Amenable actions of groups and the Poisson boundary

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1 Amenable actions

Let G be a second countable locally compact group, $(X, \mathcal{B}(X))$ a standard Borel G-space ($g \cdot x = xg$), $\mu \in \mathcal{P}(X)$ a quasi-invariant ($\forall B \in \mathcal{B}(X)\ \forall g \in G : \mu(B) = 0 \Leftrightarrow \mu(Bg) = 0$) and ergodic probability measure.
Let H be a locally compact Borel group, a Borel map $\alpha : X \times G \to H$ is a cocycle if $\forall g_1, g_2 \in G : \alpha(x, g_1 g_2) = \alpha(x, g_1)\alpha(xg_1, g_2)$ a.e. In particular, if $G = \mathbb{Z}$ then

$$\begin{aligned} \{\alpha : X \times \mathbb{Z} \to H \mid \alpha \text{ cocycle}\} &\longrightarrow \{f : X \to H \mid f \text{ Borel meas.}\} \\ \alpha &\longmapsto (x \mapsto \alpha(x, 1)) \end{aligned}$$

is a 1-1 identification of cocycles and Borel functions.

Remark 1.1. *Suppose that $\alpha : X \times G \longrightarrow H$ is a cocycle. Define the skew product G-space $X \times_\alpha H$, that is the space $X \times H$ together with the G-action $(x, h)g := (xg, h\alpha(x, g))$ (this is actually a near action but it is equal a.e. to an action)[1]. Let $\mathcal{A} := \{f \in L^\infty(X \times_\alpha H) \mid \forall g \in G : f((x, h)g) = f(x, h) \text{ a.e}\}$ be the fixed points of $L^\infty(X \times_\alpha H)$ under the G-action and note that $\mathcal{A}$ is an abelian von Neumann algebra, hence there is a standard measure space (M, ν_M) and a measure preserving Borel map $\varphi : X \times_\alpha H \longrightarrow M$ such that $\varphi^*(L^\infty(M, \nu_M)) = \mathcal{A}$. Moreover note that $H \curvearrowright X \times_\alpha H$ by $(x, h)h_0 := (x, h_0^{-1}h)$ and this action commutes with the (near) action of G. Hence $H \cdot \mathcal{A} = \mathcal{A}$ and we may choose M to be an H-space and φ to be an H-map. (M, ν_M) is called the Mackey range of α.*

Let E be a separable Banach space,

Lemma 1. [2] *$Iso(E)$ is a separable metrizable group with the strong operator topology and the induced Borel structure on $Iso(E)$ is standard and is the smallest such that all maps $T \mapsto T(\xi)$, where $\xi \in E$, are Borel.*

[1] R.J. Zimmer, Ergodic Theory and Semisimple Groups, Appendix B
[2] Zimmer, Amenable ergodic group actions and an application to Poisson boundaries of random walks

Let E^* be the dual space of E, $E_1^* = \overline{B_1^{E^*}} \subseteq E^*$ the closed unit ball in E^* and denote by $\langle\ ,\ \rangle$ the dual pairing of E^* and E. For $T \in \overline{B_1^{\mathcal{L}(E)}}$, where $\mathcal{L}(E)$ denotes the space of bounded linear operators from E to E, we have an adjoint map $T^* \in \overline{B_1^{\mathcal{L}(E^*)}}$ characterized by the following condition: $\forall \xi \in E\ \forall \lambda \in E^*\ : \langle T\xi, \lambda\rangle = \langle \xi, T^*\lambda\rangle$.

Lemma 2. *The maps*

$$\begin{aligned} Iso(E) \times E_1^* &\longrightarrow E_1^*, & \qquad Iso(E) &\longrightarrow Homeo(E_1^*), \\ (T, \lambda) &\longmapsto T^*(\lambda) & T &\longmapsto T^* \end{aligned}$$

are continuous, where $Homeo(E_1^*)$ is the group of homemorphisms with the topology of uniform convergence.

Remark 1.2. *G is amenable if any affine action of G on a weak*-compact convex subset of the dual of a (not necessarily separable) Banach space E has a fixed point, i.e. if $\alpha \in Hom(G, Iso(E))$ or equivalently if $\alpha : X \times G \longrightarrow Iso(E)$ is a cocycle, where $X = \{pt\}$, and if $K \subset E_1^*$ is a compact, convex subset such that K is G-invariant, $\alpha^*(G)K \subset K$, then there is a fixed point $\lambda \in K$, that is $\alpha^*(G)\lambda = \lambda$ or equivalently if we interpret α as a cocycle, there is a Borel map $\varphi : X = \{pt\} \longrightarrow E_1^*$ that is fixed by α^*, i.e. $\alpha^*(\{pt\}, G)\varphi(\{pt\}g) = \varphi(\{pt\})$.*

Let E be a separable Banach space and suppose for any $x \in X$ there is some $K_x \subset E_1^*$ non-empty, convex, compact set. We call $\{K_x\}_{x\in X}$ a Borel field of compact convex sets if $\{(x, \lambda) | \lambda \in K_x\} \subset \mathcal{B}(X \times E_1^*)$. Suppose $\alpha : X \times G \longrightarrow Iso(E)$ is a cocycle, then there is an adjoint cocycle

$$\begin{aligned} \alpha^* : X \times G &\longrightarrow Homeo(E_1^*), \\ (x, g) &\longmapsto (\alpha(x, g)^{-1})^* \end{aligned}$$

$\{K_x\}_{x\in X}$ is then called α-invariant if for any $g \in G$: $\alpha^*(x, g)K_{xg} = K_x$ a.e. and a Borel map $\varphi : X \longrightarrow E_1^*$ such that $\varphi(x) \in K_x$ a.e. is called an α-invariant section in $\{K_x\}_{x\in X}$ if for any $g \in G$: $\alpha^*(x, g)\varphi(xg) = \varphi(x)$ a.e.

Definition 1.3. *An action $G \curvearrowleft X$ is called amenable if for any separable Banach space E, any cocycle $\alpha : X \times G \longrightarrow Iso(E)$ and any α-invariant field $\{K_x\}_{x\in X}$ there is an α-invariant section $\varphi : X \longrightarrow E_1^*$ in $\{K_x\}_{x\in X}$.*

Proposition 1. *$G \curvearrowleft \{pt\}$ is amenable if and only if G is amenable*

Proof. If G is amenable then $G \curvearrowleft \{pt\}$ is amenable and if $G \curvearrowleft \{pt\}$ then the amenability condition of G holds for separable Banach spaces [3]. Let E be an arbitrary Banach space, $\alpha \in Hom(G, Iso(E))$ and $K \in E_1^*$ non-empty, convex, compact,

[3] see Remark 1.2

G-invariant set. G is separable, hence there is a sequence $\{E_n\}_{n\in\mathbb{N}}$ such that $E_n \subset E$ is a separable, closed, G-invariant subspace and $E = \overline{\bigcup_{n\in\mathbb{N}} E_n}$. Let $\varphi_n : E^* \longrightarrow E_n^*$ be the restriction operator and define $K_n := \varphi_n(K)$. Note that K_n is a compact convex G-invariant subspace and by separability of E_n the set of fixed points is non-empty and closed. Finally deduce by the finite intersection property the existence of a fixed point. $\square$

Proposition 2. *Suppose that $H < G$ is a closed subgroup. $G \curvearrowright G/H$ is amenable if and only if H is amenable.*

Proposition 3. *If G is amenable and $G \curvearrowright (X,\mu)$ is ergodic then $G \curvearrowright X$ amenable.*

Remark 1.4. *Let E be a separable Banach space, then*

$$L^1(X,E) := \{f : X \longrightarrow E | f \text{ Borel meas. } \int_X \|f(x)\| d\mu(x) < \infty\}/\sim$$

is a separable Banach space. A map $\lambda : X \longrightarrow E^*$ is called weakly measurable if $X \ni x \longmapsto \|\lambda(x)\|$ is measurable and

$$L^\infty(X,E^*) := \{\lambda : X \longrightarrow E^* | \lambda \text{ weakly meas. } X \ni x \longmapsto \|\lambda(x)\| \in L^\infty(X)\}$$

is a Banach space under the essential sup*-norm. Moreover we can define the pairing*

$$\begin{aligned} \langle\ ,\ \rangle : L^\infty(X,E^*) \times L^1(X,E) &\longrightarrow \mathbb{C}. \\ (\lambda, f) \longmapsto \langle \lambda, f\rangle &:= \int_X \langle \lambda(x), f(x)\rangle d\mu(x) \end{aligned}$$

Note that $L^\infty(X,E^*) \ni \lambda \longmapsto \langle \lambda,\ \rangle \in L^1(X,E)^*$ is an isometry, i.e. $L^\infty(X,E^*) \cong L^1(X,E)^*$. Hence, the closed unit ball in $L^\infty(X,E^*)$ is compact and metrizable with the $\sigma(L^\infty(X,E^*), L^1(X,E))$ topology. Moreover, if $\{K_x\}_{x\in X}$ is a Borel field of compact convex subsets of E_1^* then $B := \{\lambda \in L^\infty(X,E^*) | \lambda(x) \in K_x \text{ a.e.}\} \subset B_1^{L^\infty(X,E^*)}$ is a closed convex subset.

Proof. Let $\alpha : X \times G \longrightarrow Iso(E)$ be a cocycle, $\{K_x\}_{x\in X}$ an α-invariant Borel field. Note that $g{\cdot}\mu(B) := \mu(Bg)$ defines a.e. an equivalent measure on X so by the Radon-Nikodym theorem there exists a positive Borel density function $r : X \times G \longrightarrow \mathbb{R}^+$ such that

$$d(g \cdot \mu)(x) := d\mu(xg) = r(x,g) d\mu,$$

in fact, r is a cocycle

$$r(x, g_1 g_2) = \frac{d\mu(x g_1 g_2)}{d\mu(x)} = \frac{d\mu(x g_1 g_2)}{d\mu(x g_1)} \frac{d\mu(x g_1)}{d\mu(x)} = r(x g_1, g_2) r(x, g_1)$$

and it is called the Radon-Nikodym cocycle. Define

$$\pi : G \longrightarrow Iso(L^1(X,E)),$$
$$g \longmapsto (f \longmapsto r(\cdot,g)\alpha(\cdot,g)f(\cdot g))$$

π is a well-defined homomorphism:

$$\|\pi(g)f\| = \int_X r(x,g)\underbrace{\|\alpha(x,g)f(xg)\|}_{=\|f(xg)\|}d\mu(x) = \int_X r(xg^{-1},g)\|f(x)\| \underbrace{d\mu(xg^{-1})}_{=r(x,g^{-1})d\mu(x)} = \|f\|,$$

continuity follows from the fact that if $f \in L^1(X,E)$ and $\lambda \in L^\infty(X,E^*)$ then

$$G \ni g \longmapsto \langle \lambda, \pi(g)f\rangle = \int_X \langle \lambda(x), r(x,g)\alpha(x,g)f(xg)\rangle d\mu(x)$$

is measurable by Fubini. Moreover the adjoint action π^* on $L^1(X,E)^* \cong L^\infty(X,E^*)$ is given by $(\pi^*(g)\lambda)(x) = \alpha^*(x,g)\lambda(xg)$. The set $B := \{\lambda \in L^\infty(X,E^*)|\lambda(x) \in K_x \text{ a.e.}\} \subset \overline{B_1^{L^\infty(X,E^*)}}$ is non-empty, compact, convex and G-invariant, so by amenability of G there is a fixed point $\lambda \in B$ such that $\pi^*(G)\lambda = \lambda$, i.e. $\alpha^*(x,g)\lambda(xg) = (\pi^*(g)\lambda)(x) = \lambda(x)$ a.e., so λ is an α-invariant section in $\{K_x\}_{x\in X}$. □

Theorem 1. *If $G \curvearrowright X$ is amenable and $\alpha : X \times G \longrightarrow H$ is a cocycle then $H \curvearrowright M$ is amenable.*

2 The Poisson boundary à la Zimmer

Let $\mu \in \mathcal{P}(G)$ be étalée [4], $(\Omega,\eta) := (G^{\mathbb{Z}},\mu^{\mathbb{Z}})$ and $(\Omega_0,\eta_0) := (G^{\mathbb{N}_0},\mu^{\mathbb{N}_0})$. Let $\theta : \Omega \longrightarrow \Omega$ be the right-shift $(\theta\omega)_n = \omega_{n+1}$, so that we have an action $\mathbb{Z} \curvearrowright \Omega$ defined by $\omega n := \theta^n(\omega)$. Using this action we can define the cocycle $\alpha : \Omega \times \mathbb{Z} \longrightarrow G$ by $\alpha(\omega,1) := \omega(0) =: \alpha(\omega)$[5]. Consider the skew product G-space $X := \Omega \times_\alpha G$, then

$$\mathcal{A} = \{f \in L^\infty(X)|\ \forall n \in \mathbb{Z} :\ f((\omega,g)n) = f(\omega,g) \text{ a.e.}\} = \{f \in L^\infty(X)|\ f \circ \tilde{\theta} = f\},$$

where $\tilde{\theta} : X \longrightarrow X$ is the skew product transformation $(\tilde{\theta}(\omega,g) := (\theta(\omega), g\alpha(\omega))$, so functions on the Mackey range M of α correspond to invariant functions on the sample space of the 2-sided walk.

Let $\mathcal{H}^\infty(G,\mu) = \{f \in L^\infty(G,\mu)|Pf = f\}$ be the space of harmonic functions on G, where P is the Markov operator of the walk corresponding to the transition probability $\delta_g * \mu$, i.e. if $f \in L^1(G,\mu)$ then

$$Pf(g) := \int_G \int_G f(hk)d(\delta_g * \mu)(k,h) = \int_G f(gh)d\mu(h),$$

[4] μ^{*n} has a non-singular component with respect to the Haar measure for some n

[5] We think of α and θ also as being defined on Ω_0

Let $\Omega_0 \ni \omega \longmapsto X_n(\omega) := \omega_n \in G$ be the projections which induce the natural filtration $\mathcal{F}_n := \bigvee_{k\geq n} \sigma(X_k)$ on Ω_0. If $f \in \mathcal{H}^\infty(G,\mu)$ then for any $g \in G$

$$\mathbb{E}(f(gX_1 \dots X_{n+1})|\mathcal{F}_n) = \int_G f(gX_1 \dots X_n h) d\mu(h) = f(gX_1 \dots X_n),$$

i.e. $\{f(gX_1 \dots X_n)\}_n$ is a bounded martingale, so by Doob's martingale convergence theorem in particular the function defined by

$$\begin{aligned} \tilde{f} : \Omega_0 \times G &\longrightarrow \mathbb{C} \\ (\omega, g) &\longmapsto \lim_{n\to\infty} f(g\omega_1 \dots \omega_n), \end{aligned}$$

exists a.e. and $\tilde{f} \in \mathcal{A}_0$ (by harmonicity), so we can interpret harmonic functions on G as invariant functions on the sample space of the 1-sided random walk $X_0 := \Omega_0 \times G$.

Let[6] $\mathcal{B}ool(X_0) := \{f \in L^\infty(X_0) | f^2 = f\}$ be the Boolean σ-algebra of X_0 and consider $\mathcal{B}_0 := \mathcal{B}ool(X_0) \cap \mathcal{A}_0$. $\mathcal{B}_0$ is then isomorphic to $\mathcal{B}ool(P,m)$, where (P,m) is an ergodic G-space[7]. Let $p_0 : X_0 \longrightarrow P$[8] be a measure preserving G-map such that $p_0^*(P) \cong \mathcal{B}_0$[9] and note that $p_0 \circ \tilde{\theta} = p_0$ so that p_0 is a G-equivariant and $\tilde{\theta}$-invariant map. Let $\iota : \Omega_0 \longrightarrow X_0$ be the natural injection ($\iota(\omega) = (\omega, e)$) and define $\nu_P := (p_0 \circ \iota)_*(\eta_0) \in \mathcal{P}(P)$ [10]. The measure space (P, ν_P) is the Poisson boundary of the random walk. Similarly we can also construct a shift-invariant, measure class preserving Borel map $p : \Omega \times G \longrightarrow M$. Define

$$\begin{aligned} Z_k : \Omega_0 &\longrightarrow P, \\ \omega &\longmapsto p_0(\theta^k \omega, e) \end{aligned}$$

Let $X_k : \Omega_0 \longrightarrow G$ be as above, then $X_k \sim \mu$ and

$$Z_{k+1}X_k^{-1}(\omega) = p_0(\theta^{k+1}\omega, e)\underbrace{X_k^{-1}(\omega)}_{\omega_k^{-1}} = p_0((\theta^{k+1}\omega, e)\omega_k^{-1}) = p_0(\underbrace{\theta^{k+1}\omega, \omega_k}_{=\tilde{\theta}(\theta^k\omega, e)}) = Z_k(\omega)$$

which shows that $\{Z_k\}_k$ is a μ-process (this explains the choice of ν_P). To summarize this discussion we see that $\mathcal{B}ool(P,m) \cong \mathcal{B}_0 \hookrightarrow \mathcal{A}_0 \cong L^\infty(M, \nu_M)$ so that P is a factor of M.

(P, ν_P) corresponds to the usual notion of the Poisson boundary in the following sense: the reason why μ is étalée is that if it is, then any harmonic function is continuous and so any two harmonic functions equal a.e. must be the same, so it suffices to show that any harmonic function $h \in \mathcal{H}^\infty(G,\mu)$ can be represented

[6] $\mathcal{A}_0$ is an abelian von Neumann algebra, hence $\mathcal{A}_0 \cong L^\infty(P,m)$ for some standard measure G-space (P,m) but this space need not be ergodic!

[7] See G. W. Mackey, Point realizations of transformation groups

[8] p is defined on a conull set $\Omega_0' \times G$

[9] If $\varphi : (X,\mu) \longrightarrow (Y,\nu)$ is a measure class preserving map then we get an injective map $\varphi^* : \mathcal{B}ool(Y) \longrightarrow \mathcal{B}ool(X)$

[10] m can be recovered from ν_P by $m = \int_G (g_*\nu_P) dm_G(g)$, where m_G is a Haar measure

by a bounded measurable function $\varphi \in L^\infty(P, \nu_P)$. A short sketch of the idea is the following: a martingale-type argument shows that $h(g) = \mathbb{E}_g(H)$ for some $H \in L^\infty(X_0)$, the function

$$\begin{aligned} \Phi : X_0 = \Omega_0 \times G &\longrightarrow \Omega_0, \\ (\omega, g)_n &\longmapsto (g\omega_0 \cdots \omega_{n-1}) \end{aligned}$$

is an intertwiner of $\tilde{\theta}$ on X_0 and θ on Ω_0 and for any $\lambda \in \mathcal{P}(G)$: $\Phi_*(\eta_0 \times \lambda) = P_\lambda$, where P_λ is the Markov measure of the walk with initial distribution λ. Hence $H \circ \Phi$ is invariant on X_0, so there is a function $\varphi : P \longrightarrow \mathbb{C}$ such that $H \circ \Phi = \varphi \circ p$ a.e. and

$$h(g) = \mathbb{E}_g(H) = \int_{X_0} H \circ \Phi d(\eta_0 \times \delta_g) = \int_{\Omega_0} H \circ \Phi(\omega, g) d\eta_0(\omega) =$$

$$= \int_{\Omega_0} (\varphi \circ p)(\omega, g) d\eta_0(\omega) = \int_P \varphi(zg) d\nu_P(z).$$

3 Amenability of the Poisson boundary

Theorem 2. *$G \curvearrowright (P, \nu_P)$ is amenable*

Proof. The discussion above gives the following commutative diagram of measure preserving G-maps

$$\begin{tikzcd} \Omega \times G \arrow[r, "p_0"] \arrow[d, "r \times \mathrm{id}"'] & M \arrow[d, "t"] \\ \Omega_0 \times G \arrow[r, "p"'] & P \end{tikzcd}$$

Let $\gamma_0 : P \times G \longrightarrow Iso(E)$ be a cocycle, where E is a separable Banach space, let $\{K_z\}_{z \in P}$ be a γ_0-invariant field. We need to construct an γ_0-invariant section. Define

$$\begin{aligned} \gamma : M \times G &\longrightarrow Iso(E) \\ (m, g) &\longmapsto \gamma_0(t(m), g) \end{aligned}$$

Suppose we can construct a γ-invariant section σ that factors (mod measure) to a function on P, then we are done.

Step 1:[11] There is a cocycle $\beta : \Omega \times \mathbb{Z} \longrightarrow G$ such that β is strict on some inessential contraction[12] $\Omega^0 * \mathbb{Z}$ and $\forall n \in \mathbb{Z}$: $\beta(\omega, n) = \alpha(\omega, n)$ a.e.

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[11] R.J. Zimmer, Amenable ergodic group actions, Lemma 3.4

[12] Ω^0 is a conull subset and $\Omega^0 * \mathbb{Z} := \{(\omega, n) | \omega n \in \Omega^0\}$

Step 2:[13] There is a conull set $\Omega^0 \subset \Omega$, a measure preserving $\mathbb{Z}$-equivariant [14], measure preserving function $\tilde{p} : \Omega^0 \times G \longrightarrow M$ and a cocycle $\tilde{\alpha} : \Omega \times \mathbb{Z} \longrightarrow G$ such that for all $n \in \mathbb{N}$ $\tilde{\alpha}(\omega, n) = \alpha(\omega, n)$ a.e. and $(\tilde{p} \circ \iota, \tilde{\alpha}) : \Omega^0 * \mathbb{Z} \longrightarrow M \times G$ is a strict homomorphism.

Step 3:[15] Let $M_0 \subset M$ be a conull set. Then there is a conull set $\Omega_1 \subset \Omega^0$ and a measurable function $\vartheta : \Omega_1 \longrightarrow G$ such that $(q, \beta) : \Omega_1 * \mathbb{Z} \longrightarrow M \times G$ is a strict homomorphism such that $(q, \beta)(\Omega_1 * \mathbb{Z}) \subset M_0 * G$, where $q(\omega) = \tilde{p} \circ \iota(\omega)\vartheta(\omega)^{-1}$ and $\beta(\omega, n) = \vartheta(\omega)\tilde{\alpha}(\omega, n)\vartheta(\omega n)^{-1}$. In particular M_0 can be chosen to be $t^{-1}(P_0)$ for some conull set $P_0 \subset P$.

Step 4:[16] If $K \subset \Omega \times \mathbb{Z}$ contains a conull set and (ω, n_1), $(\omega n_1, n_2) \in K$ implies $(\omega, n_1 n_2) \in K$, then K contains an inessential constraction of $\Omega \times \mathbb{Z}$

By Step 1 we may assume that γ is strict on some inessential contraction $M_1 * G$. For any $g \in G$ we have $\gamma^*(m, g)K_{t(m)g} = K_{t(m)}$ a.e. and this holds on an inessential contraction of M [17].
Let $(q, \beta) : \Omega_1 * \mathbb{Z} \longrightarrow M_0 * G$ as in Step 3. Define $\delta = \gamma \circ (q, \beta) : \Omega_1 * \mathbb{Z} \longrightarrow Iso(E)$ then δ is a strict cocycle on $\Omega_1 * \mathbb{Z}$ and

$$\delta^*(\omega, n)K_{t(\omega)n} = \gamma^*(q(\omega), \beta(\omega, n))K_{q(t(\omega)n)} = \gamma^*(q(\omega), \beta(\omega, n))K_{q(t(\omega)\beta(\omega,n))} = K_{q(\omega)}$$

i.e. $\{K_{t(\omega)}\}_\omega$ is a δ-invariant field, hence [18] we can find a δ-invariant section $\varphi : \Omega \longrightarrow E_1^*$. Let $K := \{(\omega, n) \in \Omega_1 * \mathbb{Z} | \delta^*(\omega, n)\varphi(\omega n) = \varphi(\omega)\}$, then K is a conull set and if (ω, n_1), $(\omega n_1, n_2) \in K$ then $(\omega, n_1 n_2) \in K$, so by Step 4 K contains an inessential contraction and we may assume that $\delta^*(\omega, n)\varphi(\omega n) = \varphi(\omega)$ for all $(\omega, n) \in \Omega_1 * \mathbb{Z}$. Define following two maps

$$\begin{aligned} \psi : \Omega_1 \times G &\longrightarrow E_1^*, \\ (\omega, g) &\longmapsto \gamma^*(q(\omega), g^{-1})^{-1}\varphi(\omega) \end{aligned} \qquad \begin{aligned} w : \Omega_1 \times G &\longrightarrow E_1^*, \\ (\omega, g) &\longmapsto \psi(\omega, g\vartheta(\omega)^{-1}) \end{aligned}$$

if $(\omega, n) \in \Omega_1 * \mathbb{Z}$ then

$$\psi(\omega n, g\beta(\omega, n)) = \psi(\omega, g) \text{ a.e.}$$

and similarly

$$w(\omega n, g\tilde{\alpha}(\omega, n)) = w(\omega, g) \text{ a.e.}$$

[13]R.J. Zimmer, Amenable ergodic group actions, Lemma 3.5
[14]In this case shift-equivariant
[15]R.J. Zimmer, Amenable ergodic group actions, Lemma 3.6
[16]A. Ramsay, Virtual groups and group actions, Lemma 5.2
[17]R.J. Zimmer, Amenable ergodic group actions, Lemma 1.7 and Step 4
[18]R.J. Zimmer, Amenable ergodic group actions, Theorem 2.1

so w is essentially $\mathbb{Z}$-invariant on $\Omega \times_\alpha G$ and hence there is a map $\sigma : M \longrightarrow E_1^*$ such that $\sigma(\tilde{p}(\omega, g)) = w(\omega, g)$ a.e. σ is a γ-invariant section[19], moreover we have

$$w(\omega, g) = \gamma^*(\tilde{p} \circ \iota(\omega)\vartheta(\omega)^{-1}, \vartheta(\omega)g^{-1})^{-1}\varphi(\omega) = \gamma_0^*(t(\tilde{p} \circ \iota(\omega))\vartheta(\omega)^{-1}, \vartheta(\omega)g^{-1})^{-1}\varphi(\omega)$$

$t(\tilde{p} \circ \iota(\omega))$ factors to Ω_0 so we just have to choose ϑ and φ correctly. By Step 3 M_0 can be chosen to be $t^{-1}(P_0)$ for some conull set $P_0 \subset P$ and hence ϑ can be chosen so that it factors to a map on Ω_0. Moreover φ was chosen to be a fixed point of the action $\mathbb{Z} \curvearrowright L^\infty(\Omega, E_1^*)$, for $n \geq 0$ we see that $\delta = \gamma \circ (q, \beta)$ factors to a function on Ω_0, so under the $\mathbb{Z}$ action, $L^\infty(\Omega_0, E_1^*)$ will be invariant under $\mathbb{N}$, which is an amenable semigroup, hence we can find an invariant section, in other words φ can be chosen so that it factor to Ω_0. $\square$

[19]R.J. Zimmer, Amenable ergodic group actions Page 365