INTERSECTION PROPERTIES OF STABLE SUBGROUPS AND BOUNDED COHOMOLOGY

YAGO ANTOLÍN, MAHAN MJ, ALESSANDRO SISTO, AND SAMUEL J. TAYLOR

ABSTRACT. We show that a finite collection of stable subgroups of a finitely generated group has finite height, finite width and bounded packing. We then use knowledge about intersections of conjugates to characterize finite families of quasimorphisms on hyperbolically embedded subgroups that can be to simultaneously extended to the ambient group.

1. INTRODUCTION

There is a well-developed theory of convex cocompact subgroups of Kleinian groups. This has been generalized, first to mapping class groups [FM02, KL08, Ham08, MS12], and then to $\mathrm{Out}(F_n)$ [DT14, HH14, ADT16] over the last several years. Further extending these notions, a general theory of subgroup stability was introduced by Durham and Taylor [DT15].

One of the aims of this paper is to extend some well-known intersection properties of quasiconvex subgroups of hyperbolic or relatively hyperbolic groups [GMRS98, HW09] to the context of stable subgroups of finitely generated groups:

Theorem 1.1. *Let $H_1, \cdots, H_l$ be stable subgroups of a finitely generated group. Then the collection $\mathscr{H} = \{H_1, \cdots, H_l\}$ satisfies the following:*

(1) *$\mathscr{H}$ has finite height.*
(2) *$\mathscr{H}$ has finite width.*
(3) *$\mathscr{H}$ has bounded packing.*

The proof of finiteness of height follows the same line of argument as [GMRS98] and has already been used to observe finiteness of height in convex cocompact subgroups of mapping class groups and $\mathrm{Out}(F_n)$ [DM16]. For this, we use the treatment of stable subgroups initiated by Cordes–Durham [CD16], which identifies subgroup stability with a form of convex cocompactness. See Section 2.2 for details. The other two properties in Theorem 1.1 require an essentially different argument, and for this we follow ideas from Hruska–Wise [HW09]. In fact, we establish a somewhat stronger property than bounded packing: For a finite collection of pairwise close stable subsets, there exists a coarse barycenter, or equivalently, stable subsets satisfy a "coarse Helly property" (see Proposition 6.3 for a precise statement).

Combining Theorem 1.1 with existing work of a number of authors, we have the following (see [CD16] for an elaboration on these examples):

Corollary 1.2. *Let the pair (G, H) of a finitely generated group and a subgroup H satisfy one of the following:*

(1) *G is hyperbolic and H is quasiconvex in G.*
(2) *G is relatively hyperbolic and H is a finitely generated subgroup quasi-isometrically embedded in the coned off graph* [Far98].
(3) *$G = A(\Gamma)$ is a RAAG with Γ a finite graph which is not a join and H is a finitely generated subgroup quasi-isometrically embedded in the extension graph.*
(4) *$G = \text{Mod}(S)$ and H is a convex cocompact subgroup.*
(5) *$G = \text{Out}(F_n)$ and H is a convex cocompact subgroup.*
(6) *H is an hyperbolic, hyperbolically embedded subgroup of G.*

Then H has finite height, finite width and bounded packing.

Proof. Item (1) is the content of [GMRS98] and [HW09]. The other items follow from Theorem 1.1 along with work by various authors establishing that *H* is stable in *G*. For (2) this is due to [ADT16], for (3) this is due to [KMT14], for (4) this is due to [DT15], for (5) this is again [ADT16], and for (6) this is in [Sis16] (see Remark 2.11). □

We also give a couple of straightforward applications. Using work in [Sag95], [Sag97], [HW09], we have the following.

Corollary 1.3. *(See Corollary 6.2.) Suppose H is a finitely generated stable codimension 1 subgroup of a finitely generated group G. Then the corresponding CAT(0) cube complex is finite dimensional.*

Similarly, using work in [Sch97], [Mj08], we prove the following:

Proposition 1.4. *(See Proposition 6.7.) Let G_1, G_2 be finitely generated groups with Cayley graphs Γ_1, Γ_2, word metrics d_1, d_2, and stable subgroups H_1, H_2. Let Λ_1, Λ_2 be the limit sets of H_1, H_2 in the Morse boundaries $\partial_M G_1, \partial_M G_2$ respectively. Let $\mathscr{J}_i$, $i = 1, 2$, be the collection of translates of H_{wi}, the weak hulls of Λ_i in Γ_i. Let $\phi : \mathscr{J}_1 \to \mathscr{J}_2$ be uniformly proper. Then there exists a quasi-isometry q from Γ_1 to Γ_2 which pairs the sets $\mathscr{J}_1$ and $\mathscr{J}_2$ as ϕ does (in particular, Γ_1, Γ_2 are quasi-isometric).*

Applications to bounded cohomology. We study the case of hyperbolically embedded subgroups in more detail, with the goal of obtaining the applications to bounded cohomology explained below.

Hyperbolically embedded subgroups were introduced by Dahmani, Guirardel and Osin [DGO17] as a generalization of parabolic subgroups of relatively hyperbolic group. They are very useful to study acylindrically hyperbolic groups, and in fact they are essential, for example, in the proof of SQ-universality [DGO17], as well as in the study of bounded cohomology [HO13, FPS15, HS17]. In the specific case when *G* is hyperbolic, a subgroup *H* is hyperbolically embedded if and only if *H* is quasiconvex and almost malnormal. This extra structure allows to obtain stronger structural properties for the intersections. For example, in Proposition 5.1, we show that if H_1 and H_2 are hyperbolically embedded in (G, X), with the Cayley graph $\Gamma(G, X)$ hyperbolic, then the collection of essentially distinct intersections of H_1 with G-conjugates of H_2, is hyperbolically embedded in (G, X).

Our main application of intersections of hyperbolically embedded subgroups is related to the bounded cohomology of acylindrically hyperbolic groups via quasimorphisms. We briefly recall this relation, see [Fri16] for details.

There are natural maps from $H_b^\bullet(G,\mathbb{R})$, the **bounded cohomology groups** of G with trivial coefficients, to $H^\bullet(G,\mathbb{R})$, the standard cohomology groups of G. The kernel of theses maps is denoted by $EH_b^\bullet(G,\mathbb{R})$, and it is called the **exact bounded cohomology** of G with trivial coefficients. Thus, the problem of understanding the bounded cohomology of a group can be reduced to the problem of understanding the exact bounded cohomology. In degree 2, there is an explicit description of $EH_b^2(G,\mathbb{R})$ in terms of quasimorphisms.

Recall that a **quasimorphism** of G is a function $f\colon G \to \mathbb{R}$ satisfying the property that there is some $D \geq 0$ such that $|f(xy) - f(x) - f(y)| \leq D$ for all $x,y \in G$. A quasimorphism is **homogeneous** if $f(x^n) = nf(x)$ for all $n \in \mathbb{N}$ and all $x \in G$. Denote by $Q^h(G,\mathbb{R})$ the space of homogeneous quasimorphisms. One has that

$$Q^h(G,\mathbb{R})/Hom(G,\mathbb{R}) \cong EH_b^2(G,\mathbb{R}).$$

Brooks [Bro81] develop a combinatorial construction that gives an infinite dimensional family of quasimorphims for free groups. This construction can be viewed as extending a quasimorphism from an infinite cyclic subgroup to the whole group. There have been many generalizations of this construction, see e.g. [EF97, Fuj98, BF02, Fuj00], the most general of which being in the context of hyperbolically embedded subgroups (see [HO13, FPS15]). In this paper, we improve this construction to extend quasimorphisms from different hyperbolically embedded subgroups simultaneously.

Let $H_1,\ldots,H_l$ be subgroups of a group G. Let q_i be a homogeneous quasimorphism on H_i. We say that $\{q_i\}$ is **intersection-compatible** if whenever $x \in H_i$ is conjugate to $y \in H_j$ we have $q_i(x) = q_j(y)$. It is very easy to see that if the q_i have a common extension, then $\{q_i\}$ is intersection-compatible.

Theorem 1.5. *(See Theorem 5.5). Let G be a group with generating set X and assume that $\Gamma(G,X)$ is hyperbolic. Let $H_1,\ldots,H_l$ be hyperbolically embedded in (G,X), and let q_i be a homogeneous quasimorphism on H_i. Then there exists a homogeneous quasimorphism q on G so that $q|_{H_i} = q_i$ if and only if $\{q_i\}$ is intersection-compatible.*

As an application of this, we show that not all the exact bounded cohomology of G in degree 2 can be recovered from a given finite family of hyperbolically embedded subgroups (Corollary 5.9), despite the fact that it can be recovered from the collection of all virtually free hyperbolically embedded subgroups by [HS17].

2. PRELIMINARIES

2.1. Subgroup stability. We recall some material from [DT15, Cor15].

Definition 2.1. *Let X be a geodesic metric space and $f\colon \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function. A quasigeodesic γ in X is f–**stable** if for any (K,ε)-quasigeodesic η with endpoints on γ, we have $\eta \subset N_{f(K,\varepsilon)}(\gamma)$, the $f(K,\varepsilon)$-neighborhood of γ.*
*The function f is called the **stability function** of γ.*

Definition 2.2. *Let $\Phi\colon X \to Y$ be a quasi-isometric embedding between geodesic metric spaces. $\Phi(X)$ is called a **stable subspace** of Y and Φ is called a **stable embedding** if there exists a stability function f such that the image of any geodesic in X is an f-stable quasigeodesic in Y.*
*If $H \leq G$ are finitely generated groups, H is called a **stable subgroup** of G if the inclusion map $i\colon H \to G$ is a stable embedding.*

Note that if H is a stable subgroup of G, then the coset gH is also stable in G with the same stability function. We will need the following basic lemma, which follows from the definitions.

Lemma 2.3. *Suppose that H is a stable subgroup of a finitely generated group G. For any $D \geq 0$ there exists $R \geq 0$ such that if γ is a geodesic in G whose endpoints have distance D from H then*

$$\gamma \subset N_R(H).$$

Moreover, such a geodesic is stable with stability function depending only on D and that of H.

We conclude this section by recalling a result of Cordes:

Lemma 2.4 ([Cor15, Lemma 2.2 and 2.3])**.** *Suppose that X is a geodesic metric space and let α, β, and γ be the sides of a geodesic triangle Δ in X. If α and β are f–stable, then Δ is δ–thin and γ is f'–stable, where $\delta \geq 0$ and f' depend only on f.*

2.2. Boundary convex cocompactness.

The **Morse boundary** $\partial_M X$ of a geodesic metric space X was defined in [Cor15, CH16]. We shall not need the exact definition. Suffice to say that one can think of it as consisting of asymptote classes of sequences of points which can be connected to a fixed basepoint $o \in X$ by stable geodesics. Let H be a finitely generated group acting by isometries on a proper geodesic metric space X. The **limit set** $\Lambda_H \subset \partial_M X$ is the set of points which can be represented by sequences in $H \cdot o$. It is proved in [CD16] that Λ_H is independent of the base-point o. The **weak hull** $H_w(H)$ of Λ_H in X is the union of all geodesics with distinct endpoints in Λ_H.

Definition 2.5. [CD16] *H acts boundary convex cocompactly on X if*

  (1) *H acts properly on X;*
  (2) *For some (any) $o \in X$, Λ_H is nonempty and compact;*
  (3) *For some (any) $o \in X$, the action of H on $H_w(H)$ is cocompact.*

*Let G be a finitely generated group and $H \subset G$ a subgroup. H is **boundary convex cocompact** in G if H acts boundary convex cocompactly on any Cayley graph of G with respect to a finite generating set.*

The following theorem was proven by Cordes and Durham:

Theorem 2.6. [CD16] *Let G be a finitely generated group and H a subgroup. Then H is stable in G if and only if it is boundary convex cocompact.*

In this case, H is hyperbolic and the inclusion map $i : H \to G$ extends continuously and H-equivariantly to an embedding of the Gromov boundary ∂H into $\partial_M G$ which is a homeomorphism onto its limit set Λ_H.

2.3. Height, Width and Bounded Packing.

For a subgroup $H \leq G$, we write $H^g = gHg^{-1}$.

Definition 2.7. [GMRS98] *Let G be a group and H a subgroup.*

  - *Conjugates $H^{g_1}, \cdots, H^{g_k}$ are **essentially distinct** if the cosets $g_1 H, \cdots, g_k H$ are distinct.*
  - *H has height at most n in G if the intersection of any $(n+1)$ essentially distinct conjugates is finite. The least n for which this is satisfied is called the **height** of H in G.*
  - *The **width** of H is the maximal cardinality of the set $\{g_i H : |H^{g_i} \cap H^{g_j}| = \infty\}$, where $\{g_i H\}$ ranges over all collections of distinct cosets.*

*Similarly, given a finite collection $\mathscr{H} = \{H_1, \cdots, H_l\}$, conjugates $H_{\sigma(1)}^{g_1}, \cdots, H_{\sigma(k)}^{g_k}$ are **essentially distinct** if the cosets $g_1 H_{\sigma(1)}, \cdots, g_k H_{\sigma(k)}$ are distinct. The finite collection $\mathscr{H}$ of subgroups of G has height at most n if the intersection of any $(n+1)$ essentially distinct conjugates is finite. The*

width of H is the maximal cardinality of the set $\{g_{\sigma(i)}H : |H_{\sigma(i)}^{g_i} \cap H_{\sigma(j)}^{g_j}| = \infty\}$, *where* $\{g_{\sigma(i)}H\}$ *ranges over all collections of distinct cosets.*

Note that height (resp. width) of a finite family $\mathscr{H}$ is zero if and only if all the subgroups of $\mathscr{H}$ are finite.

A geometric analog of height was defined by Hruska and Wise.

Definition 2.8. [HW09] *Let G be a finitely generated group and* Γ *a Cayley graph with respect to a finite generating set. A subgroup H has* **bounded packing** *in G if, for all* $D \geq 0$, *there exist* $N \in \mathbb{N}$ *such that for any collection of N distinct cosets gH in G, at least two are separated by a distance of at least D.*

Similarly, a finite collection $\mathscr{H} = \{H_1, \cdots, H_l\}$ *of subgroups of G has* **bounded packing** *if, for all* $D \geq 0$, *there exist* $N \in \mathbb{N}$ *such that for any collection of N distinct cosets* $gH_i : H_i \in \mathscr{H}$, *at least two are separated by a distance of at least D.*

Equivalently, $\mathscr{H}$ has bounded packing in G if for each D there exists $N = N(D)$ bounding the cardinality of collections $\{g_1 H_{\sigma(1)}, \cdots, g_r H_{\sigma(r)}\}$ of pairwise $D-$close distinct left cosets of elements of $\mathscr{H}$. Here, closeness is with respect to minimum distance between cosets.

2.4. **Hyperbolically embedded subgroups.** Hyperbolically embedded subgroups generalize the concept of parabolic subgroups of relatively hyperbolic groups (Lemma 2.10) and every hyperbolic hyperbolically embedded subgroup is stable (Remark 2.11).

Recall also that, by a theorem of Osin [Osi16], a group is **acylindrically hyperbolic** if it contains a non-degenerate hyperbolically embedded subgroup.

We do not give the general definition of hyperbolically embedded subgroup, and we refer the reader to [DGO17, Def. 4.25]. Instead, we restrict to the case of hyperbolically embedded subgroup which are hyperbolic, the one of interest in this paper, and give the characterization that will be most useful for our purposes. The following was proved in [AMS16, Theorem 3.9] and independently proved by Hull in [Hul16, Thm. 4.13].

Theorem 2.9. *Suppose that G is a group and X is a possibly infinite generating set so that* $\mathrm{Cay}(G,X)$ *is hyperbolic.*

A family $\{Q_i\}_{i=1}^n$ *of subgroups of G is hyperbolically embedded in* (G,X), *denoted* $\{Q_i\}_{i=1}^n \hookrightarrow_h (G,X)$, *if and only if the following hold:*

(Q1) *(* $\{Q_i,\}_{i=1}^n$ *is geometrically separated) For every* $\varepsilon > 0$ *there exists* $R = R(\varepsilon)$ *such that for* $g \in G$ *if*

$$\mathrm{diam}(Q_i \cap N_\varepsilon(gQ_j)) \geq R$$

then $i = j$ *and* $g \in Q_i$ *(here the distances are measured with respect to the graph metric on* $\mathrm{Cay}(G,X)$).

(Q2) *(Finite generation) For each i, there exists a finite subset* $Y_i \subset G$ *generating* Q_i.

(Q3) *(Quasi-isometrically embedded) There exist* $\mu \geq 1$ *and* $c \geq 0$ *such that for any* $i \in \{1, \ldots, n\}$ *and all* $h \in Q_i$ *one has* $|h|_{Y_i} \leq \mu |h|_X + c$.

We write $\{Q_i\}_{i=1}^n \hookrightarrow_h G$ if there is some set X such that $\{Q_i\}_{i=1}^n \hookrightarrow_h (G,X)$.

Hyperbolically embedded subgroups generalize peripheral subgroups of relatively hyperbolic groups in the following sense:

Lemma 2.10. [DGO17, Proposition 4.28] *A group G is relatively hyperbolic with respect to* $\{H_\lambda\}_{\lambda \in \Lambda}$ *if and only if for any finite relative generating set X,* $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G,X)$.

Remark 2.11. *It was proved in [Sis16, Theorem 2] that if H is a finitely generated subgroup of G such that $H \hookrightarrow_h G$, then H is quasiconvex. That is, for every $\mu \geq 1$ and $c \geq 0$, there exists C such that all (μ,c)-quasigeodesics in G whose endpoints are in H lie in $N_C(H)$. So, H is stable if (and only if) H is hyperbolic.*

On most of our statements we will assume that $H \hookrightarrow_h (G,X)$ where $\Gamma(G,X)$ is hyperbolic. It follows from Theorem 2.9 that in this case, H is hyperbolic and, in particular, stable in G.

For later purposes we record a property of cosets of hyperbolically embedded subgroups, encoded in the following definition.

Definition 2.12. *Let G be a group and X a generating set of G. A subset H of G satisfies the WPD condition in (G,X) if for every $\varepsilon > 0$, there is $N > 0$ such that for every $h_1, h_2 \in H$ satisfying $d_X(h_1, h_2) > N$, we have*

$$|\{g \in G \mid d_X(h_1, gh_1) < \varepsilon, d_X(h_2, gh_2) < \varepsilon\}| < \infty.$$

The following is proved in [Sis16, Lemma 3.2], and in the case where (G,X) is hyperbolic, it can be deduced easily from Theorem 2.9.

Lemma 2.13. *Suppose that $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G,X)$. Then each H_λ is WPD in (G,X).*

3. Limit Sets and Height

In this section, we show that some standard arguments in the theory of Kleinian groups go through in the more general context of stable subgroups. A proof of the following lemma is sketched in [Gro93] for quasiconvex subgroups of hyperbolic groups.

Lemma 3.1 (Intersection of limit sets). *Let H_1, H_2 be stable subgroups of a finitely generated group G with limit sets Λ_1, Λ_2 in $\partial_M G$. Let Λ_3 be the limit set of $H_1 \cap H_2$. Then*

(1) *$H_1 \cap H_2$ is stable, and*
(2) *$\Lambda_3 = \Lambda_1 \cap \Lambda_2$.*

Proof. Let Γ be a Cayley graph of G with respect to a finite generating set. The first statement follows immediately from a result of Short [Sho91] which states that the intersections of quasiconvex subgroups of an arbitrary finitely generated group is quasiconvex.

Since $H_1 \cap H_2 \subset H_1$, $\Lambda_3 \subset \Lambda_1$. Similarly, $\Lambda_3 \subset \Lambda_2$. Hence $\Lambda_3 \subset \Lambda_1 \cap \Lambda_2$.

To prove the opposite inclusion, let $p \in \Lambda_1 \cap \Lambda_2$. We want to show that $p \in \Lambda_3$. Since H_1, H_2 are stable, there exist (uniform) quasigeodesics $\gamma_i \subset H_i$, $i = 1, 2$ converging to p. Without loss of generality, we may assume that γ_1, γ_2 both start at $1 \in \Gamma$. By stability, γ_1, γ_2 lie in a uniformly bounded neighborhood of each other in Γ [Cor15, Corollary 2.6]. In particular, there exists $g \in G$, and infinitely many $g_i \in \gamma_1$, $h_i \in \gamma_2$, such that $g_i g = h_i$ for all i. Hence $g = g_i^{-1} h_i$ for all i and so $g_i g_1^{-1} = h_i h_1^{-1} \in H_1 \cap H_2$ for all i, thus producing an infinite sequence of elements in $H_1 \cap H_2$ converging to p. □

The proof of the following proposition is a reprise of the elementary argument in [GMRS98] using limit sets in the Morse boundary.

Proposition 3.2 (Finite height). *Let $\mathscr{H} = \{H_1, \cdots, H_l\}$ be a finite collection of stable subgroups of a finitely generated group G. Then $\mathscr{H}$ has finite height.*

Proof. As before, let Γ be the Cayley graph of G with respect to a finite generating set. Let Λ_i denote the limit set of H_i. By Lemma 3.1, the limit set of $\cap_j H_{\sigma(j)}^{g_j}$ is the intersection of the limit

sets of $H_{\sigma(j)}^{g_j}$, which in turn is the intersection $\Lambda_\cap$ of the limit sets $g_j\Lambda_{\sigma(j)}$ of $g_jH_{\sigma(j)}$. Supposing that $\cap_j H_{\sigma(j)}^{g_j}$ is infinite, $\Lambda_\cap$ consists of at least two points.

If γ is a geodesic in the weak hull of $\Lambda_\cap$, then $g_j^{-1}\gamma$ is in the weak hull of $H_{\sigma(j)}$ and so by Theorem 2.6, $g_j^{-1}\gamma \subset N_D(H_{\sigma(j)})$, for some D depending only on $\mathscr{H}$. Hence, if x is any point along γ then $N_D(x)$ meets each $g_jH_{\sigma(j)}$. The number of cosets of any H_i is therefore bounded by the cardinality of the ball $N_D(x)$, which is finite as G is finitely generated and the Cayley graph Γ is taken with respect to a finite generating set. The conclusion follows.   □

Also, Theorem 2.6 immediately gives the following:

Proposition 3.3. *Let $H \subset G$ be stable. Then H is of finite index in the stabilizer of Λ_H and hence in the commensurator $Comm_G(H)$ of H in G.*

Proof. The stabilizer $Stab(\Lambda_H)$ stabilizes the weak hull $H_w(H)$ and acts properly on it. Since H acts cocompactly on $H_w(H)$ by Theorem 2.6 and $H \subset Stab(\Lambda_H)$, it follows that if K is a compact (weak) fundamental domain for H in $H_w(H)$ (i.e. a compact subset of $H_w(H)$, whose $H-$translates cover $H_w(H)$) there are at most finitely many elements g of $Stab(\Lambda_H)$ such that $gK \cap K \neq \emptyset$. It follows that $[Stab(\Lambda_H) : H] < \infty$. Since $Comm_G(H) \subset Stab(\Lambda_H)$, the last assertion follows.   □

4. WIDTH AND PACKING

To prove finiteness of width and bounded packing, we cannot reprise the argument in [GMRS98] which uses compactness of the boundary and global hyperbolicity in an essential way. In [HW09], Hruska and Wise give a new proof of finiteness of width for quasiconvex subgroups of hyperbolic groups. Here, we adapt their argument to stable subgroups of arbitrary finitely generated groups.

4.1. **Finiteness of width and bounded packing.** We begin by proving bounded packing for a single stable subgroup.

Theorem 4.1. *A stable subgroup H of a finitely generated group G has bounded packing.*

Proof. Let $\mathscr{C}$ be a collection of left coset of H in G whose D-neighborhoods intersect pairwise. Our goal is to bound the size of $\mathscr{C}$ in terms of D, and following [HW09], we do so by induction on the height of H. This is possible by Proposition 3.2. Recall that, for the base of induction (height zero), H is finite and trivially has bounded packing.

Translating by G, we may assume that $H \in \mathscr{C}$. Moreover, if $gH \in \mathscr{C}$, then $gH = hxH$ for $h \in H$ and $x \in B_D(1)$, since gH is $D-$close to H. We fix $x \in B_D(1)$ and bound the number of cosets of the form hxH in $\mathscr{C}$. Since the number of such x is bounded, this will establish the theorem.

Hruska and Wise show that for any $L \geq 0$ there is an $L' \geq 0$ so that

(1) $N_L(H) \cap N_L(xH) \subset N_{L'}(K),$

where $K = H \cap xHx^{-1}$ [HW09, Lemma 4.5]. Moreover, there is a bijection, $hxH \leftrightarrow hK$, and the height of K in H is strictly less than the height of H in G [HW09, Lemma 4.2]. Since K is a stable subgroup of H by Lemma 3.1, the induction hypothesis implies that K has bounded packing in H.

We finally claim that there is a D' depending only on D such that the D'-neighborhoods of the hK intersect pairwise in H. By bounded packing, this will imply that the number of such cosets is bounded. Using the bijection above, this in turn bounds the size of $\mathscr{C}$ as required. To prove the claim, let h_1, h_2 be such that $h_1xH, h_2xH \in \mathscr{C}$. Pick points $a, b, c \in G$ such that

$$a \in N_D(H) \cap N_D(h_1xH), \qquad b \in N_D(H) \cap N_D(h_2xH), \qquad c \in N_D(h_1xH) \cap N_D(h_2xH).$$

Applying Lemma 2.3, we see that for $R \geq 0$ depending on D, $[a,b] \subset N_R(H)$, $[b,c] \subset N_R(h_2xH)$, and $[a,c] \in N_R(h_1xH)$. As these form a triangle of uniformly stable geodesics (Lemma 2.3), there is a $\delta \geq 0$ depending only on D (and the stability function of H) such that this triangle is δ-thin (Lemma 2.4). Hence, there is a $w \in [a,b]$ within distance δ from both $[b,c]$ and $[a,c]$. We conclude that w is contained in the intersection of the $(2R+\delta)$–neighborhoods of H and h_ixH for $i = 1,2$. Applying the containment Eq. 1, there is a $L' \geq 0$ such that

$$w \in N_{2R+\delta}(H) \cap N_{2R+\delta}(h_ixH) \subset N_{L'}(h_iK)$$

for $i = 1,2$. We conclude that the G-distance between h_1K and h_2K is at most $2L'$. Since H is undistorted in G, we conclude that there distance in H is at most D', which depending only on D and the stability function of H. This concludes the proof. □

We now want to use bounded packing to study intersections of conjugates rather than cosets, and to pass from cosets to conjugates we will need the following lemma.

Lemma 4.2. *Let H_1, H_2 be stable subgroups. Then there exists $D \geq 0$ so that whenever $|H_1^{g_1} \cap H_2^{g_2}| = \infty$ for some $g_1, g_2 \in G$, then the cosets g_1H_1 and g_2H_2 have intersecting D–neighborhoods.*

Proof. Set $K = H_1^{g_1} \cap H_2^{g_2}$ and let $k \in K$ be an infinite order element (notice that K is hyperbolic since it is stable by Lemma 3.1.(1), and in particular K is infinite if and only if it contains an infinite order element). Since k is a Morse element of G, it produces two points k^∞ and $k^{-\infty}$ in

$$\Lambda_K = g_1\Lambda_{H_1} \cap g_2\Lambda_{H_2} \subset \partial_M G,$$

using Lemma 3.1. Let γ be a geodesic in the weak hull of K joining $k^{-\infty}$ and k^∞. Then $g_1^{-1}\gamma$ lies in the weak hull of H_1 and $g_2^{-1}\gamma$ lies in the weak hull of H_2. Since each H_i, $i = 1,2$, acts cocompactly on its weak hull by Theorem 2.6, there is a D depending only on H_1, H_2, such that γ lies in the intersection of $N_D(g_1H_1)$ and $N_D(g_2H_2)$. □

Proposition 4.3. *Let $H_1, \ldots, H_l$ be stable subgroups of a finitely generated group G. Then there exist finitely many conjugacy classes of subgroups of G of the form $H_{i(1)}^{g_1} \cap \cdots \cap H_{i(n)}^{g_n}$, where $i(j) \in \{1, \ldots, n\}$ and $g_j \in G$ and $n \in \mathbb{N}$.*

Proof. Since each H_i is hyperbolic, it contains only finitely many conjugacy classes of finite subgroups, and hence we can can restrict to considering infinite intersections. We show by induction on n that, whenever $K_1, \ldots, K_{l'}$ are stable subgroups of G, there are only finitely many conjugacy classes of infinite subgroups of G of the form $K_{i(1)}^{g_1} \cap \cdots \cap K_{i(n)}^{g_n}$. This suffices in view of finiteness of height (Proposition 3.2).

For $n = 1$ this is obvious. Let us prove it for $n = 2$. It suffices to show that there are only finitely many conjugacy classes of infinite subgroups of G of the form $K_1 \cap K_2^g$. By the Lemma 4.2 there is some $D \geq 0$ depending only on K_1, K_2 so that gK_2 intersects $N_D(K_1)$, so that up to multiplying g on the left by an element of K_1 we have that gK_2 has a representative in the ball of radius D around the identity, which is finite.

Assuming that the claim holds for some n, we can replace $K_1, \ldots, K_{l'}$ with a set $\mathscr{K}$ of representatives of the conjugacy classes of infinite subgroups of the form $K_{i(1)}^{g_1} \cap \cdots \cap K_{i(n)}^{g_n}$. This is a set of stable subgroups (by Lemma 3.1), and it is finite by the inductive hypothesis. As any intersection of $n + 1$ conjugates of the K_i is an intersection of two conjugates of elements of $\mathscr{K}$, this completes the proof. □

Theorem 4.4. *Let $\mathscr{H} = \{H_1, \cdots, H_l\}$ be a finite collection of stable subgroups of a finitely generated group G. Then $\mathscr{H}$ has bounded packing in G. Further, $\mathscr{H}$ has finite width.*

Proof. First, let $\mathscr{C}$ be a collection of coset of subgroups in $\mathscr{H}$ whose D–neighborhoods intersect pairwise. By Theorem 4.1, for each $i = 1, \ldots, l$ there is a $N_i \geq 0$ such that the number of coset of H_i in $\mathscr{C}$ is at most N_i. Hence, $\#\mathscr{C} \leq \sum_{i=1}^{l} N_i$ and so $\mathscr{H}$ has bounded packing.

Finite width comes from the fact that, by Lemma 4.2, if $H_i^{g_1} \cap H_j^{g_2}$ is infinite for $i, j \in \{1, \ldots, l\}$ and $g_1, g_2 \in G$, then the cosets $g_1 H_i$ and $g_2 H_j$ have intersecting D–neighborhoods, for D depending only on $\mathscr{H}$. Finite width now follows from bounded packing. $\qquad\square$

5. HYPERBOLICALLY EMBEDDED SUBGROUPS AND BOUNDED COHOMOLOGY

5.1. **Intersections of hyperbolically embedded subgroups.** In this subsection we prove a few results about intersections of (conjugates of) hyperbolically embedded subgroups.

Proposition 5.1. *Let G be a group with generating set X and assume that $\Gamma(G, X)$ is hyperbolic. Let H_1 and H_2 be hyperbolically embedded in (G, X). Then*

   (i) *there are finitely many H_1-conjugacy classes of subgroups of the form $H_1 \cap H_2^g$ with $g \in G$.*
   (ii) *If $K_1, \ldots, K_n$ are representatives of the H_1-conjugacy classes of $H_1 \cap H_2^g$, then H_1 is hyperbolic relative to $K_1, \ldots, K_n$.*
   (iii) *$\{K_1, \ldots, K_n\} \hookrightarrow_h (G, X)$.*

Proof. (i) follows from Proposition 4.3, which immediately gives finitely many G–conjugacy classes, and the fact that hyperbolically embedded subgroups are almost malnormal [DGO17, Proposition 2.10].

Once we prove (ii), (iii) follows from (ii) and [DGO17, Theorem 4.35]. Hence, let us prove (ii).

By Lemma 2.10, it is enough to show that $\{K_i\}_{i=1}^{n} \hookrightarrow_h (H_1, Y)$ where Y is some finite generating set of H_1. Since H_1 is quasi-isometrically embedded in (G, X), we can work with d_X rather than d_Y. Notice that since $\Gamma(G, X)$ is hyperbolic, $\Gamma(H_1, Y)$ is hyperbolic as well. Therefore, by Theorem 2.9, we need to prove that each K_i is finitely generated and quasi-isometrically embedded in (H_1, d_X), which follows from Lemma 3.1 (from which one deduces that K_i is a finitely generated undistorted subgroup of H_1). We also need to show that $\{K_i\}_{i=1}^{n}$ is geometrically separated in (H_1, d_X).

In order to prove geometric separation let t_i be so that $K_i = H_1 \cap H_2^{t_i}$. Notice that there exists $R > 0$ such that for every i, $K_i \subseteq N_R(t_i H_2)$. We claim that for every $a, b \in H_1$ if $aK_i \neq bK_j$ then $at_i H_2 \neq bt_j H_2$. Once the claim is proved, the geometric separation of $\{K_i\}_{i=1}^{n}$ follows directly from the geometric separation of H_2, since for every $g \in H_1$ we have $gK_i \subseteq N_R(gt_i H_2)$.

Suppose that $at_i H_2 = bt_j H_2$, and let $h_2 \in H_2$ such that $at_i h_2 = bt_j$, then

$$K_i^a = H_1 \cap (H_2^{t_i})^a = H_1 \cap H_2^{at_i} = H_1 \cap H_2^{at_i h_2} = H_1 \cap H_2^{bt_j} = K_j^b.$$

In particular, $i = j$ since by assumption distinct K_i's are representatives of distinct H_1-conjugacy classes.

Hence,

$$aK_i = H_1 \cap bt_i h_2^{-1} t_i^{-1} H_2^{t_i} = H_1 \cap bH_2^{t_i} = bK_j,$$

and the claim is proved. $\qquad\square$

Lemma 5.2. *Let G be a group, let X be a generating set of G so that $\Gamma(G, X)$ is hyperbolic, and let H_1, H_2 be hyperbolically embedded in (G, X). Then for each $R \geq 0$ there exists C such that there only finitely many $s \in G$ satisfying*

(2) $\qquad d_X(1, s) \leq R$ *and* $\operatorname{diam}(H_1 \cap N_R(sH_2)) \geq C$.

Proof. We can enlarge X by adding finitely many elements to ensure that $X \cap H_i$ is a generating set of H_i.

Notice that there exists $R' \geq R$, depending on R and the quasiconvexity constants of H_1, H_2 only, so that for every d if there exists $a \in H_1 \cap N_R(sH_2)$ satisfying $d_X(1,a) \geq d$ then there exists $a' \in H_1 \cap N_{R'}(sH_2)$ satisfying $d_X(1,a') = d$.

Fix $R > 0$, and let R' be as above. Since H_2 satisfies the WPD condition (see Definition 2.12), letting $\varepsilon = 2R'$, there is N such that for every $b \in H_2$ satisfying that $d_X(1,b) > N$ we have that

$$(3) \qquad |\{g \in G \mid d_X(1,g) < \varepsilon, \, d_X(b,gb) < \varepsilon\}| < \infty.$$

We put $C = N + 2R'$.

Let $S \subset G$ be the set

$$S = \{s \in G \mid d_X(1,s) \leq R, \, \mathrm{diam}(H_1 \cap N_R(sH_2)) \geq C\}.$$

Assume that S is infinite. Since the metric d_X restricted to H_1 is proper, by the definition of R' there is an infinite subset S_1 of S and $a \in H_1$, such that $a \in H_1 \cap N_{R'}(sH_2)$ for all $s \in S_1$ and $d_X(1,a) = C$. For each $s_i \in S_1$, let $t_i \in H_2$ such that $d_X(a, s_i t_i) \leq R'$. Note that

$$d_X(1,t_i) = d_X(s_i, s_i t_i) \geq d_X(1,a) - d_X(1,s_i) - d_X(a, s_i t_i) \geq C - 2R' = N.$$

Since we similarly have that $d_X(1,t_i) \leq C + 2R'$, and the metric d_X restricted to H_2 is proper, there is an infinite subset S_2 of S_1 and $b \in H_2$ such that $t_i = b$ for all $s_i \in S_2$.

In particular, for all $s_1, s_2 \in S_2$,

$$d_X(1,b) > N \quad \text{and} \quad d_X(s_1, s_2) \leq 2R \leq \varepsilon \quad \text{and} \quad d_X(s_1 b, s_2 b) \leq 2R' = \varepsilon,$$

or equivalently,

$$d_X(1,b) > N \quad \text{and} \quad d_X(1, s_1^{-1} s_2) \leq 2R \leq \varepsilon \quad \text{and} \quad d_X(b, s_1^{-1} s_2 b) \leq 2R' = \varepsilon.$$

By (3), i.e. the WPD condition of H_2, there are only finitely many different possibilities for $s_1^{-1} s_2$. $\qquad \square$

Proposition 5.3. *Let G be a group, let X be a generating set of G so that $\Gamma(G,X)$ is hyperbolic, and let H_1, H_2 be hyperbolically embedded in (G,X). Then for every $R \geq 0$, we have $d_{\mathrm{Haus}}(H_1 \cap N_R(H_2), H_1 \cap H_2) < \infty$.*

Proof. Again, we enlarge X by adding finitely many elements to ensure that $X \cap H_i$ is a generating set of H_i.

Notice that $H_1 \cap H_2 \subseteq H_1 \cap N_R(H_2)$, so we only need to prove that any point in $H_1 \cap N_R(H_2)$ is at a bounded distance from a point in $H_1 \cap H_2$.

Let $h_1 \in H_1 \cap N_R(H_2)$ and $h_2 \in H_2$ such that $d_X(h_1, h_2) \leq R$. Let $h \in H_1 \cap H_2$ such that $d_{H_1}(h, h_1)$ is minimal, where d_{H_1} is the path metric on H_1 (which is quasi-isometric to d_X restricted to H_1). Without loss of generality, we can assume that $h = 1$. Let γ_i be geodesics in H_i from 1 to h_i. By quasiconvexity and hyperbolicity, there exists R' such that $d_{\mathrm{Haus}}(\gamma_1, \gamma_2) < R'$ where R' only depends on R, δ, and the quasiconvexity constants of H_1 and H_2.

For every vertex $x \in \gamma_1$, let s_x be so that xs_x is a vertex γ_2, and $d_X(x, xs_x) \leq R'$. By Lemma 5.2, and the observation that $H_1 \cap N_{R'}(s_x H_2) = x^{-1}(H_1 \cap N_{R'}(H_2))$, there is C and N such if $d_{H_1}(1, h_1) > C$, then there at most N possibilities for s_x. If $d_{H_1}(1, h_1) > C + N + 1$, then there are distinct vertices $x, y \in \gamma_1$ such that $s_x = s_y$. We assume that x is closer than y to 1 (in the d_{H_1} metric).

Observe that $yx^{-1} \in H_1$. Observe also that $yx^{-1} \in H_2$, since $yx^{-1} = (ys_x)(s_x^{-1}x^{-1})$. Notice that

$$d(yx^{-1}, h_1) \le d_{H_1}(yx^{-1}, y) + d_{H_1}(y, h_1) = d_{H_1}(1, x) + d_{H_1}(y, h_1) < d_{H_1}(1, h_1)$$

which contradicts the minimality of $d_{H_1}(1, h_1)$.   □

5.2. Extension of quasimorphisms.
In this subsection, we shall prove a bounded cohomology application of Proposition 5.1.

Recall that, given q_i a quasimorphism on $H_i \le G$, $i \in I$, we say that $\{q_i\}$ intersection-compatible, if whenever $x \in H_i$ and $y \in H_j$ are G-conjugate one has that $q_i(x) = q_j(y)$.

Remark 5.4. *Notice that, since homogeneous quasimorphisms are conjugation invariant, if the q_i are the restriction of some homogeneous quasimorphism on G then $\{q_i\}$ is intersection-compatible.*

Theorem 5.5. *Let G be a group with generating set X and assume that $\Gamma(G,X)$ is hyperbolic. Let $H_1, \ldots, H_l$ be hyperbolically embedded in (G,X), and let q_i be a homogeneous quasimorphism on H_i. Then there exists a homogeneous quasimorphism q on G so that $q|_{H_i} = q_i$ if and only if $\{q_i\}$ is intersection-compatible.*

We briefly recall the construction of extensions of quasicocycles from [FPS15], specializing it to quasimorphisms. Let $H \hookrightarrow_h (G,X)$. Let $\hat{G}$ be obtained from $\Gamma(G,X)$ by adding, for every coset gH of H, a vertex $c(gH)$ connected to all elements of gH by an edge of length $1/4$. For $g, x \in G$, denote by $\pi_{gH}(x)$ the set of all entrance points in gH of geodesics in $\hat{G}$ from x to $c(gH)$. Let $D \ge 0$ be sufficiently large.

For $g, x \in G$, define the trace $tr_{gH}(x) \in \mathbb{R}G$ of x onto gH to be 0 if $\text{diam}_{gH}(\pi_{gH}(\{1, x\})) \le D$, and let $tr_{gH}(x)$ be the average of all $y^{-1}z$ for $y \in \pi_{gH}(1)$ and $z \in \pi_{gH}(x)$ otherwise. Notice that it does not depend on the choice of coset representative g.

Let q be an alternating quasimorphism on H, meaning that $q(h) = -q(h^{-1})$ for each $h \in H$. Then one can define

$$\Theta(q)(x) = \sum_{gH} q(tr_{gH}(x)).$$

In [FPS15] it is shown that $\Theta(q)$ is (well-defined and) an alternating quasimorphism, and $\Theta(q)|_H$ is within bounded distance of q. We will denote by $\Psi(q)$ the unique homogeneous quasimorphism within bounded distance of $\Theta(q)$. In particular, if q is homogeneous then $\Psi(q)|_H = q$.

For later purposes, we also record the following lemma about π_H. For $x \in G$, denote by $\rho_H(x)$ the set of all $h \in H$ that minimize the distance from x in $\Gamma(G,X)$ (this is a uniformly bounded set by quasiconvexity). Denote by d_H a word metric on H. We note that the d_H-diameter of $\pi_{gH}(x)$ is uniformly bounded as a consequence of [FPS15, Lemma 2.9].

Lemma 5.6. *In the notation above, there exists C so that, for each $x \in G$, $\pi_H(x)$ is within Hausdorff d_H-distance C of $\rho_H(x)$.*

Proof. We will use the fact that, by [FPS15, Lemma 2.8], there exists $B \ge 1$ so that if w, y satisfy $d_H(\pi_H(w), \pi_H(y)) \ge B$ then any geodesic in $\hat{G}$ from w to y goes through $c(H)$. Moreover, for each $g \notin H$ we have $\text{diam}_H(\pi_H(gH)) \le B$. Finally, we will also use the fact that any geodesic in $Y = \Gamma(G,X)$ is an unparametrized quasigeodesic of $\hat{G}$ with uniform constants by [KR14, Proposition 2.6].

Let $h \in \rho_h(x)$. If $d_H(h, \pi_H(x)) \le 10B$ then we are done, so suppose that this is not the case. Let γ be a geodesic in $\Gamma(G,X)$ from h to x, parametrized by arc length. Since $\text{diam}_H(\pi_H(x)) \le B$, there exist a minimal integer t so that $d_H(\pi_H(\gamma(t)), \pi_H(x)) \le B$.

We now provide a uniform bound R on $d_Y(\gamma(t), x')$, for some fixed $x' \in \pi_H(x)$. This in turn yields a bound on $d_Y(h, x')$ and hence $d_H(h, x')$. In fact, $d_Y(\gamma(t), H) = t$, so we must have $t \leq R$, and hence we get $d_Y(h, x') \leq 2R$.

Let us bound $d_Y(\gamma(t), x')$. Let α be a geodesic in $\hat{G}$ from x to x'. First of all, since γ is an unparametrized quasigeodesic of the hyperbolic space $\hat{G}$, there exists a uniform δ so that $d_{\hat{G}}(\gamma(t), x'') \leq \delta$ for some $x'' \in \alpha$. Since $d_H(\pi_H(\gamma(t-1)), \pi_H(x'')) > B$ (because $\pi_H(x'') \subseteq \pi_H(x)$), we get that any geodesic from $\gamma(t-1)$ to x'' goes through $c(H)$ and hence $d_{\hat{G}}(x'', H) \leq \delta$. Hence $d_{\hat{G}}(x'', x') \leq \delta$, because α is a geodesic from x to some point in $\pi_H(x)$. To sum up, we have $d_{\hat{G}}(x', \gamma(t)) \leq 2\delta$.

In order to conclude the proof, we now show that there does not exist any coset gH with $d_{gH}(\pi_{gH}(\gamma(t)), \pi_{gH}(x')) > 10B$. Once this is done, we can consider a geodesic β in $\hat{G}$ from $\gamma(t)$ to x' and replace its subsegments of length $1/2$ around the apices $c(gH)$ that β goes through with geodesics in gH, thereby obtaining a path of uniformly bounded length.

If we had some gH with $d_{gH}(\pi_{gH}(\gamma(t)), \pi_{gH}(x')) > 10B$, then we would have $d_{gH}(\pi_{gH}(\gamma(t-1)), \pi_{gH}(x')) > 5B$. But then any geodesic from either $\gamma(t-1)$ or $\gamma(t)$ to $c(H)$ would contain $c(gH)$, and hence we would have $\pi_H(\gamma(t)) = \pi_H(c(gH)) = \pi_H(\gamma(t-1))$, contradicting the minimality of t. $\qquad\square$

Proof of Theorem 5.5. By Remark 5.4, we only need to show that we can always extend an intersection-compatible family $\{q_1, \ldots, q_l\}$ of quasimorphisms. For $l = 1$, we can use [HO13] or Ψ as constructed above.

Let now $l \geq 2$ and let $H_1, \ldots, H_l < G$ be subgroups and suppose $H_i \hookrightarrow_h (G, X)$. Let $\hat{G}^1$ be obtained from $\Gamma(G, X)$ by adding, for every coset gH_1 of H_1, a vertex $c(gH_1)$ connected to all elements of gH_1 by an edge of length $1/4$. Let $K_1, \ldots, K_n$ are representatives of the H_1-conjugacy classes of subgroups of the form $H_1 \cap H_j^g$ where $j \geq 2$ (there are finitely many conjugacy classes by Theorem 5.1).

For $g, x \in G$, denote $\pi_{gH_1}(x)$ the set of all entrance points in gH_1 of geodesics in $\hat{G}^1$ from x to $c(gH)$.

Lemma 5.7. *There exists $C \geq 0$ with the following property. For each $g \in G$ and $j \in \{2, \ldots, l\}$ there exists $h \in G$ and $i \in \{1, \ldots, n\}$ so that $hK_i \subseteq gH_1$ and so that for all $x \in H_j$ we have $d_{gH_1}(\pi_{gH_1}(x), hK_i) \leq C$.*

Proof. In view of Lemma 5.6, we can consider the closest point projection ρ_{gH_1} instead of π_{gH_1}. If the diameter of $\rho_{gH_1}(H_j)$ is sufficiently large, then, by quasiconvexity, gH_1 and H_j are at uniformly bounded distance in Y, and in fact up to multiplying on the left by an element of h_j, we can assume that $d_X(1, g)$ is uniformly bounded. In particular, by Lemma 5.2, there are finitely many choices for g. Since $\rho_{gH_1}(H_j)$ is the coarse intersection of gH_1 and H_j, by Proposition 5.3 we have that $\rho_{gH_1}(H_j)$ is within bounded Hausdorff distance of the intersection of H_1^g and H_j, as required. $\qquad\square$

Let $\{q_i\}$ be intersection-compatible homogeneous quasimorphisms. By the inductive hypothesis there exists a homogeneous quasimorphism q' on G that extends $q_2, \ldots, q_l$.

Set $q_1' = q_1 - q'|_{H_1}$, so that q_1' is a quasimorphism on H_1. In particular, by intersection-compatibility, $q_1'|_{K_j}$ vanishes for each j, and hence $q_1'|_{B_C(1)K_jB_C(1)}$ is bounded, where $B_C(1)$ is the ball of radius C around 1 in H_1. Hence, we can perturb q_1' up to bounded error into an alternating quasimorphisms q_1'' to ensure that $q_1''(h) = 0$ whenever $h \in B_C(1)K_jB_C(1)$ for some j.

Finally, we can set $q = \Psi(q_1'') + q'$. Notice that $q|_{H_1}$ is within bounded distance from q_1. More-over, from Lemma 5.7 and the definition of Ψ we see that $\Psi(q_1'')|_{H_j}$ is identically 0 for all $j \geq 2$. Hence, again for $j \geq 2$, $q|_{H_j}$ is within bounded distance of q', whence of q_j, as required. $\square$

5.3. **Adding a hyperbolically embedded subgroup.**

By [DGO17, Theorem 6.14], any acylindri-cally hyperbolic group G contains a unique maximal finite normal subgroup $E(G)$. Let F_2 denote a non-abelian free group of rank 2.

Proposition 5.8. *Let G be a group and let $H_1, \ldots, H_l$ be subgroups of infinite index with $H_i \hookrightarrow_h (G, X_i)$ with $\Gamma(G, X_i)$ hyperbolic. Then there exists a hyperbolically embedded subgroup H so that $\{H, H_i\} \hookrightarrow_h (G, X_i)$ for every i so that $H \cong F_2 \times E(G)$, In particular $H \cap H_i^g$ is finite for every $g \in G$ and every i.*

Proof. Notice that G contains a finite-index subgroup G' that acts trivially by conjugation on $E(G)$.

By [Osi16, Theorem 5.4, Lemma 5.12], for every i there exists $Y_i \supseteq X_i$ so that $H_i \hookrightarrow_h (G, Y_i)$ and G acts on the hyperbolic space $\Gamma(G, Y_i \cup H_i)$ acylindrically and non-elementarily, and clearly H_i has bounded orbits. By [MS17], given independent random walks $(W_n), (Z_n)$ on G', the probability that $\langle W_n, Z_n, E(G) \rangle$ is $F_2 \times E(G)$ and hyperbolically embedded in $(G, Y_i \cup H_i)$ goes to 1 as n tends to infinity. Since $X_i \subseteq Y_i \cup H_i$, the identity map from (G, d_{X_i}) to $(G, d_{Y_i \cup H_i})$ is Lipschitz. From this and Theorem 2.9, it is easily seen that whenever K is hyperbolically embedded in $(G, Y_i \cup H_i)$, we have that $\{K, H_i\}$ is hyperbolically embedded in (G, X_i), and the proof is complete. $\square$

Corollary 5.9. *Let G be an acylindrically hyperbolic group and let $H_1, \ldots, H_l \hookrightarrow_h (G, X)$ be sub-groups of infinite index, where $\Gamma(G, X)$ is hyperbolic. Then the product of the restriction maps $res : EH_b^2(G, \mathbb{R}) \to \Pi_{i=1}^l EH_b^2(H_i, \mathbb{R})$ is not injective.*

Proof. Let H be as in Proposition 5.8, and let ϕ be any non-trivial homogeneous quasimorphism on H. Then by Theorem 5.5 there exists a homogeneous quasimorphism $\bar{\phi}$ on G that restricts to 0 on each H_i but restricts to ϕ on H. The class $\alpha \in EH_b^2(G, \mathbb{R})$ represented by $\bar{\phi}$ has $res(\alpha) = 0$, but α is non-trivial. $\square$

6. FURTHER APPLICATIONS

A number of applications of finite width and bounded packing can be found in the literature. In this Section, we mention a couple of these consequences for stable subgroups. Note that we have listed a number of examples of stable subgroups in the introductory section (Corollary 1.2).

(1) Dimension of cubulation
In [Sag95, Sag97] (see also [HW09]), Sageev proves the following:

Proposition 6.1. *Suppose H is a finitely generated codimension 1 subgroup of a finitely gener-ated group G satisfying bounded packing. Then the corresponding CAT(0) cube complex is finite dimensional.*

Combining with Theorem 4.4 we have as an immediate Corollary:

Corollary 6.2. *Suppose H is a finitely generated stable codimension 1 subgroup of a finitely gen-erated group G. Then the corresponding CAT(0) cube complex is finite dimensional.*

(2) Relative Rigidity
In [Mj08], the first author showed that a uniformly proper bijection between configurations of weak

hulls of limit sets of quasiconvex subgroups of hyperbolic groups comes from a quasi-isometry of the groups, thus extending a result of Schwartz [Sch97]. The same proof works here once we prove the following strengthening of bounded packing, which provides a "coarse Helly property" for stable subsets:

Proposition 6.3 (Coarse centers). *Let $\mathscr{H} = \{H_1, \cdots, H_l\}$ be a finite collection of stable subgroups of a finitely generated group G. For each $D \geq 0$ there exists $R \geq 0$ such that if $\mathscr{C}$ is a collection of pairwise D–close cosets of subgroups of $\mathscr{H}$, then there is an $x \in G$ such that,*

$$N_R(x) \cap gH_i \neq \emptyset$$

for each $gH_i \in \mathscr{C}$.

Proof. Fix a Cayley graph Γ for G and for simplicity assume that Γ contains Cayley graphs of the H_i as subgraphs. By Theorem 4.4, $\#\mathscr{C} \leq N$ for some constant $N \geq 0$ depending only on D.

Consider the connected subgraph of Γ,

$$Y = \bigcup N_D(gH_i),$$

where the union is over all $gH_i \in \mathscr{C}$.

Claim 6.4. *The subgraph Y is f'–stable in X, where f' depends only on D and the stability constants of the subgroups in $\mathscr{H}$.*

Let us see how the proposition follows from the claim. Since Y is stable in X, Y is δ–hyperbolic, with δ depending only on f' ([DT15, Lemma 3.3]) and the subgraphs gH_i are quasiconvex. Hence, we may apply either [NR03, Lemma 7] or [Mj08, Lemma 3.3] which give an $R \geq 0$, depending on N and δ, and an $x \in Y$ such that the R–ball about x in Y meets each $gH_i \in \mathscr{C}$. This will complete the proof.

Hence, it remains to establish the claim. We do so by induction on $N = \#\mathscr{C}$. For $N = 1$, there is nothing to prove. In general, write $Y = Y_N \cup Z$ where $Z = N_D(gH_i)$ for some $gH_i \in \mathscr{C}$ and Y_N is the union of D–neighborhoods of the cosets in $\mathscr{C} \setminus \{gH_i\}$. Note that Y_N is f'–stable by the induction hypothesis and Z is stable with stability function depending only on D and that of H_i. Let $a, b \in Y$ and assume (as we may) that $a \in Y_N$ and $b \in Z$. Further, pick any $c \in Y_N \cap Z$ and choose geodesics $\alpha = [a, c]$, $\beta = [b, c]$, and $\gamma = [b, c]$. By stability of Y_N and Z, α and β are uniformly stable and we conclude from Lemma 2.4, that γ is contained in a uniformly bounded neighborhood of Y and is uniformly stable. Hence, we conclude that Y is uniformly stable in Γ. (Note that "uniform" here depends on N, which is bounded by Theorem 4.4.) □

Let G_1, G_2 be finitely generated groups with Cayley graphs Γ_1, Γ_2, word metrics d_1, d_2, and stable subgroups H_1, H_2. Let Λ_1, Λ_2 be the limit sets of H_1, H_2 in $\partial_M G_1, \partial_M G_2$ respectively. Let $\mathscr{J}_i$, $i = 1, 2$, be the collection of translates of H_{wi}, the weak hulls of Λ_i in Γ_i. Each d_i induces a pseudo-metric on the collection $\mathscr{J}_i$ for $i = 1, 2$ by regarding $J, K \in \mathscr{J}_i$ as closed subsets of Γ_i. We continue calling this induced metric d_i.

Definition 6.5. *A bijective set map ϕ from $\mathscr{J}_1 \to \mathscr{J}_2$ is said to be **uniformly proper** if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that*

    (1) *For all $J, K \in \mathscr{J}_1$, $d_1(J, K)) \leq n \Rightarrow d_2(\phi(J), \phi(K)) \leq f(n)$,*
    (2) *For all $J, K \in \mathscr{J}_2$, $d_2(J, K)) \leq n \Rightarrow d_1(\phi^{-1}(J), \phi^{-1}(K)) \leq f(n)$.*

Definition 6.6. *A map f from Γ_1 to Γ_2 is said to pair the sets $\mathscr{J}_1$ and $\mathscr{J}_2$ as ϕ does if there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that for all $p \in \Gamma$, $J \in \mathscr{J}_1$, $d_1(p, J) \leq n \Rightarrow d_2(f(p), \phi(J)) \leq h(n)$.*

The axioms of [Mj08, Section 3.4] are now verified exactly as in that paper. (The main point in [Mj08] is the existence of a coarse barycenter, which in our context is established by Proposition 6.3.) We thus have the following:

Proposition 6.7. *Let G_1, G_2, Γ_1, Γ_2, d_1, d_2 and $\mathscr{J}_1, \mathscr{J}_2$ be as above. Let $\phi : \mathscr{J}_1 \to \mathscr{J}_2$ be uniformly proper. Then there exists a quasi-isometry q from Γ_1 to Γ_2 which pairs the sets $\mathscr{J}_1$ and $\mathscr{J}_2$ as ϕ does (in particular, Γ_1, Γ_2 are quasi-isometric).*

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID AND INSTITUTO DE CIENCIAS MATEMÁTICAS, CSIC-UAM-UC3M-UCM

E-mail address: yago.anpi@gmail.com

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI-400005, INDIA

E-mail address: mahan.mj@gmail.com; mahan@math.tifr.res.in

DEPARTMENT OF MATHEMATICS, ETH ZURICH, 8092 ZURICH, SWITZERLAND

E-mail address: sisto@math.ethz.ch

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, 10 HILLHOUSE AVE, NEW HAVEN, CT 06520, U.S.A.

E-mail address: s.taylor@yale.edu