On a weak* stochastic Fubini theorem

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1 Introduction

Let $Z = (Z_t)_{t \in \mathbb{R}_+}$ be an $\mathbb{R}^d$-valued semimartingale, $(\vartheta_{t,s})_{t \geq 0,\, 0 \leq s \leq t}$ a two-parameter $\mathbb{R}^d$-valued process such that $(\vartheta_{t,s})_{0 \leq s \leq t}$ is Z-integrable and consider the process

$$X_t := \int_0^t \vartheta_{t,s} dZ_s.$$

If Z is Brownian motion and $\vartheta_{t,s}$ is non-random then X is a Volterra-type integral process and there are many Gaussian-type results. If Z is a Lévy process and $\vartheta_{t,s} = g(t-s)$, where $g : \mathbb{R}_+ \to \mathbb{R}^d$ is deterministic, then X is called a moving average, which occurs in stochastic Volterra equations (e.g. Protter (1985)) and stochastic delay equations (e.g. Reiß et al. (2007)) among many. In particular, the question whether X is a semimartingale or not is of great importance, for instance if X models an asset-price, then it is of particular interest to show that integration with respect to X is possible. Basse/Pedersen (2009) show in this case that X is a semimartingale if and only if g is absolutely continuous with a density g' satisfying some integrability conditions which depend on the Lévy triplet.

If Z is a general semimartingale, then Protter (1985) reduces the problem to the standard stochastic Fubini theorem under the assumption that $t \mapsto \vartheta_{t,s}(\omega)$ is C^1 with a locally Lipschitz derivative, where the main idea is to decompose X as

$$X_t = \int_0^t \vartheta_{s,s} dZ_s + \int_0^t (\vartheta_{t,s} - \vartheta_{s,s}) dZ_s = \int_0^t \vartheta_{s,s} dZ_s + \int_0^t \left(\int_s^t \frac{\partial}{\partial r} \vartheta_{r,s}\, dr \right) dZ_s.$$

The stochastic Fubini theorem as in Theorem IV.64/65 of Protter (2005) yields

$$\int_0^t \left(\int_s^t \frac{\partial}{\partial r} \vartheta_{r,s}\, dr \right) dZ_s = \int_0^t \left(\int_0^r \frac{\partial}{\partial r} \vartheta_{r,s}\, dZ_s \right) dr,$$

which shows that X is a semimartingale, as the right-hand side is absolutely continuous with respect to the Lebesgue measure. Choulli/Schweizer (2013) go further and assume that

$t \mapsto \vartheta_{t,s}$ is for each $s \geq 0$ of finite variation but does not admit a dominating measure simultaneously for all $s \geq 0$ as above. The standard Fubini theorem cannot be used as discussed above, as the main assumption is that there is a fixed measure that depends on a parameter but not on the randomness (see van Neerven/Veraar (2005), Veraar (2012)). The main idea will be to construct a stochastic integral with respect to Z, where suitable integrands are (d-dimensional) measure-valued processes $\mu_t(\omega) \in \mathscr{M}([0,T])^d$, the integral $\int_0^t \mu_s dZ_s(\omega) \in \mathscr{M}([0,T])$ is a (1-dimensional) measure-valued process and it satisfies for any continuous function $f \in C([0,T])$ the property

$$\int_{[0,T]} f d\Big(\int_0^t \mu_s dZ_s\Big) = \int_0^t \Big(\int_{[0,T]} f d\mu_s\Big) dZ_s,$$

which we call the weak* Fubini property. We show, that under an extra integrability condition on the variation process of $(t \mapsto \vartheta_{t,s})_{s \in \mathbb{R}_+}$ the process X introduced above is a semimartingale.

Our strategy is the following: we view the space $\mathscr{M}([0,T])$ of signed Radon measures on $[0,T]$ abstractly as the dual space of a Banach space E. In the setup of the problem it is natural to think of the desired integrands for the new stochastic integral as being $\mathscr{M}([0,T])^d$-valued. However, we will interpret this space rather as the tensor product of two Banach spaces $\mathscr{M}([0,T]) \otimes \mathbb{R}^d$, which together with a suitable norm is again a Banach space. The main reason for this approach is that we can view $\mathbb{R}^d$ as the space of (bounded) linear forms $\mathscr{L}(\mathbb{R}^d, \mathbb{R})$ from $\mathbb{R}^d$ to $\mathbb{R}$. Recall that in the spirit of Métivier, natural integrands of an $\mathcal{H}$-valued semimartingale Z are $\mathscr{L}(\mathcal{H}, \mathcal{G})$-valued processes, where $\mathcal{H}$ and $\mathcal{G}$ are separable Hilbert spaces, so that if Y is an allowed integrand, then $\int Y dZ$ is a $\mathcal{G}$-valued process.

We can rephrase our abstract problem setting as follows: given a separable Banach space E, two separable Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$ and an $\mathcal{H}$-valued semimartingale Z we would like to define a new stochastic $E^* \otimes \mathcal{G}$-valued integral such that allowed integrands X are $E^* \otimes \mathscr{L}(\mathcal{H}, \mathcal{G})$-valued and the new stochastic integral $\int X dZ$ is $E^* \otimes \mathcal{G}$-valued and satisfies a compatibility condition with the old stochastic integral, namely

$$\int X dZ(\xi) = \int X(\xi) dZ$$

for all $\xi \in E$, where we interpret $\int X dZ$ as a linear operator from E^* to $\mathcal{G}$ and similarly X as an operator from E to $\mathscr{L}(\mathcal{H}, \mathcal{G})$. Note that this equation implicitly suggests that allowed integrands X of the new stochastic integral should at least pointwise, for each $\xi \in E$, be integrands of the old stochastic integral. Hence, our basic strategy will be to extend to old stochastic integral pointwise, satisfying some continuity properties that are well-suited for our purposes. To make this clear, if $E^* = \mathscr{M}([0,T])$, $\mathcal{H} = \mathbb{R}^d$ and $\mathcal{G} = R$, then a suitable integrand of the new stochastic integral should be of the form $\mu = (\mu_1, \ldots, \mu_d) \in \mathscr{M}([0,T])^d$

or in tensor notation $\mu = \sum_{i=1}^{d} \mu_i \otimes e_i \in \mathscr{M}([0,T]) \otimes \mathbb{R}^d$ such that $\int \mu dZ$ is an $\mathscr{M}([0,T])$-valued process, for any continuous function $f \in C([0,T])$ we have the weak* Fubini property anounced before and the process $\mu(f) = (\mu_1(f), \ldots, \mu_d(f)) \in \mathbb{R}^d$ is Z-integrable. The main drawback or difficulty of this extension is that allowed integrands of the new stochastic integrals are not necessarily measurable, and hence many arguments require special care when measurability is needed. Another technicality that we have to address is the tensor product of two Banach spaces E^* and F as discussed above. We will be forced to construct a suitable norm on $E^* \otimes F$ such that the completion of this space satisfies our requirements, the resulting tensor product is then innately related to the well-known injective tensor product.

In section 2 we discuss measurability of Banach space valued functions and processes and repeat the construction of the stochastic integral with respect to a Hilbert-valued semimartingale in the sense if Métivier. In this section we introduce the basic notation that will be used throughout this text and try to convey the main difficulties and known results regarding measurability of Banach space valued functions and processes. In section 3 we construct the new integral, which we call a weak* stochastic integral and discuss a measurability result that is tailor-made for the case of the two-parameter process. Section 4 contains the main result on the two-parameter process.

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2 Preliminaries

This section is divided into two parts. We start by discussing the concept of measurability for Banach space valued functions and processes. The main result contained in this part is Pettis' measurability theorem and the techniques involving the proof will be used later. In the second part of this section we recollect the construction of the stochastic integral with respect to a Hilbert space valued semimartingale via control processes in the sense of Métivier. The construction of the weak* integral that we develop later in Section 3 is an extension of this case, and as such several ideas already appear here.

2.1 Banach space valued functions and processes

Let $(E, \|\cdot\|)$ be a Banach space over $\mathbb{R}$ and $(C, \mathscr{C})$ a measure space. We start by recalling the notion of measurability of functions between measure spaces.

Definition 2.1. *Suppose that $\mathscr{B}$ is a σ-algebra on E. We say that a function $X : C \to E$ is $(\mathscr{C}, \mathscr{B})$-measurable if $X^{-1}(B) \in \mathscr{C}$ for all measurable subsets $B \in \mathscr{B}$.*

When $E = \mathbb{R}$ and $\mathscr{B} = \mathcal{B}(\mathbb{R})$ is the Borel σ-algebra generated by the standard topology on $\mathbb{R}$, any $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable function is the pointwise limit of a sequence of step functions. It is natural to expect that this result should generalize to the case of Banach valued functions, but this is not true for a general σ-algebra. However, as step functions are extremely practical, we define the set of $\mathscr{C}$-measurable E-valued step functions

$$\mathscr{E}(E, \mathscr{C}) := \left\{ \sum_{i=1}^{n} \mathbb{1}_{C_i} \xi_i \,\middle|\, n \in \mathbb{N},\, C_i \in \mathscr{C},\, \xi_i \in E \right\}.$$

Each step function satisfies the property that it is $(\mathscr{C}, \mathscr{B})$-measurable, irrespective of the σ-algebra $\mathscr{B}$ on E.

Lemma 2.2 (**Measurability of step functions**). *Let $\mathscr{B}$ be a σ-algebra on E and let $X \in \mathscr{E}(E, \mathscr{C})$ be a $\mathscr{C}$-measurable E-valued step function, then X is $(\mathscr{C}, \mathscr{B})$-measurable.*

Proof. Let $X = \sum_{i=1}^{n} \mathbb{1}_{C_i} \xi_i \in \mathscr{E}(E, \mathscr{C})$ and $B \in \mathscr{B}$ a measurable set, then either $X^{-1}(B)$ is empty or it is not. Suppose $X^{-1}(B) \neq \varnothing$, then $\mathrm{im}(X) \cap B = \{\xi_{i_1}, \ldots, \xi_{i_k}\}$, with $1 \leq i_1 < \cdots < i_k \leq n$, which implies that $X^{-1}(B) = \bigcup_{j=1}^{k} C_{i_j} \in \mathscr{C}$. □

Consider a $(\mathscr{C}, \mathscr{B})$-measurable function $X : C \to E$ for some σ-algebra $\mathscr{B}$ on E. A natural question that arises is whether it is possible to approximate X with step functions or vice versa. This question implicitly entails a choice of the topology on E, which must not necessarily be the norm topology. For instance, we rarely view the dual space E^* of a Banach space E together with the norm topology, instead we take the weak* topology on E^*, which has in some cases more convenient properties compared to the norm topology.

Definition 2.3. *Let $\mathcal{O}$ be a topology on E. We say that a function $X : C \to E$ is strongly $\mathscr{C}$-measurable with respect to $\mathcal{O}$ if there is a sequence $(X^n)_{n \in \mathbb{N}} \subset \mathscr{E}(E, \mathscr{C})$ of $\mathscr{C}$-measurable E-valued step functions such that $X^n(c) \xrightarrow[n \to \infty]{\mathcal{O}} X(c)$ for all $c \in C$.*

Note that it does not directly follow from the definition whether a strongly $\mathscr{C}$-measurable function with respect to a topology $\mathcal{O}$ is $(\mathscr{C}, \mathscr{B})$-measurable or not, even if $\mathscr{B}$ is the Borel σ-algebra generated by $\mathcal{O}$. We start by discussing the case where $\mathcal{O}$ is the norm topology on E. Denote the Borel σ-algebra on E generated by the norm topology by $\mathcal{B}(E)$. The following result due to Pettis (1938) Theorem 1.1 characterizes strongly $\mathscr{C}$-measurable functions with respect to the norm topology.

Proposition 2.4 (Pettis' measurability theorem). *Let $X : C \to E$ be a function. Then, X is strongly $\mathscr{C}$-measurable with respect to the norm topology if and only if X is separably valued and the function $\lambda(X) : C \to \mathbb{R}$ is $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable for all $\lambda \in E^*$.*

Proof. The proof we give here is basically identical to the original proof given in Pettis (1938). This following version can be found in van Neerven (2008) Theorem 1.5 up to minor changes. If X is strongly $\mathscr{C}$-measurable with respect to the norm topology we can find a sequence $(X^n)_{n\in\mathbb{N}} \subset \mathscr{E}(E, \mathscr{C})$ of $\mathscr{C}$-measurable E-valued step functions that converges pointwise to X in the norm topology. Each step function X^n has finite range $\mathrm{im}(X^n) = \{\xi_1^n, \ldots, \xi_{k_n}^n\}$. Hence, if we denote by E_0 the closure of the subspace spanned by all these vectors we obtain that $\mathrm{im}(X) \subset E_0$, which shows that X is separably valued. Moreover, if we take any $\lambda \in E^*$, the $\mathbb{R}$-valued step function $\lambda(X^n)$ is $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable by Lemma 2.2 for any $n \in \mathbb{N}$ and $\lambda(X^n(c)) \xrightarrow[n\to\infty]{} \lambda(X(c))$ for any $c \in C$, which shows that $\lambda(X)$ is $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable.

Conversely suppose that $\mathrm{im}(X) \subset E_0$ for a separable closed subspace $E_0 \subset E$. Let $\{\xi_n\}_{n\in\mathbb{N}} \subset E_0$ be a dense countable subset in E_0. Denote by E_1^* the closed unit ball in the dual space E^* and choose a countable subset $\{\lambda_n\}_{n\in\mathbb{N}} \subset E_1^*$ such that for any $\xi \in E_0$

$$\|\xi\| = \sup_{n\in\mathbb{N}} |\lambda_n(\xi)|.$$

For completeness we show how to find such a sequence: recall that

$$\|\xi\| = \sup_{\lambda\in E_1^*} |\lambda(\xi)|,$$

hence for any $n \in \mathbb{N}$ we may choose $\lambda_n \in E_1^*$ such that $0 \le \|\xi_n\| - |\lambda_n(\xi_n)| \le \frac{1}{n}$. So if $\xi \in E_0$, then for subsequence $(\xi_{n_k})_{k\in\mathbb{N}} \subset (\xi_n)_{n\in\mathbb{N}}$, such that $\|\xi - \xi_{n_k}\| \xrightarrow[k\to\infty]{} 0$, we get

$$\sup_{n\in\mathbb{N}} |\lambda_n(\xi)| \le \|\xi\| \le \|\xi - \xi_{n_k}\| + \|\xi_{n_k}\| \le \|\xi - \xi_{n_k}\| + \frac{1}{n_k} + |\lambda_{n_k}(\xi_{n_k})| \le$$

$$\|\xi - \xi_{n_k}\| + \frac{1}{n_k} + |\lambda_{n_k}(\xi - \xi_{n_k})| + |\lambda_{n_k}(\xi)| \le 2\|\xi - \xi_{n_k}\| + \frac{1}{n_k} + \sup_{n\in\mathbb{N}} |\lambda_n(\xi)| \xrightarrow[k\to\infty]{} \sup_{n\in\mathbb{N}} |\lambda_n(\xi)|,$$

which proves the existence of the countable subset $\{\lambda_n\}_{n\in\mathbb{N}}$ as claimed above.

For any $\xi \in E_0$ we notice that the function

$$\begin{aligned} &C \to [0, \infty) \\ &c \mapsto \|X(c) - \xi\| = \sup_{n\in\mathbb{N}} |\lambda_n\big(X(c) - \xi\big)|, \end{aligned}$$

is $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable, as it is the countable supremum of $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable. Hence, for any $m \in \mathbb{N}$ and $1 \leq k \leq m$ we know that the sets

$$C_k^{m,1} = \left\{ c \in C \,\middle|\, d(X^m(c), \lambda_k) = \min_{1 \leq i \leq n} d(X^m(c), \lambda_i) \right\} \text{ and}$$

$$C_k^{m,2} = \left\{ c \in C \,\middle|\, \forall\, 1 \leq l \leq k-1 : d(X^m(c), \lambda_l) > \min_{1 \leq i \leq n} d(X^m(c), \lambda_i) \right\},$$

are $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable. Let $m \in \mathbb{N}$, $\xi \in E_0$ and denote by $k_m(\xi)$ the least integer with the property that

$$k_m(\xi) := \min \left\{ 1 \leq k \leq m \,\middle|\, \|\xi - \xi_k\| = \min_{1 \leq i \leq m} \|\xi - \xi_i\| \right\}.$$

Note that $\|\xi_{k_m(\lambda)} - \xi\| \xrightarrow[m \to \infty]{} 0$, by density of the subset $\{\xi_n\}_{n \in \mathbb{N}}$ in E_0. Define the function

$$\begin{aligned} X^m : C &\to E \\ c &\mapsto \sum_{k=1}^{m} \mathbb{1}_{C_k^m}(c) \xi_k, \end{aligned}$$

where $C_k^m = C_k^{m,1} \cap C_k^{m,2} = \{c \in C \,|\, X^m(c) = \xi_k\} \in \mathscr{C}$. Let $c \in C$, then $X^m(c) \xrightarrow[m \to \infty]{} X(c)$ in the norm topology by the previous discussion, which shows that X is strongly $\mathscr{C}$-measurable with respect to the norm topology. $\square$

This powerful result due to Pettis (1938) implies that measurable Banach valued functions behave like $\mathbb{R}$-valued measurable functions under the Borel σ-algebra generated by the norm topology, in the sense that they can be approximated by step functions and pointwise limits are measurable. The next results make this clear.

Corollary 2.5 (**Pointwise limit of strongly measurable functions**). *Suppose that $(X^n)_{n \in \mathbb{N}}$ is a sequence of strongly $\mathscr{C}$-measurable functions with respect to the norm topology that converges pointwise to a function $X : C \to E$, then X is strongly $\mathscr{C}$-measurable with respect to the norm topology.*

Proof. Suppose that $(X^n)_{n \in \mathbb{N}}$ is a sequence of strongly $\mathscr{C}$-measurable functions with respect to the norm topology that converges pointwise to X. By Proposition 2.4, each X^n takes values in some separable closed subspace $E_0^n \subset E$. Hence, X takes only values in the closure of the span of these separable subspaces, which in turn is separable again. Moreover, if $\lambda \in E^*$, then each $\lambda(X^n)$ is measurable by Proposition 2.4 and $\lambda(X^n(c)) \xrightarrow[n \to \infty]{} \lambda(X(c))$ for all $c \in C$, which shows that $\lambda(X)$ is $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable. By Proposition 2.4 it follows that X is strongly $\mathscr{C}$-measurable with respect to the norm topology. $\square$

Proposition 2.6 (Strong measurability and measurability). *Let $X : C \to E$ be a function. Then, X is strongly $\mathscr{C}$-measurable with respect to the norm topology if and only if X is separably valued and $(\mathscr{C}, \mathcal{B}(E))$-measurable.*

Proof. This proof can be found in Rieffel (1970) Chapter 3 Theorem 10 and Lemma 12. Suppose that X is separably valued and $(\mathscr{C}, \mathcal{B}(E))$-measurable. Any $\lambda \in E^*$ is in particular continuous, so $\lambda(X)$ is $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable. Hence, by Proposition 2.4 we may conclude that X is strongly $(\mathscr{C}, \mathcal{B}(E))$-measurable.

Conversely, if X is strongly $\mathscr{C}$-measurable with respect to the norm topology, then in particular it is separably valued. Let $(X^n)_{n\in\mathbb{N}} \subset \mathscr{E}(E, \mathscr{C})$ be a sequence of $\mathscr{C}$-measurable E-valued step functions converging pointwise to X in the norm topology. To show that X is $(\mathscr{C}, \mathcal{B}(E))$-measurable it suffices to verify that $X^{-1}(U) \in \mathscr{C}$ for all open sets $U \in E$, since X^{-1} is a Boolean algebra homomorphism. Let $U \in E$ be open, we claim that

$$X^{-1}(U) = \bigcup_{m\geq 1} \bigcup_{n\geq 1} \bigcap_{k\geq n} (X^k)^{-1}\left(\left\{\xi \in E \,\middle|\, d(\xi, U^c) > \frac{1}{m}\right\}\right),$$

where $d(\xi, U^c) = \inf\{\|\xi - \xi'\| \,|\, \xi' \in U^c\}$. Indeed, $c \in X^{-1}(U)$ if and only if $X(c) \in U$ if and only if $X^k(c) \in \{\xi \in E |\, d(\xi, U^c) > \frac{1}{m}\}$ for some $m, n \in \mathbb{N}$ and all $k \geq n$, where we use that U is open. By Lemma 2.2 X^k is $(\mathscr{C}, \mathcal{B}(E))$-measurable for any $k \in \mathbb{N}$, hence $X^{-1}(U) \in \mathscr{C}$. $\square$

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. We start by extending the notion of strong measurability to the notion of P-strong measurability and by defining the (Bochner-)integral with respect to Banach valued functions.

Definition 2.7. *Let $\mathcal{O}$ be a topology on E. We say that a function $X : \Omega \to E$ is P-strongly $\mathscr{F}$-measurable with respect to $\mathcal{O}$ if there is a sequence $(X^n)_{n\in\mathbb{N}} \subset \mathscr{E}(E, \mathscr{C})$ of $\mathscr{F}$-measurable E-valued step functions such that $X^n(\omega) \xrightarrow[n\to\infty]{\mathcal{O}} X(\omega)$ for P-a.e. $\omega \in \Omega$.*

The results presented above extend naturally to the P-a.e. case, hence we do not repeat them. Instead we refer to Talagrand Chapter 3. The (Bochner-)integral of an E-valued function with respect to P, which as we will explain, is an extension of the naturally defined integral with respect to step functions.

Definition 2.8. *Let $X = \sum_{i=1}^n \mathbb{1}_{F_i}\xi_i \in \mathscr{E}(E, \mathscr{F})$ be an $\mathscr{F}$-measurable E-valued step function. We call*

$$\mathbb{E}(X) := \int_\Omega X dP := \sum_{i=1}^n P(F_i)\xi_i \in E,$$

the integral of X with respect to P.

Lemma 2.9. *Let $X : \Omega \to E$ be a P-strongly $\mathscr{F}$-measurable function with respect to the norm topology, then the $\mathbb{R}$-valued function $\|X\| : \Omega \to \mathbb{R}$ is $(\mathscr{F}, \mathcal{B}(\mathbb{R}))$-measurable.*

Proof. By Proposition 2.4, respectively the extension to the P-a.e case, there is a conull set $\Omega_0 \subset \Omega$ such that $X : \Omega_0 \to E$ is separably valued. Suppose that $X(\Omega_0) \subset E_0$ for a closed separable subspace $E_0 \subset E$ and choose a countable subset $\{\lambda_n\}_{n\in\mathbb{N}} \subset E_1^*$ in the unit ball of E^* as in the proof of Proposition 2.4, such that $\|\xi\| = \sup_{n\in\mathbb{N}} |\lambda_n(\xi)|$ for all $\xi \in E_0$. Note that $\lambda(X)$ is $(\mathscr{F}, \mathcal{B}(\mathbb{R}))$-measurable for all $\lambda \in E^*$ by Proposition 2.4, so $\|X\|$ is $(\mathscr{F}, \mathcal{B}(\mathbb{R}))$-measurable, as a countable supremum of measurable functions. □

Definition 2.10. *Let $\mathcal{O}$ be a topology on E. We define the class $L^0(\Omega, \mathscr{F}; E, \mathcal{O})$ of P-strongly $\mathscr{F}$-measurable functions with respect to $\mathcal{O}$, where we identify functions that are equal P-a.e. Whenever we say that an element $X \in L^0(\Omega, \mathscr{F}; E, \mathcal{O})$ has a given property, we mean that there is a representative in the class corresponding to X with this property, and we denote this representative by X.*

Whenever the topology on E is clear, we write $L^0(\Omega, \mathscr{F}; E)$ instead of $L^0(\Omega, \mathscr{F}; E, \mathcal{O})$. Note that if E is separable and $\mathcal{B}(E)$ is the Borel σ-algebra on E generated by the norm topology, then the conditions $(\mathscr{F}, \mathcal{B}(E))$-measurable and P-strongly $\mathscr{F}$-measurable with respect to the norm topology are tautological by Proposition 2.6. This discussion enables a definition of the integral with respect to P for an $(\mathscr{F}, \mathcal{B}(E))$-measurable function X. The mapping defined by

$$\begin{aligned} \|\cdot\|_{L^1} : L^0(\Omega, \mathscr{F}; E, \mathcal{B}(E)) &\to [0, \infty] \\ X &\mapsto \mathbb{E}(\|X\|), \end{aligned}$$

which is well-defined by Lemma 2.9, defines a seminorm on the vector space

$$V_0 := \left\{ X \in L^0(\Omega, \mathscr{F}; E, \mathcal{B}(E)) \,\middle|\, \mathbb{E}(\|X\|) < \infty \right\}.$$

Indeed, it is positive and if $X, Y \in V_0$ then $\|X(\omega) + Y(\omega)\| \leq \|X(\omega)\| + \|Y(\omega)\|$ for P-a.e. $\omega \in \Omega$, which proves the triangle inequality. Moreover, note that $\mathscr{E}(E, \mathscr{F}) \subset V_0$.

Definition 2.11. *We denote by $L^1(\Omega, \mathscr{F}; E, \mathcal{B}(E))$ the closure of $\mathscr{E}(E, \mathscr{F})$ in V_0 with respect to the seminorm $\|\cdot\|_{L^1}$ identifying functions that agree with respect to the seminorm $\|\cdot\|_{L^1}$.*

The next result defines the integral with respect to P for an element in $L^1(\Omega, \mathscr{F}; E, \mathcal{B}(E))$ and gives a practical characterization of this integral in terms of the dual pairing.

Lemma 2.12. *Let $X \in L^1(\Omega, \mathscr{F}; E, \mathcal{B}(E))$, then $\mathbb{E}(X) \in E$ exists and $\lambda(\mathbb{E}(X)) = \mathbb{E}(\lambda(X))$ for all $\lambda \in E^*$.*

Proof. Suppose that $(X^n)_{n\in\mathbb{N}} \subset \mathscr{E}(E, \mathscr{F})$ is a sequence $\mathscr{F}$-measurable E-valued step functions such that $\|X^n(\omega) - X(\omega)\| \xrightarrow[n\to\infty]{} 0$ for P-a.e. $\omega \in \Omega$, then

$$\|\mathbb{E}(X^n - X^m)\| \leq \mathbb{E}(\|X^n - X^m\|) \xrightarrow[n,m\to\infty]{} 0,$$

we denote the limit of this Cauchy sequence in E by $\mathbb{E}(X)$. Let $\lambda \in E^*$, then

$$\lambda(\mathbb{E}(X^n)) = \mathbb{E}(\lambda(X^n)),$$

for all $n \in \mathbb{N}$. By continuity of $\lambda \in E^*$ we have $\lambda(\mathbb{E}(X^n)) \xrightarrow[n\to\infty]{} \lambda(\mathbb{E}(X))$. Moreover, $\lambda(X^n) \xrightarrow[n\to\infty]{P\text{-a.e.}} \lambda(X)$, $\lambda(X^n) \overset{P\text{-a.e.}}{\leq} \|\lambda\|\|X^n\|$ and $\lambda(X) \overset{P\text{-a.e.}}{\leq} \|\lambda\|\|X\|$ which implies by dominated convergence that $\mathbb{E}(\lambda(X^n)) \xrightarrow[n\to\infty]{} \mathbb{E}(\lambda(X))$. □

Notice that we defined the integral with respect to P just for $(\mathscr{F}, \mathcal{B}(E))$-measurable functions, where $\mathcal{B}(E)$ is the σ-algebra generated by the norm topology. With more effort, it is possible to define an integral with respect to P if the topology that generates the σ-algebra satisfies some regularity conditions. However, we will circumvent this issue and exploit the definition and properties of this integral.

We are now able to introduce the notion of conditional expectation in this setup. Suppose that $\mathscr{G} \subset \mathscr{F}$ is a sub-σ-algebra, define the mapping

$$\begin{aligned} \mathbb{E}(\,\cdot \mid \mathscr{G}) : \mathscr{E}(E, \mathscr{F}) &\to \mathscr{E}(E, \mathscr{G}) \\ \sum_{i=1}^n \mathbb{1}_{F_i}\xi_i &\mapsto \sum_{i=1}^n \mathbb{E}(\mathbb{1}_{F_i} \mid \mathscr{G})\xi_i, \end{aligned}$$

and note that for any $\mathscr{F}$-measurable E-valued step function $X = \sum_{i=1}^n \mathbb{1}_{F_i}\xi_i$ we have

$$\mathbb{E}\Big(\|\mathbb{E}(X \mid \mathscr{G})\|\Big) \leq \sum_{i=1}^n \mathbb{E}(\mathbb{1}_{F_i})\|\xi\| = \mathbb{E}(\|X\|),$$

where we assume that the measurable sets in the representation of X are disjoint, which we may always assume. Hence, $\mathbb{E}(\,\cdot \mid \mathscr{G})$ is a continuous linear mapping from a dense subspace of $L^1(\Omega, \mathscr{F}; E, \mathcal{B}(E))$ into $L^1(\Omega, \mathscr{G}; E, \mathcal{B}(E))$, such that the operator norm satisfies $\|\mathbb{E}(\,\cdot \mid \mathscr{G})\|_{op} \leq 1$. This mapping can be uniquely extended to an operator from $L^1(\Omega, \mathscr{F}; E, \mathcal{B}(E))$ into $L^1(\Omega, \mathscr{G}; E, \mathcal{B}(E))$ satisfying the same norm. We emphasize that this conditional expectation is defined without using any Radon-Nikodym type of theorem.

Definition 2.13. *Suppose that $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is a filtration in $\mathscr{F}$. A process $M : \mathbb{R}\times\Omega \to E$ is called an E-valued martingale if $M_t \in L^1(\Omega, \mathscr{F}_t; E, \mathcal{B}(E))$ for all $t\in\mathbb{R}_+$ and for all $s<t$ $\mathbb{E}(M_t|\, \mathscr{F}_s) \overset{P\text{-a.e.}}{=} M_s$.*

Corollary 2.14. *Let $(M)_{t\in\mathbb{R}_+}$ be an E-valued martingale and $\lambda\in E^*$, then $(\lambda(M_t))_{t\in\mathbb{R}_+}$ is an $\mathbb{R}$-valued martingale.*

Proof. This is a direct application of Lemma 2.12. Indeed, if $\lambda\in E^*$, then $\lambda(M)$ is adapted and integrable. If $s<t$ and $F_s\in\mathscr{F}_s$, then

$$\mathbb{E}\Big(\mathbb{1}_{F_s}\lambda(M_t - M_s)\Big) = \lambda\Big(\mathbb{E}\Big(\mathbb{1}_{F_s}(M_t - M_s)\Big)\Big) = 0,$$

which shows that $\lambda(M)$ is an $\mathbb{R}$-valued martingale. $\square$

Note that when $E=\mathbb{R}$ the space L^1 we defined above is the regular L^1. We turn to the discussion of measurability for Banach valued processes processes. Suppose that $\mathscr{C}\subset\mathcal{B}(\mathbb{R})\otimes\mathscr{F} =: \bar{\mathscr{F}}$ is a sub-σ-algebra on $\bar{\Omega} := \Omega\times\mathbb{R}$.

Definition 2.15. *Let $\mathcal{O}$ be a topology on E. We say that an E-valued process $X=(X_t)_{t\in\mathbb{R}_+}$ is P-strongly $\mathscr{C}$-measurable with respect to $\mathcal{O}$ if there is a sequence of $\mathscr{C}$-measurable step functions $(X^n)_{n\in\mathbb{N}}\subset\mathscr{E}(E,\mathscr{C})$ that converges to X up to indistinguishability, that is*

$$P\Bigg(\Bigg\{\omega\in\Omega\,\Bigg|\,\exists t\in\mathbb{R}_+ : X_t(\omega) \overset{\mathcal{O}}{\neq} \lim_{n\to\infty} X^n_t(\omega)\Bigg\}\Bigg) = 0.$$

We define the class $L^0(\bar{\Omega},\mathscr{C};E,\mathcal{O})$ of P-strongly $\mathscr{C}$-measurable processes with respect to $\mathcal{O}$, where we identify processes that are equal up to indistinguishability. Moreover, we say that $X\in L^0(\bar{\Omega},\mathscr{C};E,\mathcal{O})$ has a given property, if there is a representative in this class that has this property and we denote this representative by X.

As above we repeat that this definition does not directly imply that a P-strongly $\mathscr{C}$-measurable process X is $(\mathscr{C},\mathscr{B})$-measurable, even when $\mathscr{B}$ is the Borel σ-algebra generated by $\mathcal{O}$. However, in the case where $\mathcal{O}$ is the norm topology on E the previous results translate in the language of processes into the following:

Corollary 2.16. *A process X is P-strongly $\mathscr{C}$-measurable with respect to the norm topology if and only if X is separably valued up to indistinguishability and the process $\lambda(X):\bar{\Omega}\to\mathbb{R}$ is P-strongly $(\mathscr{C},\mathcal{B}(\mathbb{R}))$-measurable simultaneously for all $\lambda\in E^*$.*

Proof. Suppose that X is P-strongly $\mathscr{C}$-measurable with respect to the norm topology. Let $(X^n)_{n\in\mathbb{N}}\subset\mathscr{E}(E,\mathscr{C})$ be a sequence of $\mathscr{C}$-measurable E-valued step functions such that $\|X^n_t(\omega)-X_t(\omega)\| \xrightarrow[n\to\infty]{} 0$ for all $(t,\omega)\in\mathbb{R}\times\Omega_0$, where $\Omega_0\subset\Omega$ is a conull subset. The process $\bar{X} := \mathbb{1}_{\mathbb{R}\times\Omega_0} X$ is strongly $(\mathscr{C}, \mathcal{B}(E))$-measurable, so by Proposition 2.4 it is separably valued. Suppose that $\operatorname{im}(\bar{X}) \subset E_0$ for a separable closed subspace $E_0 \subset E$, then

$$P\left(\left\{\omega \in \Omega \middle| \exists t \in \mathbb{R}_+ :\ X_t(\omega) \notin E_0\right\}\right) \leq P(\Omega_o^c) = 0,$$

which shows that X is separably valued up to indistinguishability. Let $\lambda \in E^*$, then by continuity we obtain

$$\lambda(X_t(\omega)) = \lambda(\lim_{n\to\infty} X_t^n(\omega)) = \lim_{n\to\infty} \lambda(X_t^n(\omega)),$$

for all $(t, \omega) \in \mathbb{R} \times \Omega_0$, which shows that the process $\lambda(X) : \bar{\Omega} \to \mathbb{R}$ is P-strongly $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable.

Conversely suppose that X is separably valued up to indisinguishability and let $E_0 \subset E$ be a closed separable subspace such that $X_t(\omega) \in E_0$ for all $(t, \omega) \in \mathbb{R}\times\Omega_1$, where $\Omega_1 \subset \Omega$ is a conull set. By assumption $\lambda(X) : \bar{\Omega} \to \mathbb{R}$ is P-strongly $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable simultaneously for all $\lambda \in E^*$, so there is a conull subset $\Omega_2 \subset \Omega$ such that $\mathbb{1}_{\mathbb{R}\times\Omega_2}\lambda(X)$ is $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable for all $\lambda \in E^*$. The set $\Omega_0 := \Omega_1 \cap \Omega_2$ is a conull subset such that $\bar{X} := \mathbb{1}_{\mathbb{R}\times\Omega_0} X$ is separably valued and $\lambda(\bar{X}) = \mathbb{1}_{\mathbb{R}\times\Omega_0}\lambda(X)$ is $(\mathscr{C}, \mathcal{B}(\mathbb{R}))$-measurable for all $\lambda \in E^*$, so $\bar{X}$ is strongly $(\mathscr{C}, \mathcal{B}(E))$-measurable by Proposition 2.4 and indistinguishable from X. Hence, if we let $(X^n)_{n\in\mathbb{N}} \subset \mathscr{E}(E, \mathscr{C})$ be a sequence of $\mathscr{C}$-measurable E-valued step functions such that $\|X_t^n(\omega) - \bar{X}_t(\omega)\| \xrightarrow[n\to\infty]{} 0$ for all $(t, \omega) \in \bar{\Omega}$, then

$$P\left(\left\{\omega \in \Omega \middle| \exists t \in \mathbb{R}_+ :\ X_t(\omega) \overset{\|\cdot\|}{\neq} \lim_{n\to\infty} X_t^n(\omega)\right\}\right) \leq P(\Omega_0^c) = 0,$$

which shows that X is P-strongly $\mathscr{C}$-measurable with respect to the norm topology.

□

Corollary 2.17. *Suppose that $(X^n)_{n\in\mathbb{N}}$ is a sequence of P-strongly $\mathscr{C}$-measurable processes with respect to the norm topology that converges up to indistinguishability to the process $X : \bar{\Omega} \to E$, then X is P-strongly $\mathscr{C}$-measurable with respect to the norm topology.*

Proof. Let $\Omega_0 \subset \Omega$ be a conull set such that $\|X_t(\omega) - X_t^n(\omega)\| \xrightarrow[n\to\infty]{}$ for all $(t, \omega) \in \mathbb{R} \times \Omega_0$. For each $n \in \mathbb{N}$ let $\Omega_n \subset \Omega$ be a conull set such that $\mathbb{1}_{\mathbb{R}\times\Omega_n} X^n$ is strongly $\mathscr{C}$-measurable with respect to the norm topology. The set $\Omega_* := \bigcap_{n\in\mathbb{N}_0} \Omega_n$ is still a conull set, the function $\bar{X}^n := \mathbb{1}_{\mathbb{R}\times\Omega_*} X^n$ is strongly $\mathscr{C}$-measurable with respect to the norm topology for each $n \in \mathbb{N}$ and $\|X_t(\omega) - X_t^n(\omega)\| \xrightarrow[n\to\infty]{} 0$ for all $(t, \omega) \in \mathbb{R} \times \Omega_*$. Hence, $\bar{X} := \mathbb{1}_{\mathbb{R}\times\Omega_*} X$ is strongly $\mathscr{C}$-measurable with respect to the norm topology by Corollary 2.5. Let $(Y^n)_{n\in\mathbb{N}} \subset \mathscr{E}(E, \mathscr{C})$ be a sequence of E-valued $\mathscr{C}$-measurable step functions such that $\|\bar{X}_t(\omega) - Y_t^n(\omega)\| \xrightarrow[n\to\infty]{} 0$

for all $(t,\omega) \in \bar{\Omega}$, then

$$P\left(\left\{\omega \in \Omega \,\middle|\, \exists t \in \mathbb{R}_+ :\ X_t(\omega) \overset{\|\cdot\|}{\neq} \lim_{n\to\infty} Y_t^n(\omega)\right\}\right) \leq P(\Omega_*^c) = 0,$$

which proves that X is P-strongly $\mathscr{C}$-measurable with respect to the norm topology. $\square$

For a more detailed and thorough discussion we refer to Chapter 3 in Rieffel (1970), Chapter 1 in van Neerven (2008) and Chapter 3 in Talagrand (1984) for the topic of measurability for Banach space valued functions and processes. We refer to Chapter 1.10 in Métiver (1977), Chapter 1 in Dinculeanu (2000) and Pettis (1938) for a discussion on the Bochner integral. It should be noted that all results presented here were only valid for the norm topology, respectively the σ-algebra generated by this topology. In general, it is not clear at all, whether and when the limit of strongly measurable processes is still strongly measurable and whether and when a measurable function is again strongly measurable and vice versa. The usual definition of measurability is not useful in the case of Banach valued functions, this was already noted by Bochner (1933) and Pettis (1938), who call strongly measurable functions directly measurable functions. This measurability issue will be discussed in more detail in Sections 3 and 4. However, it is beyond our scope to give a full characterization in these cases. Even in the case where the σ-algebra is generated by the norm topology, there may exist (depending on the axiomatic system being used) measurable functions that are not strongly measurable, we refer for instance to Chadwick (1982) for a discussion on this topic. In Section 3 we return to this problem when discussing measurability with respect to the σ-algebra generated by the weak* topology on E^*. It is very important to notice that many of the results presented here do generalize under some circumstances to the case where the topology on E is metrizable in a suitable way, or if the process takes values in a metrizable subspace of E.

2.2 The stochastic integral

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space satisfying the usual assumptions and set as above $\bar{\Omega} := \mathbb{R}_+ \times \Omega$. The main σ-algebra we will consider on $\bar{\Omega}$ is the predictable σ-algebra $\mathscr{P}$ generated by all adapted and left-continuous $\mathbb{R}$-valued processes.

Lemma 2.18 (**Predictable rectangles**). *The family of predictable rectangles defined by*

$$\mathscr{R} := \left\{(s,t] \times F \,\middle|\, s,t \in \mathbb{R}_+,\ s < t,\ F \in \mathscr{F}_s\right\} \cup \left\{\{0\} \times F \,\middle|\, F \in \mathscr{F}_0\right\},$$

generates the predictable σ-algebra $\mathscr{P}$.

Proof. The proof given here is standard and a general version with more generators of $\mathscr{P}$ can be found for instance in Theorem 3.3 in Métivier (1982). Suppose that $R = (s,t] \times F \in \mathscr{R}$, where $F \in \mathscr{F}_s$. The function $\mathbb{1}_R : \bar{\Omega} \to \mathbb{R}$ is adapted and left-continuous, hence $\sigma(\mathscr{R}) \subset \mathscr{P}$.

Conversely suppose that $X : \bar{\Omega} \to \mathbb{R}$ is adapted, left-continuous and bounded. Define for all $n \in \mathbb{N}$ the process

$$X_t^n(\omega) := \mathbb{1}_{\{0\}}(t)X_0(\omega) + \sum_{i\in\mathbb{N}} \mathbb{1}_{(i2^{-n},(i+1)2^{-n}]}(t)X_{i2^{-n}}(\omega).$$

Let $B \in \mathcal{B}(\mathbb{R})$ be a measurable set, then

$$(X^n)^{-1}(B) = \{0\} \times \{\omega \in \Omega \big| \ X_0(\omega) \in B\} \cup \bigcup_{i\in\mathbb{N}} \Big((i2^{-n},(i+1)2^{-n}] \times \{\omega \in \Omega \big| \ X_{i2^{-n}}(\omega) \in B\} \Big),$$

is $\mathscr{P}$ measurable, as $\{\omega \in \Omega | X_0(\omega) \in B\} \in \mathscr{F}_0$ and $\{\omega \in \Omega | X_{i2^{-n}}(\omega) \in B\} \in \mathscr{F}_{i2^{-n}}$ by adaptedness. Let $(t,\omega) \in \bar{\Omega}$ then $X_t^n(\omega) \xrightarrow[n\to\infty]{} X_t(\omega)$ by left-continuity of X, which implies that $\mathscr{P} \subset \sigma(\mathscr{R})$. $\square$

Hence, whenever $P \in \mathscr{P}$ is a predictable set, there is a sequence of step functions of the form $X^n := \sum_{i=1}^n \mathbb{1}_{R_i^n}\lambda_i^n$ with $\lambda_i^n \in \mathbb{R}$ and $R_i^n \in \mathscr{R}$ such that $X_t^n(\omega) \xrightarrow[n\to\infty]{} \mathbb{1}_P(t,\omega)$ for all $(t,\omega) \in \bar{\Omega}$. This in turn implies that it is sufficient to consider $\mathscr{R}$-measurable E-valued step functions of the form

$$\mathscr{E}(E) := \left\{ \sum_{i=1}^n \mathbb{1}_{R_i}\xi_i \,\middle|\, n \in \mathbb{N},\, R_i \in \mathscr{R},\, \xi_i \in E \right\},$$

instead of working directly with the set $\mathscr{E}(E,\mathscr{P})$ of $\mathscr{P}$-measurable E-valued step function, which we defined above. As it will become apparent in the next few lines, the structure of predictable rectangles is optimal for the definition of an integral.

We start by defining the stochastic integral of an $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step function with respect to an $\mathcal{H}$-valued process, where $\mathscr{L}(\mathcal{H},\mathcal{G})$ denotes the Banach space of bounded operators from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{G}$.

Definition 2.19. *Let $\mathcal{H}$ and $\mathcal{G}$ be two Hilbert spaces and $Z : \bar{\Omega} \to \mathcal{H}$ and $\mathcal{H}$-valued process. For an $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},G)$-valued step function $X = \sum_{i=1}^n \mathbb{1}_{(s_i,t_i]\times F_i}T_i \in \mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}))$ we define the process*

$$\int_0^t X_s dZ_s(\omega) := \sum_{i=1}^n \mathbb{1}_{F_i}(\omega)T_i\Big(Z_{t\wedge t_i}(\omega) - Z_{t\wedge s_i}(\omega)\Big) \in \mathcal{G},$$

and we call $\int XdZ : \bar{\Omega} \to \mathcal{G}$ the stochastic integral of X with respect to Z.

Let $\mathcal{H}$ be separable Hilbert space, denote the set of $\mathcal{H}$-valued semimartingales by $\mathscr{S}(\mathcal{H})$ and fix $Z \in \mathscr{S}(\mathcal{H})$. The semimartingale Z and the Hilbert space $\mathcal{H}$ will be kept unchanged throughout this part.

Definition 2.20. *We call an increasing, positive and adapted process $A = (A_t)_{t\in\mathbb{R}_+}$ a control process for Z if for any Hilbert space $\mathcal{G}$, any $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H}, G)$-valued step function $X \in \mathscr{E}(L(\mathcal{H}, \mathcal{G}))$ and any stopping time $\tau \in \mathscr{T}$ we have*

$$\mathbb{E}\bigg(\sup_{t<\tau}\bigg\|\int_0^t X_s dZ_s\bigg\|_{\mathcal{G}}^2\bigg) \leq \mathbb{E}\bigg(A_{\tau^-}\int_0^{\tau^-}\|X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})}dA_s\bigg),$$

where $A_{\tau^-}(\omega) := \lim_{t\nearrow\tau(\omega)} A_t(\omega)$. We denote by $\mathscr{A}(Z)$ the set of all control processes for Z.

An intuitive explanation of why we require τ^-, shared to me by Martin Schweizer, is that this corresponds to pre-stopping, for instance if X is a left-continuous process and τ is the first time at which the process exceeds a given constant, then the pre-stopped process X^{τ^-} will be bounded by this constant, but this may not be true for the stopped process X^τ.

To illustrate the concept of a control process we show how to construct it for a process of finite variation and a continuous locally square integrable martingale.

Example 2.21 (Processes of finite variation). *Suppose that V is an RCLL $\mathcal{H}$-valued process of finite variation. We denote by $|V|$ the increasing process such that $|V|_t(\omega)$ is the variation of $V_\cdot(\omega)$ on the interval $[0,t]$, that is*

$$|V|_t(\omega) := Var(V_\cdot(\omega), [0,t]) :=$$

$$\sup\bigg\{\sum_{i=1}^n \|V_{t_{i+1}}(\omega) - V_{t_i}(\omega)\|_{\mathcal{H}}\bigg|\, n\in\mathbb{N},\ 0\leq t_0 < t_1 < \cdots < t_n \leq t\bigg\}.$$

Let $\tau \in \mathscr{T}$ be a stopping time, $X = \sum_{i=1}^n \mathbb{1}_{(s_i,t_i]\times F_i}T_i \in \mathscr{E}(L(\mathcal{H},\mathcal{G}))$ an $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H}, G)$-valued step function and assume without loss of generality that the predictable rectangles in the representation of X are disjoint, then for P-a.e. $\omega\in\Omega$ we have

$$\sup_{t<\tau(\omega)}\bigg\|\int_0^t X_s dV_s(\omega)\bigg\|_{\mathcal{G}} = \sup_{t<\tau(\omega)}\bigg\|\sum_{i=1}^n \mathbb{1}_{F_i}(\omega)T_i\Big(V_{t\wedge t_i}(\omega) - V_{t\wedge s_i}(\omega)\Big)\bigg\|_{\mathcal{G}} \leq$$

$$\sup_{t<\tau(\omega)}\sum_{i=1}^n \mathbb{1}_{F_i}(\omega)\|T_i\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}\|V_{t\wedge t_i}(\omega) - V_{t\wedge s_i}(\omega)\|_{\mathcal{H}} \leq \int_0^{\tau(\omega)^-}\|X_s(\omega)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}d|V|_s(\omega) < \infty,$$

and the Cauchy-Schwarz inequality for the Lebesgue-Stieltjes integral in the last inequality yields

$$\sup_{t<\tau(\omega)}\bigg\|\int_0^t X_s dV_s(\omega)\bigg\|_{\mathcal{G}}^2 \leq |V|_{\tau^-}(\omega)\int_0^{\tau(\omega)^-}\|X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})}d|V|_s(\omega).$$

Taking the integral with respect to P gives

$$\mathbb{E}\left(\sup_{t<\tau}\left\| \int_0^t X_s dV_s \right\|_{\mathcal{G}}\right) \leq \mathbb{E}\left(|V|_{\tau^-} \int_0^{\tau^-} \|X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} d|V|_s\right),$$

which shows that the variation process of any RCLL $\mathcal{H}$-valued process V of finite variation is a control process for V.

Example 2.22 (**Square integrable continuous martingales**). *Suppose that $M \in \mathscr{M}^{2,c}_{loc}(\mathcal{H})$ is an $\mathcal{H}$-valued locally square integrable martingale and let $X = \sum_{i=1}^n \mathbb{1}_{(s_i,t_i]\times F_i} T_i \in \mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}))$ be an $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step function, where we can and we do assume that the rectangles in the representation of X are disjoint. Notice that if $(\tau_n)_{n\in\mathbb{N}}$ is a localizing sequence for M, then for any $n \in \mathbb{N}$*

$$\left(\int X dM\right)^{\tau_n} = \int X dM^{\tau_n} \in \mathscr{M}^{2,c}(\mathcal{G}),$$

is a $\mathcal{G}$-valued continuous square integrable martingale and so $\int X dM \in \mathscr{M}^{2,c}_{loc}(\mathcal{G})$ is a $\mathcal{G}$-valued continuous locally square integrable martingale. For notational purposes denote the process $\int X dM$ by N. We will show that $\|N\|^2_{\mathcal{G}}$ is an $\mathbb{R}$-valued submartingale. The process $\|N\|^2_{\mathcal{G}}$ is adapted by assumption and integrable. Let $s < t$ and $F_s \in \mathscr{F}_s$, then

$$0 \leq \mathbb{E}\left(\|N_t - N_s\|^2_{\mathcal{G}} \mathbb{1}_{F_s}\right) = \mathbb{E}\left(\|N_t\|^2_{\mathcal{G}} \mathbb{1}_{F_s}\right) - 2\mathbb{E}\left((N_t|\, N_s)_{\mathcal{G}} \mathbb{1}_{F_s}\right) + \mathbb{E}\left(\|N_s\|^2_{\mathcal{G}} \mathbb{1}_{F_s}\right),$$

where $(\cdot \mid \cdot)_{\mathcal{G}}$ denotes the scalar product in $\mathcal{G}$. We claim that

$$\mathbb{E}\left((N_t|\, N_s)_{\mathcal{G}} \middle| \mathscr{F}_s\right) = \left(E(N_t|\, \mathscr{F}_s) \middle| N_s\right)_{\mathcal{G}}.$$

Indeed, if $N^n_s := \sum_i^n \mathbb{1}_{G^n_i} g^n_i \in \mathscr{E}(\mathscr{F}_s, \mathcal{G})$ is a sequence of $\mathscr{F}_s$-measurable $\mathcal{G}$-valued step functions such that $\|N^n_s(\omega) - N_s(\omega)\|_{\mathcal{G}} \xrightarrow[n\to\infty]{} 0$ for P-a.e. $\omega \in \Omega$, then using the fact that $G^n_i \in \mathscr{F}_s$ and applying Lemma 2.12 yields

$$E\left((N_t|\, N^n_s)_{\mathcal{G}} \middle| \mathscr{F}_s\right) \overset{P\text{-a.e.}}{=} \sum_i^n \mathbb{E}\left((N_t|\, g^n_i)_{\mathcal{G}} \mathbb{1}_{G^n_i} \middle| \mathscr{F}_s\right) \overset{P\text{-a.e.}}{=} \sum_i^n \mathbb{E}\left((N_t|\, g^n_i)_{\mathcal{G}} \middle| \mathscr{F}_s\right) \mathbb{1}_{G^n_i} \overset{P\text{-a.e.}}{=}$$

$$\sum_i^n \left(\mathbb{E}(N_t|\, \mathscr{F}_s) \middle| g^n_i\right)_{\mathcal{G}} \mathbb{1}_{G^n_i} \overset{P\text{-a.e.}}{=} \left(\mathbb{E}(N_t|\, \mathscr{F}_s) \middle| N^n_s\right)_{\mathcal{G}},$$

for all $n \in \mathbb{N}$. Recall that conditional expectation is an L^1-contraction, this proves the claim. This argument can be found in Kunita (1970) Proposition 1 or for an alternative approach *see Métivier (1977) Satz* 10.5*. Hence we obtain that*

$$0 \leq \mathbb{E}\Big(\|N_t - N_s\|_{\mathcal{G}}^2 \mathbb{1}_{F_s} \Big) = \mathbb{E}\Big(\|N_t\|_{\mathcal{G}}^2 \mathbb{1}_{F_s} \Big) - \mathbb{E}\Big(\|N_s\|_{\mathcal{G}}^2 \mathbb{1}_{F_s} \Big),$$

for all $s < t$ and $F_s \in \mathscr{F}_s$, which means that $\|N\|_{\mathcal{G}}^2$ is an $\mathbb{R}$-valued (positive) submartingale, which is in particular continuous. So by Doob's L^p-inequality we obtain for any stopping time $\tau \in \mathscr{T}$

$$\mathbb{E}\Big(\sup_{t \leq \tau} \Big\| \int_0^t X_s dM_s \Big\|_{\mathcal{G}}^2 \Big) \leq 4\mathbb{E}\Big(\Big\| \int_0^\tau X_s dM_s \Big\|_{\mathcal{G}}^2 \Big).$$

Similarly as in the real case there is a unique up to indistinguishability, predictable, increasing, positive and continuous process $\langle N \rangle$ of finite variation and null at zero, such that $\|N\|_{\mathcal{G}}^2 - \langle N \rangle$ is an $\mathbb{R}$-valued martingale, see Métivier (1982) Corollary 15.4 (Doob-Meyer decomposition theorem). In particular this implies that

$$\mathbb{E}\Big(\Big\| \int_0^\tau X_s dM_s \Big\|_{\mathcal{G}}^2 \Big) = \mathbb{E}\Big(\Big\langle \int X dM \Big\rangle_\tau \Big) = \mathbb{E}\Big(\Big\langle \int X dM \Big\rangle_{\tau^-} \Big),$$

where in the second equality we use that $\langle N \rangle$ is continuous in this case. We claim that

$$\mathbb{E}\Big(\Big\langle \int X dM \Big\rangle_{\tau^-} \Big) \leq \mathbb{E}\Big(\int_0^{\tau^-} \|X_s\|^2 d\langle M \rangle_s \Big).$$

So let $\sigma \in \mathscr{T}$ be a stopping time such that $M^\sigma \in \mathscr{M}^{2,c}(\mathcal{H})$ is a square integrable continuous martingale, then for all $(t, \omega) \in \bar{\Omega}$ we have

$$\Big\| \int_0^t X_s dM_s^\sigma(\omega) \Big\|_{\mathcal{G}}^2 - \int_0^{t\wedge\sigma} \|X_s(\omega)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}^2 d\langle M \rangle_s(\omega) =$$

$$\sum_{i=1}^n \mathbb{1}_{F_i}(\omega) \Big(\|T_i\big(M_{t\wedge\sigma\wedge t_i}(\omega) - M_{t\wedge\sigma\wedge s_i}(\omega)\big)\|_{\mathcal{G}}^2 - \|T_i\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}^2 \big(\langle M \rangle_{t\wedge\sigma\wedge t_i}(\omega) - \langle M \rangle_{t\wedge\sigma\wedge s_i}(\omega)\big) \Big) \leq$$

$$\sum_{i=1}^n \mathbb{1}_{F_i}(\omega) \|T_i\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}^2 \Big(\|M_{t\wedge\sigma\wedge t_i}(\omega) - M_{t\wedge\sigma\wedge s_i}(\omega)\|_{\mathcal{H}}^2 - \big(\langle M \rangle_{t\wedge\sigma\wedge t_i}(\omega) - \langle M \rangle_{t\wedge\sigma\wedge s_i}(\omega)\big) \Big).$$

Hence,

$$\mathbb{E}\Big(\Big\langle \int X dM^\sigma \Big\rangle_{\tau^-} - \int_0^{\tau^-} \|X_s\|_{\mathcal{G}}^2 d\langle M^\sigma \rangle_s \Big) \leq$$

$$\mathbb{E}\Big(\Big\langle \int X dM^\sigma \Big\rangle_{\tau^-} - \Big\| \int_0^\tau X_s dM_s^\sigma \Big\|_{\mathcal{G}}^2 + \Big\| \int_0^\tau X_s dM_s^\sigma \Big\|_{\mathcal{G}}^2 - \int_0^{\tau^-} \|X_s\|_{\mathcal{G}}^2 d\langle M^\sigma \rangle_s \Big) =$$

$$\sum_{i=1}^n \|T_i\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}^2 \mathbb{E}\Big(\mathbb{1}_{F_i} \|M_{t\wedge\sigma\wedge t_i} - M_{t\wedge\sigma\wedge s_i}\|_{\mathcal{H}}^2 - \mathbb{1}_{F_i} \big(\langle M \rangle_{t\wedge\sigma\wedge t_i} - \langle M \rangle_{t\wedge\sigma\wedge s_i}\big) \Big) \leq 0,$$

since $\left(\langle \int XdM^\sigma\rangle - \| \int XdM^\sigma\|^2_{\mathcal{G}}\right)$ *and* $\left(\|M_{t\wedge\sigma\wedge t_i} - M_{t\wedge\sigma\wedge s_i}\|^2_{\mathcal{H}} - (\langle M\rangle_{t\wedge\sigma\wedge t_i} - \langle M\rangle_{t\wedge\sigma\wedge s_i})\right)$ *are* $\mathbb{R}$*-valued martingales and this in turn implies that*

$$\mathbb{E}\bigg(\Big\langle \int XdM\Big\rangle_{\tau^-}\bigg) \leq \mathbb{E}\bigg(\int_0^{\tau^-} \|X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} d\langle M\rangle_s\bigg),$$

which proves the claim. The previous discussion shows that for any stopping time $\tau \in \mathscr{T}$ *and* $X \in \mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}))$

$$\mathbb{E}\bigg(\sup_{t<\tau}\Big\|\int_0^t X_s dM_s\Big\|^2_{\mathcal{G}}\bigg) \leq \mathbb{E}\bigg(\sup_{t\leq\tau}\Big\|\int_0^t X_s dM_s\Big\|^2_{\mathcal{G}}\bigg) \leq 4\mathbb{E}\bigg(\Big\|\int_0^\tau X_s dM_s\Big\|^2_{\mathcal{G}}\bigg) =$$

$$4\mathbb{E}\bigg(\Big\langle \int XdM\Big\rangle_{\tau^-}\bigg) \leq 4\mathbb{E}\bigg(\int_0^{\tau^-} \|X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} d\langle M\rangle_s\bigg) \leq$$

$$4\mathbb{E}\bigg((1+\langle M\rangle_{\tau^-})\int_0^{\tau^-} \|X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} d\langle M\rangle_s\bigg) = \mathbb{E}\bigg(A_{\tau^-}\int_0^{\tau^-} \|X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} dA_s\bigg),$$

where we set $A := 2(1+\langle M\rangle_{\tau^-})$ *and use in the last equality that* $d(2(1+\langle M\rangle)) = d(2\langle M\rangle)$ *as Lebesgue-Stieltjes measures. This shows that* $A = 2(1+\langle M\rangle)$ *is a control process for* M*. For a general locally square integrable martingale* M*, a control process is of the form* $A = 2(1+\langle M\rangle + [\check{M}])$*, where* $\check{M}$ *is the pure jump-martingale part of* M*, and* $[\check{M}]$ *is the quadratic variation of* $\check{M}$*, see Métivier (1982) Theorem* 19.4 *or Métivier and Pellaumail (1979) Theorem* 2 *and* 2′ *for a Doob-like inequality in this general setup and see Métivier (1982) Theorem* 23.14 *or Emery* (1980) *for the construction of the control process of* M *in the general case. In particular we refer to counter-examples* 1.1.3 *and* 1.1.4 *in Métivier and Pellaumail (1979) for examples on why* "$\sup\limits_{s<\tau}$" *cannot be simply replaced by* "$\sup\limits_{s\leq\tau}$"*.*

As discussed in the previous examples, any semimartingale allows the existence of a control process and this is in fact a sufficient condition for a semimartingale, namely:

Theorem 2.23 (**Métivier**)**.** *An RCLL* $\mathcal{H}$*-valued process is a semimartingale if and only if it admits a control process.*

Proof. See Métivier (1982) Theorem 23.14. □

The extension of the stochastic integral to a suitable class of integrands requires the construction of seminorms that control the integral with respect to elementary step functions defined above. Let $\mathcal{G}$ be a separable Hilbert space. We denote by $L^0\big(\bar{\Omega}, \mathscr{P}; \mathscr{L}(\mathcal{H},\mathcal{G})\big)$ the class of P-strongly $\mathscr{P}$-measurable processes with respect to the norm topology on $\mathscr{L}(\mathcal{H},\mathcal{G})$.

For a control process $A \in \mathscr{A}(Z)$ and a stopping time $\tau \in \mathscr{T}$ such that $\mathbb{E}(A_{\tau-}^2) < \infty$ we consider the mapping

$$\begin{aligned} q_\tau^A : L^0\big(\bar{\Omega}, \mathscr{P}; \mathscr{L}(\mathcal{H}, \mathcal{G})\big) &\to [0, \infty] \\ X &\mapsto \mathbb{E}\Bigg(A_{\tau-} \int_0^{\tau-} \|X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} dA_s\Bigg)^{\frac{1}{2}}. \end{aligned}$$

We remark that an element in $L^0\big(\bar{\Omega}, \mathscr{P}; \mathscr{L}(\mathcal{H}, \mathcal{G})\big)$ is not necessarily $\mathscr{P}$-measurable, however, there is a sequence $(X^n)_{n\in\mathbb{N}} \subset \mathscr{E}(\mathscr{L}(\mathcal{H}, \mathcal{G}), \mathscr{P})$ of $\mathscr{P}$-measurable $\mathscr{L}(\mathcal{H}, \mathcal{G})$-valued step functions such that $X^n \xrightarrow[n\to\infty]{\mathscr{L}(\mathcal{H},\mathcal{G})} X$ up to indistintinguishability, which implies that $X(h) \in L^0(\bar{\Omega}, \mathscr{P}; \mathcal{G})$ for all $h \in \mathcal{H}$ and by Proposition 2.6 this means that $X(h)$ is $\mathscr{P}$-measurable for all $h \in \mathcal{H}$, since $\mathcal{G}$ is separable. The assumption that $\mathcal{H}$ is separable implies then, that $\|X\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}$ can be realized as a countable supremum of $\mathscr{P}$-measurable processes, so $\|X\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}$ is $\mathscr{P}$-measurable and hence $q_\tau^A(X)$ is well-defined. The mapping q_τ^A defines a seminorm on the vector space

$$\Lambda_\tau^A(\mathscr{L}(\mathcal{H}, \mathcal{G})) := \Bigg\{X \in L^0\big(\bar{\Omega}, \mathscr{P}; \mathscr{L}(\mathcal{H}, \mathcal{G})\big) \Bigg| \, q_\tau^A(X) < \infty\Bigg\}.$$

Indeed, if $X \in \Lambda_\tau^A(\mathscr{L}(\mathcal{H}, \mathcal{G}))$, then $(q_\tau^A(X))^2$ is the integral of $\|X\|^2$ with respect to the (finite) measure on $(\bar{\Omega}, \mathscr{P})$ defined by

$$P \otimes A_{\tau^-}(B) := \mathbb{E}\Bigg(A_{\tau-} \int_0^{\tau-} \mathbb{1}_B(s, \cdot) dA_s\Bigg),$$

for $B \in \mathscr{P}$, in other words, q_τ^A is an L^2-seminorm. An important observation is that the (quotient) space $L^2(\bar{\Omega}, \mathscr{P}, P \otimes A_{\tau^-})$ of square integrable $\mathscr{P}$-measurable $\mathbb{R}$-valued functions with respect to the measure $P \otimes A_{\tau^-}$ contains the space of $\mathscr{R}$-measurable $\mathbb{R}$-valued step functions $\mathscr{E}(\mathbb{R})$ as a dense subspace by Lemma 2.18. This implies that any $\mathscr{P}$-measurable $\mathscr{L}(\mathcal{H}, \mathcal{G})$-valued step function in $\mathscr{E}(\mathscr{L}(\mathcal{H}, \mathcal{G}), \mathscr{P})$ can be approximated with respect to q_τ^A by a sequence of $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H}, \mathcal{G})$-valued step function in $\mathscr{E}(\mathscr{L}(\mathcal{H}, \mathcal{G}))$. To be more precisely, suppose that $X = \sum_{i=1}^n \mathbb{1}_{P_i} T^i \in \mathscr{E}(\mathscr{L}(\mathcal{H}, \mathcal{G}), \mathscr{P})$, where $P_i \in \mathscr{P}$ and $T^i \in \mathscr{L}(\mathcal{H}, \mathcal{G})$. For each $1 \leq i \leq n$ there is a sequence $(X^{m,i})_{m\in\mathbb{N}} \subset \mathscr{E}(\mathbb{R})$ of $\mathscr{R}$-measurable $\mathbb{R}$-valued step functions such that

$$\|X^{m,i} - \mathbb{1}_{P_i}\|^2_{L^2(\bar{\Omega}, \mathscr{P}, P\otimes A_{\tau^-})} = \int_{\bar{\Omega}} |X^{m,i} - \mathbb{1}_{P_i}|^2 d(P \otimes A_{\tau^-}) \xrightarrow[m\to\infty]{} 0.$$

Hence,

$$q_\tau^A(X - \sum_{i=1}^n X^{m,i} T_i) \leq \sum_{i=1}^n \|T_i\|_{\mathscr{L}(\mathcal{H},\mathcal{G})} \|X^{m,i} - \mathbb{1}_{P_i}\|_{L^2(\bar{\Omega}, \mathscr{P}, P\otimes A_{\tau^-})} \xrightarrow[m\to\infty]{} 0.$$

Lemma 2.24. *The space of $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step functions $\mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}))$ is a q_τ^A-dense subspace of the seminormed vector space $\Lambda_\tau^A(\mathscr{L}(\mathcal{H},\mathcal{G}))$.*

Proof. The ideas contained in this proof can be found in the proof of Proposition 22.4 in Métivier (1982). Let $X \in \Lambda_\tau^A(\mathscr{L}(\mathcal{H},\mathcal{G}))$. By separability of $\mathcal{H}$ we may write

$$\|X\|_{\mathscr{L}(\mathcal{H},\mathcal{G})} = \sup_{n\in\mathbb{N}} \|X(h_n)\|_{\mathcal{G}},$$

for a dense countable subset $\{h_n\}_{n\in\mathbb{N}}$ in the unit ball of $\mathcal{H}$. Note that for each $n \in \mathbb{N}$ the process $\|X(h_n)\|_{\mathcal{G}} : \bar{\Omega} \to [0,\infty)$ is $\mathscr{P}$-measurable, which implies that $\|X\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}$ is $\mathscr{P}$-measurable, as it is a countable supremum of predictable processes. For any $(s,\omega) \in \bar{\Omega}$ we have $\|\mathbb{1}_{\{\|X\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}\le K\}}(s,\omega)X_s(\omega) - X_s(\omega)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})} \xrightarrow[K\to\infty]{} 0$, so by dominated convergence it holds that

$$\mathbb{E}\left(A_{\tau^-}\int_0^{\tau^-} \|\mathbb{1}_{\{\|X\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}\le K\}}(s,\cdot)X_s - X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})}dA_s\right) \xrightarrow[K\to\infty]{} 0.$$

Hence we may assume that $\sup\limits_{(s,\omega)\in\bar{\Omega}} \|X_s(\omega)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})} \le K$ for some $K \ge 0$. Let $\{h_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ and $\{g_n\}_{n\in\mathbb{N}}$ an orthonormal basis of $\mathcal{G}$. Denote by $S\text{pan}(h_1,\ldots,h_n)$ and $S\text{pan}(g_1,\ldots,g_n)$ the linear subspaces generated by $\{h_1\ldots h_n\}$ and $\{g_1,\ldots,g_n\}$ respectively. Let $\pi^n_{\mathcal{H}} : \mathcal{H} \to S\text{pan}(h_1,\ldots,h_n)$ and $\pi^n_{\mathcal{G}} : \mathcal{G} \to S\text{pan}(g_1,\ldots,g_n)$ be the orthogonal projections onto $S\text{pan}(h_1,\ldots,h_n)$ and $S\text{pan}(g_1,\ldots,g_n)$. Notice that the process $X^n := \pi^n_{\mathcal{G}} X \pi^n_{\mathcal{H}} \in \mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}),\mathscr{P})$ is a $\mathscr{P}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step function. Moreover, since $\{h_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are orthonormal bases, we obtain for each $k \in \mathbb{N}$ and $(s,\omega) \in \bar{\Omega}$

$$\left\|\left(\pi^n_{\mathcal{G}} X_s(\omega)\pi^n_{\mathcal{H}} - X_s(\omega)\right)(h_k)\right\|_{\mathcal{G}} \xrightarrow[n\to\infty]{} 0 \text{ and}$$

$$\left\|\left(\pi^n_{\mathcal{G}} X_s(\omega)\pi^n_{\mathcal{H}} - X_s(\omega)\right)(h_k)\right\|_{\mathcal{G}} \le 2K,$$

so by dominated convergence it follows that

$$\mathbb{E}\left(A_{\tau^-}\int_0^{\tau^-} \|X^n_s - X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})}dA_s\right) \xrightarrow[n\to\infty]{} 0,$$

that is $X^n \xrightarrow[n\to\infty]{q_\tau^A} X$. This is sufficient, as any X^n can be approximated by a sequence of $\mathscr{R}$-measurable $\mathscr{L}((H,\mathcal{G}))$-valued step functions as explained above, where we showed that $\mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}))$ is a q_τ^A-dense subspace of $\mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}),\mathscr{P})$. Note as an aside that we do not use the full strength of the P-strongly $\mathscr{P}$-measurability of X. Instead we just exploit the separable Hilbert-structure and require for any $h \in \mathcal{H}$ the process $X(h) : \bar{\Omega} \to \mathcal{G}$ to be $\mathscr{P}$-measurable. $\square$

Let $\Pi_\tau(\mathcal{G}) := \{\mathbb{1}_{[[0,\tau))} Y \,|\, Y \in L^0(\bar{\Omega}, \bar{\mathscr{F}}; \mathcal{G}) \text{ is RCLL}\}$, where we consider the stochastic interval $[[0,\tau)) := \{(t,\omega) \in \bar{\Omega} \,|\, 0 \le t < \tau(\omega)\}$, and

$$p_\tau : L^0(\bar{\Omega}, \bar{\mathscr{F}}; \mathcal{G}) \to [0,\infty]$$
$$Y \mapsto \mathbb{E}\bigg(\sup_{s<\tau} \|Y_s\|_{\mathcal{G}}^2\bigg)^{\frac{1}{2}}.$$

Note again that the mapping p_τ is well-defined. Indeed if $Y \in L^0(\bar{\Omega}, \bar{\mathscr{F}}; \mathcal{G})$ then $Y(g)$ is $\bar{\mathscr{F}}$-measurable by Proposition 2.4 for any $g \in \mathcal{G}$ and since $\mathcal{G}$ is separable, $\|Y\|_{\mathcal{G}}$ is $\bar{\mathscr{F}}$-measurable as a countable supremum of $\bar{\mathscr{F}}$-measurable processes.

Lemma 2.25. *The vector space $\Pi_\tau(\mathcal{G})$ is complete with respect to the seminorm p_τ.*

Proof. The proof is the same as of the real case in Lemma 24.1.1 in Métivier. The mapping p_τ is positive and if $Y^1, Y^2 \in \Pi_\tau(\mathcal{G})$ then

$$p_\tau(Y^1+Y^2)^2 = \mathbb{E}\bigg(\sup_{s<\tau}\|Y_s^1+Y_s^2\|_{\mathcal{G}}^2\bigg) \le \mathbb{E}\bigg(\sup_{s<\tau}\|Y_s^1\|_{\mathcal{G}}^2 + 2\sup_{s<\tau}\|Y_s^1\|_{\mathcal{G}}\sup_{s<\tau}\|Y_s^2\|_{\mathcal{G}} + \sup_{s<\tau}\|Y_s^2\|_{\mathcal{G}}^2\bigg) \le$$

$$\mathbb{E}\bigg(\sup_{s<\tau}\|Y_s^1\|_{\mathcal{G}}^2\bigg) + 2\mathbb{E}\bigg(\sup_{s<\tau}\|Y_s^1\|_{\mathcal{G}}^2\bigg)^{\frac{1}{2}}\mathbb{E}\bigg(\sup_{s<\tau}\|Y_s^2\|_{\mathcal{G}}^2\bigg)^{\frac{1}{2}} + \mathbb{E}\bigg(\sup_{s<\tau}\|Y_s^2\|_{\mathcal{G}}^2\bigg) = \bigg(p_\tau(Y^1)+p_\tau(Y^2)\bigg)^2,$$

where we use the Cauchy-Schwarz inequality in the second estimate and this proves that p_τ is indeed a seminorm. Let $(Y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\Pi_\tau(\mathcal{G})$ and choose a subsequence $n_k \nearrow \infty$ such that

$$P\bigg(\bigg\{\omega \in \Omega \,\bigg|\, \sup_{s<\tau}\|Y_s(\omega)^{n_k+1} - Y_s(\omega)^{n_k}\|_{\mathcal{G}}^2 > \frac{1}{2^k}\bigg\}\bigg) \le \frac{1}{2^k},$$

then for any $l \in \mathbb{N}$ we have

$$P\bigg(\bigg\{\omega \in \Omega \,\bigg|\, \sup_{s<\tau}\|Y_s(\omega)^{n_k+l} - Y_s(\omega)^{n_k}\|_{\mathcal{G}}^2 > \frac{1}{2^{k-4}}\bigg\}\bigg) \le \frac{1}{2^{k-1}},$$

and the Borel-Cantelli lemma implies that

$$P\bigg(\limsup_{k\to\infty}\bigg\{\sup_{s<\tau}\|Y_s^{n_k+l} - Y_s^{n_k}\|_{\mathcal{G}}^2 > \frac{1}{2^{k-4}}\bigg\}\bigg) = 0.$$

Hence, the mappings $t \mapsto Y_t^{n_k}(\omega)$ converge uniformly on $[0,\tau(\omega))$ to a function $t \mapsto Y_t(\omega)$ for P-a.e. $\omega \in \Omega$. The process $\mathbb{1}_{[[0,\tau))}Y$ is then RCLL and it belongs to the seminormed vector space $\Pi_\tau(\mathcal{G})$ by Corollary 2.5, since it is the pointwise limit of P-strongly $\bar{\mathscr{F}}$-measurable processes with respect to the norm topology. $\square$

Formally we can view the stochastic integral as a linear operator defined by

$$\begin{aligned}\Phi_\tau^A : \mathscr{E}(L(\mathcal{H},\mathcal{G})) &\to \Pi_\tau(\mathcal{G})\\ X &\mapsto \mathbb{1}_{[[0,\tau))}\int XdZ,\end{aligned}$$

where the notation Φ_τ^A indicates that the mapping $X \mapsto \mathbb{1}_{[[0,\tau))}\int XdZ$ is not only viewed as an algebraic map, but as a continuous operator with respect to the seminorm q_τ^A on $\mathscr{E}(L(\mathcal{H},\mathcal{G}))$ and for this it is necessary for the process to be defined on $\mathbb{1}_{[[0,\tau))}$, as we want to exploit the bounding property of the control process A, more precisely:

Lemma 2.26. *The mapping $\Phi_\tau^A : \mathscr{E}(L(\mathcal{H},\mathcal{G})) \to \Pi_\tau(\mathcal{G})$ is a bounded operator with respect to the seminorms q_τ^A on $\mathscr{E}(L(\mathcal{H},\mathcal{G}))$ and p_τ on $\Pi_\tau(\mathcal{G})$.*

Proof. Let $X \in \mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}))$ be an $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step function, then $\int XdZ \in L^0(\bar{\Omega},\bar{\mathscr{F}};\mathcal{G})$ and it is RCLL so $\mathbb{1}_{[[0,\tau))}\int XdZ \in \Pi_\tau(\mathcal{G})$. Moreover,

$$p_\tau\left(\mathbb{1}_{[[0,\tau))}\int XdZ\right)^2 = \mathbb{E}\left(\sup_{t<\tau}\left\|\int_0^t X_s dZ_s\right\|_{\mathcal{G}}^2\right) \leq \mathbb{E}\left(A_{\tau^-}\int_0^{\tau^-}\|X_s\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}^2 dA_s\right) < \infty,$$

where the second to last inequality uses the main property of the control process A and the last inequality uses that $\|X\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}$ is globally bounded as it is a step function and $\mathbb{E}(A_{\tau^-}^2) < \infty$ by assumption. □

Hence, Φ_τ^A can be uniquely extended to a bounded operator $\Phi_\tau^A : \Lambda_\tau^A(\mathscr{L}(\mathcal{H},\mathcal{G})) \to \Pi_\tau(\mathcal{G})$. However, we want a process defined on $\mathbb{R}_+ \times \Omega$ and not just on the stochastic interval $[[0,\tau))$. The basic technique to overcome this problem is to define the stochastic integral for each stopping time of a sequence of increasing stopping times $(\tau_n)_{n\in\mathbb{N}}$ and to glue the resulting processes on overlapping stochastic intervals. This requires the process Φ_τ^A to be independent of the stopping time $\tau \in \mathscr{T}$ in the following sense:

Lemma 2.27 (Independence of stopping times). *Suppose that τ, $\sigma \in \mathscr{T}$ are two stopping times such that both satisfy the property that $\mathbb{E}(A_{\tau^-}^2) < \infty$ and $\mathbb{E}(A_{\sigma^-}^2) < \infty$. If $X \in \Lambda_\tau^A(\mathscr{L}(\mathcal{H},\mathcal{G})) \cap \Lambda_\sigma^A(\mathscr{L}(\mathcal{H},\mathcal{G}))$, then*

$$\mathbb{1}_{[[0,\tau\wedge\sigma))}\Phi_\tau^A(X) = \mathbb{1}_{[[0,\tau\wedge\sigma))}\Phi_\sigma^A(X).$$

Proof. This is part of the proof of Lemma 24.1.2 in Métivier (1982). Let $X \in \Lambda_\tau^A(\mathscr{L}(\mathcal{H},\mathcal{G}))\cap \Lambda_\sigma^A(\mathscr{L}(\mathcal{H},\mathcal{G}))$ and choose a sequence $(X_1^n)_{n\in\mathbb{N}} \subset \mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}))$ of $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step functions such that $X_1^n \xrightarrow[n\to\infty]{q_\tau^A} X$. In particular this implies that $X_1^n \xrightarrow[n\to\infty]{q_{\tau\wedge\sigma}^A} X$. Note that for each $n \in \mathbb{N}$ we have

$$\mathbb{1}_{[[0,\tau\wedge\sigma))}\Phi_\tau^A(X_1^n) = \Phi_{\tau\wedge\sigma}^A(X_1^n).$$

Hence, $\Phi^A_{\tau\wedge\sigma}(X^n_1) \xrightarrow[n\to\infty]{p_{\tau\wedge\sigma}} \Phi^A_{\tau\wedge\sigma}(X)$ and $\Phi^A_{\tau\wedge\sigma}(X^n_1) \xrightarrow[n\to\infty]{p_{\tau\wedge\sigma}} \mathbb{1}_{[[0,\tau\wedge\sigma))}\Phi^A_\tau(X)$, which by uniqueness of the extension implies that $\Phi^A_{\tau\wedge\sigma}(X) = \mathbb{1}_{[[0,\tau\wedge\sigma))}\Phi^A_\tau(X)$ in $\Pi_\tau(\mathcal{G})$. Similarly, this argument can repeated for a sequence $(X^n_2)_{n\in\mathbb{N}} \subset \mathscr{E}(\mathscr{L}(\mathcal{H},\mathcal{G}))$ of $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step functions such that $X^n_2 \xrightarrow[n\to\infty]{q^A_\sigma} X$, which gives the result. □

To conclude the construction of the integral with respect to the semimartingale Z we require an integrand X to allow the existence of an increasing sequence of stopping times as described above, in order to be able to glue the process on overlapping intervals. A class of processes which admits this is described by the following result:

Theorem 2.28 (**The stochastic integral**). *Let $X \in L^0(\bar{\Omega}, \mathscr{P}; \mathscr{L}(\mathcal{H},\mathcal{G}))$ and consider the process*

$$\lambda^A_t(X) := A_t \int_0^t \|X_s\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} dA_s.$$

If $\lambda^A_t(X) < \infty$ P-a.e. for all $t \geq 0$, then there is an increasing sequence $(\tau_n)_{n\in\mathbb{N}} \subset \mathscr{T}$ of stopping times such that $\lim_{n\to\infty} \tau_n \overset{P\text{-a.e.}}{=} \infty$ and for any $n \in \mathbb{N}$ we have $\mathbb{E}(A^2_{\tau_n^-}) < \infty$ and $X \in \Lambda^A_{\tau_n}(\mathscr{L}(\mathcal{H},\mathcal{G}))$. In particular there is a unique RCLL $\mathcal{G}$-valued process $\int XdZ$ up to indistinguishability, such that for any $n \in \mathbb{N}$

$$\Phi^A_{\tau_n}(X) = \mathbb{1}_{[[0,\tau_n))} \int XdZ.$$

Moreover, the process $\int XdZ$ is independent of the control process $A \in \mathscr{A}(Z)$, whenever $\lambda^A(X) < \infty$.

Proof. This proof can be found in Lemmas 24.1.2 and 24.1.3 in Métivier (1982). The sequence $\tau_n := \inf\{t \geq 0 | A_t \wedge \lambda^A_t(X) \geq n\}$ satisfies the properties above. Define Y^A by setting

$$Y^A\big|_{[[0,\tau_n))} := \mathbb{1}_{[[0,\tau_n))}\Phi^A_{\tau_n}(X).$$

Lemma 2.27 shows that the process Y^A is defined independently of the sequence $(\tau_n)_{n\in\mathbb{N}}$. To show that this definition is independent of the control process A let A_1, $A_2 \in \mathscr{A}(Z)$ be two control processes such that $\lambda^{A^1}(X) < \infty$ and $\lambda^{A^2}(X) < \infty$. Then, $A^1 + A^2 \in \mathscr{A}(Z)$ and $\lambda^{A^1+A^2}(X) < \infty$. Lemma 2.27 shows that

$$\mathbb{1}_{[[0,\sigma_n))}Y^{A^1} = \mathbb{1}_{[[0,\sigma_n))}Y^{A^2} = \mathbb{1}_{[[0,\sigma_n))}Y^{A^1+A^2},$$

where $\sigma_n := \inf\{t \geq 0 | (A^1_t + A^2_t) \vee \lambda^{A^1+A^2}_t\}$. □

Therefore, we consider as the class of Z-integrable processes the set

$$\Lambda(\mathscr{L}(\mathcal{H},\mathcal{G})) := \bigcup_{A\in\mathscr{A}(Z)} \left\{ X \in L^0(\bar{\Omega},\mathscr{P};\mathscr{L}(\mathcal{H},\mathcal{G})) \middle| \lambda^A(X) < \infty \right\}.$$

We finalize this part with an example that shows that the construction above yields the same stochastic integral as the well-known isometric stochastic integral with respect to a continuous locally square integrable martingale.

Example 2.29 (The case of a continuous $\mathbb{R}$-valued martingale). *Suppose that $M = (M_t)_{t\in\mathbb{R}}$ is a continuous locally square integrable martingale. Denote by $L^2(\bar{\Omega},\mathscr{P},P_M)$ the L^2-space with respect to the Doléans measure P_M of M, that is*

$$\|H\|^2_{L^2(\bar{\Omega},\mathscr{P},P_M)} = \mathbb{E}\left(\int_0^\infty H_s^2 d\langle M\rangle_s\right),$$

and by $\mathscr{M}_0^{2,c}(\mathbb{R})$ the Hilbert space of continuous square integrable $\mathbb{R}$-valued martingales with respect to the norm

$$\|M\|^2_{\mathscr{M}^2} := \mathbb{E}(|M_\infty|^2) = \mathbb{E}(\langle M\rangle_\infty).$$

Then, there is a unique isometry

$$\begin{aligned} \Phi_M : L^2(\bar{\Omega},\mathscr{P},P_M) &\to \mathscr{M}_0^{2,c}(\mathbb{R}) \\ H &\mapsto H \bullet M, \end{aligned}$$

such that $\langle H \bullet M, N\rangle = \int H d\langle M,N\rangle$, for any continuous local martingale N and $H \in L^2(\bar{\Omega},\mathscr{P},P_M)$. Moreover, we know by Example 2.22 that $A := 2(1+\langle M\rangle)$ is a control process for M and any $H \in L^2(\bar{\Omega},\mathscr{P},P_M)$ satisfies $\int_0^t H_s^2 d\langle M\rangle_s < \infty$ P-a.e. for any $t \geq 0$. So that $\lambda_t^A(H) < \infty$ P-a.e. for any $t \geq 0$, which means that H is M-integrable in the sense we described above. Let $(\tau_n)_{n\in\mathbb{N}} \subset \mathscr{T}$ be a sequence of stopping times such that

$$\mathbb{E}\left(A_{\tau_n^-}\int_0^{\tau_n^-} H_s^2 dA_s\right) < \infty,$$

for all $n \in \mathbb{N}$, which exists by Theorem 2.28. It suffices to show that for each $n \in \mathbb{N}$ we have $\mathbb{1}_{[[0,\tau_n))} H \bullet M = \mathbb{1}_{[[0,\tau_n))} \int H dM$, hence let $\tau = \tau_l$ for some $l \in \mathbb{N}$. Choose a sequence $(H^n)_{n\in\mathbb{N}}$ of $\mathscr{R}$-measurable step function as in Lemma 2.24 such that

$$\mathbb{E}\left(A_{\tau^-}\int_0^{\tau^-} (H_s - H_s^n)^2 dA_s\right) \xrightarrow[n\to\infty]{} 0,$$

then $\mathbb{1}_{[[0,\tau))} H^n \xrightarrow[n\to\infty]{L^2} \mathbb{1}_{[[0,\tau))} H$ in $L^2(\bar{\Omega},\mathscr{P},P_M)$, since $A \geq 1$ and $dA = d\langle M\rangle$. Note that for *any $n \in \mathbb{N}$ we have $H^n \bullet M = \int H^n dM$, see for instance Chapter 4 Lemma 2.10 in Schweizer (2012). Hence*

$$\mathbb{E}\left(\sup_{t<\tau}\left|\int_0^t HdM - (H \bullet M)_t\right|^2\right)^{\frac{1}{2}} \leq$$

$$\mathbb{E}\left(\sup_{t<\tau}\left|\int_0^t HdM - \int_0^t H^n dM\right|^2\right)^{\frac{1}{2}} + \mathbb{E}\left(\sup_{t<\tau}\left|(H \bullet M)_t - (H^n \bullet M)_t\right|^2\right)^{\frac{1}{2}} \leq$$

$$\mathbb{E}\left(\sup_{t<\tau}\left|\int_0^t HdM - \int_0^t H^n dM\right|^2\right)^{\frac{1}{2}} + 2\mathbb{E}\left(\left|(H \bullet M)_\tau - (H^n \bullet M)_\tau\right|^2\right)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0$$

where we use that Φ_M is an isometry and $\int H^n dM \xrightarrow[n\to\infty]{p_\tau} \int HdM$. This shows that for all $H \in L^2(\bar{\Omega}, \mathscr{P}, P_M)$ the processes $\int HdM$ and $H \bullet M$ coincide up to indistinguishability. A similar argument, just using a localizing sequence of stopping times, can be repeated to show that any locally integrable process $H \in L^2_{loc}(\bar{\Omega}, \mathscr{P}, P_M)$ is also integrable with respect to M in the sense we described above.

We refer to Métivier (1982) Chapters 20 and 26 for results on martingales and the construction of the stochastic integral as we did. In particular the construction of the stochastic integral with respect to a martingale that allows unbounded operators as integrands is particularly fascinating, see Chapter 22 in Métivier (1982) for this. The construction of the control process for a general martingale can be found in Métivier (1982) as we stated above. However, this involves the development of too many techniques, the main ingredients being the notion of the dual predicable projection of a process (see Chapter 15 in Métivier (1982) or Chapter 5 §22 in Dinculeanu (2000)), the pure jump-martingale part of a martingale (see Chapter 19 in Métivier (1982)) and the stopped Doob's inequality (see Métivier and Pellaumail (1980)). Once the characterization of semimartingales through control processes is known, these results just serve the purpose of illustrating the concept of a control process, as they are not needed in our construction, which is the main reason we refrained from proving this (painstaking) result.

3 Construction of the weak* integral

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space satisfying the usual assumptions, $\mathcal{H}$ and $\mathcal{G}$ two separable Hilbert spaces and $Z \in \mathscr{S}(\mathcal{H})$ an $\mathcal{H}$-valued semimartingale. Let E be a separable Banach space and denote the set of dense countable subsets in the unit ball $E_1 := \overline{B_1^E(0)}$ by $\mathscr{D}(E_1)$. We will consider elementary step functions of the form

$$\mathscr{E}\big(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})\big) := \left\{ \sum_{i=1}^n \mathbb{1}_{R_i} u_i \middle| n \in \mathbb{N},\, R_i \in \mathscr{R}, u_i \in E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}) \right\},$$

where E^* is the dual Banach space of E and $E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})$ denotes the weak* injective tensor product of E^* and $\mathscr{L}(\mathcal{H},\mathcal{G})$, by which we mean the closure of $E^*\otimes\mathscr{L}(\mathcal{H},\mathcal{G})$ as a subspace of $\mathscr{L}(E,\mathscr{L}(\mathcal{H},\mathcal{G}))$. Recall the definition of the algebraic tensor product,

Definition 3.1. *Let R be a commutative ring and M, N two R-modules. The tensor product of M and N is an R-module T together with an R-bilinear mapping $\otimes : M \times N \to T$ such that for any R-module P and any R-bilinear mapping $\beta : M \times N \to P$, there exists a unique R-linear mapping $\iota : T \to P$ such that $\beta = \iota \circ \otimes$.*

Proposition 3.2. *Let R be a commutative ring and M, N two R-modules. The tensor product of M and N exists and is unique up to isomorphism.*

Proof. See for instance Atiyah, MacDonald (1969) Proposition 2.12. However, we sketch the existence-part of the statement. Let C be the free R-module $R^{M\times N}$, so elements in C are formal finite sums of the form $\sum_{i=1}^k r_i(m_i, n_i)$, with $k \in \mathbb{N}$, $r_i \in R$, $m_i \in M$ and $n_i \in N$. Let D be the submodule generated by elements in C of the type

$$(m + m', n) - (m, n) - (m', n), \quad (m, n + n') - (m, n) - (m, n'),$$

$$(rm, n) - r(m, n), \quad (m, rn) - r(m, n),$$

where $r \in R$, $m, m' \in M$ and $n, n' \in N$. Define $M \otimes N := T := C/D$ and denote the image of (m, n) under the projection $C \to T$ by $m \otimes n$. $M \otimes N$ is then a model for the tensor product, and it is the model that we will use. Elements in $M \otimes N$ are of the form $\sum_{i=1}^k m_i \otimes n_i$ with $k \in \mathbb{N}$, $m_i \in M$ and $n_i \in N$. Moreover for all $m, m' \in M$, $n, n' \in N$ and $r \in R$ we have the following relations

$$(m + m') \otimes n = m \otimes n + m' \otimes n, \quad m \otimes (n + n') = m \otimes n + m \otimes n'$$

$$(rm) \otimes n = r(m \otimes n) = m \otimes (rn).$$

$\square$

If F is a Banach space, we denote the algebraic tensor product over $\mathbb{R}$ of E^* and F by $E^*\otimes F$. The bilinear mapping

$$\beta : E^*\times F \to \mathscr{L}(E,F)$$
$$(\lambda,\zeta) \mapsto \Big(\xi \mapsto \lambda(\xi)\zeta\Big),$$

gives rise to a unique linear mapping $\iota : E^*\otimes F \to \mathscr{L}(E,F)$, such that the following diagram commutes

$$\begin{tikzcd}
E^*\times F \arrow[r, "\beta"] \arrow[dr, "\otimes"'] & \mathscr{L}(E,F) \\
& E^*\otimes F \arrow[u, "\iota"']
\end{tikzcd}$$

We start by justifying that we can view $E^* \otimes F$ as a subspace of $\mathscr{L}(E,F)$.

Lemma 3.3. *The linear mapping $\iota : E^* \otimes F \to \mathscr{L}(E,F)$ is injective.*

Proof. Suppose that $\sum_{i=1}^n \lambda_i \otimes \zeta_i \in \ker \iota$ and for simplicity that $n = 2$, then

$$\iota\Big(\lambda_1 \otimes \zeta_1 + \lambda_2 \otimes \zeta_2\Big)(\xi) = \lambda_1(\xi)\zeta_1 + \lambda_2(\xi) \otimes \zeta_2 = 0,$$

for all $\xi \in E$. Suppose that $\lambda_2(\xi_2) = -1$ for some $\xi_2 \in E$, otherwise this would imply that $\lambda_2 = 0$ and hence $\lambda_1 \otimes \zeta_1 = 0$. So

$$\zeta_2 = \lambda_1(\xi_2)\zeta_1,$$

and therefore

$$\lambda_1 \otimes \zeta_1 + \lambda_2 \otimes \zeta_2 = \lambda_1 \otimes \zeta_1 + \lambda_2 \otimes (\lambda_1(\xi_2)\zeta_1) = (\lambda_1 + \lambda_1(\xi_2)\lambda_2) \otimes \zeta_1 = 0.$$

This implies either $\zeta_1 = 0$ or $\lambda_1 = -\lambda_1(\xi_2)\lambda_2$. If $\zeta_1 = 0$ then $\zeta_2 = 0$ and so $\lambda_1\otimes\zeta_1+\lambda_2\otimes\zeta_2 = 0$. If $\lambda_1 = -\lambda_1(\xi_2)\lambda_2$, then

$$\lambda_1 \otimes \zeta_1 + \lambda_2 \otimes \zeta_2 = \lambda_2 \otimes (-\lambda_1(\xi_1)\zeta_1 + \zeta_2) = 0,$$

since $\zeta_2 = \lambda_1(\xi_2)\zeta_1$. This argument is sufficient for the case of a general $n \in \mathbb{N}$. Indeed, suppose that $n \geq 3$ and suppose that $\lambda_n(\xi_n) = -1$ for some $\xi_n \in E$ then

$$\zeta_n = \sum_{i=1}^{n-1} \lambda_i(\xi_n)\zeta_i,$$

and plugging this relation in yields

$$\sum_{i=1}^{n} \lambda_i \otimes \zeta_i = \sum_{i=1}^{n-1} \bar{\lambda}_i \otimes \zeta_i,$$

where $\bar{\lambda}_i := \lambda_i + \lambda_i(\xi_n)\lambda_n \in E^*$. So repeating this argument reduces to the case where $n = 2$, which we proved above. Hence, $\ker \iota = \{0\}$ and this ends the proof. $\square$

However, note that β is not injective. In particular ι induces a norm $\|\cdot\|_\epsilon$ on $E^*\otimes F$, which can be explicitly written as

$$\|u\|_\epsilon := \|\iota(u)\|_{\mathscr{L}(E,F)} = \Big\| \sum_{i=1}^{n} \beta(\lambda_i, \zeta_i) \Big\|_{\mathscr{L}(E,F)} = \sup_{\xi \in E_1} \Big\| \sum_{i=1}^{n} \lambda_i(\xi)\zeta_i \Big\|_F,$$

where $\sum_{i=1}^{n} \lambda_i \otimes \zeta_i \in E^*\otimes F$ is any representation of $u \in E^*\otimes F$. We define the space $E^*\hat{\otimes}_\epsilon F$ to be the closure of $\iota(E^* \otimes F) \subset \mathscr{L}(E, F)$ with respect to the norm $\|\cdot\|_\epsilon$. Traditionally, the subscript ϵ stands for the injective norm, but due to the similarities between this tensor product and the injective tensor product (both coincide if E is reflexive) and due to the fact that we do not use the injective tensor product, the subscript ϵ will denote the tensor product constructed above.

Example 3.4 (**Motivating example**). *Suppose that X is a compact topological space, then $C(X)^* \cong \mathscr{M}(X)$ by the Riesz representation theorem, where $C(X)$ denotes the Banach space of continuous functions on X with respect to the* sup*-norm $\|\cdot\|_\infty$ and $\mathscr{M}(X)$ denotes the Banach space of signed Radon measures on X with respect to the variation norm $\|\cdot\|_V$. Let $d \in \mathbb{N}$ and set $\mathscr{M}(X)^d := \bigoplus_{i=1}^{d} \mathscr{M}(X)$. Recall that the direct sum and the direct product of finitely many modules is the same, however we prefer to write it as a sum. For a suitable $\mathscr{M}(X)^d$-valued process μ, our aim is to define an $\mathscr{M}(X)$-valued process $\int \mu dZ$, where $Z \in \mathscr{S}(\mathbb{R}^d)$ is an $\mathbb{R}^d$-valued semimartingale, such that*

$$\int_X f d\Big(\int \mu_s dZ_s \Big) = \int \Big(\int_X f d\mu_s \Big) dZ_s,$$

whenever $f \in C(X)$ is a continuous function. Moreover, $\int(\int_X f d\mu_s) dZ_s$ should coincide with the stochastic integral constructed above for all $f \in C(X)$. As we will explain in this section, it is more natural to work with $\mathscr{M}(X) \otimes \mathbb{R}^d$ instead of $\mathscr{M}(X)^d$, as $\mathbb{R}^d = \mathscr{L}(\mathbb{R}^d, \mathbb{R})$ is the space on which we defined the stochastic integral above. More precisely, note that we have the following algebraic isomorphisms

$$\mathscr{M}(X) \otimes \mathbb{R}^d = \mathscr{M}(X) \otimes \bigoplus_{i=1}^{d} \mathbb{R} \cong \bigoplus_{i=1}^{d} \mathscr{M}(X) \otimes \mathbb{R} \cong \bigoplus_{i=1}^{d} \mathscr{M}(X) = \mathscr{M}(X)^d,$$

in fact the mapping

$$\begin{aligned} \varphi : \mathscr{M}(X)^d &\to \mathscr{M}(X) \otimes \mathbb{R}^d \\ \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_d \end{pmatrix} &\mapsto \sum_{i=1}^{d} \mu_i \otimes e_i \end{aligned}$$

is an isomorphism, where $e_1, \dots, e_d \in \mathbb{R}^d$ *is the canonical basis of* $\mathbb{R}^d$*. Note that the space* $\mathscr{M}(X) \otimes \mathbb{R}^d$ *is complete with respect to the norm* $\|\cdot\|_\epsilon$ *defined above. Indeed, if* $(u_n)_{n\in\mathbb{N}} \subset \mathscr{M}(X)\otimes\mathbb{R}^d$ *is a Cauchy sequence with respect to* $\|\cdot\|_\epsilon$ *and we fix the standard representation* $u_n = \sum_i^d \mu_i^n \otimes e_i$*, then for any* $\delta > 0$ *and* $m, n \in \mathbb{N}$ *large enough we have*

$$\|u^n - u^m\|_\epsilon = \sup_{\|f\|\leq 1} \left\| \begin{pmatrix} \int_X f d\mu_1^n - \int_X f d\mu_1^m \\ \vdots \\ \int_X f d\mu_d^n - \int_X f d\mu_d^m \end{pmatrix} \right\|_{\mathbb{R}^d} < \delta,$$

which implies that $(\mu_i^n)_{n\in\mathbb{N}} \subset \mathscr{M}(X)$ *is a Cauchy sequence with respect to the variation norm* $\|\cdot\|_V$ *for each coordinate* $1 \leq i \leq d$ *and in addition this implies that* $\mathscr{M}(X) \otimes \mathbb{R}^d = \mathscr{M}(X)\hat{\otimes}_\epsilon\mathbb{R}^d$*. The space* $\mathscr{M}(X)^d$ *is also a Banach space if we endow it with the natural norm*

$$\left\| \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_d \end{pmatrix} \right\| := \sum_{i=1}^{d} \|\mu^i\|_V.$$

It follows directly that φ *is a homeomorphism or alternatively this follows from the open mapping theorem, since* φ *is an isomorphism of vector spaces and a bounded operator between two Banach spaces. Hence, in this case we have* $\mathscr{M}(X)\hat{\otimes}_\epsilon\mathbb{R}^d = \mathscr{M}(X)\otimes\mathbb{R}^d \cong \mathscr{M}(X)^d$*, which explains the motivation behind the choice of the tensor product introduced above.*

The space $E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})$ has two natural identifications, either as described above as a subspace of $\mathscr{L}(E,\mathscr{L}(\mathcal{H},\mathcal{G}))$ or of $\mathscr{L}(\mathcal{H}, E^*\hat{\otimes}_\epsilon\mathcal{G})$. To explain the identification as a subspace of $\mathscr{L}(\mathcal{H}, E^*\hat{\otimes}_\epsilon\mathcal{G})$ suppose that $u \in E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})$ and let $(u_n)_{n\in\mathbb{N}} \subset E^*\otimes\mathscr{L}(\mathcal{H},\mathcal{G})$ be a Cauchy sequence such that

$$\sup_{\xi\in E_1} \|u(\xi) - u_n(\xi)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})} \xrightarrow[n\to\infty]{} 0.$$

Similarly as above we have a bilinear mapping

$$\begin{aligned}\tilde{\beta} : E^* \times \mathscr{L}(\mathcal{H}, \mathcal{G}) &\to \mathscr{L}(H, E^* \hat{\otimes}_\epsilon \mathcal{G}) \\ (\lambda, T) &\mapsto \Big(h \mapsto \lambda \otimes (T(h)) \Big),\end{aligned}$$

which induces a unique linear mapping $\tilde{\imath} : E^* \otimes \mathscr{L}(\mathcal{H}, \mathcal{G})$ such that the following diagram commutes

$$\begin{tikzcd} E^* \times \mathscr{L}(\mathcal{H}, \mathcal{G}) \arrow[rr, "\tilde{\beta}"] \arrow[drr, "\otimes"'] & & \mathscr{L}(H, E^* \hat{\otimes}_\epsilon \mathcal{G}) \\ & & E^* \otimes \mathscr{L}(\mathcal{H}, \mathcal{G}) \arrow[u, "\tilde{\imath}"'] \end{tikzcd}$$

The same proof as above shows that $\tilde{\imath}$ is injective. Let $u_n = \sum_{i=1}^n \lambda_i^n \otimes T_i^n$ be a representation of u_n, then

$$\tilde{\imath}(u_n)(h) = \sum_{i=1}^n \lambda_i^n \otimes (T_i^n(h)) \in E^* \otimes \mathcal{G} \subset E^* \hat{\otimes}_\epsilon \mathcal{G},$$

for all $h \in H$. Note that if we denote by $\mathcal{H}_1$ the unit sphere in $\mathcal{H}$, then

$$\|\tilde{\imath}(u_n - u_m)\|_{\mathscr{L}(\mathcal{H}, E^* \hat{\otimes}_\epsilon \mathcal{G})} = \sup_{h \in \mathcal{H}_1} \|\tilde{\imath}(u_n - u_m)(h)\|_{E^* \hat{\otimes}_\epsilon \mathcal{G}} =$$

$$\sup_{h \in \mathcal{H}_1,\, \xi \in E_1} \left\| \sum_{i=1}^n \lambda_i^n(\xi) T_i^n(h) - \sum_{i=1}^m \lambda_i^m(\xi) T_i^m(h) \right\|_{\mathcal{G}} = \sup_{\xi \in E_1} \|(u_n - u_m)(\xi)\|_{\mathscr{L}(\mathcal{H}, \mathcal{G})} \xrightarrow[n,m \to \infty]{} 0,$$

which implies that $(\tilde{\imath}(u_n))_{n \in \mathbb{N}} \subset \mathscr{L}(\mathcal{H}, E^* \hat{\otimes}_\epsilon \mathcal{G})$ is also a Cauchy sequence in $\mathscr{L}(\mathcal{H}, E^* \hat{\otimes}_\epsilon \mathcal{G})$ and we denote the limit by $\Psi(u)$, which is independent of the choice of the Cauchy sequence that approximates u. Indeed, if $(u_n^1)_{n \in \mathbb{N}}, (u_n^2)_{n \in \mathbb{N}} \subset E^* \otimes \mathscr{L}(\mathcal{H}, \mathcal{G})$ are two Cauchy sequences such that

$$\sup_{\xi \in E_1} \|u(\xi) - u_n^1(\xi)\|_{\mathscr{L}(\mathcal{H}, \mathcal{G})} \xrightarrow[n \to \infty]{} 0 \quad \text{and}$$

$$\sup_{\xi \in E_1} \|u(\xi) - u_n^2(\xi)\|_{\mathscr{L}(\mathcal{H}, \mathcal{G})} \xrightarrow[n \to \infty]{} 0,$$

then,

$$\|\tilde{\imath}(u_n^1 - u_n^2)\|_{\mathscr{L}(\mathcal{H}, E^* \hat{\otimes}_\epsilon \mathcal{G})} = \sup_{\xi \in E^1} \|(u_n^1 - u_n^2)(\xi)\|_{\mathscr{L}(\mathcal{H}, \mathcal{G})} \leq$$

$$\sup_{\xi \in E^1} \|(u_n^1 - u)(\xi)\|_{\mathscr{L}(\mathcal{H}, \mathcal{G})} + \sup_{\xi \in E^1} \|(u - u_n^2)(\xi)\|_{\mathscr{L}(\mathcal{H}, \mathcal{G})} \xrightarrow[n \to \infty]{} 0.$$

This discussion implies that we have a natural isometry $\Psi : E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}) \to \mathscr{L}(\mathcal{H}, E^* \hat{\otimes}_\epsilon \mathcal{G})$ onto the image of this map, which gives the desired identification.

So, if $\xi \in E$ then we can think of $u(\xi)$ as an operator in $\mathscr{L}(E, \mathscr{L}(\mathcal{H}, \mathcal{G}))$ or if $h \in \mathcal{H}$ then we can think of $\Psi(u)(h)$ as an element in $E^* \hat{\otimes}_\epsilon \mathcal{G}$ or as an operator in $\mathscr{L}(E, \mathcal{G})$ and we will denote $\Psi(u)(h)$ by $u(h)$ for notational purposes. In order to avoid confusion with this abuse of notation, which we need in order to keep the notation as clear as possible, we will only denote by ξ elements in E, so the only case where $u(\xi)$ is interpreted as an operator in $\mathscr{L}(E, \mathscr{L}(\mathcal{H}, \mathcal{G}))$ will be when ξ is in the argument of u. Note that with this identification we may write

$$u(\xi)(h) = u(h)(\xi) \in \mathcal{G},$$

for all $\xi \in E$ and $h \in \mathcal{H}$. Indeed, take an approximating sequence $(u_n)_{n \in \mathbb{N}} \subset E^* \otimes \mathscr{L}(\mathcal{H}, \mathcal{G})$ for u as above, then

$$u_n(\xi)(h) = \left(\sum_{i=1}^{n} \lambda_i^n(\xi) T_i^n \right)(h) = \sum_{i=1}^{n} \lambda_i^n(\xi) T_i^n(h) = \left(\sum_{i=1}^{n} \lambda_i^n \otimes T_i^n(h) \right)(\xi) = u_n(h)(\xi),$$

for all $n \in \mathbb{N}$, $\xi \in E$ and $h \in \mathcal{H}$. Therefore, if $\xi \in E_1 \setminus \{0\}$ and $h \in \mathcal{H}_1 \setminus \{0\}$ we get

$$\|u(\xi)(h) - u(h)(\xi)\|_{\mathcal{G}} \leq \|u(\xi)(h) - u_n(\xi)(h)\|_{\mathcal{G}} + \|u(h)(\xi) - u_n(h)(\xi)\|_{\mathcal{G}} \leq$$

$$\|u(\xi) - u_n(\xi)\|_{\mathscr{L}(\mathcal{H}, \mathcal{G})} + \|u(h) - u_n(h)\|_{E^* \hat{\otimes}_\epsilon \mathcal{G}} \xrightarrow[n \to \infty]{} 0.$$

Moreover, note that whenever E is reflexive, the weak* injective tensor product and the injective tensor product coincide, but the converse is not true in general, see Chapters 42 and 43 in Treves (1967) and Chapter 3 in Ryan (2002) for a discussion on the injective tensor product and tensor products on locally convex vector spaces in general.

Furthermore, we endow $E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G})$ with the weak* topology $\mathcal{O}^*$, that is the coarsest topology such that the evaluation maps $u \mapsto u(\xi) \in \mathscr{L}(\mathcal{H}, \mathcal{G})$ corresponding to $\xi \in E$ are all continuous and we take the σ-algebra on $E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G})$ generated by this topology. Note that for $\mathcal{H} = \mathcal{G} = \mathbb{R}$, this is precisely the usual weak* topology on E^*. We will write $L^0(\bar{\Omega}, \mathscr{P}; E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}))$ instead of $L^0(\bar{\Omega}, \mathscr{P}; E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}), \sigma(\mathcal{O}^*))$ for notational purposes, so a process X is of the class $L^0(\bar{\Omega}, \mathscr{P}; E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}))$ if there is a sequence of $\mathscr{P}$-measurable $E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G})$-valued step functions $(X^n)_{n \in \mathbb{N}} \subset \mathscr{E}(E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}), \mathscr{P})$ such that

$$P\left(\left\{ \omega \in \Omega \,\middle|\, \exists t \in \mathbb{R}_+ : \; X_t(\omega) \overset{w^*}{\neq} \lim_{n \to \infty} X_t^n(\omega) \right\} \right) = 0,$$

where the limit is taken with respect to the weak*-topology. However, note that X is not necessarily $(\mathscr{P}, \sigma(\mathcal{O}))$-measurable and limits in the class $L^0(\bar{\Omega}, \mathscr{P}; E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}))$ need not necessarily lie in this class.

Suppose that $X = \sum_{i=1}^n \mathbb{1}_{(s_i,t_i]\times F_i} u_i \in \mathscr{E}(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$ is an $\mathscr{R}$-measurable $E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G})$-valued step function, then, since Z is $\mathcal{H}$-valued, we can define the following process

$$\int_0^t X_s dZ_s(\omega) := \sum_{i=1}^n \mathbb{1}_{F_i}(\omega) u_i \Big(Z_{t\wedge t_i}(\omega) - Z_{t\wedge s_i}(\omega) \Big) \in E^*\hat{\otimes}_\epsilon \mathcal{G}.$$

This mapping will be extended to a suitable class of integrands taking values in $E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G})$ such that the property

$$\int_0^t X_s dZ_s(\xi) = \int_0^t X_s(\xi) dZ_s,$$

holds for all $\xi \in E$ and $\int X(\xi)dZ$ coincides with the stochastic integral constructed in Section 2, this property will be called the weak* Fubini property.

This section is divided into five parts. The first part introduces a class of processes for which the construction of this new stochastic integral is almost a pointwise extension of the stochastic integral constructed in Section 2, we call this stochastic integral a weak* stochastic integral. In part two and three we discuss the ambient space of the weak* stochastic integral and some issues regarding well-definedness of the integral. Part four gives the main result of this section, that is the construction of the integral and the statement and proof of the weak* Fubini property. In the last part of this section we discuss the subtle issue of measurability for $E^*\hat{\otimes}_\epsilon F$-valued processes and state a sufficient condition for strongly $\mathscr{P}$-measurability of an $E^*\hat{\otimes}_\epsilon \mathbb{R}^d$-valued processes, which is relatively simple to verify for the case of the two-parameter process.

3.1 The space $\Lambda(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$

Let $A \in \mathscr{A}(Z)$ be a control process for Z, $\tau \in \mathscr{T}$ a stopping time such that $\mathbb{E}(A_{\tau-}^2) < \infty$, let $\gamma \in l^1_+(\mathbb{N}) := \{\gamma \in l^1(\mathbb{N}) | \forall k \in \mathbb{N} : \gamma_k > 0\}$ and let $\underline{\xi} := \{\xi_k\}_{k\in\mathbb{N}} \in \mathscr{D}(E_1)$ be a dense countable set in E_1, we define

$$q^A_{\tau,\gamma,\underline{\xi}} : L^0(\bar{\Omega}, \mathscr{P}; E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G})) \to [0,\infty]$$
$$X \mapsto \left(\sum_{k\in\mathbb{N}} \gamma_k \, \mathbb{E}\bigg(A_{\tau-} \int_0^{\tau^-} \|X_s(\xi_k)\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} \, dA_s \bigg) \right)^{\frac{1}{2}},$$

and the vector space

$$\Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G})) := \left\{ X \in L^0(\bar{\Omega}, \mathscr{P}; E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G})) \,\middle|\, q^A_{\tau,\gamma,\xi}(X) < \infty \right\}.$$

Note that $q^A_{\tau,\gamma,\xi}(X)^2 = \sum_{k\in\mathbb{N}} \gamma_k \, q^A_\tau(X(\xi_k))^2$ is the γ-weighted sum of the processes $(X(\xi_k))_{k\in\mathbb{N}}$ with respect to the seminorm q^A_τ defined in the previous section, hence it is well-defined.

Lemma 3.5. $\mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})) \subset \Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$

Proof. Let $X = \sum_{i=1}^n \mathbb{1}_{R_i} u_i \in \mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$ be an $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step function . For any $k \in \mathbb{N}$ we have

$$\|u_i(\xi_k)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})} \leq \sup_{\xi \in E_1} \|u_i(\xi)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})} = \|u_i\|_\epsilon < \infty,$$

hence if we set $C := n^2\mathbb{E}(A^2_{\tau^-}) \max\limits_{1\leq i\leq n} \|u_i\|^2_\epsilon$ we obtain

$$q^A_{\tau,\gamma,\underline{\xi}}(X)^2 = \sum_{k\in\mathbb{N}} \gamma_k \mathbb{E}\bigg(A_{\tau^-} \int_0^{\tau^-} \|X_s(\xi_k)\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} dA_s \bigg) =$$

$$= \sum_{k\in\mathbb{N}} \gamma_k \mathbb{E}\bigg(A_{\tau^-} \int_0^{\tau^-} \bigg\| \sum_{i=1}^n \mathbb{1}_{R_i} u_i(\xi_k) \bigg\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} dA_s \bigg) \leq C\|\gamma\|_{l^1(\mathbb{N})} < \infty,$$

so $X \in \Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$. □

Lemma 3.6. *The mapping $q^A_{\tau,\gamma,\underline{\xi}}$ defines a seminorm on $\Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$.*

Proof. This argument is taken from the proof of Lemma 1.5 in Choulli/Schweizer (2013). Denote by $l^2(\mathbb{N},\gamma)$ the l^2-space with weights $(\gamma_k)_{k\in\mathbb{N}}$. If $X \in \Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$ then $q^A_{\tau,\gamma,\underline{\xi}}(X)$ can be interpreted as an l^2-seminorm in the following way:

$$q^A_{\tau,\gamma,\underline{\xi}}(X) = \left\| \left\{ \mathbb{E}\bigg(A_{\tau^-} \int_0^{\tau^-} \|X_s(\xi_k)\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})} dA_s \bigg)^{\frac{1}{2}} \right\}_{k\in\mathbb{N}} \right\|_{l^2(\mathbb{N},\gamma)} = \|\{q^A_\tau(X(\xi_k))\}_{k\in\mathbb{N}}\|_{l^2(\mathbb{N},\gamma)},$$

which shows that $q^A_{\tau,\gamma,\underline{\xi}}$ is a seminorm, since q^A_τ is a seminorm. □

Lemma 3.7. *The set of $\mathscr{R}$-measurable $E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step functions $\mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$ is dense with respect to $q^A_{\tau,\gamma,\underline{\xi}}$ in the seminormed vector space $\Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$.*

Proof. Let $X \in \Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$, then by separability of E_1 and $\mathcal{H}$ the process $\|X\|_\epsilon$ is $\mathscr{P}$-measurable. Indeed, since $\underline{\xi} \subset \mathscr{D}(E_1)$ is dense in the unit ball E_1, we can write for any dense subset $(h_n)_{n\in\mathbb{N}}$ in the unit sphere of $\mathcal{H}$

$$\|X_t(\omega)\|_\epsilon := \sup_{n\in\mathbb{N}} \|X_t(\omega)(\xi_n)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})} = \sup_{n,k\in\mathbb{N}} \|X_t(\omega)(\xi_n)(h_k)\|_{\mathcal{G}},$$

for all $(t,\omega) \in \bar{\Omega}$, where we use the identifications described above. This shows that $\|X\|_\epsilon$ is $\mathscr{P}$-measurable, since $\|X(\xi_n)(h_k)\|_{\mathcal{G}}$ is $\mathscr{P}$-measurable for all $n,k \in \mathbb{N}$. Moreover, $\|(\mathbb{1}_{\{\|X\|_\epsilon \leq K\}} X_s(\omega) - X_s(\omega))(\xi)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})} \xrightarrow[K\to\infty]{} 0$ for all $(s,\omega) \in \bar{\Omega}$ and $\xi \in E$. Hence, by dominated convergence

$$\sum_{k\in\mathbb{N}}\gamma_k\mathbb{E}\bigg(A_{\tau^-}\int_0^{\tau^-}\|\big(\mathbb{1}_{\{\|X\|_\epsilon\le K\}}X_s-X_s\big)(\xi_k)\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})}dA_s\bigg)\xrightarrow[K\to\infty]{}0.$$

So without loss of generality we may assume $\sup_{(s,\omega)\in\bar{\Omega}}\|X_s(\omega)\|_\epsilon\le K$ for some $K\ge 0$. Choose a sequence $(X^n)_{n\in\mathbb{N}}\subset\mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}),\mathscr{P})$ of $\mathscr{P}$-measurable $E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step functions such that

$$P\Bigg(\bigg\{\omega\in\Omega\,\bigg|\,\exists t\in\mathbb{R}:X_t(\omega)\overset{w^*}{\neq}\lim_{n\to\infty}X^n_t(\omega)\bigg\}\Bigg)=0,$$

where the limit is taken with respect to the weak* topology and note that we may choose this sequence such that $\|X^n\|_\epsilon\le K+1$, for instance by taking $\bar{X}^n:=\mathbb{1}_{\{\|X-X^n\|_\epsilon\le 1\}}X^n$, which is still a predictable step function and converges to X pointwise in the weak* topology, more precisely for any $(s,\omega)\in\bar{\Omega}$ and $\xi\in E$ we have

$$\|(\bar{X}^n_s(\omega)-X_s(\omega))(\xi)\|_{\mathscr{L}(\mathcal{H},\mathcal{G})}\xrightarrow[n\to\infty]{}0.$$

Therefore we obtain that

$$\sum_{k\in\mathbb{N}}\gamma_k E\bigg(A_{\tau^-}\int_0^{\tau^-}\|X_s(\xi_k)-X^n_s(\xi_k)\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})}dA_s\bigg)\xrightarrow[n\to\infty]{}0,$$

by dominated convergence. Just as in the remark preceding Lemma 2.24 we can show that $\mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$ is a $q^A_{\tau,\gamma,\underline{\xi}}$-dense subspace of $\mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}),\mathscr{P})$, which finishes this proof. $\square$

Analogously as in the previous section we introduce a class of processes that allow a suitable increasing sequence of stopping times.

Lemma 3.8 (**weak* Z-integrable processes**). *Suppose that $X\in L^0(\bar{\Omega},\mathscr{P};E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$ is such that the process*

$$\lambda_t^{A,\gamma,\underline{\xi}}(X):=\sum_{k\in\mathbb{N}}\gamma_k A_t\int_0^t\|X_s(\xi_k)\|^2_{\mathscr{L}(\mathcal{H},\mathcal{G})}dA_s<\infty$$

is finite P-a.e. for any $t>0$. Then, there is an increasing sequence of stopping times such that $\lim_{n\to\infty}\tau_n\overset{P\text{-a.e.}}{=}\infty$ and for any $n\in\mathbb{N}$ we have $\mathbb{E}(A^2_{\tau_n^-})<\infty$ and $X\in\Lambda^A_{\tau_n,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$.

Proof. Define for any $n\in\mathbb{N}$ the stopping time $\tau_n:=\inf\{t\ge 0|A_t\wedge\lambda_t^{A,\gamma,\underline{\xi}}(X)\ge n\}$. This sequence is increasing, $\lim_{n\to\infty}\tau_n=\infty$ P-a.e. and it satisfies the desired properties. $\square$

Hence, it is natural to consider as the space of integrands, which we call weak* Z-integrable processes, the class

$$\Lambda(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})) := \bigcup_{\mathscr{A}(Z)\times l^1_+(\mathbb{N})\times\mathscr{D}(E_1)} \left\{ X\in L^0(\bar{\Omega},\mathscr{P};E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))\,\middle|\, \lambda^{A,\gamma,\underline{\xi}}(X)<\infty \right\},$$

where the union is taken over all control processes $A\in\mathscr{A}(Z)$, all sequences $\gamma\in l^1_+(\mathbb{N})$ and dense countable subsets $\underline{\xi}\in\mathscr{D}(E_1)$, where $\lambda^{A,\gamma,\underline{\xi}}(X)<\infty$ means that $\lambda_t^{A,\gamma,\underline{\xi}}(X)<\infty$ P-a.e. for all $t\geq 0$. Note that if $E=\mathbb{R}$, then $\Lambda(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$ and $\Lambda(\mathscr{L}(\mathcal{H},\mathcal{G}))$ coincide, but in general these two classes of integrands are different. Moreover, the class of weak* Z-integrable processes as presented here is not as large as we would like it to be, for instance, it is not clear if the pointwise limit of P-strongly $\mathscr{P}$-measurable processes is still strongly measurable. In part four of this section we will discuss a natural extension of the class of allowed integrands.

3.2 The Space $\Pi(E^*\hat{\otimes}_\epsilon\mathcal{G})$

Let $A\in\mathscr{A}(Z)$ be a control process, $\tau\in\mathscr{T}$ a stopping time such that $\mathbb{E}(A^2_{\tau-})<\infty$, $\underline{\xi}:=\{\xi_k\}_{k\in\mathbb{N}}\in\mathscr{D}(E_1)$ a dense countable set in E_1 and $\gamma\in l^1_+(\mathbb{N})$. Define the space

$$\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G}) := \left\{ \mathbb{1}_{[[0,\tau))}Y \,\middle|\, Y:\bar{\Omega}\to E^*\hat{\otimes}_\epsilon\mathcal{G} \text{ is RCLL, } \forall\,\xi\in E: Y(\xi)\in L^0(\bar{\Omega},\bar{\mathscr{F}};\mathcal{G}) \right\},$$

and the mapping

$$\begin{aligned} p_{\tau,\gamma,\underline{\xi}} : \Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G}) &\to [0,\infty] \\ Y &\mapsto \left(\sum_{k\in\mathbb{N}}\gamma_k\mathbb{E}(\sup_{s<\tau}\|Y_s(\xi_k)\|^2_{\mathcal{G}})\right)^{\frac{1}{2}}, \end{aligned}$$

and note that $p_{\tau,\gamma,\underline{\xi}}(Y)^2=\sum_{k\in\mathbb{N}}\gamma_k p_\tau(Y(\xi_k))^2$. It is tempting to define in analogy to the last section $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$ as the space consisting of processes of the form $\mathbb{1}_{[[0,\tau))}Y$ such that $Y\in L^0(\bar{\Omega},\bar{\mathscr{F}};E^*\hat{\otimes}_\epsilon\mathcal{G})$ is RCLL. However, due to the lack of regularity of $E^*\hat{\otimes}_\epsilon\mathcal{G}$, we would not be able, in general, to show that it is complete with respect to the seminorm $p_{\tau,\gamma,\underline{\xi}}$, since the pointwise limit of a sequence in $L^0(\bar{\Omega},\mathscr{F};E^*\hat{\otimes}_\epsilon\mathcal{G})$ is not necessarily of that class. Surprisingly (or not), the class of processes $Y:\bar{\Omega}\to E^*\hat{\otimes}_\epsilon\mathcal{G}$ such that $Y(\xi)\in L^0(\bar{\Omega},\mathscr{F};\mathcal{G})$ for all $\xi\in E$, which we will later call weak* measurable processes, are more well-behaved for our purposes. We start by showing that $p_{\tau,\gamma,\underline{\xi}}$ is a seminorm and the weak* Fubini property for step functions, which will later be extended to a larger class of processes after we show that $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$ is a complete space.

Lemma 3.9. *The mapping $p_{\tau,\gamma,\underline{\xi}}$ defines a seminorm on $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$.*

Proof. If $Y \in \Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$ then $p_{\tau,\gamma,\underline{\xi}}(Y) = \|\{\mathbb{E}(\sup_{s<\tau}\|Y_s(\xi_k)\|^2_{\mathcal{G}})^{\frac{1}{2}}\}_{k\in\mathbb{N}}\|_{l^2(\mathbb{N},\gamma)}$. □

Lemma 3.10 (The weak* stochastic integral and the weak* Fubini property for step functions). *Consider $\mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$ together with the seminorm $q^A_{\tau,\gamma,\underline{\xi}}$ and the space $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$ with the seminorm $p_{\tau,\gamma,\underline{\xi}}$, then the mapping*

$$\begin{aligned}\Phi^A_{\tau,\gamma,\underline{\xi}} : \mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})) &\to \Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})\\ X &\mapsto \mathbb{1}_{[[0,\tau))}\int XdZ,\end{aligned}$$

is a bounded operator and it satisfies the weak* Fubini property, namely

$$\mathbb{1}_{[[0,\tau))}\int XdZ(\xi) = \mathbb{1}_{[[0,\tau))}\int X(\xi)dZ,$$

for any $\xi \in E$ and $X \in \mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$.

Proof. Let $X = \sum_{i=1}^n \mathbb{1}_{(s_i,t_i]\times F_i}u_i \in \mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$. Since $A \in \mathscr{A}(Z)$ is a control process for Z we get for any $\xi \in E$ the inequality

$$\mathbb{E}\bigg(\sup_{s<\tau}\|\int_0^s X(\xi)dZ\|^2_{\mathcal{G}}\bigg) \le \mathbb{E}\bigg(A_{\tau^-}\int_0^{\tau^-}\|X_s(\xi)\|^2_{\mathcal{G}}dA_s\bigg).$$

Hence $p_{\tau,\gamma,\underline{\xi}}(\Phi^A_{\tau,\gamma,\underline{\xi}}(X)) \le q^A_{\tau,\gamma,\underline{\xi}}(X) < \infty$. To verify the weak* Fubini property let $\xi \in E$, then

$$\int_0^t XdZ(\xi) = \sum_{i=1}^n \mathbb{1}_{F_i}u_i(Z_{t\wedge t_i} - Z_{t\wedge s_i})(\xi) = \sum_{i=1}^n \mathbb{1}_{F_i}u_i(\xi)(Z_{t\wedge t_i} - Z_{t\wedge s_i}) =$$

$$\int_0^t\bigg(\sum_{i=1}^n \mathbb{1}_{(s_i,t_i]\times F_i}u_i(\xi)\bigg)dZ = \int_0^t X(\xi)dZ,$$

where we use the identifications of $E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})$ described above. □

The notation $\Phi^A_{\tau,\gamma,\underline{\xi}}$ is used to stress that this is a bounded operator when we endow $\mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$ with the seminorm $q^A_{\tau,\gamma,\underline{\xi}}$, however as an algebraic map, the only A-dependence is due to τ. In order to argue that $\Phi^A_{\tau,\gamma,\underline{\xi}}$ has a unique extension we still need to prove as announced before that the space $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$ is complete.

Lemma 3.11. *The space $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$ is complete for the seminorm $p_{\tau,\gamma,\underline{\xi}}$.*

Proof. Suppose that $(Y^n)_{n\in\mathbb{N}} \subset \Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$ is a Cauchy sequence, that is

$$\sum_{k\in\mathbb{N}} \gamma_k \mathbb{E}\bigg(\sup_{s<\tau} \|Y_s^n(\xi_k) - Y_s^m(\xi_k)\|_{\mathcal{G}}^2\bigg) \xrightarrow[m,n\to\infty]{} 0.$$

Choose a subsequence $(n_m)_{m\in\mathbb{N}}$ such that the second inequality

$$P\bigg(\bigcup_{k\in\mathbb{N}}\bigg\{\sup_{s<\tau}\|Y_s^{n_{m+1}}(\xi_k) - Y_s^{n_m}(\xi_k)\|_{\mathcal{G}}^2 > \frac{1}{\gamma_k 2^m}\bigg\}\bigg) \leq$$

$$2^m \sum_{k\in\mathbb{N}} \gamma_k \mathbb{E}\bigg(\sup_{s<\tau}\|Y_s^{n_{m+1}}(\xi_k) - Y_s^{n_m}(\xi_k)\|_{\mathcal{G}}^2\bigg) \leq \frac{1}{2^m},$$

holds for all $m \in \mathbb{N}$. Hence, for all $m, l \in \mathbb{N}$ we obtain

$$P\bigg(\bigcup_{k\in\mathbb{N}}\bigg\{\sup_{s<\tau}\|Y_s^{n_{m+l}}(\xi_k) - Y_s^{n_m}(\xi_k)\|_{\mathcal{G}}^2 > \frac{1}{\gamma_k 2^{m-4}}\bigg\}\bigg) \leq \frac{1}{2^{m-1}},$$

which follows from the fact that

$$\bigcup_{k\in\mathbb{N}}\bigg\{\sup_{s<\tau}\|Y_s^{n_{m+l}}(\xi_k) - Y_s^{n_m}(\xi_k)\|_{\mathcal{G}}^2 > \frac{1}{\gamma_k 2^{m-4}}\bigg\} \subset$$

$$\bigcup_{k\in\mathbb{N}}\bigcup_{i=1}^{l}\bigg\{\sup_{s<\tau}\|Y_s^{n_{m+i}}(\xi_k) - Y_s^{n_{m+i-1}}(\xi_k)\|_{\mathcal{G}}^2 > \frac{1}{\gamma_k 2^{m+i-1}}\bigg\}.$$

Indeed, if $\omega \notin \bigcup_{k\in\mathbb{N}}\bigcup_{i=1}^{l}\bigg\{\sup_{s<\tau}\|Y_s^{n_{m+i}}(\xi_k) - Y_s^{n_{m+i-1}}(\xi_k)\|_{\mathcal{G}}^2 > \frac{1}{\gamma_k 2^{m+i-1}}\bigg\}$, then for all $k \in \mathbb{N}$ we have the following chain of inequalities

$$\sup_{s<\tau}\|Y_s^{n_{m+l}}(\omega)(\xi_k) - Y_s^{n_m}(\omega)(\xi_k)\|_{\mathcal{G}}^2 \leq \bigg(\sum_{i=1}^{l}\|Y_s^{n_{m+i}}(\omega)(\xi_k) - Y_s^{n_{m+i-1}}(\omega)(\xi_k)\|_{\mathcal{G}}\bigg)^2 \leq$$

$$\bigg(\sum_{i=1}^{l}\frac{1}{\sqrt{\gamma_k 2^{m+i-1}}}\bigg)^2 \leq \frac{1}{\gamma_k 2^{m-1}}\bigg(\sum_{i=1}^{\infty}\frac{1}{\sqrt{2^i}}\bigg)^2 = \frac{(1+\sqrt{2})^2}{\gamma_k 2^{m-1}} \leq \frac{1}{\gamma_k 2^{m-4}},$$

and this in turn gives

$$P\bigg(\bigcup_{k\in\mathbb{N}}\bigg\{\sup_{s<\tau}\|Y_s^{n_{m+l}}(\xi_k) - Y_s^{n_m}(\xi_k)\|_{\mathcal{G}}^2 > \frac{1}{\gamma_k 2^{m-4}}\bigg\}\bigg) \leq$$

$$\sum_{i=1}^{l} P\Bigg(\bigcup_{k\in\mathbb{N}}\left\{\sup_{s<\tau}\|Y_s^{n_{m+i}}(\xi_k)-Y_s^{n_{m+i-1}}(\xi_k)\|_{\mathcal{G}}^2>\frac{1}{\gamma_k 2^{m+i-1}}\right\}\Bigg)\leq\sum_{i=1}^{\infty}\frac{1}{2^{m+i-1}}=\frac{1}{2^{m-1}},$$

which proves the claim we made above. The Borel-Cantelli lemma implies that

$$\Omega\setminus\Omega_0:=\limsup_{m\to\infty}\bigcup_{k\in\mathbb{N}}\left\{\sup_{s<\tau}\|Y_s^{n_{m+l}}(\xi_k)-Y_s^{n_m}(\xi_k)\|_{\mathcal{G}}^2>\frac{1}{\gamma_k 2^{m-4}}\right\}$$

is a null-set, in particular if $\omega\in\Omega_0$ then

$$\sup_{s<\tau(\omega)}\|Y_s^{n_{m+l}}(\omega)-Y_s^{n_m}(\omega)\|_{E^*\hat{\otimes}_\epsilon\mathcal{G}}^2=\sup_{s<\tau(\omega)}\sup_{k\in\mathbb{N}}\|Y_s^{n_{m+l}}(\omega)(\xi_k)-Y_s^{n_m}(\omega)(\xi_k)\|_{\mathcal{G}}^2\xrightarrow[m\to\infty]{}0,$$

which means that the functions $s\mapsto Y_s^{n_m}(\omega)$ converge uniformly for all $\omega\in\Omega_0$ on the interval $[0,\tau(\omega))$ to a function $s\mapsto\tilde{Y}_s(\omega)\in E^*\otimes_\epsilon\mathcal{G}$. Define the process $Y:=\mathbb{1}_{(\mathbb{R}_+\times\Omega_0)\cap[[0,\tau))}\tilde{Y}$, then Y is RCLL, due to the uniform convergence, and $Y(\xi)\in L^0(\bar{\Omega},\bar{\mathscr{F}};\mathcal{G})$ for all $\xi\in E$ by Proposition 2.4 or more precisely Corollary 2.17, so $Y\in\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$. □

Corollary 3.12 (**The local weak* stochastic integral**). *The mapping defined above $\Phi^A_{\tau,\gamma,\underline{\xi}}:\mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))\to\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})$ extends uniquely to a bounded linear operator*

$$\begin{aligned}\Phi^A_{\tau,\gamma,\underline{\xi}}:\Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))&\to\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G})\\ X&\mapsto\mathbb{1}_{[[0,\tau))}\int X\,dZ,\end{aligned}$$

which satisfies the inequality $p_{\tau,\gamma,\underline{\xi}}(\Phi^A_{\tau,\gamma,\underline{\xi}}(X))\leq q^A_{\tau,\gamma,\underline{\xi}}(X)$ for any $X\in\Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$.

Hence the space

$$\Pi(E^*\hat{\otimes}_\epsilon\mathcal{G}):=\bigcup_{\mathscr{I}}\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathcal{G}),$$

where $\mathscr{I}:=\{(A,\tau,\gamma,\underline{\xi})\in\mathscr{A}(Z)\times\mathscr{T}\times l^1(\mathbb{N})\times\mathscr{D}(E_1)|\ \mathbb{E}(A^2_{\tau-})<\infty\}$, is a natural ambient space for this integral. In the next part we argue why it is possible to extend the stochastic integral using an increasing sequence of stopping times in analogy to Section 2. The main difficulty is that we have to prove prove independence of the choice of stopping times, independence of the choice of weights $\gamma\in l^1_+(\mathbb{N})$, independence of the choice of a dense countable subset $\underline{\xi}\in\mathscr{D}(E_1)$, independence of the choice of control process A in a suitable way and finally paste all independence claims together, so that we are able to glue a stochastic integral as in Section 2.

3.3 The extension

Lemma 3.13 (Independence of stopping times). *Let $A \in \mathscr{A}(Z)$ be a control process for Z, $\gamma \in l^1_+(\mathbb{N})$ and $\underline{\xi} \in \mathscr{D}(E_1)$ a dense countable subset. Suppose that τ, $\sigma \in \mathscr{T}$ are two stopping times such that $\mathbb{E}(A^2_{\tau^-}) < \infty$ and $\mathbb{E}(A^2_{\sigma^-}) < \infty$.*
If $X \in \Lambda^A_{\tau,\gamma,\underline{\xi}}(E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G})) \cap \Lambda^A_{\sigma,\gamma,\underline{\xi}}(E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}))$ then

$$\mathbb{1}_{[[0,\tau\wedge\sigma))} \Phi^A_{\tau,\gamma,\underline{\xi}}(X) = \mathbb{1}_{[[0,\tau\wedge\sigma))} \Phi^A_{\sigma,\gamma,\underline{\xi}}(X) = \Phi^A_{\tau\wedge\sigma,\gamma,\underline{\xi}}(X).$$

Proof. The assumption $X \in \Lambda^A_{\tau,\gamma,\underline{\xi}}(E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G})) \cap \Lambda^A_{\sigma,\gamma,\underline{\xi}}(E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}))$ implies in particular that $X \in \Lambda^A_{\tau\wedge\sigma,\gamma,\underline{\xi}}(E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}))$. Choose a sequence of $\mathscr{R}$-measurable $\mathscr{E}^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G})$-valued step functions $(X^n_1)_{n\in\mathbb{N}} \subset \mathscr{E}(E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}))$ such that $X^n_1 \xrightarrow[n\to\infty]{q^A_{\tau,\gamma,\underline{\xi}}} X$. Note that $X^n_1 \xrightarrow[n\to\infty]{q^A_{\tau\wedge\sigma,\gamma,\underline{\xi}}} X$ and for any $n \in \mathbb{N}$ we have the identity

$$\mathbb{1}_{[[0,\tau\wedge\sigma))} \Phi^A_{\tau,\gamma,\underline{\xi}}(X^n_1) = \mathbb{1}_{[[0,\tau\wedge\sigma))} \mathbb{1}_{[[0,\tau))} \int X^n_1 dZ = \mathbb{1}_{[[0,\tau\wedge\sigma))} \int X^n_1 dZ = \Phi^A_{\tau\wedge\sigma,\gamma,\underline{\xi}}(X^n_1).$$

Moreover, the fact that $X^n_1 \xrightarrow[n\to\infty]{q^A_{\tau\wedge\sigma,\gamma,\underline{\xi}}} X$, implies that $\Phi^A_{\tau\wedge\sigma,\gamma,\underline{\xi}}(X^n_1) \xrightarrow[n\to\infty]{p_{\tau\wedge\sigma,\gamma,\underline{\xi}}} \Phi^A_{\tau\wedge\sigma,\gamma,\underline{\xi}}(X)$ by Corollary 3.12, but $X^n_1 \xrightarrow[n\to\infty]{q^A_{\tau,\gamma,\underline{\xi}}} X$ in turn shows that $\mathbb{1}_{[[0,\tau\wedge\sigma))} \Phi_{\tau,\gamma,\underline{\xi}}(X^n_1) \xrightarrow[n\to\infty]{p_{\tau,\gamma,\underline{\xi}}} \mathbb{1}_{[[0,\tau\wedge\sigma))} \Phi_{\tau,\gamma,\underline{\xi}}(X)$ and the convergence also holds with respect to $p_{\tau\wedge\sigma,\gamma,\underline{\xi}}$. Hence, by uniqueness we obtain

$$\mathbb{1}_{[[0,\tau\wedge\sigma))} \Phi_{\tau,\gamma,\underline{\xi}}(X) = \Phi^A_{\tau\wedge\sigma,\gamma,\underline{\xi}}(X),$$

in the seminormed vector space $\Pi_{\tau,\gamma,\underline{\xi}}(E^* \hat{\otimes}_\epsilon \mathcal{G})$. The same argument can be repeated for a sequence $(X^n_2)_{n\in\mathbb{N}}$ such that $X^n_2 \xrightarrow[n\to\infty]{q^A_{\sigma,\gamma,\underline{\xi}}} X$, which proves the claim. □

Lemma 3.14 (Independence of weights and dense subsets). *Let $A \in \mathscr{A}(Z)$ be a control process and $\tau \in \mathscr{T}$ a stopping time such that $\mathbb{E}(A^2_{\tau^-}) < \infty$. Suppose that γ^1, $\gamma^2 \in l^1_+(\mathbb{N})$ and $\underline{\xi}^1$, $\underline{\xi}^2 \in \mathscr{D}(E_1)$. If $X \in \Lambda^A_{\tau,\gamma^1,\underline{\xi}^1}(E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G})) \cap \Lambda^A_{\tau,\gamma^2,\underline{\xi}^2}(E^* \hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H}, \mathcal{G}))$ then*

$$\Phi^A_{\tau,\gamma^1,\underline{\xi}^1}(X) = \Phi^A_{\tau,\gamma^2,\underline{\xi}^2}(X).$$

Proof. The ideas contained here can be found in the proof of Theorem 2.1 in Choulli/Schweizer (2013). Define the interlaced sequence $\gamma := (\gamma^1_1, \gamma^2_1, \gamma^1_2, \gamma^2_2, \dots) \in l^1_+(\mathbb{N})$ and the dense countable subset $\underline{\xi} := \{\xi^1_1, \xi^2_1, \xi^1_2, \xi^2_2, \dots\} \in \mathscr{D}(E_1)$ in E_1. Then

$$q^A_{\tau,\gamma,\underline{\xi}}(X)^2 = q^A_{\tau,\gamma^1,\underline{\xi}^1}(X)^2 + q^A_{\tau,\gamma^2,\underline{\xi}^2}(X)^2,$$

so $X \in \Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$. Choose a sequence $(X^n)_{n\in\mathbb{N}} \subset \mathscr{E}(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$ of $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step functions such that $X^n \xrightarrow[n\to\infty]{q^A_{\tau,\gamma,\underline{\xi}}} X$, then both $X^n \xrightarrow[n\to\infty]{q^A_{\tau,\gamma^1,\underline{\xi}^1}} X$ and $X^n \xrightarrow[n\to\infty]{q^A_{\tau,\gamma^2,\underline{\xi}^2}} X$. Moreover, if $Y \in \Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathcal{G})$ then

$$p_{\tau,\gamma,\underline{\xi}}(Y)^2 = p_{\tau,\gamma^1,\underline{\xi}^1}(Y)^2 + p_{\tau,\gamma^2,\underline{\xi}^2}(Y)^2.$$

Hence, since $\Phi^A_{\tau,\gamma,\underline{\xi}}(X^n) \xrightarrow[n\to\infty]{p_{\tau,\gamma,\underline{\xi}}} \Phi^A_{\tau,\gamma,\underline{\xi}}(X)$, we obtain that $\Phi^A_{\tau,\gamma^1,\underline{\xi}^1}(X^n) \xrightarrow[n\to\infty]{p_{\tau,\gamma^1,\underline{\xi}^1}} \Phi^A_{\tau,\gamma,\underline{\xi}}(X)$, but simultaneously we have $\Phi^A_{\tau,\gamma^1,\underline{\xi}^1}(X^n) \xrightarrow[n\to\infty]{p_{\tau,\gamma^1,\underline{\xi}^1}} \Phi^A_{\tau,\gamma^1,\underline{\xi}^1}(X)$, which by uniqueness of the extension implies that

$$\Phi^A_{\tau,\gamma,\underline{\xi}}(X) = \Phi^A_{\tau,\gamma^1,\underline{\xi}^1}(X),$$

in the seminormed vector space $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathcal{G})$. Repeat the argument for the triplet $(\tau,\gamma^2,\underline{\xi}^2)$ to obtain the desired result. □

Lemma 3.15 (Independence of control processes). *Let $\gamma \in l^1_+(\mathbb{N})$ and $\underline{\xi} \in \mathscr{D}(E_1)$ a dense countable set in E_1. Let A^1, $A^2 \in \mathscr{A}(Z)$ be two control processes and $\tau \in \mathscr{T}$ a stopping time such that $\mathbb{E}((A^1_{\tau^-})^2) < \infty$ and $\mathbb{E}((A^2_{\tau^-})^2) < \infty$.*
If $X \in \Lambda^{A^1}_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G})) \cap \Lambda^{A^2}_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$ then

$$\Phi^{A^1}_{\tau,\gamma,\underline{\xi}}(X) = \Phi^{A^2}_{\tau,\gamma,\underline{\xi}}(X) = \Phi^{A^1+A^2}_{\tau,\gamma,\underline{\xi}}(X).$$

Proof. Note that $A^1 + A^2 \in \mathscr{A}(Z)$ and $X \in \Lambda^{A^1+A^2}_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$. Choose a sequence of $\mathscr{R}$-measurable $\mathscr{L}(\mathcal{H},\mathcal{G})$-valued step functions $(X^n)_{n\in\mathbb{N}} \subset \mathscr{E}(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$ such that $X^n \xrightarrow[n\to\infty]{q^{A^1+A^2}_{\tau,\gamma,\underline{\xi}}} X$. Note that

$$\Phi^{A^1}_{\tau,\gamma,\underline{\xi}}(X^n) = \Phi^{A^1+A^2}_{\tau,\gamma,\underline{\xi}}(X^n),$$

holds for any $n \in \mathbb{N}$ and $\Phi^{A^1+A^2}_{\tau,\gamma,\underline{\xi}}(X^n) \xrightarrow[n\to\infty]{p_{\tau,\gamma,\underline{\xi}}} \Phi^{A^1+A^2}_{\tau,\gamma,\underline{\xi}}(X)$, which implies that the equality

$$\Phi^{A^1}_{\tau,\gamma,\underline{\xi}}(X) = \Phi^{A^1+A^2}_{\tau,\gamma,\underline{\xi}}(X),$$

holds in $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathcal{G})$ with respect to $p_{\tau,\gamma,\underline{\xi}}$. Repeating the same argument for $\Phi^{A^2}_{\tau,\gamma,\underline{\xi}}(X)$ gives the desired equality. □

Corollary 3.16 (**Independence**). *Let A^1, $A^2 \in \mathscr{A}(Z)$ be two control processes and τ, $\sigma \in \mathscr{T}$ two stopping times such that $\mathbb{E}((A^1_{\tau-})^2) < \infty$ and $\mathbb{E}((A^2_{\sigma-})^2) < \infty$. Let γ^1, $\gamma^2 \in l^1_+(\mathbb{N})$ and $\underline{\xi}^1$, $\underline{\xi}^2 \in \mathscr{D}(E_1)$.*
Suppose that $X \in \Lambda^{A^1}_{\tau,\gamma^1,\underline{\xi}^1}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})) \cap \Lambda^{A^2}_{\sigma,\gamma^2,\underline{\xi}^2}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$, then the following identity holds

$$\mathbb{1}_{[[0,\tau\wedge\sigma))}\Phi^{A^1}_{\tau,\gamma^1,\underline{\xi}^1}(X) = \mathbb{1}_{[[0,\tau\wedge\sigma))}\Phi^{A^2}_{\sigma,\gamma^2,\underline{\xi}^2}(X) = \Phi^{A^1+A^2}_{\tau\wedge\sigma,\gamma^1,\underline{\xi}^1}(X) = \Phi^{A^1+A^2}_{\tau\wedge\sigma,\gamma^2,\underline{\xi}^2}(X).$$

Proof. Note that $X \in \Lambda^{A^1+A^2}_{\tau\wedge\sigma,\gamma^1,\underline{\xi}^1}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G})) \cap \Lambda^{A^1+A^2}_{\tau\wedge\sigma,\gamma^2,\underline{\xi}^2}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$, so the result follows from the previous claims stated above. □

3.4 The weak* Fubini property

Proposition 3.17 (**The local weak* Fubini property**). *Let $\gamma \in l^1_+(\mathbb{N})$, $\underline{\xi} \in \mathscr{D}(E_1)$ a dense countable subset in E_1, $A \in \mathscr{A}(Z)$ a control process and $\tau \in \mathscr{T}$ a stopping time such that $\mathbb{E}(A^2_{\tau-}) < \infty$. The mapping $\Phi^A_{\tau,\gamma,\underline{\xi}}$ satisfies the weak* Fubini property, that is, whenever $\xi \in E$ then*

$$\mathbb{1}_{[[0,\tau))}\int XdZ(\xi) = \mathbb{1}_{[[0,\tau))}\int X(\xi)dZ,$$

for any $X \in \Lambda^A_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$.

Proof. Choose a sequence $(X^n)_{n\in\mathbb{N}} \subset \mathscr{E}(E^*\hat{\otimes}_\epsilon\mathscr{L}(\mathcal{H},\mathcal{G}))$ such that $X^n \xrightarrow[n\to\infty]{q^A_{\tau,\gamma,\underline{\xi}}} X$. For any $n \in \mathbb{N}$ we know by Lemma 3.10 that

$$\Phi^A_{\tau,\gamma,\zeta}(X^n)(\xi) = \mathbb{1}_{[[0,\tau))}\int X^n(\xi)dZ.$$

We may assume by linearity that $\xi \in E_1$ and by density of the subset $\underline{\xi} \subset E_1$ we may also assume that $\xi \in \underline{\xi}$. Since $\Phi^A_{\tau,\gamma,\underline{\xi}}(X^n) \xrightarrow[n\to\infty]{p_{\tau,\gamma,\underline{\xi}}} \Phi^A_{\tau,\gamma,\underline{\xi}}(X) = \mathbb{1}_{[[0,\tau))}\int XdZ$ we actually have that

$$\mathbb{E}\left(\sup_{s<\tau}\left\|\Phi^A_{\tau,\gamma,\underline{\xi}}(X^n)(\xi) - \Phi^A_{\tau,\gamma,\underline{\xi}}(X)(\xi)\right\|^2_{\mathcal{G}}\right) \xrightarrow[n\to\infty]{} 0.$$

Hence, there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\sup_{s<\tau}\left\|\Phi^A_{\tau,\gamma,\underline{\xi}}(X^{n_k})(\xi) - \Phi^A_{\tau,\gamma,\underline{\xi}}(X)(\xi)\right\|_{\mathcal{G}} \xrightarrow[k\to\infty]{P\text{-a.e.}} 0.$$

Moreover, $\Phi^A_{\tau,\gamma,\underline{\xi}}(X^{n_k})(\xi) = \Phi^A_\tau(X^{n_k}(\xi))$ for all $k \in \mathbb{N}$ and $\Phi^A_{\tau,\gamma,\underline{\xi}}(X^{n_k})(\xi) \xrightarrow[n\to\infty]{p_\tau} \Phi^A_\tau(X(\xi))$, where $\Phi_\tau^A(X(\xi)) = \mathbb{1}_{[[0,\tau))} \int X(\xi) dZ$ is the stochastic integral from Section 2, so

$$\mathbb{E}\left(\sup_{s<\tau} \left\| \Phi_\tau^A(X(\xi)) - \Phi_\tau^A(X^{n_k})(\xi) \right\|_{\mathcal{G}}^2 \right) \xrightarrow[n\to\infty]{} 0.$$

Therefore, for a suitable subsequence also denoted by $(n_k)_{k\in\mathbb{N}}$ we obtain that

$$\sup_{s<\tau} \left\| \Phi_\tau^A(X(\xi)) - \Phi_{\tau,\gamma,\underline{\xi}}^A(X)(\xi) \right\|_{\mathcal{G}} \overset{P\text{-a.e.}}{\leq}$$

$$\overset{P\text{-a.e.}}{\leq} \sup_{s<\tau} \left\| \Phi_\tau^A(X(\xi)) - \Phi_{\tau,\gamma,\underline{\xi}}^A(X^{n_k})(\xi) \right\|_{\mathcal{G}} + \sup_{s<\tau} \left\| \Phi_{\tau,\gamma,\underline{\xi}}^A(X^{n_k})(\xi) - \Phi_{\tau,\gamma,\underline{\xi}}^A(X)(\xi) \right\|_{\mathcal{G}} \xrightarrow[k\to\infty]{P\text{-a.e.}} 0,$$

which shows in particular, that the processes $\mathbb{1}_{[[0,\tau))} \int X(\xi) dZ$ and $\mathbb{1}_{[[0,\tau))} \int X dZ(\xi)$ coincide up to indistinguishability. □

Theorem 3.18 (The weak* stochastic integral and the weak* Fubini property). *There is a well-defined mapping*

$$\begin{aligned} \Lambda(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G})) &\to \Pi(E^*\hat{\otimes}_\epsilon \mathcal{G}) \\ X &\mapsto \int X dZ, \end{aligned}$$

called the weak* stochastic integral with respect to Z, that satisfies the weak* Fubini property.

Proof. Let $X \in \Lambda(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$. Let $A \in \mathscr{A}(Z)$ be a control process, $\gamma \in l_+^1(\mathbb{N})$ and $\underline{\xi} \in \mathscr{D}(E_1)$ such that $\lambda^{A,\gamma,\underline{\xi}}(X) < \infty$. By Lemma 3.8. there is an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}} \subset \mathscr{T}$ such that $X \in \bigcap_{n\in\mathbb{N}} \Lambda_{\tau_n,\gamma,\underline{\xi}}^A(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$. Define the process $Y^{A,\gamma,\zeta}$ by setting for any $n \in \mathbb{N}$

$$Y^{A,\gamma,\underline{\xi}}\big|_{[[0,\tau_n))} := \Phi_{\tau_n,\gamma,\underline{\xi}}^A(X).$$

Lemma 3.13 shows that $Y^{A,\gamma,\underline{\xi}}$ is independent of the sequence of stopping times. Finally Corollary 3.16 shows that if $(A^1,\gamma^1,\underline{\xi}^1)$, $(A^2,\gamma^2,\underline{\xi}^2) \in \mathscr{A}(Z)\times l_+^1(\mathbb{N})\times\mathscr{D}(E_1)$ are two triplets satisfying $\lambda^{A^1,\gamma^1,\underline{\xi}^1}(X), \lambda^{A^2,\gamma^2,\underline{\xi}^2}(X) < \infty$ then $Y^{A^1,\gamma^1,\underline{\xi}^1} = Y^{A^2,\gamma^2,\underline{\xi}^2}$, hence $Y^{A,\gamma,\underline{\xi}}$ is independent of the triplet $(A,\gamma,\underline{\xi}) \in \mathscr{A}(Z)\times l_+^1(\mathbb{N})\times\mathscr{D}(E_1)$. So the process $Y^{A,\gamma,\underline{\xi}} =: \int X dZ$ is well defined and Proposition 3.17 shows that it satisfies the weak* Fubini property. □

We finish this part by discussing an extension of the class of allowed integrands of the weak* stochastic integral to a particular class of processes that preserves the weak* Fubini property.

Lemma 3.19 (The weak* stochastic integral and the weak* Fubini property for the limit of strongly measurable processes). *Let $X : \bar{\Omega} \to E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G})$ be a process such that $X(\xi) : \bar{\Omega} \to \mathscr{L}(\mathcal{H},\mathcal{G})$ is P-strongly $\mathscr{P}$-measurable for each $\xi \in E$. Suppose that there is a control process $A \in \mathscr{A}(Z)$, an increasing sequence $\tau_n \nearrow \infty$ of stopping times such that $\mathbb{E}(A^2_{\tau_n^-}) < \infty$ for all $n \in \mathbb{N}$, a sequence $\gamma \in l^1_+(\mathbb{N})$ and a dense countable subset $\underline{\xi} \in \mathscr{D}(E_1)$ in E_1 such that*

$$\sum_{n\in\mathbb{N}} \gamma_n \mathbb{E}\left(A_{\tau_n^-} \int_0^{\tau_n^-} \|X_s(\xi_n)\|^2_{L(\mathcal{H},\mathcal{G})} dA_s \right) < \infty.$$

If a sequence $(X^n)_{n\in\mathbb{N}} \subset \Lambda(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$ of weak* Z-integrable processes, satisfying the condition $\lambda^{A,\gamma,\underline{\xi}}(X) < \infty$, converges pointwise to X in the weak* topology, then the process $\int X dZ$ is a well-defined $E^*\hat{\otimes}_\epsilon \mathcal{G}$-valued process and satisfies the weak* Fubini property.

Proof. It suffices to construct the process $\int X dZ$ locally, so let $\tau = \tau_l$ for some $l \in \mathbb{N}$. By dominated convergence we obtain for any $n \in \mathbb{N}$

$$\sum_{k\in\mathbb{N}} \gamma_k \mathbb{E}\left(A_{\tau^-} \int_0^{\tau^-} \|X(\xi_n) - X^m(\xi_n)\|^2_{L(\mathcal{H},\mathcal{G})} dA_s \right) \xrightarrow[m\to\infty]{} 0.$$

Therefore, the sequence $(X^m)_{m\in\mathbb{N}}$ is a Cauchy sequence with respect to the seminorm $q^A_{\tau,\gamma,\underline{\xi}}$ and by Corollary 3.12 this implies that $(\int X^m dZ)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathcal{G})$, which is a complete space by Lemma 3.11. We denote the limit in $\Pi_{\tau,\gamma,\underline{\xi}}(E^*\hat{\otimes}_\epsilon \mathcal{G})$ by $\mathbb{1}_{[[0,\tau))} \int X dZ$. Let $\xi \in E$, for each $m \in \mathbb{N}$ we know by Proposition 3.17 that

$$\int X^m dZ(\xi) = \int X^m(\xi) dZ.$$

Hence, as in the proof of Proposition 3.17 we may choose a suitable subsequence $(m_k)_{k\in\mathbb{N}}$ such that

$$\sup_{s<\tau} \left\| \int_0^s X dZ(\xi) - \int_0^s X(\xi) dZ \right\|_{\mathcal{G}} \overset{P-\text{a.e.}}{\leq}$$

$$\sup_{s<\tau} \left\| \int_0^s X dZ(\xi) - \int_0^s X^{m_k} dZ(\xi) \right\|_{\mathcal{G}} + \sup_{s<\tau} \left\| \int_0^s X^{m_k}(\xi) dZ - \int_0^s X(\xi) dZ \right\|_{\mathcal{G}} \xrightarrow[k\to\infty]{P-\text{a.e.}} 0,$$

which shows the claim. □

This last result is satisfactory in the sense that a reasonable pointwise limit of a sequence in $\Lambda(E^*\hat{\otimes}_\epsilon \mathscr{L}(\mathcal{H},\mathcal{G}))$ is weak* Z-integrable and satisfies the weak* Fubini property.

3.5 A note on measurability

Let E and F be two Banach spaces and define $E^*\hat{\otimes}_\epsilon F$ as above as the closure of $E^* \otimes F$ as a subspace of $\mathscr{L}(E, F)$ endowed with the σ-algebra generated by the weak* topology, that is the coarsest topology on $E^*\hat{\otimes}_\epsilon F$ such that the evaluation maps $u \mapsto u(\xi) \in F$ corresponding to $\xi \in E$ are all continuous. Let $(C, \mathscr{C})$ be a measure space. Recall that a function $X : C \to E^*\hat{\otimes}_\epsilon F$ is strongly $\mathscr{C}$-measurable with respect to the weak* topology if it is the pointwise limit with respect to the weak* topology of $\mathscr{C}$-measurable $E^*\hat{\otimes}_\epsilon F$-valued step functions. This property is in general very difficult to verify and during the case of the two-parameter process we will need to argue that a process is strongly measurable. However, we expect to simplify this task by obtaining a generalization of Pettis' measurability theorem (see Proposition 2.4) to this case. Recall that Pettis' measurability theorem states that a process is strongly measurable with respect to the norm topology if it is separably valued and the dual pairing of the process with any linear form is measurable as an $\mathbb{R}$-valued process. A sufficient condition for the strong $\mathscr{C}$-measurability with respect to the weak* topology of a function $X : C \to E^*\hat{\otimes}_\epsilon F$ should include X to be separably valued with respect to the weak* topology and $X(\xi) : C \to F$ to be strongly $\mathscr{C}$-measurable with respect to the norm topology or equivalently $(\mathscr{C}, \mathcal{B}(F))$-measurable for all $\xi \in E$, where $\mathcal{B}(F)$ is the Borel σ-algebra generated by the norm topology on F. This is a guess based on the statement of Pettis' measurability theorem. Let us first introduce a name for these kind of functions that behave well under the dual pairing.

Definition 3.20. *A function $X : (C, \mathscr{C}) \to E^*\hat{\otimes}_\epsilon F$ is called weak* $\mathscr{C}$-measurable if for any $\xi \in E$ the function $X(\xi) : (C, \mathscr{C}) \to F$ is strongly $\mathscr{C}$-measurable with respect to the norm topology on F or equivalently $(\mathscr{C}, \mathcal{B}(F))$-measurable.*

Recall that the proof of Pettis' measurability theorem is a constructive approximation argument with respect to the metric induced by the norm. However, we cannot expect a separable subspace in $E^*\hat{\otimes}_\epsilon F$ to be metrizable. We therefore restrict ourselves to the much simpler case when $F = \mathbb{R}^n$, since we understand the weak* toplogy on $E^*\hat{\otimes}_\epsilon F$ better and this is precisely the situation that will be encountered during the discussion of the two-parameter process in the next section. If $F = \mathbb{R}^n$ then $E^*\hat{\otimes}_\epsilon \mathbb{R}^n \cong (E^*)^n$, as explained above for $E^* = \mathscr{M}(X)$ (see Example 3.1). A reasonable choice of a metrizable subset of $E^*\hat{\otimes}_\epsilon F$ is just a norm-closed ball of the form $\overline{B_r^{E^*}(0)}^n$ with $r > 0$. This basically follows directly from the Tychonoff-Alaoglu theorem. However, as the proof will show, the mere existence of a metric that generates the topology is not sufficient if we only want to assume the process X to be weak* measurable, we actually need a concrete metric. The next result states this intuition formally.

Proposition 3.21. *Suppose that E is a separable Banach space, let $X : (C, \mathscr{C}) \to (E^*)^n$ be a weak* $\mathscr{C}$-measurable function and suppose that X takes its values in $\overline{B_r^{E^*}(0)}^n$ for some $r > 0$. Then, X is strongly $\mathscr{C}$-measurable with respect to the weak* topology.*

Proof. Without loss of generality assume that $n = 1$. Since E is separable, the weak* topology restricted to $\overline{B_r^{E^*}(0)}$ is metrizable. Hence $\overline{B_r^{E^*}(0)}$ is separable, as it is a compact metric space. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a weak* dense countable subset of $\overline{B_r^{E^*}(0)}$ and $\{\xi_k\}_{k\in\mathbb{N}}$ a dense subset of the unit ball E_1 of E. Recall that the metric defined by

$$\begin{aligned} d : \overline{B_r^{E^*}(0)} \times \overline{B_r^{E^*}(0)} &\to [0,\infty) \\ (\lambda_1 \ , \ \lambda_2) \qquad &\mapsto \sum_{k\in\mathbb{N}} \frac{1}{2^k} \Big| \lambda_1(\xi_k) - \lambda_2(\xi_k) \Big|, \end{aligned}$$

induces the weak* topology on $\overline{B_r^{E^*}(0)}$. This can be seen by noting that the function

$$\begin{aligned} f : \overline{B_r^{E^*}(0)} &\to \prod_{k\in\mathbb{N}} \overline{B_r^{\mathbb{R}}(0)} \\ \lambda &\mapsto (\lambda(\xi_k))_{k\in\mathbb{N}}, \end{aligned}$$

is weak* continuous and injective, hence a topological embedding and the standard metric on the product space induces then the metric d defined above. For any $\lambda \in \overline{B_r^{E^*}(0)}$ the function

$$\begin{aligned} C &\to [0,\infty) \\ c &\mapsto d(X(c),\lambda), \end{aligned}$$

is $\mathscr{C}$-measurable, since each $X(\xi_k)$ is $\mathscr{C}$-measurable and $d(X(c),\lambda)$ is the pointwise limit of $\mathscr{C}$-measurable functions. Note that we do not assume X to be $\mathscr{C}$-measurable, hence it is a priori not clear whether for any metric d the process $d(X,\lambda)$ is $\mathscr{C}$-measurable. For each $m \in \mathbb{N}$ and $\lambda \in \overline{B_r^{E^*}(0)}$ define

$$k_m(\lambda) := \min \left\{ 1 \le k \le m \,\middle|\, d(\lambda,\lambda_k) = \min_{1\le i\le m} d(\lambda,\lambda_i) \right\},$$

and note that $d(\lambda_{k_m(\lambda)},\lambda) \xrightarrow[m\to\infty]{} 0$ by density of $\{\lambda_k\}_{k\in\mathbb{N}}$. Define the function

$$\begin{aligned} X^m : C &\to E^* \\ c &\mapsto \lambda_{k_m(X(c))}, \end{aligned}$$

and notice that X^m has finite range. Moreover, for each $1 \leq k \leq m$ the sets

$$C_k^{m,1} = \left\{ c \in C \middle| \, d(X^m(c), \lambda_k) = \min_{1 \leq i \leq n} d(X^m(c), \lambda_i) \right\} \text{ and}$$

$$C_k^{m,2} = \left\{ c \in C \middle| \, \forall 1 \leq l \leq k-1 : d(X^m(c), \lambda_l) > \min_{1 \leq i \leq n} d(X^m(c), \lambda_i) \right\},$$

are $\mathscr{C}$-measurable and $C_k^m = \{c \in C \big| X^m(c) = \lambda_k\} = C_k^{m,1} \cap C_k^{m,2}$. Hence,

$$X^m = \sum_{k=1}^{m} \mathbb{1}_{C_k^m} \lambda_k \in \mathscr{E}(E^*, \mathscr{C}).$$

is a $\mathscr{C}$-measurable E^*-valued step function that converges to X pointwise for any $c \in C$ in the weak* topology, that is for any $c \in C$ we have $X^m(c) \xrightarrow[m \to \infty]{w^*} X(c)$. □

This proof relies on the assumption that E is separable and F is finite dimensional. With more effort, one could (probably) weaken the assumption that F is finite dimensional and replace it with separability. However, F has to be separable as noted above. Previously we worked with $F = \mathscr{L}(\mathcal{H}, \mathcal{G})$, where $\mathcal{H}$ and $\mathcal{G}$ are separable Hilbert spaces, but $\mathscr{L}(\mathcal{H}, \mathcal{G})$ is in general not separable (for instance let $\mathcal{H} = \mathcal{G} = l^2(\mathbb{N})$, then $l^\infty(\mathbb{N})$, which is not separable, can be isometrically embedded in $\mathscr{L}(H, G)$). As we explained before, we will need to check in the next section whether a process X is P-strongly measurable in the special case where $F = \mathbb{R}^n$. The following result gives a condition that is simple to verify and is tailor-made for the case of the two-parameter process we discuss in the next section.

Proposition 3.22. *Suppose $\mathscr{C} \subset \mathcal{B}(\mathbb{R}_+) \otimes \mathscr{F}$ is a sub-σ-algebra and $X : (\bar{\Omega}, \mathscr{C}) \to (E^*)^n$ a weak* $\mathscr{C}$-measurable process such that*

$$P\left(\left\{ \omega \in \Omega \middle| \, \exists t \in \mathbb{R}_+ : X_t(\omega) \notin \overline{B_r^{E^*}(0)}^n \right\} \right) = 0,$$

for some $r > 0$. Then, $X \in L^0(\bar{\Omega}, \mathscr{C}; (E^*)^n)$ is P-strongly $\mathscr{C}$-measurable with respect to the weak topology.*

Proof. Let $\Omega_0 \subset \Omega$ be a conull subset such that for any $\omega \in \Omega_0$ and $t \in \mathbb{R}_+$ we have $X_t(\omega) \in \overline{B_r^{E^*}(0)}^n$. Hence $\bar{X}_t(\omega) := \mathbb{1}_{\mathbb{R}_+ \times \Omega_0}(t, \omega) X_t(\omega)$, which is indistinguishable from X, is a weak* $\mathscr{C}$-measurable process and $\bar{X}_t(\omega) \in \overline{B_r^{E^*}(0)}^n$ for any $(t, \omega) \in \bar{\Omega}$. By Proposition 3.21 we can find a sequence $(X^n)_{n \in \mathbb{N}} \subset \mathscr{E}(E^*, \mathscr{C})$ of $\mathscr{C}$-measurable step functions such that $X^n \xrightarrow[n \to \infty]{w^*} \bar{X}$ pointwise, which implies that

$$P\left(\left\{ \omega \in \Omega \middle| \, \exists t \in \mathbb{R}_+ : X_t(\omega) \overset{w^*}{\neq} \lim_{n \to \infty} X_t^n(\omega) \right\} \right) \leq P(\Omega_0^c) = 0.$$

$\square$

4 Applications

4.1 A natural setting

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space satisfying the usual assumptions. We consider now the special case $\mathcal{H} = \mathbb{R}^d$, $\mathcal{G} = \mathbb{R}$ and $E = C(X)$, where X is a compact Hausdorff space. Then, $\mathscr{L}(\mathcal{H}, \mathcal{G}) = \mathbb{R}^d$ and $C(X)^* \cong \mathscr{M}(X)$ is the Banach space of Radon signed measures on X, where the norm is the variation norm, so if $\mu \in \mathscr{M}(X)$ then the variation norm of μ is given by

$$\|\mu\|_V := \sup\left\{ \int_X f d\mu \,\middle|\, f \in \overline{B_1^{C(X)}} \right\} = \sup\left\{ \sum_{i=1}^n |\mu(B_i)| \,\middle|\, B_1, \ldots, B_n \in \mathcal{B}(X) \text{ disjoint} \right\},$$

where $\mathcal{B}(X)$ denotes the Borel σ-algebra on X. Moreover, by definition of the weak* injective tensor product we may identify $\mathscr{M}(X)\hat{\otimes}_\epsilon\mathbb{R}^d \cong \bigoplus_{k=1}^d \mathscr{M}(X) = \mathscr{M}(X)^d$ (see Example 3.4).

Let $Z \in \mathscr{S}(\mathbb{R}^d)$ be an $\mathbb{R}^d$-valued semimartingale. Suppose that $\mu \in \Lambda(\mathscr{M}(X)^d)$ is a weak* Z-integrable process and consider a control process $A \in \mathscr{A}(Z)$, a sequence $\gamma \in l^1_+(\mathbb{N})$, a dense countable subset $\underline{f} \in \mathscr{D}(C(X)_1)$ in the unit ball of $C(X)$ and $\tau_n \nearrow \infty$ a sequence of stopping times such that $\mathbb{E}(A^2_{\tau_n^-}) < \infty$ and $\mu \in \Lambda^A_{\tau_n,\gamma,\underline{\xi}}(\mathscr{M}(X)^d)$ for each $n \in \mathbb{N}$. The process $\int \mu dZ$ is $\mathscr{M}(X)$-valued and for any continuous function $f \in C(X)$ we have the Fubini property

$$\int_X f d\Big(\int_0^t \mu_s dZ_s\Big) = \int_0^t \Big(\int_X f d\mu_s\Big) dZ_s.$$

A suitable class of integrands that extends the Fubini property to bounded measurable functions is the following:

Proposition 4.1. *Let $\mu \in \Lambda(\mathscr{M}(X)^d)$ be as above and suppose in addition that there is a sequence of constants $C_n > 0$ such that for any continuous function $g \in C(X)$ and $n \in \mathbb{N}$ we have $q^A_{\tau_n}(\int_X g d\mu) \leq C_n \|g\|_\infty$. Let $f \in \mathscr{L}^\infty(X)$ be a bounded measurable function, then the process $\int_X f d\big(\int \mu\, dZ\big)$ exists and satisfies the weak* Fubini property, that is*

$$\int_X f d\Big(\int \mu\, dZ\Big) = \int \Big(\int_X f d\mu\Big) dZ.$$

Proof. The argument contained here is adapted from the proof of Theorem 2.3 in Choulli and Schweizer (2013). Let $f \in \mathscr{L}^\infty(X)$ be a bounded measurable function and suppose without loss of generality that $\|f\|_\infty \leq 1$. It suffices to show the existence of the process and the weak* Fubini property locally, so let $\tau = \tau_l$ and $C = C_l$ for some $l \in \mathbb{N}$. Choose a subsequence $(f_{n_k})_{k\in\mathbb{N}} \subset \underline{f}$ such that $f_{n_k} \xrightarrow[k\to\infty]{} f$ pointwise and assume without loss of generality that $\|f_{n_k}\|_\infty \leq 1 + \|f\|_\infty$ for all $k \in \mathbb{N}$. First note that for P-a.e. $\omega \in \Omega$, if $t \geq 0$ then

$$\lambda_t^A\Big(\int_X f d\mu\Big)(\omega) = A_t(\omega)\int_0^t \Big|\int_X f d\mu_s(\omega)\Big|^2 dA(\omega)_s \leq$$

$$3A_t(\omega)\int_0^t \Big|\int_X f - f^{n_k} d\mu_s(\omega)\Big|^2 dA(\omega)_s + 3A_t(\omega)\int_0^t \Big|\int_X f_{n_k} d\mu_s(\omega)\Big|^2 dA(\omega)_s < \infty,$$

for all $n_k \in \mathbb{N}$ large enough, by dominated convergence. This implies that the process $\int_X f d\mu \in \Lambda(\mathbb{R}^d)$ is Z-integrable in the sense of Section 2. The property $q_\tau^A(\int_X f_{n_k} d\mu) \leq C\|f_{n_k}\|_\infty$ ensures by dominated convergence that $\mathbb{1}_{[[0,\tau))}\int(\int_X f_{n_k} d\mu)dZ$ converges with respect to p_τ to $\mathbb{1}_{[[0,\tau))}\int(\int_X f d\mu)dZ$ in $\Pi_\tau(\mathbb{R})$. Indeed, note by Fatou's lemma, that

$$\mathbb{E}\Big(A_{\tau^-}\int_0^{\tau^-}\Big|\int_X f d\mu_s\Big|^2 dA_s\Big) = \mathbb{E}\Big(A_{\tau^-}\int_0^{\tau^-}\lim_{k\to\infty}\Big|\int_X f_{n_k} d\mu_s\Big|^2 dA_s\Big) \leq$$

$$\mathbb{E}\Big(A_{\tau^-}\liminf_{k\to\infty}\int_0^{\tau^-}\Big|\int_X f_{n_k} d\mu_s\Big|^2 dA_s\Big) \leq \liminf_{k\to\infty}\mathbb{E}\Big(A_{\tau^-}\int_0^{\tau^-}\Big|\int_X f_{n_k} d\mu_s\Big|^2 dA_s\Big) \leq C,$$

therefore we may argue using Lemma 2.26 and dominated convergence, that

$$\mathbb{E}\Big(\sup_{t<\tau}\Big|\int_0^t\Big(\int_X (f_{n_k} - f)d\mu_s\Big)dZ_s\Big|^2\Big) \leq \mathbb{E}\Big(A_{\tau^-}\int_0^{\tau^-}\Big|\int_X (f - f_{n_k})d\mu_s\Big|^2 dA_s\Big) \xrightarrow[k\to\infty]{} 0.$$

So in particular for a subsequence also denoted by $(n_k)_{k\in\mathbb{N}}$ we obtain

$$\int_0^t\Big(\int_X f_{n_k} d\mu_s\Big)dZ(\omega)_s \xrightarrow[k\to\infty]{} \int_0^t\Big(\int_X f d\mu_s\Big)dZ(\omega)_s,$$

for P-a.e. $\omega \in \Omega$ and all $t \in [0, \tau(\omega))$. On the other hand $\mathbb{1}_{[[0,\tau))}\int \mu dZ \in \Pi_\tau(\mathscr{M}(X))$ is a signed measure valued process, so

$$\mathbb{1}_{[[0,\tau))}(t,\omega)\int_X f_{n_k} d\Big(\int_0^t \mu_s dZ(\omega)_s\Big) \xrightarrow[k\to\infty]{} \mathbb{1}_{[[0,\tau))}(t,\omega)\int_X f d\Big(\int_0^t \mu_s dZ(\omega)_s\Big),$$

for all $(t,\omega) \in \bar{\Omega}$ by dominated convergence. This implies that

$$\mathbb{1}_{[[0,\tau))}(t,\omega)\int_X f d\Big(\int_0^t \mu_s dZ(\omega)_s\Big) = \mathbb{1}_{[[0,\tau))}(t,\omega)\int_0^t\Big(\int_X f d\mu_s\Big)dZ(\omega)_s,$$

for P-a.e. $\omega \in \Omega$ and all $t \geq 0$, since for all $k \in \mathbb{N}$ we have by Proposition 3.17

$$\mathbb{1}_{[[0,\tau))}(t,\omega)\int_X f_{n_k} d\Big(\int_0^t \mu_s dZ(\omega)_s\Big) = \mathbb{1}_{[[0,\tau))}(t,\omega)\int\Big(\int_X f_{n_k} d\mu_s\Big)dZ(\omega)_s.$$

□

Let $\mu \in \Lambda(\mathscr{M}(X)^d)$ be a process as in Proposition 4.1 and $B \in \mathcal{B}(X)$ a Borel set, then

$$\int_0^t \mu_s dZ_s(B) = \int_0^t \mu_s(B) dZ_s.$$

One interpretation of this equation can be given by the following observation: suppose that $\vartheta \in \Lambda(\mathbb{R}^d)$ is a Z-integrable process viewed as a strategy, then we may consider the measure-valued process defined by

$$\begin{aligned} \mu : \mathbb{R}_+ \times \Omega &\to \mathscr{M}([0,1])^d \\ (t,\omega) &\mapsto \lambda \otimes \vartheta_t(\omega), \end{aligned}$$

where λ is the Lebesgue measure on $[0,1]$. Note that μ is predictable, weak* Z-integrable in the sense described above, and for any $f \in C([0,1])$ we have

$$q^A_{\tau_n}\left(\int_{[0,1]} f d\mu\right)^2 = \mathbb{E}\left(A_{\tau_n^-} \int_0^{\tau_n^-} \left|\int_{[0,1]} f \vartheta_s d\lambda\right|^2 dA_s\right) \leq \|\lambda\|_V^2 \|f\|_\infty^2 q^A_{\tau_n}(\vartheta)^2 \leq C_n^2 \|f\|_\infty^2,$$

where $C_n = q^A_{\tau_n}(\vartheta) < \infty$, for some control process $A \in \mathscr{A}(Z)$ and $\tau_n \nearrow \infty$ a sequence of stopping times such that both $E(A^2_{\tau_n -}) < \infty$ and $\vartheta \in \Lambda^A_{\tau_n}(\mathbb{R}^d)$, which exist since we assumed ϑ to be Z-integrable. Hence Proposition 4.1 yields

$$\nu_t([0,1]) = \int_0^t \vartheta_t(\omega) dZ_s,$$

where $\nu_t = \int_0^t \mu_s dZ_s$. So the value of a strategy at time t is the measure of $[0,1]$ for some measure-valued predictable RCLL process ν at time t.

4.2 The case of a two-parameter process

Suppose that $Z \in \mathscr{S}(\mathbb{R}^d)$ is an $\mathbb{R}^d$-valued semimartingale and $\vartheta = (\vartheta_{t,s})_{t \geq 0, 0 \leq s \leq t}$ an $\mathbb{R}^d$-valued two-parameter process. Assume that for any $t \geq 0$ the process $(\vartheta_{t,s})_{0 \leq s \leq t}$ and the diagonal process $(\vartheta_{s,s})_{s \geq 0}$ belong to the class of Z-integrable processes $\Lambda(\mathbb{R}^d)$, then we can write

$$\int_0^t \vartheta_{t,s} dZ_s = \int_0^t \vartheta_{s,s} dZ_s + \int_0^t (\vartheta_{t,s} - \vartheta_{s,s}) dZ_s.$$

Note in particular that under these assumptions, $\int \vartheta_{s,s} dZ_s$ is a well-defined $\mathbb{R}$-valued semimartingale. Suppose furthermore that for each $s \geq 0$, the process $(\vartheta_{t,s})_{t \geq 0}$ is right-continuous and of finite variation, where we set $\vartheta_{t,s} := 0$ for any $s > t$. By Carathéodory's extension theorem the process defined by

$$\mu_s([0,t]) := \mathbb{1}_{\{t \geq s\}} (\vartheta_{t,s} - \vartheta_{s,s}),$$

for $t \leq T$ extends uniquely to an $\mathscr{M}([0,T])^d$-valued process for each $T \geq 0$. Moreover, note that the process $(\mu_s([0,t]))_{s\in\mathbb{R}_+}$ is $\mathscr{P}$-measurable for any fixed $t \geq 0$, which implies that $(\int_{[0,T]} f d\mu_s)_{s\in\mathbb{R}_+}$ is $\mathscr{P}$-measurable for any continuous function $f \in C([0,T])$ and this means that $\mu = (\mu_s)_{s\in\mathbb{R}_+}$ is a weak* $\mathscr{P}$-measurable process. In particular, for any $r > 0$ the set

$$P_r := \left\{ (s,\omega) \in \Omega \,\middle|\, \|\mu_s(\omega)\|_V \leq r\mathbb{1} \right\}$$

is $\mathscr{P}$-measurable since $C(X)$ is separable, where we set

$$\|\mu_s(\omega)\|_V := \begin{pmatrix} \|\mu_s^1(\omega)\|_V \\ \vdots \\ \|\mu_s^d(\omega)\|_V \end{pmatrix}, \qquad \mathbb{1} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^d,$$

and the relation $\|\mu_s(\omega)\|_V \leq r\mathbb{1}$ is meant coordinate-wise. The weak* process $\mathbb{1}_{P_r}\mu$ is P-strongly $\mathscr{P}$-measurable with respect to the weak* topology by Proposition 3.22 for all $r > 0$. Furthermore, $\mathbb{1}_{P_r}(s,\omega)\mu_s(\omega) \xrightarrow[r\to\infty]{w^*} \mu_s(\omega)$ for all $(s,\omega) \in \bar{\Omega}$ in the weak* topology. However, we cannot argue that such a limit is P-strongly $\mathscr{P}$-measurable again, but we can make use of Lemma 3.19 under an extra assumption, which we now describe.

Fix $s \in \mathbb{R}_+$ and denote by $Var(\vartheta_{\cdot,s}(\omega),[0,t])$ the total variation of $(\vartheta_{r,s}(\omega))_{r\in\mathbb{R}}$ on the interval $[0,t]$, that is

$$Var(\vartheta_{\cdot,s}(\omega),[0,t]) := \begin{pmatrix} Var(\vartheta^1_{\cdot,s}(\omega),[0,t]) \\ \vdots \\ Var(\vartheta^d_{\cdot,s}(\omega),[0,t]) \end{pmatrix},$$

where

$$Var(\vartheta^k_{\cdot,s}(\omega),[0,t]) := \sup\left\{ \sum_{i=1}^n |\vartheta_{t_i+1,s}(\omega) - \vartheta_{t_i,s}(\omega)| \,\middle|\, n \in \mathbb{N},\ 0 \leq t_0 < t_1 < \cdots < t_n \leq t \right\},$$

is the total variation on the interval $[0,t]$ of the k-th coordinate, which is finite P-a.e. by assumption. Note that for all $(s,\omega) \in \bar{\Omega}$ we have $\|\mu_s(\omega)\|_V \leq Var(\vartheta_{\cdot,s}(\omega),[0,t])$, where the relation is again to be understood coordinate-wise. If in addition we assume that the variation process $Var(\vartheta_{\cdot,s}(\omega),[0,T])$ on $[0,T]$ is Z-integrable, then for any $f \in C([0,T])$

$$\mathbb{E}\left(A_{\tau_n^-} \int_0^{\tau_n^-} \left| \int_{[0,T]} f d\mu_s \right|^2 dA_s \right) \leq \|f\|_\infty^2 \mathbb{E}\left(A_{\tau_n^-} \int_0^{\tau_n^-} \left| Var(\vartheta_{\cdot,s},[0,T]) \right|^2 dA_s \right) < \infty,$$

for some control process $A \in \mathscr{A}(Z)$ and an increasing sequence of stopping times $\tau_n \nearrow \infty$ such that the process $(Var(\vartheta_{\cdot,s}(\omega),[0,t]))_{s\in\mathbb{R}_+} \in \Lambda^A_{\tau_n}(\mathbb{R}^d)$ is locally Z-integrable in the sense of Section 2. In particular this inequality implies that the process $\mathbb{1}_{P_r}\mu_s \in \Lambda(\mathscr{M}([0,T])^d)$ is weak* Z-integrable for any $r > 0$, where the control process is A, the sequence of stopping times is $(\tau_n)_{n\in\mathbb{N}}$, and the dense countable set $\underline{f} \in \mathscr{D}(C(X)_1)$ and the sequence $\gamma \in l^1_+(\mathbb{N})$ can be chosen arbitrarily. Lemma 3.19 implies then, that the $\mathscr{M}([0,T])$-valued process $\int \mu dZ$ is well-defined, satisfies the weak* Fubini property and by Proposition 4.1, with constants C_n given by

$$C_n = \mathbb{E}\bigg(A_{\tau_n^-}\int_0^{\tau_n^-}\bigg|Var(\vartheta_{\cdot,s},[0,t])\bigg|^2 dA_s\bigg) < \infty,$$

the weak* Fubini property extends to all bounded measurable functions on $[0,T]$. Therefore we obtain for any $u \geq 0$ and $t \in [0,T]$

$$\bigg(\int_0^u \mu_s dZ_s\bigg)[0,t] = \int_0^u \bigg(\mu_s([0,t])\bigg)dZ_s.$$

In particular for $u = T$ we observe that the process $D = (D_t)_{0\leq t\leq T}$ defined by

$$D_t := \bigg(\int_0^T \mu_s dZ_s\bigg)[0,t],$$

is adapted, RCLL and of finite variation, since $\int_0^T \mu_s dZ_s$ is $\mathscr{M}([0,T])$-valued, hence a semimartingale. Schweizer/Choulli (2013) show that D is even predictable. This proves the following result:

Theorem 4.2 (Choulli, Schweizer). *Let $Z \in \mathscr{S}(\mathbb{R}^d)$ be an $\mathbb{R}^d$-semimartingale. Let $(\vartheta_{t,s})_{t,s\geq 0}$ be a two-parameter $\mathbb{R}^d$-valued process satisfying the following properties:*
i) For all $s > t$ the process $(\vartheta_{t,s})_{t\in\mathbb{R}_+}$ satisfies $\vartheta_{t,s} = 0$.
ii) The diagonal process $(\vartheta_{s,s})_{s\in\mathbb{R}_+}$ is Z-integrable,
iii) For any $s \geq 0$, the process $(\vartheta_{t,s})_{t\in\mathbb{R}_+}$ is right-continuous and of finite variation,
iv) The variation process $(Var(\vartheta_{\cdot,s},[0,T])_{s\in\mathbb{R}_+}$ on $[0,T]$ is Z-integrable for all $T > 0$.
Then, $\bigg(\int_0^t \vartheta_{t,s}dZ_s\bigg)_{t\in\mathbb{R}_+}$ is an $\mathbb{R}$-valued semimartingale.

Choulli/Schweizer (2013) assume (translated to our setup) that there is a sequence $C_n > 0$ such that

$$\mathbb{E}\bigg(A_{\tau_n^-}\int_0^{\tau_n^-}\bigg|\int_{[0,T]} f d\mu_s\bigg|^2 dA_s\bigg) \leq \|f\|_\infty^2 C_n < \infty,$$

for any $f \in C([0,T])$. The previous discussion implies, that under the assumption that the variation process as we described above is integrable, there is a sequence C_n, namely

$$C_n = \mathbb{E}\bigg(A_{\tau_n^-}\int_0^{\tau_n^-}\bigg|Var(\vartheta_{\cdot,s},[0,t])\bigg|^2 dA_s\bigg) < \infty.$$

The previous result implies Theorem 3.2 in Protter (1985) and in particular we do not need the assumption that the derivative is Lipschitz continuous.

Corollary 4.3 (Protter). *Let $Z \in \mathscr{S}(\mathbb{R}^d)$ be an $\mathbb{R}^d$-valued semimartingale and let $(\vartheta_{t,s})_{t,s\geq 0}$ be a process satisfying i), ii) and iii) in Theorem 4.2. Suppose that $t \mapsto \frac{\partial}{\partial t}\vartheta_{t,s}(\omega)$ exists and is locally bounded uniformly in t, that is we assume that for all $t_0 \geq 0$ there is $\delta := \delta(t_0) > 0$ and $K := K(t_0) > 0$ such that for all $s \geq 0$ and $\omega \in \Omega$*

$$\left|\frac{\partial}{\partial t}\vartheta_{t,s}(\omega)\right| < K,$$

for all $t \geq 0$ such that $|t - t_0| < \delta_0$. Then, $\left(\int_0^t \vartheta_{t,s} dZ_s\right)_{t\in\mathbb{R}_+}$ is an $\mathbb{R}$-valued semimartingale.

Proof. We do this for the case $d = 1$. Let $\omega \in \Omega$, $s \in \mathbb{R}_+$ and $T \geq s$, then

$$Var(\vartheta_{\cdot,s}(\omega), [0,T]) := \sup\left\{\sum_{i=1}^n |\vartheta_{t_i+1,s}(\omega) - \vartheta_{t_i,s}(\omega)| \,\middle|\, n \in \mathbb{N},\ 0 \leq t_0 < t_1 < \cdots < t_n \leq T\right\} =$$

$$\sup\left\{\sum_{i=1}^n |\vartheta_{t_i+1,s}(\omega) - \vartheta_{t_i,s}(\omega)| \,\middle|\, n \in \mathbb{N},\ s \leq t_0 < t_1 < \cdots < t_n \leq T\right\},$$

since $\vartheta_{r,s} = 0$ for $s > r$. Let $s \leq t_0 < \cdots < t_n \leq T$, then

$$\sum_{i=1}^n |\vartheta_{t_i+1,s}(\omega) - \vartheta_{t_i,s}(\omega)| = \sum_{i=1}^n \left|\frac{\partial}{\partial t}\vartheta_{r_i,s}(\omega)\right|(t_{i+1} - t_i),$$

for some $r_i \in [t_{i-1}, t_i]$. A compactness argument gives a constant $K > 0$ such that

$$\left|\frac{\partial}{\partial t}\vartheta_{r,s}(\omega)\right| \leq K,$$

for all $s \leq r \leq T$. More precisely

$$\bigcup_{0\leq t_0\leq T} (r_0 - \delta(r_0), r_0 + \delta(r_0)) \cap [0,T],$$

is an open cover of $[0,T]$ and compactness yields finitely many $r_1, \ldots, r_k$ such that

$$[0,T] = \bigcup_{i=1}^k (r_i - \delta(r_i), r_i + \delta(r_i)) \cap [0,T],$$

finally set $K := \max_{1 \le i \le k} K(r_i) > 0$. This implies that

$$\sum_{i=1}^{n} |\vartheta_{t_i+1,s}(\omega) - \vartheta_{t_i,s}(\omega)| = \sum_{i=1}^{n} \left| \frac{\partial}{\partial t} \vartheta_{r_i,s}(\omega) \right| (t_{i+1} - t_i) \le K(T - s),$$

for any $s \le t_0 < \cdots < t_n \le t$ and therefore

$$Var(\vartheta_{\cdot,s}(\omega), [0,t]) \le K(T - s) < \infty.$$

Note that if $A \in \mathscr{A}(Z)$ is any control process for Z, then

$$\lambda_t^A(Var(\vartheta_{\cdot,s}(\omega), [0,T])) = A_t \int_0^t Var(\vartheta_{\cdot,s}(\omega), [0,T])^2 dA_s \le$$

$$A_t(\omega)^2 K^2 \int_0^t (T - s)^2 dA_s(\omega) \le A_t(\omega)^2 K^2 T^2 < \infty,$$

So $(Var(\vartheta_{\cdot,s}, [0,T]))_{s \in \mathbb{R}_+}$ is Z-integrable in the sense of Métivier. The result follows now by Theorem 4.2. □

The previous proof shows that we just need to assume an integrability condition for the bounding constant, so the corollary above is just a special case of the following result.

Corollary 4.4. *Let $Z \in \mathscr{S}(\mathbb{R}^d)$ be an $\mathbb{R}^d$-valued semimartingale and let $(\vartheta_{t,s})_{t,s \ge 0}$ be a process satisfying i), ii) and iii) in Theorem 4.2. Suppose that $t \mapsto \frac{\partial}{\partial t} \vartheta_{t,s}(\omega)$ exists for all $s \ge 0$ and P-a.e. $\omega \in \Omega$. Furthermore, assume that for all $t_0 \ge 0$ there is $\delta := \delta(t_0) > 0$ and a mapping $(s, \omega) \mapsto K(t_0, s, \omega) > 0$ satisfying the bounding condition*

$$\left| \frac{\partial}{\partial t} \vartheta_{t,s}(\omega) \right| < K(t, s, \omega)$$

for all $|t - t_0| < \delta_0$, $s \ge 0$ and P-a.e. $\omega \in \Omega$ and satisfying the following integrability condition: there is a control process $A \in \mathscr{A}(Z)$ such that for all $t_0 \ge 0$ and $t \ge 0$,

$$A_t(\omega) \int_0^t K(t_0, s, \omega)^2 dA_s(\omega),$$

exists and is finite for P-a.e. $\omega \in \Omega$. Then, $\left(\int_0^t \vartheta_{t,s} dZ_s \right)_{t \in \mathbb{R}_+}$ is an $\mathbb{R}$-valued semimartingale.

Proof. Let $T > 0$, $\omega \in \Omega$, $T \geq s \geq 0$, then

$$Var(\vartheta_{\cdot,s}(\omega), [0,T]) \leq (T-s) \max_{1 \leq i \leq k} K(r_i, s, \omega),$$

for suitable $r_1, \dots, r_k \in [0,T]$ as in the previous proof. Let $A \in \mathscr{A}(Z)$ be a control process as in the statement, then

$$\lambda_t^A(Var(\vartheta_{\cdot,s}(\omega), [0,T])) = A_t(\omega) \int_0^t Var(\vartheta_{\cdot,s}(\omega), [0,T])^2 dA_s(\omega) \leq$$

$$T^2 A_t(\omega) \int_0^t \max_{1 \leq i \leq k} K(r_i, s, \omega)^2 dA_s(\omega) \leq T^2 \sum_{i=1}^k A_t(\omega) \int_0^t K(r_i, s, \omega)^2 dA_s(\omega) < \infty,$$

for P-a.e. $\omega \in \Omega$, so $(Var(\vartheta_{\cdot,s}, [0,T]))_{s \in \mathbb{R}}$ is Z-integrable and the conclusion follows by Theorem 4.2 □

5 Conclusion

This work gives a positive answer to the intuition stated in Choulli/Schweizer (2013) that their approach generalizes to general semimartingales. The techniques used here in Section 3 are basically the same as in their paper, so the presentation does not become significantly longer and technical once we view semimartingales in the spirit of Métivier as controlled processes.

The discussion in Section 2 and at the end of Section 3 concerning measurability issues of Banach valued functions and processes deserves more attention. The literature regarding measurable Banach valued functions where the σ-algebra is not the Borel σ-algebra generated by the norm topology is nearly non-existing. In particular the case where the σ-algebra is generated by the weak*-topology should be studied further, as these kind of measurable processes arise naturally and a generalization of Proposition 3.21 should be feasible.

The construction of the weak* stochastic integral can easily be generalized to general locally convex vector spaces and dual pairs. Resulting for instance in an integral that allows a weak Fubini property instead of the weak* Fubini property we proved, which may be of independent interest.

Finally we remark that a generalization of Corollary 4.4 to the case where the derivative is only a weak derivative should be feasible. However, this would probably make the statement and the proof more technical.

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