal{H}^w_{H_i,K_i}\text{–}\mathsf{mod}\}_{i\in I} \mapsto \bigotimes_i \mathcal{C}_i.$$

Remark 7.6.2. Let $\mathsf{TateGp}^{cl} \subseteq \mathsf{TateGp}$ denote the subcategory of Tate group indschemes that are classical as prestacks.[52] Note that TateGp^{cl} is actually a $(1,1)$-category.

To simplify the exposition, we actually only give the restriction of the functor from Proposition-Construction 7.6.1 to TateGp^{cl}; this suffices for our applications. The requisite homotopy coherence needed to provide all of Proposition-Construction 7.6.1 can be given using [GR4] §V.3.4. But the

---

[52]As in [GR3] Theorem 9.3.4, any formally smooth Tate group indscheme that is weakly $\aleph_0$ in the sense of *loc. cit.* is automatically classical. In particular, this applies for a loop group, or for its formal completion at any compact open subgroup scheme.

argument (at least in the form the author has in mind) is more tedious than merits inclusion (without real applications) here.

We will give the construction after some preliminary remarks.

7.7. **Morita theory.** We will need a review of Morita categories in a higher categorical context.

7.8. First, let $\mathcal{C}$ be a symmetric monoidal category with colimits and whose monoidal product commutes with colimits in each variable.

Let $\mathcal{C}$–mod denote the 2-category of $\mathcal{C}$-module categories $\mathcal{M}$ with colimits and for which for every $\mathcal{F} \in \mathcal{C}$ and $\mathcal{G} \in \mathcal{M}$, the action functors $\mathcal{F} \star - : \mathcal{M} \to \mathcal{M}$ and $- \star \mathcal{G} : \mathcal{C} \to \mathcal{M}$ admit right adjoints[53] (so in particular, the action functor commutes with colimits in each variable).

In this case, let $\mathsf{Morita}(\mathcal{C})$ be the 2-category defined as the full subcategory of $\mathcal{C}$–mod consisting of objects of the form A–mod for $A \in \mathsf{Alg}(\mathcal{C})$. As in [Lur3] Remark 4.8.4.9, this recovers the standard Morita 2-category if $\mathcal{C}$ is a $(1, 1)$-category.

In particular, we remind from *loc. cit.* that if we have $\mathcal{M}_i \in \mathsf{Morita}(\mathcal{C})$ for $i = 1, 2$ and we choose $A_i \in \mathsf{Alg}(\mathcal{C})$ such that $\mathcal{M}_i = A_i$–mod, then the category of morphisms $F : \mathcal{M}_1 \to \mathcal{M}_2$ is canonically equivalent to the data of an (A_2, A_1)-bimodule.

7.9. In what follows, we will use the (fully-faithful) Segal functor $\mathrm{Seq}_\bullet : 2\text{–}\mathsf{Cat} \to \mathsf{Hom}(\mathbf{\Delta}^{op}, \mathsf{Cat})$; we refer to [GR4] Appendix A.1 for details.

7.10. For $\mathcal{C}$ a symmetric monoidal category that may not have colimits, there is no hope for defining its Morita category.[54] However, we can still define the associated simplicial category $\mathrm{Seq}_\bullet(\mathsf{Morita}(\mathcal{C}))$ as follows.

First, if $\mathcal{C}$ is essentially small, embed $\mathcal{C}$ into its Yoneda category $\mathsf{Yo}(\mathcal{C}) := \mathsf{Hom}(\mathcal{C}^{op}, \mathsf{Gpd})$. Note that $\mathsf{Yo}(\mathcal{C})$ is equipped with a symmetric monoidal structure commuting with colimits in each variable. In particular, $\mathsf{Morita}(\mathsf{Yo}(\mathcal{C}))$ and $\mathrm{Seq}_\bullet(\mathsf{Morita}(\mathcal{C}))$ are defined.

Now for $[n] \in \mathbf{\Delta}^{op}$, define $\mathrm{Seq}_n(\mathcal{C})$ to be the full subcategory of $\mathrm{Seq}_n(\mathsf{Yo}(\mathcal{C}))$ whose objects are sequences $\mathcal{M}_0 \to \ldots \to \mathcal{M}_n$ where each $\mathcal{M}_i$ is of the form A_i–mod for $A_i \in \mathsf{Alg}(\mathcal{C}) \subseteq \mathsf{Alg}(\mathsf{Yo}(\mathcal{C}))$ and each morphism $\mathcal{M}_i \to \mathcal{M}_{i+1}$ admits a right adjoint in the 2-category $\mathsf{Yo}(\mathcal{C})$–mod.[55] It is straightforward to see that this latter condition is equivalent to $\mathcal{M}_i \to \mathcal{M}_{i+1}$ corresponding to an (A_{i+1}, A_i)-bimodule that lies in $\mathcal{C} \subseteq \mathsf{Yo}(\mathcal{C})$.

7.11. In general, we define $\mathrm{Seq}_\bullet(\mathsf{Morita}(\mathcal{C})) : \mathbf{\Delta}^{op} \to \mathsf{Cat}$ e.g. by extending the universe so that $\mathcal{C}$ is essentially small. It is straightforward to see that we do actually obtain a simplicial category[56] in this way.

Moreover, if $\mathcal{C}$ does admit colimits and its monoidal product preserves such colimits, then the simplicial category just defined canonically coincides with the same denoted simplicial category defined by applying the Segal construction to the category $\mathsf{Morita}(\mathcal{C})$ from §7.8. Therefore, we are justified in not distinguishing the two notationally.

Notation 7.11.1. In what follows, for $\mathcal{C}$ as above and $A \in \mathsf{Alg}(\mathcal{C})$, we let $[A]$ denote the induced object in $\mathrm{Seq}_0(\mathsf{Morita}(\mathcal{C}))$.

---

[53]In the language of [Lur3] §4.2.1, $\mathcal{M}$ is cotensored and enriched over $\mathcal{C}$.

[54]Indeed, composition in a Morita category involves tensor products of bimodules, and this means certain geometric realizations need to exist.

[55]By the definition in §7.8, this means that the underlying functor admits a right adjoint that commutes with all colimits and is $\mathsf{Yo}(\mathcal{C})$-linear.

[56]By definition, a category has essentially small Homs. In principle, universe extension risks breaking this property, and the content of the claim is that this does not happen here.

7.12. Finally, we conclude by remarking that $\mathsf{Morita}(-)$ is by construction functorial for (left) lax symmetric monoidal functors.

7.13. Construction of the functor. We now return to the setting of §7.6.

Proof of Proposition-Construction 7.6.1. To simplify the exposition, we ignore the symmetric monoidal structures until the last step.

Step 1. First, define the category[57] $\mathsf{TateGp}_{fr} \subseteq \mathsf{Hom}(\Delta^1, \mathsf{TateGp})$ as the subcategory consisting of maps $K \to H$ that are the embedding of a compact open subgroup in a Tate group indscheme. Note that the forgetful functor $\mathrm{Oblv}_{fr} : \mathsf{TateGp}_{fr} \xrightarrow{(K \subseteq H) \mapsto H} \mathsf{TateGp}$ is 1-fully-faithful (i.e., the induced maps on Homs are fully-faithful morphisms of groupoids).

We claim that Oblv_{fr} is a Verdier localization functor. By definition, this means that for every $\mathcal{C} \in \mathsf{Cat}$, the functor:

$$\mathsf{Hom}(\mathsf{TateGp}, \mathcal{C}) \to \mathsf{Hom}(\mathsf{TateGp}_{fr}, \mathcal{C})$$

of restriction along Oblv_{fr} is fully-faithful functor with essential image consisting of functors sending Oblv_{fr}-local[58] morphisms to isomorphisms.

Indeed, note that the left adjoint to Oblv_{fr} is pro-representable: it maps $H \in \mathsf{TateGp}$ to the pro-object "$\lim$"$_{K \subseteq H \text{compact open}}(K \to H)$,[59] where $K \to H$ is regarded as an object in TateGp_{fr} and the quotation marks emphasize that this (filtered) limit takes place in the relevant pro-category. This pro-valued left adjoint is clearly fully-faithful, and this is well-known to imply the Verdier localization property.

Step 2. Next, we construct a *(right) lax* morphism:

$$\mathrm{Seq}_{\bullet}(\mathsf{TateGp}_{fr}^{op}) \to \mathrm{Seq}_{\bullet}(\mathsf{Morita}(\mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat}))$$

of simplicial categories sending $K \subseteq H \in \mathsf{TateGp}_{fr}$ (regarded as a 0-simplex of the left hand side) to $[K \backslash H / K]$ (regarded as a 0-simplex of the right hand side as in Notation 7.11.1). Here by *lax*, we mean that our morphism is merely a lax natural transformation, i.e., a functor over $\mathbf{\Delta}^{op}$ between the corresponding coCartesian Grothendieck fibrations.

This step is where we use Remark 7.6.2 to ignore homotopy coherence issues. That is, we treat TateGp as a $(1, 1)$-category (e.g,. by actually restricting to TateGp^{cl}).

Our lax functor assigns to every $[n] \in \mathbf{\Delta}$ and $[n]$-shaped diagram $(K_0 \subseteq H_0) \to \ldots \to (K_n \subseteq H_n)$ in TateGp_{fr} the $[n]$-simplex of $\mathrm{Seq}_{\bullet}(\mathsf{Morita}(\mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat}))$ defined by:

$$K_0 \backslash H_0 / K_0 \curvearrowright K_0 \backslash H_1 / K_1 \curvearrowleft K_1 \backslash H_1 / K_1 \curvearrowright \ldots K_{n-1} \backslash H_n / K_n \curvearrowleft K_n \backslash H_n / K_n.$$

The notation indicates that we consider $K_i \backslash H_i / K_i$ as an algebra in $\mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat}$ and $K_i \backslash H_{i+1} / K_i$ as a bimodule in this same correspondence category for $K_i \backslash H_i / K_i$ and $K_{i+1} \backslash H_{i+1} / K_{i+1}$.

We now need to specify where morphisms (in the relevant Grothendieck construction) are sent. To simplify the notation, we spell out the construction only for morphisms lying over the *active* morphism $\alpha : [1] \xrightarrow{0 \mapsto 0, 1 \mapsto 2} [2] \in \mathbf{\Delta}$. (The reader will readily see that this simplification really is only cosmetic.)

---

[57]The subscript is an abbreviation of *framed*.

[58]This phrase refers to morphisms in TateGp_{fr} that map to isomorphisms under Oblv_{fr}.

[59]The key point in verifying this formula is that for any $f : H_1 \to H_2 \in \mathsf{TateGp}$ and $K_2 \subseteq H_2$ a compact open subgroup, the category of compact open subgroups $K_1 \subseteq H_1$ mapping into K_2 is non-empty and filtered.

Then a 2-simplex of the left hand side above corresponds to a datum $(K_0 \subseteq H_0) \to (K_1 \subseteq H_1) \to (K_2 \subseteq H_2) \in \mathsf{TateGp}$, and a map to a 1-simplex $(\widetilde{K}_0 \subseteq \widetilde{H}_0)$ is equivalent to maps and $f_i : H_{\alpha(i)} \to \widetilde{H}_i$ $(i = 0, 1)$ sending $K_{\alpha(i)}$ to $\widetilde{K}_i$.

The relevant map in the right hand side is induced by the augmented simplicial diagram:

$$K_0\backslash H_1/K_1 \times K_1\backslash H_1/K_1 \times K_1\backslash H_1/K_2 \rightrightarrows K_0\backslash H_1/K_1 \times K_1\backslash H_2/K_2 \to \widetilde{K}_0\backslash\widetilde{H}_1\widetilde{K}_1$$

$$\in \mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat}.$$

Here all arrows are morphisms in the correspondence category (i.e., they represent correspondences), the underlying simplicial diagram is the bar construction for the relative tensor product of these left and right $K_1\backslash H_1/K_1$-modules, and the augmentation is given by the correspondence:

$$K_0\backslash H_1 \overset{K_1}{\times} H_2/K_2$$

$$K_0\backslash H_1/K_1 \times K_1\backslash H_2/K_2 \qquad\qquad \widetilde{K}_0\backslash\widetilde{H}_1/\widetilde{K}_1.$$

Here the left arrow is obvious, and the right arrow is the composition:

$$K_0\backslash H_1 \overset{K_1}{\times} H_2/K_2 \to K_0\backslash H_2 \overset{K_2}{\times} H_2/K_2 \xrightarrow{\mathrm{mult.}} K_0\backslash H_2/K_2 \xrightarrow{f_1} \widetilde{K}_0\backslash\widetilde{H}_1/\widetilde{K}_1.$$

Unwinding the constructions, this was exactly the sort of datum we needed to specify.

Step 3. Applying the lax symmetric monoidal functor $\mathsf{IndCoh}^*_{ren} : \mathrm{Corr}(\mathsf{PreStk}_{ren})_{indsch.lafp;loc.flat} \to \mathsf{DGCat}_{cont}$ and the functoriality from §7.12, we obtain a lax functor:

$$\mathrm{Seq}_\bullet(\mathsf{TateGp}^{op}_{fr}) \to \mathrm{Seq}_\bullet(\mathsf{Morita}(\mathsf{DGCat}_{cont})).$$

As these are each Segal categories for actual 2-categories, this is the same[60] as a lax functor $\mathsf{TateGp}^{op}_{fr} \to \mathsf{Morita}(\mathsf{DGCat}_{cont})$. The latter 2-category is by construction contained in DGCat_{cont}–mod, which is itself contained in 2–DGCat.

Therefore, we obtain a lax functor of 2-categories:

$$\mathsf{TateGp}^{op}_{fr} \to 2\text{–}\mathsf{DGCat}.$$

We claim that this lax functor is an actual functor.

Suppose we are given $(K_0 \subseteq H_0) \to (K_1 \subseteq H_1) \to (K_2 \subseteq H_2) \in \mathsf{TateGp}$. We obtain a diagram that commutes up to a natural transformation:

$$\mathcal{H}^w_{H_0,K_0}\text{–}\mathsf{mod} \longrightarrow \mathcal{H}^w_{H_2,K_2}\text{–}\mathsf{mod}$$

$$\mathcal{H}^w_{H_1,K_1}\text{–}\mathsf{mod}$$

corresponding to the map of Hecke bimodules:

$$\mathsf{IndCoh}^*_{ren}(K_0\backslash H_1/K_1) \underset{\mathcal{H}^w_{H_1,K_1}}{\otimes} \mathsf{IndCoh}^*_{ren}(K_1\backslash H_2/K_2) \to \mathsf{IndCoh}^*_{ren}(K_0\backslash H_2/K_2).$$

that we need to show is an isomorphism.

---

[60]By definition of lax functor; c.f. [GR4] Appendix A.1.3.

To verify this, note that we have a canonical monoidal functor $\mathsf{Rep}(K_1) \to \mathcal{H}^w_{H_1,K_1}$. By Lemma 5.20.1 (4), we have an equivalence:

$$\mathsf{Rep}(K_0) \underset{\mathsf{Rep}(K_1)}{\otimes} \mathcal{H}^w_{H_1,K_1} \xrightarrow{\simeq} \mathsf{IndCoh}^*_{ren}(K_0 \backslash H_1 / K_1)$$

of $\mathcal{H}^w_{H_1,K_1}$-module categories.

Therefore, we can calculate:

$$\mathsf{IndCoh}^*_{ren}(K_0 \backslash H_1 / K_1) \underset{\mathcal{H}^w_{H_1,K_1}}{\otimes} \mathsf{IndCoh}^*_{ren}(K_1 \backslash H_2 / K_2) =$$

$$\mathsf{Rep}(K_0) \underset{\mathsf{Rep}(K_1)}{\otimes} \mathcal{H}^w_{H_1,K_1} \underset{\mathcal{H}^w_{H_1,K_1}}{\otimes} \mathsf{IndCoh}^*_{ren}(K_1 \backslash H_2 / K_2) =$$

$$\mathsf{Rep}(K_0) \underset{\mathsf{Rep}(K_1)}{\otimes} \mathsf{IndCoh}^*_{ren}(K_1 \backslash H_2 / K_2) \xrightarrow{\simeq} \mathsf{IndCoh}^*_{ren}(K_0 \backslash H_2 / K_2)$$

as desired.

Step 4. Next, suppose that we are given an Oblv_{fr}-local morphism, or equivalently, $H \in \mathsf{TateGp}$ with an embedding of compact open subgroups $K_1 \subseteq K_2 \subseteq H$.

Then the functor:

$$\mathsf{TateGp}^{op}_{fr} \to 2\text{--}\mathsf{DGCat}$$

sends this datum to the functor:

$$\mathcal{H}^w_{H,K_2}\text{--}\mathsf{mod} \to \mathcal{H}^w_{H,K_1}\text{--}\mathsf{mod}$$

defined by the bimodule $\mathsf{IndCoh}^*_{ren}(K_1 \backslash H / K_2)$. By Step 1, to obtain the functor from Proposition-Construction 7.6.1, it suffices to show the above is an equivalence. As K_2 has a cofinal sequence of normal compact open subgroups, it suffices to treat the case where K_1 is normal in K_2.

Note that the composition:

$$\mathcal{H}^w_{H,K_2}\text{--}\mathsf{mod} \to \mathcal{H}^w_{H,K_1}\text{--}\mathsf{mod} \to \mathsf{DGCat}_{cont} \tag{7.13.1}$$

sends $\mathcal{D} \in \mathcal{H}^w_{H,K_2}\text{--}\mathsf{mod}$ to:

$$\mathsf{Rep}(K_1) \underset{\mathsf{Rep}(K_2)}{\otimes} \mathcal{D}$$

by the isomorphism:

$$\mathcal{H}^w_{H,K_2} \underset{\mathsf{Rep}(K_2)}{\otimes} \mathsf{Rep}(K_1) \xrightarrow{\simeq} \mathsf{IndCoh}^*_{ren}(K_2 \backslash H / K_1)$$

of $\mathcal{H}^w_{H,K_2}$-module categories (obtained as in the previous step from Lemma 5.20.1 (4)). By normality of K_1 in K_2, we may further identify:

$$\mathsf{Rep}(K_1) \underset{\mathsf{Rep}(K_2)}{\otimes} \mathcal{D} = \mathsf{Vect} \underset{\mathsf{Rep}(K_2/K_1)}{\otimes} \mathcal{D}.$$

By Theorem 5.10.1, this implies that (7.13.1) is conservative and commutes with limits, so is monadic.

Therefore, it suffices to show that the functor:

$$\mathcal{H}_{H,K_1} \to \mathsf{End}_{\mathcal{H}^w_{H,K_2}\text{--}\mathsf{mod}}(\mathsf{IndCoh}^*_{ren}(K_2 \backslash H / K_1)) \tag{7.13.2}$$

is an equivalence (as the right hand side is the monad defined by (7.13.1)). This follows by similar logic — the right hand side is:

$$\mathsf{Hom}_{\mathcal{H}_{H,K_2}^w\text{--mod}}\big(\mathcal{H}_{H,K_2}^w \underset{\mathsf{Rep}(K_2)}{\otimes} \mathsf{Rep}(K_1), \mathsf{IndCoh}_{ren}^*(K_2\backslash H/K_1)\big) =$$

$$\mathsf{Hom}_{\mathsf{Rep}(K_2)\text{--mod}}\big(\mathsf{Rep}(K_1), \mathsf{IndCoh}_{ren}^*(K_2\backslash H/K_1)\big) =$$

$$\mathsf{Hom}_{\mathsf{Rep}(K_2/K_1)\text{--mod}}\big(\mathsf{Vect}, \mathsf{IndCoh}_{ren}^*(K_2\backslash H/K_1)\big) =$$

$$\mathsf{Hom}_{\mathsf{Rep}(K_2/K_1)\text{--mod}}\big(\mathsf{Vect}, \mathsf{IndCoh}_{ren}^*(K_1\backslash H/K_1)^{K_2/K_1,w}\big) \overset{Thm.5.10.1}{=}$$

$$\mathsf{Hom}_{\mathsf{QCoh}(K_2/K_1)\text{--mod}}\big(\mathsf{QCoh}(K_2/K_1), \mathsf{IndCoh}_{ren}^*(K_1\backslash H/K_1)\big) = \mathcal{H}_{H,K_1}^w$$

as desired (it is immediate to check that this identification is compatible with the functor (7.13.2)).

Step 5. Finally, we briefly remark that all of the above immediately upgrades to the (lax) symmetric monoidal setting.

In detail, note that TateGp_{fr} is symmetric monoidal under products. and the functor $\mathsf{TateGp}_{fr} \to \mathsf{TateGp}$ is a symmetric monoidal Verdier localization.

The functor from Step 2 upgrades to a functor of simplicial symmetric monoidal categories, noting that by the construction of §7.7-7.12, $\mathsf{Morita}(\mathcal{C})$ is canonically a simplicial symmetric monoidal category.

This implies that the functor $\mathsf{TateGp}_{fr}^{op} \to \mathsf{DGCat}_{cont}\text{--mod}$ is naturally a (left) lax symmetric monoidal functor, since IndCoh^* is.[61] Finally, the forgetful functor $\mathsf{DGCat}_{cont}\text{--mod} \to 2\text{--DGCat}$ is by construction lax symmetric monoidal, giving the result. $\qquad\square$

7.14. Forgetful functors.
As in §5.15, for H a Tate group indscheme, there is a canonical non-conservative functor:

$$\mathrm{Oblv}_{gen} : H\text{--mod}_{weak} \to \mathsf{DGCat}_{cont}$$

$$\mathcal{C} \mapsto \underset{K\subseteq H \text{ compact open}}{\mathrm{colim}} \mathcal{C}^{K,w}.$$

We denote the colimit appearing on the right hand side also by $\mathcal{C}$ in a similar abuse of notation as in the profinite dimensional setting. We remark that this forgetful functor Oblv_{gen} upgrades to a functor to $H\text{--mod}_{weak,naive}$, which we also denote by Oblv_{gen}.

As in the profinite dimensional setting, where there is not confusion we often omit Oblv_{gen} from the notation, i.e., we often speak of genuine H-actions on $\mathcal{C} \in \mathsf{DGCat}_{cont}$ by which we mean that we are given an object of $H\text{--mod}_{weak}$ that maps to $\mathcal{C}$ under Oblv_{gen}.

Lemma 7.14.1. *The above forgetful functor commutes with limits and colimits.*

Proof. First note that for every $K \subseteq H$ compact open, the functor $H\text{--mod}_{weak} \xrightarrow{\mathcal{C}\mapsto\mathcal{C}^{K,w}} \mathsf{DGCat}_{cont}$ commutes with limits and colimits. Indeed, it may be calculated as the composition:

$$H\text{--mod}_{weak} \xrightarrow{\simeq} \mathcal{H}_{H,K}^w \xrightarrow{\mathrm{Oblv}} \mathsf{DGCat}_{cont}$$

and the latter functor commutes with limits and colimits (as this is always the case for a category of modules over an algebra).

---

[61]To make the implicit explicit: we are using the tensor product on $\mathsf{DGCat}_{cont}\text{--mod}$ obtained by viewing DGCat_{cont} as a commutative algebra in the symmetric monoidal category denoted $\mathcal{C}at_\infty(\mathcal{K})^\otimes$ in [Lur3] Proposition 4.8.1.14 (so $\mathsf{DGCat}_{cont}\text{--mod}$ are the modules in this category).

It immediately follows that our forgetful functor commutes with colimits. Commutation with limits follows by noting that each structural functor in the colimit admits a continuous right adjoint given by $*$-averaging, so we may also calculate it as the functor:

$$\mathcal{C} \mapsto \lim_{K \subseteq H \text{ compact open}} \mathcal{C}^{K,w}$$

(where the structural functors in the limit are these right adjoints). □

7.15. Invariants and coinvariants.
Let H be a Tate group indscheme.

Define triv : $\mathsf{DGCat}_{cont} \to H\text{-mod}_{weak}$ as the restriction functor along the homomorphism $H \to \mathrm{Spec}(k)$ (regarding the target as the trivial group).

Remark 7.15.1. By Proposition-Construction 7.6.1 (2), for $K \subseteq H$ compact open we have:

$$\mathrm{triv}(\mathsf{Vect})^{K,w} = \mathsf{Rep}(K).$$

We define the functor of (genuine, weak) invariants:

$$H\text{-mod}_{weak} \to \mathsf{DGCat}_{cont}$$
$$\mathcal{C} \mapsto \mathcal{C}^{H,w}$$

to be the right adjoint to triv, and we define (genuine, weak) the coinvariants functor $\mathcal{C} \mapsto \mathcal{C}_{H,w}$ to be the left adjoint. These may be computed explicitly after a choice of compact open subgroup K as:

$$\mathcal{C}^{H,w} \simeq \mathsf{Hom}_{\mathcal{H}_{H,K}^w\text{-mod}}(\mathsf{Rep}(K), \mathcal{C}^{K,w})$$
$$\mathcal{C}_{H,w} \simeq \mathsf{Rep}(K) \underset{\mathcal{H}_{H,K}^w}{\otimes} \mathcal{C}^{K,w}. \tag{7.15.1}$$

Remark 7.15.2. The comparison between invariants and coinvariants is more subtle in the group indscheme setting. than in the group scheme setting.[62]

7.16. Rigid monoidal categories.
Before proceeding, we review some constructions with rigid monoidal DG categories, following [Gai4] §6. We refer to *loc. cit.* for the relevant notion of rigid monoidal DG category; we remind that this is a property for some $\mathcal{A} \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$ to satisfy.[63]

We will construct a canonical morphism $\varphi_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ of monoidal categories that plays a key role.

Let $\mathcal{A}^{\vee}$ be the dual of $\mathcal{A}$ as an object of DGCat_{cont}. Note that $\mathcal{A}^{\vee}$ is canonically $\mathcal{A}$-bimodule in DGCat_{cont} (as it is the dual of the $\mathcal{A}$-bimodule $\mathcal{A}$). Therefore, we obtain a monoidal functor:

$$\mathcal{A} \to \mathsf{End}_{\mathsf{mod}-\mathcal{A}}(\mathcal{A}^{\vee}) \in \mathsf{Alg}(\mathsf{DGCat}_{cont}). \tag{7.16.1}$$

(The notation indicates endomorphisms as a right $\mathcal{A}$-module in DGCat_{cont}.) On the other hand, by definition of rigidity, the functor:

$$\mathcal{A} \to \mathcal{A}^{\vee}$$
$$\mathcal{F} \mapsto (\mathcal{G} \mapsto \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathbb{1}_{\mathcal{A}}, \mathcal{F} \star \mathcal{G})$$

---

[62]And these functors behave less well. For example, they may fail to be conservative (c.f. [Gai8] Theorem 2.5.4).

[63]If $\mathcal{A}$ is compactly generated and rigid, then the subcategory $\mathcal{A}^c \subseteq \mathcal{A}$ is closed under the monoidal operation and rigid according to the more standard notion of rigid monoidal (in terms of existence of duals).

is an equivalence of right $\mathcal{A}$-module categories (here $\mathbb{1}_\mathcal{A}$ is the unit object). Therefore, the right hand side of (7.16.1) identifies canonically with $\mathcal{A}$, and we obtain the desired morphism $\varphi_\mathcal{A}$.

Remark 7.16.1. By construction, there are natural isomorphisms:

$$\underline{\operatorname{Hom}}_\mathcal{A}(\mathbb{1}_\mathcal{A}, \mathcal{F} \star \mathcal{G}) \xrightarrow{\simeq} \underline{\operatorname{Hom}}_\mathcal{A}(\mathbb{1}_\mathcal{A}, \mathcal{G} \star \varphi_\mathcal{A}(\mathcal{F}))$$

for $\mathcal{F}, \mathcal{G} \in \mathcal{A}$.

For $\mathcal{M} \in \mathcal{A}\text{–mod} := \mathcal{A}\text{–mod}(\mathsf{DGCat}_{cont})$, we let $\varphi_{\mathcal{A},*}(\mathcal{M}) \in \mathcal{A}\text{–mod}$ denote the restriction of $\mathcal{M}$ along $\varphi_\mathcal{A}$, and we let $\varphi_\mathcal{A}^*$ denote the inverse to the equivalence $\varphi_{\mathcal{A},*}$.

By [Gai4] Corollary 6.3.3, for $\mathcal{M}, \mathcal{N} \in \mathcal{A}\text{–mod}$ with $\mathcal{M}$ dualizable in DGCat_{cont}, there is a canonical equivalence:

$$\mathsf{Hom}_{\mathcal{A}\text{–mod}}(\mathcal{M}, \mathcal{N}) \xrightarrow{\simeq} \mathcal{M}^\vee \underset{\mathcal{A}}{\otimes} \varphi_{\mathcal{A},*}(\mathcal{N}) \tag{7.16.2}$$

functorial in $\mathcal{M}$ and $\mathcal{N}$.

7.17. **Polarizations and the modular character.** We now introduce the following terminology.

Definition 7.17.1. A *polarization* of $H \in \mathsf{TateGp}$ is a compact open subgroup $K \subseteq H$ such that H/K is ind-proper. If a polarization exists, we say that H is *polarizable*.

Example 7.17.2. The loop group of a reductive group is polarizable.

Example 7.17.3. If H is formal in ind-directions, i.e., its reduced locus $H^{red} \subseteq H$ is a compact open subgroup, then H is polarizable (and equipped with the canonical polarization H^{red}). In this case, we say H is a group indscheme H *of Harish-Chandra type*.

We have the following result, which is evident from the definitions (and preservation of coherent objects under flat pullbacks and proper pushforwards):

Lemma 7.17.4. *For K a polarization of H, the genuine Hecke category $\mathcal{H}_{H,K}^w$ is rigid monoidal (in the sense of [Gai4] §6).*

Corollary 7.17.5. *If H is polarizable, the coinvariants functor $H\text{–mod}_{weak} \to \mathsf{DGCat}_{cont}$ is corepresentable.*

Proof. Let K be a polarization of H. Then we have the equivalence $H\text{–mod}_{weak} \simeq \mathcal{H}_{H,K}^w\text{–mod}$, and the right hand side is rigid monoidal. Using the notation of §7.16, we obtain functorial identifications:

$$\mathcal{C}_{H,w} \overset{(7.15.1)}{=} \mathsf{Rep}(K) \underset{\mathcal{H}_{H,K}^w}{\otimes} \mathcal{C}^{K,w} \overset{(7.16.2)}{=} \mathsf{Hom}_{\mathcal{H}_{H,K}^w\text{–mod}}(\mathsf{Rep}(K), \varphi_{\mathcal{H}_{H,K}^w}^*(\mathcal{C}^{K,w})) =$$

$$\mathsf{Hom}_{\mathcal{H}_{H,K}^w\text{–mod}}(\varphi_{\mathcal{H}_{H,K}^w,*}(\mathsf{Rep}(K)), \mathcal{C}^{K,w})$$

where we have repeatedly used that $\varphi_{\mathcal{H}_{H,K}^w,*}$ and $\varphi_{\mathcal{H}_{H,K}^w}^*$ are mutually inverse equivalences. Applying the equivalence $\mathcal{H}_{H,K}^w\text{–mod} \simeq H\text{–mod}_{weak}$ now gives the result.

$\square$

Definition 7.17.6. For H a polarizable Tate group indscheme, the *modular character* $\chi_{Tate,H} = \chi_{Tate} \in H\text{–mod}_{weak}$ is the object corepresenting the functor of coinvariants.

7.18. Next, observe that because the functor of Proposition-Construction 7.6.1 is lax symmetric monoidal, H–mod_{weak} is naturally symmetric monoidal with unit $\mathrm{triv}(\mathsf{Vect})$. We denote the tensor product by $\otimes$; explicitly, for $\mathcal{C}_1, \mathcal{C}_2 \in H$–$\mathsf{mod}_{weak}$, we have an object $\mathcal{C}_1 \otimes \mathcal{C}_2 \in (H \times H)$–$\mathsf{mod}_{weak}$ from the lax symmetric monoidal functoriality, and then we restrict along the diagonal map.

Lemma 7.18.1. *If H is a polarizable Tate group indscheme, the functor:*

$$H\text{–}\mathsf{mod}_{weak} \to H\text{–}\mathsf{mod}_{weak}$$

given by tensoring with the modular character χ_{Tate} is an equivalence.

Proof. Let $K \subseteq H$ be a polarization. By the proof of Corollary 7.17.5 (and Proposition-Construction 7.6.1 (3)), we obtain a commutative diagram:

$$
\begin{array}{ccc}
H\text{–}\mathsf{mod}_{weak} & \xrightarrow{-\otimes\chi_{Tate}} & H\text{–}\mathsf{mod}_{weak} \\
{\scriptstyle (-)^{K,w}}\downarrow & & \downarrow{\scriptstyle (-)^{K,w}} \\
\mathcal{H}_{H,K}^w\text{–}\mathsf{mod} & \xrightarrow{\varphi_{\mathcal{H}_{H,K},*}} & \mathcal{H}_{H,K}\text{–}\mathsf{mod}.
\end{array}
\tag{7.18.1}
$$

Each of the vertical arrows and the bottom arrow are equivalences, so the top arrow is as well. $\qquad\square$

Let H be a polarizable Tate group indscheme. By the lemma, there is a canonical object $\chi_{-Tate} \in H$–mod_{weak} inverse to χ_{Tate} under tensor product, i.e., we have:

$$\chi_{-Tate} \otimes \chi_{Tate} = \mathrm{triv}(\mathsf{Vect}) \in H\text{–}\mathsf{mod}_{weak}.$$

Proposition 7.18.2. *For H as above and $\mathcal{C} \in H$–mod_{weak}, there is a canonical isomorphism:*

$$\mathcal{C}_{H,w} \simeq (\mathcal{C} \otimes \chi_{-Tate})^{H,w}$$

functorial in $\mathcal{C}$.

Proof. We have:

$$(\mathcal{C} \otimes \chi_{-Tate})^{H,w} = \mathsf{Hom}_{H\text{–}\mathsf{mod}_{weak}}(\mathsf{Vect}, \mathcal{C} \otimes \chi_{-Tate}) = \mathsf{Hom}_{H\text{–}\mathsf{mod}_{weak}}(\chi_{Tate}, \mathcal{C}) = \mathcal{C}_{H,w}$$

where the last equality was the definition of χ_{Tate}. $\qquad\square$

7.19. The following result gives a somewhat non-canonical description of χ_{Tate}.

Proposition 7.19.1. *Let H be a polarizable Tate group indscheme. Then for any compact open subgroup $K \subseteq H$, there exists a canonical isomorphism:*

$$\mathrm{Oblv}_K^H(\chi_{Tate,H}) \simeq \mathsf{Vect} \in K\text{–}\mathsf{mod}_{weak}$$

for $\mathrm{Oblv}_K^H : H$–$\mathsf{mod}_{weak} \to K$–$\mathsf{mod}_{weak}$ the restriction functor and for $\mathsf{Vect} \in K$–mod_{weak} regarded with the trivial action.

Proof. For, note that Oblv_K^H admits a left adjoint $\mathrm{ind}_K^{H,w}$ which also calculates its right adjoint. Indeed, under the equivalences:

$$H\text{--mod}_{weak} \overset{(-)^{K,w}}{\simeq} \mathcal{H}_{H,K}^w\text{--mod}$$

$$K\text{--mod}_{weak} \overset{(-)^{K,w}}{\simeq} \mathsf{Rep}(K)\text{--mod}$$

Oblv_K^H corresponds to restriction along the monoidal functor $\mathsf{Rep}(K) \to \mathcal{H}_{H,K}^w$. This immediately gives the existence of the left adjoint, and the fact that it also calculates the right adjoint follows from the fact that $\mathcal{H}_{H,K}^w$ is canonically self-dual as a $\mathsf{Rep}(K)$-module category (which is the case because $\mathcal{H}_{H,K}$ is canonically self-dual as a DG category via Serre duality, and $\mathsf{Rep}(K)$ is rigid symmetric monoidal).

Now for $\mathcal{C} \in K\text{--mod}_{weak}$, we obtain:

$$\mathsf{Hom}_{K\text{--mod}_{weak}}(\mathrm{Oblv}_K^H(\chi_{Tate,H}), \mathcal{C}) = \mathsf{Hom}_{H\text{--mod}_{weak}}(\chi_{Tate,H}, \mathrm{ind}_K^{H,w}(\mathcal{C})) =$$

$$\mathrm{ind}_K^{H,w}(\mathcal{C})_{H,w} = \mathcal{C}_{K,w} = \mathcal{C}^{K,w}$$

functorially in $\mathcal{C}$, giving the claim.

$\square$

Warning 7.19.2. Suppose $K_1 \subseteq K_2 \subseteq H$. Then we obtain isomorphisms:

$$\alpha_i : \mathrm{Oblv}_{K_i}^H(\chi_{Tate,H}) \simeq \mathsf{Vect} \in K_i\text{--mod}_{weak}, \quad i = 1, 2$$

However, $\mathrm{Oblv}_{K_1}^{K_2}(\alpha_2) \neq \alpha_1$. Rather, one can check that the two isomorphisms differ by tensoring with $\det(\mathfrak{k}_2/\mathfrak{k}_1)[\dim(\mathfrak{k}_2/\mathfrak{k}_1)] \in \mathsf{Rep}(K_1) = \mathsf{End}_{K_1\text{--mod}_{weak}}(\mathsf{Vect})$.[64]

8. Strong actions

8.1.   In this section, we relate weak actions for a Tate group indschemes H to strong actions of H, as defined in [Ber].

8.2.   Let us spell out our goals more precisely. Let $D^*(H) \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$ be the monoidal DG category defined (with the same notation) in [Ras3]. Let $H\text{--mod} := D^*(H)\text{--mod}$ be the (2-)category of categories with a *strong* H-action.

In this section, we will construct a restriction functor:

$$\mathrm{Oblv} = \mathrm{Oblv}^{str \to w} : H\text{--mod} \to H\text{--mod}_{weak}$$

compatible with forgetful functors to DGCat_{cont} (where for $H\text{--mod}_{weak}$, we are considering the forgetful functor Oblv_{gen} of §7.14).

8.3.   Moreover, we will show that $\mathrm{Oblv} : H\text{--mod} \to H\text{--mod}_{weak}$ admits a left and right adjoints that are morphisms of DGCat_{cont}-module categories, and with the following property.

For $\mathcal{C} \in H\text{--mod}_{weak}$, define:

$$\mathcal{C}^{\exp(\mathfrak{h}),w} := \operatorname*{colim}_{K \subseteq H \text{ compact open}} \mathcal{C}^{H_K^\wedge, w} \in \mathsf{DGCat}_{cont}$$

under the obvious structural functors. Here $H_K^\wedge$ is the formal completion of H along K, which is necessarily a Tate group indscheme. That is, we consider the restriction of $\mathcal{C}$ along the forgetful

---

[64]This factor arises because the proof of Proposition 7.19.1 (necessarily) used *Serre* duality on $\mathsf{IndCoh}(H/K)$ to obtain the canonical self-duality for $\mathcal{H}_{H,K}^w$.

functor $H\text{–mod}_{weak} \to H_K^\wedge$, apply the invariants construction of §7.15 for $H_K^\wedge$, and pass to the colimit.

Similarly, define:

$$\mathcal{C}_{\exp(\mathfrak{h}),w} := \lim_{K \subseteq H \text{ compact open}} \mathcal{C}_{H_K^\wedge,w}.$$

As we will see, each of the structural functors in the limit (resp. colimit) defining $\mathcal{C}^{\exp(\mathfrak{h}),w}$ (resp. $\mathcal{C}_{\exp(\mathfrak{h}),w}$) admits a left adjoint (resp. continuous right adjoint), so these two expressions can be expressed as limits or colimits in DGCat_{cont}.

Then we will see that the composition of our right (resp. left) adjoint $H\text{–mod}_{weak} \to H\text{–mod}$ with the forgetful functor $H\text{–mod} \to \mathsf{DGCat}_{cont}$ sends $\mathcal{C} \in H\text{–mod}_{weak}$ to $\mathcal{C}^{\exp(\mathfrak{h}),w}$ (resp. $\mathcal{C}_{\exp(\mathfrak{h}),w}$).

In other words, we will show that H acts strongly on $\mathcal{C}^{\exp(\mathfrak{h}),w}$ and $\mathcal{C}_{\exp(\mathfrak{h}),w}$, and that these categories satisfy the evident universal properties with respect to these actions and $\mathrm{Oblv}^{str \to w}$.

Remark 8.3.1. If H is polarizable, then it is straightforward to deduce from Proposition 7.18.2 that the above functors $(-)^{\exp(\mathfrak{h}),w}, (-)_{\exp(\mathfrak{h}),w} : H\text{–mod}_{weak} \to \mathsf{DGCat}_{cont}$ are equivalent up to certain twists. We will formulate this statement precisely in Proposition 8.21.1, where we will also show that this isomorphism is strongly H-equivariant in a canonical way.

8.4. **Strategy.** To orient the reader in what follows, we give a brief overview of the approach.

To give a functor $H\text{–mod} \to H\text{–mod}_{weak}$ the commutes with colimits and is a morphism of DGCat_{cont}-module categories is equivalent to specifying an object of $H\text{–mod}_{weak}$ with a right $D^*(H)$-module structure.

In a suitable sense, this object is $D^*(H)$ considered as weakly acted on via the left action of H, and with the evident commuting strong action of H on the right.

Implementing this strategy turns out the be somewhat involved. It is not so difficult to define $D^*(H)$ as an object of $H\text{–mod}_{weak}$: this is done is §8.9. However, the commuting $D^*(H)$-action takes some work, and will be given in §8.20.

8.5. **Warmup.** First, we discuss the case where H is a classical affine group scheme. While do not rely on this special case in the general construction, it is illustrative of the main ideas.

Let $H = \lim_i H_i$ be a cofiltered limit of affine algebraic groups under smooth surjective homomorphisms. As above, to construct our functor:

$$H\text{–mod} := D^*(H)\text{–mod} \to H\text{–mod}_{weak} = \mathsf{Rep}(H)\text{–mod}$$

is suffices to construct a $(\mathsf{Rep}(H), D^*(H))$-bimodule in DGCat_{cont}.

This bimodule is $\mathfrak{h}\text{–mod} \in \mathsf{DGCat}_{cont}$ (c.f. Example 4.3.8). We have:

$$\mathfrak{h}\text{–mod} = \lim_i \mathfrak{h}_i\text{–mod}$$

where each structural functor $\mathfrak{h}_i\text{–mod} \to \mathfrak{h}_j\text{–mod}$ takes the Lie algebra invariants with respect to $\mathrm{Ker}(\mathfrak{h}_i \twoheadrightarrow \mathfrak{h}_j)$. As is standard, H_i acts strongly on $\mathfrak{h}_i\text{–mod}$, and the above structural functors are equivariant in the suitable homotopy coherent sense for the H_i-action on $\mathfrak{h}_j\text{–mod}$ induced by the homomorphism $H_i \to H_j$. Therefore, we obtain an action:

$$D^*(H) := \lim_i D(H_i) \curvearrowright \lim_i \mathfrak{h}_i\text{–mod} = \mathfrak{h}\text{–mod}.$$

Now for any map of indices $i \to j$, there is an action of $\mathsf{Rep}(H_j)$ on $\mathfrak{h}_i\text{–mod}$ commuting with the strong H_j-action: it is given by restricting an H_j-representation to H_i and then tensoring with the Lie algebra representation. Again, this is suitably homotopy coherent, so we obtain an action:

$$D^*(H) \otimes \mathsf{Rep}(H_j) \rightharpoonup \mathfrak{h}\text{–mod}.$$

Finally, these actions are suitably compatible with varying j, so we obtain:

$$D^*(H) \otimes \mathsf{Rep}(H) = D^*(H) \otimes \underset{j}{\mathrm{colim}} \, \mathsf{Rep}(H_j) \rightharpoonup \mathfrak{h}\text{–mod}$$

as desired.

8.6. **A remark on naive coinvariants.** Before proceeding, it is convenient to record the following technical result. The reader may safely skip this material and refer back to it as needed.

Let S be a reasonable indscheme and let $\mathcal{P}_K \to S$ be a K- torsor for K a classical affine group scheme. By Corollary 6.25.2, naive weak K-invariants in $\mathsf{IndCoh}^*(\mathcal{P}_K)$ are given by $\mathsf{IndCoh}^*(S)$. Moreover, the naive K-action on $\mathsf{IndCoh}^*(\mathcal{P}_K)$ clearly canonically renormalizes, and the corresponding category of genuine K-invariants is $\mathsf{IndCoh}^*(S)$ by Lemma 6.18.1. This leaves the case of naive coinvariants.

Lemma 8.6.1. *In the above setting, the IndCoh-pushforward functor $\mathsf{IndCoh}^*(\mathcal{P}_K) \to \mathsf{IndCoh}^*(S)$ induces an equivalence:*

$$\mathsf{IndCoh}^*(\mathcal{P}_K)_{K,w,naive} \to \mathsf{IndCoh}^*(S). \tag{8.6.1}$$

Proof.

Step 1. First, note that we are reduced to the case where S is a quasi-compact quasi-separated eventually coconnective scheme. Indeed, if $S = \mathrm{colim}_i S_i$ with $S_i \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ and structural maps almost finitely presented, then:

$$\mathsf{IndCoh}^*(S) = \underset{i}{\mathrm{colim}} \, \mathsf{IndCoh}^*(S_i) \in \mathsf{DGCat}_{cont}$$
$$\mathsf{IndCoh}^*(\mathcal{P}_K) = \underset{i}{\mathrm{colim}} \, \mathsf{IndCoh}^*(\mathcal{P}_K \underset{S}{\times} S_i) \in \mathsf{DGCat}_{cont}.$$

This clearly gives the reduction. In the remainder of the argument, we therefore assume $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$.

Step 2. Next, suppose $\mathcal{P}_K \to S$ is trivial, i.e., $\mathcal{P}_K \xrightarrow{\simeq} K \times S$ K-equivariantly. By Lemma 6.36.3 (applied to K), we have:

$$\mathsf{IndCoh}^*(K) \otimes \mathsf{IndCoh}^*(S) \xrightarrow{\simeq} \mathsf{IndCoh}^*(\mathcal{P}_K).$$

Observe that $\mathsf{Perf}(K) \xrightarrow{\simeq} \mathsf{Coh}(K)$ because K is a limit of smooth schemes under flat affine morphisms. Therefore, the above coincides with $\mathsf{QCoh}(K) \otimes \mathsf{IndCoh}^*(S)$. This clearly gives the result in this case.

Step 3. Next, we show the result when $\mathcal{P}_K$ is Zariski-locally trivial. For this, we first establish some general facts about IndCoh^*.

Suppose $j : U \hookrightarrow S$ is a quasi-compact open subscheme. Then the natural functor:

$$\mathsf{IndCoh}^*(S) \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(U) \to \mathsf{IndCoh}^*(U)$$

is an equivalence. Indeed, $\mathsf{IndCoh}^*(U)$ is the essential image of the functor $j_*^{\mathsf{IndCoh}} j^{*,\mathsf{IndCoh}} : \mathsf{IndCoh}^*(S) \to \mathsf{IndCoh}^*(S)$, while $\mathsf{IndCoh}^*(S) \otimes_{\mathsf{QCoh}(S)} \mathsf{QCoh}(U)$ is the essential image of:

$$\mathrm{id}_{\mathsf{IndCoh}^*(S)} \underset{\mathsf{QCoh}(S)}{\otimes} j_* j^* : \mathsf{IndCoh}^*(S) = \mathsf{IndCoh}^*(S) \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(S) \rightarrow$$

$$\mathsf{IndCoh}^*(S) \underset{\mathsf{QCoh}(S)}{\otimes} \mathsf{QCoh}(S) = \mathsf{IndCoh}^*(S).$$

These endofunctors of $\mathsf{IndCoh}^*(S)$ coincide, giving the result.

As a consequence, suppose $U_1, U_2 \subseteq S$ are quasi-compact opens covering S; then we claim that the map:

$$\mathsf{IndCoh}^*(U_1) \coprod_{\mathsf{IndCoh}^*(U_1 \cap U_2)} \mathsf{IndCoh}^*(U_2) \rightarrow \mathsf{IndCoh}^*(S) \in \mathsf{DGCat}_{cont}$$

is an equivalence (this pushout being formed in DGCat_{cont}). Indeed, it is well-known[65] that we have:

$$\mathsf{QCoh}(U_1) \coprod_{\mathsf{QCoh}(U_1 \cap U_2)} \mathsf{QCoh}(U_2) \rightarrow \mathsf{QCoh}(S) \in \mathsf{DGCat}_{cont}$$

and therefore in $\mathsf{QCoh}(S)$–mod. Tensoring $\mathsf{IndCoh}^*(S)$ over $\mathsf{QCoh}(S)$ preserves this colimit, giving the claim from the above.

Now for any $U_1, U_2 \subseteq S$ as above, we obtain:

$$\mathsf{IndCoh}^*(\mathcal{P}_K \underset{S}{\times} U_1)_{K,w,naive} \coprod_{\mathsf{IndCoh}^*(\mathcal{P}_K \underset{S}{\times} U_1 \cap U_2)_{K,w,naive}} \mathsf{IndCoh}^*(\mathcal{P}_K \underset{S}{\times} U_2)_{K,w,naive} \xrightarrow{\simeq} \mathsf{IndCoh}^*(\mathcal{P}_K)_{K_w}$$

by applying the above to the base-changed Zariski cover of $\mathcal{P}_K$ and by commuting geometric realizations with pushouts. We now clearly obtain the claim by induction on the number of opens required to trivialize $\mathcal{P}_K$.

Step 4. Next, we show the result for K prounipotent.

By the previous step, it suffices to note that any K-torsor on an affine scheme T is trivial. This is standard: prounipotent K has a lower central series $K = K^1 \unrhd K^2 \unrhd \dots$ where all subquotients are (possibly infinite) products of copies of $\mathbb{G}_a$. For such products, the claim follows from vanishing of higher (flat) cohomology of T with coefficients in its structure sheaf. By induction, any K/K^n-torsor on an affine scheme is trivial, and then we deduce the same for K using countability of this filtration and surjectivity of $\pi_0(\mathrm{Hom}(T, K/K^{n+1})) \rightarrow \pi_0(\mathrm{Hom}(T, K/K^n))$.

Step 5. Finally, we show the result in general.

Let $K \rightarrow K^{red}$ be the proreductive[66] quotient of K^{red}, and let K^u be the kernel of this homomorphism, i.e., the prounipotent radical of K.

Because representations of K^{red} are semisimple, for any $\mathcal{C}$ with a naive weak K^{red} action, the functor $\mathcal{C}_{K^{red},w,naive} \rightarrow \mathcal{C}^{K^{red},w,naive}$ is an equivalence. Indeed, the argument from [Gai8] §7.2 applies just as well in the proreductive case as in the reductive one.

We then obtain:

$$\mathsf{IndCoh}^*(\mathcal{P}_K)_{K,w,naive} = (\mathsf{IndCoh}^*(\mathcal{P}_K)_{K^u,w,naive})_{K^{red},w,naive} \xrightarrow{\simeq} \mathsf{IndCoh}^*(\mathcal{P}_{K^{red}})^{K^{red},w,naive}$$

---

[65]This identity is implicit in the proof of [Gai1] Proposition 2.3.6. One can find this statement explicitly in [Gai8] by combining Theorem 2.1.1 and Proposition 6.2.7 from *loc. cit.*

[66]Here we use *proreductive* as shorthand for *pro-(algebraic group with reductive connected components)*.

for $\mathcal{P}_{K^{red}} \to S$ the induced K^{red}-torsor (appealing to the previous step here). We now obtain the result by Corollary 6.25.2.

$\square$

8.7. **Induction.** We begin with the following general lemma.

Lemma 8.7.1. *Let $f : H_1 \to H_2$ be a morphism in* TateGp.
(1) *The forgetful functor:*

$$H_2\text{–mod}_{weak} \to H_1\text{–mod}_{weak}$$

admits a left adjoint $\text{ind}^w = \text{ind}_{H_1}^{H_2,w} : H_1\text{–mod}_{weak} \to H_2\text{–mod}_{weak}$.
(2) *Suppose there exists $K \subseteq H_1$ compact open such that f realizes K as a compact open subgroup of H_2 as well.*[67]
    Then the diagram:

$$
\begin{array}{ccc}
H_1\text{–mod}_{weak} & \xrightarrow{\text{Oblv}_{gen}} & H_1\text{–mod}_{weak,naive} \\
\downarrow{\scriptstyle \text{ind}_{H_1}^{H_2,w}} & & \downarrow{\scriptstyle \text{ind}_{H_1}^{H_2,w,naive}} \\
H_2\text{–mod}_{weak} & \xrightarrow{\text{Oblv}_{gen}} & H_2\text{–mod}_{weak,naive}
\end{array}
$$

commutes (where a priori it only commutes up to a natural transformation). Here the functor on the right is tensoring over $\mathsf{IndCoh}^*(H_1)$ *with* $\mathsf{IndCoh}^*(H_2)$.

Proof. As $H_i\text{–mod}_{weak} \to K_i\text{–mod}_{weak}$ are (by construction) monadic functors, the first claim easily reduces to the setting of §5.17.

Similarly, such considerations formally reduce the second claim to the case where $H_1 = K$. We denote H_2 simply by H in this case. So we wish to show the diagram:

$$
\begin{array}{ccc}
K\text{–mod}_{weak} & \xrightarrow{\text{Oblv}_{gen}} & K\text{–mod}_{weak,naive} \\
\downarrow{\scriptstyle \text{ind}_{K}^{H,w}} & & \downarrow{\scriptstyle \text{ind}_{H_1}^{H_2,w,naive}} \\
H\text{–mod}_{weak} & \xrightarrow{\text{Oblv}_{gen}} & H\text{–mod}_{weak,naive}
\end{array}
$$

commutes. Each of the functors involved commutes with colimits and is DGCat_{cont}-linear, so it suffices to check that the diagram commutes when evaluated on the trivial representation $\mathsf{Vect} \in K\text{–mod}_{weak}$ (since this object generates by definition). In this case, the claim is that the natural map:

$$\mathsf{IndCoh}^*(H) \underset{\mathsf{IndCoh}^*(K)}{\otimes} \mathsf{Vect} \xrightarrow{\simeq} \mathsf{IndCoh}(H/K).$$

This follows from Lemma 8.6.1.

$\square$

---

[67]Using Lemma 8.6.1, one can show that the conclusion holds more generally if there exist $K_i \subseteq H_i$ compact open subgroups such that f maps K_1 into K_2 via a closed embedding.

8.8. Now for H a Tate group indscheme and $K \subseteq H$ compact open, let $H_K^\wedge$ denote the formal completion of H along K. We can form $\mathrm{ind}_{H_K^\wedge}^{H,w}(\mathsf{Vect}) \in H\mathrm{-mod}_{weak}$ (where $\mathsf{Vect} \in H_K^\wedge\mathrm{-mod}_{weak}$ is our standard trivial object). By Lemma 8.7.1, we have:

$$\mathrm{Oblv}_{gen}(\mathrm{ind}_{H_K^\wedge}(\mathsf{Vect})) = \mathsf{IndCoh}^*(H) \underset{\mathsf{IndCoh}^*(H_K^\wedge)}{\otimes} \mathsf{Vect}.$$

This tensor product evidently maps to $D(H/K)$, and we claim that the induced functor is an equivalence. Indeed, we have a commutative diagram:

$$\mathsf{IndCoh}(H/K) = \!\!=\!\! \mathsf{IndCoh}^*(H) \underset{\mathsf{IndCoh}^*(K)}{\otimes} \mathsf{Vect} \longrightarrow \mathsf{IndCoh}^*(H) \underset{\mathsf{IndCoh}^*(H_K^\wedge)}{\otimes} \mathsf{Vect}$$

$$D(H/K).$$

The diagonal arrows admit continuous, monadic right adjoints and the induced natural transformation on monads is an isomorphism, giving the claim.

In the above setting, we use our standard abuse of notation in letting $D(H/K) \in H\mathrm{-mod}_{weak}$ denote the object $\mathrm{ind}_{H_K^\wedge}^{H,w}(\mathsf{Vect})$.

8.9. Note that this object is manifestly covariant in K, so we can form:

$$D^*(H) := \lim_K D(H/K) \in H\mathrm{-mod}_{weak}.$$

Note that under Oblv_{gen} and the equivalence of §8.8, these structural functors map to de Rham pushforward functors.

By definition (and Lemma 7.14.1), this object maps under Oblv_{gen} to the category $D^*(H) \in \mathsf{DGCat}_{cont}$ defined in [Ras3], justifying the notation.

Remark 8.9.1. Each of the structural functors in the above diagram admits a left adjoint in the 2-category $H\mathrm{-mod}_{weak}$: indeed, these functors are given by D-module $*$-pullback along the smooth maps $H/K_1 \to H/K_2$.[68] Therefore, this limit is also a colimit (in $H\mathrm{-mod}_{weak}$) under those left adjoints.

8.10. By Remark 8.9.1, $D^*(H) \in H\mathrm{-mod}_{weak}$ corepresents the functor:

$$\mathcal{C} \mapsto \lim_{K \subseteq H \text{ compact open}} \mathcal{C}^{H_K^\wedge, w} = \underset{K \subseteq H \text{ compact open}}{\mathrm{colim}} \mathcal{C}^{H_K^\wedge, w} = \mathcal{C}^{\exp(\mathfrak{h}), w}$$

where the structural functors in the colimit are the evident forgetful functors, and the structural functors in the limit are their right adjoints.

8.11. Below, we will construct an action of $D^*(H) \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$ on this object $D^*(H) \in H\mathrm{-mod}_{weak}$ encoding the right action of H on itself. As in §8.4, this would suffice to construct a functor of the desired type. By the discussion of §8.10, the formula from §8.3 for the right adjoint would be immediate, and the formula for the left adjoint would follow dually.

Therefore, we will give this construction below following a sequence of digressions.

---

[68]This discussion is a bit informal, since it really applies after applying Oblv_{gen}. But e.g., it easily follows from Lemma 8.12.2 below that the left adjoints exist in the genuine setting as well.

8.12. **Some generalities on D-modules.** We make the above construction somewhat more explicit. The reader may safely skip this material and refer back to it as needed.

First, it is convenient to extend the generality of the above construction. Let X be an indscheme locally almost of finite type, and suppose X is acted on by H. Then we there is a canonical object $D(X) \in H\text{–mod}_{weak}$ attached to X (and mapping to the category of D-modules on X under Oblv_{gen}).

We sketch the construction. A variant of Proposition-Construction 7.6.1 attaches an object $\text{IndCoh}(X) \in H\text{–mod}_{weak}$ to X such that for any congruence subgroup K, $\text{IndCoh}(X)^{K,w} = \text{IndCoh}^*_{ren}(X/K)$ as an $\mathcal{H}^w_{H,K}$-module.

Now let $X^{inf}_\bullet \in \text{IndSch}^{\Delta^{op}}_{laft}$ be the infinitesimal groupoid of X, i.e., the simplicial indscheme locally almost of finite type obtained as the Cech nerve of $X \to X_{dR}$. By functoriality, this diagram is a simplicial diagram of indschemes (locally almost of finite) acted on by H. Therefore, by the above construction, we obtain a simplicial diagram $\text{IndCoh}(X^{inf}_\bullet) \in H\text{–mod}_{weak}$. We define $D(X) \in H\text{–mod}_{weak}$ as its colimit. By Lemma 7.14.1 and [GR4] Proposition III.3.3.3.3(b), this object indeed maps to the usual category of D-modules $D(X) \in \text{DGCat}_{cont}$ under the forgetful functor $H\text{–mod}_{weak} \to \text{DGCat}_{cont}$.

Now that for any choice of compact open subgroup $K \subseteq H$, we have $D(X)^{K,w} \in \mathcal{H}^w_{H,K}\text{–mod}$. By construction $D(X)^{K,w}$ is compactly generated with compact objects induced from $\text{IndCoh}^*_{ren}(X/K) = \text{IndCoh}(X)^{K,w}$.

Lemma 8.12.1. *For $K \subseteq H$ compact open, the object $D(H/K)$ defined in §8.8 coincides with the object we have just constructed.*

Proof. This is essentially a slight refinement of the argument from §8.8.

Let $\mathcal{C}_1 \in H\text{–mod}_{weak}$ denote the object from §8.8 and let $\mathcal{C}_2 \in H\text{–mod}_{weak}$ denote the object just constructed (i.e., from the construction defined for any indscheme locally almost of finite type). There is a natural map $\mathcal{C}_1 = \text{ind}^{H,w}_{H^\wedge_K}(\text{Vect}) \to \mathcal{C}_2$, and we claim it is an isomorphism. It suffices to check this after applying weak K-invariants.

By construction, we have:

$$\mathcal{C}_1^{K,w} = \text{Rep}(K) \underset{\mathcal{H}^w_{H^\wedge_K,K}}{\otimes} \mathcal{H}^w_{H,K}.$$

This gives rise to a functor:

$$\text{IndCoh}^*_{ren}(K\backslash H/K) = \text{Rep}(K) \underset{\text{Rep}(K)}{\otimes} \mathcal{H}^w_{H,K} \to \text{Rep}(K) \underset{\mathcal{H}^w_{H^\wedge_K,K}}{\otimes} \mathcal{H}^w_{H,K} = \mathcal{C}_1^{K,w}.$$

This functor admits a right adjoint that is continuous and conservative, so monadic. The further composition with $\mathcal{C}_1^{K,w} \to \mathcal{C}_2^{K,w}$ behaves similarly (by construction), and the induced maps on monads is an isomorphism, giving the claim.   □

We now show the following result for X any indscheme locally almost of finite type.

Lemma 8.12.2. *$D(X)^{K,w}$ admits a (unique) compactly generated t-structure for which the forgetful functor $D(X)^{K,w} \to D(X)^{K,w,naive}$ is t-exact and induces an equivalence $D(X)^{K,w,+} \xrightarrow{\simeq} D(X)^{K,w,naive,+}$. An object in $D(X)^{K,w}$ is compact if and only if it is eventually coconnective and its image in $D(X)$ is compact.*

In other words, as a category with a genuine K-action, $D(X)$ is given by the canonical renormalization construction from §5.18.

Proof of Lemma 8.12.2. Define the t-structure on $D(X)^{K,w}$ by taking connective objects to be generated by objects induced from $\mathsf{IndCoh}(X)^{K,w,\leqslant 0}$. To see $D(X)^{K,w} \to D(X)^{K,w,naive}$ is t-exact, it is equivalent to see that the further forgetful functor:

$$\mathrm{Oblv} : D(X)^{K,w} \to D(X)$$

is t-exact. Clearly this functor is right t-exact. Now for $\mathcal{F} \in D(X)^{K,w,\geqslant 0}$, the underlying object of $\mathsf{IndCoh}(X)^{K,w}$ is coconnective by design, so the same is true for the underlying object of $\mathsf{IndCoh}(X)$. Therefore, $\mathrm{Oblv}(\mathcal{F}) \in D(X)$ maps to a coconnective object of $\mathsf{IndCoh}(X)$; this is equivalent to $\mathrm{Oblv}(\mathcal{F})$ being coconnective, as desired.

Because the functor $D(X)^{K,w,+} \to D(X)^+$ is t-exact and these t-structures are right complete, this functor is comonadic. The forgetful functor $D(X)^{K,w} \to D(X)^{K,w,naive}$ induces an equivalence on the corresponding comonads on $D(X)$, and the latter category maps comonadically to $D(X)$. This implies $D(X)^{K,w,+} \xrightarrow{\simeq} D(X)^{K,w,naive,+}$.

The last part is proved similarly to Lemma 5.20.1 (4). We need to show that if $\mathcal{F} \in D(X)^{K,w,+} = D(X)^{K,w,naive,+}$ has $\mathrm{Oblv}(\mathcal{F}) \in D(X)$ compact, then $\mathcal{F}$ is compact in $D(X)^{K,w}$. We are clearly reduced to the case where X is classical. In this case, X is a colimit under closed embeddings of finite type schemes acted on by K, so we are further reduced to the case where X is a finite type scheme. Moreover, we can assume $\mathcal{F}$ lies in the heart of the t-structure, since it is bounded and each of its cohomology groups satisfy the same hypothesis.

Now there exists $K' \lhd K$ compact open (i.e., K/K' is an affine algebraic group) with the action of K on X factoring through K/K'. Further, as in the proof of Lemma 5.20.1 (4), the hypothesis on $\mathcal{F}$ implies that there is a compact open subgroup $K'' \subseteq K'$ also normal in K such that $\mathcal{F}$ lies in the essential image of the functor:

$$D(X)^{K/K'',w,naive,+} \to D(X)^{K,w,naive,+}.$$

Now the result follows from Lemma 5.20.2 (applied to K/K''). $\qquad\square$

8.13. **Naive Hecke actions.** Suppose that H is a Tate group indscheme and $K \subseteq H$ is compact open. Suppose H acts naively on $\mathcal{C}$, i.e., $\mathcal{C}$ is a module for $\mathsf{IndCoh}^*(H)$. Then we claim there is an induced action of the monoidal category:[69]

$$\mathcal{H}_{H,K}^{w,naive} := \mathsf{IndCoh}^*(K\backslash H/K) = \mathsf{IndCoh}^*(H/K)^{K,w,naive}$$

on $\mathcal{C}^{K,w,naive}$.

Indeed, by Lemma 8.6.1, the $\mathsf{IndCoh}(H/K) \in H\text{--}\mathsf{mod}_{weak,naive}$ corepresents the functor of naive K-invariants, so we obtain:

$$\mathcal{H}_{H,K}^{w,naive} = \mathsf{End}_{H\text{--}\mathsf{mod}_{weak,naive}}(\mathsf{IndCoh}(H/K)) \rightarrowtail \mathsf{Hom}_{H\text{--}\mathsf{mod}_{weak,naive}}(\mathsf{IndCoh}(H/K), \mathcal{C}).$$

The following result is a formal consequence of Remark 5.15.4.

---

[69]We emphasize that the middle term uses *non-renormalized* IndCoh^*.

Lemma 8.13.1. *The functor:*

$$H\text{–mod}_{weak,naive} \to \mathcal{H}_{H,K}^{w,naive}\text{–mod}$$

constructed above is fully-faithful.

Proof. First, suppose $\mathcal{C}$ is equipped with a naive action of the compact open subgroup K. Then the natural functor:

$$\mathsf{Vect} \underset{\mathsf{Rep}_{naive}(K)}{\otimes} \mathcal{C}^{K,w,naive} \to \mathcal{C}$$

is an equivalence. Indeed, this is the content of Remark 5.15.4 (and is shown in [Ras4] Proposition 3.5.1).

In particular, we obtain:

$$\mathsf{Vect} \underset{\mathsf{Rep}_{naive}(K)}{\otimes} \mathcal{H}_{H,K}^{K,w,naive} \simeq \mathsf{IndCoh}(H/K).$$

Clearly this is an equivalence of $\mathcal{H}_{H,K}^{wm,naive}$- module categories.

Now for any $\mathcal{C}$ with a naive weak action of H, we calculate:

$$\mathsf{IndCoh}^*(H/K) \underset{\mathcal{H}_{H,K}^{w,naive}}{\otimes} \mathcal{C}^{K,w} = \mathsf{Vect} \underset{\mathsf{Rep}_{naive}(K)}{\otimes} \mathcal{H}_{H,K}^{K,w,naive} \underset{\mathcal{H}_{H,K}^{w,naive}}{\otimes} \mathcal{C}^{K,w} =$$

$$\mathsf{Vect} \underset{\mathsf{Rep}_{naive}(K)}{\otimes} \mathcal{C}^{K,w} \xrightarrow{\simeq} \mathcal{C}.$$

This map is obviously the counit for the evident adjunction $H\text{–mod}_{weak,naive} \leftrightarrows \mathcal{H}_{H,K}^{w,naive}\text{–mod}$, so we obtain the claim. $\qquad\square$

8.14. **Canonical renormalization.** We now wish to give an analogue of the construction from §5.18 in the Tate setting.

8.15. We begin with some general results about renormalizing monoidal structures and module structures.

8.16. We will need the following general constructions in what follows. The reader may safely skip this material and refer back to it as necessary.

Lemma 8.16.1. *Suppose we are given:*

- *$(\mathcal{A}, \star) \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$ a monoidal DG category.*
- *$\mathcal{A}_{ren} \in \mathsf{DGCat}_{cont}$ a compactly generated DG category with $\mathcal{A}_{ren}^c$ its subcategory of compact objects.*
- *t-structures on $\mathcal{A}$ and $\mathcal{A}_{ren}$ compatible with filtered colimits.*
- *A t-exact functor $\Psi : \mathcal{A}_{ren} \to \mathcal{A}$ commuting with colimits and inducing an equivalence $\mathcal{A}_{ren}^+ \xrightarrow{\simeq} \mathcal{A}^+$ on eventually coconnnective subcategories.*

Suppose in addition that the following properties are satisfied:

(1) *The unit object $\mathbb{1} \in \mathcal{A}$ lies in $\mathcal{A}^+$.*
(2) *$\mathcal{A}_{ren}^c$ is contained in $\mathcal{A}_{ren}^+$.*
(3) *For every $\mathcal{F} \in \mathcal{A}_{ren}^c$, the functors $\Psi(\mathcal{F}) \star - : \mathcal{A} \to \mathcal{A}$ and $- \star \Psi(\mathcal{F}) : \mathcal{A} \to \mathcal{A}$ are left t-exact up to shift.*

(4) *For every $\mathcal{F} \in \mathcal{A}_{ren}^c$, the continuous functors $\mathcal{A}_{ren} \to \mathcal{A}_{ren}$ defined by ind-extension of:*

$$\mathcal{A}_{ren}^c \xrightarrow{\Psi(\mathcal{F}) \star \Psi(-)} \mathcal{A}^+ \simeq \mathcal{A}_{ren}^+ \subseteq \mathcal{A}_{ren}$$

$$\mathcal{A}_{ren}^c \xrightarrow{\Psi(-) \star \Psi(\mathcal{F})} \mathcal{A}^+ \simeq \mathcal{A}_{ren}^+ \subseteq \mathcal{A}_{ren}$$

are left t-exact up to shift.

Then $\mathcal{A}_{ren}$ admits a unique monoidal structure such that:

- *The functor Ψ admits a monoidal structure.*
- *For every $\mathcal{F} \in \mathcal{A}_{ren}^c$, the functors $\mathcal{F} \star - : \mathcal{A}_{ren} \to \mathcal{A}_{ren}$ and $- \star \mathcal{F} : \mathcal{A}_{ren} \to \mathcal{A}_{ren}$ preserve $\mathcal{A}_{ren}^+$.*

Proof.

Step 1. We begin with a general construction.

Let us denote by $\mathsf{End}_{\mathsf{DGCat}_{cont}}^{> -\infty}(\mathcal{A}_{ren}) \subseteq \mathsf{End}_{\mathsf{DGCat}_{cont}}(\mathcal{A}_{ren})$ the subcategory of functors $F : \mathcal{A}_{ren} \to \mathcal{A}_{ren}$ that are left t-exact up to shift. Then the restriction functor:

$$\mathsf{End}_{\mathsf{DGCat}_{cont}}^{> -\infty}(\mathcal{A}_{ren}) \to \mathsf{End}_{\mathsf{DGCat}}(\mathcal{A}_{ren}^+) \tag{8.16.1}$$

is fully-faithful. For this, define $\mathsf{End}_{\mathsf{DGCat}}^{LKE}(\mathcal{A}_{ren}^+) \subseteq \mathsf{End}_{\mathsf{DGCat}}(\mathcal{A}_{ren}^+)$ to be the subcategory of functors left Kan extended from their restrictions to $\mathcal{A}_{ren}^c$. Then (8.16.1) clearly maps through this subcategory. Now the restriction functor $\mathsf{End}_{\mathsf{DGCat}}^{LKE}(\mathcal{A}_{ren}^+) \to \mathsf{Hom}_{\mathsf{DGCat}}(\mathcal{A}_{ren}^c, \mathcal{A}_{ren}^+)$ is fully-faithful, and so is its composition with (8.16.1), so (8.16.1) is fully-faithful.

We remark that the essential image of (8.16.1) consists of those DG functors $F : \mathcal{A}_{ren}^+ \to \mathcal{A}_{ren}^+$ that are left Kan extended from $\mathcal{A}_{ren}^c$ and such that the resulting ind-extended functor $\mathcal{A}_{ren} \to \mathcal{A}_{ren}$ is left t-exact up to shift.

Finally, we remark that (8.16.1) is manifestly a monoidal DG functor (between non-cocomplete DG categories).

Step 2. Next, we define an auxiliary category.

Let $\mathcal{B}^c \subseteq \mathcal{A}$ be the full subcategory Karoubi generated by objects of the form $\Psi(\mathcal{F}_1) \star \ldots \star \Psi(\mathcal{F}_n)$ for $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathcal{A}_{ren}^c$. (We allow $n = 0$, i.e., $\mathbb{1}$ is one of our generators of $\mathcal{B}$.)

Clearly $\mathcal{B}^c$ is an essentially small monoidal DG category; let $\mathcal{B} := \mathsf{Ind}(\mathcal{B}^c) \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$.

Note that $\mathcal{B}^c \subseteq \mathcal{A}^+$ by assumption. Define a continuous DG functor $\zeta : \mathcal{B} \to \mathcal{A}_{ren}$ by ind-extension from:

$$\mathcal{B}^c \subseteq \mathcal{A}^+ \simeq \mathcal{A}_{ren}^+ \subseteq \mathcal{A}_{ren}.$$

We remark that ζ is a colocalization functor, i.e., it admits a fully-faithful left adjoint. Namely, this left adjoint is the ind-extension of the fully-faithful functor $\Psi : \mathcal{A}_{ren}^c \to \mathcal{B}^c \subseteq \mathcal{A}^+$.

Step 3. We now construct a $\mathcal{B}$-bimodule structure on $\mathcal{A}_{ren}$ in DGCat_{cont}.

Let e.g. $\mathcal{B}^{mon-op}$ denote $\mathcal{B}$ with its monoidal structure reversed. So we wish to construct a continuous monoidal DG functor $\mathcal{B} \otimes \mathcal{B}^{mon-op} \to \mathsf{End}_{\mathsf{DGCat}_{cont}}(\mathcal{A}_{ren})$. This is equivalent to giving a monoidal DG functor:

$$(\mathcal{B} \otimes \mathcal{B}^{mon-op})^c = \mathcal{B}^c \overline{\otimes} \mathcal{B}^{c,mon-op} \to \mathsf{End}_{\mathsf{DGCat}_{cont}}(\mathcal{A}_{ren}).$$

(We remind that $\overline{\otimes}$ indicates the tensor product on the category of small DG categories.)

Note that $\mathcal{A}^+ \simeq \mathcal{A}_{ren}^+$ is an $\mathcal{B}^c$-bimodule (in DGCat) by our assumption (3). Therefore, we obtain a monoidal functor:

$$\mathcal{B}^c\overline{\otimes}\mathcal{B}^{c,mon-op} \to \mathsf{End}_{\mathsf{DGCat}}(\mathcal{A}^+_{ren}).$$

Moreover, by assumption (4), this functor maps into the essential image of (8.16.1). Therefore, it lifts canonically to a monoidal functor:

$$\mathcal{B}^c\overline{\otimes}\mathcal{B}^{c,mon-op} \to \mathsf{End}^{>-\infty}_{\mathsf{DGCat}_{cont}}(\mathcal{A}_{ren}) \subseteq \mathsf{End}_{\mathsf{DGCat}_{cont}}(\mathcal{A}_{ren})$$

as desired.

Step 4. Next, observe that our functor ζ from above is a morphism of $\mathcal{B}$-bimodule categories (in DGCat_{cont}) Indeed, this results from the fact that the embedding $\mathcal{B}^c \hookrightarrow \mathcal{A}^+$ is a morphism of $\mathcal{B}^c$-bimodule categories (in DGCat).

In particular, $\mathrm{Ker}(\zeta)$ is a two-sided monoidal ideal in $\mathcal{B}$. As ζ was a colocalization DG functor, this means that $\mathcal{A}_{ren}$ admits a unique monoidal structure such that ζ is monoidal. This monoidal structure clearly has the desired properties.

$\square$

Example 8.16.2. Note that the assumption (4) is automatic given the other assumptions (notably, (3)) if compact objects in $\mathcal{A}^c_{ren}$ are closed under truncations.

Example 8.16.3. By Example 8.16.2, Lemma 8.16.1 applies for $\mathcal{A}_{ren} = \mathcal{H}^w_{H,K} \to \mathcal{H}^{w,naive}_{H,K} = \mathcal{A}$. In particular, it may be used to directly construct the monoidal structure on $\mathcal{H}^w_{H,K}$ from that of $\mathcal{H}^{w,naive}_{H,K}$.

We will also need a variant of the above construction for module categories.

Lemma 8.16.4. *In the setting of Lemma 8.16.1, suppose we are additionally given:*

- $\mathcal{M} \in \mathsf{DGCat}_{cont}$ *a module category (in* DGCat_{cont}*) for* $\mathcal{A}$*.*
- $\mathcal{M}_{ren} \in \mathsf{DGCat}_{cont}$ *a compactly generated DG category.*
- *t-structures on* $\mathcal{M}$ *and* $\mathcal{M}_{ren}$ *compatible with filtered colimits and such that* $\mathcal{M}^c_{ren}$ *(the sub-category of compact objects) is contained in* $\mathcal{M}^+_{ren}$*.*
- *A t-exact functor* $\psi : \mathcal{M}_{ren} \to \mathcal{M}$ *commuting with colimits and inducing an equivalence* $\mathcal{M}^+_{ren} \xrightarrow{\simeq} \mathcal{M}^+$*.*

Suppose that:

(1) *For every* $\mathcal{F} \in \mathcal{A}^c_{ren}$*, the functor* $\Psi(\mathcal{F}) \star - : \mathcal{M} \to \mathcal{M}$ *preserves* $\mathcal{M}^+$*.*

(2) *For every* $\mathcal{F} \in \mathcal{A}^c_{ren}$*, the continuous functor* $\mathcal{M}_{ren} \to \mathcal{M}_{ren}$ *defined by ind-extension from:*

$$\mathcal{M}^c_{ren} \xrightarrow{\Psi(\mathcal{F})\star\psi(-)} \mathcal{M}^+ \simeq \mathcal{M}^+_{ren} \subseteq \mathcal{M}_{ren}$$

is left t-exact up to shift.

(3) *For every* $\mathcal{G} \in \mathcal{M}^c_{ren}$*, the functor* $- \star \psi(\mathcal{G}) : \mathcal{A} \to \mathcal{M}$ *maps* $\mathcal{A}^+$ *to* $\mathcal{M}^+$*.*

(4) *For every* $\mathcal{G} \in \mathcal{M}^c_{ren}$*, the continuous functor* $\mathcal{A}_{ren} \to \mathcal{M}_{ren}$ *defined by ind-extension from:*

$$\mathcal{A}^c_{ren} \xrightarrow{\Psi(-)\star\psi(\mathcal{G})} \mathcal{M}^+ \simeq \mathcal{M}^+_{ren} \subseteq \mathcal{M}_{ren}$$

is left t-exact up to shift.

Then there is a unique action of $\mathcal{A}_{ren}$ *on* $\mathcal{M}_{ren}$ *such that:*

- *The functor* $\psi : \mathcal{M}_{ren} \to \mathcal{M}$ *is a morphism of* $\mathcal{A}_{ren}$*-module categories, where* $\mathcal{A}_{ren}$ *acts on* $\mathcal{M}$ *by restriction along* $\Psi : \mathcal{A}_{ren} \to \mathcal{A}$*.*
- *For every* $\mathcal{F} \in \mathcal{A}^c_{ren}$*, the functor* $\mathcal{F} \star - : \mathcal{M}_{ren} \to \mathcal{M}_{ren}$ *preserves* $\mathcal{M}^+_{ren}$*.*

Proof. As in the proof of Lemma 8.16.1, the restriction functor:

$$\mathsf{End}^{>-\infty}_{\mathsf{DGCat}_{cont}}(\mathcal{M}_{ren}) \to \mathsf{End}_{\mathsf{DGCat}}(\mathcal{M}^+_{ren}) \tag{8.16.2}$$

is fully-faithful, using similar notation as in that argument.

We use the notation from the proof of Lemma 8.16.1 freely below. By assumption, the (non-cocomplete) monoidal DG category $\mathcal{B}^c$ is equipped with a monoidal DG functor to the right hand side of (8.16.2) and maps into the essential image of that functor by assumption, so we obtain an induced action of $\mathcal{B}$ on $\mathcal{M}_{ren}$. We again denote this action using the notation $\star$.

As in the proof of Lemma 8.16.1, the monoidal functor ζ admits a fully-faithful left adjoint $\xi : \mathcal{A}_{ren} \hookrightarrow \mathcal{B}$.

We will use the following observation. Let $\mathcal{G} \in \mathcal{M}^c_{ren}$. By construction, the functor $\mathcal{A}_{ren} \xrightarrow{\xi(-)\star\mathcal{G}} \mathcal{M}_{ren}$ is ind-extended from the composition:

$$\mathcal{A}^c_{ren} \to \mathcal{A}^+ \xrightarrow{-\star\psi(\mathcal{G})} \mathcal{M}^+ \simeq \mathcal{M}^+_{ren} \subseteq \mathcal{M}_{ren}.$$

Therefore, our assumptions imply that this functor is left t-exact up to shift.

Now observe that ξ is automatically left lax monoidal. Therefore, it suffices to show:

(1) For $\mathcal{G} \in \mathcal{M}_{ren}$, the natural map:

$$\xi(\mathbb{1}_{\mathcal{A}}) \star \mathcal{G} \to \mathbb{1}_{\mathcal{B}} \star \mathcal{G} = \mathcal{G}$$

is an isomorphism.

(2) For $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{A}_{ren}$ and $\mathcal{G} \in \mathcal{M}_{ren}$, the natural map:

$$\xi(\mathcal{F}_1) \star \xi(\mathcal{F}_2) \star \mathcal{G} \to \xi(\mathcal{F}_1 \star \mathcal{F}_2) \star \mathcal{G}$$

is an isomorphism.

As ξ is a left adjoint, each of the functors appearing above commutes with colimits in each variable. Therefore, we may assume $\mathcal{G} \in \mathcal{M}^c_{ren} \subseteq \mathcal{M}^+_{ren}$ in each of the above cases, and $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{A}^c_{ren}$ in (2).

For (1), note that $\xi(\mathbb{1}_{\mathcal{A}}) \star \mathcal{G} \in \mathcal{M}^+_{ren}$ by the observation above, so as the same is true for $\mathcal{G}$, it suffices to check that the map is an isomorphism after applying ψ; this is clear.

For (2), the functors $\xi(\mathcal{F}_i) \star - : \mathcal{M}_{ren} \to \mathcal{M}_{ren}$ are preserve $\mathcal{M}^+_{ren}$ by construction, so again the two terms we are comparing lie in $\mathcal{M}^+_{ren}$ so it suffices to (trivially) observe that the relevant map becomes an isomorphism after applying ψ. $\square$

Example 8.16.5. As in Example 8.16.2, assumption (2) (resp. (4)) is automatic if compact objects in $\mathcal{M}_{ren}$ (resp. $\mathcal{A}_{ren}$) are closed under truncations.

8.17. Suppose H is a Tate group indscheme and suppose $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is acted on naively by H. Suppose in addition that $\mathcal{C}$ is equipped with a t-structure.

Definition 8.17.1. The naive action of H on $\mathcal{C}$ *canonically renormalizes* (relative to the t-structure) if:

- For every compact open subgroup $K \subseteq H$, the induced naive action of K on $\mathcal{C}$ canonically renormalizes (in the sense of §5.18).

- For every compact open subgroup $K \subseteq H$, the data:

$$\Psi : \mathcal{A}_{ren} = \mathcal{H}_{H,K}^{w} \to \mathcal{A} = \mathcal{H}_{H,K}^{w,naive}$$

$$\psi : \mathcal{M}_{ren} = \mathcal{C}^{K,w} \to \mathcal{M} = \mathcal{C}^{K,w,naive}$$

  satisfy the hypotheses of Lemma 8.16.4. (Here $\mathcal{C}^{K,w}$ is defined as in §5.18.)
- For every pair $K_1 \subseteq K_2 \subseteq H$ of embedded compact open subgroups of H, the morphism:

$$\mathcal{C}^{K_2,w} \underset{\mathsf{Rep}(K_2)}{\otimes} \mathsf{Rep}(K_1) \to \mathcal{C}^{K_1,w}$$

  of Lemma 5.20.1 is an equivalence.

Remark 8.17.2. Technically there is some room for confusion: if H is itself a classical affine group scheme, then this condition is a bit more stringent than the one from §5.18. The author hopes that this will not cause any confusion.

Proposition 8.17.3. *Suppose $\mathcal{C}$ is equipped with a t-structure and a naive action of H that canonically renormalize.*

Define the category $\mathsf{Gen}(\mathcal{C})$ to consist of objects $\mathcal{D} \in H\text{–}\mathsf{mod}_{weak}$ equipped with an isomorphism $\mathrm{Oblv}_{gen}(\mathcal{D}) \simeq \mathcal{C} \in H\text{–}\mathsf{mod}_{weak,naive}$ (c.f. §7.14) and with the property that for any compact open subgroup K, $\mathcal{D}^{K,w}$ is compactly generated and the induced functor:

$$\mathcal{D}^{K,w} \to \mathcal{D}^{K,w,naive} = \mathcal{C}^{K,w,naive}$$

is fully-faithful on compact objects and induces an isomorphism $\mathcal{D}^{K,w,c} \xrightarrow{\simeq} \mathcal{C}^{K,w,c}$ (the right hand side being defined by §5.18).

Then the category $\mathsf{Gen}(\mathcal{C})$ is contractible, i.e., equivalent to $* \in \mathsf{Gpd} \subseteq \mathsf{Cat}$.

Proof. Fix a compact open subgroup $K \subseteq H$. Define $\mathsf{Gen}_K(\mathcal{C})$ to consist of $\mathcal{D} \in H\text{–}\mathsf{mod}_{weak}$ equipped with an isomorphism $\mathrm{Oblv}_{gen}(\mathcal{D}) \simeq \mathcal{C}$ and satisfying the similar property as for $\mathsf{Gen}(\mathcal{C})$, but *only for K* (not for *all* compact open subgroups). Clearly $\mathsf{Gen}(\mathcal{C}) \subseteq \mathsf{Gen}_K(\mathcal{C})$ is a full subcategory. Therefore, it suffices to show that $\mathsf{Gen}_K(\mathcal{C})$ is contractible and that $\mathsf{Gen}(\mathcal{C})$ is non-empty.

We need to check that $\mathsf{Gen}(\mathcal{C})$ is non-empty. Let $\mathcal{C}^{K,w}$ be defined as by canonical renormalization for K. By Lemma 8.16.4 (and by definition of canonical renormalization for H), there is a canonical $\mathcal{H}_{H,K}^{w}$-module structure on $\mathcal{C}^{K,w}$. Let $\iota_K(\mathcal{C}) \in H\text{–}\mathsf{mod}_{weak}$ be the corresponding object with $\iota_K(\mathcal{C})^{K,w} = \mathcal{C}^{K,w}$ (as $\mathcal{H}_{H,K}^{w}$-modules). There is an evident isomorphism $\mathrm{Oblv}_{gen}\iota_K(\mathcal{C}) = \mathcal{C} \in H\text{–}\mathsf{mod}_{weak,naive}$. This construction clearly defines an object of $\mathsf{Gen}_K(\mathcal{C})$, and contractibility of the category is immediate from Lemmas 8.16.4 and 8.13.1.

We claim moreover that the above construction defines an object of $\mathsf{Gen}(\mathcal{C})$.

For any $K' \subseteq K$, we have:

$$\iota_K(\mathcal{C})^{K',w} = \iota_K(\mathcal{C})^{K,w} \underset{\mathsf{Rep}(K)}{\otimes} \mathsf{Rep}(K') = \mathcal{C}^{K,w} \underset{\mathsf{Rep}(K)}{\otimes} \mathsf{Rep}(K') \to \mathcal{C}^{K',w}$$

and this functor satisfies the conclusions of Lemma 8.16.4 with respect to the Hecke categories relative to K'. This gives an isomorphism $\iota_K(\mathcal{C}) \simeq \iota_{K'}(\mathcal{C}) \in H\text{–}\mathsf{mod}_{weak}$.

As the intersection of compact open subgroups is again compact open, we see that for any (possibly not nested) $K, K' \subseteq H$ compact open subgroups, there exists an isomorphism $\iota_K(\mathcal{C})\iota_K(\mathcal{C}) \simeq \iota_{K'}(\mathcal{C})$. This implies that:

$$\iota_K(\mathcal{C}) \in \bigcap_{K'} \mathsf{Gen}_{K'}(\mathcal{C}) =: \mathsf{Gen}(\mathcal{C})$$

as desired.

$\square$

Notation 8.17.4. In the above setting, we follow our standard abuses of notation in letting $\mathcal{C} \in H\text{–mod}_{weak}$ denote the canonical object constructed via Proposition 8.17.3 (namely, the object defined by the forgetful functor $* = \mathsf{Gen}(\mathcal{C}) \to H\text{–mod}_{weak}$).

We also need the following variant.

Proposition 8.17.5. *Suppose[70] $\mathcal{A} \in \mathsf{Alg}(\mathsf{DGCat}_{cont})$ is equipped with a t-structure such that* id : $\mathcal{A} \to \mathcal{A}$ *satisfies the hypotheses for the functor "Ψ" from Lemma 8.16.1.[71]*

Suppose $\mathcal{C} \in \mathsf{DGCat}_{cont}$ is equipped with a t-structure and an $\mathsf{IndCoh}^*(H) \otimes \mathcal{A}$-module structure such that:

- *The underlying naive H-action canonically renormalizes.*
- *For any $K \subseteq H$ compact open, the evident $\mathcal{A}$-action on $\mathcal{C}^{K,w,naive}$ satisfies the hypotheses of Lemma 8.16.4 relative to $\psi : \mathcal{C}^{K,w} \to \mathcal{C}^{K,w,naive} \in \mathsf{DGCat}_{cont}$ (and* id : $\mathcal{A} \to \mathcal{A}$*).*

Then:

(1) *For any compact open subgroup $K \subseteq H$, the morphism:*

$$\mathcal{H}^w_{H,K} \otimes \mathcal{A} \to \mathcal{H}^{w,naive}_{H,K} \otimes \mathcal{A}$$

satisfies the hypotheses of Lemma 8.16.1, where both sides are equipped with the natural tensor product t-structures.

   Moreover, the corresponding action of $\mathcal{H}^{w,naive}_{H,K} \otimes \mathcal{A}$ on $\mathcal{C}^{K,w,naive}$ satisfies the hypotheses of Lemma 8.16.4 (relative to the above functor and $\mathcal{C}^{K,w} \to \mathcal{C}^{K,w,naive}$).

(2) *Define the category $\mathsf{Gen}_{\mathcal{A}}(\mathcal{C})$ to consist of objects[72] $\mathcal{D} \in \mathcal{A}\text{–mod}(H\text{–mod}_{weak})$ equipped with an isomorphism $\mathrm{Oblv}_{gen}(\mathcal{D}) \simeq \mathcal{C} \in \mathcal{A}\text{–mod}(H\text{–mod}_{weak,naive})$ (c.f. §7.14) and with the property that on forgetting the $\mathcal{A}$-action, $\mathcal{D}$ defines an object of $\mathsf{Gen}(\mathcal{C})$ (as in the notation of Proposition 8.17.3).*

   Then the category $\mathsf{Gen}_{\mathcal{A}}(\mathcal{C})$ is contractible, i.e., equivalent to $*$.

Proof. (1) is immediate from Lemma 4.6.2 (2). Then (2) follows by the exact same argument as in Proposition 8.17.3.

$\square$

8.18. **Preliminary remarks about D-modules.** Let H be a Tate group indscheme and fix a compact open subgroup K_0.

Then K_0 induces a t-structure on $D^*(H)$. Indeed, we have:

$$D^*(H) = \operatorname*{colim}_{K \subseteq K_0 \subseteq H \text{ compact open}} D^*(H/K)$$

under $*$-pullback functors (which are defined as each pullback here is smooth). As this colimit is filtered and these functors are all t-exact up to shift (being smooth pullbacks), we obtain the claim. Explicitly, this t-structure is normalized by the fact that for each projection $\pi_K : H \to H/K$, the functor $\pi_K^{*,dR}[-\dim(K_0/K)] : D(H/K) \to D^*(H)$ is t-exact.

---

[70]The notation is potentially misleading: this category $\mathcal{A}$ behaves more like the category $\mathcal{A}_{ren}$ from Lemma 8.16.1.

[71]This is just a convenient way to say $\mathcal{A}$ is compactly generated with the action of compact objects being given by functors that are left t-exact up to shift and with unit being eventually coconnective.

[72]Here $\mathcal{A}$-modules in $H\text{–mod}_{weak}$ are defined because $H\text{–mod}_{weak}$ is tensored over DGCat_{cont}.

Remark 8.18.1. Note that the t-structures attached to different compact open subgroups differ by shifts by a locally constant function on H, namely, their relative dimension. For our present purposes, such differences are irrelevant, so we do not emphasize the choice of K_0 in what follows.

8.19.   We will use the following basic observation.

Lemma 8.19.1. *The t-structure just constructed on $D^*(H)$ satisfies the following properties.*

- *The t-structure is right complete.*
- *Any compact $\mathcal{F} \in D^*(H)$ is eventually coconnective.*
- *Compact objects are closed under truncations.*
- *For any $\mathcal{F} \in D^*(H)$ compact, the monoidal operations $\mathcal{F} \star - : D^*(H) \to D^*(H)$ and $- \star \mathcal{F} : D^*(H) \to D^*(H)$ are left t-exact up to shift.*

Proof. The first three claims are evident from the construction. For the last one, note that there is a compact open subgroup $K \subseteq H$ and a coherent D-module $\mathcal{F}_0 \in D(H/K)$ such that $\mathcal{F} = \pi_K^{*,dR}(\mathcal{F}_0)$. Then for $\mathcal{G} \in D^*(H)$, we have:

$$\mathcal{F} \star \mathcal{G} = \mathcal{F}_0 \overset{K}{\star} \mathrm{Av}_*^K(\mathcal{G})$$

where $\star$ is convolution on $D^*(H)$, Av_*^K indicates (strong) K-averaging on the left, and $- \overset{K}{\star} - : D(H/K) \otimes D(K/H) \to D^*(H)$ is the relative convolution. The functor Av_*^K is left t-exact up to shift: it is right adjoint to a functor that is t-exact up to shift. Then the claim is evident from the fact that $\mathcal{F}_0$ has support some finite type scheme and from standard cohomological estimates.

$\square$

Remark 8.19.2. The upshot is that $D^*(H)$ *almost* satisfies the hypotheses of the monoidal DG category $\mathcal{A}$ from Proposition 8.17.5: its unit object is not eventually coconnective, but $D^*(H)$ otherwise satisfies the evident non-unital analogue.

8.20. **Main construction.** We are now equipped to give the main construction.

To avoid confusion, we let $D^*(H)^{gen} \in H\text{--}\mathsf{mod}_{weak}$ denote the object constructed in §8.9, and we use $D^*(H)$ to indicate the underlying DG category $\mathrm{Oblv}_{gen}(D^*(H)^{gen})$.

First, note that there is a canonical monoidal functor $\mathsf{IndCoh}^*(H) \to D^*(H)$; indeed, this functor is constructed in §6.20 with the monoidal structure coming from Remark 6.20.1.

In particular, $D^*(H)$ is canonically a $(\mathsf{IndCoh}^*(H), D^*(H))$-bimodule. The left $\mathsf{IndCoh}^*(H)$-module (i.e., naive weak H-module) structure here is by construction to one arising from realizing $D^*(H)$ as $\mathrm{Oblv}_{gen}(D^*(H)^{gen})$.

Next, observe that the left action action of $\mathsf{IndCoh}^*(H)$ on $D^*(H)$ canonically renormalizes in the sense of §8.17, and the corresponding object (via Proposition 8.17.3) of $H\text{--}\mathsf{mod}_{weak}$ is $D^*(H)^{gen}$. Indeed, this is a routine verification by Lemma 8.12.2 and Examples 8.16.2 and 8.16.5. The last axiom for canonical renormalization (on varying the compact open subgroups) reduces to Lemma 5.20.1 (4).

Therefore, by Proposition 8.17.5 and Lemma 8.19.1 (c.f. Remark 8.19.2) we obtain an a priori non-unital[73] action of $D^*(H)$ on $D^*(H)^{gen} \in H\text{--}\mathsf{mod}_{weak}$.

By [Lur3] Proposition 5.4.3.16, it is a property (not a structure) for $D^*(H)$ to act unitally on $D^*(H)^{gen}$. We verify this explicitly as follows.

---

[73]It is clear that our discussion goes through in a non-unital setting, but this also follows directly from the unital case by freely adjoining a unit, i.e., applying Proposition 8.17.5 with $\mathcal{A} = D^*(H) \times \mathsf{Vect}$.

Let $K_0 \subseteq H$ be a fixed compact open subgroup, which we also use to normalize the t-structure on $D^*(H)$. As $H\text{–mod}_{weak} \simeq \mathcal{H}^w_{H,K_0}\text{–mod}$, we need to verify that the induced (right) $D^*(H)$-action on $D^*(H)^{K_0,w}$ is unital.

Note that this is tautologically the case for $D^*(H)^{K_0,w,naive}$. As $D^*(H)^{K_0,w,+} \xrightarrow{\simeq} D^*(H)^{K_0,w,naive,+}$ by construction, it suffices to show that the unit object $\delta_1 \in D^*(H)$ acts by a left t-exact functor on $D^*(H)^{K_0,w}$.

Recall that $\delta_1 = \operatorname{colim}_K \delta_K$ where the colimit runs over compact open subgroups and the term δ_K indicates the δ D-module on H supported on K.[74] Each δ_K is compact, so (by the construction of Lemma 8.16.4) acts on $D^*(H)^{K_0,w}$ by a functor that is left t-exact up to shift. In fact, these functors are left t-exact as is: the induced functor $D^*(H)^{K_0,w,naive}$ is $\operatorname{Oblv} \operatorname{Av}^K_*$, which is left t-exact.[75]

By the above description of δ_1, it also acts by a left t-exact functor, so our earlier remarks we are done.

This completes the construction of a (right) $D^*(H)$-action on $D^*(H)^{gen} \in H\text{–mod}_{weak}$, and therefore (as in §8.4), induces a functor:

$$H\text{–mod} = D^*(H)\text{–mod} \to H\text{–mod}_{weak}.$$

8.21. Invariants vs. coinvariants.
We now complete the promise from Remark 8.3.1, comparing weak invariants and coinvariants for $\exp(\mathfrak{h})$ in the polarizable case.

Recall the category $D^!(H) := D^*(H)^\vee = \operatorname{Hom}_{\mathsf{DGCat}_{cont}}(D^*(H), \mathsf{Vect})$ from [Ras3]. Clearly $D^!(H)$ is canonically a $(D^*(H), D^*(H))$-bimodule. As H is placid (in the sense of *loc. cit.*), any left (resp. right) H-invariant dimension theory on H defines an equivalence $D^*(H) \xrightarrow{\simeq} D^!(H)$ of left (resp. right) $D^*(H)$-modules.

Note that invariant dimension theories do exist on H: any choice of congruence subgroup defines one (see [Ras3] Construction 6.12.6). In particular, $D^!(H)$ is invertible as a bimodule.

Proposition 8.21.1. *Let H be a polarizable Tate group indscheme. Then there is a canonical isomorphism of functors:*

$$D^!(H) \underset{D^*(H)}{\otimes} (-)_{\exp(\mathfrak{h}),w} \simeq (- \otimes \chi_{-Tate})^{\exp(\mathfrak{h}),w} : H\text{–mod}_{weak} \to H\text{–mod}.$$

Proof. In what follows, we use the symmetric monoidal structure $- \otimes -$ on $H\text{–mod}_{weak}$ from §7.18.

Let $\mathcal{C} \in H\text{–mod}_{weak}$. We claim:

$$\mathcal{C}_{\exp(\mathfrak{h}),w} = \left(\mathcal{C} \otimes \operatorname{Oblv}^{str \to w} \left(D^*(H) \right) \right)_{H,w}$$

$$\mathcal{C}^{\exp(\mathfrak{h}),w} = \left(\mathcal{C} \otimes \operatorname{Oblv}^{str \to w} \left(D^!(H) \right) \right)^{H,w}$$

as objects of[76] $H\text{–mod}$ functorially in $\mathcal{C}$. Indeed, the first identity is immediate, and the second identity follows similarly the fact that $(-)^{H,w}$ is DGCat_{cont}-linear for polarizable H.

Then the claim is straightforward:

---

[74]Note that this object of $D^*(H)$ is in cohomological degree $-\dim(K_0/K)$ if $K \subseteq K_0$.

[75]Here, of course, the averaging is taken on the right, i.e., it does not interact with the weak K_0-invariants.

[76]Here we are using the fact that $\operatorname{Oblv}^{str \to w}(D^!(H)), \operatorname{Oblv}^{str \to w}(D^*(H)) \in D^*(H)\text{–mod}(H\text{–mod}_{weak})$, where this structure arises from the bimodule structures on $D^!(H)$ and $D^*(H)$.

$$\mathcal{C}_{\exp(\mathfrak{h}),w} = \left(\mathcal{C} \otimes \mathrm{Oblv}^{str \to w}\left(D^*(H)\right)\right)_{H,w} \overset{Prop.\underline{7}.18.2}{=}$$

$$\left(\mathcal{C} \otimes \chi_{-Tate} \otimes \mathrm{Oblv}^{str \to w}\left(D^*(H)\right)\right)^{H,w} =$$

$$D^!(H)^{\otimes -1} \underset{D^*(H)}{\otimes} \left(\mathcal{C} \otimes \chi_{-Tate} \otimes \mathrm{Oblv}^{str \to w}\left(D^!(H)\right)\right)^{H,w} =$$

$$D^!(H)^{\otimes -1} \underset{D^*(H)}{\otimes} \left(\mathcal{C} \otimes \chi_{-Tate}\right)^{\exp(\mathfrak{h}),w}$$

for $D^!(H)^{\otimes -1} := \mathsf{Hom}_{D^*(H)-\mathsf{mod}}(D^!(H), D^*(H))$ the $D^*(H)$-bimodule inverse to $D^!(H)$. This clearly gives the identity.

$\square$

Example 8.21.2. Suppose $\mathcal{C} = \mathsf{IndCoh}^*(H) \in H\text{–}\mathsf{mod}_{weak}$ (i.e., the evident object that corepresents Oblv_{gen}). Then $\mathcal{C}_{\exp(\mathfrak{h}),w} \simeq D^*(H) \in H\text{–}\mathsf{mod}$. If one takes[77] $\mathsf{IndCoh}^!(H) := \mathsf{IndCoh}^*(H) \otimes \chi_{-Tate} \in H\text{–}\mathsf{mod}_{weak}$, then Proposition 8.21.1 says $\mathsf{IndCoh}^!(H)^{\exp(\mathfrak{h}),w} = D^!(H)$, as expected.

9. SEMI-INFINITE COHOMOLOGY

9.1. Construction of semi-infinite cohomology. Let H be a Tate group indscheme.

Definition 9.1.1. The *absolute semi-infinite cohomology* functor:

$$C^{\frac{\infty}{2}}(\mathfrak{h}, -) : \mathsf{Vect}_{\exp(\mathfrak{h}),w} \to \mathsf{Vect} \in H\text{–}\mathsf{mod}$$

is the counit map corresponding to the adjunction constructed in §8.

The goal for this section is to show that for H formally smooth, this functor identifies (in a suitable sense) with the classical functor of semi-infinite cohomology for Tate Lie algebras.

Remark 9.1.2. As in indicated in the notation above, $C^{\frac{\infty}{2}}(\mathfrak{h}, -)$ is strongly H-equivariant. This is a non-obvious (if widely anticipated) property from the traditional construction of semi-infinite cohomology via Clifford algebras.

Remark 9.1.3. The above functor is defined (and is strongly H-equivariant) for any Tate group indscheme H. However, for the purposes of relating this functor to classical constructions, we may assume H is polarizable; indeed, replacing H by its formal completion along any compact open subgroup manifestly does not change $C^{\frac{\infty}{2}}(\mathfrak{h}, -)$ as a morphism in DGCat_{cont}. Therefore, in the analysis of this section, H is frequently taken to be polarizable.

Remark 9.1.4. The above construction (hence our comparison theorem) only applies for those Tate Lie algebras $\mathfrak{h} \in \mathsf{ProVect}^{\heartsuit}$ arising as the Lie algebra of some formally smooth Tate group indscheme. Equivalently, there must exist $\mathfrak{k} \subseteq \mathfrak{h}$ a compact open Lie subalgebra arising that arises as the Lie algebra of some affine group scheme. Certainly this is the case whenever $\mathfrak{h}$ has a pro-nilpotent compact open subalgebra, which covers all examples of interest.

We anticipate (as indicated in the notation) that there is a theory of weak actions for the "formal group" $\exp(\mathfrak{h})$ for a general Tate Lie algebra $\mathfrak{h}$. The argument given below for the comparison theorem should then apply as is in that setup. However, as the applications we have in mind do not require such a theory, we do not develop one in this text.

---

[77]Note that $\mathsf{IndCoh}^!(H)$ is also the internal Hom in the symmetric monoidal category $H\text{–}\mathsf{mod}_{weak}$ from $\mathsf{IndCoh}^*(H)$ to the trivial object Vect.

Remark 9.1.5. We thank Gurbir Dhillon for insisting that we include this material in the present text and for helpful discussions related to it.

9.2. Throughout this section, H denotes a Tate group indscheme.

We maintain the conventions of §7.3: all quotients (including classifying stacks) are Zariski sheafified.

9.3. **Central extensions.** We begin by discussing some general constructions relating to central extensions.

9.4. We begin with the following construction.

First, note that there is a canonical action of $\mathbb{B}\mathbb{G}_m$ on Vect, i.e., an action of $\mathsf{QCoh}(\mathbb{B}\mathbb{G}_m)$ equipped with the convolution monoidal structure on Vect. Indeed, this monoidal category is canonically equivalent to $\mathbb{Z}$-graded vector spaces with degree-wise. Our monoidal functor $\mathsf{QCoh}(\mathbb{B}\mathbb{G}_m) \to \mathsf{Vect}$ takes the (-1)-st degree component.[78]

Extend this construction to an action of $\mathbb{B}\mathbb{G}_m \times \mathbb{Z}$ by having the generator $1 \in \mathbb{Z}$ act on Vect as $(-)[1] : \mathsf{Vect} \xrightarrow{\simeq} \mathsf{Vect}$.

Now for any H a Tate group indscheme equipped with a homomorphism:

$$(\varepsilon, \delta) : H \to \mathbb{B}\mathbb{G}_m \times \mathbb{Z}$$

of groups, we obtain an action of H on Vect by restriction along the monoidal pushforward functor:[79]

$$\mathsf{IndCoh}^*(H) \to \mathsf{IndCoh}^*(\mathbb{B}\mathbb{G}_m \times \mathbb{Z}) = \mathsf{QCoh}(\mathbb{B}\mathbb{G}_m \times \mathbb{Z}).$$

Remark 9.4.1. Under the above construction, any $h \in H(k)$ defines a skyscraper sheaf in $\mathsf{IndCoh}^*(H)$, so by extension, an automorphism of Vect. By construction, this automorphism is $\varepsilon(h) \otimes -[\delta(h)]$, where $\varepsilon(h)$ is the k-line defined by h and ε.

Proposition 9.4.2. *The above construction gives an equivalence of groupoids:*

$$\mathsf{Hom}_{\mathsf{Gp}}(H, \mathbb{B}\mathbb{G}_m \times \mathbb{Z}) \to \mathsf{Hom}_{\mathsf{Alg}(\mathsf{DGCat}_{cont})}(\mathsf{IndCoh}^*(H), \mathsf{Vect}). \tag{9.4.1}$$

Proof.

Step 1. First, it is convenient to dualize (in the sense of [Gai4]). For S a reasonable indscheme, we let $\mathsf{IndCoh}^!(S) \in \mathsf{DGCat}_{cont}$ denote the dual to $\mathsf{IndCoh}^*(S)$ (which exists because $\mathsf{IndCoh}^*(S)$ is compactly generated). Note that this construction is covariant in S; we denote pullback along a map $f : S \to T$ by $f^! : \mathsf{IndCoh}^!(T) \to \mathsf{IndCoh}^!(S)$.[80] In particular, there is a canonical object $\omega_S \in \mathsf{IndCoh}^!(S)$, the !-pullback of $k \in \mathsf{Vect} = \mathsf{IndCoh}^!(\mathrm{Spec}(k))$ along the structure map $S \to \mathrm{Spec}(k)$.

There is a standard natural transformation $\Upsilon_S : \mathsf{QCoh}(-) \to \mathsf{IndCoh}^!(-)$ such that for any $S \in \mathsf{IndSch}_{reas}$, $\Upsilon_S(\mathcal{O}_S) = \omega_S$. Indeed, for $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$, Υ_S is by definition dual to $\Psi_S : \mathsf{IndCoh}^*(S) \to \mathsf{QCoh}(S)$ (using the standard self-duality of $\mathsf{QCoh}(-)$ on qcqs schemes). In general, the construction is obtained by right Kan extension from this one.

---

[78]The sign here makes normalizations for later constructions more convenient: see Remark 9.4.1.

[79]This functor is defined by the formalism of §6 because H and $\mathbb{B}\mathbb{G}_m \times \mathbb{Z}$ are weakly renormalizable prestacks and this morphism is reasonable indschematic.

[80]This construction should not be confused with the one studied in the IndCoh^*-setting of §6 for a proper (or ind-proper) morphism. Because we only use this construction in the proof of the present proposition, we abuse notation by using the same notation to mean different things (in somewhat different contexts).

Note that Υ_S is fully-faithful for any $S \in \mathsf{IndSch}_{reas}$. Indeed, by construction, this reduces to $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$, and in that case it follows because the dual functor Ψ_S admits a fully-faithful left adjoint.

Step 2. Now suppose that S is *strict* in the sense of §6.35. By functoriality, $\mathsf{IndCoh}^!(S)$ is symmetric monoidal with tensor product $- \overset{!}{\otimes} -$ with unit object ω_S (c.f. [Gai5] §5.6).

We remark that the (symmetric) monoidal category $(\mathsf{IndCoh}^!(S), \overset{!}{\otimes})$ acts canonically on $\mathsf{IndCoh}^*(S)$ by duality.

In what follows, we say $\mathcal{L} \in \mathsf{IndCoh}^!(S)$ is *invertible* if $\mathcal{L} \overset{!}{\otimes} - : \mathsf{IndCoh}^!(S) \to \mathsf{IndCoh}^!(S)$ is an equivalence.

Step 3. Suppose that $S \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$. We suppose that S is strict and $\mathcal{L} \in \mathsf{IndCoh}^!(S)$ is invertible. We claim that:

 (1) $\mathcal{L}$ lies in the essential image of Υ_S.
 (2) The object $\overline{\mathcal{L}} \in \mathsf{QCoh}(S)$ with $\Upsilon_S(\overline{\mathcal{L}}) = \mathcal{L}$ (which is well-defined by the above) is invertible in $\mathsf{QCoh}(S)$ and therefore corresponds to a $\mathbb{Z}$-graded[81] line bundle on S (the $\mathbb{Z}$-grading being locally constant).

First, note that $\mathcal{L} \in \mathsf{IndCoh}^!(S)$ is compact: this follows from invertibility and compactness of ω_S (we emphasize that S is eventually coconnective). We let $\mathbb{D}\mathcal{L} \in \mathsf{Coh}(S)$ denote the corresponding object under the equivalence:

$$(\mathsf{IndCoh}^!(S)^c)^{op} \simeq \mathsf{IndCoh}^*(S)^c = \mathsf{Coh}(S).$$

For this, let $s : \mathrm{Spec}(K) \to S$ be a map with K a field. Clearly $s^!(\mathcal{L})$ is invertible in $\mathsf{IndCoh}^!(\mathrm{Spec}(K)) = K\text{–mod}$. In particular, it corresponds to a graded line. We then obtain:

$$s^!(\mathcal{L}) = \langle K, s^!(\mathcal{L}) \rangle = \langle s_*^{\mathsf{IndCoh}}(K), \mathcal{L} \rangle = \mathrm{Hom}_{\mathsf{IndCoh}^*(S)}(\mathbb{D}\mathcal{L}, s_*^{\mathsf{IndCoh}}(K)) = s^*(\mathbb{D}\mathcal{L})^\vee$$

where the brackets indicate pairings between evident dual categories, the upper-* fiber is the usual quasi-coherent fiber of $\mathbb{D}\mathcal{L} \in \mathsf{Coh}(S) \subseteq \mathsf{QCoh}(S)$, and the dual indicates the dual as a K-vector space.

In particular, the *-fibers of $\mathbb{D}\mathcal{L}$ at field-valued points are concentrated in some single cohomological degree and 1-dimensional there. Now it follows from Nakayama's lemma that $\mathbb{D}\mathcal{L}$ is perfect and a graded line bundle.

Applying duality again, this translates to saying that $\mathcal{L} \in \Upsilon_S(\mathsf{Perf}(S))$, and moreover, is Υ_S of a graded line bundle.

Step 4. Next, we observe that the above immediately generalizes to the case where $S \in \mathsf{IndSch}_{reas}$ can be expressed as a filtered colimit of strict schemes $S_i \in {}^{>-\infty}\mathsf{Sch}_{qcqs}$ under almost finitely presented closed embeddings. Indeed, this case immediately reduces to the schematic case considered above.

In particular, this applies for $S = H$ our Tate group indscheme, as observed in Remark 7.2.5.

Step 5. We now complete the argument.

By duality and strictness of H, a monoidal functor $\mathsf{IndCoh}^*(H) \to \mathsf{Vect}$ is equivalent to a comonoidal functor $\mathsf{Vect} \to \mathsf{IndCoh}^!(H)$, i.e., an object $\mathcal{L} \in \mathsf{IndCoh}^!(H)$ with isomorphisms $m^!(\mathcal{L}) \overset{\simeq}{\to} \mathcal{L} \boxtimes \mathcal{L}$, $e^!(\mathcal{L}) = k$ equipped with higher homotopical compatibilities (for $m : H \times H \to H$ the multiplication and $e : \mathrm{Spec}(k) \to H$ the unit).

---

[81]We normalize this identification by having the suspension $(-)[1]$ correspond to increasing the grading by 1.

Observe that $\mathcal{L}$ is invertible in $\mathsf{IndCoh}^!(H)$: its inverse is the pullback of $\mathcal{L}$ along the inversion map $H \xrightarrow{\simeq} H$.

Therefore, $\mathcal{L}$ defines a $\mathbb{Z}$-graded line bundle on H, or equivalently, a map $H \to \mathbb{B}\mathbb{G}_m \times \mathbb{Z}$. The comonoidal structure above is equivalent to making the map into a map of group prestacks. It is immediate to verify that this equivalence is the inverse to the functor (9.4.1).

$\square$

9.5. **The Tate canonical extension.** We now construct a canonical central extension:

$$1 \to \mathbb{G}_m \to H_{Tate} \to H \to 1$$

of any polarizable Tate group indscheme H.

Take $\chi_{Tate} \in H\text{--}\mathsf{mod}_{weak}$. Recall that $H\text{--}\mathsf{mod}_{weak}$ is naturally symmetric monoidal and that χ_{Tate} is invertible for this monoidal structure.

Recall from Proposition 7.19.1 that $\mathrm{Oblv}_{gen}(\chi_{Tate}) \in \mathsf{DGCat}_{cont}$ is a trivial gerbe, i.e., this DG category is *non-canonically* isomorphic to Vect (the identification depends on a choice of compact open subgroup K of H). In particular, we have a *canonical* isomorphism $\mathsf{End}_{\mathsf{DGCat}_{cont}}(\mathrm{Oblv}_{gen}(\chi_{Tate})) = \mathsf{Vect}$.

As H acts naively on $\mathrm{Oblv}_{gen}(\chi_{Tate})$, this defines a canonical homomorphism $\mathsf{IndCoh}^*(H) \to \mathsf{Vect}$. By Proposition 9.4.2, we obtain a homomorphism map:

$$(\varepsilon_{Tate}, \delta_{Tate}) : H \to \mathbb{B}\mathbb{G}_m \times \mathbb{Z}.$$

By definition, H_{Tate} is the central extension defined by the homomorphism ε_{Tate}.

Remark 9.5.1. Define an object[82] $\mathsf{Vect}_{\chi_{Tate}} \in H\text{--}\mathsf{mod}_{weak}$ as:

$$\mathsf{Vect}_{\chi_{Tate}} := \chi_{Tate} \otimes \mathrm{triv}(\mathrm{Oblv}_{gen}(\chi_{Tate}^{\otimes -1})).$$

Note that $\mathsf{Vect}_{\chi_{Tate}}$ maps canonically under Oblv_{gen} to Vect, and the induced naive H-action is the one constructed above. By Proposition 7.19.1, any choice of compact open subgroup $K \subseteq H$ induces an isomorphism $\chi_{Tate} \simeq \mathsf{Vect}_{\chi_{Tate}} \in H\text{--}\mathsf{mod}_{weak}$.

9.6. We now discuss basic properties of H_{Tate}.

Proposition-Construction 9.6.1. *For any compact open subgroup K of H, there is a canonical splitting of H_{Tate} over K.*

Proof. Immediate from Proposition 7.19.1.

$\square$

Corollary 9.6.2. H_{Tate} *is a Tate group indscheme.*

Warning 9.6.3. In general, for $K_1 \subseteq K_2 \subseteq H$ compact open subgroups, the canonical splitting for K_2 may not restrict to the canonical splitting for K_1 (although this is automatic if K_1 is pro-unipotent).

Proposition-Construction 9.6.4. *Suppose H has the property that δ_{Tate} is identically 0.*

Let $\mathrm{Res} : H\text{--}\mathsf{mod}_{weak} \to H_{Tate}\text{--}\mathsf{mod}_{weak}$ *denote the functor of restriction along $H_{Tate} \to H$, and let $\chi_{Tate} \in H\text{--}\mathsf{mod}_{weak}$ denote the modular character for H.*

Then $\mathrm{Res}(\chi_{Tate})$ *is canonically trivialized, i.e., there is a canonical isomorphism:*

$$\mathrm{Res}(\chi_{Tate}) \simeq \mathrm{triv}\,\mathrm{Oblv}_{gen}(\mathrm{Res}(\chi_{Tate}) \in H_{Tate}\text{--}\mathsf{mod}_{weak}.$$

---

[82]By analogy with usual representations, $\mathsf{Vect}_{\chi_{Tate}}$ is the character defined by the "1-dimensional" representation χ_{Tate}.

Proof. In the notation of Remark 9.5.1, it suffices to construct an isomorphism $\mathrm{Res}(\mathsf{Vect}_{\chi_{Tate}}) \simeq \mathsf{Vect} \in H_{Tate}\mathrm{-mod}_{weak}$.

Note that by standard cohomological estimates, the underlying naive action of H on $\mathrm{Oblv}_{gen}(\mathsf{Vect}_{\chi_{Tate}})$ canonically renormalizes in the sense of Proposition 8.17.3, and that $\mathsf{Vect}_{\chi_{Tate}}$ is obtained by this canonical renormalization procedure. The same applies for H_{Tate} in place of H. Therefore, it suffices to give the construction on underlying naive categories.

But here the result follows from Proposition 9.4.2 and the evident trivialization of the composite homomorphism:

$$H_{Tate} \to H \to \mathbb{B}\mathbb{G}_m \times \mathbb{Z}.$$

$\square$

Remark 9.6.5. By Theorem 5.10.1 for of $\mathbb{G}_m$, it is easy to see that $\mathrm{Res}(\chi_{Tate}) \in H_{Tate}\mathrm{-mod}_{weak}$ is the modular character for H_{Tate}. Therefore, the modular character of H_{Tate} is canonically trivialized.

9.7.   We now wish to formulate in a precise way the following idea: *for* $\mathcal{C} \in H\mathrm{-mod}_{weak}$, $(\mathcal{C} \otimes \mathsf{Vect}_{\chi_{Tate}})^{H,w}$ *is the subcategory of* $\mathcal{C}^{H_{Tate},w}$ *of objects on which* $\mathbb{G}_m$ *acts by homotheties.*

There is a somewhat more satisfying formulation in the naive setting than the genuine one, so we separate the two cases.

9.8.   Let $k(1) \in \mathsf{Rep}(\mathbb{G}_m)$ denote the standard representation, and for $n \in \mathbb{Z}$, let $k(n)$ denote its nth tensor power. We let $\mathsf{Vect}_{(n)} \subseteq \mathsf{Rep}(\mathbb{G}_m)$ denote the image of $\mathsf{Vect} \xrightarrow{k \mapsto k(n)} \mathsf{Rep}(\mathbb{G}_m)$, i.e., the category of graded vector spaces of pure degree n.

Suppose $\mathcal{C} \in H\mathrm{-mod}_{weak}$ and restrict $\mathcal{C}$ to $H_{Tate}\mathrm{-mod}_{weak}$; we now omit Res from the notation. As the central $\mathbb{G}_m \subseteq H_{Tate}$ acts trivially on $\mathcal{C}$, there is a forgetful functor:

$$\mathcal{C}^{H_{Tate},w} \to \mathcal{C}^{\mathbb{G}_m,w} = \mathsf{Rep}(\mathbb{G}_m) \otimes \mathrm{Oblv}_{gen}(\mathcal{C}).$$

We remark that this functor factors through $\mathcal{C}^{H_{Tate},w,naive}$, and that the corresponding functor $\mathcal{C}^{H_{Tate},w,naive} \to \mathsf{Rep}(\mathbb{G}_m) \otimes \mathrm{Oblv}_{gen}(\mathcal{C})$ is conservative.

For $n \in \mathbb{Z}$, define $\mathcal{C}^{H_{Tate},w,naive}_{(n)} \subseteq \mathcal{C}^{H_{Tate},w,naive}$ as the full subcategory:

$$\mathcal{C}^{H_{Tate},w,naive} \underset{\mathsf{Rep}(\mathbb{G}_m) \otimes \mathrm{Oblv}_{gen}(\mathcal{C})}{\times} \mathsf{Vect}_{(n)} \otimes \mathrm{Oblv}_{gen}(\mathcal{C}).$$

In words: this is the full subcategory of H_{Tate}-equivariant objects where the central $\mathbb{G}_m$ acts by the nth power of its canonical character, this notion being defined because $\mathbb{G}_m$ acts trivially on $\mathcal{C}$.

Note that:

$$\mathcal{C}^{H,w,naive} \xrightarrow{\simeq} \mathcal{C}^{H_{Tate},w}_{(0)} \tag{9.8.1}$$

by semi-simplicity of $\mathsf{Rep}(\mathbb{G}_m)$.

Proposition 9.8.1. *Let H be a Tate group indscheme with δ_{Tate} identically 0.*
   Then for any $\mathcal{C} \in H\mathrm{-mod}_{weak}$, the canonical functor:

$$(\mathcal{C} \otimes \mathsf{Vect}_{\chi_{Tate}})^{H,w,naive} \to (\mathcal{C} \otimes \mathsf{Vect}_{\chi_{Tate}})^{H_{Tate},w,naive} \overset{Prop.-Const.9.6.4}{\simeq} \mathcal{C}^{H_{Tate},w,naive}$$

is fully-faithful with essential image $\mathcal{C}^{H_{Tate},w,naive}_{(1)}$.

Proof. Consider the isomorphism $\mathsf{Vect} \xrightarrow{\simeq} \mathsf{Vect}_{\chi_{Tate}} \in H_{Tate}\text{--}\mathsf{mod}_{weak}$ from Proposition-Construction 9.6.4 restricted to $\mathbb{G}_m$. Note that $\mathsf{Vect}_{\chi_{Tate}}|_{\mathbb{G}_m}$ is canonically isomorphic to Vect with its trivial action (as it was obtained by restriction from H). Therefore, this isomorphism is an equivalence:

$$\mathsf{Vect} \xrightarrow{\simeq} \mathsf{Vect}_{\chi_{Tate}}|_{\mathbb{G}_m} = \mathsf{Vect} \in \mathbb{G}_m\text{--}\mathsf{mod}_{weak}.$$

Therefore, this isomorphism amounts to specifying an invertible object of $\mathsf{End}_{\mathbb{G}_m\text{--}\mathsf{mod}_{weak}}(\mathsf{Vect}) = \mathsf{Rep}(\mathbb{G}_m)$. It follows from the construction that this object is $k(1)$.

We now obtain the result from (9.8.1).

$\square$

9.9. We now explain how to adapt the above to the setting of genuine actions.

Note that by Theorem 5.10.1 (for $\mathbb{G}_m$), $\mathrm{Res} : H\text{--}\mathsf{mod}_{weak} \to H_{Tate}\text{--}\mathsf{mod}_{weak}$ admits a right (and left) DGCat_{cont}-linear adjoint commuting with colimits. We abuse notation somewhat in denoting this functor by $(-)^{\mathbb{G}_m,w}$.

For any $n \in \mathbb{Z}$, there is an adjunction map:

$$\mathsf{Vect}^{\otimes n}_{\chi_{Tate}} \to (\mathsf{Vect}^{\otimes n}_{\chi_{Tate}})^{\mathbb{G}_m,w} \overset{Prop.-Const.9.6.4}{\simeq} \mathsf{Vect}^{\mathbb{G}_m,w} \in H\text{--}\mathsf{mod}_{weak}.$$

The induced map:

$$\underset{n\in\mathbb{Z}}{\oplus} \mathsf{Vect}^{\otimes n}_{\chi_{Tate}} \quad \mathsf{Vect}^{\mathbb{G}_m,w} \in H\text{--}\mathsf{mod}_{weak}$$

is an isomorphism.[83]

On tensoring, for any $\mathcal{C} \in H\text{--}\mathsf{mod}_{weak}$, we obtain an isomorphism:

$$\underset{n\in\mathbb{Z}}{\oplus}(\mathcal{C} \otimes \mathsf{Vect}^{\otimes n}_{\chi_{Tate}}) \xrightarrow{\simeq} \mathrm{Res}(\mathcal{C})^{\mathbb{G}_m,w} \in H\text{--}\mathsf{mod}_{weak}.$$

Passing to invariants, we obtain:

$$\underset{n\in\mathbb{Z}}{\oplus}(\mathcal{C} \otimes \mathsf{Vect}^{\otimes n}_{\chi_{Tate}})^{H,w} \xrightarrow{\simeq} \mathcal{C}^{H_{Tate},w}.$$

We record these observations as the following analogue of Proposition 9.8.1.

Proposition 9.9.1. *For $\mathcal{C} \in H\text{--}\mathsf{mod}_{weak}$, $\mathcal{C}^{H_{Tate},w}$ is canonically $\mathbb{Z}$-graded with $(\mathcal{C} \otimes \mathsf{Vect}_{\chi_{Tate}})^{H,w}$ as its degree 1 component.*

9.10. We remark briefly on another interpretation of the above results. We allow ourselves to be slightly imprecise here in speaking about $\mathbb{B}\mathbb{G}_m$-actions on categories on equal footing with genuine actions of Tate group indschemes, although this is not formally allowed in the theory developed in §7 (though one could suitably extend the theory without difficulty).

By fiat, genuine $\mathbb{B}\mathbb{G}_m$-actions on $\mathcal{C} \in \mathsf{DGCat}_{cont}$ are the same as naive ones, i.e., $\mathsf{IndCoh}^*(\mathbb{B}\mathbb{G}_m)$-actions. As $\mathsf{IndCoh}^*(\mathbb{B}\mathbb{G}_m) = \mathsf{QCoh}(\mathbb{Z})$ with convolution on the left corresponding to tensor products on the right, such a datum is equivalent to a $\mathbb{Z}$-grading $\mathcal{C} = \oplus_{n\in\mathbb{Z}}\mathcal{C}_{(n)}$. Here $\mathcal{C}_{(0)} = \mathcal{C}^{\mathbb{B}\mathbb{G}_m,w}$.

Note that the $\mathbb{B}\mathbb{G}_m$ on Vect constructed in §9.4 has $\mathsf{Vect} = \mathsf{Vect}_{(-1)}$, i.e., it is Vect graded in pure degree -1. For simplicity, we denote this object by $\mathsf{Vect}_{(-1)} \in \mathbb{B}\mathbb{G}_m\text{--}\mathsf{mod}_{weak}$.

We have a fiber sequence of groups:

$$H_{Tate} \to H \xrightarrow{\varepsilon_{Tate}} \mathbb{B}\mathbb{G}_m$$

---

[83]The $\oplus$ denotes the coproduct in $H\text{--}\mathsf{mod}_{weak}$. Note that (even infinite) coproducts in $H\text{--}\mathsf{mod}_{weak}$ coincide with products as the same is true in DGCat_{cont}.

as δ_{Tate} is assumed to be 0. By design, the pullback of $\mathsf{Vect}_{(-1)}$ along ε_{Tate} is $\mathsf{Vect}_{\chi_{Tate}}$.

Therefore, for $\mathcal{C}$ with a genuine (or naive) action of H, $\mathbb{B}\mathbb{G}_m$ acts on $\mathcal{C}^{H_{Tate},w}$, i.e., we obtain a grading $\mathcal{C}^{H_{Tate},w} = \oplus_{n\in\mathbb{Z}} \mathcal{C}^{H_{Tate},w}_{(n)}$. We then have:

$$(\mathcal{C} \otimes \mathsf{Vect}_{\chi_{Tate}})^{H,w} = \left(\left(\oplus_{n\in\mathbb{Z}} \mathcal{C}^{H_{Tate},w}_{(n)} \right) \otimes \mathsf{Vect}_{(-1)} \right)^{\mathbb{B}\mathbb{G}_m,w} = \mathcal{C}^{H_{Tate},w}_{(1)}$$

as desired.

9.11. Representations of Tate group indschemes.

The following result describes the major structures of $\mathsf{Rep}(H)$.

Proposition 9.11.1. *Let H be polarizable.*

(1) *$\mathsf{Rep}(H)$ is compactly generated.*

(2) *Suppose H is a classical indscheme. There is a unique t-structure on $\mathsf{Rep}(H)$ such that for any compact open subgroup $K \subseteq H$, the (conservative) forgetful functor $\mathsf{Rep}(H) \to \mathsf{Rep}(K)$ is t-exact.*

(3) *Suppose that H is of Harish-Chandra type (c.f. Example 7.17.3) and formally smooth. Then $\mathsf{Rep}(H)^+$ is the bounded below derived category of $\mathsf{Rep}(H)^\heartsuit$.*

Proof. Let $K \subseteq H$ be a polarization.

For any $\mathcal{C} \in H\text{--}\mathsf{mod}_{weak}$, the forgetful functor $\mathcal{C}^{H,w} \to \mathcal{C}^{K,w}$ is conservative and admits a continuous left adjoint $\mathrm{Av}_!^w : \mathcal{C}^{K,w} \to \mathcal{C}^{H,w}$. Indeed, the former property is true for any compact open subgroup while the latter is true by ind-properness of H/K. Therefore, if $\mathcal{C}^{K,w}$ is compactly generated, then $\mathcal{C}^{H,w}$ is compactly generated. Applying this for $\mathcal{C} = \mathsf{Vect}$ gives (1).

Next, in the setting of (2), observe that it suffices to show $\mathrm{Oblv}\,\mathrm{Av}_!^w : \mathsf{Rep}(K) \to \mathsf{Rep}(K)$ is right t-exact. Indeed, we are reduced to showing this by the monadicity of Oblv shown above. (We remark that t-exactness of the restriction functor to some compact open subgroup clearly implies the same for any compact open subgroup.)

Because H is classical, the same is true of H/K, i.e., we can write $H/K = \mathrm{colim}_i S_i$ a filtered colimit of classical proper k-schemes.

Suppose $V \in \mathsf{Rep}(K)^\heartsuit$ is finite dimensional. It suffices to show that for such V, $\mathrm{Oblv}\,\mathrm{Av}_!^w(V) \in \mathsf{Rep}(K)^{\leqslant 0}$, or equivalently, that the underlying vector space of this K-representation is in $\mathsf{Vect}^{\leqslant 0}$. Let $\mathcal{E}_V \in \mathsf{QCoh}(H/K)$ be the corresponding (naively H-equivariant) vector bundle. Then:

$$\mathrm{Oblv}\,\mathrm{Av}_!^w(V) = \Gamma^{\mathsf{IndCoh}}(H/K, \mathcal{E}_V \otimes \omega_{H/K}) = \mathrm{colim}_i \Gamma^{\mathsf{IndCoh}}(S_i, \mathcal{E}_V|_{S_i} \otimes \omega_{S_i}) = \Gamma(S_i, \mathcal{E}_V^\vee|_{S_i})^\vee.$$

We have $\Gamma(S_i, \mathcal{E}_V^\vee|_{S_i}) \in \mathsf{Vect}^{\geqslant 0}$ as S_i is classical, so we obtain the claim by dualizing.

Finally, in the setting of (3), we suppose K is chosen so H is formally complete along it; note that the above argument shows that $\mathrm{Oblv}\,\mathrm{Av}_!^w$ is t-exact by formal smoothness of H/K.

In the case $H = K$, the fact that $\mathsf{Rep}(K)^+$ is the bounded below derived category of its heart is standard. Any object $\mathrm{Av}_*^w(V)$ for $V \in \mathsf{Vect}^\heartsuit$ is injective in $\mathsf{Rep}(K)^\heartsuit$. Moreover, any object of $\mathsf{Rep}(K)^\heartsuit$ admits an injective resolution by such objects. Finally, for $W \in \mathsf{Rep}(K)^\heartsuit$ and V as above, $\underline{\mathrm{Hom}}_{\mathsf{Rep}(K)}(W, \mathrm{Av}_*^w(V)) = \underline{\mathrm{Hom}}_{\mathsf{Vect}}(W, V)$ is concentrated in cohomological degree 0. These observations imply the claim.

In general, the argument follows by Lemma 9.11.2 below (or see a variant of this argument in [Ras6] Lemma A.18.1).

$\square$

We used the following result above.

Lemma 9.11.2. *Suppose* $\mathcal{C}, \mathcal{D} \in \mathsf{DGCat}$ *are equipped with t-structures. compatible with filtered colimits. Let* $G : \mathcal{C} \to \mathcal{D} \in \mathsf{DGCat}_{cont}$ *be a conservative, t-exact functor with a t-exact left adjoint* F.

Suppose $\mathcal{D}^+$ *is the bounded below derived category of* $\mathcal{D}^\heartsuit$. *Then* $\mathcal{C}^+$ *is the bounded below derived category of* $\mathcal{C}^\heartsuit$.

Proof. Let $I \in \mathcal{C}^\heartsuit$ be an injective object. We need to show that for any $\mathcal{F} \in \mathcal{C}^\heartsuit$, $\underline{\mathrm{Hom}}_\mathcal{C}(\mathcal{F}, I) \in \mathsf{Vect}^\heartsuit$. Clearly this complex is in degrees $\geqslant 0$. We will show by induction on $i > 0$ that $H^i \underline{\mathrm{Hom}}_\mathcal{C}(\mathcal{F}, I) = \mathrm{Ext}^i_\mathcal{C}(\mathcal{F}, I)$ vanishes for all $\mathcal{F}$. For $i = 1$, we have $\mathrm{Ext}^1_\mathcal{C}(\mathcal{F}, I) = \mathrm{Ext}^1_{\mathcal{C}^\heartsuit}(\mathcal{F}, I) = 0$, giving the base case. Suppose the result is true for $i \geqslant 1$, and we will deduce it for $i + 1$.

First, note that the counit map $FG(\mathcal{F}) \to \mathcal{F} \in \mathcal{C}^\heartsuit$ is an epimorphism. Indeed, we can check this after applying the conservative, t-exact functor G, and then the map splits.

Let $\mathcal{F}_0$ be the kernel of this counit. We obtain an exact sequence:

$$\mathrm{Ext}^i_\mathcal{C}(\mathcal{F}_0, I) \to \mathrm{Ext}^{i+1}_\mathcal{C}(\mathcal{F}, I) \to \mathrm{Ext}^{i+1}_\mathcal{C}(FG(\mathcal{F}), I) = \mathrm{Ext}^{i+1}_\mathcal{D}(G(\mathcal{F}), G(I)).$$

The first term vanishes by induction. The last term vanishes because $G : \mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ admits a t-exact left adjoint so preserves injectives, and by assumption on $\mathcal{D}$. This gives the claim. $\qquad\square$

Combining Propositions 9.11.1, 9.8.1 and 9.9.1, we obtain:

Corollary 9.11.3. *For polarizable* H, *the categories* $\mathsf{Rep}_{\pm Tate}(H)$ *are compactly generated. If* H *is classical, there is a unique compactly generated t-structure on* $\mathsf{Rep}_{\pm Tate}(H)$ *for which the forgetful functor to* $\mathsf{Rep}(K)$ *is t-exact for any compact open subgroup* $K \subseteq H$ *(using Proposition 7.19.1). The category* $\mathsf{Rep}_{Tate}(H)^+$ *(resp.* $\mathsf{Rep}_{-Tate}(H)^+$*) maps isomorphically onto the subcategory of* $\mathsf{Rep}(H_{Tate})^+$ *consisting of objects on which the central* $\mathbb{G}_m$ *acts by (direct sums of shifts of) its standard representation (resp. the inverse to the standard representation). If* H *is additionally of Harish-Chandra type, then* $\mathsf{Rep}_{\pm Tate}(H)^+$ *is the bounded below derived category of* $\mathsf{Rep}_{\pm Tate}(H)^\heartsuit$.

9.12. **Passage to Lie algebras.** Let H be a Tate group indscheme of Harish-Chandra type. We assume H is polarizable in what follows (although most of the discussion generalizes to the non-polarizable case by replacing H with its formal completion along some compact open subgroup).

Define $\mathfrak{h}\text{--mod} := \mathsf{Vect}^{\exp(\mathfrak{h}),w}$. Similarly, define $\mathfrak{h}_{Tate}\text{--mod}$ (resp. $\mathfrak{h}_{-Tate}\text{--mod}$) as $(\mathsf{Vect}_{\chi_{Tate}})^{\exp(\mathfrak{h}),w}$ (resp. $(\mathsf{Vect}_{\chi_{-Tate}})^{\exp(\mathfrak{h}),w}$).[84]

Definition 9.12.1. For $K \subseteq H$ a fixed compact open subgroup, the *relative semi-infinite cohomology* functor:

$$C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -) : \mathfrak{h}_{-Tate}\text{--mod} \to \mathsf{Vect} \in H\text{--mod}$$

corresponds to $C^{\frac{\infty}{2}}(\mathfrak{h}, -) : \mathsf{Vect}_{\exp(\mathfrak{h}),w} \to \mathsf{Vect}$ under the equivalence:

---

[84]The notation is potentially confusing. There is a central extension $\mathfrak{h}_{Tate}$ around (the Lie algebra of H_{Tate}), and we are in effect considering modules over it on which the central element $1 \in k \subseteq \mathfrak{h}_{Tate}$ acts by the identity (or minus the identity), of course in a suitable derived categorical sense.

This is a somewhat standard abuse, and we hope that it does not cause confusion. To be clear: we will never consider all modules over the Tate Lie algebra $\mathfrak{h}_{Tate}$.

$$\mathfrak{h}_{-Tate}\text{–mod} := (\mathsf{Vect}_{\chi_{-Tate}})^{\exp(\mathfrak{h}),w} \overset{Prop.\,8.21.1}{\simeq}$$

$$D^!(H) \underset{D^*(H)}{\otimes} (\mathsf{Vect}_{\chi_{-Tate}} \otimes \chi_{Tate})_{\exp(\mathfrak{h}),w} \simeq$$

$$\mathsf{Vect}_{\exp(\mathfrak{h}),w}$$

using the choice of K both to identify $\mathsf{Vect}_{\chi_{Tate}}$ with χ_{Tate} via Proposition 7.19.1 and to identify $D^!(H)$ with $D^*(H)$ via [Ras3] Construction 6.12.6.

Note that H acts strongly on $\mathfrak{h}$–mod by the construction of §8. For $K \subseteq H$ compact open, we have $\mathfrak{h}\text{–mod}^K = \mathsf{Rep}(H_K^\wedge)$ by construction. In particular, we have $\mathfrak{h}\text{–mod} = \mathrm{colim}_K \mathfrak{h}\text{–mod}^K$. Moreover, for H formally smooth, $\mathfrak{h}\text{–mod}^K$ has a canonical t-structure by Proposition 9.11.1.

9.13.   We now suppose that H is formally smooth. In this case, its Lie algebra $\mathfrak{h}$ is naturally a Tate Lie algebra in the sense of Example 4.3.8, and we have two possibly conflicting definitions of $\mathfrak{h}$–mod. However, we claim that they do not in fact conflict.

Below, we understand $\mathfrak{h}$–mod in the sense defined immediately above, i.e., as $\mathsf{Vect}^{\exp(\mathfrak{h}),w}$.

Lemma 9.13.1.   *(1) For each pair $K_1 \subseteq K_2 \subseteq H$ of compact open subgroups, the (conservative) restriction functor:*

$$\mathfrak{h}\text{–mod}^{K_2} \to \mathfrak{h}\text{–mod}^{K_1}$$

is t-exact. In particular, the colimit $\mathfrak{h}$–mod over all such compact open subgroups admits a canonical t-structure.

(2) The forgetful functor $\mathfrak{h}\text{–mod} := \mathsf{Vect}^{\exp(\mathfrak{h}),w} \to \mathsf{Vect}$ is t-exact and conservative on eventually coconnective subcategories. The corresponding $\overset{\to}{\otimes}$-algebra (as defined by Proposition 3.7.1) is the completed universal enveloping algebra of the Tate Lie algebra $\mathfrak{h}$. Moreover, the compact generators of $\mathfrak{h}$–mod correspond to the renormalization datum specified in Example 4.3.8.

Proof. The t-exactness of the various restriction functors is clear from Proposition 9.11.1.

Moreover, for $K_1 \subseteq K_2 \subseteq H$ compact open subgroups and for $V \in \mathfrak{h}\text{–mod}^{K_1,\geqslant 0}$, we claim that the adjunction map $\mathrm{Oblv}\,\mathrm{Av}_*^{K_1 \to K_2}(V) \to V$ induces a monomorphism in $\mathfrak{h}\text{–mod}^{K_1,\heartsuit}$ upon applying H^0. Indeed, we can test this after applying the (conservative, t-exact) forgetful functor to $\mathfrak{k}_2\text{–mod}_1^K$, where it is evident.

It follows that for any $V \in \mathfrak{h}\text{–mod}^{\geqslant 0}$ and K a congruence subgroup, the adjunction map $\mathrm{Oblv}\,\mathrm{Av}_*^K(V) \to V$ gives a monomorphism on H^0. As $V = \mathrm{colim}_K \mathrm{Oblv}\,\mathrm{Av}_*^K(V)$, this implies that $\mathrm{Oblv}: \mathfrak{h}\text{–mod} \to \mathsf{Vect}$ is conservative on eventually coconnective subcategories.

Now define an object:

$$\mathcal{P} := \lim_K \mathrm{ind}_{\mathfrak{k}}^{\mathfrak{h}}(k) \in \mathsf{Pro}(\mathfrak{h}\text{–mod}^{\heartsuit}) \subseteq \mathsf{Pro}(\mathfrak{h}\text{–mod}^+)$$

where the notation is understood as follows. First, the limit is formed in the pro-category, and is indexed by compact open subgroups $K \subseteq H$. Then $k \in \mathfrak{k}$–mod denotes the trivial representation and $\mathrm{ind}_{\mathfrak{k}}^{\mathfrak{h}}: \mathfrak{k}\text{–mod} \to \mathfrak{h}\text{–mod}^K$ is the left adjoint to the forgetful functor.

Then $\mathcal{P}$ pro-corepresents the forgetful functor $\mathfrak{h}\text{–mod}^+ \to \mathsf{Vect}^+$. Moreover, under the forgetful functor, $\mathcal{P}$ maps an object of $\mathsf{Pro}(\mathsf{Vect}^{\heartsuit})$. By Proposition 3.7.1, $\mathfrak{h}\text{–mod}^+$ is the bounded below derived category of its heart, and this heart is the category of discrete modules for $\mathrm{Oblv}(\mathcal{P})$ with respect to its natural $\overset{\to}{\otimes}$-algebra structure.

One can identify $\mathrm{Oblv}(\mathcal{P})$ with its $\overset{\rightarrow}{\otimes}$-algebra structure as follows. Let $\mathfrak{h}^{disc} \in \mathsf{LieAlg}(\mathsf{Vect})$ denote the Lie algebra obtained by forgetting the topology on $\mathfrak{h}$.[85] We have a canonical map $\mathfrak{h}^{disc} \to \mathfrak{h}$ of Tate Lie algebras, giving rise to a forgetful functor $\mathfrak{h}$–$\mathsf{mod} \to \mathfrak{h}^{disc}$–$\mathsf{mod}$. By [GR4], the (non-topological) algebra attached to $\mathfrak{h}^{disc}$ is the usual enveloping algebra $U(\mathfrak{h}^{disc})$. Moreover, the natural map:

$$\mathrm{ind}_{\mathfrak{k}^{disc}}^{\mathfrak{h}^{disc}}(k) \to \mathrm{ind}_{\mathfrak{k}}^{\mathfrak{h}}(k) \in \mathsf{Vect}$$

is an isomorphism. This immediately implies the claim.

Finally, it is immediate from the constructions to identify the compact generators.

$\square$

Corollary 9.13.2. *Under the above hypotheses, $\mathfrak{h}$–mod^+ is the bounded below derived category of $\mathfrak{h}$–$\mathsf{mod}^\heartsuit$.*

Proof. Immediate from Lemma 9.13.1 and Proposition 3.7.1.

$\square$

9.14. **Classical semi-infinite cohomology.** Let H be a formally smooth polarizable Tate group indscheme.

We let $\mathfrak{h}_{Tate_{std}}$ denote the central extension $0 \to k \to \mathfrak{h}_{Tate_{std}} \to \mathfrak{h} \to 0$ of Tate Lie algebras constructed e.g. in [BD1] §7.13. We abuse notation in letting $\mathfrak{h}_{Tate_{std}}$–$\mathsf{mod}$ denote not the category of representations as is, but the analogue where we impose the requirement that the central $1 \in k$ act by the identity. We remind that $\mathfrak{h}_{Tate_{std}}$ is canonically split over any Lie subalgebra $\mathfrak{k}_0 \subseteq \mathfrak{h}$ that is a lattice (in the usual sense of Tate vector spaces).

By Lemma 19.8.1 from [FG1] and Corollary 9.13.2 above, for $K \subseteq H$ a compact open subgroup, we have DG a functor:

$$C_{std,0}^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -) : \mathfrak{h}_{-Tate_{std}}\text{–}\mathsf{mod}^+ \to \mathsf{Vect}$$

of *standard semi-infinite cohomology* (defined in terms of Clifford algebras and spin representations) whose restriction to $\mathfrak{h}_{-Tate_{std}}$–$\mathsf{mod}^{\geq -n}$ commutes with filtered colimits for any n. We also let:

$$C_{std}^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -) : \mathfrak{h}_{-Tate_{std}}\text{–}\mathsf{mod} \to \mathsf{Vect}$$

denote the functor obtained by restricting $C_{std,0}^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -)$ to $\mathfrak{h}_{-Tate_{std}}$–$\mathsf{mod}^c$ and then ind-extending.

9.15. We will now show that the canonical natural transformation:

$$\eta : C_{std}^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -)|_{\mathfrak{h}_{-Tate_{std}}\text{–}\mathsf{mod}^+} \to C_{std,0}^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -)$$

of functors $\mathfrak{h}_{-Tate_{std}}$–$\mathsf{mod}^+ \to \mathsf{Vect}$ is an isomorphism. (Combined with Theorem 9.16.1 below, this means that $C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -)|_{\mathfrak{h}_{-Tate}\text{–}\mathsf{mod}^+}$ may be calculated using the standard semi-infinite complex.)

First, if $\mathfrak{h} = \mathfrak{k}$, this follows immediately from the fact that compact objects in $\mathfrak{k}$–mod are closed under truncations (c.f. Example 4.4.4).

In general, recall from [BD1] §7.13.29 that for $M \in \mathfrak{h}_{-Tate_{std}}$–$\mathsf{mod}^\heartsuit$, $C_{std,0}^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; M)$ has a canonical increasing filtration indexed by $\mathbb{Z}^{\geq 0}$ with associated graded terms:

---

[85]In other words, we pass to the inverse limit of the pro-vector space underlying $\mathfrak{h}$ and then apply H^0 if for some pathological reason there are higher cohomology groups.

$$\operatorname{gr}_i C^{\frac{\infty}{2}}_{std,0}(\mathfrak{h}, \mathfrak{k}; M) = C^\bullet(\mathfrak{k}, \Lambda^i(\mathfrak{h}/\mathfrak{k}) \otimes M)[i] \tag{9.15.1}$$

for $C^\bullet(\mathfrak{k}, -)$ denoting the cohomological Chevalley complex. This is functorial in M, so the functor $C^{\frac{\infty}{2}}_{std,0}(\mathfrak{h}, \mathfrak{k}; -) : \mathfrak{h}_{-Tate_{std}}\text{–mod}^+ \to \mathsf{Vect}$ upgrades to a functor valued in $\mathsf{Fil}_{\geqslant 0}\mathsf{Vect}$ the (DG) category of $\mathbb{Z}^{\geqslant 0}$-filtered vector spaces.

By construction, the functor $C^{\frac{\infty}{2}}_{std}(\mathfrak{h}, \mathfrak{k}; -)$ also upgrades to $\mathsf{Fil}_{\geqslant 0}\mathsf{Vect}$ compatibly with the above and the map η. To check η is an isomorphism, it is enough to do so on the associated graded level. As the associated graded functors above factor through the forgetful functor $\mathfrak{h}_{-Tate_{std}}\text{–mod} \to \mathfrak{k}\text{–mod}^+$ (c.f. (9.15.1)), we conclude as above.

9.16.   For the remainder of this section, H is a formally smooth polarizable Tate group indscheme with $\delta_{Tate} = 0$.

We have the following comparison theorem.

Theorem 9.16.1. *For H and K as above, there is a canonical identification of $\mathfrak{h}_{Tate}$ with $\mathfrak{h}_{Tate_{std}}$ as central extensions of $\mathfrak{h}$. Moreover, under this identification, there is a canonical isomorphism $C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -) \simeq C^{\frac{\infty}{2}}_{std}(\mathfrak{h}, \mathfrak{k}; -)$ of functors $\mathfrak{h}_{-Tate}\text{–mod} \simeq \mathfrak{h}_{-Tate_{std}}\text{–mod} \to \mathsf{Vect}$. This pair of identifications is uniquely characterized by compatibility with the given splittings of these central extensions over $\mathfrak{k}$ and with the isomorphisms Lemmas 9.17.1 and 9.18.1 as formulated and proved below.*

We will prove this result in the remainder of this section.

9.17.   First, we review an important property of standard semi-infinite infinite cohomology.

Lemma 9.17.1. *The composition:*

$$\mathfrak{k}\text{–mod} \xrightarrow{\operatorname{ind}^{\mathfrak{h}-Tate_{std}}_{\mathfrak{k}}} \mathfrak{h}_{-Tate_{std}}\text{–mod} \xrightarrow{C^{\frac{\infty}{2}}_{std}(\mathfrak{h}, \mathfrak{k}; -)} \mathsf{Vect}$$

is isomorphic to the functor $C^\bullet(\mathfrak{k}, -) := \underline{\operatorname{Hom}}_{\mathfrak{k}\text{–mod}}(k, -) : \mathfrak{k}\text{–mod} \to \mathsf{Vect}$ of Lie algebra cohomology.

Proof. By [BD1] Remark 7.13.30, there is a canonical isomorphism of between the composite functor:

$$\mathfrak{k}\text{–mod}^+ \xrightarrow{\operatorname{ind}^{\mathfrak{h}-Tate_{std}}_{\mathfrak{k}}} \mathfrak{h}_{-Tate_{std}}\text{–mod}^+ \xrightarrow{C^{\frac{\infty}{2}}_{std,0}(\mathfrak{h}, \mathfrak{k}; -)} \mathsf{Vect}.$$

with $C^\bullet(\mathfrak{k}, -)|_{\mathfrak{k}\text{–mod}^+}$. Now the result follows by construction of $C^{\frac{\infty}{2}}_{std}$. $\square$

9.18.   We now establish similar results for $C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -)$.

First, observe that we have a duality functor:

$$\mathbb{D}^{\frac{\infty}{2}}_{\mathfrak{h},K} : \mathfrak{h}\text{–mod}^\vee \simeq \mathfrak{h}_{-Tate}\text{–mod} \in \mathsf{DGCat}_{cont}$$

(depending on the choice of compact open subgroup K). Indeed, our choice of K identifies $\mathfrak{h}_{-Tate}\text{–mod} \simeq \mathsf{Vect}_{\exp(\mathfrak{h}),w}$ (c.f. §9.12). Note that this category is in fact dualizable as it is compactly generated, and its dual is:

$$\mathsf{Hom}_{\mathsf{DGCat}_{cont}}(\mathsf{Vect}_{\exp(\mathfrak{h}),w}, \mathsf{Vect}) = \mathsf{Hom}_{\mathsf{DGCat}_{cont}}(\mathsf{Vect}, \mathsf{Vect})^{\exp(\mathfrak{h}),w} = \mathfrak{h}\text{–mod}.$$

Under this duality, the functor $C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -)$ clearly corresponds to the trivial representation $k \in \mathfrak{h}\text{–mod}^\heartsuit$.

We now have:

Lemma 9.18.1. *The composition:*

$$\mathfrak{k}\text{–mod} \xrightarrow{\text{ind}_{\mathfrak{k}}^{\mathfrak{h}-Tate}} \mathfrak{h}_{-Tate}\text{–mod} \xrightarrow{C^{\frac{\infty}{2}}(\mathfrak{h},\mathfrak{k};-)} \text{Vect}$$

is canonically isomorphic to the functor $C^{\bullet}(\mathfrak{k}, -) := \underline{\text{Hom}}_{\mathfrak{k}\text{–mod}}(k, -) : \mathfrak{k}\text{–mod} \to \text{Vect}$ *of Lie algebra cohomology.*

Proof. The induction functor $\text{ind}_{\mathfrak{k}}^{\mathfrak{h}-Tate}$ is dual to $\text{Oblv} : \mathfrak{h}\text{–mod} \to \mathfrak{k}\text{–mod}$ by construction of $\mathbb{D}^{\frac{\infty}{2}}_{\mathfrak{h},K}$. We now obtain the result by duality.

$\square$

9.19.   We now prove Theorem 9.16.1.

Let $\mathfrak{h}_{-Tate+Tate_{std}}$ denote the Baer sum central extension of $\mathfrak{h}_{-Tate}$ and $\mathfrak{h}_{Tate_{std}}$. We maintain our abuse of notation regarding modules over central extensions: the category $\mathfrak{h}_{-Tate+Tate_{std}}\text{–mod}$ is set up so the element $1 \in k \subseteq \mathfrak{h}_{-Tate+Tate_{std}}$ acts by the identity on any object of it.[86] Note that this central extension is canonically split over $\mathfrak{k}$; in particular, we have a forgetful functor $\text{Oblv} : \mathfrak{h}_{-Tate+Tate_{std}}\text{–mod} \to \mathfrak{k}\text{–mod}$.

The functor $C^{\frac{\infty}{2}}_{std}(\mathfrak{h}, \mathfrak{k}; -) : \mathfrak{h}_{-Tate_{std}}\text{–mod} \to \text{Vect}$ defines by the duality $\mathbb{D}^{\frac{\infty}{2}}_{\mathfrak{h},K}$ an object $K \in \mathfrak{h}_{-Tate+Tate_{std}}\text{–mod}.$[87]

Combining Lemmas 9.17.1 and 9.18.1, we find that $\text{Oblv}(K) = k \in \mathfrak{k}\text{–mod}$, where $k \in \mathfrak{k}\text{–mod}$ indicates the trivial module. In particular, as Oblv is t-exact and conservative, K lies in $\mathfrak{h}_{-Tate+Tate_{std}}\text{–mod}^{\heartsuit}$ and corresponds to a 1-dimensional representation.

Now observe that giving a 1-dimensional representation of a central extension $0 \to k \to \mathfrak{h}^{\flat} \to \mathfrak{h} \to 0$ (on which the central element acts by the identity) is equivalent to splitting the central extension: the induced map $\mathfrak{h}^{\flat} \to \mathfrak{h} \times k$ is an isomorphism of central extensions of $\mathfrak{h}$. Under this splitting, the given 1-dimensional representation of $\mathfrak{h}^{\flat}$ maps to the trivial representation of $\mathfrak{h}$.

Therefore, we obtain a trivialization of the central extension $\mathfrak{h}_{-Tate+Tate_{std}}$ of $\mathfrak{h}$ such that K maps to the trivial object $k \in \mathfrak{h}\text{–mod}$. This is equivalent to giving an isomorphism $\mathfrak{h}_{-Tate} \simeq \mathfrak{h}_{-Tate_{std}}$ of central extensions such that the functor

$$\mathfrak{h}_{-Tate}\text{–mod} \simeq \mathfrak{h}_{-Tate_{std}}\text{–mod} \xrightarrow{C^{\frac{\infty}{2}}_{std}(\mathfrak{h},\mathfrak{k};-)} \text{Vect}$$

matches under the duality $\mathbb{D}^{\frac{\infty}{2}}_{\mathfrak{h},K}$ to the trivial module $k \in \mathfrak{h}\text{–mod}$. This gives the desired isomorphism $C^{\frac{\infty}{2}}(\mathfrak{h}, \mathfrak{k}; -) \simeq C^{\frac{\infty}{2}}_{std}(\mathfrak{h}, \mathfrak{k}; -)$.

Uniqueness follows as the map:

$$\text{Aut}_{\mathfrak{h}\text{–mod}}(k) \to \text{Aut}_{\mathfrak{k}\text{–mod}}(k)$$

is an isomorphism (both sides are $k^{\times}$, considered as group objects in $\text{Set} \subseteq \text{Gpd}$).

---

[86]This does not of course characterize the category. One can work with group indschemes and central extensions by $\mathbb{G}_m$ as above to give one quick definition. Alternatively, one can note that the centrality means $\text{QCoh}(\mathbb{A}^1) = k\text{–mod}$ (regarding k as an abelian Lie algebra in this notation) acts canonically on the category of *all* $\mathfrak{h}_{-Tate+Tate_{std}}$-modules, and we are taking the fiber of that category at $1 \in \mathbb{A}^1(k)$.

[87]As always, this notation abusively indicates that the central element $1 \in k \subseteq \mathfrak{h}_{-Tate+Tate_{std}}$ acts by the identity on our modules, understood in the appropriately derived sense.

10. Harish-Chandra data

10.1.   Suppose H is an algebraic group and $A \in \mathsf{Alg}$ is equipped with an action of H, giving a weak action of H on A–mod. It is not difficult to see in this setup that upgrading[88] this action to a strong one is equivalent to specifying a *Harish-Chandra* datum in a suitable derived sense.

We remind that this means we are given an H-equivariant map of Lie algebras $i : \mathfrak{h} \to A$ satisfying a number of "compatibilities" (which are actually extra data in a derived setting), most notably, that the corresponding adjoint action of $\mathfrak{h}$ on A coincides with the infinitesimal action of H on A.

10.2.   The goal for this section is to develop such ideas in the setting where H is a Tate group indscheme and A is an $\overrightarrow{\otimes}$-algebra.

There are a number of subtleties compared to the finite dimensional setting discussed above related to the ideas developed so far in this text.

First, A needs to be equipped with a renormalization datum compatible with the H-action in the sense of §5.4, and with the Harish-Chandra data in a suitable sense.

Second, we need to upgrade the naive action of H on A–mod_{ren} to a genuine one. We do this using the theory of canonical renormalization from §8.17.

With that said, the theory we develop has no[89] homotopical complexity for A and H classical.[90]

The main example to have in mind is $A = U(\mathfrak{h})$, the completed enveloping algebra of $\mathfrak{h}$ (i.e., the $\overrightarrow{\otimes}$-algebra assigned to the t-exact functor $\mathfrak{h}$–$\mathsf{mod} := \mathsf{Vect}^{\exp(\mathfrak{h}),w} \to \mathsf{Vect}$ via Proposition 3.7.1).

For the above to make sense, we need a key technical result, Theorem 10.8.1, that (in particular) says that the genuine weak action of H on $\mathfrak{h}$–mod comes from canonical renormalization. We need to impose two hypotheses on the group indschemes H to obtain this result: that H is polarizable, and that it *has a prounipotent tail*, i.e., there exists a prounipotent compact open subgroup in H. Therefore, these hypotheses trail us throughout this section. We remark that they are satisfied in the main example of interest: when H is the loop group of a reductive group (or a central extension of such).

10.3.   As this section is lengthy, we begin with a brief guide to its structure.

In §10.4-10.7, we introduce the notion of genuine H-action on an $\overrightarrow{\otimes}$-algebra A; roughly, this means there is a genuine H-action on A–mod_{ren} defined by canonical renormalization.

In §10.8, we formulate Theorem 10.8.1, which was mentioned above. The proof occupies §10.8-10.17.

In §10.20, we formulate our definition of Harish-Chandra data, which relies on Theorem 10.8.1.

Finally, in §10.22-10.23, we discuss Harish-Chandra data explicitly in the case where A is classical.

10.4. **Genuine actions and $\overrightarrow{\otimes}$-algebras.** In what follows, let H be an ind-affine Tate group indscheme. (These hypotheses will be strengthened in §10.18.)

Recall the notation $\mathsf{Alg}_{ren}^{\overrightarrow{\otimes},H\curvearrowright}$ from §5.7: this is the category of renormalized $\overrightarrow{\otimes}$-algebras with naive H-actions that are compatible with the renormalization.

---

[88]Although the forgetful functor A–$\mathsf{mod} \to \mathsf{Vect}$ is weakly H-equivariant, the "upgrade" in question does not interact with the forgetful functor. For example, H acts strongly on $\mathfrak{h}$–mod, but the forgetful functor $\mathfrak{h}$–$\mathsf{mod} \to \mathsf{Vect}$ is only weakly equivariant.

[89]Although the data is 1-categorical in nature, checking that an apparent Harish-Chandra datum actually defines one in our sense involves non-trivial homological algebra (as we will see).

[90]In fact, the theory essentially requires H to be classical from the start. More precisely, we require H to be formally smooth, which forces H to be classical under mild countability assumptions; see [GR3].

Definition 10.4.1. The 1-full subcategory:

$$'\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright} \subseteq \mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$$

of $\vec{\otimes}$-*algebras with nearly genuine H-actions* has objects $A \in \mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$ such that the naive action of H on A–mod_{ren} (as in §5.7) canonically renormalizes (in the sense of §8.17) with respect to the given t-structure on A–mod_{ren}.

Morphisms $A_1 \to A_2$ in $'\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$ are morphisms in $\mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$ with the property that for every $K \subseteq H$ a compact open subgroup, the functor:

$$A_2\text{–}\mathsf{mod}_{ren}^{K,w} \to A_1\text{–}\mathsf{mod}_{ren}^{K,w}$$

obtained by ind-extension from:

$$A_2\text{–}\mathsf{mod}_{ren}^{K,w,c} \to A_2\text{–}\mathsf{mod}_{ren}^{K,w,naive,+} \to A_1\text{–}\mathsf{mod}_{ren}^{K,w,naive,+} \simeq A_1\text{–}\mathsf{mod}_{ren}^{K,w,+} \subseteq A_1\text{–}\mathsf{mod}_{ren}^{K,w}$$

is t-exact (equivalently, left t-exact).

Finally, we define:

$$\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright} \subseteq '\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$$

of $\vec{\otimes}$-*algebras with genuine H-actions* as the full subcategory consisting of objects $A \in '\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$ such that the unit morphism $k \to A \in \mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$ is a morphism in the 1-full subcategory $'\mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$. (In other words, the forgetful functor A–$\mathsf{mod}_{ren} \to \mathsf{Vect}$ induces a t-exact functor A–$\mathsf{mod}_{ren}^{K,w} \to \mathsf{Rep}(K)$ for any compact open subgroup $K \subseteq H$.)

Remark 10.4.2. By construction, each of the restriction functors:

$$\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright} \to \mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright} \to \mathsf{Alg}^{\vec{\otimes},H\curvearrowright}$$

is 1-fully-faithful; the former is in addition conservative. (This is an abstract way of saying that a genuine action of H on A is equivalent to specifying a naive action and a renormalization datum for A satisfying some properties, and that genuinely equivariant morphisms are naively H-equivariant morphisms satisfying some properties.)

Definition 10.4.3. For $A_1, A_2 \in \mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$, we say a morphism $f : A_1 \to A_2$ in $\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$ is a *genuinely H-equivariant morphism*. We refer to a morphisms in $\mathsf{Alg}^{\vec{\otimes},H\curvearrowright}$ as *naively H-equivariant*.

10.5. There is an evident functor:

$$(\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright})^{op} \to H\text{–}\mathsf{mod}_{weak}$$
$$A \mapsto A\text{–}\mathsf{mod}_{ren}$$

given by canonical renormalization. Following our standard abuses for genuine H-actions, we denote this functor $A \mapsto A$–mod_{ren}. Moreover, as $k \in \mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$ is an initial object (by fiat in our definition of genuine H-action), this functor upgrades to a functor to the overcategory $(H\text{–}\mathsf{mod}_{weak})_{/\mathsf{Vect}}$: for $A \in \mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$, the structural map A–$\mathsf{mod}_{ren} \to \mathsf{Vect}$ is the forgetful functor.

Let $\mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright} \subseteq \mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$ be the full subcategory consisting of those objects whose underlying $\vec{\otimes}$-algebra is convergent.

Theorem 10.5.1. *For H polarizable, the functor:*

$$(\mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright})^{op} \to (H\text{–}\mathsf{mod}_{weak})_{/\mathsf{Vect}}$$

is 1-fully-faithful and conservative.

We defer the proof to §10.7.

Remark 10.5.2. Although the definition of the category $\mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright}$ is weighty, this result gives a way to convert algebraic data in $\mathsf{Alg}^{\vec{\otimes}}$ that may be quite concrete to abstract categorical data involving genuine H-actions.

10.6.   To prove Theorem 10.5.1, we will need the following result.

Proposition 10.6.1. *Let H be a polarizable Tate group indscheme. Let $\mathcal{C}, \mathcal{D} \in H\text{–}\mathsf{mod}_{weak}$ be equipped with t-structures compatible with the weak H-actions. Suppose the genuine H-actions on each of $\mathcal{C}$ and $\mathcal{D}$ are obtained by canonical renormalization using these t-structures and the underlying naive H-actions (c.f. §8.17).*

Let $F, G : \mathcal{C} \to \mathcal{D} \in H\text{–}\mathsf{mod}_{weak}$ be two genuinely H-equivariant functors, and suppose that G is left t-exact (at the level of its underlying functor $\mathcal{C} \to \mathcal{D} \in \mathsf{DGCat}_{cont}$).

Then the natural map:

$$\underline{\mathrm{Hom}}_{\mathrm{Hom}_{H\text{–}\mathsf{mod}_{weak}}}(F, G) \to \underline{\mathrm{Hom}}_{\mathrm{Hom}_{H\text{–}\mathsf{mod}_{weak,naive}}}(F, G) \in \mathsf{Vect}$$

(induced by Oblv_{gen}) is an equivalence. In other words, giving a genuinely H-equivariant natural transformation between F and G is equivalent to giving a naively H-equivariant natural transformation between them.

We will use the following lemma.

Lemma 10.6.2. *Let H be a Tate group indscheme and let $K \subseteq H$ be a polarization of H. Then the forgetful functors:*

$$\mathcal{C}^{H,w} \to \mathcal{C}^{K,w}$$
$$\mathcal{C}^{H,w,naive} \to \mathcal{C}^{K,w,naive}$$

admit left adjoints, denoted $\mathrm{Av}_!^w$ and $\mathrm{Av}_!^{w,naive}$ respectively. Moreover, the diagram:

$$
\begin{array}{ccc}
\mathcal{C}^{K,w} & \longrightarrow & \mathcal{C}^{K,w,naive} \\
\downarrow{\scriptstyle \mathrm{Av}_!^w} & & \downarrow{\scriptstyle \mathrm{Av}_!^{w,naive}} \\
\mathcal{C}^{H,w} & \longrightarrow & \mathcal{C}^{H,w,naive}
\end{array}
$$

commutes (a priori, it commutes up to a natural transformation).

Proof. The existence of $\mathrm{Av}_!^w$, as we have appealed to at various points earlier in this text, follows from (7.15.1) and the Beck-Chevalley formalism.

Let $\Phi : H\text{–}\mathsf{mod}_{weak,naive} \to H\text{–}\mathsf{mod}_{weak}$ denote the (non-continuous) right adjoint to Oblv_{gen}. Clearly $\Phi(-)^{H,w} = (-)^{H,w,naive}$. Moreover, passing to right adjoints in Lemma 8.7.1 (2) it follows that $\Phi(-)^{K,w} = (-)^{K,w,naive}$. Now the commutativity of the diagram follows by rewriting it as:

$$\begin{array}{ccc}
\mathcal{C}^{K,w} & \longrightarrow & \Phi(\mathrm{Oblv}_{gen}(\mathcal{C}))^{K,w} \\
\downarrow {\scriptstyle \mathrm{Av}_!^w} & & \downarrow {\scriptstyle \mathrm{Av}_!^w} \\
\mathcal{C}^{H,w} & \longrightarrow & \Phi(\mathrm{Oblv}_{gen}(\mathcal{C}))^{H,w}.
\end{array}$$

(Alternatively, the base-change follows directly by applying the Beck-Chevalley formalism in the naive setting and comparing with (7.15.1).)

$\square$

Proof of Proposition 10.6.1. In what follows, let $K \subseteq H$ be a polarization.

Step 1. Let $\mathsf{Hom}(\mathcal{C}, \mathcal{D}) \in H\text{–}\mathsf{mod}_{weak}$ denote the inner Hom object between $\mathcal{C}$ and $\mathcal{D}$ in the symmetric monoidal category $H\text{–}\mathsf{mod}_{weak}$.

Note that $\mathrm{Oblv}_{gen} \mathsf{Hom}(\mathcal{C}, \mathcal{D})$ is the category $\mathsf{Hom}(\mathcal{C}, \mathcal{D}) := \mathsf{Hom}_{\mathsf{DGCat}_{cont}}(\mathcal{C}, \mathcal{D})$ of continuous DG functors between $\mathcal{C}$ and $\mathcal{D}$. Indeed, this follows from the fact that Oblv_{gen} admits a DGCat_{cont}-continuous left adjoint $- \otimes \mathsf{IndCoh}^*(H)$ (for $\mathsf{IndCoh}^*(H) \in H\text{–}\mathsf{mod}_{weak}$ as in Example 8.21.2).

Similarly, formation of inner Homs is intertwined by the forgetful functor $H\text{–}\mathsf{mod}_{weak} \to K\text{–}\mathsf{mod}_{weak}$: this follows from the existence of the left adjoint $\mathrm{ind}_K^{H,w}$ from Lemma 8.7.1 and the (evident) version of the projection formula for this left adjoint.

Clearly $\mathsf{Hom}(\mathcal{C}, \mathcal{D})^{H,w} = \mathsf{Hom}_{H\text{–}\mathsf{mod}_{weak}}(\mathcal{C}, \mathcal{D})$. By the above, we just as well have $\mathsf{Hom}(\mathcal{C}, \mathcal{D})^{K,w} = \mathsf{Hom}_{K\text{–}\mathsf{mod}_{weak}}(\mathcal{C}, \mathcal{D})$. Finally, because $\mathrm{Oblv}^{gen} \mathsf{Hom}(\mathcal{C}, \mathcal{D})$ is the category of functors between $\mathcal{C}$ and $\mathcal{D}$, we have:

$$\mathsf{Hom}(\mathcal{C}, \mathcal{D})^{H,w} = \mathsf{Hom}_{H\text{–}\mathsf{mod}_{weak,naive}}(\mathcal{C}, \mathcal{D})$$

$$\mathsf{Hom}(\mathcal{C}, \mathcal{D})^{K,w} = \mathsf{Hom}_{K\text{–}\mathsf{mod}_{weak,naive}}(\mathcal{C}, \mathcal{D}).$$

Step 2. Consider $G \in \mathsf{Hom}(\mathcal{C}, \mathcal{D})^{K,w}$ as above (though it lifts to H-invariants).

Let $\mathrm{Oblv} : \mathsf{Hom}(\mathcal{C}, \mathcal{D})^{K,w} \to \mathsf{Hom}(\mathcal{C}, \mathcal{D}) \in \mathsf{DGCat}_{cont}$ be the forgetful functor, and let $\mathrm{Av}_*^{K,w}$ denote its right adjoint. We claim that the natural map $G \to \mathrm{Tot}((\mathrm{Av}_*^{K,w} \mathrm{Oblv})^{\bullet+1} G) \in \mathsf{Hom}(\mathcal{C}, \mathcal{D})^{K,w}$ is an equivalence. Here we emphasize that the totalization is calculated in the DG category $\mathsf{Hom}(\mathcal{C}, \mathcal{D})^{K,w}$.

Indeed, we have:

$$\mathsf{Hom}(\mathcal{C}, \mathcal{D})^{K,w} = \mathsf{Hom}_{K\text{–}\mathsf{mod}_{weak}}(\mathcal{C}, \mathcal{D}) = \mathsf{Hom}_{\mathsf{Rep}(K)\text{–}\mathsf{mod}}(\mathcal{C}^{K,w}, \mathcal{D}^{K,w}) = \tag{10.6.1}$$
$$\mathsf{Hom}_{\mathsf{Rep}(K)^c\text{–}\mathsf{mod}(\mathsf{DGCat})}(\mathcal{C}^{K,w,c}, \mathcal{D}^{K,w})$$

for $\mathcal{C}^{K,w,c} \subseteq \mathcal{C}^{K,w}$ the subcategory of compact objects: we remind that as part of the definition of $\mathcal{C}$ being obtained from canonical renormalization, $\mathcal{C}^{K,w}$ is compactly generated with compact objects being eventually coconnective.

The advantage of the last expression in (10.6.1) is that limits are manifestly computed termwise. So for $\mathcal{F} \in \mathcal{C}^{K,w,c}$, we have:

$$\mathrm{Tot}((\mathrm{Av}_*^{K,w} \mathrm{Oblv})^{\bullet+1} G)(\mathcal{F}) = \mathrm{Tot}((\mathrm{Av}_*^{K,w} \mathrm{Oblv})^{\bullet+1} G(\mathcal{F})) \in \mathcal{D}^{K,w}.$$

By Proposition 5.18.3 (4), the natural map from $G(\mathcal{F})$ to this limit is an equivalence.

Step 3. As an immediate consequence of Step 2, note that for any $\widetilde{F} : \mathcal{C} \to \mathcal{D}$ genuinely K-equivariant, the map:

$$\underline{\mathrm{Hom}}_{\mathrm{Hom}_{K-\mathsf{mod}_{weak}}}(\widetilde{F}, G) \to \underline{\mathrm{Hom}}_{\mathrm{Hom}_{K-\mathsf{mod}_{weak,naive}}}(\widetilde{F}, G)$$

is an isomorphism.

Step 4. We now complete the argument. We again view $\mathrm{Hom}(\mathcal{C}, \mathcal{D}) \in H-\mathsf{mod}_{weak}$ and F, G as objects in $\mathrm{Hom}(\mathcal{C}, \mathcal{D})^{H,w}$.

Note that the forgetful functor Oblv : $\mathrm{Hom}(\mathcal{C}, \mathcal{D})^{H,w} \to \mathrm{Hom}(\mathcal{C}, \mathcal{D})^{K,w}$ is conservative (by (7.15.1)) and admits a left adjoint $\mathrm{Av}_!^w$ (c.f. Lemma 10.6.2). Therefore, this forgetful functor is monadic. The same applies in the naively equivariant setting.

We obtain that F is a geometric realization:

$$|(\mathrm{Av}_! \, \mathrm{Oblv})^{\bullet+1}(F)| \xrightarrow{\simeq} F \in \mathrm{Hom}(\mathcal{C}, \mathcal{D})^{H,w} = \mathrm{Hom}_{H-\mathsf{mod}_{weak}}(\mathcal{C}, \mathcal{D}).$$

Therefore, we have:

$$\underline{\mathrm{Hom}}_{\mathrm{Hom}_{H-\mathsf{mod}_{weak}}(\mathcal{C},\mathcal{D})}(F, G) \xrightarrow{\simeq} \mathrm{Tot} \, \underline{\mathrm{Hom}}_{\mathrm{Hom}_{K-\mathsf{mod}_{weak}}(\mathcal{C},\mathcal{D})}(\mathrm{Oblv}(\mathrm{Av}_! \, \mathrm{Oblv})^{\bullet}(F), G) \xrightarrow{\simeq}$$
$$\mathrm{Tot} \, \underline{\mathrm{Hom}}_{\mathrm{Hom}_{K-\mathsf{mod}_{weak,naive}}(\mathcal{C},\mathcal{D})}(\mathrm{Oblv}(\mathrm{Av}_! \, \mathrm{Oblv})^{\bullet}(F), G).$$

Here we are using Step 3 in the second isomorphism, and we are implicitly using Lemma 10.6.2 to intertwine $\mathrm{Av}_!$ functors in the genuine and naive settings. By the same logic, the last term above computes $\underline{\mathrm{Hom}}_{H-\mathsf{mod}_{weak,naive}}(F, G)$, giving the claim.

$\square$

10.7.   As promised, we now prove the above theorem.

Proof of Theorem 10.5.1. First, let us verify that the functor is conservative. The composition of this functor with the forgetful functors:

$$(H-\mathsf{mod}_{weak})_{/\mathsf{Vect}} \to H-\mathsf{mod}_{weak} \xrightarrow{\mathrm{Oblv}_{gen}} \mathsf{DGCat}_{cont}$$

send $A \in \mathsf{Alg}_{gen}^{\vec{\otimes}, H \curvearrowright}$ to $A-\mathsf{mod}_{ren}$. This functor is conservative by Remark 4.2.4, giving the claim.

Now for $A_1, A_2 \in \mathsf{Alg}_{gen}^{\vec{\otimes}, H \curvearrowright}$, we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Alg}_{gen}^{\vec{\otimes}, H \curvearrowright}}(A_1, A_2) & \longrightarrow & \mathrm{Hom}'_{(H-\mathsf{mod}_{weak})_{/\mathsf{Vect}}}(A_2-\mathsf{mod}_{ren}, A_1-\mathsf{mod}_{ren}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathsf{Alg}_{ren}^{\vec{\otimes}, H \curvearrowright}}(A_1, A_2) & \longrightarrow & \mathrm{Hom}'_{(H-\mathsf{mod}_{weak,naive})_{/\mathsf{Vect}}}(A_2-\mathsf{mod}_{ren}, A_1-\mathsf{mod}_{ren}).
\end{array}
\tag{10.7.1}
$$

Here the decoration $\prime$ on the bottom left term indicates the subcategory of those functors that are t-exact after applying $(-)^{K,w}$ for any compact open subgroup K, while the similar notation on the bottom right term indicates the subcategory of t-exact functors. We wish to show that the top arrow in (10.7.1) is fully-faithful. We will do so by showing that the other three arrows are fully-faithful.

The left arrow of (10.7.1) is fully-faithful by definition of genuine H-actions.

The right arrow of (10.7.1) is fully-faithful by Proposition 10.6.1.

Finally, the bottom arrow of (10.7.1) is an equivalence by Remark 4.2.4; indeed, by *loc. cit.* (or more precisely, by Proposition 3.7.1 and Theorem 4.6.1), the functor:

$$(\mathsf{Alg}^{\vec{\otimes}}_{conv,ren})^{op} \to (\mathsf{DGCat}_{cont})_{/\mathsf{Vect}}$$
$$A \mapsto A\text{--}\mathsf{mod}_{ren}$$

is symmetric monoidal and fully-faithful.[91]

<div align="right">□</div>

10.8. **A construction of genuine H-actions.** We now formulate a key result that allows us to construct many genuine H-actions on $\vec{\otimes}$- algebras.

Suppose $A \in \mathsf{Alg}^{\vec{\otimes},H\curvearrowright}_{conv,gen}$ be given. We have the corresponding object $A\text{--}\mathsf{mod}_{ren} \in H\text{--}\mathsf{mod}_{weak}$. We define:

$$A\#U(\mathfrak{h})\text{--}\mathsf{mod}_{ren} := \mathrm{Oblv}^{str \to w}(A\text{--}\mathsf{mod}^{\exp(\mathfrak{h}),w}_{ren}) \in H\text{--}\mathsf{mod}_{weak}.$$

(The notation will be justified in what follows.) Note that this category has a canonical genuinely H-equivariant functor to Vect:

$$A\#U(\mathfrak{h})\text{--}\mathsf{mod}_{ren} = \mathrm{Oblv}^{str \to w}(A\text{--}\mathsf{mod}^{\exp(\mathfrak{h}),w}_{ren}) \to A\text{--}\mathsf{mod}_{ren} \to \mathsf{Vect}$$

where the first functor is the counit for the adjunction and the second arrow is the standard forgetful functor for $A\text{--}\mathsf{mod}_{ren}$.

Theorem 10.8.1. *Suppose that H is formally smooth Tate group indscheme that is polarizable, and has a prounipotent tail. Then $A\#U(\mathfrak{h})\text{--}\mathsf{mod}_{ren} \in (H\text{--}\mathsf{mod}_{weak})_{/\mathsf{Vect}}$ lies in the essential image of the functor from Theorem 10.5.1.*

Example 10.8.2. Taking $A = k$, this result is already quite non-trivial: it says that under the above hypotheses, the naive H-action on $\mathfrak{h}\text{--}\mathsf{mod}$ canonically renormalizes (with respect to the standard t-structure), and that $\mathfrak{h}\text{--}\mathsf{mod} := \mathrm{Oblv}^{str \to w}(\mathsf{Vect}^{\exp(\mathfrak{h}),w}) \in H\text{--}\mathsf{mod}_{weak}$ is its canonical renormalization.

The proof of Theorem 10.8.1 is involved, so is deferred to §10.17 so that we may first give some preliminary results.

Remark 10.8.3. To orient the reader, let us briefly discuss the importance of Theorem 10.8.1 for defining Harish-Chandra data. Assume the result for now. Of course, we should let $A\#U(\mathfrak{h})$ denote the corresponding object of $\mathsf{Alg}^{\vec{\otimes},H\curvearrowright}_{conv,gen}$.

As the notation indicates, $A\#U(\mathfrak{h})$ should be understood as the usual smash product. Then a morphism $A\#U(\mathfrak{h}) \to A$ restricting to the identity along the canonical embedding $A \to A\#U(\mathfrak{h})$ is the "main" part of a Harish-Chandra datum (i.e., the rest of the data is interpreted as homotopy compatibilities). We refer to §10.22-10.23 for more explicit discussion.

Remark 10.8.4. Before proving the theorem, we do not make explicit reference to the $\vec{\otimes}$-algebra $A\#U(\mathfrak{h})$, only its category of modules. That is, we treat $A\#U(\mathfrak{h})\text{--}\mathsf{mod}_{ren}$ as alternative notation to $\mathrm{Oblv}^{str \to w}(A\text{--}\mathsf{mod}^{\exp(\mathfrak{h}),w}_{ren})$. We let $\mathrm{Oblv} : A\#U(\mathfrak{h})\text{--}\mathsf{mod}_{ren} \to \mathsf{Vect}$ denote the forgetful functor constructed above.

---

[91]Explicitly, the essential image of this functor consists of those compactly generated DG categories $\mathcal{C}$ equipped with $F : \mathcal{C} \to \mathsf{Vect}$ such that for the t-structure on $\mathcal{C}$ defined by having $\mathcal{C}^{\leqslant 0}$ generated under colimits by compact objects $\mathcal{F} \in \mathcal{C}^c$ such that $F(\mathcal{F}) \in \mathsf{Vect}^{\leqslant 0}$, the functor F is t-exact and conservative on $\mathcal{C}^+$, and such that compact objects in $\mathcal{C}$ are eventually coconnective.

10.9. t-structures. As Theorem 10.8.1 concerns canonical renormalization and therefore t-structures, it is convenient to have some convenient language regarding t-structures in the presence of H-actions.

Therefore, we begin with an extended digression on this subject. The reader may safely skip ahead to §10.14 and refer back as needed.

10.10. Suppose H is a Tate group indscheme and $\mathcal{C} \in H\text{–mod}_{weak,naive}$.

Note that the action functor:

$$\text{act}_{\mathcal{C}} : \text{IndCoh}^*(H) \otimes \mathcal{C} \to \mathcal{C}$$

lifts canonically as:

$$\begin{array}{ccc}
\text{IndCoh}^*(H) \otimes \mathcal{C} & \xdashrightarrow{\ \alpha_{\mathcal{C}}\ } & \text{IndCoh}^*(H) \otimes \mathcal{C} \\
 & \searrow_{\text{act}_{\mathcal{C}}} \quad \swarrow_{\Gamma^{\text{IndCoh}}(H,-)\otimes \text{id}_{\mathcal{C}}} & \\
 & \mathcal{C} &
\end{array}$$

with $\alpha_{\mathcal{C}}$ an equivalence. Indeed, viewing $\text{IndCoh}^*(H)$ as a coalgebra in DGCat_{cont} (via pushforwards along diagonal maps, as works for any strict indscheme), there is a unique map of $\text{IndCoh}^*(H)$-comodules $\alpha_{\mathcal{C}}$ fitting into a diagram as above. That this functor is an equivalence follows from the case $\mathcal{C} = \text{IndCoh}^*(H)$, where it is follows from strictness of H.

Suppose now that $\mathcal{C}$ is equipped with a t-structure. We say this t-structure is *compatible* with the (naive, weak) action of H on $\mathcal{C}$ if it is compatible with filtered colimits and $\alpha_{\mathcal{C}}$ is t-exact when both sides are equipped with the tensor product t-structures (as in Lemma 4.6.2).

Example 10.10.1. Suppose that $A \in \text{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$. Then the induced naive, weak H-action on $A\text{–mod}_{ren}$ is compatible with the t-structure. Indeed, this is a restatement of the definition of compatibility between an H-action and a renormalization datum.

10.11. We now move to discuss t-structures in the presence of *strong H-actions*.

We will need the following result.

Lemma 10.11.1. *Suppose K is a classical affine group scheme. Suppose $\mathcal{C} \in K\text{–mod}$ is a DG category equipped with a strong K-action.*

Let $K = \lim_j K/K_j$ with K/K_j an algebraic group (so $K_j \subseteq K$ is a normal compact open subgroup).

Below, we also write $\mathcal{C}$ for the induced object of $K\text{–mod}_{weak}$ under $\text{Oblv}^{str \to w}$.

(1) The natural functor $\text{colim}_j (\mathcal{C}^{K_j})^{K/K_j,w} \to \mathcal{C}^{K,w} \in \text{DGCat}_{cont}$ is an equivalence.

(2) Each of the structural functors in this colimit admits a continuous right adjoint.

Proof. By construction, we have:

$$\mathfrak{k}\text{–mod} = \text{colim}_i (\mathfrak{k}/\mathfrak{k}_i)\text{–mod} = \text{colim}_i D(K/K_i)^{K/K_i,w} \in \text{DGCat}_{cont}$$

as $(D(K), \text{Rep}(K))$-bimodules. Moreover, each structural functor in this colimit clearly admits a continuous right adjoint (calculated as Lie algebra cohomology for the appropriate finite dimensional Lie algebra), and that right adjoint is a morphism of $(D(K), \text{Rep}(K))$-bimodules. This gives the claim by construction of $\text{Oblv}^{str \to w}$.

$\square$

10.12. Now suppose H is a Tate group indscheme with a prounipotent tail, and that $\mathcal{C} \in H\text{–mod}$ is acted on strongly by H and is equipped with a t-structure.

We say that this t-structure is *strongly compatible* with the H-action if it compatible with the underlying weak, naive action of H and for any prounipotent compact open subgroup $K \subseteq H$, the subcategory $\mathcal{C}^K \subseteq \mathcal{C}$ is closed under truncations.

Remark 10.12.1. For H an algebraic group, this condition is clearly equivalent to the t-structure being compatible with the underlying weak action. As discussed in [Ras6] §B.4, this is equivalent to any other notion of a t-structure being compatible with a strong action essentially because the forgetful functor $\mathrm{Oblv} : D(H) \to \mathsf{QCoh}(H)$ is conservative and t-exact up to shift.

From here, one deduces that in general, if a t-structure is compatible with the weak naive action of a Tate group indscheme H, it is strongly compatible if and only if the above condition holds for some prounipotent subgroup.

Remark 10.12.2. We do not know of an example of $\mathcal{C} \in H\text{–mod}$ as above and a t-structure that is compatible with the weak, naive action of H but not strongly compatible.

10.13. Suppose $\mathcal{C} \in H\text{–mod}$ is equipped with a t-structure that is strongly compatible with the H-action.

Observe that for $K \subseteq H$ any compact open, $\mathcal{C}^K$ inherits a unique t-structure such that $\mathcal{C}^K \to \mathcal{C}$ is t-exact. Indeed, if K is prounipotent, this is true by design; in general, choose $K^u \subseteq K$ a normal, prounipotent compact open subgroup, and then observe that the action of the algebraic group K/K^u on $\mathcal{C}^{K^u}$ is compatible with the t-structure there in the sense of [Ras6] Appendix B. By *loc. cit.*, we obtain the claim.

More generally, in the above notation, each $(\mathcal{C}^{K_j})^{K/K_j,w}$ inherits a canonical t-structure, and each of the structural functors in the colimit from Lemma 10.11.1 is t-exact.

Therefore, $\mathcal{C}^{K,w}$ admits a unique t-structure such that each functor $(\mathcal{C}^{K_j})^{K/K_j,w} \to \mathcal{C}^{K,w}$ is t-exact: see [Ras6] Lemma 5.4.3.

Lemma 10.13.1. *In the above setting, the forgetful functor $\mathcal{C}^{K,w,+} \to \mathcal{C}^+$ is conservative.*

Proof. For every j, let $\alpha_j : \mathcal{C}^{K,w} \to (\mathcal{C}^{K_j})^{K/K_j,w}$ be the right adjoint to the structural functor $\beta_j : (\mathcal{C}^{K_j})^{K/K_j,w} \to \mathcal{C}^{K,w}$.

By t-exactness, it suffices to show that this functor is conservative on the heart. For $\mathcal{F} \in \mathcal{C}^{K,w,\heartsuit}$, we have $\mathcal{F} = \mathrm{colim}_j \beta_j \alpha_j(\mathcal{F})$. Note that β_j is t-exact, so α_j is left t-exact. On H^0, each structural map in this colimit is a monomorphism: indeed, $H^0(\beta_j \alpha_j(\mathcal{F}))$ is the maximal subobject of $\mathcal{F}$ lying in $(\mathcal{C}^{K_j})^{K/K_j,w,\heartsuit}$ (which is a full subcategory of $\mathcal{C}^{K,w,\heartsuit}$ because we are working with abelian categories).

Now if $\mathcal{F} \neq 0$, then $H^0(\alpha_j(\mathcal{F})) \neq 0$ for some j, and as the composition $(\mathcal{C}^{K_j})^{K/K_j,w} \to \mathcal{C}^{K,w} \to \mathcal{C}$ is clearly conservative, we obtain the claim. $\square$

Corollary 10.13.2. *In the above setting, the natural functor $\mathcal{C}^{K,w,+} \to \mathcal{C}^{K,w,naive,+}$ is an equivalence.*

Proof. The t-exact conservative forgetful functor $\mathcal{C}^{K,w,+} \to \mathcal{C}^+$ is comonadic by Lemma 3.7.2. The comonads on $\mathcal{C}$ defined by $\mathcal{C}^{K,w}$ and $\mathcal{C}^{K,w,naive}$ always coincide, so we obtain the claim. $\square$

10.14. **Some results on smash products.** We now suppose we are in the setup of Theorem 10.8.1.

Lemma 10.14.1.   *(1) $A\#U(\mathfrak{h})$–mod_{ren} is compactly generated.*

  (2) There exists a unique compactly generated t-structure on $A\#U(\mathfrak{h})$–mod_{ren} such that compact objects are eventually coconnective and such that the forgetful functor $A\#U(\mathfrak{h})$–$\mathsf{mod}_{ren} \to$ Vect is t-exact and conservative on $A\#U(\mathfrak{h})$–mod_{ren}^{+}.

  (3) The above t-structure is strongly compatible with the H-action on $A\#U(\mathfrak{h})$–mod_{ren}.

  (4) For any compact open subgroup $K \subseteq H$, $A\#U(\mathfrak{h})$–$\mathsf{mod}_{ren}^{K,w}$ is compactly generated.

  (5) By (3) and §10.13, $A\#U(\mathfrak{h})$–$\mathsf{mod}_{ren}^{K,w}$ is equipped with a canonical t-structure. This t-structure is compactly generated, and compact objects are bounded from below. Moreover, the forgetful functor $A\#U(\mathfrak{h})$–$\mathsf{mod}_{ren}^{K,w} \to \mathsf{Rep}(K)$ is t-exact.

  (6) Suppose $f : A \to B$ is a morphism of $\overset{\to}{\otimes}$-algebras with genuine H-actions.
    Then for any $K \subseteq H$ compact open, the functor:

$$B\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{K,w} \to A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{K,w}$$

  is t-exact (for the t-structures from (5)).

Proof. We proceed in steps.

Step 1. Recall that by construction, the (genuine) weak H-action on $A\#U(\mathfrak{h})$–mod_{ren} canonically upgrades to a strong action. We use the same notation $A\#U(\mathfrak{h})$–mod_{ren} for the corresponding object of H–mod; in particular, for $K \subseteq H$ compact open, we let $A\#U(\mathfrak{h})$–mod_{ren}^{K} denote the strong K-invariants for the action.

  We begin by proving that $A\#U(\mathfrak{h})$–mod_{ren}^{K} is compactly generated for any compact open subgroup $K \subseteq H$.

  Note that by definition, we have:

$$A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{K} = (A\text{–}\mathsf{mod}_{ren}^{\exp(\mathfrak{h}),w})^{K} = A\text{–}\mathsf{mod}_{ren}^{H_{K}^{\wedge},w}$$

where $H_{K}^{\wedge}$ is the formal completion of H along K. The forgetful functor:

$$A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{K} = A\text{–}\mathsf{mod}_{ren}^{H_{K}^{\wedge},w} \to A\text{–}\mathsf{mod}_{ren}^{K,w}$$

is conservative and admits a left adjoint; as this forgetful functor is manifestly continuous, it is monadic.

  Because the H-action on A–mod_{ren} arises by canonical renormalization (by assumption on A), A–$\mathsf{mod}_{ren}^{K,w}$ is compactly generated. By the above, we immediately deduce the same for $A\#U(\mathfrak{h})$–mod_{ren}^{K}.

Step 2. Note that the monad on A–$\mathsf{mod}_{ren}^{K,w}$ constructed in Step 1 is t-exact for the t-structure on A–$\mathsf{mod}_{ren}^{K,w}$ (coming from Proposition 5.18.3). Indeed, as H is formally smooth, $H_{K}^{\wedge}$ is too, so $\omega_{H_{K}^{\wedge}/K} \in \mathsf{IndCoh}(H_{K}^{\wedge}/K)^{K,w}$ lies in the heart of the t-structure. The monad in question is given by convolution with this object. Convolution by an object in $\mathsf{Coh}(H_{K}^{\wedge})^{K,w,\heartsuit}$ defines a functor A–$\mathsf{mod}_{ren}^{K,w} \to A$–$\mathsf{mod}_{ren}^{K,w}$ that is left t-exact up to shift by definition of canonical renormalization, and it is t-exact by the compatibility of the naive action of H on A–mod_{ren} with the t-structure; therefore, the same applies to convolution by arbitrary objects of $\mathsf{IndCoh}(H_{K}^{\wedge})^{K,w,\heartsuit}$, giving the claim.

  It follows that A–$\mathsf{mod}_{ren}^{H_{K}^{\wedge},w} = A\#U(\mathfrak{h})$–$\mathsf{mod}_{ren}^{K}$ admits a unique t-structure such that the (monadic, with monad the one in question) forgetful functor to A–$\mathsf{mod}_{ren}^{K,w}$ is t-exact.

Step 3. We now deduce (1) and (2).
  We have:

$$A\#U(\mathfrak{h})\text{--mod}_{ren} = \operatorname*{colim}_{K\subseteq H \text{ compact open}} A\#U(\mathfrak{h})\text{--mod}_{ren}^{K} \in \mathsf{DGCat}_{cont}.$$

Each structural functor in this colimit admits a continuous right adjoint so preserves compact objects. We obtain that the colimit is compactly generated since each term is by Step 1.

Moreover, we claim that each of the structural functors in this colimit is t-exact for the t-structures from Step 2. For $K' \subseteq K \subseteq H$ compact open subgroups, we have a commutative diagram of forgetful functors:

$$\begin{array}{ccc}
A\#U(\mathfrak{h})\text{--mod}_{ren}^{K} & \longrightarrow & A\#U(\mathfrak{h})\text{--mod}_{ren}^{K'} \\
\downarrow & & \downarrow \\
A\text{--mod}_{ren}^{K,w} & \longrightarrow & A\text{--mod}_{ren}^{K',w}.
\end{array}$$

The vertical functors are t-exact and conservative by construction, and the bottom functor is clearly t-exact, so the claim follows.

Therefore, our (filtered) colimit inherits a canonical t-structure such that each functor $A\#U(\mathfrak{h})\text{--mod}_{ren}^{K} \to A\#U(\mathfrak{h})\text{--mod}_{ren}$ is t-exact. Let us show that this t-structure has the desired properties from (2).

It is clear from the construction that this t-structure is compactly generated and that compact objects are eventually coconnective.

For K any compact open subgroup of H, the functor:

$$A\#U(\mathfrak{h})\text{--mod}_{ren}^{K} \to A\text{--mod}_{ren}^{K,w} \to A\text{--mod}_{ren} \to \mathsf{Vect}$$

calculates the forgetful functor. The first functor in this sequence is t-exact and conservative by design, while the remaining functors are t-exact and conservative on eventually coconnective subcategories by assumption. It remains to show this conservativeness survives passage to the colimit in K.

First, suppose K is a fixed prounipotent compact open. Observe that $A\#U(\mathfrak{h})\text{--mod}_{ren}^{K,\heartsuit}$ admits an explicit description: it is the abelian category of discrete $H^0(A)$-modules equipped with a suitably compatible smooth $\mathfrak{h}$-action such that $\mathfrak{k} = \mathrm{Lie}(K)$ acts locally nilpotently. Indeed, this follows from the description of the category as modules over a certain monad on $A\text{--mod}_{ren}^{K,w,\heartsuit}$.

In particular, the abelian category $A\#U(\mathfrak{h})\text{--mod}_{ren}^{K,\heartsuit} \subseteq A\#U(\mathfrak{h})\text{--mod}_{ren}^{\heartsuit}$ is closed under taking subobjects.

It follows that for $\mathcal{F} \in A\#U(\mathfrak{h})\text{--mod}_{ren}^{\heartsuit}$, the map $\mathcal{F} \to \mathrm{Oblv}\,\mathrm{Av}_*(\mathcal{F})$ induces a monomorphism on H^0; here e.g. Av_* indicates the functor of strong K-averaging. For such $\mathcal{F}$, note that:

$$\mathcal{F} = \operatorname*{colim}_{K\subseteq H \text{ compact open}} \mathrm{Oblv}\,\mathrm{Av}_*^{K}(\mathcal{F}) = \operatorname*{colim}_{K\subseteq H \text{ compact open}} H^0\,\mathrm{Oblv}\,\mathrm{Av}_*^{K}(\mathcal{F})$$

with each structural map in the latter colimit a monomorphism in $A\#U(\mathfrak{h})\text{--mod}_{ren}^{\heartsuit}$. Therefore, if $\mathcal{F}$ is non-zero, $H^0(\mathrm{Av}_*^{K}(\mathcal{F}))$ is non-zero for some K. Now the desired conservativeness follows from the similar result for $A\#U(\mathfrak{h})\text{--mod}_{ren}^{K}$.

Step 4. We now mildly generalize our earlier constructions.

Let K be as before, and let $K_0 \subseteq K$ be a normal compact open subgroup. We will study the category:

$$(A\#U(\mathfrak{h})\text{--mod}_{ren}^{K_0})^{K/K_0,w}.$$

As before, this category identifies with $(A\text{–}\mathsf{mod}_{ren}^{H_{K_0}^{\wedge},w})^{K/K_0,w}$. There is a canonical forgetful functor:

$$(A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{K_0})^{K/K_0,w} = (A\text{–}\mathsf{mod}_{ren}^{H_{K_0}^{\wedge},w})^{K/K_0,w} \to (A\text{–}\mathsf{mod}_{ren}^{K_0,w})^{K/K_0,w} = A\text{–}\mathsf{mod}_{ren}^{K,w}$$

which is again monadic. Clearly the corresponding monad on $A\text{–}\mathsf{mod}_{ren}^{K,w}$ is again t-exact.

Therefore, we once again find that $(A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{K_0})^{K/K_0,w}$ is again compactly generated, and that it admits a unique t-structure for which the forgetful functor to $A\text{–}\mathsf{mod}_{ren}^{K,w}$ is t-exact.

Step 5. We now show (4) from the statement of the lemma.

By Lemma 10.11.1, and using the notation of *loc. cit.*, we have:

$$A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{K,w} = \operatorname*{colim}_{j} (A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{K_j})^{K/K_j,w} \in \mathsf{DGCat}_{cont}.$$

Each of the structural functors in this colimit admits a continuous right adjoint. Therefore, compact generation of the colimit follows from compact generation of each term, which we have shown in Step 4.

Step 6. Let us be in the general setup of §10.10, with H acting naively on $\mathcal{C}$, which is equipped with a t-structure. Suppose that the t-structure on $\mathcal{C}$ is compactly generated.

We claim that the t-structure on $\mathcal{C}$ is compatible with the naive H-action if and only if for every $\mathcal{F} \in \mathsf{Coh}^*(H)^{\leq 0}$ and $\mathcal{G} \in \mathcal{C}^{\leq 0}$ compact, $\alpha_{\mathcal{C}}(\mathcal{F} \boxtimes \mathcal{G})$ is connective.

Indeed, this condition clearly implies $\alpha_{\mathcal{C}}$ is right t-exact. It suffices to show that $(\alpha_{\mathcal{C}})^{-1}$ is similarly right t-exact. Note that $\alpha_{\mathcal{C}}^{-1}$ is obtained by conjugating $\alpha_{\mathcal{C}}$ by the automorphism $\mathsf{inv}_*^{\mathsf{IndCoh}} \otimes \mathsf{id}_{\mathcal{C}}$ for $\mathsf{inv} : H \xrightarrow{\simeq} H$ the inversion map. As this automorphism is t-exact (since $\mathsf{inv}_*^{\mathsf{IndCoh}}$ clearly is), we obtain the result.

Step 7. We will show that the t-structure on $\mathcal{C} = A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}$ is compatible with the naive H-action using the criterion of Step 6. Let $\alpha = \alpha_{\mathcal{C}}$ in what follows.

Let $K \subseteq H$ be a fixed compact open subgroup and let $\mathsf{Av}_! = \mathsf{Av}_!^{K \to H_K^{\wedge},w} : A\text{–}\mathsf{mod}_{ren}^{K,w} \to A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{K}$ denote the left adjoint to the forgetful functor. Let $\mathcal{G}_0 \in A\text{–}\mathsf{mod}_{ren}^{K,w,\leq 0}$ be compact and let $\mathcal{G} := \mathsf{Av}_!(\mathcal{G}_0)$. Note that objects of this form compactly generate $A\#U(\mathfrak{h})\text{–}\mathsf{mod}_{ren}^{\leq 0}$ (letting K vary, of course).

Let $\mathcal{F} \in \mathsf{Coh}(H)^{\heartsuit}$. By Step 6 and the above remarks, it suffices to show that $\alpha(\mathcal{F} \boxtimes \mathcal{G})$ is connective. We will show this in Step 9 after some preliminary constructions.

Let us just note one special case. Suppose $\mathcal{F} = k_h$ is the skyscraper sheaf at a k-point h of H. Then the object in question is $\mathsf{Av}_!^{\mathrm{Ad}_h(K) \to H_K^{\wedge},w}(k_h \star \mathcal{G}_0)$. Clearly $k_h \star \mathcal{G}_0 \in A\text{–}\mathsf{mod}_{ren}^{\mathrm{Ad}_h(K),w,\leq 0}$, so the same is true after !-averaging.

The general argument is similar, but has additional complications due to working in families.

Step 8. We need some constructions involving standard Chevalley-style constructions.

Let K be any classical affine group scheme and let $K_0 \subseteq K$ be a compact open subgroup. Suppose $\mathcal{C}$ is equipped with a genuine K-action.

Let $\mathcal{H} \in \mathcal{C}^{K_{K_0}^{\wedge},w}$ be given. We will construct a canonical *Chevalley* filtration on $\mathcal{H}$, which is an increasing filtration with $\mathrm{gr}_i \mathcal{H} = \mathsf{Av}_!^{K_0 \to K_{K_0}^{\wedge},w}(\Lambda^i(\mathfrak{k}/\mathfrak{k}_0) \star \mathrm{Oblv}(\mathcal{H}))[i]$. For clarity: the notation means we forget $\mathcal{H}$ down to $\mathcal{C}^{K_0,w}$, act by $\Lambda^i(\mathfrak{k}/\mathfrak{k}_0)[i] \in \mathsf{Rep}(K_0)$, then !-average back.

Indeed, there is an action of the symmetric monoidal category $\mathsf{Rep}(K_{K_0}^{\wedge}) = \mathfrak{k}\text{–}\mathsf{mod}^{K_0}$ on $\mathcal{C}^{K_{K_0}^{\wedge}}$. The trivial representation $k \in \mathfrak{k}\text{–}\mathsf{mod}^{K_0}$ has a standard filtration with $\mathrm{gr}_i(k) = \mathrm{ind}_{\mathfrak{k}_0}^{\mathfrak{k}}(\Lambda^i(\mathfrak{k}/\mathfrak{k}_0))[i]$

(see [GR4] §IV.5.2 for a much more general context for such constructions). As k is the unit for the monoidal structure here, we obtain a filtration of the desired type by functoriality.

We will actually need a more general, parametrized version of this construction. We sketch the ideas below.

Suppose S is an affine scheme. Let $\mathcal{K}$ be a *compact*[92] group scheme over S, meaning an affine group scheme over S that can be realized as a projective limit under smooth surjective structure maps of affine group schemes that are smooth over S.

Let $\mathcal{K}_0 \subseteq \mathcal{K}$ be an (S-family of) compact open subgroups of $\mathcal{K}$, meaning $\mathcal{K}_0$ is compact in the above sense and we are given $\mathcal{K}_0 \to \mathcal{K}$ a homomorphism of group schemes over S that is a closed embedding such that $\mathcal{K}/\mathcal{K}_0$ is smooth over S (in particular, of finite presentation over S).

For such $\mathcal{K}$, there is a symmetric monoidal category $\mathsf{Rep}(\mathcal{K})$, defined as the evident colimit as in the case where S is a point. For example, $\mathsf{Rep}(K \times S) = \mathsf{Rep}(K) \otimes \mathsf{QCoh}(S)$ for K a classical affine group scheme. Note that $\mathsf{Rep}(\mathcal{K})$ is a $\mathsf{QCoh}(S)$-module category, is a symmetric monoidal category over $\mathsf{QCoh}(S)$.

Let $\mathcal{C}$ be a genuine K-category, meaning we are given $\mathcal{C}^{K,w}$ a $\mathsf{Rep}(K)$-module category. We obtain $\mathcal{C}^{\mathcal{K}_0,w} := \mathcal{C}^{K,w} \otimes_{\mathsf{Rep}(\mathcal{K})} \mathsf{Rep}(\mathcal{K}_0)$.

The ideas of §7 translate into this setting in an evident way. In particular, we a parametrized category of Harish-Chandra modules $\mathsf{Rep}(\mathcal{K}^{\wedge}_{\mathcal{K}_0})$, which is equipped with a monadic forgetful functor to $\mathsf{Rep}(\mathcal{K}_0)$. This allows us to make sense of $\mathcal{C}^{\mathcal{K}^{\wedge}_{\mathcal{K}_0},w}$, again by tensoring.

In particular, the construction of Chevalley filtrations goes through in this setting.

Step 9. We now conclude the argument. We use the notation from Step 7.

Let $S \subseteq H$ be a classical affine subscheme on which $\mathcal{F}$ is scheme-theoretically supported.

In what follows, we use a subscript S to indicate a product with S. For example, $H_S = H \times S$.

Let $\mathcal{K}$ be the group scheme $K \times S$ over S; in what follows, we *always* regard $\mathcal{K}$ as a (family of) compact open subgroup(s) of H_S via the map:

$$\mathcal{K} = K \times S \xrightarrow{(k,h) \mapsto (\mathrm{Ad}_h(k), h)} H \times S = H_S.$$

By a standard Noetherian descent argument, there exists a compact open subgroup $K_0 \subseteq H$ such that $K_{0,S} \subseteq \mathcal{K} \subseteq H_S$. Note that $\mathcal{K}/K_{0,S}$ is smooth over S in this case.

Let $A\text{–}\mathsf{mod}_{ren,S} := \mathsf{QCoh}(S) \otimes A\text{–}\mathsf{mod}_{ren}$. We similarly have $A\text{–}\mathsf{mod}^{K_S,w}_{ren,S} = A\text{–}\mathsf{mod}^{K,w}_{ren} \otimes \mathsf{QCoh}(S)$ and $A\text{–}\mathsf{mod}^{\mathcal{K},w}_{ren}$, with $\alpha_{A\text{–}\mathsf{mod}_{ren}}$ inducing an isomorphism:

$$A\text{–}\mathsf{mod}^{K_S,w}_{ren,S} \xrightarrow{\simeq} A\text{–}\mathsf{mod}^{\mathcal{K},w}_{ren,S}.$$

Regarding $\mathcal{F}$ as a coherent sheaf on S, $\mathcal{F} \boxtimes \mathcal{G}_0 \in A\text{–}\mathsf{mod}^{K,w}_{ren,S}$ by definition. Let $\mathcal{H} := \alpha_{A\text{–}\mathsf{mod}_{ren}}(\mathcal{F} \boxtimes \mathcal{G}_0) \in A\text{–}\mathsf{mod}^{\mathcal{K},w}_{ren,S}$.

We clearly have:

$$\alpha(\mathcal{F} \boxtimes \mathrm{Av}^{K \to H^{\wedge}_K,w}_!(\mathcal{G}_0)) = \mathrm{Av}^{\mathcal{K} \to H^{\wedge}_{K,S},w}_!(\mathcal{H})$$

where the left hand side is what we wish to show is connective.

Applying Step 8, we obtain an increasing filtration on $\mathcal{H}$ with:

$$\mathrm{gr}_i \mathcal{H} = \mathrm{Av}^{K_{0,S} \to \mathcal{K}^{\wedge}_{K_{0,S}},w}_!(\Lambda^i(\mathrm{Lie}(\mathcal{K})/\mathrm{Lie}(K_{0,S})) \star \mathrm{Oblv}(\mathcal{H}))[i].$$

---

[92]The terminology is admittedly bad: it is meant to evoke *compact open*, nothing about properness.

We remark that $\text{Lie}(\mathcal{K})/\text{Lie}(K_{0,S})$ is a finite-rank vector bundle on S. As in the previous step, $\text{Oblv}(\mathcal{H})$ indicates that we forget down to weak $K_{0,S}$-invariants.

Therefore, $\text{Av}_!^{\mathcal{K}\to H^\wedge_{\hat{K},S},w}(\mathcal{H})$ inherits a filtration in $\text{IndCoh}^*(H)\otimes A\text{–mod}_{ren}^{H^\wedge_{K_0},w}$ with ith associated graded term:

$$(\text{id}\otimes\text{Av}_!^{K_0\to H^\wedge_{K_0},w})(\Lambda^i(\text{Lie}(\mathcal{K})/\text{Lie}(K_{0,S}))\star\text{Oblv}(\mathcal{H}))[i]\in\text{QCoh}(S)\otimes A\text{–mod}_{ren}^{H^\wedge_{K_0},w}. \quad (10.14.1)$$

Now observe that $\text{Oblv}(\mathcal{H})\in A\text{–mod}_{ren,S}^{K_{0,S},w}$ is connective: it suffices[93] to check this after applying the forgetful functor to $A\text{–mod}_{ren,S}$ where it is clear.

This clearly that implies (10.14.1) is connective (for the tensor product t-structure), giving our claim.

Step 10. To complete the proof of (3), it remains to show that the t-structure on $A\#U(\mathfrak{h})\text{–mod}_{ren}$ is *strongly* compatible with the H-action.

Fortunately, this is evident from our constructions: the t-structure on $A\#U(\mathfrak{h})\text{–mod}_{ren}$ was defined so that:

$$A\#U(\mathfrak{h})\text{–mod}_{ren}^K = A\text{–mod}_{ren}^{H^\wedge_{\hat{K}},w}\to A\#U(\mathfrak{h})\text{–mod}_{ren}$$

is t-exact for every compact open subgroup K.

Step 11. We now show (5).

The analysis of Step 4 clearly shows that for any $K_0\subseteq K$ a normal compact open subgroup, the t-structure on $(A\#U(\mathfrak{h})\text{–mod}_{ren}^{K_0})^{K/K_0,w}$ is compactly generated and compact objects are bounded from below. By construction, this implies the same for $A\#U(\mathfrak{h})\text{–mod}_{ren}^{K,w}$.

To show that $A\#U(\mathfrak{h})\text{–mod}_{ren}^{K,w}\to\text{Rep}(K)$ is t-exact, we claim that it suffices to show this for its restriction to $(A\#U(\mathfrak{h})\text{–mod}_{ren}^{K_0})^{K/K_0,w}$. Indeed, right t-exactness of this functor is evident (as the forgetful functor $A\#U(\mathfrak{h})\text{–mod}_{ren}\to\text{Vect}$ is t-exact and all our forgetful functors are conservative on eventually coconnective subcategories).

For left t-exactness, note that if $\mathcal{F}\in A\#U(\mathfrak{h})\text{–mod}_{ren}^{K,w}$, then:

$$\mathcal{F} = \operatorname*{colim}_j L_j(\mathcal{F})$$

where L_j is the comonad on $A\#U(\mathfrak{h})\text{–mod}_{ren}^{K,w}$ defined by the adjunction:

$$(A\#U(\mathfrak{h})\text{–mod}_{ren}^{K_j})^{K/K_j,w}\rightleftarrows A\#U(\mathfrak{h})\text{–mod}_{ren}^{K,w}$$

using the notation of Lemma 10.11.1. Because the left adjoint in this adjunction is t-exact, L_j is left t-exact. As this colimit is filtered, we obtain the claim.

Now observe that the composition:

$$(A\#U(\mathfrak{h})\text{–mod}_{ren}^{K_0})^{K/K_0,w} = (A\text{–mod}_{ren}^{H^\wedge_{K_0}})^{K/K_0,w}\to A\text{–mod}_{ren}^{K,w}\to\text{Rep}(K)$$

calculates the forgetful functor in question. The second arrow is t-exact by assumption on A. The same holds for the first arrow is t-exact because $A\text{–mod}_{ren}^{H^\wedge_{K_0}}\to A\text{–mod}_{ren}^{K_0,w}$ is t-exact by construction, and forgetting from K-invariants to K_0-invariants is t-exact and conservative (because K/K_0 is finite type).

---

[93]Unlike coconnectivity.

Step 12. Finally, it remains to show (6).

First, note that the functor:

$$B\#U(\mathfrak{h})\text{–mod}^K_{ren} \to A\#U(\mathfrak{h})\text{–mod}^K_{ren}$$

is t-exact. Indeed, we have a commutative diagram:

$$
\begin{array}{ccc}
B\#U(\mathfrak{h})\text{–mod}^K_{ren} & \longrightarrow & A\#U(\mathfrak{h})\text{–mod}^K_{ren} \\
\downarrow & & \downarrow \\
B\text{–mod}^{K,w} & \longrightarrow & A\text{–mod}^{K,w}
\end{array}
$$

where, as always, the vertical arrows are given by rewriting e.g. $A\#U(\mathfrak{h})\text{–mod}^K_{ren}$ as $A\text{–mod}^{H_{\hat{K}},w}_{ren}$. These vertical arrows are conservative and t-exact by construction, and the bottom horizontal arrow is t-exact by definition of morphism in $\mathsf{Alg}^{\vec{\otimes},H\curvearrowright}_{gen}$.

More generally, we find that for any $K_0 \subseteq K$ a compact open normal subgroup, the functor:

$$(B\#U(\mathfrak{h})\text{–mod}^{K_0}_{ren})^{K/K_0,w} \to (A\#U(\mathfrak{h})\text{–mod}^{K_0}_{ren})^{K/K_0,w}$$

is t-exact. We now conclude the result by the same logic as in Step 11. $\qquad\square$

10.15. **Some results on canonical renormalization.** We now develop some general results on canonical renormalization in the setting of Tate group indschemes. The ultimate result is Corollary 10.16.3, which gives a convenient way of checking the hypotheses for canonical renormalization.

Proposition 10.15.1. *Suppose that H is a Tate group indscheme with a prounipotent tail.*

Suppose $\mathcal{C} \in H\text{–mod}$ is acted on strongly by H and equipped with a t-structure strongly compatible with the H-action.

Suppose that for every compact open subgroup $K \subseteq H$, $\mathcal{C}^{K,w}$ is compactly generated with compact objects lying in $\mathcal{C}^{K,w,+}$ (with respect to the t-structure of §10.13).

Then for every compact open subgroup $K \subseteq H$, an object $\mathcal{F} \in \mathcal{C}^{K,w}$ is compact if and only if $\mathcal{F} \in \mathcal{C}^{K,w,+}$ and $\mathrm{Oblv}(\mathcal{F})$ is compact in $\mathcal{C}$.

Remark 10.15.2. We remark that the result is clearly about K, and that H is a bit of a red herring.

Also, note that the conclusion of the lemma may be stated as $\mathcal{C} \in K\text{–mod}_{weak}$ is obtained by canonical renormalization (in the sense of §5.18) from the underlying naive weak K-action on $\mathcal{C}$.

Proof of Proposition 10.15.1. Clearly compact objects satisfy this property, as Oblv admits the continuous right adjoint Av^w_*. Therefore, suppose $\mathcal{F} \in \mathcal{C}^{K,w,+}$ with $\mathrm{Oblv}(\mathcal{F})$ compact; we wish to show that $\mathcal{F}$ is compact.

Choose $N \gg 0$ such that $\mathcal{F} \in \mathcal{C}^{K,w,\geqslant-N}$. Note that $\mathcal{F}$ is compact in $\mathcal{C}^{K,w,\geqslant-N}$: see Step 3 from the proof of Lemma 6.11.2.

Write $\mathcal{F}$ as a filtered colimit $\mathrm{colim}_i \mathcal{F}_i$ with $\mathcal{F}_i \in \mathcal{C}^{K,w}$ compact. As the t-structure is compatible with filtered colimits, we obtain $\mathcal{F} = \mathrm{colim}_i \tau^{\geqslant-N}\mathcal{F}_i$. Because $\mathcal{F}$ is compact in $\mathcal{C}^{K,w,\geqslant-N}$, we see that $\mathcal{F}$ is a summand of $\tau^{\geqslant-N}\mathcal{F}_i$ for some i.

By Lemma 10.11.1 (and in the notation of *loc. cit.*), the map:

$$\mathrm{colim}_j (\mathcal{C}^{K_j})^{K/K_j,w,c} \to \mathcal{C}^{K,w,c} \in \mathsf{DGCat}$$

is an equivalence, where K_j runs over compact open prounipotent subgroups of H that are normal in K; as is usual, the superscript $(-)^c$ indicates the subcategory of compact objects. Therefore, $\mathcal{F}_i$

lifts to $(\mathcal{C}^{K_j})^{K/K_j,w,c}$ for some index j. As the forgetful functor $(\mathcal{C}^{K_j})^{K/K_j,w} \to \mathcal{C}^{K,w}$ is t-exact, the same is true of $\mathcal{F}$ itself. We abuse notation in letting $\mathcal{F}$ also denote a lift to $(\mathcal{C}^{K_j})^{K/K_j,w}$.

As the forgetful functor $(\mathcal{C}^{K_j})^{K/K_j,w} \to \mathcal{C}^{K,w}$ admits a continuous right adjoint, it suffices to show that $\mathcal{F}$ is compact as an object of $(\mathcal{C}^{K_j})^{K/K_j,w}$. Moreover, by Lemma 10.15.3, it suffices to show that $\mathcal{F}$ is compact after forgetting to $\mathcal{C}^{K_j}$. As K_j is prounipotent by assumption, the forgetful functor $\mathcal{C}^{K_j} \to \mathcal{C}$ is fully-faithful, giving the claim.

$\square$

Above, we used the following result.

Lemma 10.15.3. *Let H be an affine algebraic group (in particular, of finite type) acting weakly on $\mathcal{C}$. Then $\mathcal{F} \in \mathcal{C}^{H,w}$ is compact if and only if $\mathrm{Oblv}(\mathcal{F}) \in \mathcal{C}$ is compact.*

Proof. Clearly if $\mathcal{F} \in \mathcal{C}^{H,w}$ is compact, then $\mathrm{Oblv}(\mathcal{F}) \in \mathcal{C}$ is compact. Suppose the converse.

Recall from the proof of Lemma 5.20.2 that for $\mathcal{G} \in \mathcal{C}^{H,w}$, $\mathcal{G}$ is a summand of $\mathrm{Tot}^{\leqslant n}(\mathrm{Av}_*^w \mathrm{Oblv})^{\bullet+1}(\mathcal{G})$ for some n; moreover, this is functorial in $\mathcal{G}$ by the construction of *loc. cit.*

Therefore, the functor $\underline{\mathrm{Hom}}_{\mathcal{C}^{H,w}}(\mathcal{F}, -) : \mathcal{C}^{H,w} \to \mathsf{Vect}$ is a summand of:

$$\mathrm{Tot}^{\leqslant n} \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathrm{Oblv}(\mathcal{F}), \mathrm{Oblv}(\mathrm{Av}_*^w \mathrm{Oblv})^{\bullet}(-)).$$

A summand of a finite limit of continuous functors is continuous, giving the claim.

$\square$

10.16.   Next, we show the following result.

Proposition 10.16.1. *Suppose that H is a polarizable Tate group indscheme with a prounipotent tail. Suppose that $\mathcal{C} \in H\text{–}\mathsf{mod}$ is equipped with a compactly generated t-structure strongly compatible with the action of H on $\mathcal{C}$.*

Then for every compact open subgroup $K \subseteq H$ and every $\mathcal{F} \in \mathcal{H}_{H,K}^{w,\geqslant 0}$, the functor $\mathcal{F} \star - : \mathcal{C}^{K,w} \to \mathcal{C}^{K,w}$ is left t-exact.

Proof.

Step 1. First, we claim that the conclusion of the for a compact open subgroup K is equivalent to asking that $\mathcal{F} \in \mathcal{H}_{H,K}^w$ compact acts on $\mathcal{C}^{K,w}$ by a functor that is left t-exact up to shift. This property is clearly weaker than that in the statement of the proposition, so suppose it is satisfied.

We remind that $\mathsf{Coh}(K\backslash H/K) \subseteq \mathcal{H}_{H,K}^w$ is the subcategory of compact objects and is closed under truncations. Therefore, it suffices to show that for $\mathcal{F} \in \mathsf{Coh}(K\backslash H/K)^{\geqslant 0}$, the functor $\mathcal{F} \star - : \mathcal{C}^{K,w} \to \mathcal{C}^{K,w}$ is left t-exact. Indeed, any object of $\mathcal{H}_{H,K}^{w,\geqslant 0}$ is a filtered colimit of such coconnective compact objects. Fix $\mathcal{F} \in \mathsf{Coh}(K\backslash H/K)^{\geqslant 0}$ in what follows.

Under our assumption, $\mathcal{F} \star -$ maps $\mathcal{C}^{K,w,\geqslant 0}$ into $\mathcal{C}^{K,w,+}$. Therefore, by Lemma 10.13.1, it suffices to show that the composition:

$$\mathcal{C}^{K,w} \xrightarrow{\mathcal{F}\star-} \mathcal{C}^{K,w} \xrightarrow{\mathrm{Oblv}} \mathcal{C}$$

is left t-exact. Moreover, we can clearly replace $\mathcal{C}^{K,w}$ with $\mathcal{C}^{K,w,naive}$ here. Then the corresponding functor may be calculated as the composition:

$$\mathcal{C}^{K,w,naive} \xrightarrow{\mathcal{F}\boxtimes-} \mathsf{IndCoh}(H/K) \otimes \mathcal{C}^{K,w,naive} \to (\mathsf{IndCoh}(H) \otimes \mathcal{C})^{K,w,naive} \xrightarrow{\alpha_{\mathcal{C}}} \simeq$$

$$\mathsf{IndCoh}(H/K) \otimes \mathcal{C} \xrightarrow{\Gamma^{\mathsf{IndCoh}}(H/K,-)} \mathcal{C}.$$

There are some things to explain in the above manipulations: we are regarding $\mathcal{F} \in \mathsf{IndCoh}(H/K)$ by forgetting the left K-equivariance; in the third term, the K-equivariance is taken for the diagonal K-action mixing the given action on $\mathcal{C}$ and the right action of H on $\mathcal{C}$; and the implicit commuting of weak K-equivariance with the tensor product by $\mathcal{C}$ in the fourth term is justified by the fact that $\mathcal{C}$ is assumed compactly generated and therefore is dualizable (or one may use Lemma 8.6.1). The first functor is left t-exact by assumption on $\mathcal{F}$; the second is clearly t-exact; the third is t-exact because the naive H-action is compatible with t-structures; and the fourth by Lemma 4.6.2 (2), using that $\Gamma^{\mathsf{IndCoh}}(H/K, -)$ is left t-exact. This gives the claim.

Step 2. Next, we check the above hypothesis in the case where K is a polarization of H. In fact, a little more generally, we will show that if K is a polarization and $K_0 \subseteq H$ is any other compact open subgroup, then for any $\mathcal{F} \in \mathsf{Coh}(K_0 \backslash H/K)$, the functor:

$$\mathcal{F} \star - : \mathcal{C}^{K_0, w} \to \mathcal{C}^{K, w}$$

is left t-exact up to shift.

Let $\mathcal{G} \in \mathsf{Coh}(K \backslash H/K_0)$ be obtained by applying Serre duality on H/K to $\mathcal{F}$ (considered with its natural K_0-equivariant structure) and then pulling back along the inversion map $K \backslash H/K_0 \xrightarrow{\simeq} K_0 \backslash H/K$.

As is standard, $\mathcal{G} \star - : \mathcal{C}^{K, w} \to \mathcal{C}^{K_0, w}$ is left adjoint to $\mathcal{F} \star -$ (by ind-properness of H/K). Therefore, it suffices to show $\mathcal{G} \star -$ is right t-exact up to shift.

This is straightforward: it suffices to show the composite with $\mathcal{C}^{K_0, w} \to \mathcal{C}$ is right t-exact up to shift by Lemma 10.13.1, and this follows by a similar (in fact, simpler) argument to Step 1, using that $\mathcal{G}$ is supported on a finite type subscheme of H/K.

Step 3. Next, suppose K is a compact open subgroup of H that admits an embedding $K \subseteq K_{pol} \subseteq H$ with K_{pol} a polarization of H and K normal in K_{pol} (so K_{pol}/K is an affine algebraic group). We will prove the result for K in this case.

Let $\mathcal{F} \in \mathcal{H}_{H,K}^{w, \geqslant 0}$ and $\mathcal{G} \in \mathcal{C}^{K, w, \geqslant 0}$ be given. We need to show that $\mathcal{F} \star \mathcal{G} \in \mathcal{C}^{K, w, \geqslant 0}$.

As the functor $\mathrm{Av}_*^w : \mathcal{C}^{K, w} \to \mathcal{C}^{K_{pol}, w}$ of averaging from K to K_{pol} is conservative and t-exact (by the normality assumption), it suffices to show that $\mathrm{Av}_*^w(\mathcal{F} \star \mathcal{G}) \in \mathcal{C}^{K_{pol}, w, \geqslant 0}$.

This term may clearly be calculated by averaging $\mathcal{F} \in \mathcal{H}_{H,K}^w$ on the left to obtain $\widetilde{\mathcal{F}} \in \mathsf{IndCoh}_{ren}(K_{pol} \backslash H/K)^{\geqslant 0}$, and then convolving with $\widetilde{\mathcal{F}}$. By the previous step, that object is eventually coconnective, and by Step 1 it is honestly coconnective.

Step 4. Finally, we prove the claim for K a general compact open subgroup of H.

By the previous step, there exists a compact open subgroup $K_0 \subseteq K$ for which the conclusion of the proposition holds.

We again let $\mathcal{F} \in \mathcal{H}_{H,K}^{w, \geqslant 0}$ and $\mathcal{G} \in \mathcal{C}^{K, w, \geqslant 0}$ denote given objects, and we aim to show that their convolution is eventually coconnective.

By the proof of Lemma 5.20.2, $\mathcal{G}$ is a direct summand of $\mathrm{Tot}^{\leqslant n}(\mathrm{Av}_*^w \mathrm{Oblv})^{\bullet + 1}(\mathcal{G})$ for some n; here our functors denote the adjoint pair $\mathrm{Oblv} : \mathcal{C}^{K, w} \rightleftarrows \mathcal{C}^{K_0, w} : \mathrm{Av}_*^w$. Each term in this finite limit lies in $\mathrm{Av}_*^w(\mathcal{C}^{K_0, w, \geqslant 0})$, so we may assume $\mathcal{G} = \mathrm{Av}_*^w(\mathcal{G}_0)$ for $\mathcal{G}_0 \in \mathcal{C}^{K_0, w, \geqslant 0}$.

It suffices to check that $\mathrm{Oblv}(\mathcal{F} \star \mathcal{G}) = \mathrm{Oblv}(\mathcal{F} \star \mathrm{Av}_*^w(\mathcal{G}_0)) \in \mathcal{C}^{K_0, w}$ is eventually coconnective, since $\mathrm{Oblv} : \mathcal{C}^{K, w} \to \mathcal{C}^{K_0, w}$ is conservative.

But the above object may be calculated by mapping $\mathcal{F}$ along the functor $\mathcal{H}_{H,K}^w \to \mathcal{H}_{H,K_0}^w$ of forgetting equivariance on both sides and then convolving with $\mathcal{G}_0$; by assumption on K_0, this object is coconnective as desired.

$\square$

Remark 10.16.2. The careful reader will see that we never really used the hypothesis that the H-action on $\mathcal{C}$ is strong. Here are the actually relevant hypotheses. First, we need a genuine H-action on $\mathcal{C}$ and a t-structure on $\mathcal{C}$ naively compatible with the H-action. In addition, for every compact open subgroup $K \subseteq H$, we need a t-structure on $\mathcal{C}^{K,w}$ for which $\mathcal{C}^{K,w} \to \mathcal{C}^{K,w,naive}$ is t-exact and an equivalence on eventually coconnective subcategories. Finally, we need that for $K_1 \subseteq K_2$ compact open subgroups, $\mathcal{C}^{K_2,w} \to \mathcal{C}^{K_1,w}$ is t-exact.

Corollary 10.16.3. *Suppose H is polarizable with a prounipotent tail. Suppose $\mathcal{C} \in H$–mod is equipped with a t-structure strongly compatible with the weak H-action. Suppose that for every $K \subseteq H$ compact open, $\mathcal{C}^{K,w}$ is compactly generated by objects lying in $\mathcal{C}^{K,w,+} \cap \mathcal{C}^{K,w,\leqslant 0}$.*

Then the naive weak action of H on $\mathcal{C}$ canonically renormalizes, and $\mathrm{Oblv}^{str \to w}(\mathcal{C}) \in H$–$\mathsf{mod}_{weak}$ is its canonical renormalization.

Proof. Immediate from the definition of canonical renormalization and from Proposition 10.15.1 and Proposition 10.16.1 (and Example 8.16.5 as applied to $\mathcal{A}_{ren} = \mathcal{H}_{H,K}^w$).

$\square$

10.17. Conclusion.

We now combine the various ingredients above to conclude the proof of Theorem 10.8.1.

By Lemma 10.14.1 (1), $A\#U(\mathfrak{h})$–mod_{ren} is compactly generated. Moreover, this category has a canonical compactly generated t-structure by Lemma 10.14.1 (2). The forgetful functor $A\#U(\mathfrak{h})$–$\mathsf{mod}_{ren} \to \mathsf{Vect}$ from §10.8 is t-exact and conservative on eventually coconnective objects by Lemma 10.14.1 (3). Therefore, by Remark 4.2.4, there is a corresponding connective $\overset{\to}{\otimes}$-algebra $A\#U(\mathfrak{h}) \in \mathsf{Alg}_{ren}^{\overset{\to}{\otimes}}$.

Moreover, the naive action of H on $A\#U(\mathfrak{h})$–mod_{ren}, its naive compatibility with the t-structure (Lemma 10.14.1 (3)), and the naive H-equivariance of the forgetful functor $A\#U(\mathfrak{h})$–$\mathsf{mod}_{ren} \to \mathsf{Vect}$ define a naive H-action on $A\#U(\mathfrak{h})$ compatible with its renormalization datum. Indeed, this follows from Remark 4.2.4 (c.f. the end of the proof of Theorem 10.5.1). This upgrades $A\#U(\mathfrak{h})$ to an object of $\mathsf{Alg}_{ren}^{\overset{\to}{\otimes},H\curvearrowright}$. We claim that this action is genuine.

First, we need to show that the genuine H-action on $A\#U(\mathfrak{h})$–mod_{ren} is obtained by canonical renormalization. We do this by applying Corollary 10.16.3 to $\mathcal{C} = A\#U(\mathfrak{h})$–$\mathsf{mod}_{ren}$. We check that the various conditions from that corollary are satisfied.

By construction, the naive weak H-action on $A\#U(\mathfrak{h})$–mod_{ren} upgrades to a strong action.

By Lemma 10.14.1 (4) and (5), $A\#U(\mathfrak{h})$–$\mathsf{mod}_{ren}^{K,w}$ is compactly generated, and its t-structure is as well, and compact objects are eventually coconnective.

Therefore, the corollary applies, and we find that the H-action on $A\#U(\mathfrak{h})$ is nearly genuine (in the sense of §10.4). It is genuine by Lemma 10.14.1 (5).

10.18. Harish-Chandra data.

In the remainder of this section, we suppose that H is a formally smooth polarizable ind-affine Tate group indscheme with prounipotent tail. In particular, Theorem 10.8.1 applies.

10.19.

The reader may prefer to skip this material and refer back to it as needed.

Let $\mathcal{C}$ be a category, and suppose $T : \mathcal{C} \to \mathcal{C}$ is a comonad. Let $\mathcal{F} \in \mathcal{C}$ be a fixed object. We claim that T canonically upgrades to a comonad on the overcategory $\mathcal{C}_{/\mathcal{F}}$.

For $\mathcal{G} \in \mathcal{C}_{/\mathcal{F}}$, we have the map $T(\mathcal{G}) \to \mathcal{G} \to \mathcal{F}$ where the first map is the counit for T and the second map is the structure map for $\mathcal{G}$; this makes $T(\mathcal{G})$ into an object of $\mathcal{C}_{/\mathcal{F}}$. We denote this functor by $T_{/\mathcal{F}} : \mathcal{C}_{/\mathcal{F}} \to \mathcal{C}_{/\mathcal{F}}$.

We claim $T_{/\mathcal{F}}$ has a natural comonad structure. Consider T–$\mathsf{comod}_{/\mathcal{F}}$, the category of T-comodules $\mathcal{G}$ in $\mathcal{C}$ equipped with a map $\alpha : \mathcal{G} \to \mathcal{F} \in \mathcal{C}$ (with no hypotheses on how α interacts with the

comodule structure). The forgetful functor $T\text{–comod}_{/\mathcal{F}} \to \mathcal{C}_{/\mathcal{F}}$ is obviously conservative; we claim that it is actually comonadic. Indeed, $\mathcal{C}_{/\mathcal{F}} \to \mathcal{C}$ clearly commutes with contractible limits,[94] so the claim follows from Barr-Beck. It is clear the underlying endofunctor of this comonad on $\mathcal{C}_{/\mathcal{F}}$ is given by $T_{/\mathcal{F}}$.

Note that by construction, the data of a $T_{/\mathcal{F}}$-comodule structure on $\mathcal{G} \in \mathcal{C}_{/\mathcal{F}}$ is equivalent to a T-comodule structure on the underlying object $\mathcal{G} \in \mathcal{C}$.

10.20. We apply the above for the comonad $\text{Oblv}^{str \to w} \circ (-)^{\exp(\mathfrak{h}),w} : H\text{–mod}_{weak} \to H\text{–mod}_{weak}$ and $\mathsf{Vect} \in H\text{–mod}_{weak}$. This defines a comonad on $H\text{–mod}_{weak,/\mathsf{Vect}}$.

By Theorem 10.8.1 and Lemma 10.14.1 (6), this comonad preserves the 1-full subcategory:

$$(\mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright})^{op} \overset{Thm.10.5.1}{\subseteq} (H\text{–mod}_{weak})_{/\mathsf{Vect}}.$$

This induces a monad on $\mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright}$, which we denote by $A \mapsto A\#U(\mathfrak{h})$.

We can now make the following definition.

Definition 10.20.1. A *Harish-Chandra datum* for $A \in \mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright}$ is a structure of module for the above monad.

Remark 10.20.2. By definition, a Harish-Chandra datum gives rise to an "action" map $A\#U(\mathfrak{h}) \to A$.

10.21. We now make the following observation.

Lemma 10.21.1. *The functor* $\text{Oblv}^{str \to w} : H\text{–mod} \to H\text{–mod}_{weak}$ *is comonadic.*

Proof. This functor is conservative as the composition:

$$H\text{–mod} \to H\text{–mod}_{weak} \xrightarrow{\text{Oblv}_{gen}} \mathsf{DGCat}_{cont}$$

computes the forgetful functor for $H\text{–mod}$, which is conservative.

To conclude, we simply note that as H is polarizable, $\text{Oblv}^{str \to w}$ admits a left adjoint by Proposition 8.21.1 and therefore commutes with all limits.

$\square$

Therefore, by the discussion of §10.19, a Harish-Chandra datum for $A \in \mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright}$ is equivalent to upgrading the genuine H-action on $A\text{–mod}_{ren}$ to a strong H-action with the property[95] that the coaction functor $A\text{–mod}_{ren} \to \text{Oblv}^{str \to w}(A\text{–mod}_{ren}^{\exp(\mathfrak{h}),w}) = A\#U(\mathfrak{h})\text{–mod}_{ren}$ come from a genuinely H-equivariant morphism $A\#U(\mathfrak{h}) \to A$.

More precisely, let $\mathsf{Alg}_{HC}^{\vec{\otimes},H\curvearrowright}$ denote the category of $A \in \mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright}$ equipped with a Harish-Chandra datum (i.e., the category of modules for the monad $A \mapsto A\#U(\mathfrak{h})$). Then we have:

Lemma 10.21.2. *The above functor:*

$$(\mathsf{Alg}_{HC}^{\vec{\otimes},H\curvearrowright})^{op} \xrightarrow{A \mapsto A\text{–mod}_{ren}} H\text{–mod} \underset{H\text{–mod}_{weak}}{\times} H\text{–mod}_{weak,/\mathsf{Vect}}$$

is 1-fully-faithful.

---

[94]See [Lur2] Proposition 4.4.2.9 for a complete proof.

[95]This encodes the fact that $A\text{–mod}_{ren}$ is a comodule in the 1-full subcategory $(\mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright})^{op} \subseteq (H\text{–mod}_{weak})_{/\mathsf{Vect}}$, i.e., it has to do with the fact that this is a 1-full subcategory and not an actual subcategory.

10.22. **The classical case.** Let A be a classical, renormalized $\overrightarrow{\otimes}$-algebra equipped with a genuine H-action. As in Theorem 10.8.1, we can form the smash product $A\#U(\mathfrak{h})$. We wish to explicitly describe the category $A\#U(\mathfrak{h})\text{–mod}^{\heartsuit}_{ren}(= A\#U(\mathfrak{h})\text{–mod}^{\heartsuit}_{naive})$.

Suppose V is an object of this abelian category. The canonical map $A \to A\#U(\mathfrak{h})$ (coming as the unit for the monad structure) makes V into a (discrete) $H^0(A)$-module. Also, the fact that the unit map $k \to A$ is H-equivariant gives a map $k\#U(\mathfrak{h}) = U(\mathfrak{h}) \to A\#U(\mathfrak{h})$, so V also acquires an $\mathfrak{h}$-module structure.

To describe the compatibility between these two actions, we need the following digression. Any $\xi \in \mathfrak{h}$ defines a *derivation* $\delta_\xi : A \to A$. In detail, ξ defines a homomorphism $\text{Fun}(H) \to k[\varepsilon]/\varepsilon^2$ extending the augmentation on $\text{Fun}(H)$, so we obtain a map:

$$A \xrightarrow{\text{coact}} A \overset{!}{\otimes} \text{Fun}(H) \to A \overset{!}{\otimes} k[\varepsilon]/\varepsilon^2 \in \mathsf{Alg}^{\overrightarrow{\otimes}}$$

giving id_A mod ε. If we write the underlying map of pro-vector spaces as $\text{id}_A \times \delta_\xi \varepsilon$, the map $\delta_\xi : A \to A \in \mathsf{ProVect}^{\heartsuit}$ is by definition our derivation.

Then we claim that the difference between the two maps:

$$A \overrightarrow{\otimes} V \xrightarrow{\text{act}} V \xrightarrow{(\xi \cdot -)} V,$$

$$A \overrightarrow{\otimes} V \xrightarrow{\text{id} \overrightarrow{\otimes}(\xi \cdot -)} A \overrightarrow{\otimes} V \xrightarrow{\text{act}} V \tag{10.22.1}$$

is:

$$A \overrightarrow{\otimes} V \xrightarrow{\delta_\xi \overrightarrow{\otimes} \text{id}_V} A \overrightarrow{\otimes} V \xrightarrow{\text{act}} V. \tag{10.22.2}$$

More symbolically:[96]

$$\xi \cdot f \cdot v - f \cdot \xi \cdot v = \delta_\xi(f) \cdot v, \quad \xi \in \mathfrak{h}, f \in A, v \in V.$$

Indeed, for $K \subseteq H$ compact open, we have the canonical equivalence:

$$A\#U(\mathfrak{h})\text{–mod}^{K,\heartsuit}_{ren} = A\text{–mod}^{H^{\wedge}_K,w,\heartsuit}_{ren} = A\text{–mod}^{H_K,w,naive,\heartsuit}_{naive}.$$

This verifies the above identity for V strongly K-equivariant. Every object in $A\#U(\mathfrak{h})\text{–mod}^{K,\heartsuit}_{ren}$ is a filtered colimit of objects strongly equivariant for some congruence subgroup, so we obtain the claim in general.

In addition, this same logic implies the converse. That is, we have the following result.

Proposition 10.22.1. *For $V \in \mathsf{Vect}^{\heartsuit}$, lifting V to an object of $A\#U(\mathfrak{h})\text{–mod}^{\heartsuit}_{ren}$ is equivalent via the above constructions to specifying an action of A on V and an action of $\mathfrak{h}$ on V such that the difference between the two maps in (10.22.1) is (10.22.2) for any $\xi \in \mathfrak{h}$.*

We then obtain:

---

[96]This latter formula is only sufficient when A is a *topological vector space*. By definition, this means that there is a (non-derived) topological vector space $A^{\sim}$ with a complete, separated, linear topology such that A is the associated pro-vector space, i.e., $A = \lim A^{\sim}/U \in \mathsf{ProVect}$ where U runs over open subspaces of $A^{\sim}$. In this case, it is typically sufficient to work with elements of A.

Note that $\mathfrak{h}$ is necessarily a topological vector space, justifying working directly with its elements in some of these formulae.

Corollary 10.22.2. *Suppose $B \in \mathsf{Alg}^{\vec{\otimes}}$ is classical. Then specifying a map $A\#U(\mathfrak{h}) \to B \in \mathsf{Alg}^{\vec{\otimes}}$ is equivalent to giving maps $\varphi : A \to B \in \mathsf{Alg}^{\vec{\otimes}}$ and $i : \mathfrak{h} \to B$ compatible with brackets[97] such for any $\xi \in \mathfrak{h}$, the difference between the two maps:*

$$A \xrightarrow{\varphi} B \xrightarrow{i(\xi)\cdot -} B$$
$$A \xrightarrow{\varphi} B \xrightarrow{-\cdot i(\xi)} B$$

is the composition:

$$A \xrightarrow{\delta_\xi} A \xrightarrow{\varphi} B.$$

Proof. Immediate from Proposition 3.7.1 (and [Lur3] Theorem 1.3.3.2). $\qquad\square$

10.23. In the above setting, we now show:

Lemma 10.23.1. *Suppose that for any compact open subgroup $K \subseteq H$, K is the spectrum of a countably generated ring. Then $A\#U(\mathfrak{h})$ is classical.*

Proof. By Proposition 3.7.1, it suffices to show $A\#U(\mathfrak{h})\text{--}\mathsf{mod}^+_{ren}$ is the bounded derived category of its underlying abelian category. By [Ras6] Lemma 5.4.3 and our countability assumption, it suffices to show that for any compact open subgroup $K \subseteq H$, $A\#U(\mathfrak{h})\text{--}\mathsf{mod}^{K,+}_{ren} = A\text{--}\mathsf{mod}^{H^\wedge_K,w,+}_{ren}$ is the bounded derived category of its underlying abelian category. This category admits a monadic, t-exact restriction functor to $A\text{--}\mathsf{mod}^{K,w,+}_{ren}$, and the corresponding monad is t-exact; so Lemma 9.11.2 reduces to showing that $A\text{--}\mathsf{mod}^{K,w,+}_{ren}$ is the bounded derived category of its heart. As this category is comonadic over $A\text{--}\mathsf{mod}^+_{ren}$ with t-exact comonad, that claim follows from the similar one for $A\text{--}\mathsf{mod}^+_{ren}$. $\qquad\square$

Remark 10.23.2. It seems likely that the above result is true without the countability hypothesis.

By this lemma (and Yoneda), Corollary 10.22.2 gives a complete description of $A\#U(\mathfrak{h})$. In particular, using the 1-categorical nature of our setup, we find the following result.

Corollary 10.23.3. *Under the above assumptions, a Harish-Chandra datum for A (in the sense of §10.20) is equivalent to specifying an H-equivariant map $i : \mathfrak{h} \to A$ compatible with brackets such that for $\xi \in \mathfrak{h}$, $[i(\xi), -] = \delta_\xi$ as maps $A \to A \in \mathsf{ProVect}^\heartsuit$, and such that the induced (naively H-equivariant) map:*

$$A\#U(\mathfrak{h}) \to A$$

is genuinely H-equivariant.

Remark 10.23.4. The last condition in the corollary can be difficult to check in practice. There is one simple case though: if compact objects in $A\text{--}\mathsf{mod}_{ren}$ are closed under truncations, then for any $B \in \mathsf{Alg}^{\vec{\otimes},H\curvearrowright}_{conv,gen}$, any naively H-equivariant morphism $B \to A$ is genuinely H-equivariant. Indeed, this follows from the definition of canonical renormalization and from the discussion of §4.4.

We record the following consequence of the above discussion for later reference.

---

[97]If B is a topological vector space, then in the language of [Bei], we would say i is a map of topological Lie algebras.

Corollary 10.23.5. *Suppose $A, B \in \mathsf{Alg}_{HC}^{\vec{\otimes},H\curvearrowright}$ are classical $\vec{\otimes}$-algebras equipped with genuine H-actions. Then giving a morphism $f : A \to B \in \mathsf{Alg}_{HC}^{\vec{\otimes},H\curvearrowright}$ is equivalent to giving a genuinely H-equivariant morphism $f : A \to B \in \mathsf{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright}$ such that the diagram:*

$$
\begin{array}{ccc}
 & \mathfrak{h} & \\
 \swarrow & & \searrow \\
A & \xrightarrow{\quad f \quad} & B
\end{array}
$$

commutes in $\mathsf{ProVect}^{\heartsuit}$, where the diagonal morphisms encode the Harish-Chandra data as above.

11. Application to the critical level

11.1. In this section, we prove Theorem 11.18.1, providing a large class of symmetries for Kac-Moody representations at critical level. We also show a compatibility result, Theorem 11.19.1, with our previous work [Ras6]. The arguments are straightforward applications of the methods developed in §10.

11.2. Let us describe the contents of this section in more detail.

Let G be a split reductive group over k. Let[98] $K := k((t))$.

Recall that for any Ad-invariant symmetric bilinear form κ on $\mathfrak{g}$, we have the corresponding Kac-Moody central extension $0 \to k \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t))$. As we will discuss in §11.6, κ defines a *twisted* notion of strong $G(K)$-actions; we denote the corresponding category by $G(K)$–mod_κ. The theory is barely different from the untwisted one. A basic object is $\widehat{\mathfrak{g}}_\kappa$–$\mathsf{mod} \in G(K)$–$\mathsf{mod}_\kappa$.

Let $crit := -\frac{1}{2}\kappa_\mathfrak{g}$, for $\kappa_\mathfrak{g}$ the Killing form on $\mathfrak{g}$.

Let $\mathrm{Op}_{\check{G}}$ denote the indscheme of $\check{G}$-opers on the punctured disc; we take as the definition of opers what are called *marked* opers in [BTCZ] §A. (which is a slight modification of the definition in [BD1]).

Our goal for this section is to construct a coaction of $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$ on $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod} \in G(K)$–$\mathsf{mod}_{crit}$ via the Feigin-Frenkel isomorphism. In other words, we wish to show that in a suitable sense, $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}$ is tensored over its center compatibly with the (critical level) strong $G(K)$-action on it. This result appears as Theorem 11.18.1.

11.3. **Central extensions.** We begin by generalizing some material from §8 in the presence of central extensions and twisted D-modules.

We outline the main ideas and leave the verification that certain constructions generalize to the reader.

11.4. Fix $c \in k$ once and for all; we refer to c as the *twisting parameter*. Let $\mathbb{B}\mathbb{G}_m$ denote the Zariski sheafified version of the classifying space.

Observe that $\mathsf{AffSch}_{/\mathbb{B}\mathbb{G}_m}^{cl}$ of classical affine schemes equipped with a line bundle embeds as a full subcategory of $\mathsf{Pro}(\mathsf{AffSch}_{f.t.,/\mathbb{B}\mathbb{G}_m})$, the pro-category of (classically) finite type classical affine schemes with a line bundle; this follows by standard Noetherian approximation (specifically, [Gro] Theorem 8.5.2). Then the procedure from [Ras3] §2 produces functors:

---

[98]We abuse notation in letting K denote both Laurent series and compact open subgroups of $G(K)$. This abuse should never cause confusion, and we prefer it to various alternatives.

$$D_c^* : \mathsf{PreStk}_{/\mathbb{B}\mathbb{G}_m} \to \mathsf{DGCat}_{cont}$$

$$D_c^! : \mathsf{PreStk}_{/\mathbb{B}\mathbb{G}_m}^{op} \to \mathsf{DGCat}_{cont}.$$

These functors are given by suitable Kan extensions from the finite type setup (c.f. *loc. cit.*), and for S affine, finite type, and equipped with a line bundle $\mathcal{L}$, they each assign to S the category of $(\mathcal{L}, c)$-twisted D-modules on S (as defined e.g. in [GR2] §5).

Remark 11.4.1. We generally omit the line bundle from the notation since it can usually be taken for granted.

As in §6.20, there is a canonical natural transformation:

$$\mathsf{IndCoh}^* \to D_c^* \in \mathsf{Hom}(\mathsf{IndSch}_{reas,/\mathbb{B}\mathbb{G}_m}, \mathsf{DGCat}_{cont})$$

defined by a formal extension process from the finite type case.

11.5. Because $\mathbb{B}\mathbb{G}_m$ is a commutative group, there is a canonical symmetric monoidal structure on $\mathsf{PreStk}_{/\mathbb{B}\mathbb{G}_m}$ for which the forgetful functor to PreStk is symmetric monoidal for the Cartesian monoidal product.

Explicitly, for $(S, \mathcal{L}_S)$ and $(T, \mathcal{L}_T)$ in $\mathsf{PreStk}_{/\mathbb{B}\mathbb{G}_m}$, $S \times T$ is equipped with the line bundle $\mathcal{L}_S \boxtimes \mathcal{L}_T$.

Then each of the functors D_c^* and $D_c^!$ are naturally lax symmetric monoidal for this symmetric monoidal structure, meaning that we have *external products* in either setup. Indeed, this lax symmetric monoidal structure arises from the finite type setup by Kan extension. As in Remark 6.20.1, the natural transformation $\mathsf{IndCoh}^* \to D_c^*$ canonically upgrades (via our same extension procedure) to a natural transformation of lax symmetric monoidal functors.

11.6. Now let H be a Tate group indscheme and let $\lambda : H \to \mathbb{B}\mathbb{G}_m$ be a homomorphism; equivalently, we have a central extension:

$$1 \to \mathbb{G}_m \to H' \to H \to 1.$$

We assume that there exists a compact open subgroup $K \subseteq H$ on which λ is trivial as a homomorphism. Then note that H' is also a Tate group indscheme. If K can be taken to be a polarization of H, then H' is polarizable.

We obtain the category $D_c^*(H)$ of twisted D-modules on H for the underlying line bundle defined by λ. Because (H, λ) is an algebra object in $\mathsf{PreStk}_{/\mathbb{B}\mathbb{G}_m}$ (for the symmetric monoidal structure described above), $D_c^*(H)$ is canonically a monoidal DG category, i.e., an algebra object in DGCat_{cont}.

We let $H\text{–}\mathsf{mod}_c$ denote the category of modules for $D_c^*(H)$ in DGCat_{cont}, and we refer to these as DG categories equipped with *(strong) c-twisted H-actions*.

11.7. Before proceeding, we will need the following digression on twisted invariants and coinvariants.

Suppose $\mathcal{C} \in H\text{–}\mathsf{mod}_{weak}$. As in §9.9, we have a certain full subcategory $\mathcal{C}_{(1)}^{H',w} \subseteq \mathcal{C}^{H_{Tate},w}$

Now suppose that λ factors as $H \xrightarrow{\hat{\lambda}} \mathbb{B}\mathbb{G}_a^\wedge \xrightarrow{\exp} \mathbb{B}\mathbb{G}_m$. In this case, we can define a new homomorphism:

$$\lambda^c := \exp(c \cdot \hat{\lambda}) : H \to \mathbb{B}\mathbb{G}_m.$$

We let $\mathcal{C}^{H,w,\chi_c}$ denote the corresponding category of *twisted invariants*, i.e., we take $\mathcal{C}^{H,w,\chi_c} := \mathcal{C}^{H_c',w}_{(1)}$ for H_c' the central extension defined by λ^c.

This construction can also be understood as follows. We obtain a homomorphism $\mathsf{IndCoh}^*(H) \to \mathsf{IndCoh}(\mathbb{B}\mathbb{G}_a^\wedge) \simeq \mathsf{QCoh}(\mathbb{A}^1)$ where the right hand side is equipped with its usual tensor product structure (as opposed to convolution). Our twisting parameter $c \in k$ defines a homomorphism $\mathsf{QCoh}(\mathbb{A}^1) \to \mathsf{Vect}$ (taking the fiber at[99] $-c$), so an object $\chi_c \in H\text{–mod}_{weak,naive}$. It is easy to see that this naive weak H-action canonically renormalizes, defining $\chi_c \in H\text{–mod}_{weak}$. Then by Proposition 9.9.1, $\mathcal{C}^{H,w,\chi_c} = (\mathcal{C} \otimes \chi_c)^{H,w}$.

Similarly, we have a twisted coinvariants functor $\mathcal{C}_{H,w,\chi_c}$, defined by tensoring with χ_c and taking coinvariants.

11.8.   Combining the material of §11.4-11.5 with the methods of §8, we obtain a forgetful functor:

$$\mathrm{Oblv}^{str \to w} : H\text{–mod}_c \to H\text{–mod}_{weak}.$$

This functor admits left and right adjoints, which we denote by $(-)_{\exp(\mathfrak{h}),w,\chi_c}$ and $(-)^{\exp(\mathfrak{h}),w,\chi_c}$.

There are explicit formulae for these functors, similar to §8.3. Before giving them, suppose that $K \subseteq H$ is a compact open subgroup on which λ is trivial. In this case, $\lambda|_{H_K^\wedge}$ clearly factors through $\mathbb{B}\mathbb{G}_m^\wedge \xrightarrow{\log}{\simeq} \mathbb{B}\mathbb{G}_a^\wedge$, so the discussion of §11.7 applies.

Now we have:

$$\mathcal{C}^{\exp(\mathfrak{h}),w} := \underset{\substack{K \subseteq H \text{ compact open} \\ \lambda \text{ trivial on K}}}{\mathrm{colim}} \mathcal{C}^{H_K^\wedge,w,\chi_c} \in \mathsf{DGCat}_{cont}$$

$$\mathcal{C}_{\exp(\mathfrak{h}),w} := \underset{\substack{K \subseteq H \text{ compact open} \\ \lambda \text{ trivial on K}}}{\mathrm{lim}} \mathcal{C}_{H_K^\wedge,w,\chi_c} \in \mathsf{DGCat}_{cont}.$$

under the obvious structural functors.

Proposition 8.21.1 has an immediate counterpart in this setting: simply change $(\exp(\mathfrak{h}), w)$-invariants and coinvariants in *loc. cit.* to the corresponding c-twisted versions.

11.9.   Now suppose that H is formally smooth. Let $\mathfrak{h}'$ denote the Lie algebra of H', considered as a central extension of the Tate Lie algebra $\mathfrak{h}$ by k. Let $\mathfrak{h}_c'$ denote the central extension obtained by Baer-scaling by our twisting parameter $c \in k$.

We let $\mathfrak{h}_c'\text{–mod} := \mathsf{Vect}^{\exp(\mathfrak{h}),w,c}$.

Note that the notation is abusive: this category should be understood not as modules of the abstract Tate Lie algebra $\mathfrak{h}_c'$, but as modules over the central extension, i.e., modules on which $1 \in k \subseteq \mathfrak{h}_c'$ acts by the identity (in a suitable derived sense).

An evident version of Proposition 9.13.1 applies; this shows that $\mathfrak{h}_c'\text{–mod}$ has a natural t-structure with the expected heart, and is the "renormalized" DG category of representations considered in [FG2] §23.

Remark 11.9.1. By Proposition 8.21.1 (or its appropriate version here), $\mathfrak{h}_c'\text{–mod}$ is dualizable with dual $\mathfrak{h}_{-c-Tate}'\text{–mod}$, where the notation indicates the Baer sum of the inverse central extension to $\mathfrak{h}_c'$ and $\mathfrak{h}_{-Tate}$. By Theorem 9.16.1, the pairing:

$$\mathfrak{h}_c'\text{–mod} \otimes \mathfrak{h}_{-c-Tate}'\text{–mod} \to \mathsf{Vect}$$

───────────

[99]The minus sign here matches the sign conventions of §9.

is given on eventually coconnective objects by tensoring and applying semi-infinite cohomology. Note that the equivalence $(\mathfrak{h}'_{-c-Tate}\text{–mod})^\vee \simeq \mathfrak{h}'_c\text{–mod}$ is of categories acted on by H strongly with c-twist.[100]

11.10. Critical level. We now apply the above to $H = G(K)$ the loop group. Let $G(O) \subseteq G(K)$ be the subgroup, which is a compact open subgroup.

Let $\lambda : G(K) \to \mathbb{B}\mathbb{G}_m$ be the map defining the Tate central extension of $G(K)$, as in §9. Let $c = -\frac{1}{2}$ above. Apply Theorem 9.16.1 relative to the compact open subgroup $K = G(O) \subseteq G(K)$ and [BD2] §2.7.5, the corresponding central extension is the critical level Kac-Moody extension. Note that this Tate central extension is canonically trivialized over $G(O)$.

We let $G(K)\text{–mod}_{crit}$ denote $H\text{–mod}_c$ and $\widehat{\mathfrak{g}}_{crit}\text{–mod}$ denote $\mathfrak{h}'_c\text{–mod}$ in this setting. We let $U(\widehat{\mathfrak{g}}_{crit})$ denote the *twisted* enveloping algebra, i.e., the $\overset{\rightarrow}{\otimes}$-algebra defined by Oblv : $\widehat{\mathfrak{g}}_{crit}\text{–mod}^+ \to \mathsf{Vect}$ (in [BD1], it is usually denoted $\overline{U}'(\mathfrak{g} \otimes K)$).

We let $\mathbb{V}_{crit} \in \widehat{\mathfrak{g}}_{crit}\text{–mod}^\heartsuit$ denote the vacuum representation $\mathrm{ind}^{\widehat{\mathfrak{g}}_{crit}}_{\mathfrak{g}[[t]]}(k)$. More generally, for $n \geqslant 0$, we let $\mathbb{V}_{crit,n} = \mathrm{ind}^{\widehat{\mathfrak{g}}_{crit}}_{t^n\mathfrak{g}[[t]]}(k) \in \widehat{\mathfrak{g}}_{crit}\text{–mod}^\heartsuit$. By construction, these objects compactly generated $\widehat{\mathfrak{g}}_{crit}\text{–mod}$.

Remark 11.10.1. The above clearly applies as is to define $G(K)\text{–mod}_\kappa$, the category of categories with level κ (strong) $G(K)$-actions, κ any scalar multiple of the Killing form. A simple modification allows for arbitrary levels κ (for possibly non-simple G): see [Ras6] §1.29-30.

11.11. The center. Define $\mathfrak{Z} \in \mathsf{ComAlg}(\mathsf{ProVect}^\heartsuit, \overset{!}{\otimes})$ as the *non-derived* center of $U(\widehat{\mathfrak{g}}_{crit})$.

More precisely, $\mathfrak{Z}$ is the pro-vector space corresponding to the the topological vector space $H^0(U(\widehat{\mathfrak{g}}_{crit})^{G(K)})$ for the adjoint action of $G(K)$ on $U(\widehat{\mathfrak{g}}_{crit})$;[101] the topology is the subspace topology. It is straightforward to see that the multiplication on $\mathfrak{Z}$ is commutative, and therefore its evident $\overset{\rightarrow}{\otimes}$-algebra structure extends canonically to a commutative $\overset{!}{\otimes}$-algebra structure (c.f. [Bei] §1.5).

In what follows, we treat $\mathfrak{Z}$ and $U(\mathfrak{g}_{crit})$ interchangeably as pro-vector spaces and as topological vector spaces, following our custom from §

11.12. We now recall the finer structure of $\mathfrak{Z}$.

Define $I_n \subseteq \mathfrak{Z}$ as the ideal $\mathfrak{Z} \cap U(\mathfrak{g}_{crit}) \cdot t^n\mathfrak{g}[[t]] \subseteq U(\mathfrak{g}_{crit})$. By construction of the topology of $\mathfrak{Z}$, this ideal is open in $\mathfrak{Z}$, and the ideals I_n as n varies provide a neighborhood basis of 0. We let $\mathfrak{z}_n = \mathfrak{Z}/I_n$.[102]

We let $\mathcal{Z}_n = \mathrm{Spec}(\mathfrak{z}_n)$ and define $\mathcal{Z} = \mathrm{colim}_n \mathcal{Z}_n \in \mathsf{IndSch}$ (so $\mathcal{Z} = \mathrm{Spf}(\mathfrak{Z})$).

By [BD1] §3.7.9-10, $\mathcal{Z}$ is[103] isomorphic to $\prod_{i=1}^r \mathbb{A}^1(K)$ where r is the rank of $\mathfrak{g}$. Moreover, *loc. cit.* constructs such an isomorphism such that $\mathcal{Z}_n \subseteq \mathcal{Z}$ corresponds to the closed subscheme $\prod_{i=1}^r t^{-d_i}\mathbb{A}^1(O) \subseteq \prod_{i=1}^r \mathbb{A}^1(K)$ for t our coordinate on the formal disc and $d_1, \ldots, d_r$ the degrees of the invariant polynomials on $\mathfrak{g}$.

---

[100]A priori, $(\mathfrak{h}'_{-c-Tate}\text{–mod})^\vee$ may look like it is acted on strongly with twist $c + Tate$. But since the Tate extension is by definition integral, we can canonically identify $H\text{–mod}_{c+Tate}$ with $H\text{–mod}_c$ (though not compatibly with the forgetful functors to $H\text{–mod}_{weak}$!).

[101]See [BD1] Theorem 3.7.7 (ii). In fact, $\mathfrak{Z}$ is the non-derived invariants of $U(\widehat{\mathfrak{g}}_{crit})$ with respect to *any* subgroup of $G(K)$ containing a compact open subgroup; see [BD1] 3.7.11. We will not need this fact here.

[102]The notation follows [BD1], which denotes our $\mathfrak{z}_0$ by $\mathfrak{z}$.

[103]Mildly non-canonically: there are various choices needed to obtain these coordinates.

In particular, $\mathcal{Z}$ is a reasonable indscheme. Therefore, $\mathfrak{Z}$–$\mathsf{mod}_{ren} := \mathsf{IndCoh}^*(\mathcal{Z})$ defines a renormalization datum for $\mathfrak{Z}$; the forgetful functor to Vect is $\Gamma^{\mathsf{IndCoh}}(\mathcal{Z}, -)$. (See also Example 4.3.5.) In what follows, we always consider $\mathcal{Z}$ as equipped with this renormalization datum.

Remark 11.12.1. Using the isomorphism above, one can construct an equivalence $\mathsf{QCoh}(\mathcal{Z}) \simeq \mathsf{IndCoh}^*(\mathcal{Z})$: this is the unique morphism of $\mathsf{QCoh}(\mathcal{Z})$-module categories sending $\mathcal{O}_{\mathcal{Z}}$ to the $*$-pullback of $\omega_{\prod_{i=1}^r \mathbb{A}^1(K)/\mathbb{A}^1(O)} \in \mathsf{IndCoh}(\prod_{i=1}^r \mathbb{A}^1(K)/\mathbb{A}^1(O))$ along the evident projection (using the additive structure on $\mathbb{A}^1$ to form the quotient). However, this construction is highly non-canonical and does not behave well with respect to the changes of coordinates (which are non-linear).

11.13.   Being the center of $U(\mathfrak{g}_{crit})$, $\mathfrak{Z}$ acts on $U(\mathfrak{g}_{crit})$.

More precisely, we can consider $\mathfrak{Z}$ as a commutative algebra object in the symmetric monoidal category $(\mathsf{Alg}^{\vec{\otimes}}, \overset{!}{\otimes})$; then $U(\mathfrak{g}_{crit})$ is a module for it in the usual sense of monoidal categories. For example, the action map $\mathrm{act} : \mathfrak{Z} \overset{!}{\otimes} U(\mathfrak{g}_{crit}) \to U(\mathfrak{g}_{crit}) \in \mathsf{Alg}^{\vec{\otimes}}$ sends $z \otimes \xi \mapsto z \cdot \xi$.

Lemma 11.13.1. *The morphism* act *is compatible with renormalization data, i.e., it upgrades (necessarily uniquely) to a morphism in* $\mathsf{Alg}^{\vec{\otimes}}_{ren}$.

Proof. Let:

$$\mathrm{coact}_{naive} : \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{naive} \to \mathfrak{Z} \overset{!}{\otimes} U(\mathfrak{g}_{crit})\text{–}\mathsf{mod}^+_{naive}$$

denote the standard forgetful functor.[104] Define:

$$\mathrm{coact}^L_{naive} : \mathfrak{Z} \overset{!}{\otimes} U(\mathfrak{g}_{crit})\text{–}\mathsf{mod}^+_{naive} \to \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{naive})$$

as its pro-left adjoint.

Let $n, m \geqslant 0$. We claim that:

$$\mathrm{coact}^L_{naive}(\mathfrak{z}_n \otimes \mathbb{V}_{crit,m}) = \lim_{r \geqslant n,m} \mathfrak{z}_n \underset{\mathfrak{z}_r}{\otimes} \mathbb{V}_{crit,m} \in \mathsf{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{naive}). \qquad (11.13.1)$$

Here the limit is formed in this pro-category, and the terms make sense because the action of $\mathfrak{Z}$ on $\mathbb{V}_{crit,m}$ factors through $\mathfrak{z}_m$ by definition. We emphasize that this is a derived tensor product, i.e., we view $\mathfrak{z}_n \otimes \mathbb{V}_{crit,m}$ as an $\mathfrak{z}_r$-bimodule in $\mathfrak{g}_{crit}\text{–}\mathsf{mod}_{naive}$ and then form its Hochschild homology in this category; because the kernel of $\mathfrak{z}_r \to \mathfrak{z}_n$ is generated by a regular sequence (by §11.12), this tensor product does honestly lie in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+_{naive}$. We note that this reasoning also shows that this object actually lies in $\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^c \subseteq \widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}^+$.

Indeed, there is a canonical map from the left hand side of (11.13.1) to the right hand side. It suffices to show this map is an isomorphism when evaluated on objects in the heart of the t-structure. We can explicitly calculate both sides using Koszul and Chevalley complexes, giving the claim.

Now because the objects $\mathfrak{z}_n \otimes \mathbb{V}_{crit,m}$ compactly generate $\mathfrak{Z} \overset{!}{\otimes} U(\mathfrak{g}_{crit})\text{–}\mathsf{mod}^{\leqslant 0}_{ren}$ by definition (see §4.6), the fact that $\mathrm{coact}^L_{naive}(\mathfrak{z}_n \otimes \mathbb{V}_{crit,m})$ is pro-(compact and connective) immediately implies that coact_{naive} renormalizes (i.e., is left t-exact), i.e., it gives the conclusion of the lemma.

$\square$

Therefore, we find that $U(\mathfrak{g}_{crit}) \in \mathsf{Alg}^{\vec{\otimes}}_{ren}$ is canonically a module for $\mathfrak{Z} \in \mathsf{Alg}^{\vec{\otimes}}_{ren}$.

---

[104]The notation is motivated by Remark 11.13.2.

Remark 11.13.2. Applying the symmetric monoidal functor $\mathsf{Alg}_{ren}^{\vec{\otimes},op} \xrightarrow{A \mapsto A-\mathsf{mod}_{ren}} \mathsf{DGCat}_{cont}$, we find that $\widehat{\mathfrak{g}}_{crit}-\mathsf{mod}$ is canonically a comodule for $\mathfrak{Z}-\mathsf{mod}_{ren} = \mathsf{IndCoh}^*(\mathcal{Z})$.[105]

11.14. Next, we include $G(K)$-actions.

For H an ind-affine group indscheme the category $\mathsf{Alg}^{\vec{\otimes},H\curvearrowright}$ from §5.5. Note that this category is canonically a module category for the (symmetric) monoidal category $(\mathsf{Alg}^{\vec{\otimes}}, \overset{!}{\otimes})$: this is a general feature for co/module categories in monoidal categories. The same discussion applies verbatim in the setting of renormalized $\vec{\otimes}$-algebras.

Now take $H = G(K)$. Recall that $G(K)$ has an adjoint action on $U(\widehat{\mathfrak{g}}_{crit})$ as a renormalized $\vec{\otimes}$-algebra encoding the naive, weak $G(K)$-action on $\widehat{\mathfrak{g}}_{crit}-\mathsf{mod}$.

The action of $\mathfrak{Z} \in \mathsf{ComAlg}(\mathsf{Alg}^{\vec{\otimes}}, \overset{!}{\otimes})$ on $U(\widehat{\mathfrak{g}}_{crit}) \in \mathsf{Alg}^{\vec{\otimes}}$ above clearly upgrades to an action in the symmetric monoidal category $\mathsf{Alg}^{\vec{\otimes},G(K)\curvearrowright}$. Moreover, tracing the definitions, Lemma 11.13.1 immediately implies that this action upgrades to an action of $\mathfrak{Z} \in \mathsf{ComAlg}(\mathsf{Alg}_{ren}^{\vec{\otimes}}, \overset{!}{\otimes})$ on $U(\widehat{\mathfrak{g}}_{crit}) \in \mathsf{Alg}_{ren}^{\vec{\otimes}}$.

Remark 11.14.1. In parallel to Remark 11.13.2, the above discussion implies that $\mathsf{IndCoh}^*(\mathcal{Z})$ coacts on $\widehat{\mathfrak{g}}-\mathsf{mod}_{crit} \in G(K)-\mathsf{mod}_{weak,naive}$, where the latter category is considered as a module category for $(\mathsf{DGCat}_{cont}, \otimes)$.

11.15. We now extend the discussion to the setting of genuine actions.

Suppose now that H is an ind-affine Tate group indscheme. Recall from §10.4 that we have the 1-full subcategory $\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$ of $\mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$. It is immediate from the definitions and Lemma 4.6.2 that this 1-full subcategory is closed under the $(\mathsf{Alg}_{ren}^{\vec{\otimes}}, \overset{!}{\otimes})$-action. Therefore, there is a unique action of $(\mathsf{Alg}_{ren}^{\vec{\otimes}}, \overset{!}{\otimes})$ on $\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$ compatible with the embedding into $\mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$.

We remark that the functor $\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright,op} \xrightarrow{A \mapsto A-\mathsf{mod}_{ren}} H-\mathsf{mod}_{weak}$ is $(\mathsf{Alg}_{ren}^{\vec{\otimes}}, \overset{!}{\otimes})$-linear, where $(\mathsf{Alg}_{ren}^{\vec{\otimes}}, \overset{!}{\otimes})$ acts on $H-\mathsf{mod}_{weak}$ through its canonical symmetric monoidal functor to DGCat_{cont}.

11.16. Recall from Theorem 10.8.1 that the $G(K)$-action on $U(\widehat{\mathfrak{g}}_{crit}) \in \mathsf{Alg}_{ren}^{\vec{\otimes},G(K)\curvearrowright}$ is genuine. We now have the following upgraded version of Lemma 11.13.1.

Lemma 11.16.1. *The morphism* $\mathrm{act} : \mathfrak{Z} \overset{!}{\otimes} U(\mathfrak{g}_{crit}) \to U(\mathfrak{g}_{crit}) \in \mathsf{Alg}_{ren}^{\vec{\otimes},H\curvearrowright}$ *is genuinely H-equivariant, i.e., it is a morphism in* $\mathsf{Alg}_{gen}^{\vec{\otimes},H\curvearrowright}$.

Proof. As in Remark 11.14.1, $\widehat{\mathfrak{g}}_{crit}-\mathsf{mod}$ has commuting[106] $\mathsf{IndCoh}^*(G(K))$-module and $\mathfrak{Z}-\mathsf{mod}_{ren}$-comodule structures.

For $K \subseteq G(K)$ a compact open subgroup, we need to show that the coaction functor:[107]

$$\mathrm{coact}^{K,w,naive} : \widehat{\mathfrak{g}}_{crit}-\mathsf{mod}^{K,w,naive} \to (\mathsf{IndCoh}^*(\mathcal{Z}) \otimes \widehat{\mathfrak{g}}_{crit}-\mathsf{mod})^{K,w,naive} = \mathsf{IndCoh}^*(\mathcal{Z}) \otimes \widehat{\mathfrak{g}}_{crit}-\mathsf{mod}^{K,w,naive}$$

---

[105]In geometric terms, note that diagonal pushforward equips $\mathsf{IndCoh}^*(\mathcal{Z})$ with a canonical coalgebra structure in DGCat_{cont}, using the fact that $\mathcal{Z}$ is a strict indscheme.

[106]In homotopically precise terms, we should say this category is an $(\mathsf{IndCoh}^*(G(K)), \mathsf{IndCoh}^!(\mathcal{Z}))$-bimodule, where $\mathsf{IndCoh}^!(\mathcal{Z}) := \mathsf{IndCoh}^*(\mathcal{Z})^\vee$ as usual.

[107]We can commute the weak invariants with the tensor product as $\mathsf{IndCoh}^*(\mathcal{Z})$ is dualizable.

renormalizes to a left t-exact functor:

$$\text{coact}^{K,w} : \widehat{\mathfrak{g}}_{crit}\text{–mod}^{K,w} \to \text{IndCoh}^*(\mathcal{Z}) \otimes \widehat{\mathfrak{g}}_{crit}\text{–mod}^{K,w}$$

(where the right hand side is equipped with the usual tensor product t-structure).

As restriction from genuine weak K-invariants to invariants for a small compact open subgroup above is t-exact and conservative, we can assume for simplicity that K is prounipotent and contained in $G(O)$.

Now the argument is essentially the same as in Lemma 11.13.1. Consider $\mathfrak{z}_n \boxtimes \mathbb{V}_{crit,m} \in \text{IndCoh}^*(\mathcal{Z}) \otimes \widehat{\mathfrak{g}}_{crit}\text{–mod}^{K,w,naive,+}$. Note that these objects compactly generate $\big(\text{IndCoh}^*(\mathcal{Z}) \otimes \widehat{\mathfrak{g}}_{crit}\text{–mod}^{K,w}\big)^{\leq 0}$ as K is prounipotent. Moreover, the pro-left adjoint (on eventually coconnective subcategories):

$$\text{coact}^{K,w,+,L} : (\text{IndCoh}^*(\mathcal{Z}) \otimes \widehat{\mathfrak{g}}_{crit}\text{–mod}^{K,w,naive})^+ \text{Pro}(\widehat{\mathfrak{g}}_{crit}\text{–mod}^{K,w,naive,+})$$

sends $\mathfrak{z}_n \boxtimes \mathbb{V}_{crit,m}$ to $\lim\limits_{r \geq n,m} \mathfrak{z}_n \underset{\mathfrak{z}_r}{\otimes} \mathbb{V}_{crit,m}$ (the limit being formed in the pro-category), which is again seen using Koszul/Chevalley resolutions. As each $\mathfrak{z}_n \underset{\mathfrak{z}_r}{\otimes} \mathbb{V}_{crit,m}$ is connective and compact in $\widehat{\mathfrak{g}}_{crit}\text{–mod}^{K,w}$, we obtain the claim.

$\square$

The above result implies that $\mathfrak{Z} \in \text{ComAlg}(\text{Alg}_{ren}^{\vec{\otimes}}, \overset{!}{\otimes})$ on $U(\widehat{\mathfrak{g}}_{crit}) \in \text{Alg}_{gen}^{\vec{\otimes}}$.

Remark 11.16.2. As with Remarks 11.13.2 and 11.14.1, the above discussion implies that $\text{IndCoh}^*(\mathcal{Z})$ coacts on $\widehat{\mathfrak{g}}\text{–mod}_{crit} \in G(K)\text{–mod}_{weak}$.

11.17.   We now include Harish-Chandra data to extend to strong actions.

Suppose H is a Tate group indscheme satisfying the hypotheses of Theorem 10.8.1.

Recall that in §10.20, we defined a monad $A \mapsto A \# U(\mathfrak{h})$ on $\text{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright}$. We claim that this monad canonically upgrades to a $(\text{Alg}_{conv,ren}^{\vec{\otimes}}, \overset{!}{\otimes})$-linear monad, i.e., it canonically lifts along the forgetful map:

$$\text{Alg}\Big(\text{End}_{\text{Alg}_{conv,ren}^{\vec{\otimes}}\text{–mod}}(\text{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright})\Big) \to \text{Alg}\Big(\text{End}(\text{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright})\Big) = \{\text{monads on } \text{Alg}_{conv,gen}^{\vec{\otimes},H\curvearrowright}\}.$$

Indeed, this follows immediately from the constructions and the observation that the functor $(-)^{\exp(\mathfrak{h}),w}$ is DGCat_{cont}-linear for H, as is clear from Proposition 8.21.1.

11.18.   Let $\text{Alg}_{HC,crit}^{\vec{\otimes},G(K)\curvearrowright}$ be defined as the category of $\vec{\otimes}$-algebras with genuine $G(K)$-actions and critical level Harish-Chandra data (as in §10.21). By §11.17, $\text{Alg}_{HC,crit}^{\vec{\otimes},G(K)\curvearrowright}$ is canonically a module category for $(\text{Alg}_{conv,ren}^{\vec{\otimes}}, \overset{!}{\otimes})$.

We claim that our earlier constructions upgrade to an action of $\mathfrak{Z} \in \text{ComAlg}(\text{Alg}_{conv,ren}^{\vec{\otimes}}, \overset{!}{\otimes})$ on $U(\widehat{\mathfrak{g}}_{crit}) \in \text{Alg}_{HC,crit}^{\vec{\otimes},G(K)\curvearrowright}$ (where $U(\widehat{\mathfrak{g}}_{crit})$ is equipped with its tautological critical level Harish-Chandra datum).

Indeed, as all of the $\vec{\otimes}$-algebras here are classical, by Corollary 10.23.5, this amounts to the evident commutativity of the diagram:

$$\widehat{\mathfrak{g}}_{crit}$$

$$\xi \mapsto 1 \otimes \xi \swarrow \qquad \searrow$$

$$\mathfrak{Z} \overset{!}{\otimes} U(\widehat{\mathfrak{g}}_{crit}) \xrightarrow{\quad \text{act} \quad} U(\widehat{\mathfrak{g}}_{crit}).$$

Therefore, we obtain:

Theorem 11.18.1. *There is a canonical coaction of* $\mathsf{IndCoh}^*(\mathfrak{Z})$ *on* $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod}$ *considered as an object of the (*DGCat_{cont}*-enriched category)* $G(K)$–mod_{crit}.

Recalling the Feigin-Frenkel isomorphism $\mathfrak{Z} \simeq \mathrm{Op}_{\check{G}}$ *(see [BD1] §3.7 for this form of the isomorphism), we in particular obtain a coaction of* $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$ *on* $\widehat{\mathfrak{g}}_{crit}$–$\mathsf{mod} \in G(K)$–$\mathsf{mod}_{crit}$.

11.19. **A compatibility.** We now establish a compatibility with a related result from [Ras6]. We use the notation from *loc. cit.* without further mention.

Theorem 11.19.1. *The equivalence:*

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}) \xrightarrow{\simeq} \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$$

from [Ras6] Corollary 7.8.1 canonically upgrades to an equivalence of $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$*-comodules, where the comodule structure on the left hand side comes from Theorem 11.18.1 and the comodule structure on the right hand side is the tautological one.*

Remark 11.19.2. Despite the appearance of the Langlands dual group in the statement, this is essentially notational: we are just choosing to write $\mathrm{Op}_{\check{G}}$ instead of $\mathfrak{Z}$.

Proof of Theorem 11.19.1. In [Ras6] §5, we showed that $\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})$ is compactly generated, and we equipped it with a certain canonical t-structure for which compact objects are bounded. By *loc. cit.* Corollary 7.8.1, compact objects are even closed under truncation functors in $\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})$ (this is special to critical level).

Moreover, *loc. cit.* shows that the natural functor $\Psi : \mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}) \to \mathsf{Vect}$ (induced by Drinfeld-Sokolov reduction) is t-exact, and $\Psi|_{\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})^+}$ is shown to be conservative.

By Proposition 3.7.1, there is a canonical convergent, connective $\overset{\rightarrow}{\otimes}$-algebra $\mathcal{W}_{crit}$ with $\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})^+ \xrightarrow{\simeq} \mathcal{W}_{crit}$–$\mathsf{mod}^+$. By [Ras6] §5, this algebra identifies with the usual critical level affine $\mathcal{W}$-algebra (and in particular, it is classical); this isomorphism is canonical, and has to do with the description of the functor Ψ via Drinfeld-Sokolov reduction.

Now by Theorem 11.18.1 and functoriality, $\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})$ is canonically a $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$-comodule. It follows that the functor Ψ factors through a unique morphism:

$$\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod}) \overset{\Psi^{enh}}{\dashrightarrow} \mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$$

$$\Psi \searrow \qquad \swarrow \Gamma^{\mathsf{IndCoh}}(\mathrm{Op}_{\check{G}}, -)$$

$$\mathsf{Vect}$$

with Ψ^{enh} a morphism of $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$-comodule categories. Note that Ψ^{enh} is t-exact: as compact objects in $\mathsf{Whit}(\widehat{\mathfrak{g}}_{crit}\text{–}\mathsf{mod})$ are closed under truncations, this follows as Ψ and $\Gamma^{\mathsf{IndCoh}}(\mathrm{Op}_{\check{G}}, -)$ are t-exact and conservative on eventually coconnective subcategories.

Therefore, there is a canonical map $\alpha : \mathfrak{Z} \to \mathcal{W}_{crit}$ corresponding to the functor Ψ^{enh}. By construction, α is the standard map arising by functoriality of Drinfeld-Sokolov reduction. As in [FF], α is an isomorphism. Moreover, the equivalence $\mathfrak{Z} \simeq \mathrm{Fun}(\mathrm{Op}_{\check{G}})$ from [BD1] §3.7 corresponds

to the identification $\alpha : \mathfrak{Z} \xrightarrow{\simeq} \mathcal{W}_{crit}$ and the identification $\mathcal{W}_{crit} = \mathrm{Fun}(\mathrm{Op}_{\check{G}})$ from Feigin-Frenkel duality [FF].

It follows that Ψ^{enh} is an equivalence on bounded below subcategories. As compact objects on both sides are exactly almost compact objects, Ψ^{enh} is actually an equivalence. Moreover, by the above discussion, Ψ^{enh} is canonically isomorphic to the equivalence [Ras6] Corollary 7.8.1.

As Ψ^{enh} was a morphism of $\mathsf{IndCoh}^*(\mathrm{Op}_{\check{G}})$-comodules by construction, we obtain the claim.

$\square$

References

[AB]   Maurice Auslander and David A Buchsbaum. Homological dimension in noetherian rings. *Transactions of the American Mathematical Society*, 88(1):194–206, 1958.

[ABC$^+$] Dima Arinkin, Dario Beraldo, Justin Campbell, Lin Chen, Yuchen Fu, Dennis Gaitsgory, Quoc Ho, Sergey Lysenko, Sam Raskin, Simon Riche, Nick Rozenblyum, James Tao, David Yang, and Yifei Zhao. Notes from the winter school on local geometric Langlands, 2018. Available at `https://sites.google.com/site/winterlanglands2018/notes-of-talks`.

[AG1]   Dima Arinkin and Dennis Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Mathematica*, 21(1):1–199, 2015.

[AG2]   S. Arkhipov and D. Gaitsgory. Differential operators on the loop group via chiral algebras. *Int. Math. Res. Not.*, (4):165–210, 2002.

[AG3]   S Arkhipov and D Gaitsgory. Localization and the long intertwining operator for representations of affine Kac-Moody algebras. *Preprint, http://www. math. harvard. edu/~ gaitsgde/GL/Arkh. pdf*, 2015.

[Ark1]   Sergei Arkhipov. Semiinfinite cohomology of Tate Lie algebras. *Moscow Mathematical Journal*, 2(1):35–40, 2002.

[Ark2]   Sergey Arkhipov. Algebraic construction of contragradient quasi-Verma modules in positive characteristic. In *Representation Theory of Algebraic Groups and Quantum Groups*, pages 27–68. Mathematical Society of Japan, 2004.

[Ati]   MF Atiyah. Algebraic topology and operators in Hilbert space. In *Lectures in Modern Analysis and applications I*, pages 101–121. Springer, 1969.

[BBE]   Alexander Beilinson, Spencer Bloch, and Hélène Esnault. ε-factors for Gauss-Manin determinants. *Mosc. Math. J.*, 2(3):477–532, 2002. Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

[BD1]   Sasha Beilinson and Vladimir Drinfeld. Quantization of Hitchin's integrable system and Hecke eigensheaves. Available at `http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf`.

[BD2]   Sasha Beilinson and Vladimir Drinfeld. *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[BDGH]   Mathew Bullimore, Tudor Dimofte, Davide Gaiotto, and Justin Hilburn. Boundaries, mirror symmetry, and symplectic duality in 3d $\mathcal{N} = 4$ gauge theory. *Journal of High Energy Physics*, 2016(10):108, 2016.

[Bei]   Sasha Beilinson. Remarks on topological algebras. *Moscow Math. Journ*, 8(1):1–20, 2008.

[Ber]   Dario Beraldo. Loop group actions on categories and Whittaker invariants. *arXiv preprint arXiv:1310.5127*, 2013.

[Bez]   Roman Bezrukavnikov. On two geometric realizations of an affine Hecke algebra. *Publ. Math. Inst. Hautes Études Sci.*, 123:1–67, 2016.

[BF1]   Roman Bezrukavnikov and Michael Finkelberg. Equivariant Satake category and Kostant–Whittaker reduction. *Mosc. Math. J*, 8(1):39–72, 2008.

[BF2]   Alexander Braverman and Michael Finkelberg. Coulomb branches of 3-dimensional gauge theories and related structures. In *Geometric Representation Theory and Gauge Theory*, pages 1–52. Springer, 2019.

[BFN]   Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. *Advances in Theoretical and Mathematical Physics*, 22(5):1071–1147, 2018.

[BGS]   Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel. Koszul duality patterns in representation theory. *Journal of the American Mathematical Society*, 9(2):473–527, 1996.

[BLL$^+$] Christopher Beem, Madalena Lemos, Pedro Liendo, Wolfger Peelaers, Leonardo Rastelli, and Balt C van Rees. Infinite chiral symmetry in four dimensions. *Communications in Mathematical Physics*, 336(3):1359–1433, 2015.

[Bot1]   Raoul Bott. An application of the Morse theory to the topology of Lie-groups. *Bulletin de la Société Mathématique de France*, 84:251–281, 1956.

[Bot2]   Raoul Bott. The stable homotopy of the classical groups. *Annals of Mathematics*, pages 313–337, 1959.

[BTCZ]   Roman Bezrukavnikov, Roman Travkin, Tsao-Hsien Chen, and Xinwen Zhu. Quantization of Hitchin integrable system via positive characteristic. *arXiv preprint arXiv:1603.01327*, 2016.

[BZN]   David Ben-Zvi and David Nadler. Loop spaces and connections. *Journal of Topology*, 5(2):377–430, 2012.

[CDR]   Justin Campbell, Gurbir Dhillon, and Sam Raskin. Fundamental local equivalences in quantum geometric Langlands. *arXiv preprint arXiv:1907.03204*, 2019.

[CG]   Kevin Costello and Davide Gaiotto. Vertex Operator Algebras and 3d $\mathcal{N} = 4$ gauge theories.

[CR]   Justin Campbell and Sam Raskin. Derived Satake via factorization. 2020. Forthcoming.

[CW]   Sabin Cautis and Harold Williams. Canonical bases for Coulomb branches of $4d$ $\mathcal{N} = 2$ gauge theories. Forthcoming.

[DGGH]   Tudor Dimofte, Niklas Garner, Michael Geracie, and Justin Hilburn. Mirror symmetry and line operators. *Journal of High Energy Physics*, 2020(1908.00013):1–150, 2020.

[Dhi]   Gurbir Dhillon. Semi-infinite cohomology and the linkage principle for $\mathcal{W}$-algebras. *arXiv preprint arXiv:1905.06477*, 2019.

[Die]   Jean Alexandre Dieudonné. History of algebraic geometry: an outline of the history and development of algebraic geometry. 1985.

[Dri]   Vladimir Drinfeld. Infinite-dimensional vector bundles in algebraic geometry. In *The unity of mathematics*, pages 263–304. Springer, 2006.

[Fei]   Boris Lvovich Feigin. The semi-infinite homology of Kac-Moody and Virasoro Lie algebras. *Russian Mathematical Surveys*, 39(2):155, 1984.

[FF]   Boris Feigin and Edward Frenkel. Duality in $\mathcal{W}$-algebras. *International Mathematics Research Notices*, 1991(6):75–82, 1991.

[FG1]   Edward Frenkel and Dennis Gaitsgory. Local geometric Langlands correspondence and affine Kac-Moody algebras. In *Algebraic geometry and number theory*, pages 69–260. Springer, 2006.

[FG2]   Edward Frenkel and Dennis Gaitsgory. D-modules on the affine flag variety and representations of affine Kac-Moody algebras. *Represent. Theory*, 13:470–608, 2009.

[Flo]   Andreas Floer. Morse theory for Lagrangian intersections. *Journal of differential geometry*, 28(3):513–547, 1988.

[Gai1]   Dennis Gaitsgory. Quasi-coherent sheaves on stacks. Available at `http://math.harvard.edu/~gaitsgde/GL/QCohtext.pdf`.

[Gai2]   Dennis Gaitsgory. The notion of category over an algebraic stack. *arXiv preprint math/0507192*, 2005.

[Gai3]   Dennis Gaitsgory. Whittaker categories. Available at `http://math.harvard.edu/~gaitsgde/GL/LocalWhit.pdf`. Formerly titled *Les jours et les travaux*, 2008.

[Gai4]   Dennis Gaitsgory. Generalities on DG categories. Available at `http://math.harvard.edu/~gaitsgde/GL/textDG.pdf`, 2012.

[Gai5]   Dennis Gaitsgory. Ind-coherent sheaves. *Moscow Mathematical Journal*, 13(3):399–528, 2013.

[Gai6]   Dennis Gaitsgory. Outline of the proof of the geometric Langlands conjecture for GL (2). *arXiv preprint arXiv:1302.2506*, 2013.

[Gai7]   Dennis Gaitsgory. Kac-Moody representations (Day IV, Talk 3). Available at `https://sites.google.com/site/geometriclanglands2014/notes`. Unpublished conference notes, 2014.

[Gai8]   Dennis Gaitsgory. Sheaves of categories and the notion of 1-affineness. In T. Pantev, C. Simpson, B. Toën, M. Vaquié, and G. Vezzosi, editors, *Stacks and Categories in Geometry, Topology, and Algebra:*, Contemporary Mathematics. American Mathematical Society, 2015.

[Gai9]   Dennis Gaitsgory. A conjectural extension of the Kazhdan-Lusztig equivalence. *arXiv preprint arXiv:1810.09054*, 2018.

[GK]   Dennis Gaitsgory and David Kazhdan. Representations of algebraic groups over a 2-dimensional local field. *Geometric & Functional Analysis GAFA*, 14(3):535–574, 2004.

[GR1]   Davide Gaiotto and Miroslav Rapčák. Vertex algebras at the corner. *Journal of High Energy Physics*, 2019(1):160, 2019.

[GR2]   Dennis Gaitsgory and Nick Rozenblyum. Crystals and D-modules. *Pure and Applied Mathematics Quarterly*, 10(1):57–154, 2014.

[GR3]   Dennis Gaitsgory and Nick Rozenblyum. DG indschemes. *Contemp. Math*, 610:139–251, 2014.

[GR4]   Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry*. American Mathematical Soc., 2017.

[Gro]   Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, page 361, 1967.

[GW]   Davide Gaiotto and Edward Witten. S-duality of boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory. *Advances in Theoretical and Mathematical Physics*, 13(3):721–896, 2009.

[HR]   Justin Hilburn and Sam Raskin. Tate's thesis in the de Rham setting. 2020. Forthcoming.

[Hum]   James E. Humphreys. *Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$.*

[HW]   Amihay Hanany and Edward Witten. Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics. *Nuclear Physics B*, 492(1-2):152–190, 1997.

[IS]   Kenneth Intriligator and Nathan Seiberg. Mirror symmetry in three dimensional gauge theories. *Physics Letters B*, 387(3):513–519, 1996.

[KV]   Mikhail Kapranov and Eric Vasserot. Vertex algebras and the formal loop space. *Publications mathématiques de l'IHÉS*, 100:209–269, 2004.

[Liu]   Chia-Cheng Liu. Semi-infinite cohomology and kazhdan-lusztig equivalence at positive level. *arXiv preprint arXiv:1807.01773*, 2018.

[Lur1]   Jacob Lurie. A survey of elliptic cohomology. In *Algebraic topology*, pages 219–277. Springer, 2009.

[Lur2]   Jacob Lurie. *Higher Topos Theory*, volume 189. Princeton University Press, 2009.

[Lur3]   Jacob Lurie. Higher algebra. Available at `http://math.harvard.edu/~lurie/papers/HigherAlgebra.pdf`, 2012.

[Lur4]   Jacob Lurie. Spectral algebraic geometry. *Preprint, available at www. math. harvard. edu/~ lurie/papers/SAG-rootfile. pdf*, 2016.

[Neg]   Andrei Negut. Groups acting on categories. Notes from a talk, available at `http://people.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/April6(GrpActCat).pdf`., 2010.

[Pos1]   Leonid Positselski. *Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures*, volume 70. Springer Science & Business Media, 2010.

[Pos2]   Leonid Positselski. *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence.* American Mathematical Soc., 2011.

[PS]   AN Pressley and AN Segal. *Loop groups and their representations.* Oxford University Press, Oxford, 1985.

[Ras1]   Sam Raskin. A geometric proof of the Feigin-Frenkel theorem. *Representation Theory of the American Mathematical Society*, 16(13):489–512, 2012.

[Ras2]   Sam Raskin. Chiral categories. Available at `https://web.ma.utexas.edu/users/sraskin/chiralcats.pdf`, 2015.

[Ras3]   Sam Raskin. D-modules on infinite dimensional varieties. Available at `https://web.ma.utexas.edu/users/sraskin/dmod.pdf`, 2015.

[Ras4]   Sam Raskin. On the notion of spectral decomposition in local geometric Langlands. arXiv preprint arXiv:1511.01378, 2015.

[Ras5]   Sam Raskin. Chiral principal series categories II: the factorizable Whittaker category. Available at `https://web.ma.utexas.edu/users/sraskin/cpsii.pdf`, 2016.

[Ras6]   Sam Raskin. W-algebras and Whittaker categories. Available at `https://web.ma.utexas.edu/users/sraskin/whit.pdf`, 2016.

[Ras7]   Sam Raskin. Affine Beilinson-Bernstein localization at the critical level for GL_2. 2020.

[Ras8]   Sam Raskin. Categorical Weil representations. In preparation, 2020.

[Roz]   Nick Rozenblyum. Localization of Kac-Moody representations (Day V, Talk 4). Available at `https://sites.google.com/site/geometriclanglands2014/notes`. Unpublished conference notes, 2014.

[Sch]   Peter Scholze. Lectures on condensed mathematics. *Notes available at `https://www.math.unibonn.de/people/scholze/Condensed.pdf`*, 2019.

[Seg]   Graeme Segal. Elliptic cohomology. *Séminaire Bourbaki*, 88:161–162, 1987.

[Ser1]   Jean-Pierre Serre. Faisceaux algébriques cohérents. *Annals of Mathematics*, pages 197–278, 1955.

[Ser2]   Jean-Pierre Serre. Sur la dimension homologique des anneaux et des modules noethériens. In *Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko*, volume 1956, pages 175–189, 1955.

[Vor]   Alexander A Voronov. Semi-infinite homological algebra. *Inventiones mathematicae*, 113(1):103–146, 1993.

[Wit1]   Edward Witten. Elliptic genera and quantum field theory. *Communications in Mathematical Physics*, 109(4):525–536, 1987.

[Wit2]   Edward Witten. An $SU(2)$ anomaly. In *Instantons In Gauge Theories*, pages 230–234. World Scientific, 1994.

The University of Texas at Austin, Department of Mathematics, RLM 8.100, 2515 Speedway Stop C1200, Austin, TX 78712

E-mail address: sraskin@math.utexas.edu