SMALL INTERSECTION NUMBERS IN THE CURVE GRAPH

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ABSTRACT. Let $S_{g,p}$ denote the genus g orientable surface with $p \geq 0$ punctures, and let $\omega(g,p) = 3g + p - 3 > 1$. We prove the existence of infinitely long geodesic rays $(v_0, v_1, v_2, ...)$ in the curve graph satisfying the following optimal intersection property: for any natural numbers i and k, the endpoints v_i, v_{i+k} of any length k subsegment intersect at most $f_{i,k}(\omega)$ times, where $f_{i,k}(x)$ is $O(x^{k-2})$. This answers a question of Dan Margalit.

1. Introduction

Let $S = S_{g,p}$ denote the orientable surface of genus $g \geq 0$ with $p \geq 0$ punctures and say that S has *complexity* $\omega(S) = \omega(g,p) = 3g + p - 3$. Throughout the paper we assume that $\omega(S) > 1$. The *curve graph* for S, denoted $\mathcal{C}_1(S)$, is the graph whose vertices correspond to isotopy classes of essential, non-peripheral simple closed curves on S, and whose edges join vertices that represent curves whose union is a 2-component multi-curve. (See Section 2 for definitions.) Denote distance in this graph by d_S (or simply d when the surface is clear from context). The subscript 1 denotes the fact that $\mathcal{C}_1(S)$ is the 1-skeleton of a $(3g+p-4)$-dimensional flag simplicial complex, in which the k-simplices correspond to $(k+1)$-component multi-curves. We denote by $\mathcal{C}_0$ the vertices of the graph $\mathcal{C}_1$.

The curve graph was introduced by Harvey in [Har81] and has since become a central tool to understand the mapping class group of a surface and hyperbolic structures on surfaces and 3-manifolds. In particular, the geometry of the curve graph played a significant role in proving quasi-isometric rigidity of mapping class groups [BKMM12, Ham05], the rank conjecture [BM08, Ham05], and the Ending Lamination Theorem [Min10, BCM12]. These results rely on the work of Masur and Minsky [MM99, MM00] who showed that the curve graph is hyperbolic and provided a formula to coarsely compute word length in the mapping class group using distance in curve graphs. Despite the importance of curve graph distance to these applications, explicit examples of curves that have exactly distance k, for $k \geq 4$,

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are difficult to construct (See remark 1 of [BM12]). In this note, we construct such curves that are as simple as possible in terms of their geometric intersection number on the surface (see section 2 for details).

By an argument going back to Lickorish [Lic62] and stated explicitly for closed surfaces by Hempel [Hem01] and more generally by Bowditch [Bow06], the geometric intersection number strongly controls the distance d_S. Concretely, given a pair of curves α, β on $S_{g,p}$,

$$d_S(\alpha, \beta) \leq 2\log_2(i(\alpha, \beta)) + 2.$$

A complexity-dependent version of this bound was obtained by the first author [Aou12]. For any $\lambda \in (0, 1)$, there exists $N = N(\lambda)$ such that for all S with $\omega(S) > N$, if $\alpha, \beta \in \mathcal{C}_0(S)$,

$$d_S(\alpha, \beta) \geq k \Rightarrow i(\alpha, \beta) > \omega^{\lambda(k-2)}. \tag{1.1}$$

The purpose of this note is to establish a corresponding upper bound on the minimal number of times a pair of distance k simple closed curves intersect. We show:

Theorem 1.1. *For any g, p with $\omega = \omega(g, p) > 1$, there exists an infinite geodesic ray $\gamma = (v_0, v_1, v_2, ...)$ in $\mathcal{C}_1(S_{g,p})$ such that for any $i \leq j$,*

$$i(v_i, v_j) \leq \epsilon(\epsilon B)^{2j-5}\omega^{|j-i|-2} + f_{i,j}(\omega),$$

where $f_{i,j}(x)$ is $O(x^{j-i-4})$, B is a universal constant and $\epsilon = 1$ if $g \geq 2$ and $\epsilon = 4$ otherwise.

For convenience, denote

$$i_{k,(g,p)} = \min\{i(\alpha, \beta) : \alpha, \beta \in \mathcal{C}_0(S_{g,p}),\ d_S(\alpha, \beta) = k\},$$

and write $i_{k,g}$ for $i_{k,(g,0)}$. Then Theorem 1.1 implies $i_{k,(g,p)}$ is bounded above by a polynomial function of ω with degree $k - 2$.

We remark that Theorem 1.1 was proven in response to the following question, formulated by Dan Margalit:

Question 1 (Margalit). Is it the case that for fixed k, the function $i_{k,g}$ is $O(g^{k-2})$?

By inequality 1.1, we see that for k fixed, $i_{k,g}$ is not $O(g^{k-3})$.

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2. Preliminaries

We briefly recall the definition of the curve graph for an annulus, and we review the properties of subsurface projections to this graph. See [MM00] for the general definition of subsurface projections and additional details.

First, for the surface $S = S_{g,p}$, a simple closed curve is *essential* if it does not bound a disk on S and is *non-peripheral* if it does not bound a once-punctured disk on S. As is common in the literature, we call an essential, non-peripheral simple closed curve simply a *curve*. A *multi-curve* is a collection of disjoint curves on S no two of which are isotopic. The complexity $\omega(S) = 3g + p - 3$ is equal to the maximum number of components of a multi-curve on S. Hence, our assumption that $\omega(S) > 1$ ensures that $\mathcal{C}_0(S)$ is not discrete, and in this case it is well known that $\mathcal{C}_0(S)$ is connected. Surfaces for which the complexity is less than or equal to 1 are called *sporadic*.

Since for $\omega(S) > 1$, $\mathcal{C}_1(S)$ is connected, we may define a distance d_S between vertices $\alpha, \beta \in C_0(S)$ using the standard path metric. This distance is the minimal number of edges crossed in any path between α and β in $\mathcal{C}_1(S)$. If $d_S(\alpha, \beta) > 1$, then any curve representatives of these vertices intersect, and we say that α and β *meet*. For any vertices $\alpha, \beta \in \mathcal{C}_0(S)$, we can define their *geometric intersection number* $i(\alpha, \beta)$ as the minimum number of intersections between curves that represent the isotopy classes of α and β. Any representatives of these curves that intersect minimally are said to be in *minimal position*. As is standard in the subject, we will sometimes blur the distinction between a curve and its isotopy class, but when dealing with pairs of curves we will always choose representatives that are in minimal position. If A and B are subsets of vertices, we follow [MM00] and define

$$d_S(A, B) = \operatorname{diam}_{\mathcal{C}(S)}(A \cup B).$$

For a closed annulus $Y \subset S$ whose core curve α is essential, let $\tilde{Y}$ be the cover of S corresponding to Y. Denote by $\overline{Y}$ the compactification of $\tilde{Y}$ obtained in the usual way, for example by choosing a hyperbolic metric on S. The curve graph $\mathcal{C}(Y)$ is the graph whose vertices are homotopy classes of properly embedded, simple arcs of $\overline{Y}$ with endpoints on distinct boundary components. Edges of $\mathcal{C}(Y)$ correspond to pairs of vertices that have representatives with disjoint interiors. The projection π_Y from the curve graph of S to the curve graph of Y is defined as follows: for any $\beta \in \mathcal{C}_0(S)$ first realize α and β with minimal intersection. If β is disjoint from α then $\pi_Y(\beta) = \emptyset$. Otherwise, the complete preimage of β in $\tilde{Y}$ contains arcs with well-defined endpoints on distinct components of $\partial\overline{Y}$. Define $\pi_Y(\beta) \subset \mathcal{C}(Y)$ to be this collection of arcs in $\overline{Y}$.

If α is a curve in S, we also denote by $\mathcal{C}(\alpha)$ the curve graph for the annulus Y with core curve α and we denote its path metric by d_α. Let $\pi_\alpha : \mathcal{C}(S) \setminus N_1(\alpha) \to \mathcal{C}_\alpha$ be the associated subsurface projection, where $N_1(\cdot)$ denotes the closed 1-neighborhood in $\mathcal{C}_1(S)$. From [MM00], we note that when $\omega(S) > 1$ the diameter of $\pi_\alpha(\beta)$ is ≤ 1 for any curve β that meets α. Further, π_α is coarsely 1-Lipschitz along paths in $\mathcal{C}(S) \setminus N_1(\alpha)$, i.e if $\gamma_0, \gamma_1, \ldots, \gamma_n$ is a path in $\mathcal{C}_0$ with $\pi_\alpha(\gamma_i) \neq \emptyset$ for each i, then $d_\alpha(\gamma_1, \gamma_n) \leq n+1$. Here, we are using the convention that the distance between sets in $\mathcal{C}(\alpha)$ is the diameter of their union. Also recall that if T_α denotes the Dehn twist about α then

$$d_\alpha(\gamma, T_\alpha^N(\gamma)) \geq N - 2.$$

Here, $d_\alpha(\beta, \gamma)$ is short-hand for $d_\alpha(\pi_\alpha(\beta), \pi_\alpha(\gamma))$.

As a consequence of the Lipschitz condition of the projection, note that if $\beta, \gamma \in \mathcal{C}_0(S)$ both meet α and

$$d_\alpha(\beta, \gamma) \geq d_S(\beta, \gamma) + 2,$$

then any geodesic in $\mathcal{C}(S)$ from β to γ contains a vertex adjacent to α. In fact, a much stronger result, known as the bounded geodesic image theorem, is true. This was first proven by Masur and Minsky in [MM00], but the version we state here is due to Webb and gives a uniform, computable constant [Web13]. It is stated below for general subsurfaces, although we will use it only for annuli.

Theorem 2.1 (Bounded geodesic image theorem)**.** *There is a $M \geq 0$ so that for any surface S and any geodesic g in $\mathcal{C}(S)$, if each vertex of g meets the subsurface Y then* $\mathrm{diam}(\pi_Y(g)) \leq M$*.*

We end this section with the following well known fact, see [Iva92]. Let $\alpha, \beta, \gamma \in \mathcal{C}_0(S)$, then

$$|i(\gamma, T_\alpha^N(\beta)) - N \cdot i(\alpha, \gamma) i(\alpha, \beta)| \leq i(\beta, \gamma).$$

We refer to this as the twist inequality.

In the next section, we will briefly make use of the *arc and curve graph* $\mathcal{AC}_1(S)$, a 1-complex associated to a surface with boundary or punctures where the vertices are properly embedded essential arcs (modulo isotopy rel boundary) together with $\mathcal{C}_0(S)$, and edges correspond to pairs of vertices that can be realized disjointly on the surface. Recall that a properly embedded arc a in S is *essential* if it is not homotopic into the boundary of (or a puncture of) S rel ∂a. Let $\mathcal{AC}_0(S)$ denote the vertices of $\mathcal{AC}_1(S)$.

A non-annular subsurface $\Sigma \subset S$ is called *essential* if all of its boundary components are essential curves in S, and a properly embedded arc is *essential* if it can not be homotoped into the boundary or a neighborhood of a puncture. Then there is a projection map $\pi_\Sigma^{\mathcal{AC}} : \mathcal{C}_0(S) \to \mathcal{P}(\mathcal{AC}_0(\Sigma))$, where $\mathcal{P}(\cdot)$ denotes the power set, defined as follows: for a vertex $v \in \mathcal{C}_0$, first let c be a curve that represents the isotopy class v and intersects $\partial\Sigma$ minimally. Then send v to the collection of vertices in $\mathcal{AC}_0(\Sigma)$ that represent components of the intersection of c and Σ.

3. Minimal intersecting filling curves

Curves $\alpha, \beta \in \mathcal{C}_0(S)$ *fill* S if, after choosing minimally intersecting representatives, $S \setminus (\alpha \cup \beta)$ consists of disks and once-punctured disks. It is immediate from the definition of distance in $\mathcal{C}_0(S)$ that α and β fill S if and only if $d_S(\alpha, \beta) \geq 3$. The proof of Theorem 1.1 proceeds by beginning with curves α_3, β_3 in $S_{g,n}$ that fill and have intersection number bounded linearly by $\omega(g, n)$. In all but the genus 2 case we find α_3 and β_3 whose intersection number is the minimal possible.

Lemma 3.1. *Given $S_{g,p}$ with $\omega(g,p) > 1$, the following holds:*

(1) *If $g \neq 2, 0$ and $p = 0$,*
$$i_{3,(g,p)} = 2g - 1.$$
(2) *If $g \neq 2, 0$ and $p \geq 1$,*
$$i_{3,(g,p)} = 2g + p - 2.$$
(3) *If $g = 0$ and $p \geq 6$ even,*
$$i_{3,(g,p)} = p - 2,$$
and for p odd,
$$i_{3,(g,p)} = p - 1.$$
(4) *If $g = 2$ and $p \leq 2$,*
$$i_{3,(g,p)} = 4.$$
(5) *If $g = 2$ and $p \geq 2$ even,*
$$i_{3,(g,p)} = 2g + p - 2,$$
and for $p \geq 3$ odd,
$$2g + p - 2 \leq i_{3,(g,p)} \leq 2g + p - 1.$$

Proof. Before beginning a case-by-case analysis, we first observe some general facts. Suppose that α and β are curves on $S = S_{g,p}$ that are in minimal position and fill. Let D denote the number of topological disks in $S \setminus (\alpha \cup \beta)$. Note that if $p = 0$ then $D \geq 1$ since there are no punctured disks in the complement of $\alpha \cup \beta$. Using that fact that $\alpha \cup \beta$ is a 4-valent graph whose vertices are the intersection of α and β and whose complementary regions are either disks or punctured disks, we compute that $i(\alpha, \beta) = 2g + p - 2 + D$. Hence, if $p = 0$ then $i_{3,(g,p)} \geq 2g - 1$ and if $p \geq 1$ then $i_{3,(g,p)} \geq 2g + p - 2$. This, plus the additional fact that when $g = 0$ any two curves intersect an even number of times, gives all the lower bounds appearing in $(1) - (5)$.

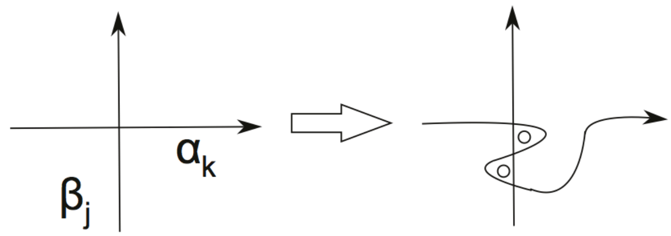

FIGURE 1. Pushing α_k across β_j and back over creates 2 bigons; puncturing each produces a filling pair on $S_{g,p+2}$.

For (1), [AH13] produce pairs of filling curves on a closed surface of genus $g \geq 3$ that intersect $2g-1$ times by an explicit construction. For (2), first note that when $p=0$ any pair of curves that fills and intersects $2g-1$ times must have a single disk as its complementary region. This follows from the Euler characteristic argument above. By puncturing this disk, we obtain a filling pair on $S_{g,1}$ that intersects $2g-1 = 2g+1-2$ times. When $g=1$, it is easy to find two curves intersecting exactly p times, for any $p \geq 1$. The complement of these two curves is p topological disks and puncturing these disks gives a pair of filling curves on $S_{1,p}$ that intersect $p = 2g+p-2$ times.

Before completing case (2), we introduce a procedure that produces a filling pair for $S_{g,p+2}$ from a filling pair for $S_{g,p}$ at the expense of two additional intersection points. Let α and β be a filling pair for $S_{g,p}$; orient α and β, and label the arcs of α (resp. β) separated by intersection points from $\alpha_1, ..., \alpha_{i(\alpha,\beta)}$ (resp. $\beta_1, ..., \beta_{i(\alpha,\beta)}$) with respect to the chosen orientation, and a choice of initial arc. Suppose that the initial point of α_k coincides with the terminal point of β_j, as seen on the left hand side of Figure 1.

Then pushing α_k across β_j and back produces a pair of bigons; puncturing each of these bigons produces a filling pair intersecting $i(\alpha,\beta)+2$ times on $S_{g,p+2}$. Thus if $p = 2k+1$ is odd and $g > 2$, by (1) there exists a filling pair whose complement is connected, and we can puncture this single complementary region to obtain a filling pair on $S_{g,1}$. Then performing the operation pictured above k times yields a filling pair on $S_{g,p}$ intersecting $2g+p-2$ times. The Euler characteristic argument above yields a lower bound of $2g+p-2$ for $i_{3,(g,p)}$, and this proves (3) in the case p is odd.

If p is even, the same argument can work if there exists a filling pair (α_g, β_g) on $S_{g,0}$ intersecting $2g$ times, which is equivalent to the complement of $\alpha \cup \beta$ consisting of two topological disks. Assuming such a filling pair exists, we obtain a filling pair on $S_{g,2}$ intersecting $2g = 2g+p-2$ times by puncturing both disks. Then the double bigon procedure described above produces the desired filling pair for any larger number of even punctures.

Therefore, to finish the proof of (2) it suffices to exhibit a filling pair on $S_{g,0}, g > 2$, intersecting $2g$ times. Consider the polygonal decomposition of $S_{2,0}$ shown in Figure 2, originally constructed in [AH13].

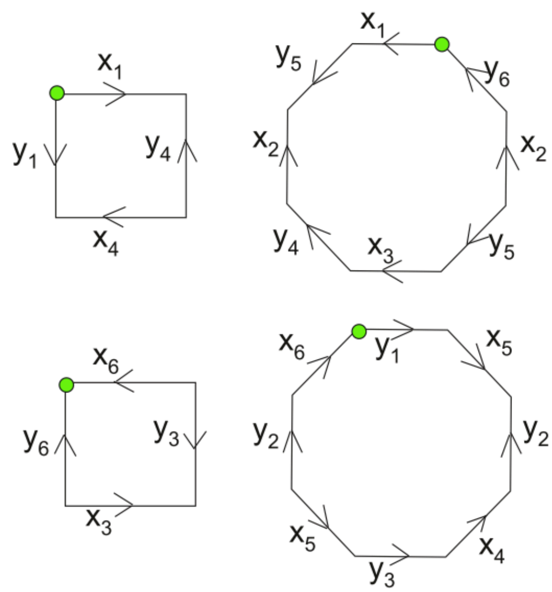

FIGURE 2. Gluing the polygons together with respect to the oriented edge labeling yields $S_{2,0}$, and the x-arcs concatenate in the quotient to form a simple closed curve x which fills $S_{2,0}$ with the curve y, the concatenation of the y-arcs. The 4 green points are all identified together in $S_{2,0}$.

The boundary of these polygons project to a filling pair (x, y) on $S_{2,0}$ intersecting 6 times. Take $S_{1,0}$, equipped with the filling pair described above intersecting twice, and cut out a small disk centered around either of these two intersection points to obtain $\tilde{S}_1$, a torus with one boundary component equipped with arcs $\tilde{\alpha}_1, \tilde{\beta}_1$.

Then given $S_{2,0}$ equipped with (x, y), cut out a small disk centered around the green intersection point above in Figure 2 to obtain $\tilde{S}_2$, a genus two surface with one boundary component equipped with arcs $\tilde{x}, \tilde{y}$. Then glue $\tilde{S}_1$ to $\tilde{S}_2$ by identifying boundary components, while concatenating the endpoints of $\tilde{\alpha}_1$ to $\tilde{x}$, and the endpoints of $\tilde{\beta}_1$ to $\tilde{y}$.

This yields a pair of simple closed curves (α_3, β_3) on $S_{3,0}$ intersecting $2g$ times, and we claim that this is a filling pair. Indeed, let γ be any simple closed curve on $S_{3,0}$ and assume γ is disjoint from both α_3 and β_3. Consider the projections $\pi^{\mathcal{AC}}_{\tilde{S}_1}(\gamma), \pi^{\mathcal{AC}}_{\tilde{S}_2}(\gamma)$ of γ to the arc and curve graph $\mathcal{AC}$ of the subsurfaces $\tilde{S}_1, \tilde{S}_2$. By assumption the arc $\pi^{\mathcal{AC}}_{\tilde{S}_2}(\gamma)$ is disjoint from the arcs $\tilde{x}, \tilde{y}$.

It then follows that this arc must be homotopic into $\partial\tilde{S}_2$, because the arcs $\tilde{x}, \tilde{y}$ are distance at least 3 in $\mathcal{AC}(\tilde{S}_2)$. Hence γ is homotopic into $\tilde{S}_1$; however, this contradicts the fact that $\tilde{\alpha}, \tilde{\beta}$ fill $\tilde{S}_1$, and therefore γ can not be disjoint from both α_3 and β_3.

Then to obtain a filling pair $(\alpha_{2k+1}, \beta_{2k+1})$ intersecting $2(2k+1)$ times on any odd genus surface, we simply iterate this procedure by choosing a filling pair intersecting $2(2(k-1)+1)$ times on $S_{2(k-1)+1,0}$, cutting out a disk centered at any intersection point, and gluing on a copy of $\tilde{S}_2$. Thus, the existence of the desired pair for any even genus follows from the same argument by the existence of such a pair on $S_{2,0}$- see Figure 6 below. This completes the proof of (2).

For (3), it suffices to give a pair of curves intersecting 4 times on each of $S_{0,5}$ and $S_{0,6}$. For then we can apply the double bigon construction to increase intersection number by two while adding two additional punctures. Filling pairs of curves on these surfaces are shown in Figure 5.

When $g = 2$, there exists a filling pair intersecting 4 times [FM12]. It is shown in [AH13] that $i_{3,(2,0)} > 3$, but we give a short argument here that was communicated to us by Dan Margalit. Recall that for $S = S_{2,0}$ there is a homeomorphism $h : S \to S$ called the hyperelliptic involution such that h^2 is the identity map, h fixes each vertex of $\mathcal{C}(S)$, and the quotient S/h is a sphere with 6 marked points, which can be treated as punctures. See [FM12] for details. If α and β are curves on S that fill and intersect 3 times, then in S/h, α and β descend to arcs a and b, respectively, whose interiors intersect at most once. It follows that each of a and b must have an endpoint on a common marked point. However, it is easy to see that no such arcs can fill a sphere with 6 marked points. We conclude that $i_{3,(2,0)} = 4$. Since two curves on $S_{2,0}$ that fill and intersect 4 times have 2 disks in their complement, puncturing one or two of these disks give filling pairs that complete case (4). Case (5) comes from a final application of the double bigon construction, starting with a minimal intersecting filling pair on either $S_{2,1}$ or $S_{2,2}$, depending on the parity of p. Only in the case where p is even does this construction achieve the lower bound calculated in the first paragraph. □

To prove the main result, we will first exhibit the existence of a length 3 geodesic segment satisfying the property that any subsegment has endpoints intersecting close to minimally for their respective curve graph distances. The main theorem is then proved by carefully extending such a segment and inducting on curve graph distance. Thus, we conclude this section with the following lemma:

Lemma 3.2. *Given $S_{g,p}$ with $\omega(g,p) > 1$, there exists a length 3 geodesic segment (v_0, v_1, v_2, v_3) in $\mathcal{C}(S_{g,p})$ such that:*

(1) *If $g \neq 2, 1$, then for any $k, j, 0 \leq k, j \leq 3$,*

$$i(v_k, v_j) = i_{|k-j|,(g,p)};$$

(2) *If $g = 1, p > 1$, then*

$$i(v_0, v_3) = i_{3,(1,p)},\ i(v_0, v_2), i(v_1, v_3) \leq 3.$$

(3) *If* $g = 2$ *and* p *is even, then the conclusion from* (1) *holds. If* p *is odd, then*

$$i(v_0, v_3) \leq i_{3,(2,p)} + 1, \ i(v_0, v_2) = i(v_1, v_3) = 1.$$

Proof. For (1), assume first that $p = 0$ and $g > 2$. Then by (1) of Lemma 3.1, there exists a filling pair (α, β) on $S_{g,p}$ whose complement consists of a single connected component. As in the proof of Lemma 3.1, orient both α and β and label the arcs along α (resp. β) $\alpha_1, ..., \alpha_{2g-1}$ (resp. $\beta_1, ..., \beta_{2g-1}$). Then cutting along $\alpha \cup \beta$ produces a single polygon P with $(8g-4)$ sides, whose edges are labeled from the set

$$A(g) := \left\{ \alpha_1^{\pm}, ..., \alpha_{2g-1}^{\pm}, \beta_1^{\pm}, ..., \beta_{2g-1}^{\pm} \right\}.$$

$(\alpha_k, \alpha_k^{-1})$ is referred to as an inverse pair; these edges project down to the same arc of α on the surface. Note that the edges of P alternate between belonging to α and β.

Consider the map $M : A(g) \to A(g)$ which sends an edge e to the inverse of the edge immediately following e along P in the clockwise direction. We claim that M has order 4. Indeed, the map M is combinatorially an order 4 rotation about an intersection point of $\alpha \cup \beta$, as pictured below.

Now, suppose that every inverse pair constitutes a pair of opposite edges of P; that is to say, the complement of any inverse pair in the edge set of P consists of two connected components with the same number of edges. Then M induces a rotation of P by $2\pi/(4g-1)$, which is not an order 4 rotation, a contradiction.

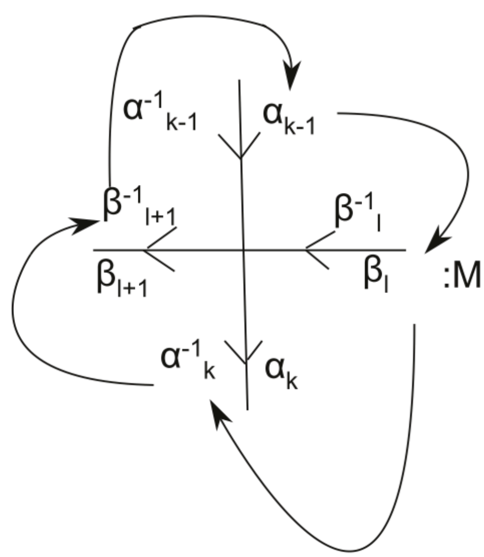

FIGURE 3. M sends the arc β_l to α_k^{-1}. The arrows demonstrate the order 4 action of M around the vertex.

Therefore, there must be at least one inverse pair comprised of edges which are not opposite on P. Without loss of generality, this pair is of the form $(\alpha_k, \alpha_k^{-1})$. Let R be the connected component of the complement of $\alpha_k \cup \alpha_k^{-1}$ in the edge set of P containing more than $4g-3$ edges.

Then there must exist an inverse pair of the form (β_j, β_j^{-1}) contained in R, since the edges of P alternate between belonging to α and β, and thus there must be a strictly larger number of β edges in R than in the other component.

Then there is an arc connecting the edges $(\alpha_k, \alpha_k^{-1})$ which projects down to a simple closed curve v_2 disjoint from β and intersecting α exactly once. Similarly, there is an arc connecting (β_j, β_j^{-1}) projecting down to a simple closed curve v_1 which is disjoint from both v_1 and α, and which intersects β exactly once. Then define $v_0 := \alpha, v_3 := \beta$; this concludes the proof of (1) in the case $p = 0$.

If $g > 2$ and $p > 0$ is odd, then the double bigon construction introduced in the proof of Lemma 3.1 can be used again here to obtain a length 3 geodesic $\{v_0, v_1, v_2, v_3\}$ in $\mathcal{C}_1(S_{g,p})$ satisfying the desired property.

However if p is even, we can not simply appeal to the construction for $p = 0$ and use the double bigon construction, because the minimally intersecting filling pair for even p constructed in Lemma 3.1 is obtained by starting with a pair intersecting 4 times on $S_{2,0}$, gluing on genus 2 pieces as in Figure 2, and applying the double bigon procedure as necessary to acommodate for more punctures. We postpone this case until the end of the proof.

When $g = 1$ and $p > 1$, note that $i_{3,g,p} = 2g + p - 2 = p$, since on $S_{1,0}$, any two simple closed curves intersecting n times have the property that the complement of their union consists of n simply connected components. Recall also that a free homotopy class containing a simple closed representative on $S_{1,0}$ is determined uniquely by a pair of coprime integers. Then starting with the $(1, 0)$ and $(1, p)$ curves on $S_{1,0}$, simply puncture each of the p complementary regions to obtain a pair of curves on $S_{1,p}$ which fill and intersect minimally.

The figure below shows that when $p \neq 3$, there exists a geodesic (v_0, v_1, v_2, v_3) satisfying the requirements of the lemma, and such that

$$i(v_0, v_2) = i(v_1, v_3) = 2.$$

If $p = 3$, a similar construction to those shown in Figure 4 exists: v_3 becomes the $(1, 3)$-curve and v_0 is still the $(1, 0)$-curve. The curve v_1 is as in the $p = 2$ case, in that it connects the vertical edges of the square and weaves between the punctures (so as to guarantee that it is not homotopic to v_0). The curve v_2 is as in both cases pictured above: it bounds a twice punctured sphere with 1 boundary component, is disjoint from v_1, and intersects v_0 twice. Note that $i(v_1, v_3) = 3$.

For $g = 0$, we have the length three geodesic in $\mathcal{C}(S_{0,5})$ and $\mathcal{C}(S_{0,6})$ as in Figure 5. Again, applying the double bigon construction to the filling

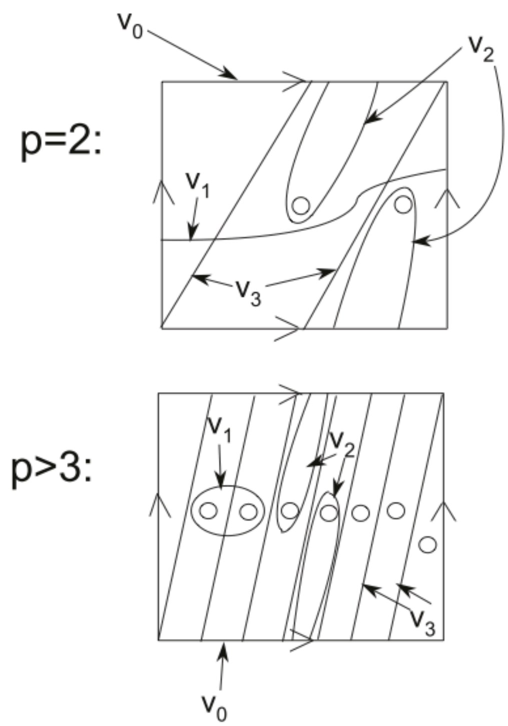

FIGURE 4. If $g = 1$ and $p \neq 3$, there exists a geodesic $\{v_0, v_1, v_2, v_3\}$ such that the end points of any length 2 subsegment intersect twice, and $i(v_0, v_3) = i_{3,(1,p)}$.

curves does not change the intersection number with the other curves in the geodesic. This completes the argument for $g = 0$.

If $g = 2$, the existence of the desired geodesic segment in $\mathcal{C}_1(S_{2,0})$ will imply the existence of the corresponding segment in $\mathcal{C}_1(S_{2,p})$ for $p > 1$ by another application of the double bigon construction. The filling pair (α, β) on $S_{2,0}$ shown on Page 41 of [FM12] is obtained by gluing together a pair of octagons in accordance with the gluing pattern pictured below in Figure 6.

Note that both α_1 and α_1^{-1} are on the left octagon, and β_2, β_2^{-1} are both edges of the right octagon. Therefore, let v_3 be a simple closed curve whose lift to the disjoint union of octagons pictured above is an arc connecting α_1 to α_1^{-1}, and let v_2 be a curve whose lift is an arc connecting β_2 to β_2^{-1}. Then $(v_0, v_1, v_2, v_3 = \beta)$ is the desired geodesic segment in $\mathcal{C}(S_{2,0})$.

Finally, if p is even and $g > 2$, observe that in the bottom octagon of Figure 6, there is an arc connecting the edges labeled x_5 which projects to a curve v_1 disjoint from the y curve, and intersecting the x curve only once. Furthermore, there is an arc in the upper octagon connecting the two y_5 edges which projects to a simple closed curve v_2 on S_2 disjoint from the x curve, and intersecting the y curve only once. In the construction

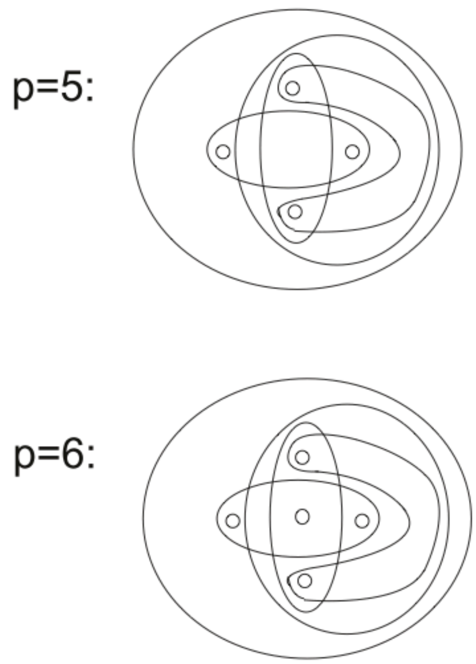

FIGURE 5. Length 3 geodesics in $\mathcal{C}_1(S_{0,5})$ and $\mathcal{C}_1(S_{0,6})$.

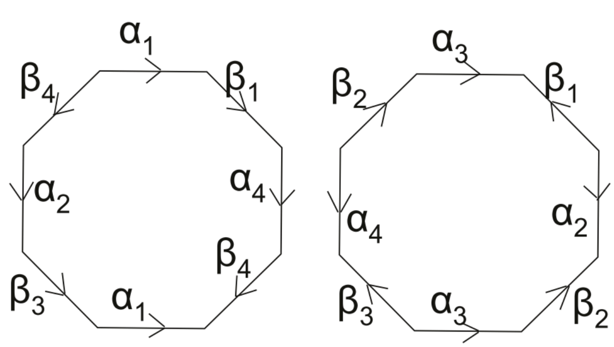

FIGURE 6. Gluing the octagons together by pairing together sides with the same label produces $S_{2,0}$; the β-arcs concatenate in order to form a simple closed curve β, which fills $S_{2,0}$ with the simple closed curve α- the concatenation of the α-arcs.

outlined in Lemma 3.1, when p is even, we glue a copy of this genus 2 piece to some $S_{g,p}$ equipped with a minimally intersecting filling pair $\{\alpha, \beta\}$, such that $S_{g,p} \setminus (\alpha \cup \beta)$ has two connected components which are not punctured bigons.

After gluing, we concatenate the complement of the green vertex in the x curve to α to obtain a simple closed curve v_0 on $S_{g+2,p}$, and similarly the complement of the green vertex in the y curve becomes a sub-arc of some curve v_3 on $S_{g+2,p}$; $S_{g+2,p} \setminus (v_0 \cup v_3)$ will still have exactly two connected components that are not punctured bigons. Note that the curves v_1, v_2 are embedded in the complement of the green vertex on S_2, and therefore we can also think of them as curves on $S_{g+2,p}$ after gluing on the genus 2 piece. The path (v_0, v_1, v_2, v_3) in $\mathcal{C}_1(S_{g,p+2})$ is the desired length 3 geodesic. □

Since the intersection numbers determined in Lemma 3.2 are the basis for our construction in the next sections, we make the following notation: if $\{v_0, v_1, v_2, v_3\}$ is the geodesic in $\mathcal{C}(S_{g,p})$ determined by Lemma 3.2, then set $\eta_{3,(g,p)} = i(v_0, v_3)$ and $\eta_{2,(g,p)} = \max\{i(v_1, v_3), i(v_0, v_2)\}$. Note that in most cases these are the minimum possible intersection numbers given their distance.

4. Warm-up

To give the idea of the general argument, we present a simplified argument to prove that $i_{4,(g,p)}$ is $O(\omega^2)$, where $\omega = \omega(g,p)$. The idea is that for small curve graph distance we can bypass the bounded geodesic image theorem using the simple fact that the projection from the curve graph to the curve graph of an annulus is coarsely 1- Lipschitz.

Begin with curves α_3 and β_3 that have distance 3 in the curve graph and intersect $i_{3,(g,p)}$ times. Let δ be a curve that has distance 2 from β_3 and distance 1 from α_3. Set β_4 equal to $T^8_{\alpha_3}(\beta_3)$ and α_4 equal to β_3. Note that $d(\alpha_4, \beta_4) \leq 4$ since each of these curves has distance 2 from $\delta = T^8_{\alpha_3}(\delta)$. If there is a geodesic from α_4 to β_4 all of whose vertices intersect α_3 then since the projection to $\mathcal{C}(\alpha_3)$ is Lipschitz

$$d_{\alpha_3}(\alpha_4, \beta_4) \leq 4 + 1 = 5.$$

This, however, contradicts our choice of Dehn twist $T^8_{\alpha_3}$ since

$$d_{\alpha_3}(\alpha_4, \beta_4) = d_{\alpha_3}(\beta_3, T^8_{\alpha_3}(\beta_3)) \geq 8 - 2 = 6.$$

We conclude that any geodesic from α_4 to β_4 must enter the one neighborhood of α_3 and so $d(\alpha_4, \beta_4) = 4$. Finally, by the twist inequality

$$i(\alpha_4, \beta_4) = i(\beta_3, T^{10}_{\alpha_3}(\beta_3)) = 10 \cdot i(\alpha_3, \beta_3)^2 = 10 \cdot (i_{3,(g,p)})^2,$$

as required.

This process can be repeated, however at each step we require a twist whose power grows linearly with curve graph distance. To avoid this, we use the bounded geodesic image theorem (Theorem 2.1).

5. Minimal intersection rays

Set $B = M + 3$, where M is as in Theorem 2.1. Fix a surface $S = S_{g,p}$ and begin with the length 3 geodesic (v_0, v_1, v_2, v_3) in $\mathcal{C}_1(S_{g,p})$ with $i(v_i, v_j) = \eta_{|j-i|,(g,p)}$ as in Lemma 3.2 and the final paragraph of Section 3. Set $\eta = \eta_{3,(g,p)}$. What's important here is the fact that $i(v_0, v_3)$ is bounded linearly in the complexity of S, while $i(v_0, v_2)$ and $i(v_1, v_3)$ are uniformly bounded, independent of complexity. From this, we construct a geodesic ray whose vertices have optimal intersection number given their distance, in the sense described in the introduction. We begin by defining a sequence of geodesics γ^k in $\mathcal{C}_1(S)$ whose lengths grow exponentially in k and have the property that all but the last vertex of γ^k is contained in γ^{k+1}. We refer to k as the *level* of γ^k.

Set $\gamma^0 = (v_0^0, v_1^0, v_2^0, v_3^0) := (v_0, v_1, v_2, v_3)$ and let $n_k = 2^k + 2$. We define $\gamma^{k+1} = (v_i^{k+1})_{i=0}^{n_{k+1}}$ from $\gamma^k = (v_i^k)_{i=0}^{n_k}$ as follows:

$$v_i^{k+1} = \begin{cases} v_i^k & 0 \le i \le n_k - 1 \\ T^B_{v^k_{n_k}}(v^k_{n_{k+1}-i}) & n_k - 1 \le i \le n_{k+1}. \end{cases}$$

Since $v^k_{n_k-1} = T^B_{v^k_{n_k}}(v^k_{n_k-1})$, γ^{k+1} is a path of adjacent vertices in $\mathcal{C}_0(S)$ of length $\ell(\gamma^{k+1}) = n_{k+1}$. For example,

$$\gamma^1 = (v_0^0,\ v_1^0,\ v_2^0 = T^B_{v_3^0}(v_2^0),\ T^B_{v_3^0}(v_1^0),\ T^B_{v_3^0}(v_0^0))$$

represents a length 4 path in $\mathcal{C}_1(S)$.

To simplify notation we note that $v_0^k = v_0^0$ for all k and so we denote this vertex of $\mathcal{C}_1(S)$ simply by v_0. Also, for each k we define $e_k := v^k_{n_k}$. Hence, the endpoints of γ^k are v_0 and e_k, and we orient γ^k from v_0 to e_k. If we denote by $\gamma^k_\circ$ the initial subsegment of γ^k containing all but the last vertex e_k of γ^k, then $\gamma^k_\circ$ is an initial subsegment of γ^{k+1} and we write

$$\gamma^{k+1} = \gamma^k_\circ \cup T^B_{e_k}(\gamma^k_\circ).$$

Lemma 5.1. *For $k \ge 0$, γ^k is a geodesic in $\mathcal{C}_1(S)$.*

Proof. For $k = 0$ this is by construction. Assume that the lemma holds for γ^k and recall that $\gamma^{k+1} = \gamma^k_\circ \cup T^B_{v^k_{n_k}}(\gamma^k_\circ))$ has length $n_{k+1} = 2^{k+1} + 2$. Note that

$$d_{e_k}(v_0, T^B_{e_k}(v_0)) \ge B - 2 > M,$$

so by Theorem 2.1 any geodesic between these vertices must pass through a 1-neighborhood of e_k. Hence,

$$d(v_0, T^B_{e_k}(v_0)) \ge 2\ell(\gamma^k) - 2 \ge 2(2^k + 2) - 2 = 2^{k+1} + 2 = \ell(\gamma^{k+1}).$$

Hence, γ^{k+1} is a geodesic. $\square$

The following theorem is our main technical result. It gives the desired intersection number, by *level*. The corollary following it removes the dependence on level.

Theorem 5.2. *For $k \geq 0$ and $\gamma^k = (v_i^k)_{i=0}^{n_k}$ the following inequality holds for all $0 \leq i \leq j \leq n_k$:*

$$i(v_i, v_j) \leq \epsilon(\epsilon B)^{2^k-1}\eta^{|j-i|-2} + f_{i,j,k}(\eta),$$

where $f_{i,j,k}(x)$ is $O(x^{|j-i|-4})$, $\epsilon = 1$ if $g \geq 2$ and $\epsilon = 4$ otherwise.

Proof. The proof is by induction on k. For $k = 0$, this holds by our choice of (v_0, v_1, v_2, v_3) using Lemma 3.2. Assume that the result holds for γ^k and let $1 \leq i \leq j \leq n_{k+1}$. If $j \leq n_k - 1$ then $i(v_i^{k+1}, v_j^{k+1}) = i(v_i^k, v_j^k)$ and the result holds by induction. Similarly, if $n_k - 1 \leq i$ then $v_i^{k+1} = T_{e_k}^B(v_m^k)$ and $v_j^{k+1} = T_{e_k}^B(v_l^k)$ for some $1 \leq m, l \leq n_k - 1$. In this case,

$$i(v_i^{k+1}, v_j^{k+1}) = i(T_{e_k}^B(v_m^k), T_{e_k}^B(v_l^k)) = i(v_m^k, v_l^k).$$

Since $|j - i| = |l - m|$, we are done by induction.

So we may assume that $1 \leq i \leq n_k - 2$ and that $v_j^{k+1} = T_{e_k}^B(v_l^k)$ for $l \leq n_k - 2$. By definition of v_j^{k+1}, we see that $l = n_{k+1} - j$. Then

$$i(v_i^{k+1}, v_j^{k+1}) = i(v_i^k, T_{e_k}^B(v_l^k))$$

and so we must bound $i(v_i^k, T_{e_k}^B(v_l^k))$. Since this expression involves only vertices of γ^k, we will drop the superscript k from the notation. By the twist inequality,

$$i(v_i, T_{e_k}^B(v_l)) \leq Bi(e_k, v_l)i(e_k, v_i) + i(v_i, v_l), \tag{5.1}$$

where e_k is v_{n_k}, the terminal vertex of γ^k. Applying the induction hypothesis to γ^k we have the following inequalities:

$$\begin{aligned} i(v_{n_k}, v_l) &\leq \epsilon(\epsilon B)^{2^k-1}\eta^{n_k-l-2} + f_{l,n_k,k}(\eta), \\ i(v_{n_k}, v_i) &\leq \epsilon(\epsilon B)^{2^k-1}\eta^{n_k-i-2} + f_{i,n_k,k}(\eta), \\ i(v_i, v_l) &\leq \epsilon(\epsilon B)^{2^k-1}\eta^{|i-l|-2} + f_{i,l,k}(\eta), \end{aligned}$$

where $f_{a,b,k}(x)$ is $O(x^{|b-a|-4})$. Plugging these inequalities into Inequality 5.1, we obtain

$$i(v_i, T_{v_{n_k}}^B(v_l)) \leq \epsilon(\epsilon B)^{2^{k+1}-1}\eta^{(2n_k-l-i-4)} + f_{i,j,k+1}(\eta),$$

where $f_{i,j,k+1}(x)$ is defined to be

$$\begin{aligned} f_{i,j,k+1}(x) := \epsilon(\epsilon B)^{2^k-1}x^{n_k-l-2}f_{i,n_k,k}(x) &+ \\ \epsilon(\epsilon B)^{2^k-1}x^{n_k-i-2}f_{l,n_k,k}(x) &+ \\ \epsilon(\epsilon B)^{2^k-1}x^{|l-i|-2} + f_{i,l,k}(x)&. \end{aligned}$$

Since

$$2n_k - l - i - 4 = 2^{k+1} - l - i = j - i - 2,$$

it only remains to show that $f_{i,j,k+1}(x)$ is $O(x^{|j-i|-4})$. To see this, we note that $f_{i,j,k+1}(x)$ is $O(h(x))$ where

$$h(x) = \max\{x^{2n_k-l-i-6}, x^{|l-i|-2}\}.$$

By our assumption, $l \leq n_k - 2 = 2^k$ so that

$$2n_k - l - i - 6 \geq 2^{k+1} - l - i - 2 \geq l - i - 2.$$

This implies that $f_{i,j,k+1}$ is $O(x^{2n_k-l-i-6})$ and observing that $2n_k-l-i-6 = j-i-4$ completes the proof.

□

Now set $\gamma = \cup_k \gamma_\circ^k$. This is an infinite geodesic ray with endpoint v_0. For convenience, relabel the vertices of γ so that $\gamma = (v_0, v_1, v_2, \ldots)$.

Corollary 5.3. *Let γ be the geodesic ray in $\mathcal{C}_1(S_{g,p})$ as described above. Then for any $i \leq j$*

$$i(v_i, v_j) \leq \epsilon(\epsilon B)^{2j-5}\eta^{|j-i|-2} + f_{i,j}(\eta),$$

where $f_{i,j}(x)$ is $O(x^{|j-i|-4})$, $\epsilon = 1$ if $g \geq 2$ and $\epsilon = 4$ otherwise.

Proof. Take k so that $2^k + 2 \leq j < 2^{k+1} + 2$. Then v_j is a vertex of γ that first appears at the $k+1$th level. That is, $v_j = v_j^{k+1} \in \gamma^{k+1}$. Then $2^{k+1} - 1 = (2^{k+1} + 4) - 5 \leq 2j - 5$. Now apply Theorem 5.2 to γ^{k+1} to conclude the proof. □

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