A STABILITY THEORY BEYOND THE CO-ROTATIONAL SETTING FOR CRITICAL WAVE MAPS BLOW UP

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Abstract. We exhibit non-equivariant perturbations of the blowup solutions constructed in [18] for energy critical wave maps into $\mathbb{S}^2$. Our admissible class of perturbations is an open set in some sufficiently smooth topology and vanishes near the light cone. We show that the blowup solutions from [18] are rigid under such perturbations, including the space-time location of blowup. As blowup is approached, the dynamics agree with the classification obtained in [7], and all six symmetry parameters converge to limiting values. Compared to the previous work [16] in which the rigidity of the blowup solutions from [18] under equivariant perturbations was proved, the class of perturbations considered in the present work does not impose any symmetry restrictions. Separation of variables and decomposing into angular Fourier modes leads to an infinite system of coupled nonlinear equations, which we solve for small admissible data. The nonlinear analysis is based on the distorted Fourier transform, associated with an infinite family of Bessel type Schrödinger operators on the half-line indexed by the angular momentum n. A semi-classical WKB-type spectral analysis relative to the parameter $\hbar = \frac{1}{n+1}$ for large $|n|$ allows us to effectively determine the distorted Fourier basis for the entire infinite family. Our linear analysis is based on the global Liouville-Green transform as in the earlier works [4, 5].

Contents

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1. Introduction

This work is dedicated to the energy critical Wave Maps equation $u : \mathbb{R}^{2+1} \longrightarrow \mathbb{S}^2$, and more specifically, to developing a theory which allows to establish the stability (and in fact, rigidity) of a certain class of its finite time blow up solutions *without any symmetry restrictions*. Specifically, the final result established shows that the finite time blow up solutions constructed in [10, 18] *in the co-rotational setting*, and with blow up scaling $\lambda(t) = t^{-1-\nu}$ sufficiently close to the self similar rate, i.e. $0 < \nu < \nu_*$ and $\nu_* > 0$ sufficiently small, are stable under *arbitrary* sufficiently smooth and small perturbations, *without any symmetry restrictions*. The precise result, stated below, will show that the perturbations result in slightly modulated profiles in perfect agreement with the general classification result due to Duyckaerts-Jia-Kenig-Merle [7], and it appears to be the first result which shows the full stability of a non-scattering solution in the setting of Wave Maps (as usual, the open set of perturbations is relative to a stronger topology than energy). A peculiarity of the solutions constructed in [10,18] is the fact that even before blow up, they are only of finite regularity (in fact $H^{1+\nu-}$-regularity, with ν as before), on account of a shock they display on the light cone centered at the singularity. This shock appears to endow the solutions with a strong rigidity, provided ν is sufficiently small, which may be interpreted as the reason why the space-time location of the singularity for the perturbed solution is un-changed, a feature which would be clearly false for blow up solutions which are smooth before the blow up time. It is important to keep in mind that the small perturbations added to the original data are much smoother than the original data. It is by now known that many different types of hyperbolic equations admit blow up solutions of similar character, and it is hoped that the present work may provide techniques to analyze their stability properties as well. In fact, the asymptotic Fourier methods developed here may be of use for much more general problems, beyond the narrow context of highly specific blow up solutions considered in this work. Before stating the main theorem, we recall the basic setup

1.1. Blow up in the co-rotational setting.

Recall that a Wave Map $u : \mathbb{R}^{2+1} \longrightarrow \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is a vector valued solution to the system

$$\Box u = \left(|u_t|^2 - |\nabla_x u|^2\right) u, \quad \Box = -\partial_t^2 + \Delta_x \tag{1.1}$$

This model is known to admit a rich class of static solutions of finite energy, due to the large class of finite energy harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$. Amongst these, the *stereographic projection map* $\mathcal{Q} : \mathbb{R}^2 \longrightarrow \mathbb{S}^2$ is the one of minimal energy amongst the non-trivial harmonic maps. It is an example of a so-called *co-rotational harmonic map*, in that it admits the representation

$$\mathcal{Q}(r,\theta) = \begin{pmatrix} \cos\theta \sin Q \\ \sin\theta \sin Q \\ \cos Q \end{pmatrix},$$

where Q is only dependent on r, and in fact for the stereographic projection we have $Q(r) = 2\arctan r$. Making the co-rotational ansatz for (1.1) leads to a remarkable simplification of the equation, removing the derivatives. In fact, making the ansatz

$$u(t,r) = \begin{pmatrix} \cos\theta\sin v \\ \sin\theta\sin v \\ \cos v \end{pmatrix},$$

where now $v = v(t,r)$ is a *scalar unknown*, one derives the equation

$$\Box v = \frac{\sin 2v}{2r^2}, \quad \Box = -\partial_t^2 + \partial_r^2 + \frac{1}{r}\partial_r. \tag{1.2}$$

The fact that $v(t,r) = 2\arctan r$ is a static solution of this problem is then easily verified by direct calculation. The simplified model (1.2) has been the subject of intense numerical and theoretical investigations, see for example [26–28]. In particular, the crucial role that harmonic maps play in the formation of singularities for this model was first stressed in [29], and numerical investigations showing the bubbling-off of co-rotational harmonic maps were done in [2]. The first rigorous construction of blow-up solutions within the co-rotational setting is due to [18], later followed by a different blow up construction in [22]. While the former constructed a *continuum of blow up solutions* of the form

$$v(t,r) = Q(\lambda(t)r) + \epsilon(t,r)$$

where $\lambda(t) = t^{-1-\nu}$, $\nu > \frac{1}{2}$ (later refined to $\nu > 0$ in [10], see also [17]), and moreover with $\epsilon(t,\cdot) \in H^{1+\nu-}$, the solutions in [22] are C^∞ before formation of the singularity, and were shown to be *stable within the co-rotational class* (as usual, the open set of perturbations is relative to a stronger topology than energy), a feature missing in [10, 18]. Co-rotational stability of the solutions in [10, 18] for ν small enough was proved recently in [16]. It is to be noted that the technique for showing stability in [16] are completely different than the ones giving stability in [22]. In particular, the work [16] relies heavily on the use of the distorted Fourier transform. In the present work, Fourier techniques will also play an all-encompassing role.

1.2. **Beyond the co-rotational setting.** As is clear from comparison of (1.2), (1.1), the stability problem outside of the co-rotational context is of a quite different character than the co-rotational one: the nonlinear source terms display derivatives, and in particular they lose the smoothing property which the purely semi-linear source terms in the co-rotational setting possess. In fact, the classical local well-posedness theory as developed for example in [12–14], reveals that in order to get strong local well-posedness all the way to the critical scaling level (which is essentially required for us since the solutions we perturb are only of regularity $H^{1+\nu-}$, with $\dot{H}^1$ the scaling critical space), a special algebraic cancellation feature in the nonlinearity is generally required (so-called null-structure); the Wave Maps problem has such a structure. Problems with this structure can be handled by means of the $H^{s,\delta}$ (also referred to as $X^{s,b}$-spaces) functional framework, which, however, appears intimately adapted to the free wave equation. Perturbing around the co-rotational blow ups leads to a very different kind of operator than the standard d'Alembertian. While it would be very interesting to develop abstract function spaces similar to the $H^{s,\delta}$-spaces to handle more general hyperbolic problems encompassing in particular our setup, our strategy in this work is to pursue a reductive ansatz, employing suitable coordinates and a separation of variables to arrive at a countable family of $1+1$-dimensional wave operators indexed by the angular frequency, and which we analyze asymptotically to reduce the analysis to a setup similar to the one in [16]. The asymptotic analysis of the infinite family of wave operators uses techniques developed in [4, 5]. This allows us to develop sufficiently precise parametrices for the wave operators in order to analyze the propagation of the shock on the light cone in the various nonlinear interactions and

at various angular frequencies. An abstract $H^{s,\delta}$ framework is then substituted for in this work by a precise concept of *admissible singular functions*, see Definition 7.8, which is both compatible with the family of wave propagators at the various angular frequencies, as well as the structure of the nonlinear source terms (since, being formulated on the distorted Fourier side, it allows a natural translation to the physical side). In addition to handling an infinite family of wave propagators at different angular frequencies, one also needs to deal with the inherent instabilities at three exceptional angular momenta, namely $n = 0, \pm 1$. The Fourier representations at these angular momenta are exceptional, due to the presence of resonances/eigenvectors at frequency zero. These cause degeneracies in the corresponding spectral measures, and require a separate analysis of the *continuous spectral part* and the *unstable (discrete) spectral part*. We then rely on the fact that all these wave operators admit a super-symmetric partner, owing to a remarkable product structure already taken advantage of in the co-rotational setting in [16, 22], which reduces an analysis of the wave equations at exceptional angular momenta to an analysis of the super-symmetric partner equation (which no longer exhibits a singular spectral measure), as well as the ODE governing the evolution of the instability.
It turns out that the conjunction of the shock on the light cone and the nonlinear source terms involving derivatives produce source terms for the evolution of the unstable modes which would lead to divergence. Handling this issue requires the use of some form of modulation theory, which, however, differs from the more standard one in that it is the *action of internal symmetries on the singular part of the original co-rotational blow up* which is being used to counteract other singular terms arising from the interaction of the perturbation and the original co-rotational singularity (i.e., a shock along the light cone). The only relevant symmetries here are scaling, three types of rotations on the target, as well as Lorentz transforms, which are in fact all symmetries leaving the light cone invariant.

1.3. **Statement of the main theorem.** The following is the result of the developments of this work, proved at the very end. Recall that the co-rotational finite time blow up solutions in [10, 18] are of the form (in polar coordinates)

$$\Phi(t,\theta,r) = \begin{pmatrix} \cos\theta\sin U \\ \sin\theta\sin U \\ \cos U \end{pmatrix}$$

where $U = U(t,r) = Q(\lambda(t)r) + \epsilon(t,r)$ where $\lambda(t) = t^{-1-\nu}$, $Q(r) = 2\arctan r$, and the error term $\epsilon(t,r)$ vanishes asymptotically in a suitable sense in the light cone centred at the singularity $(t,r) = (0,0)$ as t approaches 0. The following theorem will be rendered completely precise in the ensuing analysis, with a delicate description of the radiation term $\delta\Psi$ in the light cone.

Theorem 1.1. *Let* $\Phi = \begin{pmatrix} \cos\theta\sin U \\ \sin\theta\sin U \\ \cos U \end{pmatrix}$, $U = U(t,r)$ *be one of the finite time co-rotational blow up solutions constructed in [10, 18], with* $\nu > 0$ *sufficiently small, and restricted to the space time slab* $(0,t_0] \times \mathbb{R}^2$, *where* $t_0 = t_0(\nu)$ *is sufficiently small. Then there is* $\delta_* = \delta_*(\nu, t_0) > 0$ *with the following property: let* $(\delta\Phi_1, \delta\Phi_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *with the property that*

$$\left\|(\delta\Phi_1, \delta\Phi_2)\right\|_{H^{100}(\mathbb{R}^2)} < \delta_*,$$

and such that

$$\Psi[t_0] := (\Phi(t_0,\cdot) + \delta\Phi_1, \partial_t\Phi(t_0,\cdot) + \delta\Phi_2) : \mathbb{R}^2 \longrightarrow \mathbb{S}^2 \times T\mathbb{S}^2$$

constitutes a data set for (1.1)*, and furthermore* $\delta\Phi_j$*,* $j = 1, 2$*, is supported inside the disk* $r = |x| < \frac{t_0}{2}$*. Then the solution for* (1.1)*, with data* $\Psi[t_0]$ *at time* $t = t_0$ *admits a solution* Ψ *on the space-time slab* $(0, t_0] \times \mathbb{R}^2$ *which can be written in the form*

$$\Psi = \mathcal{L}_{v(t)} \mathcal{R}_{h(t)}^{\alpha(t),\beta(t)} \mathcal{S}_{c(t)} \left(\Phi + \delta\Psi \right),$$

where $\mathcal{L}_{v(t)}$ *is a time dependent Lorentz transform,* $\mathcal{R}_{h(t)}^{\alpha(t),\beta(t)}$ *a general rotation on the target* $\mathbb{S}^2$ *parametrized as the product of three elementary rotations, and* $\mathcal{S}_{c(t)}$ *a continuously varying change of scale. The radiation part* $\delta\Psi$ *vanishes asymptotically in the light cone near the singularity in the energy topology, and the modulation parameters* $v(t), \alpha(t), \beta(t), h(t), c(t)$ *approach finite limiting values as* $t \to 0$*.*

2. Summary of the sections of the paper

Very roughly speaking, the paper is divided into an elaborate linear part, which provides the basis for the distorted Fourier analysis at arbitrary angular frequencies, as well as a nonlinear part, where a machinery is developed to handle the various nonlinear source terms, in particular the null-form terms, and with irregular "inputs". The nonlinear part also encompasses the analysis of the unstable modes arising at angular frequencies $n = 0, \pm 1$, which forces us to employ the modulation techniques. Here we give a brief description of the linear as well as the nonlinear part. To get things started, we of course need the right formulation of the perturbation problem, which is accomplished in section 3 following an ansatz by Davila, DelPino, Wei [6]. Specifically, letting as before Φ denote the unperturbed co-rotational blow up solution, and introducing the frame (which constitutes an orthonormal basis for $T_\Phi \mathbb{S}^2$)

$$E_1 = \begin{pmatrix} \cos\theta \cos U \\ \sin\theta \cos U \\ -\sin U \end{pmatrix}, \quad E_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix},$$

we make the ansatz

$$\Psi = \mathcal{L}_{v(t)} \mathcal{R}_{h(t)}^{\alpha(t),\beta(t)} \mathcal{S}_{c(t)} \left(\Phi + \sum_{j=1,2} \varphi_j E_j + \mathfrak{q} \right),$$

where $\mathfrak{q}$ is a multiple of Φ, of quadratic size in (φ_1, φ_2) (provided the latter are small).
In section 3, in a first approximation, we ignore the modulations, and derive the leading order wave equations for the φ_j. For this, we decompose these functions into discrete Fourier series in relation to the angular parameter θ, i. e. we write $\varphi_j = \sum_{n \in Z} \varphi_j(n) e^{in\theta}$. In order to extract wave operators whose spatial part is time independent, we change variables to $\tau = \int_t^\infty \lambda(s)\, ds$, $R = \lambda(t) \cdot r$, $\lambda(t) = t^{-1-\nu}$. Furthermore, we introduce the variables

$$\varepsilon_\pm(n) = \varphi_1(n) \mp i\varphi_2(n), \quad n \in \mathbb{Z},$$

which satisfy wave equations with elliptic part given by the family of operators $H_n^\pm$ in (2.1). This is then the family of operators whose spectral theory is analyzed asymptotically for large values of n (the small values can be studied by more standard methods) in the linear part of the paper.

2.1. **The linear part.** This encompasses sections 3.1 to 5.4. The operators governing the linearized flow, restricted to a fixed angular frequency n, are given by

$$H_n^+ := \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) + g_n(R), \quad H_n^- := \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) - g_n(R). \tag{2.1}$$

where

$$f_n(R) = \frac{n^2+1}{R^2} - \frac{8}{(R^2+1)^2}, \quad g_n(R) = \frac{2n}{R^2} - \frac{4n}{R^2+1},$$
$$-f_n(R) + g_n(R) = -\frac{(n-1)^2}{R^2} - \frac{4n}{R^2+1} + \frac{8}{(R^2+1)^2},$$
$$-f_n(R) - g_n(R) = -\frac{(n+1)^2}{R^2} + \frac{4n}{R^2+1} + \frac{8}{(R^2+1)^2}$$

These operators are symmetric in $L^2(R\,dR)$ with domain consisting of $C^2((0,\infty))$ functions of compact support in $(0,\infty)$. Conjugating $H_n^\pm$ by the weights $R^{\frac{1}{2}}$ yields the Schrödinger operators

$$\mathcal{H}_n^+ = R^{\frac{1}{2}} H_n^+ R^{-\frac{1}{2}} = \partial_R^2 + \frac{1}{4R^2} - \frac{(n-1)^2}{R^2} - \frac{4n}{R^2+1} + \frac{8}{(R^2+1)^2},$$
$$\mathcal{H}_n^- = R^{\frac{1}{2}} H_n^- R^{-\frac{1}{2}} = \partial_R^2 + \frac{1}{4R^2} - \frac{(n+1)^2}{R^2} + \frac{4n}{R^2+1} + \frac{8}{(R^2+1)^2}.$$

which are symmetric in $L^2(dR)$ over the same domain. The potentials here are strongly singular, meaning that they are not integrable on $(0,1)$. The nonlinear analysis will be based on the (distorted) Fourier transform associated with these operators, see Gesztesy, Zinchenko [11] for the general theory of this Fourier transform, as well as Section 4.2 below. All operators $-\mathcal{H}_n^\pm$ are nonnegative, and for $|n| \geq 2$ they do not possess any 0 energy modes. However, $\mathcal{H}_n^+ = \mathcal{H}_{-n}^-$ for $n = 0, \pm 1$ each exhibit a 0 energy state, which is either an eigenvalue or resonance. In Section 9.2 below we relate these 0 modes to six modes of the matrix operator $\mathfrak{L}$ of (3.20) which are generated by all available symmetries: translations of $\mathbb{R}^2$, Lorentz transforms of Minkowski space $\mathbb{R}^{1+2}_{t,x}$, dilations, and rotations of the sphere $\mathbb{S}^2$. This is an 8-parameter family, but translations and Lorentz transforms lead to the same modes. This is an essential feature of our main theorem since translations do not appear in the asymptotic description of the blowup solutions. Even though separating variables might seem like a heavy-handed approach, it appears necessary to proceed in this fashion since the physical symmetries do not affect any frequencies $|n| \geq 2$ in the angle.

In order to effectively diagonalize each of the the linearized operators (for fixed angular frequency) by means of its associated distorted Fourier transform, we need to gain precise control on the solutions of the equations $-\mathcal{H}_n^\pm \phi = E^2\phi$, in the entire range of all three variables n, E, R. We treat $|n| \leq N$ separately and by different methods from the regime $|n| \geq N$, where N is some large absolute constant. Small $|n|$ are handled by the perturbative techniques of [18] and [24, 25]. The large $|n|$ regime, which is intrinsically a singular perturbation problem, is reduced to a semi-classical spectral problem $\left(-\hbar^2\partial_R^2 + V(R;\hbar)\right)f = (\hbar E)^2 f$ with $\hbar = (n+1)^{-1}$ and $V(R;\hbar) = R^{-2} + \hbar W(R;\hbar)$. A change of variables $x = \hbar E R$ moves the turning point $V(R;\hbar) - (\hbar E)^2 = 0$ to the vicinity of $x = 1$. As in [5], we need to carry out the so-called Langer correction by adding $\hbar^2/(4x^2)$ to the potential. Otherwise our WKB type approximations will diverge as $E \to 0$ (see [5] for more on this). The resulting problem is of the form, see (4.65) and (4.66),

$$-\hbar^2 \widetilde{f}''(x) + Q_0(x)\widetilde{f}(x) = 0, \quad Q_0(x) := x^{-2} - 1 + \text{corrections}.$$

To this we apply the global Liouville-Green transform (4.72), viz.

$$\tau(x,\alpha;\hbar) := \operatorname{sign}(x - x_t(\alpha;\hbar)) \left| \frac{3}{2} \int_{x_t(\alpha;,\hbar)}^{x} \sqrt{|Q_0(u,\alpha;\hbar)|}\, du \right|^{\frac{2}{3}}$$

also known as Langer transform, which turns the previous equation into a perturbed Airy equation in the variable $\tau \in (-\infty, \infty)$, see (4.76). In Section 4.6 we solve this perturbed Airy equation so as to obtain a fundamental system in τ. Our strategy is inspired by the one of [4, 5]. It is absolutely essential for us that our approximation is accurate up to a *multiplicative correction* of the form $1 + \hbar a(\tau; E, \hbar)$ where a is uniformly bounded in the entire three-parameter regime. In addition, we control the derivatives in E and τ. In particular, the functions a cannot contain any rapidly oscillating factors, and all oscillations have to be included in the main term. This appears to be in stark contrast to the vast body of the semi-classical literature in which the errors are additive, or only the limit $\hbar \to 0$ is considered. Anything but the asymptotic descriptions of the fundamental system and the spectral measure as we derive them below uniformly in all three parameters, would be insufficient for our nonlinear analysis. One surprising feature of our analysis, which does not appear in [4, 5] is a parameter regime in which $a(\tau; E, \hbar)$ can form arbitrarily long plateaus without any decay, see Lemma 4.31.

2.2. **The nonlinear part.** Roughly speaking, the nonlinear part is divided into three parts. In the first part, section 6, we develop the basic bilinear and multilinear estimates allowing us to control the source terms *away from the light cone*, i.e., in the region where all source terms have a regularity like that of the perturbation of the initial data. Here the main difficulty consists in proving estimates which experience no losses in the angular frequency, and with respect to a space which is consistent with the wave parametrix and the *transference operator* which comes up in the solution of the linear homogeneous problem associated to $H_n^{\pm}$. We shall refer this space as the *smooth space*, since elements in it have a high degree of differentiability, but on the flip side they have little structure beyond the smoothness. In terms of the distorted Fourier variables $\overline{x}(\tau, \xi)$, these norms are of the form

$$\left\| \left(\hbar^2\xi\right)^{1-\frac{\delta}{2}} \left\langle \hbar^2\xi \right\rangle^{\delta+\frac{3}{2}} \overline{x}(\tau,\xi) \right\|_{L^2_{d\xi}},$$

where $\hbar = \frac{1}{|n|+1}$ is in terms of the angular frequency n, and $\langle x \rangle = \sqrt{1+x^2}$. It is important that these norms include the weight $\left(\hbar^2\xi\right)^{1-\frac{\delta}{2}}$ for small frequencies, and the different weight $\left(\hbar^2\xi\right)^{\frac{5}{2}+\frac{\delta}{2}}$ for very large frequencies. The fact that one uses the variable $\hbar^2\xi$ rather than ξ comes from the fine structure of the eigenfunctions and the transference operator at angular frequency n, and more specifically the fact that the separation point between the non-oscillatory and oscillatory regimes (so-called turning point) is defined in terms of $R\xi^{\frac{1}{2}}\hbar$. We observe that other than some technical but non-essential issues arising near the spatial origin $R = 0$, the small angular frequencies and in particular the exceptional ones can be handled very similarly to the large ones in this setting.

In the second part, covering section 7, section 8, we develop a functional framework which encompasses the shock singularity along the light cone, and which allows us to handle the null-form nonlinear source

terms, as well as the remaining terms, for angular frequencies $|n| \geq 2$. Due to the complexity of the function space, we do this in several stages, starting with a basic *prototype singular term* (Definition 7.2), which gets refined to the more general *admissible singular function* concept in Definition 7.8, and this gets combined with the *smooth space* concept from before in the final Definition 8.3 for angular frequencies $|n| \geq 2$. This gets complemented by an analogous definition for the exceptional modes, which also includes the desired bounds for the unstable modes. Completing the required bounds for the angular frequencies $|n| \geq 2$ then amounts to proving estimates for all the source terms as in Prop. 8.12, as well as solving the inhomogeneous wave equation in its Fourier realization. The latter is accomplished in section 8, where the same method as in [19] is used to iterate away the linear non-local source terms due to the transference operator. The reiteration method used here relies crucially on the asymptotic bounds for the transference operator kernel established in section 5.
It then remains to handle the exceptional modes. Here we trade in the difficulty involved in handling large angular frequencies (these only occur in pairs in the corresponding source terms, meaning they become harmless due to the rapid decay in terms of the angular frequency) for the difficulty controlling the unstable part. Precisely, it is here we have to take advantage of the freedom to continuously modulate in the symmetries in order to neutralize certain top order singular terms, the details being carried out in section 9. Finally, the iteration scheme leading to the desired solution is outlined in the section 10.

3. Perturbing equivariant blowup solutions

We consider critical wave maps from $\mathbb{R}^{1+2}_{t,x} \to \mathbb{S}^2$, not necessarily equivariant. We recall the extrinsic description of wave maps of this type. A smooth map $u := (u^1, u^2, u^3) : \mathbb{R}^{1+2} \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is a wave map if at every point $z = (t, x) \in \mathbb{R}^{1+2}$

$$\Box u \perp T_{u(z)}\mathbb{S}^2, \quad \Box u(z) = \Big(\Box u^1(z), \Box u^2(z), \Box u^3(z)\Big). \tag{3.1}$$

Here $\Box$ denotes the operator $\Box := \partial^\alpha \partial_\alpha = -\partial_t^2 + \partial_1^2 + \partial_2^2$. Formally speaking, (3.1) is the Euler-Lagrange equation of critical points of the action

$$\mathcal{L}(\partial u) = \frac{1}{2}\int_{\mathbb{R}^{1+2}_{t,x}} \big(- |u_t|_g^2 + |u_{x_1}|_g^2 + |u_{x_2}|_g^2\big)\, dtdx$$

and $|\cdot|_g$ is the metric on the target manifold. For $\mathbb{S}^2$, note that u is a normal vector field, therefore there is a scalar function $\lambda : \mathbb{R}^{1+2} \to \mathbb{R}$ such that

$$\Box u = \lambda u. \tag{3.2}$$

On the other hand, u takes values on $\mathbb{S}^2$, which implies $\langle u, u\rangle = 1$. Here $\langle\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$. Hence

$$\lambda = \langle \Box u, u\rangle = \partial^\alpha \langle \partial_\alpha u, u\rangle - \langle \partial_\alpha u, \partial^\alpha u\rangle = -\langle \partial_\alpha u, \partial^\alpha u\rangle = |\partial_t u|^2 - |\nabla u|^2.$$

Therefore the extrinsic description of a wave map from $\mathbb{R}^{1+2}$ to $\mathbb{S}^2$ is

$$\Box u = \Big(|\partial_t u|^2 - |\nabla u|^2\Big)\, u, \quad \text{or} \quad S(u) := -\partial_t^2 u + \Delta u + \Big(|\nabla u|^2 - |\partial_t u|^2\Big)\, u = 0. \tag{3.3}$$

For more general surfaces than $\mathbb{S}^2$ the nonlinearity is replaced by the second fundamental form contracted with the Minkowski tensor, i.e., $A(\partial_\alpha, \partial_\beta)\eta^{\alpha\beta}$. As a wave equation, (3.3) conserves energy.

Proposition 3.1. *Let* $u : \mathbb{R}^2 \times [0,T) \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ *be a smooth solution to* (3.3) *such that* $(u(0,\cdot), u_t(0,\cdot)) \in \dot{H}^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. *Then the energy*

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\partial_t u|^2 + |\nabla u|^2\right) dx \tag{3.4}$$

is constant for all times $t \in [0,T)$.

Proof. The proof proceeds by inner product with $\partial_t u$ on both sides of (3.3), and integrating by parts. The right-hand side contributes nothing since

$$\left(|\partial_t u|^2 - |\nabla u|^2\right) u \cdot \partial_t u = \frac{1}{2}\left(|\partial_t u|^2 - |\nabla u|^2\right)\partial_t |u|^2 = 0,$$

due to $|u| \equiv 1$. □

Let $\Phi(t,r,\theta)$ be a smooth 1-equivariant wave map written in spherical coordinates.

$$\Phi(t,r,\theta) := (\cos\theta \sin U(t,r), \sin\theta \sin U(t,r), \cos U(t,r)). \tag{3.5}$$

We introduce the orthonormal frame

$$\begin{aligned} E_1 &= \partial_U = (\cos\theta\cos U, \sin\theta\cos U, -\sin U) \\ E_2 &= (\sin U)^{-1}\partial_\theta = (-\sin\theta, \cos\theta, 0) \end{aligned} \tag{3.6}$$

on the sphere (treating the azimuth angle U as an independent variable). We write non-equivariant perturbations of Φ in the form, cf. [6],

$$\Psi := \Phi + \Pi_{\Phi^\perp}\varphi + \mathfrak{q} \tag{3.7}$$

We expand the tangential vector field $\Pi_{\Phi^\perp}\varphi$ in the form

$$\begin{aligned} \Pi_{\Phi^\perp}\varphi(t,r,\theta) :=& \varphi_1(t,r,\theta)(\cos\theta\cos U, \sin\theta\cos U, -\sin U) + \varphi_2(t,r,\theta)(-\sin\theta,\cos\theta,0) \\ :=& \varphi_1(t,r,\theta)E_1 + \varphi_2(t,r,\theta)E_2. \end{aligned} \tag{3.8}$$

The term $\mathfrak{q}$ in (3.7) is a correction which insures that $|\Psi| = 1$. We will specify it later. It is quadratic in the size of $\Pi_{\Phi^\perp}\varphi$, which we assume small. Both Φ and Ψ satisfy (3.3). Subtracting them yields

$$\begin{aligned} &\Box\Psi - \Box\Phi = \Box\Pi_{\Phi^\perp}\varphi + \Box\mathfrak{q} = \left(|\partial_t\Psi|^2 - |\nabla\Psi|^2\right)\Psi - \left(|\partial_t\Phi|^2 - |\nabla\Phi|^2\right)\Phi \\ &= \left(|\partial_t\Phi + \partial_t\Pi_{\Phi^\perp}\varphi + \partial_t\mathfrak{q}|^2 - |\nabla\Phi + \nabla\Pi_{\Phi^\perp}\varphi + \nabla\mathfrak{q}|^2\right)(\Phi + \Pi_{\Phi^\perp}\varphi + \mathfrak{q}) - \left(|\partial_t\Phi|^2 - |\nabla\Phi|^2\right)\Phi \\ &= 2(\partial_t\Phi\cdot\partial_t\Pi_{\Phi^\perp}\varphi - \nabla\Phi\cdot\nabla\Pi_{\Phi^\perp}\varphi)\Phi + 2(\partial_t\Phi\cdot\partial_t\mathfrak{q} - \nabla\Phi\cdot\nabla\mathfrak{q})\Phi \\ &\quad + (|\partial_t\Pi_{\Phi^\perp}\varphi + \partial_t\mathfrak{q}|^2 - |\nabla\Pi_{\Phi^\perp}\varphi + \nabla\mathfrak{q}|^2)\Phi + \left(|\partial_t\Phi + \partial_t\Pi_{\Phi^\perp}\varphi + \partial_t\mathfrak{q}|^2 - |\nabla\Phi + \nabla\Pi_{\Phi^\perp}\varphi + \nabla\mathfrak{q}|^2\right)(\Pi_{\Phi^\perp}\varphi + \mathfrak{q}) \end{aligned}$$

Moving all terms which depend linearly on $\Pi_{\Phi^\perp}\varphi$ to the left yields

$$\begin{aligned} &\Box\Pi_{\Phi^\perp}\varphi + \left(|\nabla\Phi|^2 - |\Phi_t|^2\right)\Pi_{\Phi^\perp}\varphi + 2\left(\nabla\Phi\cdot\nabla\Pi_{\Phi^\perp}\varphi - \Phi_t\cdot(\Pi_{\Phi^\perp}\varphi)_t\right)\Phi \\ &= -\Box\mathfrak{q} + 2(\partial_t\Phi\cdot\partial_t\mathfrak{q} - \nabla\Phi\cdot\nabla\mathfrak{q})\Phi + (|\partial_t\Pi_{\Phi^\perp}\varphi + \partial_t\mathfrak{q}|^2 - |\nabla\Pi_{\Phi^\perp}\varphi + \nabla\mathfrak{q}|^2)\Phi \\ &\quad + \left(|\partial_t\Phi + \partial_t\Pi_{\Phi^\perp}\varphi + \partial_t\mathfrak{q}|^2 - |\Phi_t|^2 + |\nabla\Phi|^2 - |\nabla\Phi + \nabla\Pi_{\Phi^\perp}\varphi + \nabla\mathfrak{q}|^2\right)\Pi_{\Phi^\perp}\varphi \\ &\quad + \left(|\partial_t\Phi + \partial_t\Pi_{\Phi^\perp}\varphi + \partial_t\mathfrak{q}|^2 - |\nabla\Phi + \nabla\Pi_{\Phi^\perp}\varphi + \nabla\mathfrak{q}|^2\right)\mathfrak{q} =: \mathfrak{Q} \end{aligned} \tag{3.9}$$

Here $\mathfrak{Q}$ are terms of second and higher orders in φ_1, φ_2. From the definitions we compute the following first order derivatives:

$$\partial_r E_1 = -U_r \Phi, \quad \partial_r \Phi = U_r E_1, \quad \partial_r E_2 = 0 \text{ (same for } \partial_t)$$
$$\partial_\theta E_1 = \cos U E_2, \quad \partial_\theta E_2 = -\cos U E_1 - \sin U \Phi, \quad \partial_\theta \Phi = \sin U E_2$$

Further, with $\nabla = \widehat{e_r}\partial_r + r^{-1}\widehat{e_\theta}\partial_\theta$ the gradient in polar coordinates:

$$\begin{aligned}
\nabla\Phi =& \widehat{e_r} U_r E_1 + \widehat{e_\theta}\frac{\sin U}{r} E_2, \quad \Phi_t = U_t E_1, \\
|\nabla\Phi|^2 - |\Phi_t|^2 =& U_r^2 - U_t^2 + \frac{1}{r^2}\sin^2 U, \\
(\Pi_{\Phi^\perp}\varphi)_t =& \varphi_{1,t} E_1 - \varphi_1 U_t \Phi + \varphi_{2,t} E_2, \quad \Phi_t \cdot (\Pi_{\Phi^\perp}\varphi)_t = U_t \varphi_{1,t}, \\
\nabla \Pi_{\Phi^\perp}\varphi =& (\varphi_{1,r} E_1 - \varphi_1 U_r \Phi + \varphi_{2,r} E_2)\widehat{e_r} + \frac{1}{r}(\varphi_{1,\theta} E_1 + \varphi_1 \cos U E_2 + \varphi_{2,\theta} E_2 - \varphi_2(\cos U E_1 + \sin U \Phi))\widehat{e_\theta}, \\
\nabla \Pi_{\Phi^\perp}\varphi \cdot \nabla\Phi =& \varphi_{1,r} U_r + \frac{1}{2r^2}\sin(2U)\varphi_1 + \frac{1}{r^2}\sin U \varphi_{2,\theta}.
\end{aligned} \tag{3.10}$$

The d'Alembertian $\Box \Pi_{\Phi^\perp}\varphi$ is obtained as follows (using $\Delta = \operatorname{div}\nabla$ in polar coordinates):

$$\begin{aligned}
(\Pi_{\Phi^\perp}\varphi)_{tt} &= (\varphi_{1,tt} - \varphi_1 U_t^2)E_1 + \varphi_{2,tt} E_2 - (2\varphi_{1,t} U_t + \varphi_1 U_{tt})\Phi, \\
\Delta \Pi_{\Phi^\perp}\varphi &= \frac{1}{r}(r(\varphi_{1,r} E_1 + \varphi_{2,r} E_2 - \varphi_1 U_r \Phi))_r + \frac{1}{r^2}((\varphi_{1,\theta} - \varphi_2 \cos U)E_1 + (\varphi_{2,\theta} + \varphi_1 \cos U)E_2 - \varphi_2 \sin U \Phi)_\theta \\
&= (\varphi_{1,rr} + \frac{1}{r}\varphi_{1,r} + \frac{1}{r^2}\varphi_{1,\theta\theta} - \frac{2\varphi_{2,\theta}}{r^2}\cos U - \varphi_1 U_r^2 - \frac{\varphi_1 \cos^2 U}{r^2})E_1 \\
&\quad + (\varphi_{2,rr} + \frac{1}{r}\varphi_{2,r} + \frac{1}{r^2}\varphi_{2,\theta\theta} + \frac{2\cos U}{r^2}\varphi_{1,\theta} - \frac{\varphi_2}{r^2})E_2 \\
&\quad + (-2\varphi_{1,r} U_r - \varphi_1 U_{rr} - \frac{1}{r}\varphi_1 U_r - \frac{\sin 2U}{2r^2}\varphi_1 - \frac{2\sin U}{r^2}\varphi_{2,\theta})\Phi
\end{aligned}$$

Now we are ready to plug in all the above formulas into the left hand side of (3.9). We have:

$$\begin{aligned}
\Box \Pi_{\Phi^\perp}\varphi =& \left(-\varphi_{1,tt} + \varphi_{1,rr} + \frac{1}{r}\varphi_{1,r} + \frac{1}{r^2}\varphi_{1,\theta\theta} + \varphi_1(U_t^2 - U_r^2) - \frac{\cos^2 U}{r^2}\varphi_1 - \frac{2\cos U}{r^2}\varphi_{2,\theta}\right)E_1 \\
&+ \left(-\varphi_{2,tt} + \varphi_{2,rr} + \frac{1}{r}\varphi_{2,r} + \frac{1}{r^2}\varphi_{2,\theta\theta} - \frac{1}{r^2}\varphi_2 + \frac{2\cos U}{r^2}\varphi_{1,\theta}\right)E_2 \\
&+ \left(2\varphi_{1,t} U_t + \varphi_1 U_{tt} - 2\varphi_{1,r} U_r - \varphi_1 U_{rr} - \frac{1}{r}\varphi_1 U_r - \frac{\sin 2U}{2r^2}\varphi_1 - \frac{2\sin U}{r^2}\varphi_{2,\theta}\right)\Phi
\end{aligned} \tag{3.11}$$

Now we substitute (3.10) and (3.11) into the left-hand side of (3.9). The coefficient for E_1 gives

$$\Box\varphi_1 - \frac{\cos 2U}{r^2}\varphi_1 - \frac{2\cos U}{r^2}\varphi_{2,\theta} = \mathfrak{Q}\cdot E_1 \tag{3.12}$$

The coefficient for E_2 is

$$\Box\varphi_2 - \frac{\cos^2 U}{r^2}\varphi_2 + (U_r^2 - U_t^2)\varphi_2 + \frac{2\cos U}{r^2}\varphi_{1,\theta} = \mathfrak{Q}\cdot E_2 \tag{3.13}$$

The coefficient for Φ on the left-hand side of (3.9) is

$$\Big(U_{tt} - U_{rr} - \frac{1}{r}U_r + \frac{\sin(2U)}{2r^2}\Big)\varphi_1 = 0$$

due to the equivariant wave map equation satisfied by U. Therefore, the right-hand side of (3.9) also vanishes, $\mathfrak{Q} \cdot \Phi = 0$.

3.1. **The linear operators governed by the rescaled harmonic map, separation of variables.** We let U be the equivariant blowup solution from [17, 18] which takes the form

$$U(t,r) = Q(\lambda(t)r) + \epsilon(t,r) = Q(R) + \epsilon(t,r) = 2\arctan R + \epsilon(t,r), \qquad \lambda(t) = t^{-1-\nu}, \nu > 0 \tag{3.14}$$

Here Q is the unique 1-equivariant harmonic map, and the energy of ϵ interior to a light cone $0 \le r \le t$ vanishes as $t \to 0+$:

$$\lim_{t\to 0+} \int_{r\le t} (\epsilon_t^2 + \epsilon_r^2)(t,r)\, r dr = 0$$

Later we will need much finer properties of ϵ, but for now we simplify the left-hand sides of (3.12) and (3.13) by replacing all occurrences of U with Q. The errors thus created will be treated perturbatively in the main nonlinear analysis. We introduce the new independent variables

$$R := \lambda(t)r = t^{-1-\nu}r, \quad \tau := \nu^{-1}t^{-\nu} = \int_t^\infty \lambda(s)\, ds. \tag{3.15}$$

The spatial Laplacian is given by

$$\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 = \lambda^2\left(\partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2\right), \tag{3.16}$$

and we have with $Q = Q(R)$

$$\cos Q = \frac{1-R^2}{1+R^2}, \quad \cos^2 Q = \frac{1-2R^2+R^4}{1+2R^2+R^4}, \quad \cos(2Q) = \frac{R^4-6R^2+1}{R^4+2R^2+1}. \tag{3.17}$$

On the other hand, we have (with $\dot\lambda(t) = \frac{d\lambda(t)}{dt}$)

$$\begin{aligned} \partial_r Q(R) &= \frac{2\lambda(t)}{1+R^2}, \quad (\partial_r Q(R))^2 = \frac{4\lambda(t)^2}{R^4+2R^2+1}, \\ \partial_t Q(R) &= \frac{2R}{1+R^2}\frac{\dot\lambda(t)}{\lambda(t)}, \quad (\partial_t Q(R))^2 = \frac{4R^2}{R^4+2R^2+1}\frac{(1+\nu)^2}{t^2} = \lambda^2\frac{4R^2}{R^4+2R^2+1}\frac{(1+\nu)^2}{\nu^2\tau^2} \end{aligned} \tag{3.18}$$

In these new variables and after dividing by λ^2, we arrive at the new spatial linear operator acting on $\varphi = (\varphi_1, \varphi_2)$,

$$\mathcal{L}\varphi := \begin{pmatrix} \Big(\partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2 - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}\Big)\varphi_1 - \frac{2-2R^2}{R^2(1+R^2)}\partial_\theta\varphi_2 \\ \Big(\partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2 - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)} - \frac{(1+\nu)^2}{\nu^2\tau^2}\frac{4R^2}{1+2R^2+R^4}\Big)\varphi_2 + \frac{2-2R^2}{R^2(1+R^2)}\partial_\theta\varphi_1 \end{pmatrix}. \tag{3.19}$$

The term $-\frac{(1+\nu)^2}{\nu^2\tau^2}\frac{4R^2}{1+2R^2+R^4}\varphi_2$ derives from the term $-U_t^2\varphi_2$ in (3.13), and would be absent if Φ were a harmonic map in the variable R. On the other hand, due to the decay factor τ^{-2} we will be able to place this term on the right-hand side of the equation and treat it perturbatively. Therefore, the linear operator driving the entire analysis is

$$\mathfrak{L}\,\varphi := \begin{pmatrix} \left(\partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2 - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}\right)\varphi_1 - \frac{2-2R^2}{R^2(1+R^2)}\partial_\theta\varphi_2 \\ \left(\partial_R^2 + \frac{1}{R}\partial_R + \frac{1}{R^2}\partial_\theta^2 - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}\right)\varphi_2 + \frac{2-2R^2}{R^2(1+R^2)}\partial_\theta\varphi_1 \end{pmatrix}. \tag{3.20}$$

The change of variables in time is given by

$$\begin{aligned} \partial_t &= \lambda(\partial_\tau + \beta R\partial_R), \qquad \beta(\tau) = \frac{\lambda'(\tau)}{\lambda(\tau)} = (1+1/\nu)\tau^{-1}, \quad \lambda' = \frac{d\lambda}{d\tau} \\ \partial_t^2 &= \lambda^2\left[(\partial_\tau + \beta R\partial_R)^2 + \beta(\partial_\tau + \beta R\partial_R)\right] \end{aligned}$$

whence the full dynamical problem in (τ, R) is of the form

$$-\Big[(\partial_\tau + \beta R\partial_R)^2 + \beta(\partial_\tau + \beta R\partial_R)\Big]\varphi + \mathfrak{L}\,\varphi = \lambda^{-2}\begin{pmatrix}\mathfrak{Q}\cdot E_1 \\ \mathfrak{Q}\cdot E_2\end{pmatrix} + \mathfrak{E} + \begin{pmatrix} 0 \\ \frac{(1+\nu)^2}{\nu^2\tau^2}\frac{4R^2}{1+2R^2+R^4}\varphi_2\end{pmatrix} \tag{3.21}$$

where $\mathfrak{E}$ is the error term obtained by replacing U with Q in (3.12) and (3.13). The precise form of the right-hand side of (3.21) will be determined in Section 3.4. To analyze the linear dynamics, we (formally) expand φ_1 and φ_2 into Fourier series

$$\begin{aligned} \varphi_1(t,R,\theta) &= \sum_n \widehat{\varphi_1}(n,t,R)e^{in\theta}, \quad \widehat{\varphi_1}(n,t,R) := \int_0^{2\pi} \varphi_1(t,R,\theta)e^{-in\theta}\,\frac{d\theta}{2\pi}, \quad n\in\mathbb{Z}, \\ \varphi_2(t,R,\theta) &= \sum_n \widehat{\varphi_2}(n,t,R)e^{in\theta}, \quad \widehat{\varphi_2}(n,t,R) := \int_0^{2\pi} \varphi_2(t,R,\theta)e^{-in\theta}\,\frac{d\theta}{2\pi}, \quad n\in\mathbb{Z}. \end{aligned} \tag{3.22}$$

For fixed $n \in \mathbb{Z}$, the operator applied on the Fourier coefficients is given by

$$\mathfrak{L}_n\,\widehat{\varphi}(n) := \begin{pmatrix} \left(\partial_R^2 + \frac{1}{R}\partial_R - \frac{n^2}{R^2} - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}\right)\widehat{\varphi_1}(n) - in\frac{2-2R^2}{R^2(1+R^2)}\widehat{\varphi_2}(n) \\ \left(\partial_R^2 + \frac{1}{R}\partial_R - \frac{n^2}{R^2} - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)}\right)\widehat{\varphi_2}(n) + in\frac{2-2R^2}{R^2(1+R^2)}\widehat{\varphi_1}(n) \end{pmatrix}. \tag{3.23}$$

We can write this operator as a 2×2 matrix

$$\begin{aligned} \mathfrak{L}_n\,\widehat{\varphi}(n) &:= \mathfrak{A}_n \begin{pmatrix} \widehat{\varphi_1}(n,t,R) \\ \widehat{\varphi_2}(n,t,R) \end{pmatrix} \\ \mathfrak{A}_n &:= \begin{pmatrix} \partial_R^2 + \frac{1}{R}\partial_R - \frac{n^2}{R^2} - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)} & -in\frac{2-2R^2}{R^2(1+R^2)} \\ in\frac{2-2R^2}{R^2(1+R^2)} & \partial_R^2 + \frac{1}{R}\partial_R - \frac{n^2}{R^2} - \frac{R^4-6R^2+1}{R^2(R^4+2R^2+1)} \end{pmatrix} \end{aligned} \tag{3.24}$$

3.2. **Symmetries and zero modes.** We now exhibit nonzero solutions of $\mathfrak{L}\,\varphi = 0$ in relation to the symmetries of the nonlinear problem. First, the dilation symmetry $s \mapsto Q(e^s R)$ yields

$$\partial_{s=0}\,Q(e^s R) = \frac{R}{1+R^2} =: \psi(R)$$

which leads to two modes

$$\mathfrak{L}\begin{pmatrix}\psi \\ 0\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}, \qquad \mathfrak{L}\begin{pmatrix}0 \\ \psi\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix} \tag{3.25}$$

This is the resonance mode of the linearized equivariant model [16, 18], i.e., of $\mathfrak{A}_0$.

To determine the zero modes obtained from space translations, we first write the ground state harmonic map $\Xi(R,\theta)$ in the form

$$\begin{aligned}\Xi(R,\theta) &= (\cos\theta\sin Q, \sin\theta\sin Q, \cos Q)\\ &= (\cos\theta\frac{2R}{1+R^2}, \sin\theta\frac{2R}{1+R^2}, \frac{1-R^2}{1+R^2}).\end{aligned} \tag{3.26}$$

The two translations are given by

$$\partial_1 = \cos\theta\,\partial_R - \frac{\sin\theta}{R}\partial_\theta, \quad \partial_2 = \sin\theta\,\partial_R + \frac{\cos\theta}{R}\partial_\theta. \tag{3.27}$$

We have the following derivatives on Ξ (setting $Q(R) = 2\arctan R$)

$$\begin{aligned}\partial_R\Xi &= \left(\cos\theta\frac{2-2R^2}{(1+R^2)^2}, \sin\theta\frac{2-2R^2}{(1+R^2)^2}, -\frac{4R}{(1+R^2)^2}\right)\\ &= \frac{2}{1+R^2}(\cos\theta\cos Q, \sin\theta\cos Q, -\sin Q) = \frac{2}{1+R^2}E_1,\\ \partial_\theta\Xi &= \left(-\sin\theta\frac{2R}{1+R^2}, \cos\theta\frac{2R}{1+R^2}, 0\right) = \frac{2R}{1+R^2}E_2\end{aligned} \tag{3.28}$$

and thus

$$\begin{aligned}\partial_1\Xi &= \cos\theta\,\partial_R\Xi - \frac{\sin\theta}{R}\partial_\theta\Xi = \frac{2}{1+R^2}\cos\theta E_1 - \sin\theta\frac{2}{1+R^2}E_2,\\ \partial_2\Xi &= \sin\theta\,\partial_R\Xi + \frac{\cos\theta}{R}\partial_\theta\Xi = \frac{2}{1+R^2}\sin\theta E_1 + \frac{2}{1+R^2}\cos\theta E_2.\end{aligned} \tag{3.29}$$

Therefore, two linearly independent solutions of $\mathcal{L}\varphi = 0$ corresponding to all space translations are given by

$$\begin{aligned}(\varphi_1(R), \varphi_2(R)) &= \left(\cos\theta\frac{2}{1+R^2}, -\sin\theta\frac{2}{1+R^2}\right),\\ (\varphi_1(R), \varphi_2(R)) &= \left(\frac{2}{1+R^2}\sin\theta, \frac{2}{1+R^2}\cos\theta\right).\end{aligned} \tag{3.30}$$

On the level of fixed angular momenta these solutions generate the following zero modes

$$\mathfrak{A}_1\begin{pmatrix}(1+R^2)^{-1}\\ i(1+R^2)^{-1}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \qquad \mathfrak{A}_{-1}\begin{pmatrix}(1+R^2)^{-1}\\ -i(1+R^2)^{-1}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Next, we turn our attention to Lorentz transforms which are of the form

$$L_\alpha: \quad (t, x_1, x_2) \mapsto (t\cosh\alpha + x_1\sinh\alpha, t\sinh\alpha + x_1\cosh\alpha, x_2), \quad \alpha\in\mathbb{R}$$

whence $Q(R)$ transforms into

$$Q_\alpha(t,x_1,x_2) = Q(R_\alpha(t,x_1,x_2)), \quad R_\alpha(t,x_1,x_2) := \sqrt{(t\sinh\alpha + x_1\cosh\alpha)^2 + x_2^2}$$

with derivative

$$\partial_\alpha\Big|_{\alpha=0} Q_\alpha(t,x_1,x_2) = \frac{2x_1 t}{R(1+R^2)} = \frac{2t\cos\theta}{1+R^2}$$

The stationary equivariant wave map associated with azimuth angle $Q(R)$ is

$$\begin{aligned} u(x_1, x_2) &= (\cos\theta \sin Q(R), \sin\theta \sin Q(R), \cos Q(R)) \\ &= \left(\frac{x_1}{R}\sin Q(R), \frac{x_2}{R}\sin Q(R), \cos Q(R)\right) \end{aligned} \tag{3.31}$$

which transforms into

$$u_\alpha(t, x_1, x_2) = \left(\frac{t\sinh\alpha + x_1\cosh\alpha}{R_\alpha}\sin Q(R_\alpha), \frac{x_2}{R_\alpha}\sin Q(R_\alpha), \cos Q(R_\alpha)\right)$$

Differentiating in α we obtain

$$\begin{aligned} \partial_\alpha\Big|_{\alpha=0} u_\alpha(t, x_1, x_2) &= t\left(\frac{x_2^2}{R^3}\sin Q + \frac{2x_1^2}{R^2(1+R^2)}\cos Q, -\frac{x_1x_2}{R^3}\sin Q + \frac{2x_1x_2}{R^2(1+R^2)}\cos Q, -\frac{2x_1}{R(1+R^2)}\sin Q\right) \\ &= t\left(\frac{\sin^2\theta}{R}\sin Q + \frac{2\cos^2\theta}{1+R^2}\cos Q, -\frac{\sin\theta\cos\theta}{R}\sin Q + \frac{2\sin\theta\cos\theta}{1+R^2}\cos Q, -\frac{2\cos\theta}{1+R^2}\sin Q\right) \\ &= t\frac{2\cos\theta}{1+R^2}E_1 - t\frac{\sin\theta\sin Q}{R}E_2 = t\frac{2\cos\theta}{1+R^2}E_1 - t\frac{2\sin\theta}{1+R^2}E_2 \end{aligned}$$

This is exactly $t\varphi$ where φ agrees with the first line of (3.30). The second line of (3.30) is obtained by the Lorentz transform in (t, x_2). Thus, we do not obtain any new solutions to $\mathcal{L}\varphi = 0$ from Lorentz transforms.

Next we consider the rotations acting on the target. The linearized kernels corresponding to the three rotations are given by

$$(-\sin\theta\sin Q, \cos\theta\sin Q, 0), \quad (0, -\cos Q, \sin\theta\sin Q), \quad (\cos Q, 0, -\cos\theta\sin Q). \tag{3.32}$$

These are obtained by acting on u as in (3.31) with the infinitesimal rotation matrices

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

respectively. We expand these tangent maps in the basis (E_1, E_2, Φ). We have

$$\begin{aligned} (-\sin\theta\sin Q, \cos\theta\sin Q, 0) &= \sin Q E_2 = \frac{2R}{1+R^2}E_2, \\ (0, -\cos Q, \sin\theta\sin Q) &= -\sin\theta E_1 - \cos\theta\cos Q E_2 = -\sin\theta E_1 - \cos\theta\frac{1-R^2}{1+R^2}E_2, \\ (-\cos Q, 0, \cos\theta\sin Q) &= \cos\theta E_1 - \sin\theta\cos Q E_2 = \cos\theta E_1 - \sin\theta\frac{1-R^2}{1+R^2}E_2. \end{aligned} \tag{3.33}$$

The first line is one of the two modes from (3.25). The others are new 0 modes which we write as

$$\varphi = \left(\sin\theta, \cos\theta\frac{1-R^2}{1+R^2}\right), \quad \varphi = \left(\cos\theta, -\sin\theta\frac{1-R^2}{1+R^2}\right) \tag{3.34}$$

Finally we consider the space rotation acting on the domain $\mathbb{R}_x^2 \times \mathbb{R}_t^1$. Since we are only concerned with the space rotation, we consider the stationary wave map (3.31). If we represent the rotation r_α by the matrix

$$r_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

then

$$(r_\alpha u)(x_1,x_2) = \left(\frac{\cos\alpha x_1 - \sin\alpha x_2}{R}\sin Q(R), \frac{\sin\alpha x_1 + \cos\alpha x_2}{R}\sin Q(R), \cos Q(R)\right)$$

$$\Rightarrow \quad \frac{d}{d\alpha}(r_\alpha u)|_{\alpha=0} = \left(-\frac{x_2}{R}\sin Q(R), \frac{x_1}{R}\sin Q(R), 0\right) = \sin Q E_2,$$

which was already considered in (3.33). Therefore we do not obtain any new solutions to $\mathcal{L}\varphi = 0$ from the space rotation in $\mathbb{R}^2_x \times \mathbb{R}^1_t$.

Next, we diagonalize the operator from (3.24) and thus relate the 0 modes from (3.25), (3.30), and (3.34) to eigenvalues or resonances of certain scalar self-adjoint Schrödinger operators in $L^2(R\,dR)$. In this way we shall moreover see that there are no other 0 modes than those generated from symmetries.

3.3. **Diagonalizing the linear operators.** Let

$$f_n(R) := \frac{n^2}{R^2} + \frac{R^4 - 6R^2 + 1}{R^2(R^4 + 2R^2 + 1)}, \quad g_n(R) := n\frac{2 - 2R^2}{R^2(R^2 + 1)} \tag{3.35}$$

Writing the matrix (3.24) in the form

$$\mathfrak{A}_n := \begin{pmatrix} a & -ib \\ ib & a \end{pmatrix}, \quad a := \partial_R^2 + \frac{1}{R}\partial_R - f_n(R), \quad b := g_n(R), \tag{3.36}$$

we change variables to reduce it to a diagonal matrix:

$$\mathfrak{D}_n := \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} = P\mathfrak{A}_n P^{-1}, \quad P = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{i}{2} \end{pmatrix}$$

Coming back to (3.24), we transform $(\widehat{\varphi}_1, \widehat{\varphi}_2)$ into

$$\varepsilon := (\varepsilon_+, \varepsilon_-)^T := (\widehat{\varphi}_1 - i\widehat{\varphi}_2, \widehat{\varphi}_1 + i\widehat{\varphi}_2)^T, \tag{3.37}$$

so that the operator acting on ε is given by

$$P\mathfrak{A}_n\begin{pmatrix}\widehat{\varphi}_1\\ \widehat{\varphi}_2\end{pmatrix} = \begin{pmatrix} \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) + g_n(R) & 0 \\ 0 & \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) - g_n(R) \end{pmatrix}\begin{pmatrix}\varepsilon_+ \\ \varepsilon_-\end{pmatrix} \tag{3.38}$$

We denote the operators on the diagonal

$$H_n^+ := \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) + g_n(R), \quad H_n^- := \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) - g_n(R). \tag{3.39}$$

Next, we transform the 0 modes from (3.25), (3.30), and (3.34) which we found above according to (3.37). This yields

$$H_0^\pm \frac{R}{1+R^2} = 0, \quad H_1^+\frac{1}{1+R^2} = H_{-1}^-\frac{1}{1+R^2} = 0, \quad H_1^-\frac{R^2}{1+R^2} = H_{-1}^+\frac{R^2}{1+R^2} = 0 \tag{3.40}$$

These relations will play an important role in our analysis, especially the middle one due to $\frac{1}{1+R^2} \in L^2(R\,dR)$. In other words, 0 is an eigenvalue of $H_1^+ = H_{-1}^-$. In contrast, for $H_0^\pm$ and $H_{-1}^+ = H_1^-$ we only have a zero energy resonance since the associated functions are not in $L^2(R\,dR)$. While these resonances affect the spectral theory, cf. Section 4, they do not appear explicitly in the dynamical analysis.

3.4. **Determination of perturbative corrections and the precise nonlinearity.** In this section we compute the precise nonlinearity for the equation of (φ_1, φ_2). Recall (3.7), viz.

$$\Psi := \Phi + \Pi_{\Phi^\perp}\varphi + \mathfrak{q}$$

The term $\mathfrak{q}$ is not uniquely determined from this ansatz. We now choose it to be parallel to Φ itself. This can always be done provided φ is small. Thus, we set

$$\Psi := \Phi + \Pi_{\Phi^\perp}\varphi + a\left(\Pi_{\Phi^\perp}\varphi\right)\Phi, \quad a\left(\Pi_{\Phi^\perp}\varphi\right) = \sqrt{1 - |\Pi_{\Phi^\perp}\varphi|^2} - 1. \tag{3.41}$$

Equation (3.9) then transforms into the following one

$$\Box\Pi_{\Phi^\perp}\varphi + \left(|\nabla\Phi|^2 - |\Phi_t|^2\right)\Pi_{\Phi^\perp}\varphi + 2\left(\nabla\Phi\cdot\nabla\Pi_{\Phi^\perp}\varphi - \Phi_t\cdot(\Pi_{\Phi^\perp}\varphi)_t\right)\Phi := N_p, \tag{3.42}$$

with right-hand side

$$\begin{aligned} N_p =& \left(|\partial_t\Psi|^2 - |\nabla\Psi|^2\right) a\left(\Pi_{\Phi^\perp}\varphi\right)\Phi + \left(|\partial_t\left(a(\Pi_{\Phi^\perp}\varphi)\Phi\right)|^2 - |\nabla\left(a(\Pi_{\Phi^\perp}\varphi)\Phi\right)|^2\right)\Phi \\ &+ \left(\partial_t\left(\Phi + \Pi_{\Phi^\perp}\varphi\right)\cdot\partial_t\left(a(\Pi_{\Phi^\perp}\varphi)\Phi\right) - \nabla\left(\Phi + \Pi_{\Phi^\perp}\varphi\right)\cdot\nabla\left(a(\Pi_{\Phi^\perp}\varphi)\Phi\right)\right)\Phi \\ &+ \left(|\partial_t\Psi|^2 - |\partial_t\Phi|^2 - |\nabla\Psi|^2 + |\nabla\Phi|^2\right)\Pi_{\Phi^\perp}\varphi - \Box\left(a(\Pi_{\Phi^\perp}\varphi)\Phi\right). \end{aligned} \tag{3.43}$$

Next, we expand N_p in the frame Φ, E_1, E_2 and determine the coefficients for E_1 and E_2. For the latter it suffices to retain the terms in the last two lines in (3.43), as the others are parallel to Φ. In fact, of these two final terms we can discard another term parallel to Φ, and expand the following expression:

$$\left(|\partial_t\Psi|^2 - |\partial_t\Phi|^2 - |\nabla\Psi|^2 + |\nabla\Phi|^2\right)\Pi_{\Phi^\perp}\varphi - 2\nabla\left(a(\Pi_{\Phi^\perp}\varphi)\right)\cdot\nabla\Phi + 2\left(a(\Pi_{\Phi^\perp}\varphi)\right)_t\Phi_t - a(\Pi_{\Phi^\perp}\varphi)\Box\Phi. \tag{3.44}$$

Now we compute each term in (3.44):

$$\Psi_t = U_tE_1 + \varphi_{1,t}E_1 - \varphi_1U_t\Phi + \varphi_{2,t}E_2 + a(\Pi_{\Phi^\perp}\varphi)U_tE_1 + \left(a(\Pi_{\Phi^\perp}\varphi)\right)_t\Phi, \tag{3.45}$$

which yields

$$\begin{aligned} |\partial_t\Psi|^2 - |\partial_t\Phi|^2 =& \left(\left(a(\Pi_{\Phi^\perp}\varphi)\right)_t - U_t\varphi_1\right)^2 + \varphi_{2,t}^2 \\ &+ \left(\varphi_{1,t} + a(\Pi_{\Phi^\perp}\varphi)U_t\right)^2 + 2U_t\cdot\left(\varphi_{1,t} + a(\Pi_{\Phi^\perp}\varphi)U_t\right). \end{aligned} \tag{3.46}$$

For $\nabla\Psi$, we obtain

$$\begin{aligned} \nabla\Psi =& \nabla\Phi + \nabla\left(\Pi_{\Phi^\perp}\varphi\right) + \nabla\left(a(\Pi_{\Phi^\perp}\varphi)\Phi\right) \\ =& \begin{pmatrix} U_rE_1 \\ \frac{\sin U}{r}E_2 \end{pmatrix} + \begin{pmatrix} \partial_r\left(a(\Pi_{\Phi^\perp}\varphi)\right)\Phi \\ \frac{1}{r}\partial_\theta\left(a(\Pi_{\Phi^\perp}\varphi)\right)\Phi \end{pmatrix} + a(\Pi_{\Phi^\perp}\varphi)\begin{pmatrix} U_rE_1 \\ \frac{\sin U}{r}E_2 \end{pmatrix} \\ &+ \begin{pmatrix} \varphi_{1,r}E_1 - \varphi_1U_r\Phi + \varphi_{2,r}E_2 \\ \frac{1}{r}\varphi_{1,\theta}E_1 + \frac{\cos U}{r}\varphi_1E_2 + \frac{1}{r}\varphi_{2,\theta}E_2 - \frac{1}{r}\varphi_2\left(\sin U\Phi + \cos UE_1\right) \end{pmatrix}, \end{aligned} \tag{3.47}$$

which gives

$$
\begin{aligned}
& -|\nabla\Psi|^2+|\nabla\Phi|^2\\
=& -\left(2a(\Pi_{\Phi^\perp}\varphi)+(a(\Pi_{\Phi^\perp}\varphi))^2\right)\left(U_r^2+\frac{\sin^2 U}{r^2}\right)-\left((a(\Pi_{\Phi^\perp}\varphi))_r^2+\frac{1}{r^2}\,(a(\Pi_{\Phi^\perp}\varphi))_\theta^2\right)\\
& -\varphi_{1,r}^2-U_r^2\varphi_1^2-\varphi_{2,r}^2-\frac{1}{r^2}\,(\varphi_{1,\theta}-\cos U\varphi_2)^2-\frac{1}{r^2}\,(\cos U\varphi_1+\varphi_{2,\theta})^2-\frac{\sin^2 U}{r^2}\varphi_2^2\\
& -2U_r\varphi_{1,r}-\frac{2\sin U}{r^2}\,(\cos U\varphi_1+\varphi_{2,\theta})+2\,(a(\Pi_{\Phi^\perp}\varphi))_r\,U_r\varphi_1+\frac{2}{r^2}\,(a(\Pi_{\Phi^\perp}\varphi))_\theta\sin U\varphi_2\\
& -2a(\Pi_{\Phi^\perp}\varphi)U_r\varphi_{1,r}-2\frac{a(\Pi_{\Phi^\perp}\varphi)}{r^2}\sin U\,(\cos U\varphi_1+\varphi_{2,\theta})\,.
\end{aligned}
\tag{3.48}
$$

Combining (3.46) and (3.48), we obtain

$$
\begin{aligned}
& |\partial_t\Psi|^2-|\partial_t\Phi|^2-|\nabla\Psi|^2+|\nabla\Phi|^2\\
=& -\left(2a(\Pi_{\Phi^\perp}\varphi)+(a(\Pi_{\Phi^\perp}\varphi))^2\right)\left(U_r^2-U_t^2+\frac{\sin^2 U}{r^2}\right)+(a(\Pi_{\Phi^\perp}\varphi))_t^2-(a(\Pi_{\Phi^\perp}\varphi))_r^2-\frac{1}{r^2}\,(a(\Pi_{\Phi^\perp}\varphi))_\theta^2\\
& +\varphi_{1,t}^2-\varphi_{1,r}^2-\frac{1}{r^2}\varphi_{1,\theta}^2+\varphi_{2,t}^2-\varphi_{2,r}^2-\frac{1}{r^2}\varphi_{2,\theta}^2-\frac{1}{r^2}\varphi_2^2+2\varphi_1\left((a(\Pi_{\Phi^\perp}\varphi))_r\,U_r-(a(\Pi_{\Phi^\perp}\varphi))_t\,U_t\right)\\
& +\varphi_1\left(U_t^2-U_r^2-\frac{\cos^2 U}{r^2}\right)\varphi_1^2+2\,(1+a(\Pi_{\Phi^\perp}\varphi))\left(U_t\varphi_{1,t}-U_r\varphi_{1,r}-\frac{\sin U}{r^2}\cos U\varphi_1\right)\\
& +\frac{2\cos U}{r^2}\,(\varphi_{1,\theta}\varphi_2-\varphi_1\varphi_{2,\theta})-\frac{2\sin U}{r^2}\varphi_{2,\theta}+\frac{2\sin U}{r^2}\left((a(\Pi_{\Phi^\perp}\varphi))_\theta\,\varphi_2-a(\Pi_{\Phi^\perp}\varphi)\varphi_{2,\theta}\right).
\end{aligned}
\tag{3.49}
$$

We write $-2\nabla\,(a(\Pi_{\Phi^\perp}\varphi))\cdot\nabla\Phi$ in the form

$$
\begin{aligned}
& -2\nabla\,(a(\Pi_{\Phi^\perp}\varphi))\cdot\nabla\Phi\\
=&\frac{2\,(\Pi_{\Phi^\perp}\varphi\cdot\nabla\Pi_{\Phi^\perp}\varphi)\cdot\nabla\Phi}{\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\\
=&\frac{2}{\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\left(U_r\,(\varphi_1\varphi_{1,r}+\varphi_2\varphi_{2,r})\,E_1+\frac{\sin U}{r^2}\,(\varphi_1\varphi_{1,\theta}+\varphi_2\varphi_{2,\theta})\,E_2\right)
\end{aligned}
\tag{3.50}
$$

and $2\,(a(\Pi_{\Phi^\perp}\varphi))_t\,\Phi_t$ turns into

$$
2\,(a(\Pi_{\Phi^\perp}\varphi))_t\,\Phi_t=-2\frac{\Pi_{\Phi^\perp}\varphi\cdot(\Pi_{\Phi^\perp}\varphi)_t\,\Phi_t}{\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}=-\frac{2}{\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\,(\varphi_1\varphi_{1,t}+\varphi_2\varphi_{2,t})\,U_tE_1. \tag{3.51}
$$

The final term in (3.44) can be written as

$$
a(\Pi_{\Phi^\perp}\varphi)\left(|\nabla\Phi|^2-|\Phi_t|^2\right)\Phi=a(\Pi_{\Phi^\perp}\varphi)\left(U_r^2-U_t^2+\frac{\sin^2 U}{r^2}\right)\Phi, \tag{3.52}
$$

since Φ is a wave map. Now we can write the equations (3.12)–(3.13) in the form

$$\begin{aligned}&\Box\varphi_1-\frac{\cos 2U}{r^2}\varphi_1-\frac{2\cos U}{r^2}\varphi_{2,\theta}\\=&\left(|\Psi_t|^2-|\Phi_t^2|-|\nabla\Psi|^2+|\nabla\Phi|^2\right)\varphi_1\\&+\frac{2}{\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\left(U_r(\varphi_1\varphi_{1,r}+\varphi_2\varphi_{2,r})-U_t(\varphi_1\varphi_{1,t}+\varphi_2\varphi_{2,t})\right):=N(\varphi_1),\end{aligned}\tag{3.53}$$

and

$$\begin{aligned}&\Box\varphi_2-\frac{\cos^2 U}{r^2}\varphi_2+(U_r^2-U_t^2)\varphi_2+\frac{2\cos U}{r^2}\varphi_{1,\theta}\\=&\left(|\Psi_t|^2-|\Phi_t^2|-|\nabla\Psi|^2+|\nabla\Phi|^2\right)\varphi_2\\&+\frac{2\sin U}{r^2\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\left(\varphi_1\varphi_{1,\theta}+\varphi_2\varphi_{2,\theta}\right):=N(\varphi_2).\end{aligned}\tag{3.54}$$

In (3.53)-(3.54), $|\Psi_t|^2-|\Phi_t^2|-|\nabla\Psi|^2+|\nabla\Phi|^2$ is given by (3.49). To write down the exact equation in (τ,R)-variable, we need to put the error between U and $2\arctan R$ as well as the term $-\frac{(1+\nu)^2}{\nu^2\tau^2}\frac{4R^2}{R^4+2R^2+1}$ on the right hand side. Let the background equivariant solution be given by $U=Q+\epsilon$. We start by computing

$$\begin{aligned}\cos 2U-\cos 2Q=&-2\sin(U+Q)\sin\epsilon=-2\sin(2Q+\epsilon)\sin\epsilon\\\cos^2 U-\cos^2 Q=&\frac{1}{2}(\cos 2U-\cos 2Q)=-\sin(2Q+\epsilon)\sin\epsilon\\2\cos U-2\cos Q=&-4\sin\left(\frac{2Q+\epsilon}{2}\right)\sin\frac{\epsilon}{2}\end{aligned}\tag{3.55}$$

For the difference from the term $U_r^2-U_t^2$, we arrive at

$$\begin{aligned}&U_r^2-U_t^2-Q_r^2+Q_t^2\\=&(Q+\epsilon)_r^2-Q_r^2-(Q+\epsilon)_t^2+Q_t^2=2Q_r\epsilon_r+\epsilon_r^2-2Q_t\epsilon_t-\epsilon_t^2\\=&\frac{2\lambda^2}{1+R^2}\partial_R\epsilon+\lambda^2(\partial_R\epsilon)^2\\&-2\lambda^2\left(\partial_\tau Q+\frac{\lambda'(\tau)}{\lambda}R\partial_R Q\right)\left(\partial_\tau\epsilon+\frac{\lambda'(\tau)}{\lambda}R\partial_R\epsilon\right)-\lambda^2\left(\partial_\tau\epsilon+\frac{\lambda'(\tau)}{\lambda}R\partial_R\epsilon\right)^2\\=&\lambda^2\left(\frac{2\partial_R\epsilon}{1+R^2}+(\partial_R\epsilon)^2-\frac{\lambda'(\tau)}{\lambda}\frac{8R}{1+R^2}\left(\partial_\tau\epsilon+\frac{\lambda'(\tau)}{\lambda}R\partial_R\epsilon\right)-\left(\partial_\tau\epsilon+\frac{\lambda'(\tau)}{\lambda}R\partial_R\epsilon\right)^2\right).\end{aligned}\tag{3.56}$$

Therefore, equations (3.53)–(3.54) take the form

$$\begin{aligned}&\Box\varphi_1-\frac{\cos 2Q}{r^2}\varphi_1-\frac{2\cos Q}{r^2}\varphi_{2,\theta}\\=&N(\varphi_1)-\frac{2\sin(2Q+\epsilon)\sin\epsilon}{r^2}\varphi_1-\frac{4\sin\left(\frac{2Q+\epsilon}{2}\right)\sin\frac{\epsilon}{2}}{r^2}\varphi_{2,\theta}=:\mathfrak{N}(\varphi_1)\end{aligned}\tag{3.57}$$

and

$$\begin{aligned}
&\Box\varphi_2 - \frac{\cos^2 Q}{r^2}\varphi_2 + \left(Q_r^2 - Q_t^2\right)\varphi_2 + \frac{2\cos Q}{r^2}\varphi_{1,\theta} \\
=&N(\varphi_2) - \frac{\sin(2Q+\epsilon)\sin\epsilon}{r^2}\varphi_2 + \frac{4\sin\left(\frac{2Q+\epsilon}{2}\right)\sin\frac{\epsilon}{2}}{r^2}\varphi_{1,\theta} \\
&-\lambda^2\left(\frac{2\partial_R\epsilon}{1+R^2} + (\partial_R\epsilon)^2 - \frac{\lambda'(\tau)}{\lambda}\frac{8R}{1+R^2}\left(\partial_\tau\epsilon + \frac{\lambda'(\tau)}{\lambda}R\partial_R\epsilon\right) - \left(\partial_\tau\epsilon + \frac{\lambda'(\tau)}{\lambda}R\partial_R\epsilon\right)^2\right)\varphi_2 =: \mathfrak{N}(\varphi_2).
\end{aligned} \tag{3.58}$$

Passing to (τ, R)-variables, we obtain for $\varphi =: (\varphi_1, \varphi_2)^T$,

$$\begin{aligned}
-\left(\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)^2 + \frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\right)\varphi + \mathfrak{L}\varphi = \mathfrak{N}(\varphi), \\
\mathfrak{N}(\varphi) =: \lambda^{-2}\begin{pmatrix} \mathfrak{N}(\varphi_1) \\ \mathfrak{N}(\varphi_2) + \frac{(1+\nu)^2\lambda^2}{\nu^2\tau^2}\frac{4R^2}{1+2R^2+R^4}\varphi_2 \end{pmatrix}.
\end{aligned} \tag{3.59}$$

The nonlinearity takes the form

$$\begin{aligned}
\lambda^{-2}\mathfrak{N}(\varphi_1) =&\lambda^{-2}N(\varphi_1) - \frac{2\sin(2Q+\epsilon)\sin\epsilon}{R^2}\varphi_1 - \frac{4\sin\left(\frac{2Q+\epsilon}{2}\right)\sin\frac{\epsilon}{2}}{R^2}\varphi_{2,\theta} \\
\lambda^{-2}\mathfrak{N}(\varphi_2) =&\lambda^{-2}N(\varphi_2) - \frac{\sin(2Q+\epsilon)\sin\epsilon}{R^2}\varphi_2 + \frac{4\sin\left(\frac{2Q+\epsilon}{2}\right)\sin\frac{\epsilon}{2}}{R^2}\varphi_{1,\theta} \\
&-\left(\frac{2\partial_R\epsilon}{1+R^2} + (\partial_R\epsilon)^2 - \frac{\lambda'(\tau)}{\lambda}\frac{8R}{1+R^2}\left(\partial_\tau\epsilon + \frac{\lambda'(\tau)}{\lambda}R\partial_R\epsilon\right) - \left(\partial_\tau\epsilon + \frac{\lambda'(\tau)}{\lambda}R\partial_R\epsilon\right)^2\right)\varphi_2.
\end{aligned} \tag{3.60}$$

We will effectively solve the equations (3.59) in the backward light cone centered at the singularity by means of an iterative scheme, solving linear inhomogeneous problems consecutively. Assuming the data to be C^∞ in the light cone (which is not really necessary), the iterates $\Pi_{\Phi^\perp}\varphi^{(j)}$, say, will also be C^∞ when interpreted as functions on $I \times \mathbb{R}^2$. Now write this as

$$\Pi_{\Phi^\perp}\varphi = \varphi_1\begin{pmatrix} \cos\theta\cos U \\ \sin\theta\cos U \\ -\sin U \end{pmatrix} + \varphi_2\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

Expanding the functions $\varphi_j = \sum_n \hat{\varphi}_j(n)e^{in\theta}$, $j = 1, 2$, we infer the conditions that the functions

$$\hat{\varphi}_1(n)\sin U \cdot e^{in\theta} \in C^\infty, \quad (\hat{\varphi}_1(n)\cos U + i\hat{\varphi}_2(n))\, e^{i(n+1)\theta} \in C^\infty,$$

where we have summarized the condition for the first two components by reverting to complex notation. These imply that $\hat{\varphi}_1(n)$ has to vanish at least of order $|n| - 1$ at the origin and can be expanded in a power series in R with all powers congruent 2 modulo $|n| - 1$, while $\hat{\varphi}_1(n) + i\hat{\varphi}_2(n) = \varepsilon_-(n)$ has to vanish to order at least $|n + 1|$, with power series around $R = 0$ only containing powers congruent 2 modulo $|n + 1|$. Using the fact that $\overline{\hat{\varphi}_1(n) - i\hat{\varphi}_2(n)} = \hat{\varphi}_1(-n) + i\hat{\varphi}_2(-n)$ we know that $\varepsilon_+(n)$ may not vanish at $R = 0$ only for $n = 1$. This of course implies that the source terms for the equations of $\varepsilon_+(n), \varepsilon_-(n)$ need to have the Taylor expansions around $R = 0$ with the same properties (since the operators $H_n^{\pm}$ kill the lowest order term $R^{n\mp 1}$). Thus recalling (3.37), (3.38) and writing the equations for $\varepsilon_+(n), \varepsilon_-(n)$ in the form

$$\left(-\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 - \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right) + H_n^{\pm}\right)\varepsilon_\pm(n) = F_\pm, \tag{3.61}$$

the right-hand sides will have Taylor expansions around $R = 0$ with the same properties. Crucially there is no reason at all for the individual terms that we derive later on in the formula for $F_\pm$ to have these properties at $R = 0$, but we do know a priori that *their sum* will have it.

4. Spectral analysis of the linear operators

This section is devoted entirely to the spectral analysis of the linear operators on the diagonal on the left-hand side of (3.38). This refers to the determination of the spectral measures and the generalized Fourier transform of each of these operators for all angular momenta $n \in \mathbb{Z}$, the case $n = 0$ already having been dealt with in the equivariant case [16, 18]. The most demanding aspect will be analyzing the asymptotic behavior as $n \to \pm\infty$. As in [4, 5] this will be accomplished by means of a Liouville-Green transform applied to an equivalent semiclassical problem. However, we will need to go considerably further than these aforementioned references. We begin with elementary properties of the shape of the spectrum and the behavior at zero energy, followed by the spectral analysis of $n = \pm 1$, and finally we present the more delicate turning point analysis for $|n| \geq 2$, viewed as a semi-classical problem.

4.1. **Selfadjointness and resolvent.** The operators arising in (3.38) are

$$H_n^+ := \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) + g_n(R), \quad H_n^- := \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) - g_n(R). \tag{4.1}$$

where

$$\begin{aligned} f_n(R) &= \frac{n^2+1}{R^2} - \frac{8}{(R^2+1)^2}, \quad g_n(R) = \frac{2n}{R^2} - \frac{4n}{R^2+1}, \\ -f_n(R) + g_n(R) &= -\frac{(n-1)^2}{R^2} - \frac{4n}{R^2+1} + \frac{8}{(R^2+1)^2}, \\ -f_n(R) - g_n(R) &= -\frac{(n+1)^2}{R^2} + \frac{4n}{R^2+1} + \frac{8}{(R^2+1)^2} \end{aligned}$$

If $n = 0$ (linearized equivariant operator) we will simply write H_0 in place of $H_0^+ = H_0^-$. These operators are symmetric in $L^2(R\,dR)$ with domain given by $C^2((0,\infty))$ functions of compact support in $(0,\infty)$. In the following we will use standard terminology and techniques from spectral theory, see [9, 11, 23, 30]. For that purpose it is convenient to also consider the conjugations of $H_n^{\pm}$ by the weights $R^{\frac{1}{2}}$, viz.

$$\begin{aligned} \mathcal{H}_n^+ &= R^{\frac{1}{2}} H_n^+ R^{-\frac{1}{2}} = \partial_R^2 + \frac{1}{4R^2} - \frac{(n-1)^2}{R^2} - \frac{4n}{R^2+1} + \frac{8}{(R^2+1)^2}, \\ \mathcal{H}_n^- &= R^{\frac{1}{2}} H_n^- R^{-\frac{1}{2}} = \partial_R^2 + \frac{1}{4R^2} - \frac{(n+1)^2}{R^2} + \frac{4n}{R^2+1} + \frac{8}{(R^2+1)^2}. \end{aligned} \tag{4.2}$$

These operators are clearly symmetric in $L^2(dR)$ over the same dense family as before.

Proposition 4.1. *$\mathcal{H}_n^+$ and $\mathcal{H}_n^- = \mathcal{H}_{-n}^+$ with $n \in \mathbb{N}$ are nonpositive symmetric operators in the Hilbert space $L^2((0,\infty))$, with domain given by C^2 functions compactly supported in $(0,\infty)$. Each $\mathcal{H}_n^+$ is strongly singular at $R = 0$ and in the limit point case at $R = \infty$. Moreover, for all $n \neq 1$ these operators are in the limit point case at $R = 0$, whereas $\mathcal{H}_1^+$ is limit circle at the origin. Thus, $\mathcal{H}_n^+$ is essentially self-adjoint for $n \neq 1$, while $\mathcal{H}_1^+$ is not. We denote by $\mathcal{H}_n^+ = \mathcal{H}_{-n}^-$, the self-adjoint operator in $L^2(dR)$ given by the Friedrichs extensions.*

The spectrum of each $\mathcal{H}_n^+$ is $(-\infty, 0]$, which is also the essential spectrum. At the threshold 0 one has the following properties: $\mathcal{H}_0^+$ has a resonance with state $R^{\frac{3}{2}}/(1+R^2)$, $\mathcal{H}_1^+ = \mathcal{H}_{-1}^-$ has an eigenvalue with eigenfunction $\sqrt{R}/(1+R^2)$, and $\mathcal{H}_{-1}^+ = \mathcal{H}_1^-$ has a resonance with state $R^{\frac{5}{2}}/(1+R^2)$. These are precisely the modes originating from symmetries, see (3.40).

Finally, the resolvent kernel of $(-H_n^+ + 1)^{-1}$ satisfies, for any $f \in C([0,\infty))$ with compact support,

$$((-H_n^+ + 1)^{-1} f)(R) = -R^{-\frac{1}{2}} \int_0^\infty G_n(R, R'; -1)\sqrt{R'}\, f(R')\, dR' = cR^{|n-1|}(1 + O(R^2)) \tag{4.3}$$

as $R \to 0$. Here G_n is the resolvent kernel of $(\mathcal{H}_n^+ - 1)^{-1}$. In particular, unless $n = 1$, the left-hand side vanishes at $R = 0$.

Proof. Let $f \in C_0^2((0,\infty))$ with compact support and note that

$$\begin{aligned}\langle H_n^+ f, f\rangle &= \int_0^\infty \partial_R (R\partial_R f)\, f\, dR - \int_0^\infty \frac{(n-1)^2}{R} f^2\, dR + \int_0^\infty \frac{8R}{(R^2+1)^2} f^2\, dR - \int_0^\infty \frac{4nR}{R^2+1} f^2\, dR \\ &\leq -\int_0^\infty (f'(R))^2 R\, dR - \int_0^\infty \frac{(n-1)^2}{R} f^2\, dR < 0\end{aligned}$$

for all $n \geq 2$. Similarly, for H_n^- with $n \geq 2$ we compute that

$$\langle H_n^+ f, f\rangle = -\int_0^\infty (f'(R))^2 R\, dR + \int_0^\infty U(R) f^2(R) R\, dR < 0$$

since

$$\begin{aligned}U(R) &= -\frac{(n+1)^2}{R^2} + \frac{4n}{R^2+1} + \frac{8}{(R^2+1)^2} \\ &= \frac{-(n+1)^2 R^4 - \Big(2(n+1)^2 - 4n(R^2+1) - 8\Big) R^2 - (n+1)^2}{R^2(R^2+1)^2} \\ &= \frac{-(n-1)^2 R^4 + (-2n^2+6)R^2 - (n+1)^2}{R^2(R^2+1)^2} < 0,\end{aligned}$$

if $n \geq 2$. Hence, the quadratic forms of $H_n^\pm$ over $L^2(R\, dR)$ are nonpositive for all $|n| \geq 2$ on a dense family. So we can pass to the standard Friedrichs extensions for these operators (and equivalently, for their conjugates $\mathcal{H}_n^\pm$ over $L^2(dR)$). The equivariant case $n = 0$ was treated in [18], and one has a negative operator here as well with a 0 energy resonance.

One checks that $\mathcal{H}_1^+(\sqrt{R}/(1+R^2)) = 0$. Since $\sqrt{R}/(1+R^2) \in L^2((0,\infty))$ is a nonnegative eigenfunction, it must be the ground state and $\mathcal{H}_1^+$ has no discrete spectrum. Thus, $\mathcal{H}_1^+ \leq 0$ as a symmetric operator.

On the other hand $\mathcal{H}_{-1}^+(R^{\frac{5}{2}}/(1+R^2)) = 0$. The function $R^{\frac{5}{2}}/(1+R^2)$ fails to lie in $L^2(dR)$ and is referred to as a zero energy resonance. The meaning of this is that its asymptotic behavior is *subordinate* at both

$R = 0$ and $R = \infty$. This refers to the fact that near $R = \infty$, $\mathcal{H}^+_{-1}$ has a fundamental system with asymptotic behavior $\sqrt{R}$ and $\sqrt{R}\log R$, respectively. On the other hand, near $R = 0$ the fundamental basis behaves as $R^{\frac{5}{2}}$ and $R^{-\frac{3}{2}}$, resp. Using this function we conclude that

$$\mathcal{H}^+_{-1} = -D^*D, \quad D = \frac{d}{dR} + U, \quad U(R) := -\frac{5}{2R} + \frac{2R}{1+R^2}$$

from which it again follows that $\mathcal{H}^+_{-1} \leq 0$.

The limit point behavior at $R = \infty$ of all $\mathcal{H}^+_n$ follows from Theorem X.8 in [23]. The equation $\mathcal{H}^+_n f = zf$, with $z \in \mathbb{C}$ has a fundamental system $f_1(R,z), f_2(R,z)$ with the asymptotic behaviors $f_1(R,z) \sim R^{n-\frac{1}{2}}$ and $f_2(R,z) \sim R^{-n+\frac{3}{2}}$, respectively, as $R \to 0+$ provided $n \neq 1$. Note that one of these is not in $L^2((0,1))$, which we discard. We are left with one which we call the regular solution. So if $n \neq 1$, then $\mathcal{H}^+_n$ is in the limit point case at both $R = 0$ and $R = \infty$. Therefore by Theorem X.7 in [23], the closure of $\mathcal{H}^+_n$ for all $n \neq 1$ is self-adjoint. In particular, this unique self-adjoint extension must agree with the Friedrichs extension. If $n = 1$, then these solutions satisfy $f_1(R,z) \sim R^{\frac{1}{2}}$ and $f_2(R,z) \sim R^{\frac{1}{2}}\log R$, respectively, as $R \to 0+$ (for any $z \in \mathbb{C}$). We now show that the Friedrichs extension of $\mathcal{H}^+_1$ does not allow a logarithmic singularity as $R \to 0_+$. From the fact $\mathcal{H}^+_1\left(\frac{\sqrt{R}}{1+R^2}\right) = 0$ we conclude that

$$\mathcal{H}^+_1 = -\mathcal{D}^*\mathcal{D}, \quad \mathcal{D} = \frac{d}{dR} + \mathcal{V}(R), \quad \mathcal{V}(R) := \frac{3R^2-1}{2R(1+R^2)}.$$

By its construction, the domain of the Friedrichs extension of $\mathcal{H}^+_1$ is a subspace of the completion of $C^2_0((0,\infty))$ under the norm

$$\left\langle -\mathcal{H}^+_1 f, f\right\rangle = \|\mathcal{D}f\|^2_{L^2(0,\infty)} = \int_0^\infty \left| f(R) + \frac{3R^2-1}{2R(1+R^2)} f(R)\right|^2 dR \tag{4.4}$$

By explicit calculation, one verifies that for any function with the property that $f_2(R) \sim \sqrt{R}\log R$ as $R \to 0_+$, with the corresponding asymptotic behavior of the derivative, this integral over $0 < R < 1$ is infinite. This means that only the solution $f_1(R,z) \sim R^{\frac{1}{2}}$ as $R \to 0+$ is regular for $\mathcal{H}^+_1$. This uniquely determines the resolvent kernel. Indeed, denoting the regular solution by $\phi_n(R,z)$, and the Weyl solution $\psi_n(R,z) \in L^2((1,\infty))$ for $z \in \mathbb{C} \setminus \mathbb{R}$, we have the resolvent kernel of $(\mathcal{H}^+_n - z)^{-1}$ satisfying

$$G_n(R,R';z) = \frac{\phi_n(R,z)\psi_n(R',z)}{W_n(z)}, \qquad 0 < R < R' \tag{4.5}$$

with $W_n(z)$ the Wronskian of ϕ_n, ψ_n. The asymptotic behavior (4.3) follows from the preceding. □

The importance of (4.3) for $n = 1$ lies with the fact that the resolvent kernel $\left(\mathcal{H}^+_1 - 1\right)^{-1}(R,R')$ has no logarithmic singularity as $R \to 0_+$. As the previous proof shows, this is due to the Friedrichs extension of $\mathcal{H}^+_1$. Note that one cannot naively impose boundary conditions at $R = 0$ due to the strong singularity of the potential. The operator H^+_1 plays a special role in our analysis, and is the only one with a zero energy eigenfunction.

Henceforth, we can consider both $H^\pm_n$ and $\mathcal{H}^\pm_n$ as selfadjoint operators on $L^2(R\,dR)$, respectively $L^2(dR)$, without further mention. The relevance of this lies with the distorted Fourier transform which these operators therefore possess [9, 11] and which will play a decisive role in our entire nonlinear analysis. The main point of this section is to exhibit the Fourier bases and the spectral measures of each $\mathcal{H}_n^{\pm}$. As already noted, the limit $n \to \pm\infty$ here is particularly delicate and will be treated as a semiclassical problem. We will see that the spectrum of all $\mathcal{H}_n^+$ is purely absolutely continuous if $n \neq 1$, and $\mathcal{H}_1^+$ is absolutely continuous with a simple eigenvalue at 0. Finally, one can easily prove by truncation and the quadratic form computations appearing in the proof of Proposition 4.1 that no $\mathcal{H}_n^+$ with $|n| \geq 2$ exhibits a 0 energy resonance. But we have no need for this result so we skip it.

4.2. **The distorted Fourier transform.** We close this introductory discussion by recalling the distorted Fourier transform associated with an operator

$$T = -\frac{d^2}{dR^2} + V(R) \;\text{ on }\; L^2((0,\infty)), \tag{4.6}$$

see [8, 9, 11]. Here V is real-valued, continuous, decaying as $R \to \infty$, and strongly singular at $R = 0$. In this paper we only encounter potentials V which are of the form $V(R) = R^{-2}(a_0 + a_1 R^2 + O(R^4))$ as $R \to 0$, with an analytic function $O(R^4)$. And $V(R) = O(R^{-2})$ as $R \to \infty$. While we are not necessarily assuming that T is in the limit point case at $R = 0$, we suppose so first for simplicity. Then let $\phi(\cdot, z), \theta(\cdot, z)$ be a fundamental system of $Tf = zf$, analytic near the real line, and real-valued on it. Assume the Wronskian $W(\theta(\cdot, z), \phi(\cdot, z)) = 1$ and that $\phi(\cdot, z) \in L^2((0, 1))$. Note that θ is unique only up to addition of a multiple of $k(z)\phi(\cdot, z)$ with a constant $k(z)$ that is analytic and real on $\mathbb{R}$. And $\phi(\cdot, z)$ is unique only up to multiplication by a nonzero *analytic function* near $\mathbb{R}$ which is real on $\mathbb{R}$. Furthermore, let $\psi(R, z)$ be a Weyl-Titchmarsh solution for $\operatorname{Im} z > 0$, i.e., it lies in $L^2((1, \infty))$ for $\operatorname{Im} z > 0$. The normalization of $\psi(R, z)$ is not too relevant, but it can be convenient to assume that ψ has asymptotic behavior $z^{-\frac{1}{4}} e^{iz^{\frac{1}{2}} R}$ as $R \to \infty$. Because of the decay of V this asymptotic behavior can be achieved. This normalization implies $W(\overline{\psi(\cdot, z)}, \psi(\cdot, z)) = 2i$ provided $\operatorname{Im} z = 0$. The (generalized) Weyl-Titchmarsh m function is then defined as

$$C\,\psi(\cdot, z) = \theta(\cdot, z) + m(z)\phi(\cdot, z), \quad C \neq 0 \tag{4.7}$$

with some constant C. Therefore,

$$m(z) = \frac{W(\theta(\cdot, z), \psi(\cdot, z))}{W(\psi(\cdot, z), \phi(\cdot, z))} \tag{4.8}$$

This does not depend on the normalization of the Weyl-Titchmarsh solution. In view of the degrees of freedom we have in defining ϕ, θ the m-function is far from unique. A spectral measure of T is obtained as the limit

$$\rho((\lambda_1, \lambda_2]) = \frac{1}{\pi} \lim_{\delta \to 0+} \lim_{\epsilon \to 0+} \int_{\lambda_1+\delta}^{\lambda_2+\delta} \operatorname{Im} m(\lambda + i\epsilon)\, d\lambda \tag{4.9}$$

The distorted Fourier transform of $f \in C((0, \infty))$ with compact support,

$$\hat{f}(\xi) = \int_0^\infty \phi(R, \xi) f(R)\, dR \tag{4.10}$$

is a unitary transformation $L^2((0, \infty)) \to L^2(\mathbb{R}, \rho)$, with inverse transform

$$f(R) = \int_{\mathbb{R}} \phi(R, \xi) \hat{f}(\xi)\, \rho(d\xi),$$

see [11] for the details and the existence of these integrals. Note that no assumption is made on the measure. In fact, if T has eigenvalues, then ρ will have atoms those points and the unitarity will contain the projections onto the eigenfunctions. Once this representation is obtained, we can derive equivalent ones as follows: if $h \in C(\mathbb{R})$ is positive, then replacing $\phi(R,\xi)$ with $h(\xi)\phi(R,\xi)$ and $\rho(d\xi)$ with $h(\xi)^{-2}\rho(d\xi)$ leads to another – equivalent – distorted Fourier transform. The spectral measures in our case will be purely absolutely continuous, with the possible exception of an atom at 0 (as for $\mathcal{H}_1^+$). But there cannot be any other atoms, or a singular continuous part.

In practice we will compute the density of the spectral measure as follows. First, we connect the solutions ϕ with the Weyl-Titchmarsh solutions, i.e.,

$$\phi(\cdot,\xi) = a(\xi)\psi(\cdot,\xi) + \overline{a(\xi)\psi(\cdot,\xi)}, \quad \xi > 0 \tag{4.11}$$

Recall that we normalize $W(\theta,\phi) = 1$ but we do not assume a normalization of the Weyl-Titchmarsh solutions ψ. Then $C(\xi)\psi(\cdot,\xi) = \theta(\cdot,\xi) + m(\xi)\phi(\cdot,\xi)$, see (4.7), and

$$\begin{aligned} CW(\psi(\cdot,\xi),\phi(\cdot,\xi)) &= 1 \\ |C(\xi)|^2 W(\psi(\cdot,\xi),\overline{\psi(\cdot,\xi)}) &= W(\theta(\cdot,\xi) + m(\xi)\phi(\cdot,\xi), \theta(\cdot,\xi) + \overline{m}(\xi)\phi(\cdot,\xi)) \\ &= \overline{m}(\xi) - m(\xi) = -2i\operatorname{Im} m(\xi) \end{aligned}$$

In view of (4.9), the density of the spectral measure is

$$\begin{aligned} \frac{d\rho}{d\xi}(\xi) = \frac{1}{\pi}\operatorname{Im} m &= \frac{W(\psi(\cdot,\xi),\overline{\psi(\cdot,\xi)})}{-2i\pi|W(\psi(\cdot,\xi),\phi(\cdot,\xi))|^2} \\ &= \frac{1}{2i\pi|a(\xi)|^2 W(\psi(\cdot,\xi),\overline{\psi(\cdot,\xi)})} \end{aligned} \tag{4.12}$$

This relation was essential in [16, 18] since we can easily find $W(\psi(\cdot,\xi),\overline{\psi(\cdot,\xi)})$, say by computing the Wronskian at $R = \infty$, and $a(\xi)$ is found by matching representations of ϕ (such as (4.27)) with an expansion such as (4.29). More precisely, we obtain an upper bound on $|a(\xi)|$ in this way by matching at $\xi^{\frac{1}{2}}R \simeq 1$, and the lower bound follows directly from (4.11) and a lower bound on ϕ since $|\phi(\cdot,\xi)| \le 2|a(\xi)||\psi(\cdot,\xi)|$.

Finally, if T is in the limit-circle case at $R = 0$, then we need to select a selfadjoint extension from an infinite family of possibilities, for example by means of a boundary condition. In the strongly singular case this cannot be done naively. In Proposition 4.1 we found that the Friedrichs extension turned out to be the correct choice since it guarantees the regularity property (4.3). The remainder of this section is devoted to the determination of a distorted Fourier transform for each of the angular momenta $n \in \mathbb{Z}$. In this regard the low modes $n = 0, \pm 1$ play a special role due to the appearance of a threshold resonance or eigenvalue.

We will freely switch between the self-adjoint operators H_n^+ in $L^2(R\,dR)$ and their versions $\mathcal{H}_n^+$ which are self-adjoint in $L^2(dR)$. The former are more natural as they arise in the linearized wave map. In the context of (4.6), with $(Mf)(R) := R^{-\frac{1}{2}}f(R)$, we have $T = M^{-1} \circ S \circ M$ where

$$S = -\frac{d^2}{dR^2} - \frac{1}{R}\frac{d}{dR} + \frac{1}{4R^2} + V(R)$$

If $\phi(\cdot,\xi)$ is the Fourier basis from (4.10) relative to T, then $\tilde\phi(R,\xi) := R^{-\frac12}\phi(R,\xi)$ is the Fourier basis relative to S with the same spectral measure, and $S(\tilde\phi(\cdot,\xi)) = \xi\tilde\phi(\cdot,\xi)$. In fact, one has

$$\begin{aligned} \hat f(\xi) &= \int_0^\infty \tilde\phi(R,\xi) f(R) R\,dR = \int_0^\infty \phi(R,\xi) f(R) R^{\frac12}\,dR \\ R^{\frac12} f(R) &= \int_0^\infty \phi(R,\xi)\hat f(\xi)\,\rho(d\xi), \qquad f(R) = \int_0^\infty \tilde\phi(R,\xi)\hat f(\xi)\,\rho(d\xi) \end{aligned} \tag{4.13}$$

The unitarity in this context means that $\|f\|_{L^2(R\,dR)} = \|\hat f\|_{L^2(d\rho)}$. We will formulate conditions below under which the integrals in (4.13) converge.

4.3. **Linearized operators at low angular momenta** $n = 0, \pm 1$**.** In this section, we determine the Fourier transform associated with the nonpositive operators

$$\begin{aligned} H_1^+ &= \partial_R^2 + \frac1R\partial_R - \frac{4}{R^2+1} + \frac{8}{(R^2+1)^2} = H_{-1}^- \\ H_1^- &= \partial_R^2 + \frac1R\partial_R - \frac{4}{R^2} + \frac{4}{R^2+1} + \frac{8}{(R^2+1)^2} = H_{-1}^+. \end{aligned} \tag{4.14}$$

As noted in Proposition 4.1, 0 is an eigenvalue of $H_1^+ = H_{-1}^-$ with eigenfunction $1/(1+R^2)$, and a resonance of $H_{-1}^+ = H_1^-$ with resonance function $R^2/(1+R^2)$. These explicit solutions permit us to factorize the operators as follows. From

$$\mathcal{D}_+\left(\frac{1}{1+R^2}\right) := \left(\partial_R + \frac{2R}{1+R^2}\right)\left(\frac{1}{1+R^2}\right) = 0, \tag{4.15}$$

we infer that $H_1^+ = -\mathcal{D}_+^*\mathcal{D}_+$ where

$$\mathcal{D}_+^* := -\partial_R + \frac{R^2-1}{R(1+R^2)}. \tag{4.16}$$

Similarly for H_1^-, we have

$$H_1^- = -\mathcal{D}_-^*\mathcal{D}_-, \quad \mathcal{D}_- = \partial_R - \frac2R + \frac{2R}{1+R^2}, \quad \mathcal{D}_-^* = -\partial_R - \frac3R + \frac{2R}{1+R^2}. \tag{4.17}$$

We now associate with these second order operators their "super-symmetric" versions which do not exhibit zero energy modes, viz.

$$\tilde H_1^+ := \mathcal{D}_+\mathcal{D}_+^* = -\partial_R^2 - \frac1R\partial_R + \frac{1}{R^2}, \quad \tilde H_1^- := \mathcal{D}_-\mathcal{D}_-^* = -\partial_R^2 - \frac1R\partial_R + \frac{9}{R^2} - \frac{8}{R^2+1}. \tag{4.18}$$

We will develop the Fourier transform for these operators, and not for the original ones. This appears to be essential in our technique since the spectral measures of the super-symmetric versions are much less singular at threshold energies. The super-symmetric cousin of the linearized equivariant operator H_0 already played a key role in [16], see in particular their Section 4. The dynamical implications of the switching will not be apparent until much later when we solve the dynamical problem. $\tilde H_1^+$ is a pure Bessel operator whose Fourier transform is explicitly known. We begin with that operator. The analysis of $n = 0$ was already carried out in [18, Section 5]. See also [16, Section 4] for the supersymmetric treatment of $\mathcal{H}_0$.

4.3.1. $n = 1$ *mode.* We recall the following facts about the Bessel operator

$$\mathcal{H} = -\frac{d^2}{dR^2} + \frac{3}{4R^2} = R^{\frac{1}{2}}\tilde{H}_1^+ R^{-\frac{1}{2}}$$

from [11, Section 4]. A fundamental system of solutions of

$$\mathcal{H}f = zf \tag{4.19}$$

is given by

$$R^{\frac{1}{2}}J_1(z^{\frac{1}{2}}R), \quad R^{\frac{1}{2}}Y_1(z^{\frac{1}{2}}R), \quad z \in \mathbb{C}\setminus\{0\}, \quad R \in (0,\infty), \tag{4.20}$$

with $J_1(\cdot)$ and $Y_1(\cdot)$ the usual Bessel functions of order 1. A fundamental system of the type described in Section 4.2 is given by

$$\begin{aligned}\Phi_1(R,z) =& \frac{\pi}{2} z^{-\frac{1}{2}} R^{\frac{1}{2}} J_1(z^{\frac{1}{2}}R),\\ \Theta_1(R,z) =& z^{\frac{1}{2}}R^{\frac{1}{2}}(-Y_1(z^{\frac{1}{2}}R) + \pi^{-1}\log(z)J_1(z^{\frac{1}{2}}R)).\end{aligned} \tag{4.21}$$

These two functions in (4.21) extend to entire functions with respect to $z \in \mathbb{C}$ for fixed $R \neq 0$ and they are real for $z \in \mathbb{R}$. Moreover, the normalizations are such that

$$W(\Theta_1(\cdot,z),\Phi_1(\cdot,z)) = \frac{\pi}{2}W(R^{\frac{1}{2}}J_1(z^{\frac{1}{2}}R), R^{\frac{1}{2}}Y_1(z^{\frac{1}{2}}R)) = 1, \quad z \in \mathbb{C}. \tag{4.22}$$

A Weyl-Titchmarsh solution to (4.19) is given by

$$R^{\frac{1}{2}}H_1^{(1)}(z^{\frac{1}{2}}R) = R^{\frac{1}{2}}(J_1(z^{\frac{1}{2}}R) + iY_1(z^{\frac{1}{2}}R)), \quad z \in \mathbb{C}\setminus[0,\infty), \quad R \in (0,\infty), \tag{4.23}$$

with $H_1^{(1)}$ the Hankel function of order 1. To account for the normalizations in (4.20), we modify (4.23) to

$$\begin{aligned}\psi(R,z) =& z^{\frac{1}{2}} i R^{\frac{1}{2}} H_1^{(1)}(z^{\frac{1}{2}}R)\\ =& z^{\frac{1}{2}}R^{\frac{1}{2}}(-Y_1(z^{\frac{1}{2}}R) + iJ_1(z^{\frac{1}{2}}R))\\ =& \Theta_1(z,R) + m(z)\Phi_1(z,R), \quad z \in \mathbb{C}\setminus[0,\infty), \quad R \in (0,\infty).\end{aligned} \tag{4.24}$$

In particular, the generalized Weyl-Titchmarsh function is

$$m(z) = \frac{2}{\pi}z(i - \frac{1}{\pi}\log z), \quad z \in \mathbb{C}\setminus[0,\infty). \tag{4.25}$$

The branch of $\log z$ is such that it is real for $z > 0$. In particular, $m(\xi + i0) = 0$ if $\xi < 0$ and the spectral measure $\rho(\xi)$ is absolutely continuous and of the explicit form

$$\rho(d\xi) = \frac{1}{\pi}\mathrm{Im}\, m(\xi + i0)\mathbb{1}_{[\xi>0]}\, d\xi = \frac{2}{\pi^2}\,\xi\,\mathbb{1}_{[\xi>0]}\, d\xi \tag{4.26}$$

We have therefore obtained the following representation of the distorted Fourier transform associated with the super-symmetric cousin $\tilde{H}_1^+$ of H_1^+. As noted in Section 4.2 the Fourier representation is not unique, and can be normalized in infinitely many ways.

The power series representation for $J_1(u)$ is, see [1],

$$J_1(u) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+2)}\left(\frac{u}{2}\right)^{2m+1} = \frac{u}{2} - \frac{u^3}{16} + O(u^5) \qquad u \to 0 \tag{4.27}$$

This implies that the Fourier basis of Lemma 4.2 admits the following convergent expansion

$$\frac{\pi}{2}\xi^{-\frac{1}{2}}J_1(\xi^{\frac{1}{2}}R) = \frac{\pi}{4}R + R\sum_{\ell=1}^{\infty} b_\ell(\xi R^2)^\ell \tag{4.28}$$

for all arguments. However, it is only useful for small argument $R\xi^{\frac{1}{2}} \ll 1$. For large arguments the solutions exhibit oscillatory behavior. In fact, the Weyl-Titchmarsh solution from (4.24) admit the asymptotic series expansion, see [1],

$$\psi(R,\xi) = \xi^{\frac{1}{2}} i R^{\frac{1}{2}} H_1^{(1)}(\xi^{\frac{1}{2}}R) \sim e^{-\frac{\pi i}{4}}\,\xi^{\frac{1}{4}} e^{i\xi^{\frac{1}{2}}R}\Big(\sqrt{\frac{2}{\pi}} + \frac{a_1}{\xi^{\frac{1}{2}}R} + \frac{a_2}{(\xi^{\frac{1}{2}}R)^2} + \dots\Big) \tag{4.29}$$

as $\xi R^2 \to \infty$. This means that $\psi(R,\xi) = \xi^{\frac{1}{4}} e^{i\xi^{\frac{1}{2}}R}\sigma(\xi^{\frac{1}{2}}R)$ where $\sigma(q)$ is a smooth function of $q \geq 1$ so that for all $\ell \geq 0$, $m \geq 1$,

$$\Big|(q\partial_q)^\ell\Big(e^{\frac{\pi i}{4}}\,\sigma(q) - \sqrt{\frac{2}{\pi}} - \frac{a_1}{q} - \frac{a_2}{q^2} - \dots - \frac{a_m}{q^m}\Big)\Big| \leq C_{m,\ell}\, q^{-m-\ell-1} \tag{4.30}$$

We now connect $\Phi_1(R,\xi)$ and $\psi(R,\xi)$ as in (4.11), i.e.,

$$\Phi_1(\cdot,\xi) = a(\xi)\psi(\cdot,\xi) + \overline{a(\xi)\psi(\cdot,\xi)}, \quad \xi > 0 \tag{4.31}$$

Using (4.12) we find that

$$W(\psi(\cdot,\xi),\overline{\psi(\cdot,\xi)}) = -\frac{4i\xi}{\pi}, \quad a(\xi) = -\frac{\pi}{4i\xi}, \quad \frac{d\rho}{d\xi}(\xi) = \frac{1}{8\xi|a(\xi)|^2} = \frac{2\xi}{\pi^2}\mathbb{1}_{[\xi>0]} \tag{4.32}$$

which agrees with (4.26).

Lemma 4.2. *The distorted Fourier transform associated with $\mathcal{H} = -\frac{d^2}{dR^2} + \frac{3}{4R^2}$ has the following property: for any $f \in C^2((0,\infty))$ with*

$$\int_0^\infty \Big(R^{-1}|f(R)| + |f'(R)| + R|f''(R)|\Big)\,dR \leq M < \infty$$

the Fourier transform

$$\hat{f}(\xi) = \lim_{L\to\infty}\int_0^L \Phi_1(R,\xi)f(R)\,dR \tag{4.33}$$

with $\phi(R,\xi)$ as in (4.21)*, exists for all $\xi > 0$ and*

$$\int_0^\infty |\hat{f}(\xi)||\Phi_1(R,\xi)|\,\xi\,d\xi \lesssim M \tag{4.34}$$

Thus,

$$f(R) = \frac{2}{\pi^2}\int_0^\infty \hat{f}(\xi)\Phi_1(R,\xi)\xi\,d\xi, \quad \forall\, R > 0 \tag{4.35}$$

with absolutely convergent integrals.

Proof. In view of (4.21), (4.27), (4.29), (4.31), and (4.32),

$$|\Phi_1(R,\xi)| \lesssim \min(R^{\frac{3}{2}}, \xi^{-\frac{3}{4}}) \qquad \forall\, \xi > 0,\ R > 0 \tag{4.36}$$

Thus, the left-hand side of (4.34) is bounded by $\int_0^\infty |\hat{f}(\xi)|\, \xi^{\frac{1}{4}}\, d\xi$ which we now proceed to estimate. Consider a partition of unity

$$1 = \chi_0(u) + \sum_{j=0}^{\infty} \chi(2^{-j}u) \qquad \forall\, u \geq 0 \tag{4.37}$$

with smooth functions χ_0 supported on $[0,1]$ and χ supported on $[\frac{1}{2}, 2]$, respectively. For any $\xi > 0$ we define

$$\begin{aligned} A(\xi) &= \int_0^\infty \chi_0(R^2\xi)\Phi_1(R,\xi)\, f(R)\, dR \\ B_j(\xi) &= \int_0^\infty \chi(2^{-j}R^2\xi)\Phi_1(R,\xi)\, f(R)\, dR \end{aligned} \tag{4.38}$$

so that at least formally

$$\hat{f}(\xi) = A(\xi) + \sum_{j=0}^{\infty} B_j(\xi) \tag{4.39}$$

Then

$$|A(\xi)| \lesssim \int_0^{\xi^{-\frac{1}{2}}} R^{\frac{3}{2}} |f(R)|\, dR. \tag{4.40}$$

and

$$\int_0^\infty |A(\xi)| \xi^{\frac{1}{4}} d\xi \lesssim \int_0^\infty \int_0^{R^{-2}} \xi^{\frac{1}{4}}\, d\xi\, |f(R)| R^{\frac{3}{2}}\, dR \lesssim \int_0^\infty R^{-1} |f(R)|\, dR \tag{4.41}$$

To bound B_j we integrate by parts twice. To do so, we write

$$\begin{aligned} B_j(\xi) = &\int_0^\infty \chi(2^{-j}R^2\xi)\, \xi^{\frac{1}{4}} e^{i\xi^{\frac{1}{2}}R} \sigma(R\xi^{\frac{1}{2}}) a(\xi) f(R)\, dR \\ &+ \int_0^\infty \chi(2^{-j}R^2\xi)\, \xi^{\frac{1}{4}} e^{-i\xi^{\frac{1}{2}}R}\, \overline{\sigma(R\xi^{\frac{1}{2}}) a(\xi)} f(R)\, dR \end{aligned}$$

It suffices to deal with the first line. Thus, by (4.32),

$$\begin{aligned} |B_j(\xi)| &\lesssim \xi^{-\frac{7}{4}} \int_0^\infty |\partial_R^2(\chi(2^{-j}R^2\xi)\sigma(R\xi^{\frac{1}{2}}) f(R))|\, dR \\ &\lesssim \xi^{-\frac{7}{4}} \int_0^\infty \mathbb{1}_{[R^2\xi \simeq 2^j]}(|f''(R)| + R^{-1}|f'(R)| + R^{-2}|f(R)|)\, dR \end{aligned} \tag{4.42}$$

Summing in $j \geq 0$ and inserting this bound into $\int_0^\infty |\hat{f}(\xi)|\xi^{\frac{1}{4}}\,d\xi$ yields

$$\int_0^\infty \sum_{j=0}^\infty |B_j(\xi)|\xi^{\frac{1}{4}}\,d\xi \lesssim \int_0^\infty \sum_{j=0}^\infty \xi^{-\frac{3}{2}} \int_0^\infty \mathbb{1}_{[R^2\xi\simeq 2^j]}\big(|f''(R)| + R^{-1}|f'(R)| + R^{-2}|f(R)|\big)\,dR\,d\xi$$
$$\lesssim \int_0^\infty \big(R|f''(R)| + |f'(R)| + R^{-1}|f(R)|\big)\,dR$$

In combination with (4.41) this proves (4.34).

The existence of the limit (4.33) pointwise in $\xi > 0$ follows from the convergence of the series (4.39) for fixed $\xi > 0$. Indeed, summing in (4.42) one obtains

$$\sum_{j=0}^\infty |B_j(\xi)| \lesssim \xi^{-\frac{7}{4}} \int_0^\infty \mathbb{1}_{[R^2\xi\gtrsim 1]}\big(|f''(R)| + R^{-1}|f'(R)| + R^{-2}|f(R)|\big)\,dR < \infty$$

The finiteness in last line follows from the fact that $R \gtrsim \xi^{-\frac{1}{2}} > 0$ is bounded away from 0, and from our assumption on f. □

4.3.2. $n = -1$ *mode.* None of the linearized operators associated with angular momenta other than $n = 1$ can be explicitly reduced to an exact Bessel equation. However, they are perturbations of exact Bessel operators. We use the technique of [18] to obtain the asymptotic form of the Fourier basis and the spectral measure. The goal will be to obtain expansions similar to (4.28), (4.29) for the operator

$$-\mathcal{H}_{-1}^+ = -\partial_R^2 + \frac{15}{4R^2} - \frac{4}{R^2+1} - \frac{8}{(R^2+1)^2}.$$

We exhibit a fundamental system of $\mathcal{H}_{-1}^+ f = 0$. From Proposition 4.1 one solution is $\Phi_0^{-1}(R) := \frac{R^{\frac{5}{2}}}{1+R^2}$. We seek another one with $W(\Theta_0^{-1}, \Phi_0^{-1}) = 1$. This leads to the ODE

$$-\partial_R\Theta_0^{-1} + \left(\frac{5}{2R} - \frac{2R}{1+R^2}\right)\Theta_0^{-1} = \frac{1+R^2}{R^{\frac{5}{2}}}$$

with general solution

$$\Theta_0^{-1}(R) = \frac{1 + 4R^2 + CR^4 - 4R^4\log(R)}{4R^{\frac{3}{2}}(1+R^2)}$$

We fix the constant $C = -5$ so that $\Theta_0^{-1}(1) = 0$:

$$\Theta_0^{-1}(R) = \Phi_0^{-1}(R)\left(\frac{4R^2+1}{4R^4} - \log R - \frac{5}{4}\right).$$

Following [18], we use this fundamental system at energy $\xi = 0$ to construct the Fourier basis $\Phi^{-1}(R,\xi)$ of the operator $-\mathcal{H}_{-1}^+$ perturbatively.

Proposition 4.3. *For all $R \geq 0, \xi \geq 0$, we have the following expansion for $\Phi^{-1}(R,\xi)$*

$$\Phi^{-1}(R,\xi) = \Phi_0^{-1}(R) + R^{\frac{1}{2}}\sum_{j=1}^\infty (-R^2\xi)^j\Phi_j(R^2)$$

which converges absolutely. It converges uniformly if $R\xi^{\frac{1}{2}}$ remains bounded. Here $\Phi_j(u) \geq 0$ are smooth functions of $u \geq 0$ satisfying

$$\Phi_j(u) \leq \frac{1}{j!}\frac{u}{1+u}, \quad \textit{for all} \quad u \geq 0,\ j \geq 1 \tag{4.43}$$

and $\Phi_1(u) \geq c_1 \frac{u}{1+u}$ for all $u \geq 0$ with some absolute constant $c_1 > 0$.

Proof. We make the ansatz

$$\Phi^{-1}(R,\xi) = R^{\frac{1}{2}}\sum_{j=0}^{\infty}(-\xi)^j \mathbf{f}_j(R), \qquad \mathbf{f}_0(R) := \frac{R^2}{1+R^2}$$

with

$$\mathcal{H}_{-1}^{+}\left(R^{\frac{1}{2}}\mathbf{f}_j\right) = R^{\frac{1}{2}}\mathbf{f}_{j-1}, \quad \mathbf{f}_{-1} := 0.$$

This leads to the recursion

$$\begin{aligned}\mathbf{f}_j(R) &= \int_0^R R^{-\frac{1}{2}}s^{\frac{1}{2}}\left(\Phi_0^{-1}(R)\Theta_0^{-1}(s) - \Theta_0^{-1}(R)\Phi_0^{-1}(s)\right)\mathbf{f}_{j-1}(s)\,ds\\ &= \int_0^R \frac{-s^4R^4\log s^4 + s^4R^4\log R^4 - s^4(4R^2+1) + R^4(4s^2+1)}{4sR^2(1+R^2)(1+s^2)}\mathbf{f}_{j-1}(s)\,ds.\end{aligned}$$

Introducing the new variables $v := s^2, u := R^2$, as well as the new functions $\widetilde{\mathbf{f}}_j(s^2) := \mathbf{f}_j(s)$, we obtain

$$\widetilde{\mathbf{f}}_j(u) = \int_0^u \frac{-u^2v^2\log v^2 + u^2v^2\log u^2 - v^2(4u+1) + u^2(4v+1)}{8uv(1+u)(1+v)}\widetilde{\mathbf{f}}_{j-1}(v)\,dv. \tag{4.44}$$

Substituting $v = tu$ with $0 < t \leq 1$ yields

$$\widetilde{\mathbf{f}}_j(u) = u\int_0^1 \frac{4tu + 1 - 2u^2t^2\log t - t^2(4u+1)}{8t(1+u)(1+tu)}\widetilde{\mathbf{f}}_{j-1}(tu)\,dt$$

Setting $h_j(u) := \widetilde{\mathbf{f}}_j(u)/u^{j+1}, u > 0, j \geq 0$ one has for all $k \geq 0$,

$$h_{k+1}(u) = \int_0^1 \frac{1 - t^2 + 4tu(1-t) - 2u^2t^2\log t}{8(1+u)(1+tu)}t^k h_k(tu)\,dt, \quad h_0(v) = \frac{1}{1+v} \tag{4.45}$$

Inductively this shows that $h_j(u) \in C^\infty([0,\infty))$. Since the kernel is positive and $h_0 \geq 0$, we also have $h_j \geq 0$. Returning to the original variables, we infer that

$$\Phi_j(R^2) = R^{-2j}f_j(R) = R^2h_j(R^2), \quad \Phi_j(u) = uh_j(u) \geq 0$$

is smooth in $u \geq 0$. To bound h_j from above we make the following claim

$$0 \leq \int_0^1 \frac{1 - t^2 + 4ut(1-t) - 2u^2t^2\log t}{(1+tu)^2}t^k\,dt \leq \frac{7}{k+1}, \qquad \forall\, k \geq 0 \tag{4.46}$$

for all $u \geq 0$. Assuming the claim we obtain inductively from (4.45) that

$$h_k(u) \leq \frac{1}{k!(1+u)}, \quad k \geq 0,\ u > 0$$

which gives us (4.43). To prove the claim we make the following estimates

$$\int_0^1 \frac{1-t^2+4ut(1-t)-2u^2t^2\log t}{(1+tu)^2} t^k\,dt \le \int_0^1 5t^k\,dt - 2\int_0^1 t^k \log t\,dt = \frac{5}{k+1} + \frac{2}{(k+1)^2} \le \frac{7}{k+1}.$$

For the lower bound on $h_1(u)$ we compute

$$h_1(u) = \int_0^1 \frac{1-t^2+4tu(1-t)-2u^2t^2\log t}{8(1+u)(1+tu)^2}\,dt$$

In particular, $h_1(0) = \frac{1}{12}$ and for any $u \ge 0$ and with some absolute positive constants c_0, c_1

$$h_1(u) \ge \int_{\frac14}^{\frac34} \frac{1-t^2+4tu(1-t)-2u^2t^2\log t}{8(1+u)(1+tu)^2}\,dt \ge c_0 \int_{\frac14}^{\frac34} \frac{1+u+u^2}{(1+u)(1+u)^2}\,dt \ge \frac{c_1}{1+u}$$

so that $\Phi_1(u) \ge c_1 \frac{u}{1+u}$ as claimed. □

Inspection of the proof shows that we can obtain similar lower bounds on all Φ_j, but we have no need for them. Now we turn to the case when $R\xi^{\frac12} \gtrsim 1$. Our goal is to obtain an asymptotic expansion similar to the one for the Hankel function (4.29).

Proposition 4.4. *For $R^2\xi \gtrsim 1$, $\xi > 0$ a Weyl-Titchmarsh function of $\mathcal{H}^+_{-1}$ is given by*

$$\Psi^+_{-1}(R,\xi) := \xi^{-\frac14} e^{iR\xi^{\frac12}} \sigma_{-1}(R\xi^{\frac12}, R) \tag{4.47}$$

where σ_{-1} is smooth in $q \gtrsim 1, R > 0$ and admits the asymptotic approximation

$$\sigma_{-1}(q,R) \sim \sum_{j=0}^{\infty} q^{-j}\Psi^+_j(R),$$

in the sense that for all large integers j_0, and all indices α,β

$$\sup_{R>0} \Big|(R\partial_R)^\alpha (q\partial_q)^\beta [\sigma_{-1}(q,R) - \sum_{j=0}^{j_0} q^{-j}\Psi^+_j(R)]\Big| \le C_{j_0,\alpha,\beta}\, q^{-j_0-1}$$

for all $q \gtrsim 1$. One has

$$\Psi^+_0(R) = 1, \quad \Psi^+_1(R) = \frac{15i}{8} - R\int_R^\infty \frac{2i}{1+s^2}\,ds = -\frac{i}{8} + O\Big(\frac{1}{1+R^2}\Big) \tag{4.48}$$

as $R \to \infty$. For all $j \ge 0$ the coefficient functions $\Psi^+_j(R)$ are zero order symbols, i.e.,

$$\sup_{R>0} \Big|(R\partial_R)^k \Psi^+_j(R)\Big| < \infty$$

and they are analytic at infinity.

Proof. The argument is basically a verbatim repetition of the proof of [18, Proposition 5.6]. From (4.47),

$$\sigma_{-1}(R\xi^{\frac12}, R) = \Psi^+_{-1}(R,\xi)\xi^{\frac14} e^{-iR\xi^{\frac12}}$$

which satisfies the conjugated equation

$$\left(-\partial_R^2 - 2i\xi^{\frac{1}{2}}\partial_R + \frac{15}{4R^2} - \frac{4}{R^2+1} - \frac{8}{(R^2+1)^2}\right)\sigma_{-1}(R\xi^{\frac{1}{2}}, R) = 0$$

This replaces eq. (5.8) in [18]. The ansatz

$$\sigma_{-1}(R\xi^{\frac{1}{2}}, R) = \sum_{j=0}^{\infty} \xi^{-\frac{j}{2}} f_j(r)$$

leads to the recursion

$$2i\partial_R f_j(R) = \left(-\partial_R^2 + \frac{15}{4R^2} - \frac{4}{R^2+1} - \frac{8}{(R^2+1)^2}\right) f_{j-1}$$

for all $j \geq 1$ with $f_0 \equiv 1$. Solving from $R = \infty$ where f_j vanishes for $j \geq 1$ we conclude that

$$f_j(R) = \frac{i}{2}\partial_R f_{j-1}(R) + \frac{i}{2}\int_R^{\infty} \Big(\frac{15}{4s^2} - \frac{4}{s^2+1} - \frac{8}{(s^2+1)^2}\Big) f_{j-1}(s)\, ds$$

For $j = 1$ we obtain the formula for $\Psi_1^+(R) := Rf_1(R)$ stated in (4.48). The remainder of the proof of [18, Proposition 5.6] does not depend on the specific form of the potential and carries over verbatim to the case at hand. □

To find the spectral measure associated with $\mathcal{H}_{-1}^+ = \mathcal{H}_1^-$ we now follow [18, Proposition 5.7].

Proposition 4.5. *We have*

$$\Phi^{-1}(R,\xi) = 2\mathrm{Re}\left(a_{-1}(\xi)\Psi_{-1}^+(R,\xi)\right)$$

where $a(\xi)$ is smooth, always nonzero, and is of size

$$|a_{-1}(\xi)| \simeq \langle\xi\rangle^{-1}$$

Moreover, it satisfies the symbol-type bounds

$$\left|(\xi\partial_\xi)^k a_{-1}(\xi)\right| \leq c_k \quad \textit{uniformly in} \quad \xi > 0.$$

The spectral measure of $-\mathcal{H}_{-1}^+$ is absolutely continuous on $\xi \geq 0$ with density

$$\frac{d\rho_{-1}(\xi)}{d\xi} = \frac{1}{4\pi}|a_{-1}(\xi)|^{-2} \simeq \langle\xi\rangle^2$$

with the same zero order symbol bounds.

Proof. We use the formalism introduced in (4.11)–(4.12). I.e., we set

$$\Phi^{-1}(R,\xi) = a_{-1}(\xi)\Psi_{-1}^+(R,\xi) + \overline{a_{-1}(\xi)\Psi_{-1}^+(R,\xi)} \tag{4.49}$$

whence

$$a_{-1}(\xi) = \frac{i}{2}W(\Phi^{-1}, \overline{\Psi_{-1}^+}). \tag{4.50}$$

We evaluate this Wronskian in the regime where both the $\Phi^{-1}(R,\xi)$ and $\Psi^+_{-1}(R,\xi)$ asymptotics are useful, namely, $R\xi^{\frac12}\simeq 1$. By Proposition 4.3, $|\Phi^{-1}(R,\xi)|\lesssim \xi^{-\frac14}$ if $R\simeq\xi^{-\frac12}\geq 1$ and $|\Phi^{-1}(R,\xi)|\lesssim\xi^{-\frac54}$ if $R\simeq\xi^{-\frac12}\leq 1$. From the equation $\mathcal{H}^+_{-1}\Phi^{-1}(R,\xi)=-\xi\Phi^{-1}(R,\xi)$ whence

$$|\partial_R^2\Phi^{-1}(R,\xi)|\lesssim(\xi+R^{-2})|\Phi^{-1}(R,\xi)|$$

and thus $|\partial_R^2\Phi^{-1}(R,\xi)|\lesssim\xi|\Phi^{-1}(R,\xi)|$ for all $R\simeq\xi^{-\frac12}$. By calculus, $|\partial_R\Phi^{-1}(R,\xi)|\lesssim\xi^{\frac14}$ if $R\simeq\xi^{-\frac12}\geq 1$ and $|\partial_R\Phi^{-1}(R,\xi)|\lesssim\xi^{-\frac34}$ if $R\simeq\xi^{-\frac12}\leq 1$. On the other hand, by Proposition 4.4,

$$|\Psi^+_{-1}(\xi^{-\frac12},\xi)|\lesssim\xi^{-\frac14},\qquad |\partial_R\Psi^+_{-1}(\xi^{-\frac12},\xi)|\lesssim\xi^{\frac14}$$

whence

$$|a_{-1}(\xi)|=\Big|\Phi^{-1}(\xi^{-\frac12},\xi)\partial_R\Psi^+_{-1}(\xi^{-\frac12},\xi)-\partial_R\Phi^{-1}(\xi^{-\frac12},\xi)\Psi^+_{-1}(\xi^{-\frac12},\xi)\Big|\lesssim\langle\xi\rangle^{-1}$$

which gives the upper bound on $|a_{-1}(\xi)|$. The derivative bounds are obtained by differentiating the Wronskian (4.50) in ξ. For Ψ^+_{-1} one uses the symbol bounds from Proposition 4.4, and for $\Phi^{-1}(R,\xi)$ we use Proposition 4.3 as well as the equation $\mathcal{H}^+_{-1}\Phi^{-1}(R,\xi)=-\xi\Phi^{-1}(R,\xi)$.

For the lower bound on $|a_{-1}(\xi)|$, we use a simpler argument than the one in [18]. Directly from (4.49),

$$|\Phi^{-1}(R,\xi)|\leq 2|a_{-1}(\xi)||\Psi^+_{-1}(R,\xi)|,\qquad |a_{-1}(\xi)|\geq\frac{|\Phi^{-1}(R,\xi)|}{2|\Psi^+_{-1}(R,\xi)|}$$

for all $R>0$. On the one hand, $|\Psi^+_{-1}(R,\xi)|\leq C\xi^{-\frac14}$ for all $\xi>0$ and $R^2\xi\geq\frac12$ (say). From Proposition 4.3 we have the lower bound

$$|\Phi^{-1}(R,\xi)|\geq\Phi_0^{-1}(R)\left(1-\sum_{j=1}^\infty\frac{(R^2\xi)^j}{j!}\right)\geq\Phi_0^{-1}(R)\big(2-e^{R^2\xi}\big)$$

So if, $R^2\xi=\frac12$, then

$$|a_{-1}(\xi)|\gtrsim\xi^{\frac14}\Phi_0^{-1}(1/\sqrt{2\xi})\simeq\langle\xi\rangle^{-1}$$

matching the upper bound. For the spectral measure, see (4.12). □

Given $f\in L^2(R\,dR)$, let $\widetilde{f}(R)=R^{\frac12}f(R)\in L^2(dR)$. Computing the distorted Fourier transform of $\widetilde{f}$ relative to $-\mathcal{H}_1^-$ gives, at least formally

$$\widetilde{f}(R)=\int_0^\infty x_{-1}(\xi)\Phi^{-1}(R,\xi)\,\rho_{-1}(d\xi),\qquad x_{-1}(\xi)=\int_0^\infty\widetilde{f}(R)\Phi^{-1}(R,\xi)\,dR \tag{4.51}$$

The convergence of the first integral holds if f is smooth and compactly supported in $(0,\infty)$ since then $x_{-1}(\xi)$ is rapidly decaying in ξ, and bounded for small $\xi>0$. But in fact much less is needed.

Lemma 4.6. *Let $f\in L^2(R\,dR)\cap C^2((0,\infty))$ so that*

$$\int_0^\infty\big(R^{\frac32}|f''(R)|+R^{\frac12}|f'(R)|+R^{-\frac12}|f(R)|\big)\,dR=M<\infty \tag{4.52}$$

Then for all $\xi>0$ the limit

$$x_{-1}(\xi)=\lim_{L\to\infty}\int_0^L f(R)\Phi^{-1}(R,\xi)\,R^{\frac12}\,dR \tag{4.53}$$

exists and satisfies

$$\int_0^\infty |x_{-1}(\xi)||\Phi^{-1}(R,\xi)|\,\rho_{-1}(d\xi) \lesssim M \tag{4.54}$$

In particular, the first integral in (4.51) *converges absolutely to* $R^{\frac{1}{2}} f(R)$ *pointwise in* $R > 0$*.*

Proof. By Propositions 4.3, 4.4, and 4.5 we have

$$\sup_{R>0} |\Phi^{-1}(R,\xi)| \lesssim \xi^{-\frac{1}{4}}\langle\xi\rangle^{-1} \tag{4.55}$$

for all $\xi > 0$. Indeed, Propositions 4.3 implies that

$$|\Phi^{-1}(R,\xi)| \lesssim \frac{R^{\frac{5}{2}}}{1+R^2} \lesssim \frac{\xi^{-\frac{5}{4}}}{1+\xi} \simeq \xi^{-\frac{1}{4}}\langle\xi\rangle^{-1}$$

for $R^2\xi \lesssim 1$, whereas Propositions 4.4, and 4.5 immediately imply the bound (4.55) in the range $R^2\xi \gtrsim 1$. It follows that

$$\int_0^\infty |x_{-1}(\xi)||\Phi^{-1}(R,\xi)|\,\rho_{-1}(d\xi) \lesssim \int_0^\infty |x_{-1}(\xi)|\xi^{-\frac{1}{4}}\langle\xi\rangle\, d\xi \tag{4.56}$$

For any $\xi > 0$ we define

$$\begin{aligned} A(\xi) &= \int_0^\infty \chi_0(R^2\xi)\Phi^{-1}(R,\xi)\, R^{\frac{1}{2}} f(R)\, dR \\ B_j(\xi) &= \int_0^\infty \chi(2^{-j}R^2\xi)\Phi^{-1}(R,\xi)\, R^{\frac{1}{2}} f(R)\, dR \end{aligned} \tag{4.57}$$

relative to a partition of unity as in (4.37).Thus, at least formally,

$$x_{-1}(\xi) = A(\xi) + \sum_{j=0}^\infty B_j(\xi)$$

We will show that the series does converge absolutely pointwise in $\xi > 0$. Then

$$|A(\xi)| \lesssim \int_0^{\xi^{-\frac{1}{2}}} R^{\frac{5}{2}}\langle R\rangle^{-2}|f(R)|R^{\frac{1}{2}}\, dR. \tag{4.58}$$

Inserting the right-hand side into the upper bound of (4.56) yields

$$\begin{aligned} \int_0^\infty |A(\xi)|\xi^{-\frac{1}{4}}\langle\xi\rangle\, d\xi &\lesssim \int_0^\infty \int_0^{R^{-2}} \xi^{-\frac{1}{4}}\langle\xi\rangle\, d\xi R^3\langle R\rangle^{-2}|f(R)|\, dR \\ &\lesssim \int_0^\infty R^{-\frac{1}{2}}|f(R)|\, dR \end{aligned}$$

In the oscillatory regime we use the previous two propositions and integrate by parts twice. First,

$$\begin{aligned} B_j(\xi) = &\int_0^\infty \chi(2^{-j}R^2\xi)\,\xi^{-\frac{1}{4}} e^{i\xi^{\frac{1}{2}}R}\sigma_{-1}(R\xi^{\frac{1}{2}},R)a_{-1}(\xi)R^{\frac{1}{2}} f(R)\, dR \\ &+ \int_0^\infty \chi(2^{-j}R^2\xi)\,\xi^{-\frac{1}{4}} e^{-i\xi^{\frac{1}{2}}R}\,\overline{\sigma_{-1}(R\xi^{\frac{1}{2}},R)a_{-1}(\xi)}\, R^{\frac{1}{2}} f(R)\, dR \end{aligned}$$

It suffices to deal with the first line. Thus,

$$
\begin{aligned}
|B_j(\xi)| &\lesssim \xi^{-\frac{5}{4}}\langle\xi\rangle^{-1}\int_0^\infty |\partial_R^2(\chi(2^{-j}R^2\xi)\sigma_{-1}(R\xi^{\frac{1}{2}},R)R^{\frac{1}{2}}f(R))|\,dR \\
&\lesssim \xi^{-\frac{5}{4}}\langle\xi\rangle^{-1}\int_0^\infty \mathbb{1}_{[R^2\xi\simeq 2^j]}(R^{\frac{1}{2}}|f''(R)|+R^{-\frac{1}{2}}|f'(R)|+R^{-\frac{3}{2}}|f(R)|)\,dR
\end{aligned}
\tag{4.59}
$$

Summing in $j\geq 0$ and inserting this bound into (4.56) we obtain

$$
\begin{aligned}
\int_0^\infty \sum_{j=0}^\infty |B_j(\xi)|\xi^{-\frac{1}{4}}\langle\xi\rangle\,d\xi &\lesssim \int_0^\infty\sum_{j=0}^\infty \xi^{-\frac{3}{2}}\int_0^\infty \mathbb{1}_{[R^2\xi\simeq 2^j]}(R^{\frac{1}{2}}|f''(R)|+R^{-\frac{1}{2}}|f'(R)|+R^{-\frac{3}{2}}|f(R)|)\,dR\,d\xi \\
&\lesssim \int_0^\infty (R^{\frac{3}{2}}|f''(R)|+R^{\frac{1}{2}}|f'(R)|+R^{-\frac{1}{2}}|f(R)|)\,dR
\end{aligned}
$$

In summary, (4.54) holds. On the other hand, the existence of the limit (4.53) is implicit in the argument, and in fact follows from the convergence of the series (4.57) for fixed $\xi>0$. Indeed, summing in (4.59) one obtains

$$
\sum_{j=0}^\infty |B_j(\xi)|\lesssim \xi^{-\frac{5}{4}}\langle\xi\rangle^{-1}\int_0^\infty \mathbb{1}_{[R^2\xi\gtrsim 1]}(R^{\frac{1}{2}}|f''(R)|+R^{-\frac{1}{2}}|f'(R)|+R^{-\frac{3}{2}}|f(R)|)\,dR<\infty
$$

The finiteness in last line follows from the fact that $R\gtrsim\xi^{-\frac{1}{2}}>0$ is bounded away from 0, and from (4.52).

□

We now derive the spectral representation of $\tilde{H}_1^- = \mathcal{D}_-\mathcal{D}_-^*$, cf. (4.18). The generalized eigenbasis of H_1^- is $R^{-\frac{1}{2}}\Phi^{-1}(R,\xi)$, viz.

$$
H_1^-\Big(R^{-\frac{1}{2}}\Phi^{-1}(R,\xi)\Big) = \ -\xi R^{-\frac{1}{2}}\Phi^{-1}(R,\xi)
$$

Returning to $f\in L^2(R\,dR)$ as in Lemma 4.6, we have in the sense stipulated by the lemma,

$$
f(R)=\int_0^\infty x_{-1}(\xi)R^{-\frac{1}{2}}\Phi^{-1}(R,\xi)\,\rho_{-1}(d\xi),\quad x_{-1}(\xi)=\Big\langle R^{-\frac{1}{2}}\Phi^{-1}(R,\xi),f(R)\Big\rangle_{L^2_{RdR}}.
$$

Applying $\mathcal{D}_-$ to the first of the above relations, we obtain at least formally

$$
\begin{aligned}
\mathcal{D}_-f(R) &= \int_0^\infty x_{-1}(\xi)\mathcal{D}_-\Big(R^{-\frac{1}{2}}\Phi^{-1}(R,\xi)\Big)\,\rho_{-1}(d\xi) \\
&= \int_0^\infty x_{-1}(\xi)\xi^{-1}\mathcal{D}_-\Big(R^{-\frac{1}{2}}\Phi^{-1}(R,\xi)\Big)\,\widetilde{\rho}_{-1}(d\xi)
\end{aligned}
$$

where $\widetilde{\rho}_{-1}(d\xi) := \xi\rho_{-1}(d\xi)$. In view of Proposition 4.3 and the fact $\mathcal{D}_-(R^{-\frac{1}{2}}\Phi_0^{-1}(R)) = 0$ it is natural to introduce the new basis

$$
\phi_{-1}(R,\xi) := \xi^{-1}\mathcal{D}_-\Big(R^{-\frac{1}{2}}\Phi^{-1}(R,\xi)\Big),
\tag{4.60}
$$

which satisfies $\lim_{\xi\to 0+}\phi_{-1}(R,\xi) = -\mathcal{D}_-(R^2\Phi_1(R^2))$ and

$$
\tilde{H}_{-1}^+\phi_{-1}(R,\xi) = \xi\phi_{-1}(R,\xi).
$$

So we arrive at the Fourier representation formula for $\mathcal{D}_- f$ relative to the operator $\tilde{H}_1^- = \tilde{H}_{-1}^+$:

$$\mathcal{D}_- f(R) = \int_0^\infty x_{-1}(\xi)\phi_{-1}(R,\xi)\widetilde{\rho}_{-1}(d\xi), \quad x_{-1}(\xi) = \langle \mathcal{D}_- f, \phi_{-1}(\cdot,\xi)\rangle_{L^2_{RdR}}. \tag{4.61}$$

In the following proposition we will state conditions on f under which this Fourier transform converges. To obtain the second identity in (4.61), we compute

$$\begin{aligned}\langle \mathcal{D}_- f, \phi_{-1}(\cdot,\xi)\rangle_{L^2_{RdR}} &= \xi^{-1}\left\langle \mathcal{D}_- f, \mathcal{D}_-\left(R^{-\frac12}\Phi^{-1}(\cdot,\xi)\right)\right\rangle_{L^2_{RdR}} \\ &= -\xi^{-1}\left\langle f, H_1^-\left(R^{-\frac12}\Phi^{-1}(\cdot,\xi)\right)\right\rangle_{L^2_{RdR}} = \left\langle f, R^{-\frac12}\Phi^{-1}(\cdot,\xi)\right\rangle_{L^2_{RdR}} = x_{-1}(\xi).\end{aligned}$$

which is justified if $f \in C^1((0,\infty))$ has compact support.

Proposition 4.7. *Let $f \in C^3((0,\infty))$ so that*

$$\int_0^\infty \left(R|f'''(R)| + |f''(R)| + R^{-1}|f'(R)| + R^{-2}|f(R)|\right) dR = M < \infty \tag{4.62}$$

Then the Fourier coefficient

$$y(\xi) := \langle \mathcal{D}_- f, \phi_{-1}(\cdot,\xi)\rangle_{L^2_{RdR}} = \lim_{L\to\infty}\int_0^L \mathcal{D}_- f(R)\phi_{-1}(R,\xi)R\,dR$$

exists pointwise in $\xi > 0$, and

$$\mathcal{D}_- f(R) = \int_0^\infty y(\xi)\phi_{-1}(R,\xi)\widetilde{\rho}_{-1}(d\xi)$$

converges absolutely, pointwise in $R > 0$. Here $\phi_{-1}(R,\xi)$ as in (4.60) satisfies $\tilde{H}_{-1}^+\phi_{-1}(R,\xi) = \xi\phi_{-1}(R,\xi)$. The spectral measure $\widetilde{\rho}_{-1}(\xi)$ associated to $\phi_{-1}(R,\xi)$ satisfies $\frac{d\widetilde{\rho}_{-1}(\xi)}{d\xi} \simeq \xi\langle\xi\rangle^2$ for all $\xi > 0$. Moreover, we have the unitarity

$$\|\mathcal{D}_- f\|_{L^2(R\,dR)} = \left\| \langle \mathcal{D}_- f, \phi_{-1}(\cdot,\xi)\rangle_{L^2_{RdR}} \right\|_{L^2(\tilde{\rho}_{-1})}$$

provided the left-hand side is finite.

Proof. The properties of the spectral measure follow directly from Proposition 4.5. Recall from (4.17) that

$$\mathcal{D}_- f(R) = f'(R) - \frac{2}{R}f(R) + \frac{2R}{1+R^2}f(R)$$

and $|\mathcal{D}_- f(R)| \lesssim |f'(R)| + R^{-1}|f(R)|$. From Proposition 4.3 and $\mathcal{D}_-(R^{-\frac12}\Phi_0^{-1}(R)) = 0$, we infer that

$$|\phi_{-1}(R,\xi)| \lesssim R^3\langle R^2\rangle^{-1} \lesssim \xi^{-\frac12}\langle\xi\rangle^{-1} \text{ if } R^2\xi \le 1$$

Similarly, from Propositions 4.5 and 4.4, we obtain the same bound in the regime $R^2\xi \geq 1$. The remainder of the proof now is analogous to that of Lemma 4.6 and we do not write it out. The unitarity is proved in [11]. □

4.4. **Turning point analysis and the spectral measure for $n \geq 2$.** The goal of this subsection is to find the asymptotic behavior of the spectral measures for small and large energies. This will be done by means of a careful asymptotic analysis of a fundamental system for all energies. We begin with $\mathcal{H}_n^+$, and will present the needed modifications for $\mathcal{H}_n^- = \mathcal{H}_{-n}^+$ later. We will use a perturbation argument developed in the papers [5] and [4]. As in those papers, we write the eigenvalue problem $-\mathcal{H}_n^+ f = E^2 f$ as follows:

$$\begin{aligned} -\frac{1}{(n+1)^2}\partial_R^2 f + V(R)f =& \frac{E^2}{(n+1)^2} f, \\ V(R) :=& \frac{1}{R^2} - \frac{1}{4(n+1)^2R^2} - \frac{4n}{(n+1)^2}\frac{1}{R^2(R^2+1)} - \frac{8}{(n+1)^2}\frac{1}{(R^2+1)^2}. \end{aligned} \tag{4.63}$$

Switching to semiclassical notation, we introduce $\hbar := \frac{1}{n+1}$ and write $V(R) = V_n(R) = V(R;\hbar)$ as

$$\begin{aligned} V(R) = V(R;\hbar) =& \frac{1}{R^2}\left(1 - \frac{\hbar^2}{4} - \frac{4\hbar}{R^2+1} + \frac{4\hbar^2(1-R^2)}{(R^2+1)^2}\right) \\ :=& \frac{1}{R^2}\left(1 + \frac{15\hbar^2}{4} - 4\hbar\right) + \varepsilon(R^2;\hbar) \end{aligned} \tag{4.64}$$

Here

$$\varepsilon(R^2;\hbar) := \frac{4\hbar}{R^2+1} - \frac{4\hbar^2(R^2+3)}{(R^2+1)^2}$$

is a bounded smooth function on $[0,\infty)$. Henceforth, it will be understood that $\hbar \in (0, \frac{1}{3}]$.

We will construct a fundamental system for (4.63) on $R > 0$ for all $E > 0$. We first scale E out by introducing $x := \hbar E R$. If we define $\widetilde{f}(x) := f(R)$, then (4.63) becomes

$$-\hbar^2 \widetilde{f}''(x) + Q(x)\widetilde{f}(x) = 0, \quad Q(x) := \hbar^{-2}E^{-2}V\left(\frac{x}{\hbar E}\right) - 1. \tag{4.65}$$

More precisely, we have, with $\alpha := \hbar E$,

$$Q(x,\alpha;\hbar) = x^{-2}\left(1 + \frac{15\hbar^2}{4} - 4\hbar\right) + \alpha^{-2}\varepsilon\left(\frac{x^2}{\alpha^2};\hbar\right) - 1.$$

As usual, see for example [5], we need to modify the potential by adding the *Langer correction*:

$$Q_0(x;\alpha,\hbar) := Q(x;\alpha,\hbar) + \frac{\hbar^2}{4x^2} = x^{-2}(1-2\hbar)^2 + \alpha^{-2}\varepsilon\left(\frac{x^2}{\alpha^2};\hbar\right) - 1. \tag{4.66}$$

This modification is crucial in order to obtain the correct asymptotics as $E \to 0+$ in the ensuing WKB analysis. In the following lemma we consider Q_0 as a function of x^2 rather than x, and α^2 instead of α. This lemma is sharp in the sense that it does not hold in the full range $\hbar \in (0, \frac{1}{2}]$.

Lemma 4.8. *The function $Q_1(x^2;\alpha^2,\hbar) = Q_0(x;\alpha,\hbar)$ satisfies*

$$\begin{aligned} -(\partial_y Q_1)(y;a,\hbar) \simeq y^{-2}, &\quad (\partial_a Q_1)(y;a,\hbar) < 0 \\ |\partial_a Q_1(y;a,\hbar)| \lesssim \hbar(y+a)^{-2} & \end{aligned}$$

uniformly in $y > 0, a > 0$ *and* $\hbar \in (0, \frac{1}{3}]$. *The implied constants are absolute. The higher order derivatives satisfy, uniformly in* $\hbar \in (0, \frac{1}{3}]$

$$|(\partial_y^k Q_1)(y; a, \hbar)| \leq C_k\, y^{-k-1} \ \ \textit{for all} \quad y > 0, a > 0, k \geq 2$$

and

$$|\partial_y^k \partial_a^\ell Q_1(y; a, \hbar)| \leq C_{k,\ell}\, \hbar\, (y + a)^{-k-\ell-1}$$

for all $y > 0, a > 0,\ k \geq 0, \ell \geq 1$.

Proof. With

$$Q_1(y, a) = (2\hbar - 1)^2 y^{-1} + \frac{4\hbar(1 - 3\hbar)}{y + a} + \frac{8\hbar^2 y}{(y + a)^2} - 1$$

we have, with $\partial_u Q_1$ denoting the partial derivative of Q_1 with respect to its first variable,

$$\begin{aligned} -a^2(\partial_u Q_1)(ay, a) &= (2\hbar - 1)^2 y^{-2} + \frac{4\hbar(1 - 3\hbar)}{(1 + y)^2} + \frac{8\hbar^2(y - 1)}{(1 + y)^3} \\ &= y^{-2}\Big[(2\hbar - 1)^2 + 4\hbar y^2(1 + y)^{-3}\big((1 - 3\hbar)(1 + y) + 2\hbar(y - 1)\big)\Big] \\ &= y^{-2}\Big[(2\hbar - 1)^2 + 4\hbar y^2(1 + y)^{-3}(1 - 5\hbar + (1 - \hbar)y)\Big] \end{aligned} \tag{4.67}$$

We claim that the term in brackets on the right-hand side is $\simeq 1$ uniformly in $y > 0$ and $\hbar \in (0, \frac{1}{3}]$. By inspection, this is the case if $y \geq 1$ or $y > 0$ and $\hbar \in (0, \frac{1}{5}]$. It thus suffices to check that the polynomial

$$P_\hbar(y) = (1 - 2\hbar)^2(1 + y)^3 + 4\hbar y^2(1 - 5\hbar + (1 - \hbar)y) \simeq 1$$

uniformly on the rectangle $(y, \hbar) \in [0, 1] \times [\frac{1}{5}, \frac{1}{3}]$. We have

$$P'_\hbar(y) = 3(1 + y)^2 - 4\hbar^2(-3 + 4y) - 4\hbar(3 + 4y)$$

The discriminant of the quadratic polynomial P'_h is

$$D = -3\hbar - 5\hbar^2 + 32\hbar^3 + 16\hbar^4$$

On checks that $D < 0$ for all $\hbar \in [\frac{1}{5}, \frac{1}{3}]$. Thus $P'_\hbar(y)$ does not change sign in $y \in \mathbb{R}$ for all such $\hbar$. Clearly, $P'_h(y) > 0$ for large y whence $P_\hbar(y)$ is increasing for all $\hbar \in [\frac{1}{5}, \frac{1}{3}]$. In summary, for all $y \in [0, 1]$

$$P_\hbar(0) = (1 - 2\hbar)^2 \leq P_\hbar(y) \leq P_\hbar(1) = 8(1 - 2\hbar)^2 + 8\hbar(1 - 3\hbar)$$

Thus the claim above holds and from (4.67) we have

$$-(\partial_y Q_1)(y, a) \simeq y^{-2}$$

uniformly in $y > 0, a > 0$ and $\hbar \in (0, \frac{1}{3}]$.

For the a-derivative we compute

$$0 < -(\partial_a Q_1)(y, a) = \frac{4\hbar(1 - 3\hbar)}{(y + a)^2} + \frac{16\hbar^2 y}{(y + a)^3} \leq 6\hbar(y + a)^{-2}$$

uniformly in $y > 0, a > 0$ and $\hbar \in (0, \frac{1}{3}]$. The estimates for the higher order derivatives follow by using the fact that $Q_1(y; a, \hbar)$ is a rational function for both y and a. □

In the original variables the previous lemma takes the following form.

Corollary 4.9. *One has*

$$\begin{aligned} -(\partial_x Q_0)(x;\alpha,\hbar) &\simeq x^{-3}, \quad (\partial_\alpha Q_0)(x;\alpha,\hbar) < 0 \\ |\partial_\alpha Q_0(x;\alpha,\hbar)| &\lesssim \hbar\,\alpha(x^2+\alpha^2)^{-2} \end{aligned} \tag{4.68}$$

uniformly in $x>0, \alpha>0$ *and* $\hbar \in (0, \frac{1}{3}]$. *The implied constants are absolute. The higher order derivatives satisfy, uniformly in* $\hbar \in (0,\frac{1}{3}]$

$$|(\partial_x^k Q_0)(x;\alpha,\hbar)| \le C_k\, x^{-k-2} \text{ for all } \quad x>0, \alpha>0, k\ge 2$$

and

$$|\partial_x^k \partial_\alpha^\ell Q_0(x;\alpha,\hbar)| \le C_{k,\ell}\,\hbar\,(x+\alpha)^{-k-\ell-2} \tag{4.69}$$

for all $x>0, \alpha>0$, $k\ge 0, \ell\ge 1$.

Proof. We set $a=\alpha^2$ and $y=x^2$. By the chain rule

$$(2\alpha)^{-1}\partial_\alpha = \partial_a, \qquad (2x)^{-1}\partial_x = \partial_y$$

and the corollary follows from the previous lemma. For example, the second resp. third derivatives are

$$\partial_\alpha^2 = 2\partial_a + 4a\partial_a^2, \quad \partial_\alpha^3 = \alpha(12\partial_a^2 + 8\alpha^2\partial_a^3)$$

and similarly in x. □

We remark that the estimate in (4.69) is not optimal if k or ℓ are odd. Indeed, in that case one has vanishing at $x=0$, respectively $\alpha=0$ as in (4.68). The following lemma introduces the unique turning point $x_t(\alpha,\hbar)$, i.e., the root of $Q_0(x;\alpha,\hbar)=0$. The same non-optimality remark again applies to (4.70) below for odd ℓ.

Lemma 4.10. $Q_0(x;\alpha,\hbar)=0$ *has a unique root* $x_t(\alpha;\hbar)\in(1-2\hbar,1)$. *It is strictly monotone decreasing and smooth in* $\alpha>0$. *Moreover,*

$$\begin{aligned} -\partial_\alpha x_t(\alpha;\hbar) = |\partial_\alpha x_t(\alpha;\hbar)| &\lesssim \hbar\,\alpha(1+\alpha)^{-4} \\ |\partial_\alpha^\ell x_t(\alpha;\hbar)| &\lesssim \hbar\,(1+\alpha)^{-\ell-2}, \qquad \ell\ge 1 \end{aligned} \tag{4.70}$$

for all $\alpha>0$ *with a uniform constant in* $\hbar\in(0,\frac{1}{3}]$.

Proof. By the previous lemma $Q_0(x;\alpha,\hbar)$ is strictly monotone decreasing in both x and α. Thus, x_t exists uniquely and is strictly decreasing in α. Thus, to determine the range of x_t as a function of α, it suffices to consider the limits $\alpha\to 0^+$ and $\alpha\to\infty$. When $\alpha=0$, we have

$$Q_0(x;0,\hbar) = x^{-2}-1, \qquad x_t(0;\hbar)=1.$$

When $\alpha=\infty$, we have

$$Q_0(x;\infty,\hbar) = (2\hbar-1)^2x^{-2}-1, \qquad x_t(\infty;\hbar) = 1-2\hbar.$$

Taking an α derivative of

$$Q_0(x_t(\alpha;\hbar);\alpha,\hbar) = Q_1(x_t(\alpha;\hbar)^2;\alpha^2,\hbar) = 0$$

yields

$$\partial_\alpha x_t(\alpha;\hbar)(\partial_x Q_0)(x_t(\alpha;\hbar);\alpha,\hbar) + \partial_\alpha Q_0(x_t(\alpha;\hbar);\alpha,\hbar) = 0 \tag{4.71}$$

whence

$$-\partial_\alpha x_t(\alpha;\hbar) = |\partial_\alpha x_t(\alpha;\hbar)| \lesssim \hbar\alpha(1+\alpha^2)^{-2}$$

Taking another α derivative in (4.71) yields

$$|\partial_\alpha^2 x_t(\alpha;\hbar)| \lesssim \hbar(1+\alpha)^{-4}$$

as claimed. By Leibnitz's rule, $\partial_\alpha^\ell x_t(\alpha;\hbar)$ is a linear combination of all terms of the form

$$\partial_\alpha^{\nu_1} x_t \cdot \ldots \cdot \partial_\alpha^{\nu_r} x_t \; \partial_x^r \partial_\alpha^m Q_0$$

evaluated at $x = x_t(\alpha;\hbar)$, where $\nu_1 + \ldots + \nu_r + m = \ell$. By induction, if (4.70) holds, then if $\nu_i < \ell$ for all i, and if $r \geq 1$,

$$\begin{aligned} |\partial_\alpha^{\nu_1} x_t \cdot \ldots \cdot \partial_\alpha^{\nu_r} x_t \; \partial_x^r \partial_\alpha^m Q_0(x_t;\alpha,\hbar)| &\lesssim \hbar^r (1+\alpha)^{-(\nu_1+\ldots+\nu_r+2r)} (1+\alpha)^{-r-m-2} \\ &\lesssim \hbar^r (1+\alpha)^{-\ell-3r-2} \end{aligned}$$

which is largest when $r = 1$ (if $m > 0$, then one has $\hbar^{r+1}$ and not $\hbar^r$). Thus,

$$\begin{aligned} |\partial_\alpha^\ell x_t(\alpha,\hbar)\partial_x Q_0(x_t(\alpha;\hbar);\alpha,\hbar)| &\lesssim \hbar(1+\alpha)^{-\ell-5} + |\partial_\alpha^\ell Q_0(x_t(\alpha;\hbar);\alpha,\hbar)| \\ &\lesssim \hbar(1+\alpha)^{-\ell-2}. \end{aligned}$$

By induction, (4.70) holds for all ℓ as claimed. □

4.5. **Liouville-Green transform, reduction to Airy's equation.** We will now apply a global Liouville-Green transform [1] to the equation (4.65), cf. [21, Chapter 6]. As usual, this refers to a change of both the independent and dependent variables. The new independent variable $\tau = \tau(x,\alpha;\hbar)$ is given by

$$\tau(x,\alpha;\hbar) := \operatorname{sign}(x - x_t(\alpha;\hbar)) \left| \frac{3}{2} \int_{x_t(\alpha;,\hbar)}^x \sqrt{|Q_0(u,\alpha;\hbar)|}\, du \right|^{\frac{2}{3}}. \tag{4.72}$$

Before we rigorously analyze τ, we first take a look at how the equation (4.65) transforms under this change of variables. As we have seen, for $\hbar \leq \frac{1}{3}$, $Q_0(x)$ is strictly monotone for $x > 0$, so is τ by its definition. Therefore the map $\tau(\cdot,\alpha,\hbar) : (0,\infty) \to \mathbb{R}$ is injective. When $x > x_t(\alpha,\hbar)$, we have

$$\begin{aligned} \tau(x;\alpha,\hbar) &= \left(\frac{3}{2} \int_{x_t(\alpha,\hbar)}^x \sqrt{-Q_0(u;\alpha,\hbar)}\, du \right)^{\frac{2}{3}} \\ \frac{d\tau}{dx} &= \left(\frac{3}{2} \int_{x_t(\alpha,\hbar)}^x \sqrt{-Q_0(u;\alpha,\hbar)}\, du \right)^{-\frac{1}{3}} \cdot \sqrt{-Q_0(x;\alpha,\hbar)} \end{aligned}$$

and

$$\begin{aligned} \frac{d^2\tau}{dx^2} =& \frac{1}{2} \left(\frac{3}{2} \int_{x_t(\alpha,\hbar)}^x \sqrt{-Q_0(u;\alpha,\hbar)}\, du \right)^{-\frac{4}{3}} Q_0(x;\alpha,\hbar) \\ & - \left(\frac{3}{2} \int_{x_t(\alpha,\hbar)}^x \sqrt{-Q_0(u;\alpha,\hbar)} du \right)^{-\frac{1}{3}} \frac{1}{2\sqrt{-Q_0(x;\alpha,\hbar)}} \frac{dQ_0(x;\alpha,\hbar)}{dx} \end{aligned}$$

[1] For more background on this Langer transform and turning point theory in general, see [20] and [31].

For simplicity, we will use "$'$" to denote the derivative with respect to x and "$\cdot$" to denote the derivative with respect to τ. Also, we will sometimes suppress the dependence on α and $\hbar$ when there is no confusion. Following [5], we define

$$q := -\frac{Q_0}{\tau} \text{ so that } \frac{d\tau}{dx} = \tau' = \sqrt{q}, \quad \frac{d}{d\tau} = q^{-\frac{1}{2}}\frac{d}{dx}. \tag{4.73}$$

The new dependent variable is given by $w := \sqrt{\tau'}\widetilde{f}$. Then, with $\dot{q}^{\frac{1}{4}} = \partial_\tau(q^{\frac{1}{4}})$, $\ddot{q}^{\frac{1}{4}} = \partial_\tau^2(q^{\frac{1}{4}})$ we have

$$\begin{aligned} w &:= \sqrt{\tau'}\widetilde{f} = q^{\frac{1}{4}}\widetilde{f}, \quad \dot{w} = \dot{q}^{\frac{1}{4}}\widetilde{f} + q^{-\frac{1}{4}}\widetilde{f}' \\ \ddot{w} &= \ddot{q}^{\frac{1}{4}}\widetilde{f} + \dot{q}^{\frac{1}{4}}q^{-\frac{1}{2}}\widetilde{f}' - q^{-\frac{1}{2}}\dot{q}^{\frac{1}{4}}\widetilde{f}' + q^{-\frac{3}{4}}\widetilde{f}'' = q^{-\frac{3}{4}}\widetilde{f}'' + \ddot{q}^{\frac{1}{4}}\widetilde{f}. \end{aligned}$$

Using the equation (4.65), we obtain

$$\begin{aligned} \ddot{w} &= q^{-\frac{3}{4}}\left(\frac{Q_0}{\hbar^2} - \frac{1}{4x^2}\right)\widetilde{f} + \ddot{q}^{\frac{1}{4}}\widetilde{f} = -\tau\hbar^{-2}w + \ddot{q}^{\frac{1}{4}}q^{-\frac{1}{4}}w - \frac{1}{4x^2}q^{-1}w \\ -\hbar^2\ddot{w}(\tau) &= \tau w(\tau) - \hbar^2\ddot{q}^{\frac{1}{4}}q^{-\frac{1}{4}}w(\tau) + \hbar^2\frac{1}{4x^2q}w \\ -\hbar^2\ddot{w}(\tau) &=: \tau w(\tau) - \hbar^2\widetilde{V}(\tau;\alpha,\hbar)w(\tau) \end{aligned} \tag{4.74}$$

where

$$\widetilde{V}(\tau;\alpha,\hbar) = \ddot{q}^{\frac{1}{4}}q^{-\frac{1}{4}} - \frac{1}{4x^2q}. \tag{4.75}$$

In order to use the results in [4], we still need to modify the last equation in (4.74) such that it is consistent with the equation (D.9) in [4]. To this end, we introduce the new variable $\zeta := -\tau$ and the last equation in (4.74) becomes

$$\hbar^2\ddot{w}(\zeta) = \zeta w(\zeta) + \hbar^2\widetilde{V}(-\zeta,\alpha;\hbar)w(\zeta). \tag{4.76}$$

The case for $x < x_t(\alpha,\hbar)$ can be handled similarly and results in the same equation. Below we will analyze the behavior of τ in terms of x in different regimes. In particular, there are three regimes to consider:

- When x is close to $x_t(\alpha,\hbar)$. More precisely, $|x - x_t(\alpha,\hbar)| \le \frac{1}{2}x_t(\alpha;\hbar)$.
- When $x \to 0^+$. More precisely, $0 < x \le \frac{1}{2}x_t(\alpha;\hbar)$.
- When $x \to \infty$. More precisely, $x \ge \frac{3}{2}x_t(\alpha;\hbar)$.

On the other hand, the size of $\alpha = \hbar E$ affects the position of $x_t(\alpha,\hbar)$. Sometimes we distinguish the two different cases: $\alpha \ll 1$ and $\alpha \gtrsim 1$. This will play a crucial role when $x \in (0, \frac{1}{2}]$.

Now we can start our analysis on the behavior of τ and the potential $\widetilde{V}$ in the perturbed Airy equation (4.76).

Lemma 4.11. *For $x \in \left[\frac{1}{2}x_t(\alpha;\hbar), \frac{3}{2}x_t(\alpha;\hbar)\right] =: J_1(\alpha;\hbar)$, the function τ defined in* (4.72) *satisfies*

$$\tau(x;\alpha,\hbar) = (x - x_t(\alpha;\hbar))\Phi(x;\alpha,\hbar)$$

where $\Phi(x;\alpha,\hbar) \simeq 1$ uniformly in $x \in J_1$, $\alpha > 0$ and $\hbar \in (0, \frac{1}{3}]$. Moreover, uniformly in that range of parameters,

$$|\partial_x^k\Phi(x;\alpha,\hbar)| \lesssim 1, \quad |\partial_x^k\partial_\alpha^\ell\Phi(x;\alpha,\hbar)| \lesssim \hbar\,(1+\alpha)^{-\ell-2} \tag{4.77}$$

for all $k \ge 0$, $\ell \ge 1$. Finally, $q = q(x;\alpha,\hbar)$ as in (4.73) *satisfies the exact same properties.*

Proof. We have

$$\begin{aligned} -Q_0(x;\alpha,\hbar) &= Q_0(x_t(\alpha;\hbar);\alpha,\hbar) - Q_0(x;\alpha,\hbar) \\ &= (x - x_t(\alpha;\hbar)) \int_0^1 (-\partial_x Q_0)(x_t(\alpha;\hbar) + s(x - x_t(\alpha;\hbar));\alpha,\hbar)\, ds \\ &= (x - x_t(\alpha;\hbar)) g(x;\alpha,\hbar) \end{aligned} \tag{4.78}$$

By Corollary 4.9 and Lemma 4.10 we have $g(x;\alpha,\hbar) \simeq 1$ uniformly in $x \in J_1$, $\alpha > 0$ and $\hbar \in (0, \frac{1}{3}]$. Moreover, uniformly in that range of parameters,

$$|\partial_x^k g(x;\alpha,\hbar)| \lesssim 1, \quad |\partial_x^k \partial_\alpha^\ell g(x;\alpha,\hbar)| \lesssim \hbar\,(1+\alpha)^{-\ell-2} \tag{4.79}$$

for all $k \geq 0$, $\ell \geq 1$. We next show how the estimates for higher order derivatives of g are derived, in the case of pure x or α derivatives, the case of mixed derivatives following similarly:

$$\begin{aligned} \partial_x g(x;\alpha,\hbar) &= \int_0^1 (-s\partial_x^2 Q_0)(x_t(\alpha,\hbar) + s(x - x_t(\alpha,\hbar));\alpha,\hbar) ds \\ \Rightarrow \quad \partial_x^k g(x;\alpha,\hbar) &= \int_0^1 (-s^k \partial_x^{k+1} Q_0)(x_t(\alpha,\hbar) + s(x - x_t(\alpha,\hbar));\alpha,\hbar) ds. \end{aligned}$$

The estimate on $|\partial_x^k g(x;\alpha,\hbar)|$ then follows from Corollary 4.9. The α-derivatives are given as follows:

$$\begin{aligned} \partial_\alpha g(x;\alpha,\hbar) &= \int_0^1 (-(1-s)\partial_x^2 Q_0)(x_t(\alpha,\hbar) + s(x - x_t(\alpha,\hbar));\alpha,\hbar)(\partial_\alpha x_t)(\alpha,\hbar) ds \\ &\quad - \int_0^1 (\partial_x \partial_\alpha Q_0)(x_t(\alpha,\hbar) + s(x - x_t(\alpha,\hbar));\alpha,\hbar) ds, \end{aligned}$$

and more generally, using Faà di Bruno's formula as well as Leibniz's rule

$$\begin{aligned} &\partial_\alpha^\ell g(x;\alpha,\hbar) \\ &= -\sum_{\substack{0 \leq p' \leq p \leq \ell, \\ \sum p_j = p}} C_{\{p_j\},p,\ell} \int_0^1 \left((1-s)^{p'} \partial_x^{p'+1} \partial_\alpha^{\ell-p} Q_0\right)(x_t(\alpha,\hbar) + s(x - x_t(\alpha,\hbar));\alpha,\hbar) \prod_{j=1}^{p'} (\partial_\alpha^{p_j} x_t)(\alpha,\hbar) ds \end{aligned}$$

Therefore the desired estimates follow from Corollary 4.9 as well as Lemma 4.10. Inserting (4.78) into (4.72) yields

$$\begin{aligned} \tau(x;\alpha,\hbar) &= (x - x_t(\alpha;\hbar)) \Big| \frac{3}{2} \int_0^1 \sqrt{s g(x_t(\alpha;\hbar) + s(x - x_t(\alpha;\hbar));\alpha,\hbar)}\, ds \Big|^{\frac{2}{3}} \\ &=: (x - x_t(\alpha;\hbar)) \Phi(x;\alpha,\hbar) \end{aligned} \tag{4.80}$$

where Φ satisfies the same estimates as g, namely $\Phi \simeq 1$ uniformly in the parameter range above and the derivatives satisfy (4.79). As a result,

$$q(x;\alpha,\hbar) = g(x;\alpha,\hbar)/\Phi(x;\alpha,\hbar), \quad \text{and} \quad \tau'(x;\alpha,\hbar) = \sqrt{q}$$

also satisfies these exact same properties. □

For the potential function

$$\widetilde{V} = -\frac{1}{4x^2 q} + q^{-\frac{1}{4}}\ddot{q}^{\frac{1}{4}}, \tag{4.81}$$

see (4.76) and (4.75) we have the following immediate corollary.

Proposition 4.12. *Uniformly in the parameter range of Lemma 4.11 we have*

$$|\partial_\tau^k \widetilde{V}(\tau;\alpha,\hbar)| \lesssim 1, \quad |\partial_\tau^k \partial_\alpha^\ell \widetilde{V}(\tau;\alpha,\hbar)| \lesssim \hbar\,(1+\alpha)^{-\ell-2} \tag{4.82}$$

for all $k \geq 0$, $\ell \geq 1$.

Proof. This follows from the properties of q stated in the previous result. □

Next we turn to the case when $x \in J_2 := [\frac{3}{2}x_t(\alpha;\hbar), \infty)$.

Lemma 4.13. *For* $x \in J_2$ *we write* $\tau = \left(\frac{3}{2}\widetilde{\tau}\right)^{\frac{2}{3}}$. *Then* $\widetilde{\tau} > 0$ *satisfies*

$$\begin{aligned}
\widetilde{\tau}(x;\alpha,\hbar) &= x - y(\alpha;\hbar) + \rho(x;\alpha,\hbar)\\
y(\alpha;\hbar) &\simeq 1, \quad |\partial_\alpha^\ell y(\alpha;\hbar)| \lesssim \hbar\,(1+\alpha)^{-\ell-2}, \quad \ell \geq 1\\
\rho(x;\alpha,\hbar) &\simeq x^{-1}, \quad |\partial_x^k \rho(x;\alpha,\hbar)| \lesssim x^{-k-1}, \quad k \geq 0\\
|\partial_x^k \partial_\alpha^\ell \rho(x;\alpha,\hbar)| &\lesssim \hbar\,(x+\alpha)^{-\ell-k-1}, \quad k \geq 0, \ell \geq 1
\end{aligned}$$

uniformly in $\alpha > 0, 0 < \hbar \leq \frac{1}{3}$.

Proof. We have

$$-Q_0(x;\alpha,\hbar) = 1 - x^{-2}(1-2\hbar)^2 - \frac{4\hbar}{x^2+\alpha^2} + \frac{4\hbar^2(x^2+3\alpha^2)}{(x^2+\alpha^2)^2} \tag{4.83}$$

whence

$$\begin{aligned}
\widetilde{\tau}(x;\alpha,\hbar) &:= \int_{x_t(\alpha;\hbar)}^x \sqrt{-Q_0(u;\alpha,\hbar)}\,du = x - x_t(\alpha;\hbar) + \int_{x_t(\alpha;\hbar)}^x \left(\sqrt{-Q_0(u;\alpha,\hbar)} - 1\right) du\\
&= x - x_t(\alpha;\hbar) - \kappa(\alpha;\hbar) + \rho(x;\alpha,\hbar)
\end{aligned}$$

where

$$\rho(x;\alpha,\hbar) := \int_x^\infty \frac{u^{-2}(1-2\hbar)^2 + \frac{4\hbar(1-3\hbar)}{u^2+\alpha^2} + \frac{8\hbar^2 u^2}{(u^2+\alpha^2)^2}}{1+\sqrt{-Q_0(u;\alpha,\hbar)}}\,du \tag{4.84}$$

and

$$\begin{aligned}
\kappa(\alpha;\hbar) &:= \int_{x_t(\alpha;\hbar)}^\infty \frac{u^{-2}(1-2\hbar)^2 + \frac{4\hbar(1-3\hbar)}{u^2+\alpha^2} + \frac{8\hbar^2 u^2}{(u^2+\alpha^2)^2}}{1+\sqrt{-Q_0(u;\alpha,\hbar)}}\,du\\
&= \frac{1}{x_t(\alpha;\hbar)} \int_1^\infty \frac{v^{-2}(1-2\hbar)^2 + \frac{4\hbar(1-3\hbar)}{v^2+\beta^2} + \frac{8\hbar^2 v^2}{(v^2+\beta^2)^2}}{1+\sqrt{-Q_0(x_t(\alpha;\hbar)v;\alpha,\hbar)}}\,dv
\end{aligned}$$

with $\beta(\alpha;\hbar) = \frac{\alpha}{x_t(\alpha;\hbar)}$. By the results of Section 4.5, uniformly in $\alpha > 0$ and $\hbar \in (0, \frac{1}{3}]$,

$$\kappa(\alpha;\hbar) \simeq 1, \quad |\partial_\alpha^\ell \kappa(\alpha;\hbar)| \lesssim \hbar\,(1+\alpha)^{-\ell-2}, \quad \ell \geq 1$$

and

$$\rho(x;\alpha,\hbar) \simeq x^{-1}, \quad |\partial_x^k \rho(x;\alpha,\hbar)| \lesssim x^{-k-1},$$
$$|\partial_x^k \partial_\alpha^\ell \rho(x;\alpha,\hbar)| \lesssim \hbar\,(x+\alpha)^{-\ell-k-1}$$

as claimed. □

Remark 4.14. *The previous analysis covers a larger interval, not just J_1. For example, we can set $J_1 := [x_0, x_1]$ where $0 \ll x_0 \ll 1 \ll x_1$ are fixed.*

Based on Lemma 4.13, we can now describe the behavior of the potential $\widetilde{V}$ in the same regime of parameters.

Proposition 4.15. *For all $x \in J_2$, $\alpha > 0$, and $\hbar \in (0, \frac{1}{3}]$*

$$\begin{aligned} |\partial_\tau^k \widetilde{V}(\tau;\alpha,\hbar)| &\leq C_k\,\tau^{-2-k} \\ |\partial_\tau^k \partial_\alpha^\ell \widetilde{V}(\tau;\alpha,\hbar)| &\leq C_{k,\ell}\,\hbar\,\tau^{-2-k}(1+\alpha)^{-\ell-1} \end{aligned} \tag{4.85}$$

for all $k \geq 0$, $\ell \geq 1$.

Proof. By Lemma 4.13, with some constant c,

$$\tau^3 = c(x - y(\alpha;\hbar) + \rho(x;\alpha,\hbar))^2 = (\xi + \tilde{\rho}(\xi;\alpha,\hbar))^2$$

where $\xi = c^{\frac{1}{2}}(x - y(\alpha;\hbar))$. By the previous remark it suffices to consider the case $\xi \geq 1$. Then $\tilde{\rho}$ satisfies the same estimates relative to ξ as ρ does as a function of x. It is convenient to introduce the new variable

$$\eta = \xi + \tilde{\rho}(\xi;\alpha,\hbar)$$

Thus, $\tau^3 = \eta^2$ and

$$\begin{aligned} 3\tau^2 \frac{\partial \tau}{\partial \eta} = 2\eta, \quad \Big(\frac{\partial \tau}{\partial \eta}\Big)^2 &= \frac{4}{9}\tau^{-1}, \quad \frac{\partial \eta}{\partial \tau} = \frac{3}{2}\tau^{\frac{1}{2}} \\ q(\tau;\alpha,\hbar) = \Big(\frac{\partial \tau}{\partial x}\Big)^2 &= \tau^{-1}(1+\rho'(x;\alpha,\hbar))^2 \\ x^2 q(\tau;\alpha,\hbar) &= (c^{-\frac{1}{2}}\xi + y(\alpha;\hbar))^2(1+\rho'(x;\alpha,\hbar))^2\tau^{-1} \\ &= \tau^2\,(c^{-\frac{1}{2}}(1+\xi^{-1}\tilde{\rho}(\xi;\alpha,\hbar))^{-1} + y(\alpha;\hbar)/\eta)^2(1+\rho'(x;\alpha,\hbar))^2 \end{aligned} \tag{4.86}$$

In view of (4.81), and with $\sigma := q^{-1}\dot{q}$,

$$\widetilde{V}(\tau;\alpha,\hbar) = -\frac{1}{4}\tau^{-2}(c^{-\frac{1}{2}}(1+\xi^{-1}\tilde{\rho}(\xi;\alpha,\hbar))^{-1} + y(\alpha;\hbar)/\eta)^{-2}(1+\rho'(x;\alpha,\hbar))^{-2} + \frac{1}{4}\dot{\sigma} + \frac{1}{16}\sigma^2 \tag{4.87}$$

On the one hand,

$$\begin{aligned} \xi^{-1}\tilde{\rho}(\xi;\alpha,\hbar) &= O(\xi^{-2}) = O(\tau^{-3}) \\ y(\alpha;\hbar)/\eta &= O(\tau^{-\frac{3}{2}}) \\ \rho'(x;\alpha,\hbar) &= O(x^{-2}) = O(\tau^{-3}) \end{aligned}$$

On the other hand, by (4.73), $\tau'(x) = \sqrt{q} = O(\tau^{-\frac{1}{2}})$ (recall $'$ refers to x derivatives)

$$\sigma = -\tau^{-1} + 2(1+\rho'(x;\alpha,\hbar))^{-1} q^{-\frac{1}{2}}\rho'', \qquad q^{-\frac{1}{2}}\rho'' = O(\tau^{-4})$$
$$\dot\sigma = \tau^{-2} + 2q^{-\frac{1}{2}}((1+\rho'(x;\alpha,\hbar))^{-1} q^{-\frac{1}{2}}\rho'')' = \tau^{-2} + O(\tau^{-5})$$

as $\tau \to \infty$. Inserting these estimates into (4.87) implies the first line of (4.85), at least for $k = 0$. As for the derivatives in τ, we have

$$\frac{\partial}{\partial\tau} = q^{-\frac{1}{2}}\frac{\partial}{\partial x} = (1+\rho'(x;\alpha,\hbar))^{-1}\tau^{\frac{1}{2}}\frac{\partial}{\partial x}$$

Each derivative in x gains a power of $x \simeq \tau^{\frac{3}{2}}$, but we then lose a $\tau^{\frac{1}{2}}$ factor by the previous line, resulting in a total gain of a τ^{-1} factor for each application of $\frac{\partial}{\partial\tau}$ to $\widetilde{V}$.

For the α derivatives we compute

$$|\partial_\alpha q(\tau;\alpha,\hbar)| = 2\,\tau^{-1}|(1+\rho'(x;\alpha,\hbar))\partial_x\partial_\alpha\rho(x;\alpha,\hbar)| \lesssim \hbar\tau^{-1}(1+\alpha)^{-3}$$
$$q(\tau;\alpha,\hbar)^{-1}|\partial_\alpha q(\tau;\alpha,\hbar)| \lesssim \hbar(1+\alpha)^{-3}$$

whence

$$|\partial_\alpha\sigma| \lesssim |\partial_\alpha\rho'(x;\alpha,\hbar)||q^{-\frac{1}{2}}|\rho''| + q^{-\frac{1}{2}}|\rho''|q^{-1}|\partial_\alpha q| + q^{-\frac{1}{2}}|\partial_x^2\partial_\alpha\rho(x;\alpha,\hbar)|$$
$$\lesssim \hbar(1+\alpha)^{-3}\tau^{-4} + \hbar\tau^{\frac{1}{2}}(x+\alpha)^{-4} \lesssim \hbar\tau^{-1}(1+\alpha)^{-3}$$
$$|\partial_\alpha\,\sigma^2| \lesssim \hbar\tau^{-2}(1+\alpha)^{-3}$$

Similarly, one checks that $|\partial_\alpha\,\dot\sigma| \lesssim \hbar\tau^{-2}(1+\alpha)^{-3}$. However, if we take ∂_α of the first term in (4.87), then we obtain τ^{-2} multiplied by each of these three terms

$$\xi^{-1}\partial_\alpha\tilde\rho(\xi;\alpha,\hbar), \quad \partial_\alpha(y(\alpha;\hbar)/\eta), \quad \partial_\alpha\rho'(x;\alpha,\hbar)$$

which are on the order of, respectively,

$$\hbar\tau^{-\frac{3}{2}}(1+\alpha)^{-2}, \quad \hbar\tau^{-\frac{3}{2}}(1+\alpha)^{-3}, \quad \hbar(x+\alpha)^{-3} = O(\hbar\tau^{-\frac{3}{2}}(1+\alpha)^{-2})$$

uniformly in the range of parameters under consideration. In view of all these contributions,

$$|\partial_\alpha\widetilde{V}(\tau;\alpha,\hbar)| \lesssim \hbar\tau^{-2}(1+\alpha)^{-2},$$

as claimed. The higher derivatives are controlled similarly. □

It remains to analyze the interval $J_0 := (0, \frac{1}{2}x_t(\alpha;\hbar)]$.

Lemma 4.16. *For $\tau < 0$ one has the representation*

$$\begin{aligned}\widetilde{V}(\tau;\alpha,\hbar) &= \frac{5}{16\tau^2} - \tau\varphi(x;\alpha,\hbar)\\ \varphi(x;\alpha,\hbar) &= \frac{1}{4Q_2(x;\alpha,\hbar)}\Big(x\mu'(x;\alpha,\hbar) - \frac{1}{4}\mu^2(x;\alpha,\hbar)\Big)\\ &= \frac{xQ_2'(x)}{4Q_2(x)^2} + \frac{x^2Q_2''(x)}{4Q_2(x)^2} - \frac{5x^2Q_2'(x)^2}{16Q_2(x)^3}.\end{aligned} \tag{4.88}$$

where $Q_2(x;\alpha,\hbar) := x^2Q_0(x;\alpha,\hbar)$ and $\mu(x;\alpha,\hbar) = xQ_2'(x;\alpha,\hbar)/Q_2(x;\alpha,\hbar)$.

Proof. By (4.87), and suppressing the $\alpha, \hbar$ dependence from the notation,

$$\widetilde{V}(x) = -\frac{1}{4x^2q(x)} + \frac{1}{4}\dot{\sigma}(x) + \frac{1}{16}\sigma^2(x), \qquad \sigma = \dot{q}/q$$

Now,

$$\begin{aligned}
\sigma &= \frac{-\tau}{Q_0(x)}\left(\frac{Q_0(x)}{\tau^2} - \frac{Q_0'(x)}{\tau}q^{-\frac{1}{2}}\right) = -\frac{1}{\tau} + (-\tau)^{\frac{1}{2}}\frac{Q_0'(x)}{Q_0^{\frac{3}{2}}(x)}\\
\sigma^2 &= \frac{1}{\tau^2} + 2(-\tau)^{-\frac{1}{2}}\frac{Q_0'(x)}{Q_0^{\frac{3}{2}}(x)} - \tau\frac{Q_0'(x)^2}{Q_0(x)^3}\\
\dot{\sigma} &= \frac{1}{\tau^2} - \frac{1}{2}(-\tau)^{-\frac{1}{2}}\frac{Q_0'(x)}{Q_0^{\frac{3}{2}}(x)} - \tau Q_0(x)^{-\frac{1}{2}}\left(\frac{Q_0'(x)}{Q_0^{\frac{3}{2}}(x)}\right)'\\
&= \frac{1}{\tau^2} - \frac{1}{2}(-\tau)^{-\frac{1}{2}}\frac{Q_0'(x)}{Q_0^{\frac{3}{2}}(x)} - \tau\left(\frac{Q_0''(x)}{Q_0(x)^2} - \frac{3}{2}\frac{Q_0'(x)^2}{Q_0(x)^3}\right)
\end{aligned}$$

whence

$$\frac{1}{4}\dot{\sigma}(x) + \frac{1}{16}\sigma^2(x) = \frac{5}{16\tau^2} + \tau\left(-\frac{1}{4}\frac{Q_0''(x)}{Q_0(x)^2} + \frac{5}{16}\frac{Q_0'(x)^2}{Q_0(x)^3}\right)$$

and

$$\widetilde{V}(x) = \frac{5}{16\tau^2} + \frac{\tau}{4Q_2(x)}\left(1 - \frac{x^4Q_0''(x)}{Q_2(x)} + \frac{5}{4}\frac{(x^3Q_0'(x))^2}{Q_2(x)^2}\right) \tag{4.89}$$

Inserting

$$\begin{aligned}
Q_0(x) &= x^{-2}Q_2(x), \quad x^3Q_0'(x) = xQ_2'(x) - 2Q_2(x)\\
x^4Q_0''(x) &= 6Q_2(x) - 4xQ_2'(x) + x^2Q_2''(x)
\end{aligned}$$

into (4.89) yields

$$\widetilde{V}(x) = \frac{5}{16\tau^2} - \frac{\tau}{4Q_2(x)}\frac{4xQ_2(x)Q_2'(x) - 5x^2Q_2'(x)^2 + 4x^2Q_2(x)Q_2''(x)}{4Q_2(x)^2}$$

Setting $\mu(x) = xQ_2'(x)/Q_2(x)$ we have

$$x\mu'(x) = x^2Q_2''(x)/Q_2(x) + \mu(x) - \mu(x)^2$$

and thus

$$\widetilde{V}(x) = \frac{5}{16\tau^2} - \frac{\tau}{4Q_2(x)}(x\mu'(x) - \mu(x)^2/4)$$

as claimed. □

From (4.83),

$$\begin{aligned} Q_2(x;\alpha,\hbar) &= (1-2\hbar)^2 + \frac{4\hbar x^2}{x^2+\alpha^2} - \frac{4\hbar^2 x^2(x^2+3\alpha^2)}{(x^2+\alpha^2)^2} - x^2 \\ xQ_2'(x;\alpha,\hbar) &= 8\hbar\alpha^2 \frac{x^2}{(x^2+\alpha^2)^2} + 8\hbar^2\alpha^2 \frac{x^2(x^2-3\alpha^2)}{(x^2+\alpha^2)^3} - 2x^2 \end{aligned} \tag{4.90}$$

We begin the analysis of τ and $\varphi(x;\alpha,\hbar)$ in case $\alpha \gtrsim 1$.

Lemma 4.17. *If $0 < x \le \frac{1}{2}x_t(\alpha,\hbar)$, then the function τ defined by (4.72) has the form*

$$\frac{2}{3}\left(-\tau(x;\alpha,\hbar)\right)^{\frac{3}{2}} = -(1-2\hbar)\log x + \varepsilon_1(x;\alpha,\hbar)$$

with ε_1 satisfying for all $k \ge 0$, $\ell \ge 1$, uniformly in $x \in J_0(\alpha,\hbar)$

$$|\partial_x^k \varepsilon_1(x;\alpha,\hbar)| \le C_{k,\ell}, \qquad |\partial_\alpha^\ell \partial_x^k \varepsilon_1(x;\alpha,\hbar)| \le C_{k,\ell}\,\hbar\,\langle\alpha\rangle^{-2-\ell}$$

for all $\alpha \gtrsim 1$ and $0 < \hbar \le \frac{1}{3}$. Moreover, ε_1 is analytic as a function of x in a neighborhood of 0 and $\partial_x \varepsilon_1(0;\alpha,\hbar) = 0$.

Proof. In this case we have

$$\begin{aligned} \sqrt{Q_0(u;\alpha,\hbar)} &= u^{-1}\sqrt{Q_2(u;\alpha,\hbar)} \\ &= u^{-1}\sqrt{(1-2\hbar)^2 + \frac{4\hbar u^2}{u^2+\alpha^2} - \frac{4\hbar^2 u^2(u^2+3\alpha^2)}{(u^2+\alpha^2)^2} - u^2} \\ &= \frac{1-2\hbar}{u} + f(u;\alpha,\hbar), \end{aligned}$$

where f is analytic in small u, and bounded uniformly (with all derivatives in u) in $\alpha \gtrsim 1$. Moreover, $f(0;\alpha,\hbar) = 0$ and

$$|\partial_\alpha^\ell \partial_u^k f(u,\alpha;\hbar)| \le C_{k,\ell}\,\hbar\langle\alpha\rangle^{-2-\ell}, \quad \ell \ge 1,\ k \ge 0.$$

Note that f does not decay in α, since

$$\lim_{\alpha\to\infty} \sqrt{Q_0(u;\alpha,\hbar)} = u^{-1}\sqrt{(1-2\hbar)^2 - u^2}$$

but derivatives of f relative to α do decay in α. We split the integral in the definition of τ in the form, with $0 < x_0 \ll 1$ fixed independently of $\alpha, \hbar$,

$$\int_x^{x_t(\alpha,\hbar)} = \int_x^{x_0} + \int_{x_0}^{x_t(\alpha,\hbar)}$$

and obtain

$$\begin{aligned} \int_x^{x_t} \sqrt{Q_0(u;\alpha,\hbar)}\,du &= -(1-2\hbar)\log x + (1-2\hbar)\log x_0 + \int_0^{x_0} f(u;\alpha,\hbar)\,du - \int_0^x f(u;\alpha,\hbar)\,du \\ &\quad + \int_{x_0}^{x_t(\alpha,\hbar)} \sqrt{Q_0(u;\alpha,\hbar)}\,du \\ &= -(1-2\hbar)\log x + \varepsilon_1(x;\alpha,\hbar) \end{aligned} \tag{4.91}$$

The final integral here is governed by (4.78) and (4.79). In fact, from (4.78)

$$\int_{x_0}^{x_t(\alpha,\hbar)} \sqrt{Q_0(u;\alpha,\hbar)}\,du = \int_0^{x_t(\alpha;\hbar)-x_0} \sqrt{g(x_t(\alpha;\hbar)-v;\alpha,\hbar)}\,\sqrt{v}\,dv$$

which obeys the desired bounds as a function of α by Lemma 4.10 and (4.79). The term

$$\int_0^{x_0} f(u;\alpha,\hbar)\,du$$

satisfies the same properties, as does

$$\int_0^{x} f(u;\alpha,\hbar)\,du = O(x^2) \text{ as } x \to 0$$

Adding up these different contributions in (4.91) concludes the proof. □

To express x as a function of τ we invert the function in the previous proposition.

Corollary 4.18. *There exists a constant $x_0 \in J_0(\alpha;\hbar)$ so that for all $x \in (0, x_0]$, with the parameters as in the previous proposition, there is a representation*

$$x = X\left(\exp\left(-\frac{2}{3(1-2\hbar)}\,(-\tau(x;\alpha,\hbar))^{\frac{3}{2}}\right);\alpha,\hbar\right)$$

where $\tau \le \tau_0(\alpha,\hbar)$, and $\tau_0(\alpha,\hbar)$ is the value of the map $\tau = \tau(x;\alpha,\hbar)$ at $x = x_0$. Here X is a diffeomorphism $[0, y_0(\alpha;\hbar)] \to [0, x_0]$, where $y_0(\alpha,\hbar)$ is defined such that $X(y_0(\alpha,\hbar);\alpha,\hbar) = x_0$, with

$$X'(0;\alpha,\hbar) = \exp\left(\frac{1}{1-2\hbar}\varepsilon_1(0;\alpha,\hbar)\right).$$

We have $X'(y;\alpha,\hbar) \gtrsim 1$ for all $y \in [0, y_0]$ and $\alpha \gtrsim 1$ and $\hbar \in (0, \frac{1}{3}]$, as well as

$$|\partial_y^k X(y;\alpha,\hbar)| \le C_k, \quad |\partial_y^k \partial_\alpha^\ell X(y;\alpha,\hbar)| \le C_{k,\ell}\,\hbar\,\alpha^{-2-\ell} \tag{4.92}$$

for all $k \ge 0$, $\ell \ge 1$.

Proof. We set

$$\log Y(x;\alpha,\hbar) = \log x - \frac{1}{1-2\hbar}\varepsilon_1(x;\alpha,\hbar)$$
$$Y(x;\alpha,\hbar) = x\exp\left(-\frac{1}{1-2\hbar}\varepsilon_1(x;\alpha,\hbar)\right).$$

By Lemma 4.17, $Y(x;\alpha,\hbar)$ is smooth as a function of x on a neighborhood of 0 containing $J_0(\alpha,\hbar)$. Moreover,

$$Y'(0;\alpha,\hbar) = \exp\left(-\frac{1}{1-2\hbar}\varepsilon_1(0;\alpha,\hbar)\right)$$

is bounded below uniformly in $\alpha \gtrsim 1$ and $\hbar \in (0, \frac{1}{3}]$. It follows that we can smoothly invert $y = Y(x;\alpha,\hbar)$ so that $x = X(y;\alpha,\hbar)$, uniformly in the parameters. Finally, we have the bounds

$$|\partial_x^k Y(x;\alpha,\hbar)| \le C_k, \quad |\partial_x^k \partial_\alpha^\ell Y(x;\alpha,\hbar)| \le C_{k,\ell}\,\hbar\,\alpha^{-2-\ell} \tag{4.93}$$

for all $k \geq 0$ and $\ell \geq 1$. These bounds imply (4.92). Indeed, from $X(Y(x;\alpha,\hbar);\alpha,\hbar) = x$ we deduce

$$\begin{aligned} \partial_y X \partial_x Y &= 1 \\ -(\partial_x Y)^{-1}(\partial_y^2 X \partial_\alpha Y \partial_x Y + \partial_y X \partial_x \partial_\alpha Y) &= \partial_y \partial_\alpha X \\ -(\partial_x Y)^{-1}(\partial_y^3 X (\partial_\alpha Y)^2 \partial_x Y + \partial_y^2 \partial_\alpha X \partial_\alpha Y \partial_x Y + \partial_y^2 X \partial_\alpha^2 Y \partial_x Y + \partial_y^2 X \partial_\alpha Y \partial_x \partial_\alpha Y + & \\ \partial_y^2 \partial_\alpha X \partial_\alpha Y \partial_x Y + 2\partial_y \partial_\alpha X \partial_x \partial_\alpha Y + \partial_y^2 X \partial_\alpha Y \partial_x \partial_\alpha Y + \partial_y X \partial_x \partial_\alpha^2 Y) &= \partial_y \partial_\alpha^2 X \end{aligned} \tag{4.94}$$

which (4.92). The higher derivatives follow inductively. In fact, if we differentiate both sides of the first equation in (4.94) with respect to x, we obtain

$$\begin{aligned} &\partial_y^2 X (\partial_x Y)^2 + \partial_y X \partial_x^2 Y = 0, \\ &\partial_y^3 X (\partial_x Y)^3 + 3\partial_y^2 X \partial_x^2 Y \partial_x Y + \partial_y X \partial_x^3 Y = 0, \\ &... \end{aligned}$$

Inductively, we have

$$\partial_y^k X (\partial_x Y)^k + P_k(\partial_y^{k-1} X, ..., \partial_y X, \partial_x Y, \partial_x^2 Y, ..., \partial_x^k Y) = 0, \quad \text{for} \quad k \geq 2. \tag{4.95}$$

Here $P_k(\partial_y^{k-1} X, ..., \partial_y X, \partial_x Y, \partial_x^2 Y, ..., \partial_x^k Y)$ is a polynomial in the variables $\partial_y^{k-1} X, ..., \partial_y X$ and $\partial_x Y, ..., \partial_x^k Y$. Then we use an induction argument and the lower bound on $\partial_x Y$ for small x as well as the bounds (4.93) to conclude

$$|\partial_y^k X(y,\alpha;\hbar)| \leq C_k, \quad \text{for all} \quad k \geq 0. \tag{4.96}$$

For the α-derivatives of X, we differentiate (4.95) in α on both sides to obtain

$$\partial_y^k \partial_\alpha X (\partial_x Y)^k + k \partial_y^k X \partial_x \partial_\alpha Y (\partial_x Y)^{k-1} + \partial_\alpha \left(P_k(\partial_y^{k-1} X, ..., \partial_y X, \partial_x Y, ..., \partial_x^k Y) \right) = 0, \quad \text{for} \quad k \geq 2. \tag{4.97}$$

Here $\partial_\alpha \left(P_k(\partial_y^{k-1} X, ..., \partial_y X, \partial_x^2 Y, ..., \partial_x^k Y) \right)$ is a polynomial containing factors $\partial_y^{k-1}\partial_\alpha X, ...,$ and $\partial_y \partial_\alpha X, \partial_x \partial_\alpha Y, ..., \partial_x^k \partial_\alpha Y$, as well as $\partial_y^{k-1} X, ..., \partial_y X,\ \partial_x Y, ..., \partial_x^k Y$. Therefore the estimate on $\partial_y^k \partial_\alpha X$ follows from the estimates on the higher order y-derivatives of X, and the higher order derivatives of Y. Again, inductively we have

$$\partial_y^k \partial_\alpha^\ell X \partial_x Y + Q_{k,\ell} = 0. \tag{4.98}$$

Here $Q_{k,\ell}$ is a polynomial with factors $\partial_y^{k'} \partial_\alpha^{\ell'} X$, as well as higher order derivatives of Y, where $k' \leq k, \ell' \leq \ell$ and at least one of $k' < k, \ell' < \ell$ holds. Therefore the estimates on higher order derivatives $\partial_y^k \partial_\alpha^\ell X$ follow from an induction argument. □

Based on Lemma 4.17 and the previous corollary, we now analyze the behavior of $\widetilde{V}$ for $0 < x \leq \frac{1}{2} x_t(\alpha;\hbar)$ and $\alpha \gtrsim 1$.

Proposition 4.19. *For $0 < x \leq \frac{1}{2} x_t(\alpha;\hbar)$ and $\alpha \gtrsim 1$, we have*

$$|\partial_\tau^k \widetilde{V}(\tau;\alpha,\hbar)| \leq C_k\, \tau^{-2-k}, \qquad |\partial_\tau^k \partial_\alpha^\ell \widetilde{V}(\tau;\alpha,\hbar)| \leq C_{k,\ell} \hbar\, e^\tau \alpha^{-2-\ell}$$

for all $k \geq 0$ and $\ell \geq 1$.

Proof. From Lemma 4.16 and (4.90)

$$\begin{aligned}\widetilde{V}(\tau;\alpha,\hbar) &= \frac{5}{16\tau^2} + \tau\, O_{\alpha,\hbar}\left(X\left(\exp\left(-\frac{2}{3(1-2\hbar)}\,(-\tau)^{\frac{3}{2}}\right);\alpha,\hbar\right)^2\right) \\ &= \frac{5}{16\tau^2} + \tau\, O_{\alpha,\hbar}\left(\exp\left(-\frac{4}{3(1-2\hbar)}\,(-\tau)^{\frac{3}{2}}\right)\right)\end{aligned}$$

By inspection of the function μ in Lemma 4.16 and (4.90) one sees that the $O_{\alpha,\hbar}$-term here depends on α only through terms which are on the order of $\hbar$. Any derivative in α of the order ℓ will gain $\hbar\alpha^{-2-\ell}$ by the structure of μ and Corollary 4.18. □

The exponential decay stated in the proposition is not optimal, it was chosen for convenience.

Next we turn to the case when $0 < \alpha \ll 1$. An important difference arises here with respect to the potential $\widetilde{V}(x;\alpha,\hbar)$. Indeed, one checks that

$$x\mu'(x;\alpha,\hbar) - \mu(x;\alpha,\hbar)^2/4\Big|_{x=\alpha} = -\frac{\alpha^4 + \hbar^2(1-2\hbar+5\hbar^2) + 2\alpha^2(2-9\hbar+7\hbar^2)}{(1-\alpha^2-2\hbar)^2} < 0$$

for small $\hbar$ and α. This means that the term $\tau\varphi(x;\alpha,\hbar)$ in (4.88) is large for small α when $x \simeq \alpha$, and dominates $5/(16\tau^2)$. We distinguish three parameter regimes: $0 < x \ll \alpha \ll 1$, $0 < \alpha \ll x \ll 1$, and $\alpha \simeq x \ll 1$. We begin with the latter case.

Lemma 4.20. *Let $0 < k_0 \ll 1 \ll K_0$ be fixed constants. Then for all $x \in [k_0\alpha, K_0\alpha]$,*

$$\frac{2}{3}\,(-\tau(x;\alpha,\hbar))^{\frac{3}{2}} = -\log x + \sigma(x;\alpha,\hbar) \tag{4.99}$$

where

$$|\partial_x^k\partial_\alpha^\ell\sigma(x;\alpha,\hbar)| \le C_{k,\ell}\,\alpha^{-k-\ell}$$

for all $k,\ell \ge 0$. If $\ell \ge 1$, then a factor of $\hbar$ is gained on the right-hand side. The constants depend on k_0, K_0, and $K_0\alpha \ll 1$.

Proof. We have

$$\begin{aligned}\frac{2}{3}\,(-\tau(x;\alpha,\hbar))^{\frac{3}{2}} &= \int_x^{x_t}\left((1-2\hbar)^2 + \frac{4\hbar u^2}{u^2+\alpha^2} - \frac{4\hbar^2u^2(u^2+3\alpha^2)}{(u^2+\alpha^2)^2} - u^2\right)^{\frac{1}{2}}\,\frac{du}{u} \\ &= \int_{x/\alpha}^{x_t/\alpha}\left((1-2\hbar)^2 + \frac{4\hbar v^2}{1+v^2} - \frac{4\hbar^2v^2(v^2+3)}{(1+v^2)^2} - \alpha^2v^2\right)^{\frac{1}{2}}\,\frac{dv}{v} \\ &= \int_{x/\alpha}^{x_t/\alpha}\left(1 - 4\hbar\frac{1-\hbar+(1+\hbar)v^2}{(1+v^2)^2} - \alpha^2v^2\right)^{\frac{1}{2}}\,\frac{dv}{v}\end{aligned}$$

Since for $\hbar \in (0,\frac{1}{3}]$ and $v > 0$

$$\frac{d}{dv}\frac{1-\hbar+(1+\hbar)v^2}{(1+v^2)^2} = -2v(1+v^2)^{-3}(1-3\hbar+(1+\hbar)v^2) < 0$$

it follows that

$$4\hbar\frac{1-\hbar+(1+\hbar)v^2}{(1+v^2)^2} \le 4\hbar(1-\hbar) \le \frac{8}{9}$$

Thus, if $v \le \frac{1}{4\alpha}$, then

$$4\hbar\frac{1-\hbar+(1+\hbar)v^2}{(1+v^2)^2} + \alpha^2 v^2 \le \frac{8}{9} + \frac{1}{16} < 1$$

and

$$\left(1 - 4\hbar\frac{1-\hbar+(1+\hbar)v^2}{(1+v^2)^2} - \alpha^2 v^2\right)^{\frac{1}{2}} = 1 - \sum_{k=1}^{\infty}(-1)^{k-1}\binom{\frac{1}{2}}{k}\left(4\hbar\frac{1-\hbar+(1+\hbar)v^2}{(1+v^2)^2} + \alpha^2 v^2\right)^k$$

converges uniformly on $0 \le v \le \frac{1}{4\alpha}$, and $\hbar \in (0, \frac{1}{3}]$. It is, however, not easy to integrate term-wise and estimate each term separately in the whole range of $\hbar$. Instead, we proceed as follows: if $4K_0\alpha < 1$, then

$$\begin{aligned}
\frac{2}{3}\,(-\tau(x;\alpha,\hbar))^{\frac{3}{2}} &= -\log(4x) + \omega_1(\alpha,\hbar) + \omega_2(x;\alpha,\hbar) \\
\omega_1(\alpha,\hbar) &:= \int_{1/(4\alpha)}^{x_t/\alpha}\left(1 - 4\hbar\frac{1-\hbar+(1+\hbar)v^2}{(1+v^2)^2} - \alpha^2 v^2\right)^{\frac{1}{2}}\,\frac{dv}{v} \\
\omega_2(x;\alpha,\hbar) &:= -\int_{x/\alpha}^{1/(4\alpha)} \frac{4\hbar\frac{1-\hbar+(1+\hbar)v^2}{(1+v^2)^2} + \alpha^2 v^2}{1 + \left(1 - 4\hbar\frac{1-\hbar+(1+\hbar)v^2}{(1+v^2)^2} - \alpha^2 v^2\right)^{\frac{1}{2}}}\,\frac{dv}{v}
\end{aligned}$$

Note that by Lemma 4.10, $x_t(\alpha;\hbar) > \frac{1}{3}$. Lemma 4.11 applies to $\omega_1(\alpha,\hbar)$, i.e.,

$$0 \le \omega_1(\alpha,\hbar) \lesssim 1, \qquad |\partial_\alpha^\ell \omega_1(\alpha,\hbar)| \le C_\ell\,\hbar \ \text{ for all } \ \ell \ge 1$$

uniformly in the regime of parameters under consideration. Furthermore,

$$\begin{aligned}
|\omega_2(x;\alpha,\hbar)| &\le \int_{x/\alpha}^{1/(4\alpha)} \left(4\hbar(1+\hbar)(1+v^2)^{-1} + \alpha^2 v^2\right)\,\frac{dv}{v} \\
&= 2\hbar(1+\hbar)\log\frac{v^2}{1+v^2} + \frac{1}{2}\alpha^2 v^2\Big|_{x/\alpha}^{1/(4\alpha)} \\
&= 2\hbar(1+\hbar)\left(-\log(1+16\alpha^2) + \log(1+\alpha^2/x^2)\right) + \frac{1}{2}\left(\frac{1}{16} - x^2\right) = O(1)
\end{aligned}$$

uniformly in $x \simeq \alpha \ll 1$. Next, by inspection, for all $k \ge 0$,

$$|\partial_x^k \omega_2(x;\alpha,\hbar)| \le C_k \alpha^{-k}$$

Since $x \simeq \alpha$, we can replace α^{-k} with x^{-k}. For the derivatives relative to α it is convenient to undo the scaling of ω_2, to wit

$$\omega_2(x;\alpha,\hbar) := \int_x^{1/4} \frac{4\hbar\alpha^2\frac{(1-\hbar)\alpha^2+(1+\hbar)u^2}{(\alpha^2+u^2)^2} + u^2}{1 + \left(1 - 4\hbar\alpha^2\frac{(1-\hbar)\alpha^2+(1+\hbar)u^2}{(\alpha^2+u^2)^2} - u^2\right)^{\frac{1}{2}}}\,\frac{du}{u}$$

From this expression one can derive that

$$|\partial_x^k \partial_\alpha^\ell \omega_2(x;\alpha,\hbar)| \le C_{k,\ell}\,\hbar\alpha^{-k-\ell}$$

if $k \ge 0$ and $\ell \ge 1$. □

Next, we describe $\widetilde{V}$ in the regime of Lemma 4.20.

Proposition 4.21. *Using the representation of Lemma 4.16, we have*

$$\widetilde{V}(\tau;\alpha,\hbar) = \frac{5}{16\tau^2} - \tau\varphi(x;\alpha,\hbar) \tag{4.100}$$

where $|\varphi(x;\alpha,\hbar)| \le C(\hbar+\alpha^2)$ *uniformly in* $x \in [k_0\alpha, K_0\alpha]$, $0 < \alpha \ll 1$. *Moreover,*

$$|\partial_\tau^k \partial_\alpha^\ell \varphi(x;\alpha,\hbar)| \le C_{k,\ell}\,(\hbar+\alpha^2)(-\tau)^{\frac{k}{2}}\alpha^{-\ell} \tag{4.101}$$

for all $k,\ell \ge 0$. *If* $\ell \ge 1$, *then a factor of* $\hbar$ *is gained in* (4.101). *The constants depend on* k_0, K_0, *and* $\alpha_0 > 0$ *where* $K_0\alpha \le \alpha_0 \ll 1$.

Proof. By Lemma 4.16,

$$\varphi(x;\alpha,\hbar) = \frac{1}{4Q_2(x;\alpha,\hbar)}\Big(x\mu'(x;\alpha,\hbar) - \frac{1}{4}\mu^2(x;\alpha,\hbar)\Big)$$

where $Q_2(x;\alpha,\hbar) := x^2 Q_0(x;\alpha,\hbar)$ and $\mu(x;\alpha,\hbar) = xQ_2'(x;\alpha,\hbar)/Q_2(x;\alpha,\hbar)$. Formulas (4.90) determine μ. By inspection, $|\mu| \lesssim \hbar+\alpha^2$ and $|x\mu'| \lesssim \hbar+\alpha^2$ for all $x \simeq \alpha$. For the derivatives, recall that

$$\frac{\partial}{\partial\tau} = q^{-\frac12}\frac{\partial}{\partial x} = (-\tau)^{\frac12}\frac{1}{\sqrt{Q_2(x;\alpha,\hbar)}}\,x\frac{\partial}{\partial x} \tag{4.102}$$

The $x\frac{\partial}{\partial x}$ operator does not change the bound on μ or $\varphi(x;\alpha,\hbar)$. The largest contribution to the higher derivatives comes from

$$\frac{\partial^k}{\partial\tau^k} = \left(q^{-\frac12}\frac{\partial}{\partial x}\right)^k = (-\tau)^{\frac{k}{2}}\left(\sqrt{Q_2(x;\alpha,\hbar)}\right)^{-k}\left(x\frac{\partial}{\partial x}\right)^k + \text{ lower order} \tag{4.103}$$

where "lower order" refers to terms involving fewer x-derivatives. The derivatives with respect to α bring out a (single) factor of $\hbar$, and lose a factor of α^{-1} each. This yields (4.101). □

Next, we analyze the case $0 < x \ll \alpha \ll 1$.

Lemma 4.22. *If* $0 < x \ll \alpha \ll 1$, *then the function* τ *defined by* (4.72) *has the form*

$$\frac{2}{3}(-\tau(x;\alpha,\hbar))^{\frac32} = -\log x + 2\hbar\log(x/\alpha) + \psi(x/\alpha;\alpha,\hbar) + \rho(\alpha;\hbar)$$

where $\psi(z;\alpha,\hbar)$ *is analytic near* 0, *and uniformly in* $0 < \alpha \ll 1$, *and* $|z| \le r_0$ *for some absolute constant* $0 < r_0 \ll 1$,

$$|\partial_z^k \partial_\alpha^\ell \psi(z;\alpha,\hbar)| \le C_{k,\ell}\,(\hbar+\alpha^2), \quad |\partial_\alpha^\ell \rho(\alpha;\hbar)| \le C_\ell\,\alpha^{-\ell}$$

for all $k,\ell \ge 0$.

Proof. We have

$$\begin{aligned}\frac{2}{3}(-\tau(x;\alpha,\hbar))^{\frac{3}{2}} &= \frac{2}{3}(-\tau(k_0\alpha;\alpha,\hbar))^{\frac{3}{2}} + \int_x^{k_0\alpha}\left((1-2\hbar)^2 + \frac{4\hbar u^2}{u^2+\alpha^2} - \frac{4\hbar^2 u^2(u^2+3\alpha^2)}{(u^2+\alpha^2)^2} - u^2\right)^{\frac{1}{2}}\frac{du}{u} \\ &= \frac{2}{3}(-\tau(k_0\alpha;\alpha,\hbar))^{\frac{3}{2}} + \int_{x/\alpha}^{k_0}\left((1-2\hbar)^2 + \frac{4\hbar v^2}{1+v^2} - \frac{4\hbar^2 v^2(v^2+3)}{(1+v^2)^2} - \alpha^2 v^2\right)^{\frac{1}{2}}\frac{dv}{v}\end{aligned} \tag{4.104}$$

We write, with analytic f_1 and f_2 on the unit disk,

$$\frac{v^2}{1+v^2} = v^2 f_1(v^2), \quad \frac{v^2(v^2+3)}{(1+v^2)^2} = 3v^2 f_2(v^2), \quad f_1(0) = f_2(0) = 1$$

whence

$$\begin{aligned}&\left((1-2\hbar)^2 + \frac{4\hbar v^2}{1+v^2} - \frac{4\hbar^2 v^2(v^2+3)}{(1+v^2)^2} - \alpha^2 v^2\right)^{\frac{1}{2}} \\ &= (1-2\hbar)\left(1 + 4\hbar(1-2\hbar)^{-2}v^2 f_1(v^2) - 12\hbar^2(1-2\hbar)^{-2}v^2 f_2(v^2) - \alpha^2(1-2\hbar)^{-2}v^2\right)^{\frac{1}{2}} \\ &= (1-2\hbar)\left(1 + v^2 f(v^2;\alpha,\hbar)\right)^{\frac{1}{2}} = (1-2\hbar)\left(1 + \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n} v^{2n} f(v^2;\alpha,\hbar)^n\right)\end{aligned}$$

with

$$\begin{aligned}f(v^2;\alpha,\hbar) &= 4\hbar(1-2\hbar)^{-2}f_1(v^2) - 12\hbar^2(1-2\hbar)^{-2}f_2(v^2) - \alpha^2(1-2\hbar)^{-2} \\ f(0;\alpha,\hbar) &= 4\hbar(1-2\hbar)^{-2} - 12\hbar^2(1-2\hbar)^{-2} - \alpha^2(1-2\hbar)^{-2}\end{aligned}$$

as well as in the complex plane

$$\max_{|z|\le\frac{1}{2}} |f(z;\alpha,\hbar)| \le C(\hbar+\alpha^2) \tag{4.105}$$

where C is an absolute constant. With $n \ge 1$,

$$F_n(y;\alpha,\hbar) := \int_0^y v^{2n-1} f(v^2;\alpha,\hbar)^n\, dv$$

$$\int_{x/\alpha}^{k_0} v^{2n-1} f(v^2;\alpha,\hbar)^n\, dv = F_n(k_0;\alpha,\hbar) - F_n(x/\alpha;\alpha,\hbar)$$

By (4.105), if $0 \le y \le \frac{1}{2}$, then

$$|F_n(y;\alpha,\hbar)| \le C^n(\hbar+\alpha^2)^n y^{2n}$$

Thus, by (4.104), and Lemma 4.20, and with k_0 some fixed small constant,

$$\begin{aligned}\frac{2}{3}\,(-\tau(x;\alpha,\hbar))^{\frac{3}{2}} &= \frac{2}{3}\,(-\tau(k_0\alpha;\alpha,\hbar))^{\frac{3}{2}} + \int_{x/\alpha}^{k_0}(1-2\hbar)\Big(1+\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}v^{2n}f(v^2;\alpha,\hbar)^n\Big)\,\frac{dv}{v}\\ &= \frac{2}{3}\,(-\tau(k_0\alpha;\alpha,\hbar))^{\frac{3}{2}} + (1-2\hbar)\log k_0 - (1-2\hbar)\log\frac{x}{\alpha}\\ &\quad + (1-2\hbar)\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}\big(F_n(k_0;\alpha,\hbar)-F_n(x/\alpha;\alpha,\hbar)\big)\\ &= -\log x + 2\hbar\log(x/\alpha) + \psi(x/\alpha;\alpha,\hbar) + \rho(\alpha;\hbar)\end{aligned}$$

where ψ and ρ are given by

$$\begin{aligned}\psi\,(z;\alpha,\hbar) &:= -\,(1-2\hbar)\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}F_n\,(z;\alpha,\hbar)\,,\\ \rho(\alpha;\hbar) &:= \frac{2}{3}\,(-\tau(k_0\alpha;\alpha,\hbar))^{\frac{3}{2}} + (1-2\hbar)\log k_0 + \log\alpha + (1-2\hbar)\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}F_n(k_0;\alpha,\hbar),\end{aligned}$$

and have the properties stated in the lemma. In particular, the estimate on $\tau(k_0\alpha;\alpha,\hbar)$ follows from the result in Lemma 4.20. □

Based on Lemma 4.22, we now describe the potential $\widetilde{V}(\tau;\alpha,\hbar)$ in the regime $0 < x \ll \alpha \ll 1$.

Proposition 4.23. *In the parameter regime of the previous lemma, and using the representation* (4.100)*,*

$$\begin{aligned}|\partial_\tau^k\partial_\alpha^\ell\varphi(x;\alpha,\hbar)| &\le C_{k,\ell}\,(\hbar+\alpha^2)\,\alpha^{-\ell}(-\log x + 2\hbar\log(x/\alpha))^{\frac{k}{3}}(x/\alpha)^2\\ &\le C_{k,\ell}\,(\hbar+\alpha^2)\,\alpha^{-\ell}(-\tau)^{\frac{k}{2}}(x/\alpha)^2\end{aligned}$$

for all $k,\ell \ge 0$.

Proof. By (4.90), with $\xi := x/\alpha$,

$$\begin{aligned}Q_2(x;\alpha,\hbar) &= (1-2\hbar)^2 + \frac{4\hbar x^2}{x^2+\alpha^2} - \frac{4\hbar^2x^2(x^2+3\alpha^2)}{(x^2+\alpha^2)^2} - x^2\\ &= (1-2\hbar)^2 + \frac{4\hbar\xi^2}{1+\xi^2} - \frac{4\hbar^2\xi^2(\xi^2+3)}{(1+\xi^2)^2} - \alpha^2\xi^2\\ xQ_2'(x;\alpha,\hbar) &= 8\hbar\alpha^2\frac{x^2}{(x^2+\alpha^2)^2} + 8\hbar^2\alpha^2\frac{x^2(x^2-3\alpha^2)}{(x^2+\alpha^2)^3} - 2x^2\\ &= 8\hbar\frac{\xi^2}{(1+\xi^2)^2} + 8\hbar^2\frac{\xi^2(\xi^2-3)}{(1+\xi^2)^3} - 2\alpha^2\xi^2\end{aligned}$$

Thus, we have $|\mu| \le C(\hbar+\alpha^2)\xi^2$ and therefore

$$|\varphi(x;\alpha,\hbar)| \le C(\hbar+\alpha^2)\xi^2, \qquad |\partial_\alpha^\ell\varphi(x;\alpha,\hbar)| \le C(\hbar+\alpha^2)\alpha^{-\ell}\,\xi^2$$

uniformly in the parameter regime of the proposition, for all $\ell \geq 0$. In view of (4.102), (4.103), $x\partial_x = \xi\partial_\xi$, and Lemma 4.22,

$$\begin{aligned} |\partial_\tau^k \varphi(x;\alpha,\hbar)| &\leq C_k\,(\hbar+\alpha^2)\,(\sqrt{-\tau})^k\xi^2 \\ &\leq C_k\,(\hbar+\alpha^2)\,(-\log x + 2\hbar\log(x/\alpha))^{\frac{k}{3}}\xi^2 \end{aligned}$$

for all $k \geq 0$, as well as

$$|\partial_\tau^k\partial_\alpha^\ell \varphi(x;\alpha,\hbar)| \leq C_{k,\ell}\,(\hbar+\alpha^2)\,\alpha^{-\ell}(-\log x + 2\hbar\log(x/\alpha))^{\frac{k}{3}}\xi^2$$

as claimed. □

Finally we discuss the case when $0 < \alpha \ll x \ll 1$.

Lemma 4.24. *The function τ defined by* (4.72) *has the form, with $\eta = \alpha/x$,*

$$\frac{2}{3}\,(-\tau(x;\alpha,\hbar))^{\frac{3}{2}} = -(1-\alpha^2 f_3(\alpha;\hbar))\log x + C(\alpha;\hbar) + x^2 f_1(x,\eta,\alpha;\hbar) + \hbar\eta^2 f_2(x,\eta,\alpha;\hbar) \tag{4.106}$$

for all $0 < \alpha \ll x \ll 1$ and $0 < \hbar \leq \frac{1}{3}$. Here f_1 and f_2 are analytic in small (complex) x, η, α and bounded on a small polydisc in $\mathbb{C}^3$ centered at $(0,0,0)$, uniformly in $0 < \hbar \leq \frac{1}{3}$. The function f_3 is analytic and bounded on a small disk of small complex α, uniformly in the same range of $\hbar$. Finally, for all $\ell \geq 0$ we have $|\partial_\alpha^\ell C(\alpha;\hbar)| \leq C_\ell\,\alpha^{-\ell}$ uniformly in small α and $\hbar$ in the same range as before.

Proof. With $\eta = \frac{\alpha}{u}$, we have

$$\begin{aligned} Q_2(u;\alpha,\hbar) &= 1 - \frac{4\hbar(1-\hbar)\alpha^2}{u^2+\alpha^2} - \frac{8\hbar^2\alpha^2u^2}{(u^2+\alpha^2)^2} - u^2 \\ &= 1 - \frac{4\hbar(1-\hbar)\eta^2}{1+\eta^2} - \frac{8\hbar^2\eta^2}{(1+\eta^2)^2} - u^2 \\ &= 1 - \hbar\eta^2 g(\eta^2;\hbar) - u^2 \end{aligned} \tag{4.107}$$

with

$$g(z;\hbar) = \frac{4(1-\hbar)}{1+z} + \frac{8\hbar}{(1+z)^2}, \quad g(0;\hbar) = 4(1+\hbar)$$

Thus, fixing some $0 < x_0 \ll 1$ so that $0 \ll \alpha \ll x \leq x_0$, we have

$$\begin{aligned} \frac{2}{3}\,(-\tau(x;\alpha,\hbar))^{\frac{3}{2}} &= \frac{2}{3}\,(-\tau(x_0;\alpha,\hbar))^{\frac{3}{2}} + \log x_0 - \log x \\ &\quad - \int_x^{x_0}\Big[\Big(1-\hbar\eta^2 g(\eta^2;\hbar)-u^2\Big)^{\frac{1}{2}} - 1\Big]\frac{du}{u} \\ &= \frac{2}{3}\,(-\tau(x_0;\alpha,\hbar))^{\frac{3}{2}} + \log x_0 - \log x \\ &\quad + \sum_{n=1}^{\infty}(-1)^{n-1}\binom{\frac{1}{2}}{n}\sum_{\ell=0}^{n}\binom{n}{\ell}\hbar^\ell\alpha^{2\ell}\int_x^{x_0} g(\alpha^2u^{-2};\hbar)^\ell u^{2(n-2\ell)-1}\,du \end{aligned} \tag{4.108}$$

Plugging the Taylor expansion of g into the integral leads to expressions of the form (because of smallness, uniform convergence holds and integrations and summations can be exchanged), with $j \geq 0, 0 \leq \ell \leq n$,

$$\begin{aligned}\hbar^\ell \alpha^{2(\ell+j)} \int_x^{x_0} u^{2(n-2\ell-j)-1}\, du &= \frac{\hbar^\ell \alpha^{2(\ell+j)}}{2(n-2\ell-j)}(x_0^{2(n-2\ell-j)} - x^{2(n-2\ell-j)}) \\ &= O(\hbar^\ell \alpha^{2(\ell+j)} x_0^{2(n-2\ell-j)}) + O(\hbar^\ell x^{2(n-\ell)} \eta^{2(\ell+j)})\end{aligned}$$

if $n - 2\ell - j \neq 0$. Observe that

$$\begin{aligned}O(\hbar^\ell x^{2(n-\ell)} \eta^{2(\ell+j)}) &= O(x^2) \text{ if } \ell = 0 \\ O(\hbar^\ell x^{2(n-\ell)} \eta^{2(\ell+j)}) &= \hbar\, O(\eta^2) \text{ if } \ell > 0\end{aligned} \tag{4.109}$$

If $n - 2\ell - j = 0$, then

$$\begin{aligned}\hbar^\ell \alpha^{2(\ell+j)} \int_x^{x_0} u^{2(n-2\ell-j)-1}\, du &= \hbar^\ell \alpha^{2(n-\ell)}(\log x_0 - \log x) \\ &= \hbar^\ell \alpha^{2(\ell+j)}(\log x_0 - \log x)\end{aligned}$$

We have $n - \ell = \ell + j \geq 1$ (if $\ell = 0$, then $j = n \geq 1$), so the $-\log x$ is multiplied with the small factor α^2. In summary, in view of (4.108), (4.109), and Lemma 4.11 we conclude that

$$\frac{2}{3}(-\tau(x;\alpha,\hbar))^{\frac{3}{2}} = -\log x + C(\alpha;\hbar) + x^2 f_1(x,\eta,\alpha;\hbar) + \hbar\eta^2 f_2(x,\eta,\alpha;\hbar) + \alpha^2 f_3(\alpha;\hbar)\log x$$

where $\eta = \alpha/x$, and all functions have the properties stated in the lemma. In particular, in order to estimate $\tau(x_0;\alpha,\hbar)$ contributing to $C(\alpha,\hbar)$, we use the results in Lemma 4.11, by choosing $x_0 \in J_1$. □

Based on Lemma 4.24, we are finally able to describe $\widetilde{V}$ in the one remaining case.

Proposition 4.25. *In the parameter regime of the previous lemma, and using the representation* (4.100)*,*

$$\begin{aligned}|\partial_\tau^k \partial_\alpha^\ell \varphi(x;\alpha,\hbar)| &\leq C_{k,\ell}\,(\hbar\alpha^2/x^2 + x^2)\,\alpha^{-\ell}(\sqrt{-\tau})^k \\ &\leq C_{k,\ell}\,(\hbar\alpha^2/x^2 + x^2)\,\alpha^{-\ell}(-\log x)^{\frac{k}{3}}\end{aligned}$$

for all $k, \ell \geq 0$.

Proof. By (4.90), with $\eta := \alpha/x$,

$$\begin{aligned}Q_2(x;\alpha,\hbar) &= (1-2\hbar)^2 + \frac{4\hbar x^2}{x^2+\alpha^2} - \frac{4\hbar^2 x^2(x^2+3\alpha^2)}{(x^2+\alpha^2)^2} - x^2 \\ &= (1-2\hbar)^2 + \frac{4\hbar}{1+\eta^2} - \frac{4\hbar^2\eta^2(\eta^2+3)}{(1+\eta^2)^2} - x^2 \\ xQ_2'(x;\alpha,\hbar) &= 8\hbar\alpha^2 \frac{x^2}{(x^2+\alpha^2)^2} + 8\hbar^2\alpha^2 \frac{x^2(x^2-3\alpha^2)}{(x^2+\alpha^2)^3} - 2x^2 \\ &= 8\hbar \frac{\eta^2}{(1+\eta^2)^2} + 8\hbar^2 \frac{\eta^2(1-3\eta^2)}{(1+\eta^2)^3} - 2x^2\end{aligned}$$

Thus, recalling Lemma 4.16 for the definition of μ, we have $|\mu| \leq C(\hbar\eta^2 + x^2)$ and therefore

$$|\varphi(x;\alpha,\hbar)| \leq C(\hbar\eta^2 + x^2), \qquad |\partial_\alpha^\ell \varphi(x;\alpha,\hbar)| \leq C(\hbar\eta^2 + x^2)\alpha^{-\ell}$$

uniformly in the parameter regime of the proposition, for all $\ell \geq 0$. In view of (4.102), (4.103), $x\partial_x = -\eta\partial_\eta$, and Lemma 4.24,

$$|\partial_\tau^k \varphi(x;\alpha,\hbar)| \leq C_k\,(\hbar\eta^2 + x^2)\,(\sqrt{-\tau})^k \\ \leq C_k\,(\hbar\eta^2 + x^2)\,(-\log x)^{\frac{k}{3}}$$

for all $k \geq 0$, as well as

$$|\partial_\tau^k \partial_\alpha^\ell \varphi(x;\alpha,\hbar)| \leq C_{k,\ell}\,(\hbar\eta^2 + x^2)\,\alpha^{-\ell}(\sqrt{-\tau})^k \\ \leq C_{k,\ell}\,(\hbar\eta^2 + x^2)\,\alpha^{-\ell}(-\log x)^{\frac{k}{3}}$$

as claimed. □

Propositions 4.12, 4.15, 4.19, 4.23 and 4.25 provide the following complete description for the potential $\widetilde{V}(\tau;\alpha,\hbar)$. We ignore a factor of $\hbar$ which might appear upon differentiation with respect to α.

Proposition 4.26. *There exists a constant $\tau_* > 0$ and a small constant $0 < \alpha_* \ll 1$ so that uniformly in $\hbar \in (0, \frac{1}{3}]$,*

$$|\partial_\tau^k \partial_\alpha^\ell \widetilde{V}(\tau;\alpha,\hbar)| \leq C_{k,\ell}\,\langle\alpha\rangle^{-\ell-1}\langle\tau\rangle^{-2-k}, \qquad \forall\, \tau \geq -\tau_* \tag{4.110}$$

for all $k, \ell \geq 0$ and $\alpha > 0$. Moreover, (4.110) *holds for $-\infty < \tau \leq -\tau_*$, all $k, \ell \geq 0$ and $\alpha \geq \alpha_*$. Finally, if $0 < \alpha \leq \alpha_*$ and $-\infty < \tau \leq -\tau_*$, then*

$$\widetilde{V}(\tau;\alpha,\hbar) = \frac{5}{16\tau^2} - \tau\varphi(x;\alpha,\hbar)$$

where for all $k, \ell \geq 0$

$$|\partial_\tau^k \partial_\alpha^\ell \varphi(x;\alpha,\hbar)| \leq C_{k,\ell}\,\min\left(\hbar\,\alpha^2 x^{-2} + x^2, \hbar x^2/\alpha^2 + x^2\right)\alpha^{-\ell}(-\tau)^{\frac{k}{2}} \tag{4.111}$$

Here $x = x(\tau;\alpha,\hbar)$ is the inverse of the diffeomorphism $\tau = \tau(x;\alpha,\hbar)$ defined in (4.72)*, and satisfies*

$$\frac{2}{3}\,(-\tau(x;\alpha,\hbar))^{\frac{3}{2}} = \begin{cases} -(1 - O(\alpha^2))\log x + O_1(1), & \text{if } 0 < \alpha \leq x \leq x_* := x(-\tau_*;\alpha,\hbar) \\ -\log x + 2\hbar\log(x/\alpha) + O_2(1), & \text{if } 0 < x \leq \alpha \leq \alpha_* \end{cases} \tag{4.112}$$

Here $O(\alpha^2)$ is analytic in complex $|\alpha| \leq \alpha_*$, and bounded uniformly in $\hbar \in (0, \frac{1}{3}]$. Furthermore, the two terms $O_1(1)$, resp. $O_2(1)$ refer to smooth functions of τ, α (and thus also of x), uniformly bounded in $0 < \alpha \leq \alpha_*$, $-\infty < \tau \leq -\tau_*$, and so that for all $k, \ell \geq 0$ one has $\partial_x^k \partial_\alpha^\ell O_1(1) = O(x^{-k}\alpha^{-\ell})$ in the parameter regime of the first line of (4.112)*, resp. $\partial_x^k \partial_\alpha^\ell O_2(1) = O(\alpha^{-k-\ell})$ in the regime of the second line of* (4.112).

4.6. **Fundamental system for the perturbed Airy equation.** In this section we will analyze a fundamental system of solutions of the perturbed Airy equation (4.76), viz.

$$\hbar^2 \ddot{w}(\zeta;\hbar) = \zeta w(\zeta;\hbar) + \hbar^2 \widetilde{V}(-\zeta,\alpha;\hbar)w(\zeta;\hbar).$$

Here $\widetilde{V}$ is as in Proposition 4.26, but we switched to the independent variable $\zeta = -\tau$. With τ_* as in that proposition, we set $\zeta_* := -(-\tau_*) = \tau_*$.

Recall that the unperturbed semiclassical Airy equation

$$\hbar^2 \ddot{w}_0(\zeta;\hbar) = \zeta w_0(\zeta;\hbar) \tag{4.113}$$

has a fundamental system $\mathrm{Ai}(\hbar^{-\frac{2}{3}}\zeta), \mathrm{Bi}(\hbar^{-\frac{2}{3}}\zeta)$ which are positive for all $\zeta \geq 0$. For $\zeta \leq 0$ we switch to the complex system

$$\mathrm{Ai}(\hbar^{-\frac{2}{3}}\zeta) \pm i\mathrm{Bi}(\hbar^{-\frac{2}{3}}\zeta)$$

which does not vanish. Throughout this section, we shall frequently use the following standard facts about Volterra integral equations of the form

$$f(x) = g(x) + \int_x^\infty K(x,s) f(s)\,ds, \tag{4.114}$$

or

$$f(x) = g(x) + \int_a^x K(x,s) f(s)\,ds, \tag{4.115}$$

with some $g(x) \in L^\infty$ and $a \in \mathbb{R}$. These equations are solved by means of an iteration which crucially relies on the directedness of the variables. This refers to $s > x$ in (4.114) and $s < x$ in (4.115). This ordering leads to a gain of $n!$ after n iterations, as for the exponential series.

Lemma 4.27 (Lemma 2.4 in [24])**.** *Suppose $g(x) \in L^\infty([a,\infty))$ and*

$$\mu := \int_a^\infty \sup_{a<x<s} |K(x,s)|\,ds < \infty$$

There exists a unique solution to (4.114) *of the form*

$$f(x) = g(x) + \sum_{n=1}^\infty \int_a^\infty \dots \int_a^\infty \prod_{i=1}^n \chi_{[x_{i-1}<x_i]} K(x_{i-1}, x_i)\; g(x_n)\,dx_n \dots dx_1. \tag{4.116}$$

with $x_0 := x$. It satisfies the bound

$$\|f\|_{L^\infty(a,\infty)} \leq e^\mu \|g\|_{L^\infty(a,\infty)}.$$

An analogous statement holds for (4.115).

We refer the reader to [24] (or elsewhere) for the elementary proof.

Lemma 4.28. *Let $w_0(\zeta;\hbar) := Ai(\hbar^{-\frac{2}{3}}\zeta)$ for $\zeta \geq 0$ and $w_1(\zeta;\hbar) := Ai(\hbar^{-\frac{2}{3}}\zeta) + iBi(\hbar^{-\frac{2}{3}}\zeta)$ for $\zeta \leq 0$. Then the Volterra integral equation*

$$\begin{aligned} a_0(\zeta;\alpha,\hbar) &:= \int_\zeta^\infty K_0(\zeta,s;\alpha,\hbar)(1+\hbar a_0(s;\alpha,\hbar))\,ds \\ K_0(\zeta,s;\alpha,\hbar) &= \hbar^{-1}\widetilde{V}(-s;\alpha,\hbar) w_0^2(s;\hbar) \int_\zeta^s w_0^{-2}(t;\hbar)\,dt \end{aligned} \tag{4.117}$$

has a unique bounded solution $a_0(\zeta;\alpha,\hbar)$ for all $\hbar \in (0,\frac{1}{3}]$ and $\alpha > 0$, $\zeta \geq 0$. One has $\lim_{\zeta\to\infty} a_0(\zeta;\alpha,\hbar) = 0$ and $w(\zeta;\alpha,\hbar) := w_0(\zeta;\hbar)(1+\hbar a_0(\zeta;\alpha,\hbar))$ is the unique solution of (4.76) *on* $[0,\infty)$ *with $w(\zeta;\alpha,\hbar) \sim w_0(\zeta;\hbar)$ as $\zeta \to \infty$. Analogously, the Volterra integral equation*

$$\begin{aligned} a_1(\zeta;\alpha,\hbar) &:= \int_{-\infty}^\zeta K_1(\zeta,s;\alpha,\hbar)(1+\hbar a_1(s;\alpha,\hbar))\,ds \\ K_1(\zeta,s;\alpha,\hbar) &= \hbar^{-1}\widetilde{V}(-s;\alpha,\hbar) w_1^2(s;\hbar) \int_s^\zeta w_1^{-2}(t;\hbar)\,dt \end{aligned} \tag{4.118}$$

has a unique bounded solution $a_1(\zeta;\alpha,\hbar)$ *for all* $\hbar \in (0,\frac{1}{3}]$ *and* $\alpha > 0$, $\zeta \le 0$. *One has* $\lim_{\zeta\to-\infty} a_1(\zeta;\alpha,\hbar) = 0$ *and* $w(\zeta;\alpha,\hbar) := w_1(\zeta;\hbar)(1+\hbar a_1(\zeta;\alpha,\hbar))$ *is the unique solution of* (4.76) *on* $(-\infty,0]$ *with* $w(\zeta;\alpha,\hbar) \sim w_1(\zeta;\hbar)$ *as* $\zeta \to -\infty$.

Proof. For simplicity, we suppress the parameters $\alpha,\hbar$ in the notation since they are fixed for the purposes of this lemma. Suppose $w(\zeta) := w_0(\zeta)(1+\hbar a_0(\zeta))$ solves (4.76). Then

$$\begin{aligned} \ddot{w} &= \ddot{w}_0(1+\hbar a_0) + 2\hbar \dot{w}_0\dot{a}_0 + \hbar w_0\ddot{a}_0 \\ \hbar^2\ddot{w} &= \zeta w(\zeta) + \hbar^3(2\dot{w}_0\dot{a}_0 + w_0\ddot{a}_0) \\ &= \zeta w(\zeta) + \hbar^2\widetilde{V}(-\zeta)w_0(\zeta)(1+\hbar a_0(\zeta)) \end{aligned} \tag{4.119}$$

whence

$$(w_0^2(\zeta)\dot{a}_0(\zeta))^{\cdot} = \hbar^{-1}\widetilde{V}(-\zeta)w_0^2(\zeta)(1+\hbar a_0(\zeta)) \tag{4.120}$$

If a bounded solution to (4.120) exists, then it is given by

$$\begin{aligned} w_0^2(\zeta)\dot{a}_0(\zeta) &= -\hbar^{-1}\int_\zeta^\infty \widetilde{V}(-s)w_0^2(s)(1+\hbar a_0(s))\,ds \\ a_0(\zeta) &= \hbar^{-1}\int_\zeta^\infty w_0^{-2}(t)\int_t^\infty \widetilde{V}(-s)w_0^2(s)(1+\hbar a_0(s))\,ds\,dt \\ &= \hbar^{-1}\int_\zeta^\infty\int_\zeta^s w_0^{-2}(t)\,dt\,\widetilde{V}(-s)w_0^2(s)(1+\hbar a_0(s))\,ds \end{aligned} \tag{4.121}$$

Recall the well-known asymptotic behavior

$$\mathrm{Ai}(x) = (4\pi)^{-\frac12}x^{-\frac14}e^{-\frac23 x^{\frac32}}(1+\alpha(x)), \quad |\alpha^{(k)}(x)| \le C_k x^{-\frac32-k}, \quad x \ge 1 \tag{4.122}$$

Therefore, for large $s \ge \zeta \ge \hbar^{\frac23}$,

$$\begin{aligned} \int_\zeta^s w_0^2(s)w_0^{-2}(t)\,dt &\le C\int_\zeta^s e^{-\frac{4}{3\hbar}(s^{\frac32}-t^{\frac32})}(t/s)^{\frac12}\,dt \\ &\le Cs^{-\frac12}\int_\zeta^s e^{-\frac{4}{3\hbar}(s^{\frac32}-t^{\frac32})}dt^{\frac32} \\ &\le C\hbar s^{-\frac12}. \end{aligned} \tag{4.123}$$

By Proposition 4.26 we have $\widetilde{V}(-s) = O(s^{-2})$ as $s \to \infty$ and thus

$$\max_{\zeta\le s}\Big|\int_\zeta^s w_0^{-2}(t)\,dt\,\widetilde{V}(-s)w_0^2(s)\Big| = O(\hbar s^{-\frac52}) \qquad s\to\infty$$

By a standard Volterra iteration, (4.121) has a unique bounded solution, and in fact $a_0(\zeta) = O(\zeta^{-\frac32})$ as $\zeta \to \infty$. The argument can be reversed, and defining $w(\zeta) := w_0(\zeta)(1+\hbar a_0(\zeta))$ gives a solution of (4.76) with $w(\zeta) \sim w_0(\zeta)$ as $\zeta \to \infty$. Conversely, any solution of this type satisfies $w(\zeta) = w_0(\zeta)(1+b(\zeta))$ with $b = o(1)$ as $\zeta \to \infty$. Repeating the previous calculations they yields $b = \hbar a_0$, as claimed.

For negative ζ we proceed in an analogous fashion, arriving at the same equation (4.119) with a_1, w_1 in place of a_0, w_0. Integrating it from $-\infty$ to $\zeta \leq 0$ yields (4.118). The well-known asymptotic behavior of w_1 is given by, see for example Corollary C.4 in [4],

$$\begin{aligned} \mathrm{Ai}(-x) + i\mathrm{Bi}(-x) &= cx^{-\frac{1}{4}} e^{-\frac{2i}{3}x^{\frac{3}{2}}}[1 + b(x)], \qquad x \geq 1 \\ |b^{(k)}(x)| &\leq C\langle x\rangle^{-\frac{3}{2}-k} \quad \forall\, k \geq 0 \end{aligned} \tag{4.124}$$

whence for all $s \leq \zeta \leq -\hbar^{\frac{2}{3}}$

$$\begin{aligned} \Big| w_1^2(s;\alpha,\hbar) \int_s^\zeta w_1^{-2}(t;\alpha,\hbar)\, dt \Big| &\leq \hbar |s|^{-\frac{1}{2}} \Big| \int_{-\hbar^{-\frac{2}{3}}\zeta}^{-\hbar^{-\frac{2}{3}}s} t^{\frac{1}{2}} e^{\frac{4i}{3}t^{\frac{3}{2}}} [1 + b(t)]^{-2}\, dt \Big| \\ &\leq \hbar |s|^{-\frac{1}{2}} \Big| \int_{-\hbar^{-\frac{2}{3}}\zeta}^{-\hbar^{-\frac{2}{3}}s} [1 + b(t)]^{-2}\, d(e^{\frac{4i}{3}t^{\frac{3}{2}}}) \Big| \leq \hbar |s|^{-\frac{1}{2}} \end{aligned}$$

where the final inequality follows by integration by parts. This is the same bound as (4.123). Hence, the argument proceeds as before. □

The previous lemma introduces the two fundamental systems of the perturbed Airy equation (4.76). We have for $\zeta \leq 0$ the oscillatory solutions

$$\{w_1(\zeta;\hbar)(1 + \hbar a_1(\zeta;\alpha,\hbar)),\ \overline{w_1(\zeta;\hbar)}(1 + \hbar\overline{a_1(\zeta;\alpha,\hbar)})\} \tag{4.125}$$

as well as on $\zeta \geq 0$ the real-valued solutions

$$\{f_1(\zeta;\alpha,\hbar) := w_0(\zeta;\hbar)(1 + \hbar a_0(\zeta;\alpha,\hbar)),\ f_2(\zeta;\alpha,\hbar)\} \tag{4.126}$$

where f_2 is the growing solution obtained from the decaying one f_1 by the usual reduction ansatz: $f_2 = g f_1$ where $f_1\ddot{g} + 2\dot{g}\dot{f}_1 = 0$. Thus,

$$g(\zeta;\alpha,\hbar) = \int_{\zeta_0}^\zeta f_1(t;\alpha,\hbar)^{-2}\, dt, \quad f_2(\zeta;\alpha,\hbar) = f_1(\zeta;\alpha,\hbar) \int_{\zeta_0}^\zeta f_1(t;\alpha,\hbar)^{-2}\, dt$$

Here $\zeta_0 \geq 0$ is chosen such that $f_1(\zeta;\alpha,\hbar) > 0$ for all $\zeta \geq \zeta_0$. The remainder of this section analyses the fundamental systems (4.126), (4.125) in more detail. For example, we need to show that a_0, a_1 remain uniformly bounded in all variables, obtain their decay in ζ, and we also need to bound the derivatives of these functions. The general treatment of Volterra equations with Airy kernels in the appendices in [4] does not cover our problem due to the more delicate behavior of $\widetilde{V}(-\zeta;\alpha,\hbar)$ in the regime $0 < \alpha < 1$ and $\zeta \geq 0$. We remark that the equations (4.118) hold not just on $\zeta \leq 0$ but also on, say, $\zeta \leq \zeta_*$. In the following lemma we treat the case $\zeta \leq 0$ using the methods from the appendices in [4].

Lemma 4.29. *The functions $a_1(\zeta;\alpha,\hbar)$ from Lemma 4.28 satisfy the bounds*

$$\begin{aligned} |\partial_\alpha^\ell a_1(\zeta;\alpha,\hbar)| &\leq C_\ell \langle\alpha\rangle^{-\ell} \langle\zeta\rangle^{-\frac{3}{2}}, \quad \zeta \leq 0, \\ |\partial_\zeta^k \partial_\alpha^\ell a_1(\zeta;\alpha,\hbar)| &\leq C_{k,\ell}\, \langle\alpha\rangle^{-\ell} \begin{cases} |\zeta|^{-\frac{3}{2}-k} & -\infty < \zeta \leq -1 \\ |\zeta|^{\frac{1}{2}-k} & -1 < \zeta \leq -\hbar^{\frac{2}{3}} \\ \hbar^{\frac{1-2k}{3}} & -\hbar^{\frac{2}{3}} \leq \zeta \leq 0, \end{cases} \end{aligned} \tag{4.127}$$

for all $\ell \geq 0, k \geq 1$, $\alpha > 0$, *and* $\hbar \in (0, \frac{1}{3}]$.

Proof. In view of (4.118) and (4.124), we have for all $s \leq \zeta \leq -\hbar^{\frac{2}{3}}$,

$$|K_1(\zeta, s; \alpha, \hbar)| \leq C\hbar^{-1}\langle s\rangle^{-2}\langle \hbar^{-\frac{2}{3}} s\rangle^{-\frac{1}{2}}(1 + |b(-\hbar^{-\frac{2}{3}} s)|)^2 \Big| \int_s^\zeta \langle \hbar^{-\frac{2}{3}} t\rangle^{\frac{1}{2}} e^{\frac{4i}{3\hbar}(-t)^{\frac{3}{2}}}(1 + b(-\hbar^{-\frac{2}{3}} t))^{-2}\, dt\Big|$$

$$= C\hbar^{-\frac{1}{3}}\langle s\rangle^{-2}\langle \hbar^{-\frac{2}{3}} s\rangle^{-\frac{1}{2}}(1 + |b(-\hbar^{-\frac{2}{3}} s)|)^2 \Big| \int_{\hbar^{-\frac{2}{3}} s}^{\hbar^{-\frac{2}{3}}\zeta} \langle t\rangle^{\frac{1}{2}} e^{\frac{4i}{3}(-t)^{\frac{3}{2}}}(1 + b(-t))^{-2}\, dt\Big|$$

Next,

$$\int_{t_0}^{t_1} \langle t\rangle^{\frac{1}{2}} e^{\frac{4i}{3}t^{\frac{3}{2}}}(1 + b(t))^{-2}\, dt = \frac{1}{2i}\int_{t_0}^{t_1} t^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}}(1 + b(t))^{-2}\, d\Big(e^{\frac{4i}{3}t^{\frac{3}{2}}}\Big)$$

$$= t^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}}(1 + b(t))^{-2} e^{\frac{4i}{3}t^{\frac{3}{2}}}\Big|_{t_0}^{t_1} - \frac{1}{2i}\int_{t_0}^{t_1} e^{\frac{4i}{3}t^{\frac{3}{2}}}\, d(t^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}}(1 + b(t))^{-2}) = O(1)$$

uniformly in $1 \leq t_0 \leq t_1$. Here we used that $\frac{d}{dt}(t^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}}(1 + b(t))^{-2}) = O(t^{-1})$ as $t \to \infty$. It follows from the preceding that

$$|K_1(\zeta, s; \alpha, \hbar)| \leq C\hbar^{-\frac{1}{3}}\langle s\rangle^{-2}\langle \hbar^{-\frac{2}{3}} s\rangle^{-\frac{1}{2}} \leq C\langle s\rangle^{-2}|s|^{-\frac{1}{2}}$$

for all $s \leq \zeta \leq -\hbar^{\frac{2}{3}}$. If $-\hbar^{\frac{2}{3}} \leq s \leq \zeta \leq 0$, then

$$|K_1(\zeta, s; \alpha, \hbar)| \leq C\hbar^{-1}\int_s^\zeta dt \leq C\hbar^{-\frac{1}{3}} \tag{4.128}$$

If $s \leq -\hbar^{\frac{2}{3}} \leq \zeta \leq 0$, we split the integral $\int_s^\zeta$ in the form

$$\int_s^\zeta = \int_s^{-\hbar^{\frac{2}{3}}} + \int_{-\hbar^{\frac{2}{3}}}^\zeta .$$

The contribution from the first integral is treated in the same way as the case $s \leq \zeta \leq -\hbar^{\frac{2}{3}}$, and the contribution from the second integral is bounded by (4.128). By a standard Volterra iteration applied to the equation (4.118) we obtain from this bound

$$|a_1(\zeta; \alpha, \hbar)| \leq C\langle\zeta\rangle^{-\frac{3}{2}} \qquad \forall\, \zeta \leq 0,$$

uniformly in $\alpha > 0$, and $\hbar \in (0, \frac{1}{3}]$. The loss of $\hbar^{-\frac{1}{3}}$ in (4.128) is absorbed by an integration interval of length $\hbar^{\frac{2}{3}}$. Combining with the contribution from the range $-\infty < s \leq -\hbar^{\frac{2}{3}}$, we obtain the above estimate. Taking derivatives in α yields

$$\partial_\alpha^\ell a_1(\zeta; \alpha, \hbar) := \sum_{j=0}^{\ell}\binom{\ell}{j}\int_{-\infty}^\zeta \partial_\alpha^j K_1(\zeta, s; \alpha, \hbar)\partial_\alpha^{\ell-j}(1 + \hbar a_1(s; \alpha, \hbar))\, ds \tag{4.129}$$

We have

$$\partial_\alpha^j K_1(\zeta, s; \alpha, \hbar) = \hbar^{-1}\partial_\alpha^j \widetilde{V}(-s; \alpha, \hbar) w_1^2(s; \hbar)\int_s^\zeta w_1^{-2}(t; \hbar)\, dt$$

and by Proposition 4.26,

$$\left|\partial_\alpha^j K_1(\zeta, s; \alpha, \hbar)\right| \le C_j \langle s\rangle^{-2}|s|^{-\frac{1}{2}}\langle\alpha\rangle^{-j}$$

for all $j \ge 0$ and all $s \le \zeta \le 0$. These give (4.127) for $k = 0$ by induction in ℓ. Indeed, (4.129) is a Volterra equation for $\partial_\alpha^\ell a_1(\zeta; \alpha, \hbar)$ and the lower order derivatives $\partial_\alpha^j a_1(\zeta; \alpha, \hbar)$ with $0 \le j < \ell$ are estimated by means of the induction assumption. See [4, Proposition B.1] for more details.

The derivatives in ζ are more delicate and we use the method from [4, Proposition C.5]. Denote $S(x) := \mathrm{Ai}(x) + i\mathrm{Bi}(x)$ and $\tilde{a}_1(\zeta) = \tilde{a}_1(\zeta; \alpha, \hbar) := a_1(\hbar^{\frac{2}{3}}\zeta)$. Then

$$\begin{aligned}
\tilde{a}_1(\zeta) &:= \hbar^{\frac{2}{3}} \int_{-\infty}^{\zeta} K_1(\hbar^{\frac{2}{3}}\zeta, \hbar^{\frac{2}{3}}s; \alpha, \hbar)(1 + \hbar\tilde{a}_1(s))\, ds \\
&= \hbar^{\frac{1}{3}} \int_{-\infty}^{\zeta} \widetilde{V}(-\hbar^{\frac{2}{3}}s; \alpha, \hbar) w_1^2(\hbar^{\frac{2}{3}}s; \hbar) \int_s^{\zeta} w_1^{-2}(\hbar^{\frac{2}{3}}t; \hbar)\, dt\ (1 + \hbar\tilde{a}_1(s))\, ds \\
&= \hbar^{\frac{1}{3}} \int_{-\infty}^{\zeta} \widetilde{V}(-\hbar^{\frac{2}{3}}s; \alpha, \hbar) S(s)^2 \int_s^{\zeta} S(t)^{-2}\, dt\ (1 + \hbar\tilde{a}_1(s))\, ds
\end{aligned}$$

If $\zeta \le -1$, then by (4.124) we have

$$\tilde{a}_1(\zeta) = c\hbar^{\frac{1}{3}} \int_{-\zeta}^{\infty} \widetilde{V}(\hbar^{\frac{2}{3}}s; \alpha, \hbar) s^{-\frac{1}{2}} e^{-\frac{4i}{3}s^{\frac{3}{2}}} (1 + b(s))^2 \int_{-\zeta}^{s} t^{\frac{1}{2}} e^{\frac{4i}{3}t^{\frac{3}{2}}} (1 + b(t))^{-2}\, dt\ (1 + \hbar\tilde{a}_1(-s))\, ds$$

Changing variables $s^{\frac{3}{2}} = \sigma$, $t^{\frac{3}{2}} = \tau$ and setting $a_2(u) = \tilde{a}_1(-u^{\frac{2}{3}}) = a_1(-(\hbar u)^{\frac{2}{3}})$, $\beta(\sigma) := b(\sigma^{\frac{2}{3}})$, and suppressing $\alpha, \hbar$ as arguments from the notation, we obtain for all $u \ge 1$

$$\begin{aligned}
a_2(u) &= c\hbar^{\frac{1}{3}} \int_u^{\infty} \widetilde{V}((\hbar\sigma)^{\frac{2}{3}})(1 + \beta(\sigma))^2 \int_u^{\sigma} e^{\frac{4i}{3}(\tau - \sigma)} (1 + \beta(\tau))^{-2}\, d\tau\ (1 + \hbar a_2(\sigma))\, \sigma^{-\frac{2}{3}}\, d\sigma \\
&= c\hbar^{\frac{1}{3}} \int_0^{\infty} \widetilde{V}((\hbar(u + v))^{\frac{2}{3}})(1 + \beta(u + v))^2 \int_0^{v} e^{\frac{4i}{3}(w - v)} (1 + \beta(w + u))^{-2} dw\, (1 + \hbar a_2(u + v))\, (u + v)^{-\frac{2}{3}}\, dv
\end{aligned}$$

Note that the exterior variable u does not appear in the phase of the complex exponential, as in [4, Proposition B.1]. This is important as we differentiate in u. In fact,

$$\begin{aligned}
a_2'(u) = {} & c_1|\widetilde{\hbar}| \int_0^{\infty} \widetilde{V}'((\hbar(u + v))^{\frac{2}{3}})(1 + \beta(u + v))^2 \int_0^{v} e^{\frac{4i}{3}(w - v)} (1 + \beta(w + u))^{-2}\, dw\, (1 + \hbar a_2(u + v))\, (u + v)^{-1}\, dv \\
& + c_2\hbar^{\frac{1}{3}} \int_0^{\infty} \widetilde{V}((\hbar(u + v))^{\frac{2}{3}})(1 + \beta(u + v))\beta'(u + v) \int_0^{v} e^{\frac{4i}{3}(w - v)} (1 + \beta(w + u))^{-2}\, dw\, (1 + \hbar a_2(u + v))\, (u + v)^{-\frac{2}{3}}\, dv \\
& + c_3\hbar^{\frac{1}{3}} \int_0^{\infty} \widetilde{V}((\hbar(u + v))^{\frac{2}{3}})(1 + \beta(u + v))^2 \int_0^{v} e^{\frac{4i}{3}(w - v)} (1 + \beta(w + u))^{-3} \beta'(u + w)\, dw\, (1 + \hbar a_2(u + v))\, (u + v)^{-\frac{2}{3}}\, dv \\
& + c_4\hbar^{\frac{1}{3}} \int_0^{\infty} \widetilde{V}((\hbar(u + v))^{\frac{2}{3}})(1 + \beta(u + v))^2 \int_0^{v} e^{\frac{4i}{3}(w - v)} (1 + \beta(w + u))^{-2}\, dw\, (1 + \hbar a_2(u + v))\, (u + v)^{-\frac{5}{3}}\, dv \\
& + c_5\hbar^{\frac{4}{3}} \int_0^{\infty} \widetilde{V}((\hbar(u + v))^{\frac{2}{3}})(1 + \beta(u + v))^2 \int_0^{v} e^{\frac{4i}{3}(w - v)} (1 + \beta(w + u))^{-2}\, dw\, a_2'(u + v)\, (u + v)^{-\frac{2}{3}}\, dv
\end{aligned}$$

By (4.124) we have $|\beta^{(k)}(\sigma)| \leq C_k\, \sigma^{-1-k}$ for all $k \geq 0$ and $\sigma \geq 1$. Integrating by parts yields

$$\sup_{v\geq 0}\Big|\int_0^v e^{\frac{4i}{3}(w-v)}(1+\beta(w+u))^{-2}\,dw\Big| \lesssim 1$$
$$\sup_{v\geq 0}\Big|\int_0^v e^{\frac{4i}{3}(w-v)}(1+\beta(w+u))^{-3}\beta'(u+w)\,dw\Big| \lesssim u^{-2}$$

for all $u \geq 1$. It follows from these bounds, Proposition 4.26, and the previous uniform bound on a_1, and thus on a_2, that for all $u \geq 1$,

$$\begin{aligned}
|a_2'(u)| &\lesssim \hbar \int_0^\infty \langle(\hbar(u+v))^{\frac{2}{3}}\rangle^{-3}\,(u+v)^{-1}\,dv + \hbar^{\frac{1}{3}}\int_0^\infty \langle(\hbar(u+v))^{\frac{2}{3}}\rangle^{-2}\,u^{-2}\,(u+v)^{-\frac{2}{3}}\,dv \\
&\quad + \hbar^{\frac{1}{3}}\int_0^\infty \langle(\hbar(u+v))^{\frac{2}{3}}\rangle^{-2}(u+v)^{-\frac{5}{3}}\,dv + \hbar^{\frac{4}{3}}\int_0^\infty \langle(\hbar(u+v))^{\frac{2}{3}}\rangle^{-2}\,(u+v)^{-\frac{2}{3}}\,|a_2'(u+v)|\,dv \\
&\lesssim \int_u^\infty \big[\hbar\langle(\hbar w)^{\frac{2}{3}}\rangle^{-3}\,w^{-1} + \hbar^{\frac{1}{3}}\langle(\hbar w)^{\frac{2}{3}}\rangle^{-2}\,(u^{-2}\,w^{-\frac{2}{3}} + w^{-\frac{5}{3}})\big]\,dw + \hbar^{\frac{4}{3}}\int_u^\infty \langle(\hbar w)^{\frac{2}{3}}\rangle^{-2}\,w^{-\frac{2}{3}}\,|a_2'(w)|\,dw
\end{aligned}$$

If $u \gtrsim \hbar^{-1}$, we therefore have

$$\begin{aligned}
|a_2'(u)| &\lesssim \hbar^{-1}\int_u^\infty (w^{-3} + u^{-2}\,w^{-2})\,dw + \int_u^\infty w^{-2}\,|a_2'(w)|\,dw \\
&\lesssim \hbar^{-1}u^{-2} + \int_u^\infty w^{-2}\,|a_2'(w)|\,dw
\end{aligned}$$

which yields upon iteration that $|a_2'(u)| \lesssim \hbar^{-1}u^{-2}$ for all $u \geq \hbar^{-1}$. On the other hand, if $1 \leq u \ll \hbar^{-1}$, then

$$\begin{aligned}
|a_2'(u)| &\lesssim \int_u^{\hbar^{-1}} \big[\hbar w^{-1} + \hbar^{\frac{1}{3}}(u^{-2}w^{-\frac{2}{3}} + w^{-\frac{5}{3}})\big]\,dw + \hbar + \hbar^{\frac{4}{3}}\int_u^{\hbar^{-1}} w^{-\frac{2}{3}}|a_2'(w)|\,dw + \int_{\hbar^{-1}}^\infty w^{-2}|a_2'(w)|\,dw \\
&\lesssim -\hbar\log(\hbar u) + \hbar^{\frac{1}{3}}u^{-\frac{2}{3}} + \hbar^{\frac{4}{3}}\int_u^{\hbar^{-1}} w^{-\frac{2}{3}}|a_2'(w)|\,dw \\
&\lesssim \hbar^{\frac{1}{3}}u^{-\frac{2}{3}} + \hbar^{\frac{4}{3}}\int_u^{\hbar^{-1}} w^{-\frac{2}{3}}|a_2'(w)|\,dw
\end{aligned}$$

By Volterra iteration, $|a_2'(u)| \lesssim \hbar^{\frac{1}{3}}u^{-\frac{2}{3}}$ for all $1 \leq u \ll \hbar^{-1}$. We have

$$a_2'(u) = \frac{2}{3}a_1'(-(\hbar u)^{\frac{2}{3}})(-\hbar^{\frac{2}{3}})u^{-\frac{1}{3}} \tag{4.130}$$

Thus, redefining $\zeta := -\hbar^{\frac{2}{3}}u^{\frac{2}{3}}$, we obtain

$$|\dot{a}_1(\zeta)| \lesssim \begin{cases} |\zeta|^{-\frac{1}{2}} & \forall -1 \leq \zeta \leq -\hbar^{\frac{2}{3}} \\ |\zeta|^{-\frac{5}{2}} & \forall\ \zeta \leq -1 \end{cases}$$

as claimed in (4.127) for $\ell = 0$. Next, we need to discuss the case $-\hbar^{\frac{2}{3}} \leq \zeta \leq 0$. For this we go back to the ODE for $\dot{a}_1$, i.e., the analogue of (4.120) for a_1, w_1. Integrating once we arrive at (again, not writing $\alpha, \hbar$ as arguments)

$$\begin{aligned}\dot{a}_1(\zeta) &= w_1^{-2}(\zeta)w_1^2(-\hbar^{\frac{2}{3}})\dot{a}_1(-\hbar^{\frac{2}{3}}) + \hbar^{-1}w_1^{-2}(\zeta)\int_{-\hbar^{\frac{2}{3}}}^{\zeta}\tilde{V}(-s)w_1^2(s)(1+\hbar a_1(s))\,ds \\ &= w_1^{-2}(\zeta)w_1^2(-\hbar^{\frac{2}{3}})\dot{a}_1(-\hbar^{\frac{2}{3}}) + \hbar^{-\frac{1}{3}}w_1^{-2}(\zeta)\int_{-\hbar^{-\frac{2}{3}}\zeta}^{1}\tilde{V}(\hbar^{\frac{2}{3}}s)S^2(-s)(1+\hbar a_1(-\hbar^{\frac{2}{3}}s))\,ds \\ &= O(\hbar^{-\frac{1}{3}})\end{aligned} \tag{4.131}$$

by the already established bound for $\dot{a}_1(-\hbar^{\frac{2}{3}})$. This concludes the proof of (4.127) for $\ell = 0$. The previous method extends to all $\ell \geq 1$, simply by combining the argument for the derivatives relative to α, see (4.129), with the previous derivation. Next we turn to the estimates for higher order derivatives in ζ. The method is very similar to that for estimating $a_2'(u)$. Using Leibniz's rule, we have

$$\begin{aligned}a_2^{(k)}(u) = &\sum_{p+q+r+s+m=k} C_{p,q,r,s,m}\,\hbar^{\frac{1}{3}}\int_0^\infty \frac{d^p}{du^p}\Big(\widetilde{V}((\hbar(u+v))^{\frac{2}{3}})\Big)\frac{d^q}{du^q}\Big((1+\beta(u+v))^2\Big) \\ &\cdot\int_0^v e^{\frac{4i}{3}(w-v)}\frac{d^r}{du^r}\Big((1+\beta(w+u))^{-2}\Big)dw\frac{d^s}{du^s}\left((1+\hbar a_2(u+v))\right)\frac{d^m}{du^m}\Big((u+v)^{-\frac{2}{3}}\Big)dv.\end{aligned}$$

For each factor of the above formula, we have the following pointwise estimate:

$$\begin{aligned}&\left|\frac{d^m}{du^m}\Big((u+v)^{-\frac{2}{3}}\Big)\right| \lesssim (u+v)^{-\frac{2}{3}-m}, \quad m \geq 0, \\ &\frac{d^s}{du^s}(1+\hbar a_2(u+v)) = \hbar a_2^{(s)}(u+v), \quad s \geq 1, \\ &\left|\frac{d^q}{du^q}\Big((1+\beta(u+v))^2\Big)\right| \lesssim (u+v)^{-1-q}, \quad q \geq 1, \\ &\left|\frac{d^p}{du^p}\Big(\widetilde{V}((\hbar(u+v))^{\frac{2}{3}})\Big)\right| \lesssim \Big\langle \hbar^{\frac{4}{3}}(u+v)^{\frac{4}{3}}\Big\rangle^{-1}(u+v)^{-p}, \\ &\left|\int_0^v e^{\frac{4i}{3}(w-v)}\frac{d^r}{du^r}\Big((1+\beta(w+u))^{-2}\Big)dw\right| \lesssim u^{-r-1}, \quad r \geq 1.\end{aligned}$$

Therefore we obtain the following estimate:

$$\begin{aligned}|a_2^{(k)}(u)| \lesssim &G^{(k)}(u) + \hbar^{\frac{4}{3}}\int_0^\infty \Big\langle \hbar^{\frac{2}{3}}(u+v)^{\frac{2}{3}}\Big\rangle^{-2}(u+v)^{-\frac{2}{3}}\left|a_2^{(k)}(u+v)\right|dv \\ = &G^{(k)}(u) + \hbar^{\frac{4}{3}}\int_u^\infty \langle \hbar w\rangle^{-\frac{4}{3}}w^{-\frac{2}{3}}\left|a_2^{(k)}(w)\right|dw.\end{aligned}$$

Here $G^{(k)}(u)$ is given by

$$\begin{aligned}G^{(k)}(u) =&G_1^{(k)}(u) + G_2^{(k)}(u) \\ =:&\sum_{p+q+r+m=k} C_{p,q,r,m}\,\hbar^{\frac{1}{3}}\int_0^\infty \frac{d^p}{du^p}\Big(\widetilde{V}((\hbar(u+v))^{\frac{2}{3}})\Big)\frac{d^q}{du^q}\Big((1+\beta(u+v))^2\Big) \\ &\cdot\int_0^v e^{\frac{4i}{3}(w-v)}\frac{d^r}{du^r}\Big((1+\beta(w+u))^{-2}\Big)dw\,(1+\hbar a_2(u+v))\frac{d^m}{du^m}\Big((u+v)^{-\frac{2}{3}}\Big)dv\end{aligned}$$

$$+\sum_{p+q+r+s+m=k,1\le s<k} C_{p,q,r,s,m}\hbar^{\frac{1}{3}}\int_0^\infty \frac{d^p}{du^p}\left(\widetilde{V}((\hbar(u+v))^{\frac{2}{3}})\right)\frac{d^q}{du^q}\left((1+\beta(u+v))^2\right)$$
$$\cdot\int_0^v e^{\frac{4i}{3}(w-v)}\frac{d^r}{du^r}\left((1+\beta(w+u))^{-2}\right)dw\frac{d^s}{du^s}(1+\hbar a_2(u+v))\frac{d^m}{du^m}\left((u+v)^{-\frac{2}{3}}\right)dv.$$

If $u\gtrsim \hbar^{-1}$, $G_1^{(k)}(u)$ is bounded by

$$\begin{aligned}|G_1^{(k)}(u)| \lesssim&\hbar^{\frac{1}{3}}\int_u^\infty \left\langle \hbar^{\frac{2}{3}}w^{\frac{2}{3}}\right\rangle^{-2} w^{-\frac{2}{3}-k}dw+\sum_{1\le r\le k}\hbar^{\frac{1}{3}}u^{-r-1}\int_u^\infty \left\langle \hbar^{\frac{2}{3}}w^{\frac{2}{3}}\right\rangle^{-2} w^{-\frac{2}{3}-k+r}dw\\ \lesssim&\hbar^{-1}\int_u^\infty w^{-2-k}dw+\sum_{1\le r\le k}\hbar^{-1}u^{-r-1}\int_u^\infty w^{-2-k+r}dw\\ \lesssim&\hbar^{-1}u^{-1-k}.\end{aligned}$$

This computation shows that, for $u\gtrsim\hbar^{-1}$, we expect to obtain an estimate for $a_2^{(k)}(u)$ as $\left|a_2^{(k)}(u)\right|\lesssim \hbar u^{-k-1}$. Therefore, by induction, we can assume that $G_2^{(k)}(u)$, when $u\gtrsim\hbar^{-1}$, enjoys the same estimate as for $G_1^{(k)}(u)$. So we finally obtain

$$\left|a_2^{(k)}(u)\right|\lesssim \hbar u^{-k-1},\quad u\gtrsim\hbar^{-1}. \tag{4.132}$$

When $1\le u\ll\hbar^{-1}$, we have

$$\begin{aligned}|G_1^{(k)}(u)| \lesssim&\hbar^{\frac{1}{3}}\int_u^{\hbar^{-1}} w^{-\frac{2}{3}-k}dw+\sum_{1\le r\le k}C_r\hbar^{\frac{1}{3}}u^{-r-1}\int_u^{\hbar^{-1}} w^{-\frac{2}{3}-k+r}dw+\hbar^k\\ \lesssim&\hbar^{\frac{1}{3}}u^{\frac{1}{3}-k}+\hbar^k\lesssim \hbar^{\frac{1}{3}}u^{\frac{1}{3}-k}.\end{aligned}$$

Again, this computation shows that, for $1\le u\ll\hbar^{-1}$, we expect to obtain an estimate for $a_2^{(k)}(u)$ as $\left|a_2^{(k)}(u)\right|\lesssim \hbar^{\frac{1}{3}}u^{\frac{1}{3}-k}$. In fact, one verifies this as before by means of an induction argument. Based on the above discussion, we claim

$$|\partial_\zeta^k a_1(\zeta)|\lesssim\begin{cases}|\zeta|^{-\frac{1}{2}-k+1} & \forall -1\le\zeta\le-\hbar^{\frac{2}{3}}\\ |\zeta|^{-\frac{5}{2}-k+1} & \forall\ \zeta\le -1\end{cases}$$

This can be proved using an induction argument. The case for $k=0$ is already proved. We assume that the estimate holds for $k'<k$, and prove the estimates for $k'=k$. In view of the relation (4.130), we have

$$a_2^{(k)}(u)=c_k a_1^{(k)}\left(-(\hbar u)^{\frac{2}{3}}\right)\left(-\hbar^{\frac{2k}{3}}\right)u^{-\frac{k}{3}}+\sum_{1\le k'<k}c_{k'}a_1^{(k')}\left(-(\hbar u)^{\frac{2}{3}}\right)\left(-\hbar^{\frac{2k'}{3}}\right)u^{-\frac{k'}{3}}\cdot u^{-k+k'} \tag{4.133}$$

Substituting the estimates for $a_1^{(k')}\left(-(\hbar u)^{\frac{2}{3}}\right)$ into the second term on the right hand side above, we obtain the desired result. Finally we consider the regime $-\hbar^{\frac{2}{3}}\le\zeta\le 0$. To this end, we simply differentiate the equation (4.131) with respect to ζ:

$$\begin{aligned}\ddot{a}_1(\zeta) =&\left(w_1^{-2}(\zeta)\right)^{\cdot} w_1^2(-\hbar^{\frac{2}{3}})\dot{a}_1(-\hbar^{\frac{2}{3}}) + \hbar^{-1}\widetilde{V}(-\zeta)(1+\hbar a_1(\zeta))\\ &+ \hbar^{-1}\left(w_1^{-2}(\zeta)\right)^{\cdot}\int_{-\hbar^{\frac{2}{3}}}^{\zeta}\widetilde{V}(-s)w_1^2(s)(1+\hbar a_1(s))ds\\ =&O(\hbar^{-1}).\end{aligned} \tag{4.134}$$

The point here is that differentiating $w_1^{-2}(\zeta)$ once gives a factor of $\hbar^{-\frac{2}{3}}$, while integration over $[-\hbar^{\frac{2}{3}}, \zeta]$ gains at least another factor $\hbar^{\frac{2}{3}}$. Therefore using an induction argument, we have

$$a_1^{(k)}(\zeta) = O(\hbar^{\frac{1-2k}{3}}). \tag{4.135}$$

This completes the proof of the lemma. □

The argument for $\zeta \geq 0$ and $\alpha \geq 1$ is very similar.

Lemma 4.30. *The functions $a_0(\zeta; \alpha, \hbar)$ from Lemma 4.28 satisfy the bounds*

$$\begin{aligned}|\partial_\alpha^\ell a_0(\zeta;\alpha,\hbar)| &\leq C_\ell\, \alpha^{-\ell}\langle\zeta\rangle^{-\frac{3}{2}}, \quad \zeta \geq 0,\\ |\partial_\zeta^k\partial_\alpha^\ell a_0(\zeta;\alpha,\hbar)| &\leq C_{k,\ell}\,\alpha^{-\ell}\begin{cases}\zeta^{-\frac{3}{2}-k} & 1<\zeta<\infty\\ \zeta^{\frac{1}{2}-k} & \hbar^{\frac{2}{3}}<\zeta\leq 1\\ \hbar^{\frac{1-2k}{3}} & 0\leq\zeta\leq\hbar^{\frac{2}{3}}\end{cases}\end{aligned} \tag{4.136}$$

for all $\ell \geq 0, k \geq 1$, $\alpha \gtrsim 1$, and $\hbar \in (0, \frac{1}{3}]$.

Proof. In view of (4.117) and (4.122), we have for all $s \geq \zeta \geq \hbar^{\frac{2}{3}}$,

$$\begin{aligned}|K_0(\zeta,s;\alpha,\hbar)| &\leq C\hbar^{-1}\langle s\rangle^{-2}\langle\hbar^{-\frac{2}{3}}s\rangle^{-\frac{1}{2}}(1+a(\hbar^{-\frac{2}{3}}s))^2e^{-\frac{4}{3\hbar}s^{\frac{3}{2}}}\left|\int_\zeta^s\langle\hbar^{-\frac{2}{3}}t\rangle^{\frac{1}{2}}e^{\frac{4}{3\hbar}t^{\frac{3}{2}}}(1+a(\hbar^{-\frac{2}{3}}t))^{-2}\,dt\right|\\ &= C\hbar^{-\frac{1}{3}}\langle s\rangle^{-2}\langle\hbar^{-\frac{2}{3}}s\rangle^{-\frac{1}{2}}(1+a(\hbar^{-\frac{2}{3}}s))^2e^{-\frac{4}{3\hbar}s^{\frac{3}{2}}}\int_{\hbar^{-\frac{2}{3}}\zeta}^{\hbar^{-\frac{2}{3}}s}\langle t\rangle^{\frac{1}{2}}e^{\frac{4}{3}t^{\frac{3}{2}}}(1+a(t))^{-2}\,dt\end{aligned}$$

Next,

$$\begin{aligned}e^{-\frac{4}{3}t_1^{\frac{3}{2}}}\int_{t_0}^{t_1}\langle t\rangle^{\frac{1}{2}}e^{\frac{4}{3}t^{\frac{3}{2}}}(1+a(t))^{-2}\,dt &= \frac{e^{-\frac{4}{3}t_1^{\frac{3}{2}}}}{2}\int_{t_0}^{t_1}t^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}}(1+a(t))^{-2}\,d\Big(e^{\frac{4}{3}t^{\frac{3}{2}}}\Big)\\ = \frac{e^{-\frac{4}{3}t_1^{\frac{3}{2}}}}{2}t^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}}(1+a(t))^{-2}e^{\frac{4}{3}t^{\frac{3}{2}}}\Big|_{t_0}^{t_1} &- \frac{e^{-\frac{4}{3}t_1^{\frac{3}{2}}}}{2}\int_{t_0}^{t_1}e^{\frac{4}{3}t^{\frac{3}{2}}}\,d\big(t^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}}(1+a(t))^{-2}\big) = O(1)\end{aligned}$$

uniformly in $1 \leq t_0 \leq t_1$. Here we used that $\frac{d}{dt}(t^{-\frac{1}{2}}\langle t\rangle^{\frac{1}{2}}(1+a(t))^{-2}) = O(t^{-1})$ as $t \to \infty$. It follows from the preceding that

$$|K_0(\zeta,s;\alpha,\hbar)| \leq C\hbar^{-\frac{1}{3}}\langle s\rangle^{-2}\langle\hbar^{-\frac{2}{3}}s\rangle^{-\frac{1}{2}} \leq C\langle s\rangle^{-2}|s|^{-\frac{1}{2}} \tag{4.137}$$

for all $s \geq \zeta \geq \hbar^{\frac{2}{3}}$. If $\hbar^{\frac{2}{3}} \geq s \geq \zeta \geq 0$, then

$$|K_0(\zeta, s; \alpha, \hbar)| \leq C\hbar^{-1} \int_\zeta^s dt \leq C\hbar^{-\frac{1}{3}} \tag{4.138}$$

If $s \geq \hbar^{\frac{2}{3}} \geq \zeta \geq 0$, we split the integral $\int_\zeta^s$ in the form

$$\int_\zeta^s = \int_{\hbar^{\frac{2}{3}}}^s + \int_\zeta^{\hbar^{\frac{2}{3}}}.$$

The contribution from the first integral is treated in the same way as the case $s \geq \zeta \geq \hbar^{\frac{2}{3}}$, and the contribution from the second integral is bounded by (4.138). As in the oscillatory case, a standard Volterra iteration applied to the equation (4.117) implies

$$|a_0(\zeta; \alpha, \hbar)| \leq C\langle\zeta\rangle^{-\frac{3}{2}} \qquad \forall\, \zeta \geq 0,$$

uniformly in $\alpha \gtrsim 1$, and $\hbar \in (0, \frac{1}{3}]$. Derivatives in α are handled as in the previous lemma.

For the derivatives in ζ, denote $\tilde{a}_0(\zeta) = \tilde{a}_0(\zeta; \alpha, \hbar) := a_0(\hbar^{\frac{2}{3}}\zeta)$. Then

$$\begin{aligned}
\tilde{a}_0(\zeta) &:= \hbar^{\frac{2}{3}} \int_\zeta^\infty K_0(\hbar^{\frac{2}{3}}\zeta, \hbar^{\frac{2}{3}}s; \alpha, \hbar)(1 + \hbar\tilde{a}_0(s))\, ds \\
&= \hbar^{\frac{1}{3}} \int_\zeta^\infty \widetilde{V}(-\hbar^{\frac{2}{3}}s; \alpha, \hbar) w_0^2(\hbar^{\frac{2}{3}}s; \hbar) \int_\zeta^s w_0^{-2}(\hbar^{\frac{2}{3}}t; \hbar)\, dt\ (1 + \hbar\tilde{a}_0(s))\, ds \\
&= \hbar^{\frac{1}{3}} \int_\zeta^\infty \widetilde{V}(-\hbar^{\frac{2}{3}}s; \alpha, \hbar) \mathrm{Ai}(s)^2 \int_\zeta^s \mathrm{Ai}(t)^{-2}\, dt\ (1 + \hbar\tilde{a}_0(s))\, ds
\end{aligned}$$

If $\zeta \geq 1$, then by (4.122) we have

$$\tilde{a}_0\,(\zeta) = c\hbar^{\frac{1}{3}} \int_\zeta^\infty \widetilde{V}(-\hbar^{\frac{2}{3}}s; \alpha, \hbar) s^{-\frac{1}{2}} e^{-\frac{4}{3}s^{\frac{3}{2}}} (1 + a(s))^2 \int_\zeta^s t^{\frac{1}{2}} e^{\frac{4}{3}t^{\frac{3}{2}}} (1 + a(t))^{-2}\, dt\ (1 + \hbar\tilde{a}_0(s))\, ds$$

Changing variables $s^{\frac{3}{2}} = \sigma$, $t^{\frac{3}{2}} = \tau$ and setting $a_3(u) = \tilde{a}_0(u^{\frac{2}{3}}) = a_0((\hbar u)^{\frac{2}{3}})$, $\gamma(\sigma) := a(\sigma^{\frac{2}{3}})$, and suppressing $\alpha, \hbar$ as arguments from the notation, we obtain for all $u \geq 1$

$$\begin{aligned}
a_3(u) &= c\hbar^{\frac{1}{3}} \int_u^\infty \widetilde{V}(-(\hbar\sigma)^{\frac{2}{3}})(1 + \gamma(\sigma))^2 \int_u^\sigma e^{\frac{4}{3}(\tau - \sigma)} (1 + \gamma(\tau))^{-2}\, d\tau\ (1 + \hbar a_3(\sigma))\, \sigma^{-\frac{2}{3}}\, d\sigma \\
&= c\hbar^{\frac{1}{3}} \int_0^\infty \widetilde{V}(-(\hbar(u + v))^{\frac{2}{3}})(1 + \gamma(u + v))^2 \int_0^v e^{\frac{4}{3}(w - v)} (1 + \gamma(w + u))^{-2}\, dw\, (1 + \hbar a_3(u + v))\, (u + v)^{-\frac{2}{3}}\, dv
\end{aligned}$$

Note that the exterior variable u does not appear in the phase of the complex exponential, as [4, Proposition B.1]. This is important as we differentiate in u. In fact, $a_3'(u)$ fulfills the following Volterra equation

$$a_3'(u) = F(u) + c_5|\widetilde{\hbar}|^{\frac{4}{3}} \int_0^\infty \widetilde{V}(-(\hbar(u + v))^{\frac{2}{3}})(1 + \gamma(u + v))^2 \int_0^v e^{\frac{4}{3}(w - v)} (1 + \gamma(w + u))^{-2}\, dw\, a_3'(u + v)\, (u + v)^{-\frac{2}{3}}\, dv$$

where

$$
\begin{aligned}
F(u) &= c_1\hbar \int_0^\infty \widetilde{V}'(-(\hbar(u+v))^{\frac{2}{3}})(1+\gamma(u+v))^2\int_0^v e^{\frac{4}{3}(w-v)}(1+\gamma(w+u))^{-2}\,dw\,(1+\hbar a_3(u+v))\,(u+v)^{-1}\,dv\\
&+ c_2\hbar^{\frac{1}{3}} \int_0^\infty \widetilde{V}(-(\hbar(u+v))^{\frac{2}{3}})(1+\gamma(u+v))\gamma'(u+v)\int_0^v e^{\frac{4}{3}(w-v)}(1+\gamma(w+u))^{-2}\,dw\,(1+\hbar a_3(u+v))\,(u+v)^{-\frac{2}{3}}\,dv\\
&+ c_3\hbar^{\frac{1}{3}} \int_0^\infty \widetilde{V}(-(\hbar(u+v))^{\frac{2}{3}})(1+\gamma(u+v))^2\int_0^v e^{\frac{4}{3}(w-v)}(1+\gamma(w+u))^{-3}\gamma'(u+w)\,dw\,(1+\hbar a_3(u+v))\,(u+v)^{-\frac{2}{3}}\,dv\\
&+ c_4\hbar^{\frac{1}{3}} \int_0^\infty \widetilde{V}(-(\hbar(u+v))^{\frac{2}{3}})(1+\gamma(u+v))^2\int_0^v e^{\frac{4}{3}(w-v)}(1+\gamma(w+u))^{-2}\,dw\,(1+\hbar a_3(u+v))\,(u+v)^{-\frac{5}{3}}\,dv
\end{aligned}
$$

By (4.122) we have $|\gamma^{(k)}(\sigma)| \le C_k\,\sigma^{-1-k}$ for all $k \ge 0$ and $\sigma \ge 1$. Integrating by parts yields

$$
\begin{aligned}
\sup_{v\ge 0}\Big|\int_0^v e^{\frac{4}{3}(w-v)}(1+\gamma(w+u))^{-2}\,dw\Big| &\lesssim 1\\
\sup_{v\ge 0}\Big|\int_0^v e^{\frac{4}{3}(w-v)}(1+\gamma(w+u))^{-3}\gamma'(u+w)\,dw\Big| &\lesssim u^{-2}
\end{aligned}
\tag{4.139}
$$

for all $u \ge 1$.

For higher order derivatives in ζ, we again use Leibniz's rule to obtain

$$
\begin{aligned}
a_3^{(k)}(u) = \sum_{p+q+r+s+m=k} & C_{p,q,r,s,m}\,\hbar^{\frac{1}{3}} \int_0^\infty \frac{d^p}{du^p}\Big(\widetilde{V}(-(\hbar(u+v))^{\frac{2}{3}})\Big)\frac{d^q}{du^q}\Big((1+\gamma(u+v))^2\Big)\\
&\cdot \int_0^v e^{\frac{4}{3}(w-v)}\frac{d^r}{du^r}\Big((1+\gamma(w+u))^{-2}\Big)\,dw\frac{d^s}{du^s}\,(1+\hbar a_3(u+v))\,\frac{d^m}{du^m}\Big((u+v)^{-\frac{2}{3}}\Big)dv,
\end{aligned}
$$

and we have the estimates

$$
\begin{aligned}
&\Big|\frac{d^m}{du^m}\Big((u+v)^{-\frac{2}{3}}\Big)\Big| \lesssim (u+v)^{-\frac{2}{3}-m}, \quad m\ge 0\\
&\frac{d^s}{du^s}\,(1+\hbar a_3(u+v)) = \hbar a_3^{(s)}(u+v), \quad s \ge 1,\\
&\Big|\frac{d^q}{du^q}\Big((1+\gamma(u+v))^2\Big)\Big| \lesssim (u+v)^{-1-q}, \quad q\ge 1,\\
&\Big|\int_0^v e^{\frac{4}{3}(w-v)}\frac{d^r}{du^r}\Big((1+\gamma(w+u))^{-2}\Big)\,dw\Big| \lesssim u^{-r-1}, \quad r\ge 1.
\end{aligned}
\tag{4.140}
$$

as well as

$$
\Big|\frac{d^p}{du^p}\Big(\widetilde{V}(-(\hbar(u+v))^{\frac{2}{3}})\Big)\Big| \lesssim \Big\langle \hbar^{\frac{2}{3}}(u+v)^{\frac{2}{3}}\Big\rangle^{-2}(u+v)^{-p}, \quad p \ge 0.
$$

The proof now concludes as in the previous lemma. □

Finally, we consider the scenario which truly differs from the one in [4].

Lemma 4.31. *The functions $a_0(\zeta;\alpha,\hbar)$ from Lemma 4.28 satisfy the bounds*

$$
|\partial_\alpha^\ell a_0(\zeta;\alpha,\hbar)| \le C_\ell\,\alpha^{-\ell}[\langle\zeta\rangle^{-\frac{3}{2}} + \min(1, x(-\zeta;\alpha,\hbar)^2/\alpha^2)], \quad \zeta \ge 0,
\tag{4.141}
$$

for all $\ell \geq 0$, $0 < \alpha \ll 1$, and $\hbar \in (0, \frac{1}{3}]$. Here $x(\tau; \alpha, \hbar)$ is the diffeomorphism from Proposition 4.26. Furthermore, in the same parameter regime,

$$|\partial_\alpha^\ell \partial_\zeta^k a_0(\zeta; \alpha, \hbar)| \leq C_{k,\ell}\, \alpha^{-\ell} \zeta^{\frac{k}{2}} [\langle\zeta\rangle^{-\frac{3}{2}} + \min(1, x(-\zeta; \alpha, \hbar)^2/\alpha^2)], \quad \zeta \geq 1, \tag{4.142}$$

and

$$|\partial_\alpha^\ell \partial_\zeta^k a_0(\zeta; \alpha, \hbar)| \leq C_{k,\ell}\, \alpha^{-\ell} \begin{cases} \zeta^{\frac{1}{2}-k} & \hbar^{\frac{2}{3}} < \zeta \leq 1 \\ \hbar^{\frac{1-2k}{3}} & 0 \leq \zeta \leq \hbar^{\frac{2}{3}} \end{cases} \tag{4.143}$$

Proof. By Proposition 4.26, for all $-\tau \gtrsim 1$

$$\widetilde{V}(\tau; \alpha, \hbar) = \frac{5}{16\tau^2} - \tau\varphi(x; \alpha, \hbar) \tag{4.144}$$

with (4.111) and (4.112) describing $\varphi(x; \alpha, \hbar)$ and the relation between x and τ. In particular, from (4.112),

$$(-\tau(x))^{\frac{1}{2}} \tau'(x) = O(x^{-1}) \tag{4.145}$$

uniformly in the parameters. By the preceding lemma, cf. (4.137), (4.138),

$$k_0(s; \alpha, \hbar) := \sup_{0 \leq \zeta \leq s} |K_0(\zeta, s; \alpha, \hbar)| \leq C(\langle s\rangle^{-2} + s|\varphi(x; \alpha, \hbar)|) \min(|s|^{-\frac{1}{2}}, \hbar^{-\frac{1}{3}})$$

We adopt the convention that $\varphi(x; \alpha, \hbar) = 0$ if $0 \leq \zeta \lesssim 1$ and we used that $\widetilde{V}(\tau; \alpha, \hbar)$ is a smooth bounded function of $|\tau| \lesssim 1$ uniformly in the parameters, see (4.110). The s^{-2} here contributes the exact same amount as Lemma 4.30, so we only need to consider $s\varphi$. From

$$|\varphi(x; \alpha, \hbar)| \lesssim \min(\hbar\,\alpha^2 x^{-2} + x^2, (\hbar + \alpha^2)x^2/\alpha^2)$$

we conclude from (4.145) writing $ds = s'(x)\,dx$,

$$\begin{aligned} &\int_\zeta^\infty |K_0(\zeta, s; \alpha, \hbar)|\,ds \leq \int_\zeta^\infty k_0(s; \alpha, \hbar)\,ds \\ &\lesssim \int_\zeta^\infty (\langle s\rangle^{-2} + s|\varphi(x; \alpha, \hbar)|) \min(|s|^{-\frac{1}{2}}, \hbar^{-\frac{1}{3}})\,ds \\ &\lesssim \langle\zeta\rangle^{-\frac{3}{2}} + \int_0^{x(-\zeta)} s \min(\hbar\,\alpha^2 x^{-2} + x^2, (\hbar + \alpha^2)x^2/\alpha^2) \min(|s|^{-\frac{1}{2}}, \hbar^{-\frac{1}{3}})\, s^{-\frac{1}{2}} x^{-1}\,dx \\ &\lesssim \langle\zeta\rangle^{-\frac{3}{2}} + \int_0^{x(-\zeta)} \min(\hbar\,\alpha^2 x^{-3} + x, (\hbar + \alpha^2)x/\alpha^2)\,dx \\ &\lesssim \langle\zeta\rangle^{-\frac{3}{2}} + \min(1, x(-\zeta)^2/\alpha^2). \end{aligned} \tag{4.146}$$

Recall that $x(-\zeta) \lesssim \exp(-c\,\zeta^{\frac{3}{2}})$, thus $\langle\zeta\rangle^{-\frac{3}{2}}$ decays more slowly than the final term as $\zeta \to \infty$. This estimate controls the first term in the Volterra iteration computing the solution of (4.117). The full Volterra series now provides the bound

$$\begin{aligned}|a_0(\zeta;\alpha,\hbar)| &\le \sum_{n=1}^{\infty}\frac{\hbar^{n-1}}{n!}\Big(\int_\zeta^\infty k_0(s;\alpha,\hbar)\,ds\Big)^n\\ &\le \int_\zeta^\infty k_0(s;\alpha,\hbar)\,ds\exp\Big(\hbar\int_\zeta^\infty k_0(s;\alpha,\hbar)\,ds\Big)\\ &\lesssim \langle\zeta\rangle^{-\frac32}+\min(1,x(-\zeta)^2/\alpha^2)\end{aligned}$$

as claimed, see (4.141) with $\ell=0$. The derivatives in α are estimated in the same fashion, noting that each derivative in α loses a factor of α by Proposition 4.26. For the ζ derivatives, we use the same changes of variables as in the previous lemma. I.e., setting $a_3(u)=\tilde a_0(u^{\frac23})=a_0((\hbar u)^{\frac23})$ we obtain

$$\begin{aligned}a_3'(u) &= c_1\hbar\int_0^\infty \widetilde{V}'(-(\hbar(u+v))^{\frac23})(1+\gamma(u+v))^2\int_0^v e^{\frac43(w-v)}(1+\gamma(w+u))^{-2}\,dw\,(1+\hbar a_3(u+v))\,(u+v)^{-1}\,dv\\
&+c_2\hbar^{\frac13}\int_0^\infty \widetilde{V}(-(\hbar(u+v))^{\frac23})(1+\gamma(u+v))\gamma'(u+v)\int_0^v e^{\frac43(w-v)}(1+\gamma(w+u))^{-2}\,dw\,(1+\hbar a_3(u+v))\,(u+v)^{-\frac23}\,dv\\
&+c_3\hbar^{\frac13}\int_0^\infty \widetilde{V}(-(\hbar(u+v))^{\frac23})(1+\gamma(u+v))^2\int_0^v e^{\frac43(w-v)}(1+\gamma(w+u))^{-3}\gamma'(u+w)\,dw\,(1+\hbar a_3(u+v))\,(u+v)^{-\frac23}\,dv\\
&+c_4\hbar^{\frac13}\int_0^\infty \widetilde{V}(-(\hbar(u+v))^{\frac23})(1+\gamma(u+v))^2\int_0^v e^{\frac43(w-v)}(1+\gamma(w+u))^{-2}\,dw\,(1+\hbar a_3(u+v))\,(u+v)^{-\frac53}\,dv\\
&+c_5\hbar^{\frac43}\int_0^\infty \widetilde{V}(-(\hbar(u+v))^{\frac23})(1+\gamma(u+v))^2\int_0^v e^{\frac43(w-v)}(1+\gamma(w+u))^{-2}\,dw\,a_3'(u+v)\,(u+v)^{-\frac23}\,dv\end{aligned}$$

As before, (4.122) implies that $|\gamma^{(k)}(\sigma)|\le C_k\,\sigma^{-1-k}$ for all $k\ge 0$ and $\sigma\ge 1$. Moreover, (4.139) holds for all $u\ge 1$. Hence, for all $u\ge 1$,

$$|a_3'(u)|\lesssim \hbar\int_u^\infty |\widetilde{V}'(-(\hbar w)^{\frac23})|w^{-1}\,dw+\hbar^{\frac13}\int_u^\infty |\widetilde{V}(-(\hbar w)^{\frac23})|(w^{-1}+\hbar|a_3'(w)|)w^{-\frac23}\,dw$$

Changing variables $u=\hbar^{-1}t^{\frac32}$, and $w=\hbar^{-1}s^{\frac32}$, and $|a_3'(u)|=:f(t)$ yields

$$f(t)\lesssim \hbar\int_t^\infty |\widetilde{V}'(-s)|s^{-1}\,ds+\hbar\int_t^\infty |\widetilde{V}(-s)|(s^{-\frac32}+f(s))s^{-\frac12}\,ds \tag{4.147}$$

for all $t\ge\hbar^{\frac23}$. By Proposition 4.26,

$$|\widetilde{V}'(-\zeta)|\lesssim\langle\zeta\rangle^{-3}+\langle\zeta\rangle^{\frac32}\min(\hbar\alpha^2x^{-2}+x^2,(\hbar+\alpha^2)x^2/\alpha^2)$$

for all $\zeta\ge 0$. If $t\ge 1$, we rewrite (4.147) in the form using (4.145),

$$\begin{aligned}f(t) &\lesssim \hbar\int_t^\infty(\xi^{-4}+\xi^{\frac12}\min(\hbar\alpha^2x^{-2}+x^2,(\hbar+\alpha^2)x^2/\alpha^2))\,d\xi\\ &\quad+\hbar\int_t^\infty(\xi^{-\frac52}+\xi^{\frac12}\min(\hbar\alpha^2x^{-2}+x^2,(\hbar+\alpha^2)x^2/\alpha^2))f(\xi)\,d\xi \qquad (4.148)\\ &\lesssim f_0(t)+\hbar\int_t^\infty L(\xi;\alpha,\hbar)f(\xi)\,d\xi\end{aligned}$$

where

$$f_0(t) := \hbar(t^{-3} + \min(1, x(t)^2/\alpha^2)), \quad L(\xi;\alpha,\hbar) := \xi^{-\frac{5}{2}} + \xi^{\frac{1}{2}} \min\left(\hbar\,\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2/\alpha^2\right).$$

By Volterra iteration, (4.148) implies

$$f(t) \lesssim \hbar(t^{-\frac{3}{2}} + \min(1, x(t)^2/\alpha^2)) \qquad t \geq 1$$

If $\hbar^{\frac{2}{3}} \leq t \leq 1$, then (4.147) becomes

$$f(t) \lesssim \hbar \int_t^1 \xi^{-2}\, d\xi + \hbar \int_t^1 f(s)\, \frac{ds}{s^{\frac{1}{2}}} \lesssim \hbar t^{-1}$$

The changes of variables above amount to

$$f(t) = |a_3'(\hbar^{-1}t^{\frac{3}{2}})| = \frac{2}{3}\hbar^{\frac{2}{3}}(\hbar^{-1}t^{\frac{3}{2}})^{-\frac{1}{3}}|a_0'(t)| = \frac{2}{3}\hbar t^{-\frac{1}{2}}|a_0'(t)|$$

whence

$$|a_0'(t)| \lesssim t^{\frac{1}{2}}[t^{-\frac{3}{2}} + \min(1, x(t)^2/\alpha^2)] \qquad \forall\ t \geq 1$$

and

$$|a_0'(t)| \lesssim t^{-\frac{1}{2}} \qquad \forall\ \hbar^{\frac{2}{3}} \leq t \leq 1$$

Finally, if $0 \leq t \leq \hbar^{\frac{2}{3}}$, then $|a_0'(t)| \lesssim \hbar^{-\frac{1}{3}}$, as in Lemma 4.30.

Next we turn to the higher order ζ-derivatives. We again use the formula

$$\begin{aligned} a_3^{(k)}(u) = \sum_{p+q+r+s+m=k} C_{p,q,r,s,m}\,\hbar^{\frac{1}{3}} \int_0^\infty \frac{d^p}{du^p}\left(\widetilde{V}(-(\hbar(u+v))^{\frac{2}{3}})\right)\frac{d^q}{du^q}\left((1+\gamma(u+v))^2\right) \\ \cdot \int_0^v e^{\frac{4}{3}(w-v)}\frac{d^r}{du^r}\left((1+\gamma(w+u))^{-2}\right)dw\,\frac{d^s}{du^s}\left(1+\hbar a_3(u+v)\right)\frac{d^m}{du^m}\left((u+v)^{-\frac{2}{3}}\right)dv, \end{aligned} \tag{4.149}$$

as well as the estimates (4.140). The true difference in this case are the bounds on higher order derivatives of $\widetilde{V}$. Let us denote:

$$\begin{aligned} \widetilde{V}\left(-(\hbar(u+v))^{\frac{2}{3}}\right) =& \widetilde{V}_1\left(-(\hbar(u+v))^{\frac{2}{3}}\right) + \widetilde{V}_2\left(-(\hbar(u+v))^{\frac{2}{3}}\right) \\ :=& \frac{5}{16(\hbar(u+v))^{\frac{4}{3}}} - (\hbar(u+v))^{\frac{2}{3}}\,\varphi(x;\alpha,\hbar). \end{aligned}$$

By our convention, the contribution from $\widetilde{V}_2$ appears only for $\hbar(u+v) \geq 1$. The contribution from $\widetilde{V}_1$ is bounded the same way as in Lemma 4.30. For the contribution from $\widetilde{V}_2$, we have, in view of Proposition 4.26,

$$\left|\frac{d^p}{du^p}\widetilde{V}_2\left(-(\hbar(u+v))^{\frac{2}{3}}\right)\right| \lesssim \sum_{p'+p''=p} \hbar^{\frac{2}{3}} \cdot \hbar^{p'}(u+v)^{\frac{2}{3}-p''} \min\{\hbar\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2\alpha^{-2}\}, \quad \text{for} \quad p \geq 0.$$

Based on these estimates, the contribution from $\widetilde{V}_2$ is bounded by, for $\hbar u \geq 1$,

$$\begin{aligned} \left|a_3^{(k)}(u)\right| \lesssim& \hbar \sum_{0\leq p'\leq k} \int_u^\infty \hbar^{p'}\left(w^{-k+p'} + \sum_{1\leq r\leq k-p'} u^{-r-1}w^{-k+p'+r}\right) \\ & \cdot \min\{\hbar\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2\alpha^{-2}\}\,dw \end{aligned}$$

$$
\begin{aligned}
&+\hbar^2 \sum_{0\le p'\le k'<k, k'\ge 1} \int_u^\infty \hbar^{p'} \left(w^{-k'+p'} + \sum_{1\le r\le k'-p'} u^{-r-1} w^{-k'+p'+r} \right) \\
&\qquad \cdot \min\{\hbar\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2\alpha^{-2}\} \left| a_3^{(k-k')}(w) \right| dw \\
&+\hbar^2 \int_u^\infty \min\{\hbar\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2\alpha^{-2}\} \left| a_3^{(k)}(w) \right| dw \\
&=: I + II + III.
\end{aligned}
$$

The first term here is the contribution of no derivative falling on $a_3(u+v)$ in (4.149), the second one is if at least one falls on $a_3(u+v)$, but no more than $k-1$, and the third, if all k derivatives fall on $a_3(u+v)$. Changing variables $u = \hbar^{-1} t^{\frac{3}{2}}, w = \hbar^{-1} s^{\frac{3}{2}}$, and $|a_3^{(k)}(u)| =: f^{(k)}(t)$ yields

$$
\begin{aligned}
I \lesssim& \hbar \sum_{0\le p'\le k} \int_t^\infty \left(\hbar^{k-1} s^{-\frac{3(k-p')-1}{2}} + \hbar^k \sum_{1\le r\le k-p'} t^{-\frac{3r+3}{2}} s^{-\frac{3k-3p'-3r-1}{2}} \right) \\
&\qquad \cdot \min\{\hbar\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2\alpha^{-2}\}\, ds, \\
II \lesssim& \hbar^2 \sum_{0\le p'\le k'<k, k'\ge 1} \int_t^\infty \left(\hbar^{k'-1} s^{-\frac{3(k'-p')-1}{2}} + \hbar^{k'} \sum_{1\le r\le k'-p'} t^{-\frac{3r+3}{2}} s^{-\frac{3k'-3p'-3r-1}{2}} \right) \\
&\qquad \cdot \min\{\hbar\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2\alpha^{-2}\} f^{(k-k')}(s)\, ds, \\
III \lesssim& \hbar \int_t^\infty s^{\frac{1}{2}} \min\{\hbar\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2\alpha^{-2}\} f^{(k)}(s)\, ds.
\end{aligned}
$$

Recall the bound

$$
\int_t^\infty s^{\frac{1}{2}} \min\{\hbar\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2\alpha^{-2}\} ds \lesssim \min\left\{1, \frac{x(t)^2}{\alpha^2}\right\},
$$

Thus, for $t \ge 1$,

$$
I \lesssim \hbar^k \int_t^\infty s^{\frac{1}{2}} \min\{\hbar\alpha^2 x^{-2} + x^2, (\hbar+\alpha^2)x^2\alpha^{-2}\} ds \lesssim \hbar^k \min\left\{1, \frac{x(t)^2}{\alpha^2}\right\}. \tag{4.150}
$$

For the contribution from $\widetilde{V}_1$, we have

$$
\left| \frac{d^p}{du^p} \widetilde{V}_1 \left(-(\hbar(u+v))^{\frac{2}{3}} \right) \right| \lesssim \hbar^{-\frac{4}{3}} (u+v)^{-\frac{4}{3}-p}.
$$

Therefore this contribution is bounded in the same way as in Lemma 4.29 and 4.30.

For the case when $\hbar^{\frac{2}{3}}\zeta \le 1$ and $0 \le \zeta \le \hbar^{\frac{2}{3}}$, by adopting the convention $\varphi = 0, 0 \le \zeta \le 1$, the corresponding argument in the proof of Lemma 4.30 applies here. □

4.7. **Distorted Fourier transform and the spectral measures for all $n \ge 2$.** The machinery developed starting with Section 4.4 up until this point allows us to determine the spectral measures associated to all operators $\mathcal{H}_n$ with n positive and large. In other words, we need $\hbar = \frac{1}{n+1}$ small, and not just $0 < \hbar \le \frac{1}{3}$. The reason for this lies with the correction factors $1 + \hbar a_j$, $j = 0, 1$ which are only useful if $\hbar|a_j| \ll 1$. The finitely many remaining n will be treated below by the same analysis from [18] which have already applied to $n = 0, \pm 1$.

4.7.1. $n \geq N_0$ *with* N_0 *large and fixed.* We wish to find the distorted Fourier transform as in Section 4.2 associated with the spectral problem $-\mathcal{H}_n^+ f = \xi f$, $\xi > 0$, cf. (4.19). Recalling equations (4.63), (4.65), (4.73), (4.72), (4.76), and Lemma 4.28, the half-line problem $-\mathcal{H}_n^+ f = E^2 f = \xi f$ in $L^2(dR)$, $R > 0$ is converted into the perturbed Airy equation (4.76) on the whole line

$$\hbar^2 \ddot{w}(\tau) = -\tau w(\tau) + \hbar^2 \tilde{V}(\tau; \alpha, \hbar) w(\tau)$$

with $\alpha = \hbar E = \hbar \xi^{\frac{1}{2}}$. Here τ and $x = \hbar E R = \hbar \xi^{\frac{1}{2}} R$ are related by (4.72). See (4.112) for the asymptotic relation between x and τ. Throughout, $\hbar = (n+1)^{-1}$. We also freely use the results and notations of the previous section, in particular Lemma 4.28 and the quantitative control on a_0 and a_1 obtained above.

Proposition 4.32. *The distorted Fourier transform associated with* $-\mathcal{H}_n^+$, $n \gg 1$, *takes the following form:*

$$\hat{f}(\xi) = \int_0^\infty \phi_n(R, \xi) f(R)\, dR, \qquad f(R) = \int_0^\infty \phi_n(R, \xi) \hat{f}(\xi) \rho_n(d\xi) \tag{4.151}$$

with

$$\begin{aligned} \phi_n(R; \xi) &= \hbar^{\frac{1}{3}} \alpha^{-\frac{1}{2}} q^{-\frac{1}{4}}(\tau) Ai(-\hbar^{-\frac{2}{3}} \tau)(1 + \hbar a_0(-\tau; \alpha, \hbar)) \\ &= \hbar^{-\frac{1}{6}} \xi^{-\frac{1}{4}} q^{-\frac{1}{4}}(\tau) Ai(-\hbar^{-\frac{2}{3}} \tau)(1 + \hbar a_0(-\tau; \alpha, \hbar)) \\ \tau &= \tau(x; \alpha, \hbar), \quad x = \alpha R \leq x_t, \quad \alpha = \hbar \xi^{\frac{1}{2}} \end{aligned} \tag{4.152}$$

Here $\tau = \tau(x; \alpha, \hbar)$ *as in* (4.72)*,* q *is defined in* (4.73)*, and* $x_t = x_t(\alpha; \hbar)$ *the turning point as in Lemma 4.10. One has* $\phi_n(R; \xi) \sim \hbar^{n-\frac{1}{2}} \xi^{\frac{n-1}{2}} R^{n-\frac{1}{2}}$ *as* $R\xi^{\frac{1}{2}} \to 0+$*. To the right of the turning point we have the representation*

$$\begin{aligned} \phi_n(R; \xi) &= -c_1 \hbar^{-\frac{1}{6}} \xi^{-\frac{1}{4}} q^{-\frac{1}{4}}(\tau) \mathrm{Re}\left((1 + \hbar \Xi(\xi; \hbar)) (Ai(-\hbar^{-\frac{2}{3}} \tau) + i Bi(-\hbar^{-\frac{2}{3}} \tau))(1 + \hbar a_1(-\tau; \alpha, \hbar)) \right) \\ &\sim -c_2 \xi^{-\frac{1}{4}} \mathrm{Re}\left((1 + \hbar \Xi(\xi; \hbar)) e^{i\frac{\pi}{4}} e^{i\xi^{\frac{1}{2}} R} \right) \ \textit{as} \ R\xi^{\frac{1}{2}} \to \infty, \end{aligned} \tag{4.153}$$

where $c_1, c_2 > 0$ *are absolute constants and* $|\partial_\xi^k \Xi(\xi; \hbar)| \leq C_k\, \xi^{-k}$ *for all* $k \geq 0$ *uniformly in* $\hbar$*. The coefficients* a_0, a_1 *satisfy the bounds in Lemma 4.29–Lemma 4.31. The spectral measure* ρ_n *is purely absolutely continuous with density satisfying*

$$\frac{1}{2} \leq \frac{d\rho_n(\xi)}{d\xi} \leq 2, \qquad \Big| \frac{d^\ell}{d\xi^\ell} \frac{d\rho_n(\xi)}{d\xi} \Big| \leq C_\ell\, \xi^{-\ell}, \qquad \forall\, \xi > 0,\ \ell \geq 0 \tag{4.154}$$

uniformly in n. For the higher order derivatives of $\phi_n(R, \xi)$ *we have*

$$(R\partial_R)^k \phi_n(R, \xi) \sim \hbar^{n-\frac{1}{2}} \xi^{\frac{n-1}{2}} R^{n-\frac{1}{2}}, \quad (\xi \partial_\xi)^k \phi_n(R, \xi) \sim \hbar^{n-\frac{1}{2}} \xi^{\frac{n-1}{2}} R^{n-\frac{1}{2}}, \quad \textit{as} \quad \hbar R \xi^{\frac{1}{2}} \to 0_+,$$

and

$$(R\partial_R)^k \left(e^{-iR\xi^{\frac{1}{2}}} \phi_n(R, \xi) \right) \sim c_3 \xi^{-\frac{1}{4}}, \quad (\xi \partial_\xi)^k \left(e^{-iR\xi^{\frac{1}{2}}} \phi_n(R, \xi) \right) \sim c_4 \xi^{-\frac{1}{4}} \quad \textit{as} \quad R\xi^{\frac{1}{2}} \to \infty.$$

Proof. We fix $n \geq N_0$ large (the latter will be determined). By Section 4.2, specifically (4.8),

$$m(z;\hbar) = \frac{W(\theta(\cdot,z),\psi(\cdot,z))}{W(\psi(\cdot,z),\phi(\cdot,z))}$$

where $\psi(\cdot,z)$ is the exponentially decaying solution of

$$-\mathcal{H}_n^+ f = zf \quad \text{for} \quad \operatorname{Im} z > 0, \quad \text{and} \quad |W(\psi,\overline{\psi})| \simeq 1. \tag{4.155}$$

Taking the limit $\xi = \xi + i0$ for $\xi > 0$ we obtain

$$m(\xi + i0;\hbar) = \frac{W(\theta(\cdot,\xi),\psi_+(\cdot,\xi))}{W(\psi_+(\cdot,\xi),\phi(\cdot,\xi))} \tag{4.156}$$

as density of the spectral measure, i.e., $\rho_n(d\xi) = m(\xi + i0)\,d\xi$. Let

$$w(-\tau;\alpha,\hbar) = w_0(-\tau;\hbar)(1 + \hbar a_0(-\tau;\alpha,\hbar)) = \mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)(1 + \hbar a_0(-\tau;\alpha,\hbar)) \tag{4.157}$$

be the solution from Lemma 4.28 (recall the change of sign $\zeta = -\tau$ which occurred at the beginning of Section 4.6). As $\tau \to -\infty$, which corresponds to $R\xi^{\frac{1}{2}} \to 0+$, this solution satisfies

$$w(-\tau;\alpha,\hbar) \sim c\hbar^{\frac{1}{6}}(-\tau)^{-\frac{1}{4}}e^{-\frac{2}{3\hbar}(-\tau)^{\frac{3}{2}}} \sim c'\hbar^{\frac{1}{6}}\left(-\log\left(\hbar R\xi^{\frac{1}{2}}\right)\right)^{-\frac{1}{6}}\left(\hbar R\xi^{\frac{1}{2}}\right)^{\frac{1-2\hbar}{\hbar}} \tag{4.158}$$

with some absolute constants c, c', cf. (4.112). We note that $w(-\tau;\alpha,\hbar) > 0$ for $\tau \leq 0$. Take $\hbar$ small enough so that $\hbar|a_0(-\tau;\alpha,\hbar)| \leq \frac{1}{2}$ on $(-\infty,0]$. Note that by the results of the previous section this then holds uniformly in $\tau \leq 0$ and $\alpha > 0$. Define another solution of the perturbed Airy equation (4.76)

$$\tilde{w}(\zeta;\alpha,\hbar) := w(\zeta;\alpha,\hbar)\int_0^\zeta w(u;\alpha,\hbar)^{-2}\,du, \qquad \zeta \geq 0 \tag{4.159}$$

Their Wronskian is $W(\tilde{w},w) = -1$ as can be seen by evaluating it at $\tau = 0$. We pass between solutions of $-\mathcal{H}_n f = \xi f$ on the one hand, and (4.76) viz.

$$\hbar^2\ddot{w}(\tau) = -\tau w(\tau) + \hbar^2\tilde{V}(\tau;\alpha,\hbar)w(\tau)$$

on the other hand, by means of the relation $f(R) = \tilde{f}(x) = q^{-\frac{1}{4}}w(\tau)$, see (4.63), (4.65), (4.76), (4.73). In particular, $d\tau = \sqrt{q}\,dx$, $q = -Q_0/\tau$, and $x = \alpha R$. Therefore

$$\begin{aligned}\frac{d}{dR}f(R) &= \alpha\widetilde{f'}(x) = \alpha\left(q^{-\frac{1}{4}}w\right)' = \alpha\partial_\tau\left(q^{-\frac{1}{4}}w\right)\cdot\frac{d\tau}{dx}\\ &= \alpha q^{\frac{1}{2}}\left(q^{-\frac{1}{4}}\dot{w} - \frac{1}{4}q^{-\frac{5}{4}}\dot{q}w\right).\end{aligned}$$

and $W(f_1,f_2) = \alpha W(w_1,w_2)$ for any pairs of related solutions of (4.63), respectively, (4.76).

We claim that, with $w, \tilde{w}$ as in (4.157), (4.159)

$$\begin{aligned}\phi(R;\xi) &= \phi_n(R,\xi) = \alpha^{-\frac{1}{2}}\hbar^{\frac{1}{3}}q^{-\frac{1}{4}}w(-\tau;\alpha,\hbar)\\ \theta(R;\xi) &= \theta_n(R,\xi) = \alpha^{-\frac{1}{2}}\hbar^{-\frac{1}{3}}q^{-\frac{1}{4}}\tilde{w}(-\tau;\alpha,\hbar)\end{aligned} \tag{4.160}$$

are admissible choices for the pair ϕ,θ in Section 4.2. In fact, they solve $-\mathcal{H}_n f = \xi f$, ϕ is L^2 near $R = 0$, they are real-valued for $\xi > 0$, and their Wronskian is

$$W(\theta,\phi) = -W(\tilde{w},w) = 1$$

From (4.66) and (4.158), up to multiplicative constants,

$$\phi_n(R,\xi) \sim (\hbar\xi^{\frac12})^{-\frac12}\hbar^{\frac13}(-\tau/Q_0)^{\frac14}\hbar^{\frac16}(-\tau)^{-\frac14}e^{-\frac{2}{3\hbar}(-\tau)^{\frac32}} \sim \hbar^{n-\frac12}\xi^{\frac{n-1}{2}}R^{n-\frac12}$$

as $R\xi^{\frac12} \to 0+$. The Weyl-Titchmarsh solution $\psi_+(R,\xi;\hbar)$ defined in (4.155) is given by

$$\psi_+(R,\xi;\hbar) = \hbar^{\frac13}\alpha^{-\frac12}q^{-\frac14}\overline{w_+(-\tau;\alpha,\hbar)} \tag{4.161}$$

where with $\zeta = -\tau \le 0$

$$w_+(\zeta;\alpha,\hbar) := w_1(\zeta;\hbar)(1+\hbar a_1(\zeta;\alpha,\hbar)), \quad w_1(\zeta;\hbar) := \mathrm{Ai}(\hbar^{-\frac23}\zeta) + i\mathrm{Bi}(\hbar^{-\frac23}\zeta)$$

is the oscillatory solution from Lemma 4.28. Here we select $\hbar$ so small that $\hbar|a_1(\zeta;\alpha,\hbar)| \le \frac12$, uniformly in $\zeta \le 0$ and $\alpha > 0$. It follows from the asymptotic behavior

$$\mathrm{Ai}(-\tau) + i\mathrm{Bi}(-\tau) \sim \pi^{-\frac12}\tau^{-\frac14}e^{i\left(-\frac23\tau^{\frac32}+\frac{\pi}{4}\right)} \qquad \tau\to\infty$$

and Lemma 4.13 that

$$\psi_+(R;\xi,\hbar) \sim \pi^{-\frac12}e^{i\theta}\xi^{-\frac14}e^{i\xi^{\frac12}R} \qquad R\to\infty.$$

for some $\theta = \theta(\alpha,\hbar)$. In particular, evaluating at $R=\infty$ we find the Wronskian relation

$$W(\psi_+(\cdot;\xi,\hbar),\overline{\psi_+(\cdot;\xi,\hbar)}) = -2i\pi^{-1} \qquad \forall\ \xi>0 \tag{4.162}$$

with a positive absolute constant C_+. Returning to the generalized m function of (4.156), we conclude that (suppressing $\alpha,\hbar$ from a_j for simplicity)

$$\begin{aligned} m(\xi+i0;\hbar) &= \hbar^{-\frac23}\frac{w_+(0;\alpha,\hbar)}{w(0;\alpha,\hbar)(-w_+(0;\alpha,\hbar)w'(0;\alpha,\hbar)+w_+'(0;\alpha,\hbar)w(0;\alpha,\hbar))} \\ &= \frac{i(1+i\mathrm{Bi}(0)/\mathrm{Ai}(0))(1+\hbar a_0(0))^{-2}}{W(\mathrm{Ai},\mathrm{Bi})+i\hbar^{\frac53}\mathrm{Ai}(0)(\mathrm{Ai}(0)+i\mathrm{Bi}(0))(a_0'(0)(1+\hbar a_0(0))^{-1}-a_1'(0)(1+\hbar a_1(0))^{-1})} \\ &= \frac{\pi(-\mathrm{Bi}(0)/\mathrm{Ai}(0)+i)(1+\hbar a_0(0))^{-2}}{1+i\pi\hbar^{\frac53}\mathrm{Ai}(0)(\mathrm{Ai}(0)+i\mathrm{Bi}(0))(a_0'(0)(1+\hbar a_0(0))^{-1}-a_1'(0)(1+\hbar a_1(0))^{-1})} \end{aligned}$$

By the previous section, $a_j(0;\alpha,\hbar) = O(1)$, $a_j'(0;\alpha,\hbar) = O(\hbar^{-\frac13})$ uniformly in $\xi>0$ and $0<\hbar\le\frac13$. Thus,

$$\begin{aligned} \frac1\pi m(\xi+i0;\hbar) &= (i-\mathrm{Bi}(0)/\mathrm{Ai}(0))(1+i\hbar^{\frac43}\Omega(\alpha;\hbar))(1+\hbar a_0(0;\alpha,\hbar))^{-2} \\ &= [i-\mathrm{Bi}(0)/\mathrm{Ai}(0)-\hbar^{\frac43}\Omega(\alpha;\hbar)(1+i\mathrm{Bi}(0)/\mathrm{Ai}(0))](1+\hbar a_0(0;\alpha,\hbar))^{-2} \\ |\partial_\alpha^\ell\Omega(\alpha;\hbar)| &\le C_\ell\,\alpha^{-\ell}, \qquad |\partial_\xi^\ell\Omega(\alpha;\hbar)| \le C_\ell\,\xi^{-\ell} \qquad \forall \ell\ge 0 \end{aligned}$$

and with $\alpha = \hbar\xi^{\frac12}$

$$\frac{d\rho_n(\xi)}{d\xi} = \frac1\pi\mathrm{Im}\,m(\xi+i0;\hbar) = [1-\hbar^{\frac43}\mathrm{Im}\,(\Omega(\alpha;\hbar)(1+i\mathrm{Bi}(0)/\mathrm{Ai}(0)))](1+\hbar a_0(0;\alpha,\hbar))^{-2}$$

We conclude that (4.154) holds as claimed. From (4.11), (4.12), and (4.162) we conclude that

$$\begin{aligned}\phi_n(R;\xi) &= 2\mathrm{Re}\,(a(\xi;\hbar)\psi_+(R;\xi,\hbar))\\ |a(\xi;\hbar)|^{-2} &= 4\frac{d\rho_n(\xi)}{d\xi} \simeq 1\end{aligned} \tag{4.163}$$

uniformly in $0 < \hbar \ll 1$, $\alpha > 0$ and $\xi > 0$. In fact,

$$\begin{aligned}a(\xi;\hbar) &= \frac{W(\phi,\overline{\psi_+})}{W(\psi_+,\overline{\psi_+})} = -iC_+^{-1}\hbar^{\frac{2}{3}}\overline{W(w,w_+)}\\ &= -iC_+^{-1}\hbar^{\frac{2}{3}}(w(0)\overline{w_+'(0)} - w'(0)\overline{w_+(0)})\\ &= -C_+^{-1}(1+\hbar a_0(0;\alpha,\hbar))(1+\hbar a_1(0;\alpha,\hbar))W(\mathrm{Ai},\mathrm{Bi}) + \hbar^{\frac{4}{3}}\Xi_1(\xi;\hbar)\\ &= -(\pi C_+)^{-1} + \hbar\Xi_2(\xi;\hbar)\end{aligned}$$

where $|\partial_\xi^k\Xi_2(\xi;\hbar)| \le C_k\,\xi^{-k}$ for all $k \ge 0$ uniformly in $\hbar$. This implies (4.153). For the bounds on derivatives of $\phi_n(R,\xi)$, we observe that the behavior of the principal part follows by a direct calculation, and we only need to look at the contribution from $a_0(-\tau;\alpha,\hbar)$ and $a_1(\tau;\alpha,\hbar)$. When $\hbar R\xi^{\frac{1}{2}} \to 0_+$, we have $\zeta = -\tau \to \infty$. Therefore we have, in view of Proposition 4.26,

$$\frac{\partial a_0}{\partial R} = \frac{\partial a_0}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x}\cdot\frac{\partial x}{\partial R} \simeq \zeta^{-\frac{1}{2}}R^{-1}\frac{\partial a_0}{\partial \zeta}.$$

For ξ-derivative, we have

$$\frac{\partial a_0}{\partial \xi} = \frac{\partial a_0}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial a_0}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial \xi} \simeq \zeta^{-\frac{1}{2}}\xi^{-1}\cdot\frac{\partial a_0}{\partial \zeta} + \hbar\xi^{-\frac{1}{2}}\cdot\frac{\partial a_0}{\partial \alpha}.$$

Then Lemma 4.30 and Lemma 4.31 imply

$$\left|R\frac{\partial a_0}{\partial R}\right|,\quad \left|\xi\frac{\partial a_0}{\partial \xi}\right| \lesssim \Gamma_0,$$

where Γ_0 denotes the bound on $|a_0|$. Therefore the bounds on $R\partial_R\phi_n(R,\xi)$ and $\xi\partial_\xi\phi_n(R,\xi)$ follow when $\hbar R\xi^{\frac{1}{2}} \to 0_+$. Now we turn to the case when $\hbar R\xi^{\frac{1}{2}} \to \infty$. In this case $\zeta = -\tau \to -\infty$, and we have, in view of Lemma 4.13,

$$\frac{\partial a_1}{\partial R} = \frac{\partial a_1}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x}\cdot\frac{\partial x}{\partial R} \simeq -x^{-\frac{1}{3}}\cdot\hbar\xi^{\frac{1}{2}}\cdot\frac{\partial a_1}{\partial \zeta}.$$

For ξ-derivative, we have

$$\frac{\partial a_1}{\partial \xi} = \frac{\partial a_1}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial \xi} + \frac{\partial a_1}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x}\cdot\frac{\partial x}{\partial \alpha} \simeq \hbar\xi^{-\frac{1}{2}}\frac{\partial a_1}{\partial \alpha} - x^{-\frac{1}{3}}\cdot\hbar\xi^{\frac{1}{2}}\cdot\frac{\partial a_1}{\partial \zeta}.$$

Then Lemma 4.29 gives

$$\left|R\frac{\partial a_1}{\partial R}\right|,\quad \left|\xi\frac{\partial a_1}{\partial \xi}\right| \lesssim \Gamma_1,$$

where Γ_1 denotes the bound on $|a_1|$. Therefore the bounds on $R\partial_R\phi_n(R,\xi)$ and $\xi\partial_\xi\phi_n(R,\xi)$ follow when $\hbar R\xi^{\frac{1}{2}} \to \infty$. The estimates on higher order derivatives follow in a similar way. □

Remark 4.33. *The representations* (4.152) *and* (4.153)*, respectively, both extend* $\lesssim \hbar^{\frac{3}{2}}$ *across the turning point.*

4.7.2. *The case of finitely many* $2 \le n \le N_0$*,* N_0 *large and fixed.* Next we analyze the behavior of the spectral measure for $\hbar \simeq 1$. We directly work with the operator appearing in (4.63):

$$-\mathcal{H}_n^+ = -\partial_R^2 + (n+1)^2 V(R), \quad 2 \le n \le N_0. \tag{4.164}$$

Lemma 4.34. *There exists a fundamental system* ϕ_0, θ_0 *of solutions for* $\mathcal{H}_n f = 0$ *with the asymptotic behavior*

$$\phi_0(R) \sim R^{n-\frac{1}{2}}, \quad \theta_0(R) \sim (2n-2)^{-1} R^{-n+\frac{3}{2}}, \quad as \quad R \to 0+,$$

and such that we have

$$\phi_0(R) \sim c_{1,n} R^{n+\frac{3}{2}}, \quad \theta_0(R) \sim c_{2,n} R^{-n-\frac{1}{2}}, \quad as \quad R \to \infty \tag{4.165}$$

for some positive constants $c_{i,n}, i = 1, 2$.

Proof. It suffices to construct ϕ_0 and then define $\theta_0(R)$ by means of $W(\theta_0, \phi_0) = 1$. We start by constructing a fundamental system f_0, f_1 near $R = +\infty$. Make the ansatz $f_0(R) = R^{n+\frac{3}{2}} + \epsilon(R)$ where

$$\left(-\partial_R^2 + \frac{(n+1)^2 - \frac{1}{4}}{R^2}\right) f_0 = \left(\frac{4n}{R^2(1+R^2)} + \frac{8}{(R^2+1)^2}\right) f_0. \tag{4.166}$$

We solve for ϵ using the two-sided Green function constructed from fundamental system $\{R^{n+\frac{3}{2}}, R^{-n-\frac{1}{2}}\}$ for the operator on the left. Thus, ϵ solves the Fredholm integral equation

$$\begin{aligned}\epsilon(R) =& \frac{1}{2n+2} R^{n+\frac{3}{2}} \int_R^\infty s^{-n-\frac{1}{2}} \left(\frac{4n}{s^2(1+s^2)} + \frac{8}{(s^2+1)^2}\right)\left(s^{n+\frac{3}{2}} + \epsilon(s)\right) ds \\ &+ \frac{1}{2n+2} R^{-n-\frac{1}{2}} \int_{R_0}^R s^{n+\frac{3}{2}} \left(\frac{4n}{s^2(1+s^2)} + \frac{8}{(s^2+1)^2}\right)\left(s^{n+\frac{3}{2}} + \epsilon(s)\right) ds\end{aligned} \tag{4.167}$$

where $R \ge R_0 \ge 1$. Here R_0 is large enough to guarantee smallness of the integral operator in a suitable norm. In fact, (4.167) can be rewritten as follows:

$$\epsilon(R) = g_n(R) + (T_n\epsilon)(R), \tag{4.168}$$

where

$$\begin{aligned}g_n(R) :=& \frac{1}{2n+2} R^{n+\frac{3}{2}} \int_R^\infty s^{-n-\frac{1}{2}} \left(\frac{4n}{s^2(1+s^2)} + \frac{8}{(s^2+1)^2}\right) s^{n+\frac{3}{2}}\, ds \\ &+ \frac{1}{2n+2} R^{-n-\frac{1}{2}} \int_{R_0}^R s^{n+\frac{3}{2}} \left(\frac{4n}{s^2(1+s^2)} + \frac{8}{(s^2+1)^2}\right) s^{n+\frac{3}{2}}\, ds, \\ (T_n\epsilon)(R) :=& \frac{1}{2n+2} R^{n+\frac{3}{2}} \int_R^\infty s^{-n-\frac{1}{2}} \left(\frac{4n}{s^2(1+s^2)} + \frac{8}{(s^2+1)^2}\right) \epsilon(s)\, ds \\ &+ \frac{1}{2n+2} R^{-n-\frac{1}{2}} \int_{R_0}^R s^{n+\frac{3}{2}} \left(\frac{4n}{s^2(1+s^2)} + \frac{8}{(s^2+1)^2}\right) \epsilon(s)\, ds.\end{aligned} \tag{4.169}$$

We introduce a Banach space X such that $f \in X$ if $R^{-n+\frac{1}{2}} f(R) \in L^\infty([R_0, \infty))$, and

$$\|f\|_X := \|R^{-n+\frac{1}{2}} f(R)\|_{L^\infty([R_0,\infty))}. \tag{4.170}$$

Then $\|g_n\|_X \leq C$ uniformly in n. Writing $(T_n f)(R) = \int_{R_0}^\infty K_n(R, s) f(s)\, ds$ one has with some absolute constant C

$$0 \leq R^{-n+\frac{1}{2}} K_n(R, s) s^{n-\frac{1}{2}} \leq C(R^{-2n} s^{2n-3} \mathbb{1}_{[R_0<s<R]} + R^2 s^{-5} \mathbb{1}_{[s>R]}) \tag{4.171}$$

This immediately implies that $\|T_n f\|_X \leq C R_0^{-2} \|f\|_X$. Choosing R_0 large enough T_n is a contraction and there is a unique solution $\epsilon \in X$ to (4.168). Therefore we obtain a unique solution f_0 to (4.166) defined for all $R > 0$ such that

$$f_0(R) = R^{n+\frac{3}{2}} + \epsilon(R) = R^{n+\frac{3}{2}} + O(R^{n-\frac{1}{2}}), \quad \text{for} \quad R \to \infty.$$

We can then define f_1 for large R by

$$f_1(R) = f_0(R) \cdot \int_R^\infty f_0^{-2}(s)\, ds,$$

which behaves like $f_1(R) \sim c_n R^{-n-\frac{1}{2}}$ for $R \to \infty$ and some positive constant c_n. This gives the asymptotic description (4.165). We next construct the solution $\phi_0(R)$ of

$$\left(-\partial_R^2 + \frac{(n-1)^2 - \frac{1}{4}}{R^2}\right)\phi_0 = \left(-\frac{4n}{(1+R^2)} + \frac{8}{(R^2+1)^2}\right)\phi_0,$$

near the origin $R = 0$ by means of the ansatz $\phi_0(R) = R^{n-\frac{1}{2}} + \gamma(R)$. Here γ is a solution of the Volterra equation

$$\begin{aligned} \gamma(R) = &-\frac{1}{2n-2} R^{n-\frac{1}{2}} \int_0^R s^{-n+\frac{3}{2}} \left(-\frac{4n}{(1+s^2)} + \frac{8}{(s^2+1)^2}\right)\left(s^{n-\frac{1}{2}} + \gamma\right) ds \\ &+ \frac{1}{2n-2} R^{-n+\frac{3}{2}} \int_0^R s^{n-\frac{1}{2}} \left(-\frac{4n}{(1+s^2)} + \frac{8}{(s^2+1)^2}\right)\left(s^{n-\frac{1}{2}} + \gamma\right) ds \\ =&\gamma_0(R) + \int_0^R \tilde{K}_n(R, s)\gamma(s)\, ds \end{aligned}$$

One checks that $0 \leq \gamma_0(R) \leq C R^{n+\frac{3}{2}}$ and

$$0 \leq R^{-n-\frac{3}{2}} \tilde{K}_n(R, s) s^{n+\frac{3}{2}} \leq Cs$$

for all $0 < s < R$. By a standard Volterra iteration, $0 \leq \gamma(R) \lesssim R^{n+\frac{3}{2}}$ for $R \ll 1$. We claim that this ϕ_0, when continued up to $R = +\infty$ has the desired asymptotics and is positive everywhere. For the latter assertion use that from the equation we have that

$$\partial_R^2 \phi_0 > 0$$

as long as $\phi_0(R) > 0$. In fact, $\phi_0, \partial_R \phi_0$ and $\partial_R^2 \phi_0$ are all positive initially. Suppose that $\phi_0(R)$ becomes zero at $R = R_0 > 0$ for the first time. Then there is a $R_1 \in (0, R_0)$ such that $\partial_R \phi_0(R_1) = 0$. This means that ϕ_0 loses convexity even before $R = R_1$. Therefore there is a $R_2 \in (0, R_1)$ such that $\partial_R^2 \phi_0(R_2) = 0$. In view of the equation satisfied by ϕ_0, we have $\phi(R_2) = 0$, which contradicts the fact that R_0 is the first point where ϕ_0 vanishes. Therefore $\phi_0(R) > 0$ for all $R > 0$.

Since ϕ_0 is a linear combination of f_0, f_1 for large R, we also immediately get the large R asymptotics, due to the convexity just observed. □

Next we construct a solution to $(-\mathcal{H}_n^+ - \xi) f = 0$ for all $R, \xi > 0$.

Lemma 4.35. *There exists a smooth function $\phi(R,\xi)$ on $R, \xi > 0$ satisfying $-\mathcal{H}_n^+\phi(R,\xi) = \xi\phi(R,\xi)$ of the form*

$$\phi(R,\xi) = \phi_0(R)[1 + \sum_{j\geq 1} \phi_j(R)(R^2\xi)^j],$$

as an absolutely convergent series where ϕ_j are smooth functions of $R > 0$ satisfying the bounds $\left|(R\partial_R)^m\phi_j(R)\right| \leq \frac{C_m^j}{j!}$ for all $m \geq 0$, $R > 0$.

Proof. We find $\phi(R,\xi)$ by solving the Volterra equation for all $R > 0$ (recall that $1 = W(\theta_0, \phi_0)$)

$$\phi(R,\xi) = \phi_0(R) - \xi\phi_0(R)\int_0^R \theta_0(s)\phi(s,\xi)\,ds + \xi\theta_0(R)\int_0^R \phi_0(s)\phi(s,\xi)\,ds.$$

Inserting the preceding ansatz, we obtain the identity

$$\sum_{j\geq 1}\phi_j(R)(R^2\xi)^j = -\xi\int_0^R \theta_0(s)\phi_0(s)[1 + \sum_{j\geq 1}\phi_j(s)(s^2\xi)^j]\,ds + \xi\frac{\theta_0(R)}{\phi_0(R)}\int_0^R \phi_0^2(s)[1 + \sum_{j\geq 1}\phi_j(s)(s^2\xi)^j]\,ds$$

This then defines the $\phi_j(R)$ inductively. To begin with, we have

$$\phi_1(R) = -R^{-2}\int_0^R \theta_0(s)\phi_0(s)\,ds + R^{-2}\frac{\theta_0(R)}{\phi_0(R)}\int_0^R \phi_0^2(s)\,ds.$$

and the other coefficients are then determined by the following recursive formula:

$$\phi_j(R) = -\frac{1}{R^{2j}}\int_0^R \theta_0(s)\phi_0(s)\phi_{j-1}(s)s^{2j-2}ds + \frac{\theta_0(R)}{R^{2j}\phi_0(R)}\int_0^R \phi_0^2(s)\phi_{j-1}(s)s^{2j-2}ds.$$

By construction for all $s > 0$ we have $0 \leq \theta_0(s)\phi_0(s) \leq Cs$. For the second term on the right hand side above, we bound

$$0 < \frac{\theta_0(R)}{\phi_0(R)}\phi_0^2(s) \lesssim \begin{cases} R^{-2n+2}\cdot s^{2n-1} & 0 < s < R < 1 \\ R^{-2n-2}\cdot s^{2n+3} & 1 < s < R \\ R^{-2n-2}s^{2n-1} & 0 < s < 1 < R \end{cases}$$
$$\lesssim s, \quad \text{for all} \quad 0 < s < R.$$

Therefore the desired bound on $\phi_j(R)$ follows from an induction argument. □

We next construct the Weyl-Titchmarsh solutions $\psi_\pm(R,\xi)$ for $R\xi^{\frac{1}{2}} \gg 1$.

Lemma 4.36. *There exists a pair of smooth functions $\psi_\pm(R,\xi)$ of $R, \xi > 0$ and $R\xi^{\frac{1}{2}} \gtrsim 1$ satisfying $-\mathcal{H}_n^+\psi_\pm(R,\xi) = \xi\psi_\pm(R,\xi)$ of the form*

$$\psi_\pm(R,\xi) = \frac{e^{\pm iR\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{4}}}\left(1 + g_\pm(R,\xi)\right).$$

Here $g_-(R,\xi) = \overline{g_+(R,\xi)}$ *and they satisfy the bounds (for some constants* $c_k > 0$*)*

$$|(R\partial_R)^k g_\pm(R,\xi)| \le c_k (R\xi^{\frac{1}{2}})^{-1}, \quad \textit{and} \quad |(\xi\partial_\xi)^k g_\pm(R,\xi)| \le c_k (R\xi^{\frac{1}{2}})^{-1} \quad \textit{for} \quad R\xi^{\frac{1}{2}} \gtrsim 1,$$

and have the following asymptotic profile with some constant $C \in \mathbb{R}\setminus\{0\}$

$$g_\pm(R,\xi) \sim \frac{\psi_{1,\pm}(R)}{R\xi^{\frac{1}{2}}} + O\left(\frac{1}{R^2\xi}\right), \quad \textit{for} \quad R\xi^{\frac{1}{2}} \to \infty,$$

where

$$\psi_{1,\pm}(R) = \pm C\, i + O\left(\frac{1}{1+R^2}\right).$$

In particular we have $W(\psi_+,\psi_-) = -2i$*.*

Proof. We discuss $g_+(R,\xi)$ in detail and the argument for $g_-(R,\xi)$ is similar, therefore we omit it. Plugging the definition of $g_+(R,\xi)$ into the equation satisfied by $\psi_+(R,\xi)$, we have

$$\begin{aligned} g_+(R,\xi) &= \int_R^\infty \frac{1-e^{2i(s-R)\xi^{\frac{1}{2}}}}{2i\xi^{\frac{1}{2}}} (1+g_+(s,\xi)) \cdot \left(\frac{(n+1)^2-\frac{1}{4}}{s^2} - \frac{4n}{s^2(1+s^2)} - \frac{8}{(1+s^2)^2}\right) ds \\ &:= g_{+,0}(R,\xi) + \int_R^\infty K_{+,n}(R,s;\xi) g_+(s,\xi)\, ds \end{aligned}$$

where

$$\begin{aligned} g_{+,0}(R,\xi) &= \int_R^\infty \frac{1-e^{2i(s-R)\xi^{\frac{1}{2}}}}{2i\xi^{\frac{1}{2}}} \cdot \left(\frac{(n+1)^2-\frac{1}{4}}{s^2} - \frac{4n}{s^2(1+s^2)} - \frac{8}{(1+s^2)^2}\right) ds, \\ K_{+,n}(R,s;\xi) &= \frac{1-e^{2i(s-R)\xi^{\frac{1}{2}}}}{2i\xi^{\frac{1}{2}}} \cdot \left(\frac{(n+1)^2-\frac{1}{4}}{s^2} - \frac{4n}{s^2(1+s^2)} - \frac{8}{(1+s^2)^2}\right). \end{aligned}$$

A direct calculation shows

$$|g_{+,0}(R,\xi)| \lesssim (R\xi^{\frac{1}{2}})^{-1}, \quad \sup_{R\le s} (R\xi^{\frac{1}{2}}) \cdot |K_{+,n}(R,s;\xi)| (s\xi^{\frac{1}{2}})^{-1} \lesssim \xi^{-\frac{1}{2}} s^{-2}.$$

A standard Volterra iteration gives the existence and uniqueness of such $g_+(R,\xi)$ and the estimates for $k=0$. To derive the bounds on derivatives, we introduce the notation $U_n(s) := \frac{(n+1)^2-\frac{1}{4}}{s^2} - \frac{4n}{s^2(1+s^2)} - \frac{8}{(1+s^2)^2}$, and the new variable $r := s-R$. Therefore

$$\begin{aligned} \left(R\partial_R g_{+,0}\right)(R,\xi) &= \int_0^\infty \frac{1-e^{2ir\xi^{\frac{1}{2}}}}{2i\xi^{\frac{1}{2}}} \left(R\partial_R U_n\right)(r+R) dr, \\ \Rightarrow \quad |\left(R\partial_R g_{+,0}\right)(R,\xi)| &\lesssim \int_0^\infty \xi^{-\frac{1}{2}} (r+R)^{-2} dr \lesssim (R\xi^{\frac{1}{2}})^{-1}. \end{aligned}$$

For the term involving $g_+(s,\xi)$, we have

$$R\partial_R\left(\int_R^\infty K_{+,n}(R,s;\xi) g_+(s,\xi)\, ds\right) = \int_0^\infty \frac{1-e^{2ir\xi^{\frac{1}{2}}}}{2i\xi^{\frac{1}{2}}} R\partial_R \left(U_n(r+R) g_+(r+R,\xi)\right) dr$$

$$
\begin{aligned}
&= \int_0^\infty \frac{1-e^{2ir\xi^{\frac12}}}{2i\xi^{\frac12}} R\partial_R\left(U_n(r+R)\right) g_+(r+R,\xi)dr \\
&\quad + \int_0^\infty \frac{1-e^{2ir\xi^{\frac12}}}{2i\xi^{\frac12}} U_n(r+R)\, R\partial_R\left(g_+(r+R,\xi)\right) dr \\
&:= I_R + II_R.
\end{aligned}
$$

Therefore

$$
|I_R| \lesssim \int_R^\infty \xi^{-1} s^{-3}\, ds \lesssim (R^2\xi)^{-1},
$$

which is desired. For II_R we have

$$
|II_R| \lesssim \int_R^\infty \xi^{-\frac12} s^{-2} |s\partial_s g_+(s,\xi)|\, ds.
$$

Then a Volterra iteration argument gives the desired bound on

$$
R\partial_R g_+(R,\xi) = R\partial_R g_{+,0}(R,\xi) + R\partial_R\left(\int_R^\infty K_{+,n}(R,s;\xi) g_+(s,\xi)\, ds\right).
$$

Next we turn to $\xi\partial_\xi g_+(R,\xi)$. We have

$$
\begin{aligned}
\xi\partial_\xi g_{+,0}(R,\xi) &= \int_0^\infty \frac{i}{2}\xi\partial_\xi\left(e^{2ir\xi^{\frac12}}\right)\xi^{-\frac12} U_n(r+R)\, dr - \frac{i}{4}\int_0^\infty e^{2ir\xi^{\frac12}}\xi^{-\frac12} U_n(r+R)\, dr \\
&= I_\xi + II_\xi.
\end{aligned}
$$

For II_ξ we change back to s-variable:

$$
II_\xi = -\frac{i}{4}\int_R^\infty e^{2i(s-R)\xi^{\frac12}}\xi^{-\frac12} U_n(s)\, ds, \quad \Rightarrow \quad |II_\xi| \lesssim (R\xi^{\frac12})^{-1}.
$$

For I_ξ, we rewrite it as

$$
\begin{aligned}
I_\xi &= \frac{i}{4}\int_0^\infty r\partial_r\left(e^{2ir\xi^{\frac12}}\right)\xi^{-\frac12} U_n(r+R)\, dr \\
&= -\frac{i}{4}\int_0^\infty e^{2ir\xi^{\frac12}}\xi^{-\frac12} U_n(r+R)\, dr - \frac{i}{4}\int_0^\infty r\xi^{-\frac12}\partial_r U_n(r+R)\, dr
\end{aligned}
$$

A direct calculation implies $|I_\xi| \lesssim (R\xi^{\frac12})^{-1}$. For the term involving $g_+(s,\xi)$, we have

$$
\xi\partial_\xi\left(\int_R^\infty K_{+,n}(R,s;\xi) g_+(s,\xi) ds\right) =: I'_\xi + II'_\xi + III'_\xi,
$$

where

$$
\begin{aligned}
I'_\xi &= \frac{1}{2i}\int_0^\infty (1-e^{2ir\xi^{\frac12}})\xi\partial_\xi(\xi^{-\frac12}) U_n(r+R) g_+(r+R,\xi)\, dr, \\
II'_\xi &= \int_0^\infty \frac{1-e^{2ir\xi^{\frac12}}}{2i\xi^{\frac12}} U_n(r+R)\xi\partial_\xi\left(g_+(r+R,\xi)\right)\, dr
\end{aligned}
$$

$$III'_\xi = \frac{i}{2\xi^{\frac{1}{2}}}\int_0^\infty \xi\partial_\xi\left(e^{2ir\xi^{\frac{1}{2}}}\right)U_n(r+R)g_+(r+R,\xi)\,dr.$$

Based on the estimate for $g_+(s,\xi)$, the estimate for I'_ξ is straightforward: $|I'_\xi| \lesssim (R^2\xi)^{-1}$. For II'_ξ we use the Volterra iteration. For III'_ξ we rewrite as

$$\frac{i}{4\xi^{\frac{1}{2}}}\int_0^\infty r\partial_r\left(e^{2ir\xi^{\frac{1}{2}}}\right)U_n(r+R)g_+(r+R)\,dr.$$

As in the estimate for I_ξ, we integrate by parts in r and use the estimate for $(r+R)\partial_r g_+(r+R,\xi)$, which has already been proved, to obtain the desired bound on $|III'_\xi| \lesssim (R^2\xi)^{-1}$. The higher order derivative estimates follow in a similar vein.

Next we turn to the asymptotic profile for $g_+(R,\xi)$. First, based on the estimate of the kernel and $g_+(s,\xi)$ we have

$$\left|\int_R^\infty K_{+,n}(R,s;\xi)g_+(s,\xi)\,ds\right| \lesssim (R\xi^{\frac{1}{2}})^{-1}\cdot\xi^{-\frac{1}{2}}\int_R^\infty s^{-2}\,ds \lesssim (R\xi^{\frac{1}{2}})^{-2}.$$

Therefore this contribution can be grouped into $O\left(\frac{1}{R^2\xi}\right)$. Next we turn to the contribution from $g_{+,0}(R,\xi)$. The contribution from $\int_R^\infty \frac{1}{2\,i\xi^{\frac{1}{2}}}U_n(s)\,ds$ is already in the desired form of $\frac{\psi_{\pm,1}(R)}{R\xi^{\frac{1}{2}}}$. For the contribution from $-\frac{1}{2i\xi^{\frac{1}{2}}}\int_R^\infty e^{2i(s-R)\xi^{\frac{1}{2}}}U_n(s)d\,s$, we write it as

$$\begin{aligned}
-\frac{1}{2\,i\xi^{\frac{1}{2}}}\int_0^\infty e^{2ir\xi^{\frac{1}{2}}}U_n(r+R)d\,r &= -\frac{1}{2i\xi^{\frac{1}{2}}}\int_0^\infty \frac{1}{2i\xi^{\frac{1}{2}}}\partial_r\left(e^{2ir\xi^{\frac{1}{2}}}\right)U_n(r+R)d\,r \\
&= -\frac{1}{4\xi}U_n(R) - \frac{1}{4\xi}\int_0^\infty e^{2ir\xi^{\frac{1}{2}}}\partial_r U_n(r+R)d\,r.
\end{aligned}$$

The boundary term $-\frac{1}{4\xi}U_n(R)$ can be grouped into $O\left(\frac{1}{R^2\xi}\right)$. The integral term is bounded by

$$\left|\frac{1}{4\xi}\int_0^\infty e^{2ir\xi^{\frac{1}{2}}}\partial_r U_n(r+R)d\,r\right| \lesssim \xi^{-1}\int_0^\infty (r+R)^{-3}d\,r \lesssim (R^2\xi)^{-1},$$

and therefore is also grouped into $O\left(\left(R^2\xi\right)^{-1}\right)$. □

Next we compare the Fourier basis constructed in Proposition 4.32 and the solutions constructed in Lemma 4.35 and Lemma 4.36.

Proposition 4.37. *Let $\phi(R,\xi)$ be as in Lemma 4.35. Then there exists a smooth complex nonvanishing function $w_n(\xi)$ which has the following asymptotic behavior:*

$$|w_n(\xi)| \sim \xi^{\frac{n+1}{2}}, \quad \xi \to 0+, \quad \text{and} \quad |w_n(\xi)| \sim \xi^{\frac{n-1}{2}}, \quad \xi \to \infty.$$

such that for any fixed but large $n \le N_0$, $w_n(\xi)\phi(R,\xi)$ and its higher order derivatives have the same asymptotic behavior as $\phi_n(R,\xi)$ constructed in Proposition 4.32 for $R\xi^{\frac{1}{2}} \to 0+$ and $R\xi^{\frac{1}{2}} \to \infty$. In particular, *if* $\frac{d\rho_n(\xi)}{d\xi}$ *is the spectral measure density associated to* $w_n(\xi)\phi(R,\xi)$*, then there exist absolute constants* $C_\ell > 0$ *and* $C > 1$*, such that*

$$C^{-1} \leq \frac{d\rho_n(\xi)}{d\xi} \leq C, \quad \textit{and} \quad \left|\frac{d^\ell}{d\xi^\ell}\frac{d\rho_n(\xi)}{d\xi}\right| \leq C_\ell \xi^{-\ell} \quad \textit{for all} \quad \xi > 0, \ell > 0.$$

Proof. Since $\phi(R,\xi), \psi_\pm(R,\xi)$ are all solutions of $(\mathcal{H}_n^+ + \xi) f = 0$, and $\phi(R,\xi)$ is real, there exists a complex function $a(\xi)$ such that

$$\phi(R,\xi) = a(\xi)\psi_+(R,\xi) + \overline{a(\xi)}\psi_-(R,\xi),$$

and thence

$$a(\xi) \simeq W(\phi(R,\xi), \psi_-(R,\xi)).$$

To obtain an estimate on $a(\xi)$, we choose R, ξ such that $R\xi^{\frac{1}{2}} \simeq 1$. Therefore if $\xi \ll 1$, then $R \gg 1$, and when $\xi \gtrsim 1$, we have $R \lesssim 1$. We start with the case when $\xi \ll 1$. In order to get an upper bound on $|a(\xi)|$, we use that $|\phi(\xi^{-\frac{1}{2}},\xi)| \lesssim \xi^{-\frac{n}{2}-\frac{3}{4}}$, $|(R\partial_R)\phi(\xi^{-\frac{1}{2}},\xi)| \lesssim \xi^{-\frac{n}{2}-\frac{3}{4}}$, as well as

$$|\psi_\pm(R,\xi)| + |R\partial_R\psi_\pm(R,\xi)| \lesssim \xi^{-\frac{1}{4}},$$

whence

$$|a(\xi)| \simeq |W(\phi(\xi^{-\frac{1}{2}},\xi), \psi_-(\xi^{-\frac{1}{2}},\xi))| \lesssim \xi^{-\frac{n}{2}-\frac{1}{2}}. \tag{4.172}$$

In order to get the lower bound on $|a(\xi)|$, we evaluate our functions at $R \simeq \xi^{-\frac{1}{2}}$. Then one finds a bound

$$|a(\xi)| \geq \frac{\left|\phi(\xi^{-\frac{1}{2}},\xi)\right|}{2\left|\psi_\pm(\xi^{-\frac{1}{2}},\xi)\right|} \gtrsim \frac{\xi^{-\frac{n}{2}-\frac{3}{4}}}{\xi^{-\frac{1}{4}}} = \xi^{-\frac{n+1}{2}}. \tag{4.173}$$

This implies that the spectral measure of $w_n(\xi)\phi(R,\xi)$ behaves like

$$\frac{d\rho_n(\xi)}{d\xi} \simeq \frac{1}{|w_n(\xi)a(\xi)|^2} \simeq 1, \quad \text{for} \quad \xi \ll 1. \tag{4.174}$$

Next we turn to the case when $\xi \gtrsim 1$, we have $\phi(\xi^{-\frac{1}{2}},\xi) \lesssim \xi^{-\frac{n}{2}+\frac{1}{4}}$ and $(R\partial_R)\phi(\xi^{-\frac{1}{2}},\xi) \lesssim \xi^{-\frac{n}{2}+\frac{1}{4}}$, and $|\psi_\pm(R,\xi)| \simeq \xi^{-\frac{1}{4}}$. Therefore we have

$$|a(\xi)| \simeq |W(\phi(\xi^{-\frac{1}{2}},\xi), \psi_-(\xi^{-\frac{1}{2}},\xi))| \lesssim \xi^{-\frac{n}{2}+\frac{1}{2}}, \tag{4.175}$$

and

$$|a(\xi)| \gtrsim \frac{|\phi(\xi^{-\frac{1}{2}},\xi)|}{2|\psi_\pm(\xi^{-\frac{1}{2}},\xi)|} \gtrsim \frac{\xi^{-\frac{n}{2}+\frac{1}{4}}}{\xi^{-\frac{1}{4}}} = \xi^{-\frac{n}{2}+\frac{1}{2}}. \tag{4.176}$$

This again gives

$$\frac{d\rho_n(\xi)}{d\xi} \simeq \frac{1}{|w_n(\xi)a(\xi)|^2} \simeq 1, \quad \text{for} \quad \xi \gtrsim 1. \tag{4.177}$$

Now (4.174) and (4.177) give the desired estimates on spectral measure. The bounds on the derivatives $\frac{d^\ell}{d\xi^\ell}\frac{d\rho_n(\xi)}{d\xi}$ follow directly from the construction of the function $a(\xi)$.

Next we analyze the asymptotic behavior of $w_n(\xi)\phi(R,\xi)$. Since now $\hbar$ has a positive lower bound, we equivalently consider the asymptotic regimes $R\xi^{\frac{1}{2}} \to 0+$ and $R\xi^{\frac{1}{2}} \to \infty$. When $R\xi^{\frac{1}{2}} \to 0+$, Lemma 4.34 and Lemma 4.35 imply, for $R\xi^{\frac{1}{2}} \to 0+$,

$$w_n(\xi)\phi(R,\xi) \sim \begin{cases} R^{n-\frac{1}{2}}\xi^{\frac{n+1}{2}} \lesssim R^{n-\frac{1}{2}}\xi^{\frac{n-1}{2}}, \quad R \to 0_+, \quad \text{and} \quad \xi \to 0+, \\ c_{1,n}R^{n+\frac{3}{2}}\xi^{\frac{n+1}{2}} \lesssim R^{n-\frac{1}{2}}\xi^{\frac{n-1}{2}}, \quad R \to \infty, \quad \text{and} \quad \xi \to 0+, \\ R^{n-\frac{1}{2}}\xi^{\frac{n-1}{2}}, \quad R \to 0_+, \quad \text{and} \quad \xi \to \infty. \end{cases} \tag{4.178}$$

Therefore this agrees with the bound on $\phi_n(R,\xi)$ for $\hbar R\xi^{\frac{1}{2}} \to 0+$ in Proposition 4.32. Now we look at the case when $R\xi^{\frac{1}{2}} \to \infty$. We have, in view of Lemma 4.36, and simply choosing $w_n(\xi) = \frac{1}{|a(\xi)|}$,

$$\begin{aligned} w_n(\xi)\phi(R,\xi) =& w_n(\xi)\left(a(\xi)\psi_+(R,\xi) + \overline{a(\xi)}\psi_-(R,\xi)\right) \\ =& \xi^{-\frac{1}{4}}\mathrm{Re}\left(\frac{a(\xi)}{|a(\xi)|}e^{iR\xi^{\frac{1}{2}}}(1+g_+(R,\xi))\right), \end{aligned} \tag{4.179}$$

which again agrees with the asymptotic behavior of $\phi_n(R,\xi)$ in Proposition 4.32 as $\hbar R\xi^{\frac{1}{2}} \to \infty$. The estimates on the higher order derivatives follow from the profile on $\phi(R,\xi)$ in Lemma 4.35 and the derivative bounds on $g_+(R,\xi)$ given in Lemma 4.36. □

Based on the analysis in Proposition 4.32 and Proposition 4.37, we can justify the existence of the Fourier transform and its inverse transform appearing in Proposition 4.32.

Proposition 4.38. *The distorted Fourier transform associated with $\mathcal{H}_n^+, n \geq 2$ has the following property: for any $f \in C^2((0,\infty))$ with*

$$\int_0^\infty \left(R^{-1}|f(R)| + |f'(R)| + R|f''(R)|\right)dR \leq M < \infty$$

the Fourier transform

$$\hat{f}(\xi) = \lim_{L\to\infty}\int_0^L \phi_n(R,\xi)f(R)dR \tag{4.180}$$

with $\phi_n(R,\xi)$ as in Proposition 4.32 for $n \geq N_0$ and $w_n(\xi)\phi(R,\xi)$ as in Proposition 4.37 for $2 \leq n \leq N_0$, exists for all $\xi > 0$ and

$$\int_0^\infty |\hat{f}(\xi)||\phi_n(R,\xi)|d\xi \lesssim M. \tag{4.181}$$

Proof. Based on the analysis in Proposition 4.32 and Proposition 4.37, we have

$$|\phi_n(R,\xi)| \lesssim \min\{R^{\frac{1}{2}}, \xi^{-\frac{1}{4}}\}, \quad \forall \xi > 0, R > 0.$$

Thus, the left-hand side of (4.181) is bounded by $\int_0^\infty |\hat{f}(\xi)|\xi^{-\frac{1}{4}}d\xi$ which we now proceed to estimate. Consider the partition of unity in (4.37) and define

$$A(\xi) := \int_0^\infty \chi_0(R^2\xi)\phi_n(R,\xi)f(R)dR, \quad B_j(\xi) := \int_0^\infty \chi(2^{-j}R^2\xi)\phi_n(R,\xi)f(R)dR.$$

Therefore we have

$$
\begin{aligned}
&|A(\xi)| \lesssim \int_0^{\xi^{-\frac{1}{2}}} R^{\frac{1}{2}}|f(R)|dR,\\
\Rightarrow \quad &\int_0^\infty \xi^{-\frac{1}{4}}|A(\xi)|d\xi \lesssim \int_0^\infty \int_0^{R^{-2}} \xi^{-\frac{1}{4}}d\xi|f(R)|R^{\frac{1}{2}}dR \lesssim \int_0^\infty R^{-1}|f(R)|dR.
\end{aligned}
$$

To bound $B_j(\xi)$, we note that by Proposition 4.32 and Proposition 4.37, for $R^2\xi \simeq 2^j$, one write $\phi_n(R,\xi)$ as

$$
\phi_n(R,\xi) = \xi^{-\frac{1}{4}}\mathrm{Re}\left(e^{iR\xi^{\frac{1}{4}}}\sigma(R,\xi)\right),
$$

and $\sigma(R,\xi)$ satisfies

$$
|(R\partial_R)^k\sigma(R,\xi)| \lesssim 1, \quad |(\xi\partial_\xi)^k\sigma(R,\xi)| \lesssim 1.
$$

Thus, we write

$$
B_j(\xi) = \xi^{-\frac{1}{4}}\int_0^\infty \chi(2^{-j}R^2\xi)e^{iR\xi^{\frac{1}{2}}}\sigma(R,\xi)f(R)dR + \xi^{-\frac{1}{4}}\int_0^\infty \chi(2^{-j}R^2\xi)e^{-iR\xi^{\frac{1}{2}}}\sigma(R,\xi)f(R)dR.
$$

Using integration by parts, we have

$$
\begin{aligned}
|B_j(\xi)| &\lesssim \xi^{-\frac{5}{4}}\int_0^\infty |\partial_R^2\left(\chi(2^{-j}R^2\xi)\sigma(R,\xi)f(R)\right)|dR\\
&\lesssim \xi^{-\frac{5}{4}}\int_0^\infty \mathbb{1}_{[R^2\xi\simeq 2^j]}\left(|f''(R)| + R^{-1}|f'(R)| + R^{-2}|f(R)|\right)dR.
\end{aligned}
$$

Starting from this point we simply proceed as in the proof for Lemma 4.2 to obtain the finiteness of $\int_0^\infty \sum_{j=0}^\infty |B_j(\xi)|\xi^{-\frac{1}{4}}d\xi$ and $\sum_{j=0}^\infty |B_j(\xi)|$. Therefore the proof is completed. □

4.8. **The analysis for negative angular frequencies.** In this section we discuss the spectral theory for the operator $\mathcal{H}_{-n}^+ = \mathcal{H}_n^-$ for $n \geq 2$. We start with the equation (4.2). With the same notations as in (4.63), we consider the following eigenvalue problem associated to $\mathcal{H}_{-n}^+$ for $n \geq 2$:

$$
\begin{aligned}
-\frac{1}{(n-1)^2}\partial_R^2 f + V(R)f =& \frac{E^2}{(n-1)^2}f,\\
V(R) :=& \frac{1}{R^2} - \frac{1}{4(n-1)^2R^2} + \frac{4n}{(n-1)^2R^2(R^2+1)} - \frac{8}{(n-1)^2(R^2+1)^2}.
\end{aligned} \tag{4.182}
$$

By convention, we set $E < 0$. Introducing the notation $\widetilde{\hbar} := \frac{1}{1-n}$, we rewrite (4.182) as

$$
\begin{aligned}
-\widetilde{\hbar}^2\partial_R^2 f + V(R)f =& \widetilde{\hbar}^2E^2 f,\\
V(R) =& \frac{1}{R^2}\left(1 + \frac{15\widetilde{\hbar}^2}{4} - 4\widetilde{\hbar}\right) + \varepsilon\left(R^2,\widetilde{\hbar}\right),
\end{aligned} \tag{4.183}
$$

where recall

$$
\varepsilon\left(R^2,\widetilde{\hbar}\right) := \frac{4\widetilde{\hbar}}{R^2+1} - \frac{4\widetilde{\hbar}^2(R^2+3)}{(R^2+1)^2}.
$$

The potential function looks identical to the one in (4.64), with $\hbar$ replaced by $\widetilde{\hbar}$, and $\widetilde{\hbar} = -1, -\frac{1}{2}, -\frac{1}{3}, \ldots$. As for the positive angular frequencies, we distinguish between the regimes for $2 \le n \le N_0$ and $n \ge N_0$, where N_0 is a large but fixed number. For $2 \le n \le N_0$, we directly construct solutions to the eigenvalue problem associated to the operator

$$-\mathcal{H}^+_{-n} = -\partial_R^2 + (n-1)^2 V(R), \quad n \ge 2. \tag{4.184}$$

Following the same procedures as in the proofs of Lemma 4.34, Lemma 4.35, and Lemma 4.36, we obtain the following:

Proposition 4.39. *For $2 \le n \le N_0$, there exists a solution $\phi_{-n}(R,\xi)$ to the eigenvalue problem $(\mathcal{H}^+_{-n}+\xi)f = 0$, which has the following asymptotic behavior for $R\xi^{\frac{1}{2}} \to 0_+$:*

$$\phi_{-n}(R,\xi) \sim \begin{cases} R^{n+\frac{3}{2}}\xi^{\frac{n}{2}-\frac{1}{2}} \le R^{n-\frac{1}{2}}\xi^{\frac{n}{2}-\frac{1}{2}}, & R \to 0_+ \quad \text{and} \quad \xi \to 0_+, \\ R^{n+\frac{3}{2}}\xi^{\frac{n}{2}+\frac{1}{2}} \le R^{n-\frac{1}{2}}\xi^{\frac{n}{2}-\frac{1}{2}}, & R \to 0_+ \quad \text{and} \quad \xi \to \infty, \\ c_5 R^{n-\frac{1}{2}}\xi^{\frac{n}{2}-\frac{1}{2}}, \quad R \to \infty \quad \text{and} \quad \xi \to 0_+. \end{cases}$$

For higher order derivatives, we have

$$|(R\partial_R)^k \phi_{-n}(R,\xi)| \lesssim R^{n-\frac{1}{2}}\xi^{\frac{n-1}{2}}, \quad \text{and} \quad |(\xi\partial_\xi)^k \phi_{-n}(R,\xi)| \lesssim R^{n-\frac{1}{2}}\xi^{\frac{n-1}{2}}, \quad \text{as} \quad R\xi^{\frac{1}{2}} \to 0_+.$$

When $R\xi^{\frac{1}{2}} \to \infty$, we have, for some complex function $a_-(\xi)$ which behaves $|a_-(\xi)| \simeq 1$,

$$\phi_{-n}(R,\xi) = 2\xi^{-\frac{1}{4}}\mathrm{Re}\left(a_-(\xi)e^{iR\xi^{\frac{1}{2}}}(1+g^-(R,\xi))\right).$$

Here$g^-(R,\xi)$ satisfies

$$g^-(R,\xi) = \frac{\psi_2(R)}{R\xi^{\frac{1}{2}}} + O\left(\frac{1}{R^2\xi}\right), \quad \text{as} \quad R\xi^{\frac{1}{2}} \to \infty,$$

and the function $\psi_2(R)$ satisfies, for some constant $C \in \mathbb{R}\setminus\{0\}$,

$$\psi_2(R) = C\,i + O\left(\frac{1}{1+R^2}\right).$$

For higher order derivatives, we have

$$|(R\partial_R)^k g^-(R,\xi)| \le c_k(R\xi^{\frac{1}{2}})^{-1}, \quad \text{and} \quad |(\xi\partial_\xi)^k g^-(R,\xi)| \le c_k(R\xi^{\frac{1}{2}})^{-1}, \quad \text{for} \quad R\xi^{\frac{1}{2}} \gtrsim 1.$$

This implies

$$(R\partial_R)\left(e^{-iR\xi^{\frac{1}{2}}}\phi_{-n}(R,\xi)\right) \sim c_6\xi^{-\frac{1}{4}}, \quad (\xi\partial_\xi)^k\left(e^{-iR\xi^{\frac{1}{2}}}\phi_{-n}(R,\xi)\right) \sim c_7\xi^{-\frac{1}{4}}, \quad \text{as} \quad R\xi^{\frac{1}{2}} \to \infty.$$

For $n \ge N_0$, we introduce the notations

$$\underline{x} := \widetilde{\hbar}ER, \quad \alpha := \widetilde{\hbar}E, \quad \widetilde{f}(\underline{x}) := f(R).$$

Then the equation (4.183) transfers to

$$
\begin{aligned}
-\widetilde{\hbar}^2\widetilde{f}''(\underline{x}) + \widetilde{Q}(\underline{x})\widetilde{f}(\underline{x}) = 0, \quad \widetilde{Q}(\underline{x}) :=& \widetilde{\hbar}^{-2}E^{-2}V^{-}\left(\frac{\underline{x}}{\widetilde{\hbar}E}\right) - 1 \\
=& \underline{x}^{-2}\left(1 + \frac{15\widetilde{\hbar}^2}{4} - 4\widetilde{\hbar}\right) + \alpha^{-2}\varepsilon_{-}\left(\frac{\underline{x}^2}{\underline{\alpha}^2};\hbar\right) - 1.
\end{aligned}
\tag{4.185}
$$

As in (4.66), we introduce

$$
\widetilde{Q}_0(\underline{x};\underline{\alpha},\widetilde{\hbar}) := \widetilde{Q}(\underline{x};\underline{\alpha},\widetilde{\hbar}) + \frac{\widetilde{\hbar}^2}{4\underline{x}^2} = \underline{x}^{-2}(1-2\widetilde{\hbar})^2 + \underline{\alpha}^{-2}\varepsilon_{-}\left(\frac{\underline{x}^2}{\underline{\alpha}^2};\widetilde{\hbar}\right) - 1. \tag{4.186}
$$

As in (4.72) and (4.73), we introduce the variable

$$
\overline{\tau}(\underline{x},\alpha;\widetilde{\hbar}) := \operatorname{sign}(\underline{x} - \underline{x}_t(\alpha;\widetilde{\hbar}))\left|\frac{3}{2}\int_{\underline{x}_t}^{\underline{x}} \sqrt{|\widetilde{Q}_0(u,\alpha;\widetilde{\hbar})|}du\right|^{\frac{2}{3}}, \tag{4.187}
$$

and the quantity

$$
\overline{q} := \frac{\widetilde{Q}_0}{\overline{\tau}}, \quad \Rightarrow \quad \frac{d\overline{\tau}}{d\underline{x}} = \overline{\tau}' = \sqrt{\overline{q}}, \quad \frac{d}{d\overline{\tau}} = \overline{q}^{-\frac{1}{2}}\frac{d}{d\underline{x}}. \tag{4.188}
$$

We use "$'$" to denote the derivative with respect to $\underline{x}$ and "$\cdot$" to denote the derivative with respect to $\overline{\tau}$. Then with the new variable $\overline{w} := \sqrt{\overline{\tau}'}\,\widetilde{f}$, (4.185) transfers to the following perturbed Airy equation:

$$
\widetilde{\hbar}^2\ddot{\overline{w}}(\overline{\tau}) = \overline{\tau}\overline{w}(\overline{\tau}) + \widetilde{\hbar}^2\underline{\widetilde{V}}(\overline{\tau};\alpha,\widetilde{\hbar})\overline{w}(\overline{\tau}) \tag{4.189}
$$

where

$$
\underline{\widetilde{V}}(\overline{\tau};\alpha,\widetilde{\hbar}) = \ddot{\overline{q}}^{\frac{1}{4}}\overline{q}^{-\frac{1}{4}} - \frac{1}{4\underline{x}^2\overline{q}}. \tag{4.190}
$$

We argue as in Section 4.4–4.6 to construct the Fourier basis associated to the operator $-\mathcal{H}^{+}_{-n}$ for $n \geq N_0$. Since the argument is identical, we omit the details here. Similar to Lemma 4.28, we have the following (with $\overline{\zeta} := -\overline{\tau}$):

Lemma 4.40. *Let* $w_0(\overline{\zeta};\widetilde{\hbar}) := Ai(|\widetilde{\hbar}|^{-\frac{2}{3}}\overline{\zeta})$ *for* $\overline{\zeta} \geq 0$ *and* $w_1(\overline{\zeta};\widetilde{\hbar}) := Ai(|\widetilde{\hbar}|^{-\frac{2}{3}}\overline{\zeta}) + iBi(|\widetilde{\hbar}|^{-\frac{2}{3}}\overline{\zeta})$ *for* $\overline{\zeta} \leq 0$. *Then the Volterra integral equation*

$$
\begin{aligned}
a_0(\overline{\zeta};\alpha,\widetilde{\hbar}) &:= \int_{\overline{\zeta}}^{\infty} K_0(\overline{\zeta},s;\alpha,\widetilde{\hbar})(1 + |\widetilde{\hbar}|a_0(s;\alpha,|\widetilde{\hbar}|))\,ds \\
K_0(\overline{\zeta},s;\alpha,\widetilde{\hbar}) &= |\widetilde{\hbar}|^{-1}\underline{\widetilde{V}}(s;\alpha,\widetilde{\hbar})w_0^2(s;\widetilde{\hbar})\int_{\overline{\zeta}}^{s} w_0^{-2}(t;\widetilde{\hbar})\,dt
\end{aligned}
\tag{4.191}
$$

has a unique bounded solution $a_0(\overline{\zeta};\alpha,\widetilde{\hbar})$ *for all* $\widetilde{\hbar} = -\frac{1}{N_0-1}, -\frac{1}{N_0}, -\frac{1}{N_0+1}, \ldots$ *and* $\alpha > 0$, $\overline{\zeta} \geq 0$. *One has* $\lim_{\overline{\zeta}\to\infty} a_0(\overline{\zeta};\alpha,\widetilde{\hbar}) = 0$ *and* $w(\overline{\zeta};\alpha,\widetilde{\hbar}) := w_0(\overline{\zeta};\widetilde{\hbar})(1 + \widetilde{\hbar}a_0(\overline{\zeta};\alpha,\widetilde{\hbar}))$ *is the unique solution of* (4.189) *on* $[0,\infty)$

with $w(\overline{\zeta};\alpha,\widetilde{\hbar}) \sim w_0(\overline{\zeta};\widetilde{\hbar})$ *as* $\overline{\zeta} \to \infty$. *Analogously, the Volterra integral equation*

$$\begin{aligned} a_1(\overline{\zeta};\alpha,\widetilde{\hbar}) &:= \int_{-\infty}^{\overline{\zeta}} K_1(\overline{\zeta},s;\alpha,\widetilde{\hbar})(1+\widetilde{\hbar}a_1(s;\alpha,\widetilde{\hbar}))\,ds \\ K_1(\overline{\zeta},s;\alpha,\widetilde{\hbar}) &= |\widetilde{\hbar}|^{-1}\underline{\widetilde{V}}(s;\alpha,\widetilde{\hbar})w_1^2(s;\widetilde{\hbar})\int_s^{\overline{\zeta}} w_1^{-2}(t;\widetilde{\hbar})\,dt \end{aligned} \tag{4.192}$$

has a unique bounded solution $a_1(\overline{\zeta};\alpha,\widetilde{\hbar})$ *for all* $\widetilde{\hbar} = -\frac{1}{N_0-1}, -\frac{1}{N_0}, -\frac{1}{N_0+1}, \ldots$ *and* $\alpha > 0$, $\overline{\zeta} \le 0$. *One has* $\lim_{\overline{\zeta}\to-\infty} a_1(\overline{\zeta};\alpha,\widetilde{\hbar}) = 0$ *and* $w(\overline{\zeta};\alpha,\widetilde{\hbar}) := w_1(\overline{\zeta};\widetilde{\hbar})(1+\widetilde{\hbar}a_1(\overline{\zeta};\alpha,\widetilde{\hbar}))$ *is the unique solution of* (4.189) *on* $(-\infty,0]$ *with* $w(\overline{\tau};\alpha,\widetilde{\hbar}) \sim w_1(\overline{\zeta};\widetilde{\hbar})$ *as* $\overline{\zeta} \to -\infty$.

We also have the analogue of Proposition 4.26:

Proposition 4.41. *There exists a constant* $\overline{\tau}_* > 0$ *and a small constant* $0 < \alpha_* \ll 1$ *so that uniformly in* $\widetilde{\hbar} = -\frac{1}{N_0-1}, -\frac{1}{N_0}, -\frac{1}{N_0+1}, \ldots$,

$$|\partial_{\overline{\tau}}^k \partial_\alpha^\ell \underline{\widetilde{V}}(\overline{\tau};\alpha,\widetilde{\hbar})| \le C_{k,\ell}\,\langle\alpha\rangle^{-\ell-1}\langle\overline{\tau}\rangle^{-2-k}, \qquad \forall\, \overline{\tau} \ge -\overline{\tau}_* \tag{4.193}$$

for all $k,\ell \ge 0$ *and* $\alpha > 0$. *Moreover,* (4.193) *holds for* $-\infty < \overline{\tau} \le -\overline{\tau}_*$, *all* $k,\ell \ge 0$ *and* $\alpha \ge \alpha_*$. *Finally, if* $0 < \alpha \le \alpha_*$ *and* $-\infty\overline{\tau} \le -\tau_*$, *then*

$$\underline{\widetilde{V}}(\overline{\tau};\alpha,\widetilde{\hbar}) = \frac{5}{16\overline{\tau}^2} - \overline{\tau}\varphi(\underline{x};\alpha,\widetilde{\hbar})$$

where for all $k,\ell \ge 0$

$$|\partial_{\overline{\tau}}^k \partial_\alpha^\ell \varphi(\underline{x};\alpha,\widetilde{\hbar})| \le C_{k,\ell}\,\min\left(|\widetilde{\hbar}|\,\alpha^2\underline{x}^{-2} + \underline{x}^2, |\widetilde{\hbar}|\underline{x}^2/\alpha^2 + \underline{x}^2\right)|\alpha|^{-\ell}(-\overline{\tau})^{\frac{k}{2}} \tag{4.194}$$

Here $\underline{x} = \underline{x}(\overline{\tau};\alpha,\widetilde{\hbar})$ *is the inverse of the diffeomorphism* $\overline{\tau} = \overline{\tau}(\underline{x};\alpha,\widetilde{\hbar})$ *defined in* (4.187), *and satisfies*

$$\frac{2}{3}\left(-\overline{\tau}(\underline{x};\alpha,\widetilde{\hbar})\right)^{\frac{3}{2}} = \begin{cases} -(1-O(\alpha^2))\log(\underline{x}) + O_1(1), & \text{if } 0 < \alpha \le \underline{x} \le \underline{x}_* := \underline{x}(-\tau_*;\alpha,\hbar) \\ -\log(\underline{x}) + 2\widetilde{\hbar}\log(\underline{x}/\alpha) + O_2(1), & \text{if } 0 < \underline{x} \le \alpha \le \alpha_* \end{cases} \tag{4.195}$$

Here $O(\alpha^2)$ *is analytic in complex* $|\alpha| \le \alpha_*$, *and bounded uniformly in* $\widetilde{\hbar} = -\frac{1}{N_0-1}, -\frac{1}{N_0}, -\frac{1}{N_0+1}$. *Furthermore, the two terms* $O_1(1)$, *resp.* $O_2(1)$ *refer to smooth functions of* $\overline{\tau}, \alpha$ *(and thus also of* $\underline{x}$*), uniformly bounded in* $0 < \alpha \le \alpha_*$, $-\infty < \tau \le -\tau_*$, *and so that for all* $k,\ell \ge 0$ *one has* $\partial_{\underline{x}}^k \partial_\alpha^\ell O_1(1) = O(\underline{x}^{-k}\alpha^{-\ell})$ *in the parameter regime of the first line of* (4.195), *resp.* $\partial_{\underline{x}}^k \partial_\alpha^\ell O_2(1) = O(\alpha^{-k-\ell})$ *in the regime of the second line of* (4.195).

Based on Proposition 4.41, we have the analogues of Lemma 4.29–Lemma 4.31.

Lemma 4.42. *The functions* $a_1(\zeta;\alpha,\widetilde{\hbar})$ *from Lemma 4.40 satisfy the bounds*

$$\begin{aligned} |\partial_\alpha^\ell a_1(\overline{\zeta};\alpha,\widetilde{\hbar})| &\le C_\ell \langle\alpha\rangle^{-\ell}\langle\overline{\zeta}\rangle^{-\frac{3}{2}}, \quad \overline{\zeta} \le 0, \\ |\partial_{\overline{\zeta}}^k \partial_\alpha^\ell a_1(\overline{\zeta};\alpha,\widetilde{\hbar})| &\le C_{k,\ell}\,\langle\alpha\rangle^{-\ell} \begin{cases} |\overline{\zeta}|^{-\frac{3}{2}-k} & -\infty < \overline{\zeta} \le -1 \\ |\overline{\zeta}|^{\frac{1}{2}-k} & -1 < \overline{\zeta} \le -|\widetilde{\hbar}|^{\frac{2}{3}} \\ |\widetilde{\hbar}|^{\frac{1-2k}{3}} & -|\widetilde{\hbar}|^{\frac{2}{3}} \le \overline{\zeta} \le 0, \end{cases} \end{aligned} \tag{4.196}$$

for all $\ell \ge 0, k \ge 1$, $\alpha > 0$, *and* $\widetilde{\hbar} = -\frac{1}{N_0-1}, -\frac{1}{N_0}, -\frac{1}{N_0+1}, \ldots$.

Lemma 4.43. *The functions $a_0(\overline{\zeta};\alpha,\widetilde{\hbar})$ from Lemma 4.40 satisfy the bounds*

$$|\partial_\alpha^\ell a_0(\overline{\zeta};\alpha,\widetilde{\hbar})| \le C_\ell\,\alpha^{-\ell}\langle\overline{\zeta}\rangle^{-\frac{3}{2}}, \quad \overline{\zeta}\ge 0,$$

$$|\partial_{\overline{\zeta}}^k\partial_\alpha^\ell a_0(\overline{\zeta};\alpha,\widetilde{\hbar})| \le C_{k,\ell}\,\alpha^{-\ell}\begin{cases} \overline{\zeta}^{-\frac{3}{2}-k} & 1<\overline{\zeta}<\infty \\ \overline{\zeta}^{\frac{1}{2}-k} & |\widetilde{\hbar}|^{\frac{2}{3}}<\overline{\zeta}\le 1 \\ |\widetilde{\hbar}|^{\frac{1-2k}{3}} & 0\le\overline{\zeta}\le|\widetilde{\hbar}|^{\frac{2}{3}} \end{cases} \tag{4.197}$$

for all $\ell\ge 0, k\ge 1$, $\alpha\gtrsim 1$, and $\widetilde{\hbar}=-\frac{1}{N_0-1},-\frac{1}{N_0},-\frac{1}{N_0+1},\ldots$.

and

Lemma 4.44. *The functions $a_0(\overline{\zeta};\alpha,\widetilde{\hbar})$ from Lemma 4.40 satisfy the bounds*

$$|\partial_\alpha^\ell a_0(\overline{\zeta};\alpha,\widetilde{\hbar})| \le C_\ell\,\alpha^{-\ell}[\langle\overline{\zeta}\rangle^{-\frac{3}{2}}+\min(1,\underline{x}(\overline{\zeta};\alpha,\widetilde{\hbar})^2/\alpha^2)], \quad \overline{\zeta}\ge 0, \tag{4.198}$$

for all $\ell\ge 0$, $0<\alpha\ll 1$, and $\widetilde{\hbar}=-\frac{1}{N_0-1},-\frac{1}{N_0},-\frac{1}{N_0+1},\ldots$. Here $\underline{x}(\overline{\zeta};\alpha,\widetilde{\hbar})$ is the diffeomorphism from Proposition 4.41. Furthermore, in the same parameter regime,

$$|\partial_\alpha^\ell\partial_{\overline{\zeta}}^k a_0(\overline{\zeta};\alpha,\widetilde{\hbar})| \le C_{k,\ell}\,\alpha^{-\ell}\overline{\zeta}^{\frac{k}{2}}[\langle\overline{\zeta}\rangle^{-\frac{3}{2}}+\min(1,\underline{x}(\overline{\zeta};\alpha,\widetilde{\hbar})^2/\alpha^2)], \quad \overline{\zeta}\ge 1, \tag{4.199}$$

and

$$|\partial_\alpha^\ell\partial_{\overline{\zeta}}^k a_0(\overline{\zeta};\alpha,\widetilde{\hbar})| \le C_{k,\ell}\,\alpha^{-\ell}\begin{cases} \overline{\zeta}^{\frac{1}{2}-k} & |\widetilde{\hbar}|^{\frac{2}{3}}<\overline{\zeta}\le 1 \\ |\widetilde{\hbar}|^{\frac{1-2k}{3}} & 0\le\overline{\zeta}\le|\widetilde{\hbar}|^{\frac{2}{3}} \end{cases} \tag{4.200}$$

With the above preparations, we can now state the result on Fourier basis for $n\ge N_0$.

Proposition 4.45. *The distorted Fourier transform associated with $-\mathcal{H}_{-n}^+$, $n\gg 1$, takes the following form:*

$$\hat{f}(\xi)=\int_0^\infty \phi_{-n}(R,\xi)f(R)\,dR, \qquad f(R)=\int_0^\infty \phi_{-n}(R,\xi)\hat{f}(\xi)\rho_{-n}(d\xi) \tag{4.201}$$

with

$$\begin{aligned}\phi_n(R;\xi)&=|\widetilde{\hbar}|^{\frac{1}{3}}\alpha^{-\frac{1}{2}}\overline{q}^{-\frac{1}{4}}(\overline{\tau})Ai(-|\widetilde{\hbar}|^{-\frac{2}{3}}\overline{\tau})(1+|\widetilde{\hbar}|a_0(-\overline{\tau};\alpha,\widetilde{\hbar}))\\ &=|\widetilde{\hbar}|^{-\frac{1}{6}}\xi^{-\frac{1}{4}}\overline{q}^{-\frac{1}{4}}(\overline{\tau})Ai(-|\widetilde{\hbar}|^{-\frac{2}{3}}\overline{\tau})(1+|\widetilde{\hbar}|a_0(-\overline{\tau};\alpha,\widetilde{\hbar}))\\ \overline{\tau}&=\overline{\tau}(\underline{x};\alpha,\widetilde{\hbar}), \quad \underline{x}=\alpha R\le 1, \quad \alpha=|\widetilde{\hbar}|\xi^{\frac{1}{2}}\end{aligned} \tag{4.202}$$

Here $\overline{\tau}=\overline{\tau}(\underline{x};\alpha,\widetilde{\hbar})$ as in (4.187)*, $\overline{q}$ is defined in* (4.188)*. One has $\phi_{-n}(R;\xi)\sim|\widetilde{\hbar}|^{n-\frac{1}{2}}\xi^{\frac{n-1}{2}}R^{n-\frac{1}{2}}$ as $R\xi^{\frac{1}{2}}\to 0+$. For $\underline{x}\ge 1$*

$$\begin{aligned}\phi_{-n}(R;\xi)&=-c_1|\widetilde{\hbar}|^{-\frac{1}{6}}\xi^{-\frac{1}{4}}\overline{q}^{-\frac{1}{4}}(\overline{\tau})\mathrm{Re}\Big((1+|\widetilde{\hbar}|\Xi(\xi;\widetilde{\hbar}))(Ai(-|\widetilde{\hbar}|^{-\frac{2}{3}}\overline{\tau})+iBi(-|\widetilde{\hbar}|^{-\frac{2}{3}}\overline{\tau}))(1+|\widetilde{\hbar}|a_1(\overline{\tau};\alpha,\widetilde{\hbar}))\Big)\\ &\sim -c_2\xi^{-\frac{1}{4}}\mathrm{Re}\Big((1+|\widetilde{\hbar}|\Xi(\xi;\widetilde{\hbar}))e^{i\frac{\pi}{4}}e^{i\xi^{\frac{1}{2}}R}\Big) \text{ as } R\xi^{\frac{1}{2}}\to\infty,\end{aligned} \tag{4.203}$$

where $c_1, c_2 > 0$ are absolute constants and $|\partial_\xi^k \Xi(\xi;\widetilde{\hbar})| \le C_k \xi^{-k}$ for all $k \ge 0$ uniformly in $\widetilde{\hbar}$. The spectral measure ρ_{-n} is purely absolutely continuous with density satisfying

$$\frac{1}{2} \le \frac{d\rho_{-n}(\xi)}{d\xi} \le 2, \qquad \Big|\frac{d^\ell}{d\xi^\ell}\frac{d\rho_{-n}(\xi)}{d\xi}\Big| \le C_\ell\, \xi^{-\ell}, \qquad \forall\, \xi > 0,\ \ell \ge 0 \tag{4.204}$$

uniformly in n. For the higher order derivatives of $\phi_{-n}(R,\xi)$ we have

$$(R\partial_R)^k \phi_{-n}(R,\xi) \sim |\widetilde{\hbar}|^{n-\frac{1}{2}} \xi^{\frac{n-1}{2}} R^{n-\frac{1}{2}}, \quad (\xi\partial_\xi)^k \phi_{-n}(R,\xi) \sim |\widetilde{\hbar}|^{n-\frac{1}{2}} \xi^{\frac{n-1}{2}} R^{n-\frac{1}{2}}, \quad as \quad R\xi^{\frac{1}{2}} \to 0+,$$

and

$$(R\partial_R)^k \left(e^{-iR\xi^{\frac{1}{2}}} \phi_{-n}(R,\xi)\right) \sim c_8 \xi^{-\frac{1}{4}}, \quad (\xi\partial_\xi)^k \left(e^{-iR\xi^{\frac{1}{2}}} \phi_{-n}(R,\xi)\right) \sim c_9 \xi^{-\frac{1}{4}} \quad as \quad R\xi^{\frac{1}{2}} \to \infty.$$

Remark 4.46. *Proposition 4.39 and Proposition 4.45 show that the Fourier basis constructed for large and small $|\widetilde{\hbar}|$ have the same asymptotic behavior as $R\xi^{\frac{1}{2}} \to 0_+$ and $R\xi^{\frac{1}{2}} \to \infty$.*

Remark 4.47. *One can easily follow the same procedure in Proposition 4.38 to show that the Fourier transform and its inverse transform exist for C^2-functions f satisfying*

$$\int_0^\infty \left(R^{-1}|f(R)| + |f'(R)| + R|f''(R)|\right) dR < \infty.$$

Remark 4.48. *Based on Proposition 4.39 and Proposition 4.45, one can follow similar procedures in the proof of these two propositions to show that the spectral measure density $\frac{d\rho_{-n}(\xi)}{d\xi}$ associated to $\phi_{-n}(R,\xi)$ satisfies*

$$C^{-1} \le \frac{d\rho_{-n}(\xi)}{d\xi} \le C, \quad \left|\frac{d^\ell}{d\xi^\ell}\frac{d\rho_{-n}(\xi)}{d\xi}\right| \le C_\ell \xi^{-\ell}, \quad \forall \xi > 0, \ell > 0, n \ge 2.$$

Here the spectral measure density $\frac{d\rho_{-n}(\xi)}{d\xi}$ is given by

$$\frac{d\rho_{-n}(\xi)}{d\xi} = \frac{1}{4\pi|a_-(\xi)|^2}.$$

5. Transference identities

5.1. The identities for angular momenta $n \ge 2$ and $n \le -2$.

5.1.1. $n \ge 2$. Let us define the operator $\mathcal{K}_\hbar$ as

$$\widehat{R\partial_R u} = -2\xi\partial_\xi \widehat{u} + \mathcal{K}_\hbar \widehat{u}. \tag{5.1}$$

Here the Fourier transform is taken with respect to the Fourier basis $\phi(R,\xi;\hbar) = R^{-\frac{1}{2}}\phi_*(R,\xi;\hbar)$ in $L^2(R\,dR)$ introduced in the previous section, with $\phi_*(R,\xi;\hbar)$ being the Fourier basis in $L^2(dR)$. Let $\rho_n(d\xi)$ be the spectral measure associated with $\phi_*(R,\xi;\hbar)$, where $\hbar = \frac{1}{n+1}$. We have

$$\overline{x}(\xi) := \mathcal{F}_*\left(R^{\frac{1}{2}} f(R)\right)(\xi) = \langle R^{-\frac{1}{2}}\phi_*(R,\xi;\hbar), f(R)\rangle_{L^2_{RdR}},$$
$$f(R) = \int_0^\infty \overline{x}(\xi) R^{-\frac{1}{2}}\phi_*(R,\xi;\hbar)\rho_n(d\xi).$$

Then one can easily check that the following Plancherel identity holds:

$$\|f\|^2_{L^2_{RdR}} = \|\langle f, R^{-\frac{1}{2}}\phi_*\rangle_{L^2_{RdR}}\|^2_{L^2_{\rho_n(d\xi)}}.$$

Therefore $\rho_n(d\xi)$ is also the spectral measure for $\phi(R,\xi;\hbar)$. The main result of this section is:

Proposition 5.1. *For any* $0 < \hbar = \frac{1}{n+1} \le \frac{1}{3}$ *the operator* $\mathcal{K}_\hbar$ *is given by*

$$\mathcal{K}_\hbar f(\xi) = -\left(2f(\eta) + \frac{\eta\left(\frac{d\rho_n}{d\eta}(\eta)\right)'}{\frac{d\rho_n(\eta)}{d\eta}} f(\eta)\right)\delta(\eta-\xi) + \left(\mathcal{K}_\hbar^{(0)} f\right)(\xi)$$

where the off-diagonal part $\mathcal{K}_\hbar^{(0)}$ *has a kernel* $K_0(\xi,\eta;\hbar)$ *given by*

$$K_0(\xi,\eta;\hbar) = \frac{\frac{d\rho_n(\eta)}{d\eta}}{\xi-\eta} F(\xi,\eta;\hbar) \tag{5.2}$$

and the symmetric function $F(\xi,\eta;\hbar)$ *satisfies (for any* $0 \le k \le k_0$ *and sufficiently small* $\hbar = \hbar(k_0)$*, where* k_0 *is arbitrary but fixed, and* $\xi \le \eta$*)*

$$|F(\xi,\eta;\hbar)| \lesssim \left(\hbar\xi^{\frac{1}{2}}\right)^{-1} \min\left\{1, \left(\hbar\xi^{\frac{1}{2}}\right)^3\right\} \cdot G := \Gamma_\hbar. \tag{5.3}$$

with

$$G := \begin{cases} \min\left\{1, \left(\hbar\xi^{\frac{1}{2}}\right)^{\frac{1}{4}} \left|\eta^{\frac{1}{2}} - \xi^{\frac{1}{2}}\right|^{-\frac{1}{4}}\right\}, & \text{for} \quad \left|\frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}} - 1\right| \lesssim 1, \\ \hbar\left(\hbar\xi^{\frac{1}{2}}\right)^{-1} \min\left\{1, \hbar\xi^{\frac{1}{2}}\right\} \cdot \left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^k, & \text{for} \quad \left|\frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}} - 1\right| \gg 1. \end{cases} \tag{5.4}$$

For $\hbar \gtrsim 1$*, we have, for any* $N > 0$

$$|F(\xi,\eta;\hbar)| \lesssim \Gamma_n := \begin{cases} \eta|\log\eta| \cdot \frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}, & \text{for} \quad \eta \ll 1, \\ \max\{\xi^{-\frac{1}{2}}\eta^{-\frac{1}{2}}, \eta^{-\frac{1}{4}}\xi^{-\frac{1}{4}}\} \cdot \left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^n \cdot \left(\frac{\langle\xi\rangle}{\eta}\right)^N, & \text{for} \quad \eta \gtrsim 1 \end{cases}. \tag{5.5}$$

For the derivatives of $F(\xi,\eta;\hbar)$*, we have, for* $k_1 + k_2 \le 2$

$$\left|\partial^{k_1}_{\xi^{\frac{1}{2}}} \partial^{k_2}_{\eta^{\frac{1}{2}}} F(\xi,\eta;\hbar)\right| \lesssim \max\left\{1, \left(\hbar\xi^{\frac{1}{2}}\right)^{-k_1}\right\} \cdot \max\left\{1, \left(\hbar\eta^{\frac{1}{2}}\right)^{-k_2}\right\} \cdot \Gamma \tag{5.6}$$

Here Γ *is* $\Gamma_\hbar$ *if* $\hbar \ll 1$*, and* Γ_n *if* $\hbar \gtrsim 1$*. The following estimate holds for trace-type derivatives for* $\hbar \ll 1$*:*

$$\left|\left(\partial_{\xi^{\frac{1}{2}}} + \partial_{\eta^{\frac{1}{2}}}\right)^k F(\xi,\eta;\hbar)\right| \le C_k\, \xi^{-\frac{k}{2}} \cdot \Gamma, \quad \text{if} \quad \xi \simeq \eta,\ |\eta-\xi| \le \hbar^{\frac{2}{3}}\xi \tag{5.7}$$

for all $k \ge 0$*.*

Proof. For simplicity, we will often drop $\hbar$ from the notation. For example, we write $\phi(R,\xi)$ for $\phi(R,\xi;\hbar)$ etc. We also write $\rho_n(d\xi) = \rho(\xi)\,d\xi$. For $f \in C_0^\infty((0,\infty))$ define

$$u(R) = \int_0^\infty \phi(R,\xi) f(\xi)\rho(\xi)\,d\xi$$

This function decays rapidly in R due to the oscillations of $\phi(R,\xi)$, see Proposition 4.32. Thus,

$$\begin{aligned}\widehat{R\partial_R u}(\xi) &= \left\langle \int_0^\infty R\partial_R\phi(R,\eta)f(\eta)\rho(\eta)\,d\eta, \phi(R,\xi)\right\rangle_{L^2(R\,dR)} \\ &= \left\langle \int_0^\infty [R\partial_R - 2\eta\partial_\eta]\phi(R,\eta)f(\eta)\rho(\eta)\,d\eta, \phi(R,\xi)\right\rangle_{L^2(R\,dR)} \\ &\quad - 2\left\langle \int_0^\infty \phi(R,\eta)\partial_\eta\left(\eta f(\eta)\rho(\eta)\right)d\eta, \phi(R,\xi)\right\rangle_{L^2(R\,dR)} \\ &= \left\langle \int_0^\infty [R\partial_R - 2\eta\partial_\eta]\phi(R,\eta)f(\eta)\rho(\eta)\,d\eta, \phi(R,\xi)\right\rangle_{L^2(R\,dR)} \\ &\quad - 2f(\xi) - 2\frac{\xi\rho'(\xi)}{\rho(\xi)}f(\xi) - 2\xi f'(\xi)\end{aligned} \tag{5.8}$$

We used that as limit of distributions

$$\lim_{L\to\infty}\langle\chi(R/L)\phi(R,\eta),\phi(R,\xi)\rangle_{L^2(R\,dR)} = \rho(\xi)^{-1}\delta(\xi-\eta) \tag{5.9}$$

where χ is compactly supported smooth bump function which equals 1 on $[0,1]$. It follows that

$$\begin{aligned}\mathcal{K}f(\eta) = &\left\langle \int_0^\infty f(\xi)[R\partial_R - 2\xi\partial_\xi]\phi(R,\xi)\rho(\xi)d\xi, \phi(R,\eta)\right\rangle_{L^2_{RdR}} \\ &- 2\left(1 + \frac{\eta\rho'(\eta)}{\rho(\eta)}\right)f(\eta).\end{aligned} \tag{5.10}$$

To extract any $\delta(\xi-\eta)$ appearing from the RdR integral in the first line here we recall from Proposition 4.32 and (4.161) that

$$\begin{aligned}\phi(R,\xi) &= 2R^{-\frac12}\mathrm{Re}\,\left(a(\xi)\psi_+(R;\alpha)\right) \\ &= 2\hbar^{-\frac16}(R\xi^{\frac12})^{-\frac12}q^{-\frac14}(\tau)\mathrm{Re}\,\left(a(\xi)\left(\mathrm{Ai}(-\hbar^{-\frac23}\tau) - i\mathrm{Bi}(-\hbar^{-\frac23}\tau)\right)(1+\hbar\overline{a_1(-\tau;\alpha)})\right) \\ \psi_+(R,\xi;\hbar) &= \hbar^{\frac13}\alpha^{-\frac12}q^{-\frac14}(\mathrm{Ai}(-\hbar^{-\frac23}\tau) - i\mathrm{Bi}(-\hbar^{-\frac23}\tau))(1+\hbar\overline{a_1(-\tau;\alpha,\hbar)})\end{aligned} \tag{5.11}$$

By Lemma 4.29 we have for all $\tau \ge 0$

$$|\partial_\alpha^\ell\partial_\tau^k a_1(\tau;\alpha)| \le C_{\ell,k}\langle\alpha\rangle^{-\ell}\langle\tau\rangle^{-\frac32-k} \qquad \forall\ k,\ell\ge 0 \tag{5.12}$$

and by (4.124),

$$\mathrm{Ai}(-\hbar^{-\frac23}\tau) - i\mathrm{Bi}(-\hbar^{-\frac23}\tau) = \pi^{-\frac12}e^{-i\frac{\pi}{4}}\hbar^{\frac16}\tau^{-\frac14}e^{i\frac{2}{3\hbar}\tau^{\frac32}}(1+\hbar O(\tau^{-\frac32})) \qquad \tau\to\infty$$

Inserting this into (5.11) we obtain

$$\phi(R,\xi) = 2\pi^{-\frac12}(R\xi^{\frac12})^{-\frac12}(\tau q(x;\alpha,\hbar))^{-\frac14}\mathrm{Re}\left(a(\xi)e^{i\frac{2}{3\hbar}\tau^{\frac32}}e^{-i\frac{\pi}{4}}(1+\hbar\tilde{a}_1(-\tau;\alpha))\right) \tag{5.13}$$

where $\tilde{a}_1$ satisfies the same bounds (5.12) as a_1. Here $x = \hbar\xi^{\frac{1}{2}}R$ and Q_0 is defined in (4.66). By (4.86) and Lemma 4.13

$$\begin{aligned}\tau^{-\frac{1}{4}}q^{-\frac{1}{4}} &= (1+\rho'(x;\alpha,\hbar))^{-\frac{1}{2}}\\ \frac{2}{3}\tau^{\frac{3}{2}} &= x - y(\alpha;\hbar) + \rho(x;\alpha,\hbar),\end{aligned}$$

Writing a_1 in place of $\tilde{a}_1$ for simplicity (since both satisfy (5.12)) we conclude that

$$\phi(R,\xi) = 2\pi^{-\frac{1}{2}}(R\xi^{\frac{1}{2}})^{-\frac{1}{2}}(1+\rho'(\hbar\xi^{\frac{1}{2}}R;\alpha,\hbar))^{-\frac{1}{2}}\mathrm{Re}\Big(a(\xi)e^{i\xi^{\frac{1}{2}}R}e^{\frac{i}{\hbar}[-y(\alpha;\hbar)+\rho(\hbar\xi^{\frac{1}{2}}R;\alpha,\hbar)]}(1+\hbar a_1(-\tau;\alpha))\Big) \tag{5.14}$$

For a function g of the form $g = g(x;\alpha,\hbar)$, we have

$$R\partial_R g - 2\xi\partial_\xi g = -\hbar\xi^{\frac{1}{2}}\partial_\alpha g = -\alpha\partial_\alpha g. \tag{5.15}$$

Thus, applying this differential operator to (5.14) yields

$$\begin{aligned}&(R\partial_R - 2\xi\partial_\xi)\phi(R,\xi) = -2\pi^{-\frac{1}{2}}(R\xi^{\frac{1}{2}})^{-\frac{1}{2}}\mathrm{Re}\Big(2\xi a'(\xi)e^{i\xi^{\frac{1}{2}}R}e^{\frac{i}{\hbar}[-y(\alpha;\hbar)+\rho(\hbar\xi^{\frac{1}{2}}R;\alpha,\hbar)]}(1+\hbar a_1(-\tau;\alpha))\Big)\\ &+2\hbar^{-1}\pi^{-\frac{1}{2}}\alpha(R\xi^{\frac{1}{2}})^{-\frac{1}{2}}\mathrm{Re}\Big(i\partial_\alpha y(\alpha;\hbar)a(\xi)e^{i\xi^{\frac{1}{2}}R}e^{\frac{i}{\hbar}[-y(\alpha;\hbar)+\rho(\hbar\xi^{\frac{1}{2}}R;\alpha,\hbar)]}(1+\hbar a_1(-\tau;\alpha))\Big)\\ &-2\hbar\pi^{-\frac{1}{2}}\alpha(R\xi^{\frac{1}{2}})^{-\frac{1}{2}}\mathrm{Re}\Big(a(\xi)e^{i\xi^{\frac{1}{2}}R}e^{\frac{i}{\hbar}[-y(\alpha;\hbar)+\rho(\hbar\xi^{\frac{1}{2}}R;\alpha,\hbar)]}\partial_\alpha a_1(-\tau;\alpha)\Big) + O(R^{-\frac{5}{2}})\\ &= -2\pi^{-\frac{1}{2}}(R\xi^{\frac{1}{2}})^{-\frac{1}{2}}\mathrm{Re}\Big\{e^{i\hbar^{-1}\Psi(R;\xi,\hbar)}\Big([2\xi a'(\xi) - i\xi^{\frac{1}{2}}a(\xi)\partial_\alpha y(\alpha,\hbar)](1+\hbar a_1(-\tau;\alpha)) + \hbar\alpha a(\xi)\partial_\alpha a_1(-\tau;\alpha)\Big)\Big\}\\ &\quad + O(R^{-\frac{5}{2}})\end{aligned}$$

as $R\to\infty$ where

$$\Psi(R;\xi,\hbar) = \hbar\xi^{\frac{1}{2}}R - y(\alpha;\hbar) + \rho(\hbar\xi^{\frac{1}{2}}R;\alpha,\hbar)$$

The final $O(R^{-\frac{5}{2}})$ is a result of the derivatives $\alpha\partial_\alpha$ falling onto the ρ terms. In addition, we used that $(1+\rho'(\hbar\xi^{\frac{1}{2}}R;\alpha,\hbar))^{-\frac{1}{2}} = 1 + O(R^{-2})$. It does not contribute to the diagonal $\xi = \eta$ and we therefore ignore it. Using that $\mathrm{Re}\, z\mathrm{Re}\, w = \frac{1}{2}(\mathrm{Re}\,(zw) + \mathrm{Re}\,(z\overline{w}))$ we infer that the δ measure on the diagonal in the integral, cf. (5.8)

$$\lim_{A\to\infty}\int_0^A [R\partial_R - 2\xi\partial_\xi]\phi(R,\xi)\phi(R,\eta)R\,dR$$

comes from the expression

$$\begin{aligned}-2\pi^{-1}(\xi\eta)^{-\frac{1}{4}}\lim_{L\to\infty}\mathrm{Re}\int_0^\infty &e^{i\hbar^{-1}[\Psi(R;\xi,\hbar)-\Psi(R;\eta,\hbar)]}\Big([2\xi a'(\xi) - i\xi^{\frac{1}{2}}a(\xi)\partial_\alpha y(\alpha,\hbar)](1+\hbar a_1(-\tau;\alpha))\\ &+\hbar\alpha a(\xi)\partial_\alpha a_1(-\tau;\alpha)\Big)\overline{a(\eta)}(1+\hbar\overline{a_1(-\sigma;\beta)})\chi_1(R)\chi_2(R/L)\,dR =: I(\xi,\eta;\hbar)\end{aligned} \tag{5.16}$$

which is to be evaluated as a distributional limit as $L\to\infty$. Here χ_1 is a s smooth cutoff which equals 1 near ∞ and which vanishes near 0, and $\chi_2 = 1-\chi_1$. It depends on the compact subset of $(0,\infty)$ which holds ξ,η. Furthermore, α is above, $\beta = \hbar\eta^{\frac{1}{2}}$, and σ is associated with $y = \hbar\eta^{\frac{1}{2}}R$ the same way that τ is to x. We claim that we can discard all terms involving a_1 from $I(\xi,\eta;\hbar)$ as they do not contribute to δ on the diagonal. We note that by the relation between τ and R we have $a_1(-\tau;\alpha) = O(R^{-1})$ as $R\to\infty$, with symbol behavior under differentiation in R. Moreover this bound is uniform in ξ in a compact set in $(0,\infty)$.

We first rewrite the phase as follows

$$\begin{aligned}\hbar^{-1}[\Psi(R;\xi,\hbar)-\Psi(R;\eta,\hbar)] &= R(\xi^{\frac12}-\eta^{\frac12})[1+\Omega(R;\xi,\eta,\hbar)]-\hbar^{-1}[y(\hbar\xi^{\frac12};\hbar)-y(\hbar\eta^{\frac12};\hbar)]\\ \Omega(R;\xi,\eta,\hbar) &= \hbar^{-1}R^{-1}(\xi^{\frac12}-\eta^{\frac12})^{-1}[\rho(\hbar\xi^{\frac12}R;\hbar\xi^{\frac12},\hbar)-\rho(\hbar\eta^{\frac12}R;\hbar\eta^{\frac12},\hbar)]\\ &= \int_0^1[\rho_x(R\gamma(s);\gamma(s),\hbar)+R^{-1}\rho_\alpha(R\gamma(s);\gamma(s),\hbar)]\,ds\\ \gamma(s) &= \hbar\eta^{\frac12}+s\hbar(\xi^{\frac12}-\eta^{\frac12})\end{aligned} \tag{5.17}$$

By Lemma 4.13, one has $|\Omega(R;\xi,\eta,\hbar)|\lesssim R^{-2}$. We introduce the new variable

$$\tilde R := R[1+\Omega(R;\xi,\eta,\hbar)],\quad \frac{d\tilde R}{dR}=1+\partial_R(R\Omega(R;\xi,\eta,\hbar))=1+O(R^{-2})$$

in (5.16). The $O(R^{-2})$ here as $R\to\infty$ is uniform in ξ,η from a compact subset of $(0,\infty)$. Hence, this error term makes a bounded contribution to (5.16) and cannot contribute to any δ measure. Consequently, we can ignore the Ω term in the phase of $I(\xi,\eta;\hbar)$, see (5.17).

Returning to the claim about a_1, note that an oscillatory integral of the form

$$\int_0^\infty e^{iR\zeta}\omega(R)\chi_1(R)\chi_2(R/L)\,dR =: f_L(\zeta) \tag{5.18}$$

where $\omega(R)=O(R^{-1})$ has the property that f_L converges strongly in $L^2(\mathbb{R})$ to some $f\in L^2(\mathbb{R})$. But this means in particular that any such expression cannot converge in the sense of distributions to an expression with a $\delta_0(\zeta)$ component.

In summary, any δ measure arising in (5.16) is already present in the simpler expression

$$-2\pi^{-1}(\xi\eta)^{-\frac14}\lim_{L\to\infty}\mathrm{Re}\int_0^\infty e^{iR(\xi^{\frac12}-\eta^{\frac12})}[2\xi a'(\xi)-i\xi^{\frac12}a(\xi)\partial_\alpha y(\alpha,\hbar)]\overline{a(\eta)}\chi_1(R)\chi_2(R/L)\,dR =: I_0(\xi,\eta;\hbar)$$

Note that we have also discarded the factor $e^{-i\hbar^{-1}[y(\hbar\xi^{\frac12};\hbar)-y(\hbar\eta^{\frac12};\hbar)]}$ in the integral since it equals 1 on the diagonal. By standard Fourier analysis, since y is real-valued

$$\begin{aligned}I_0(\xi,\eta;\hbar) &= -2\pi^{-1}\pi(\xi\eta)^{-\frac14}\mathrm{Re}\,[2\xi a'(\xi)-i\xi^{\frac12}a(\xi)\partial_\alpha y(\alpha,\hbar)]\overline{a(\eta)}\delta(\xi^{\frac12}-\eta^{\frac12})+O(1)\\ &= -8\xi\mathrm{Re}\,a'(\xi)\overline{a(\xi)}\delta(\xi-\eta)+O(1) = -4\xi\frac{d}{d\xi}(|a(\xi)|^2)\delta(\xi-\eta)+O(1)\end{aligned}$$

as $\xi\to\eta$. By (4.12) and the asymptotics of ψ_+ in (5.11),

$$\begin{aligned}W(\psi(\cdot,\xi),\overline{\psi(\cdot,\xi)}) &= -2i\pi^{-1}\\ \rho(\xi) &= \frac{1}{2i\pi|a(\xi)|^2W(\psi(\cdot,\xi),\overline{\psi(\cdot,\xi)})}=(4|a(\xi)|^2)^{-1}\end{aligned}$$

Recall that we are denoting the density of the spectral measure by ρ. Returning to (5.10) we finally obtain the exact δ-measure contribution to the operator $\mathcal{K}$. With an operator $\mathcal{K}^{(0)}$ whose Schwartz kernel contains no δ measure on the diagonal,

$$\begin{aligned}\mathcal{K}f(\eta) &= -\rho(\eta)\eta\frac{d}{d\eta}(4|a(\eta)|^2)f(\eta) - 2\left(1+\frac{\eta\rho'(\eta)}{\rho(\eta)}\right)f(\eta) + (\mathcal{K}^{(0)}f)(\eta)\\ &= -\rho(\eta)\eta\frac{d}{d\eta}(\rho(\eta)^{-1})f(\eta) - 2\left(1+\frac{\eta\rho'(\eta)}{\rho(\eta)}\right)f(\eta) + (\mathcal{K}^{(0)}f)(\eta)\\ &= -\left(2+\frac{\eta\rho'(\eta)}{\rho(\eta)}\right)f(\eta) + (\mathcal{K}^{(0)}f)(\eta)\end{aligned} \tag{5.19}$$

in agreement with the statement of the proposition.

To determine the off-diagonal part of the operator $\mathcal{K}$, choose a function $f \in C_0^\infty((0,\infty))$ and define $u(R)$ to be the function

$$u(R) := \int_0^\infty f(\xi)\left(R\partial_R - 2\xi\partial_\xi\right)\phi(R,\xi)\rho(\xi)d\xi,$$

With $\phi(R,\xi)$ the Fourier basis of H_n^+, we have $-H_n^+\phi(R,\xi) = \xi\phi(R,\xi)$. Therefore

$$\begin{aligned}&\eta\,(\mathcal{K}f)\,(\eta) - \mathcal{K}\,(\xi f(\xi))\,(\eta)\\ &= \langle u, \eta\phi(R,\eta)\rangle_{L^2_{RdR}} - \left\langle \int_0^\infty \xi f(\xi)\left(R\partial_R - 2\xi\partial_\xi\right)\phi(R,\xi)\rho(\xi)d\xi, \phi(R,\eta)\right\rangle_{L^2_{RdR}}\\ &= -\langle H_n^+u, \phi(R,\eta)\rangle_{L^2_{RdR}} - \left\langle \int_0^\infty \xi f(\xi)\left(R\partial_R - 2\xi\partial_\xi\right)\phi(R,\xi)\rho(\xi)d\xi, \phi(R,\eta)\right\rangle_{L^2_{RdR}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^\infty \xi f(\xi)\left(R\partial_R - 2\xi\partial_\xi\right)\phi(R,\xi)\rho(\xi)d\xi\\ &= -\int_0^\infty f(\xi)\left(R\partial_R - 2\xi\partial_\xi\right)\left(H_n^+\phi(R,\xi)\right)\rho(\xi)d\xi + 2\int_0^\infty \xi f(\xi)\phi(R,\xi)\rho(\xi)d\xi\\ &= \int_0^\infty f(\xi)\,[H_n^+, R\partial_R]\,\phi(R,\xi)\rho(\xi)d\xi + 2\int_0^\infty \xi f(\xi)\phi(R,\xi)\rho(\xi)d\xi - H_n^+u.\end{aligned}$$

The commutator $[H_n^+, R\partial_R]$ is given by

$$[H_n^+, R\partial_R] = 2H_n^+ + R(f_n - g_n)' + 2f_n - 2g_n := 2H_n^+ + W_n^+(R),$$

and by the explicit expressions (3.35) for $f_n(R)$ and $g_n(R)$, we have

$$\begin{aligned}&R\partial_R f_n + 2f_n = \frac{16}{(R^2+1)^2} - \frac{32}{(R^2+1)^3}, \quad R\partial_R g_n + 2g_n = -\frac{8n}{(R^2+1)^2},\\ &W_n^+(R) = \frac{16}{(R^2+1)^2} - \frac{32}{(R^2+1)^3} + \frac{8n}{(R^2+1)^2}.\end{aligned} \tag{5.20}$$

Therefore we conclude that

$$\eta(\mathcal{K}f)(\eta) - [\mathcal{K}(\xi f(\xi))](\eta) = -\int_0^\infty f(\xi)\langle W_n^+(R)\phi(R,\xi), \phi(R,\eta)\rangle_{L^2_{RdR}}\,\rho(\xi)\,d\xi$$

The kernel function $F(\xi,\eta;\hbar)$ from (5.2) is therefore given by

$$F(\xi,\eta;\hbar) := \int_0^\infty W_n^+(R)\phi(R,\xi;\hbar)\phi(R,\eta;\hbar)R\,dR \tag{5.21}$$

Recall from (5.13) and (5.14) the following representation of the oscillatory regime

$$\begin{aligned}\phi(R,\xi) &= 2\hbar^{-\frac16}(R\xi^{\frac12})^{-\frac12}q^{-\frac14}(\tau)\mathrm{Re}\Big(a(\xi)(\mathrm{Ai}(-\hbar^{-\frac23}\tau) - i\mathrm{Bi}(-\hbar^{-\frac23}\tau))(1+\hbar\overline{a_1(-\tau;\alpha)})\Big)\\ &= 2\pi^{-\frac12}(R\xi^{\frac12})^{-\frac12}(\tau q(x;\alpha,\hbar))^{-\frac14}\mathrm{Re}\Big(a(\xi)e^{i\frac{2}{3\hbar}\tau^{\frac32}}e^{-i\frac{\pi}{4}}(1+\hbar\tilde a_1(-\tau;\alpha))\Big)\\ &= 2\pi^{-\frac12}(R\xi^{\frac12})^{-\frac12}(1+\rho'(\hbar\xi^{\frac12}R;\alpha))^{-\frac12}\mathrm{Re}\Big(a(\xi)e^{i\xi^{\frac12}R}e^{\frac{i}{\hbar}[-y(\alpha)+\rho(\hbar\xi^{\frac12}R;\alpha)]}(1+\hbar\tilde a_1(-\tau;\alpha))\Big)\end{aligned} \tag{5.22}$$

where $\rho'(\hbar\xi^{\frac12}R;\alpha)$ is governed by (4.86) and Lemma 4.13, and $a(\xi)$ is as in (4.163). Here in view of (4.124) and Lemma 4.29, $\tilde a_1(-\tau;\alpha)$ satisfies $|\tilde a_1(-\tau;\alpha)| \lesssim |\tau|^{-\frac32}$ as well as $\left|\partial_\tau^\ell \tilde a_1(-\tau;\alpha)\right| \lesssim |\tau|^{-\frac32-\ell}$ for all $\hbar^{\frac23} \lesssim \tau$. For ease of notation we suppress $\hbar$ from some of the notation (i.e., we write $\phi(R,\xi)$ in place of $\phi(R,\xi;\hbar)$, $a(\xi)$ instead of $a(\xi;\hbar)$ etc.).

On the other hand, to the left of the turning point, i.e., for $x = \hbar\xi^{\frac12}R \le x_t$, one has, cf. (4.152) which we need to multiply by $R^{-\frac12}$,

$$\begin{aligned}\phi(R;\xi) &= \hbar^{\frac13}x^{-\frac12}q^{-\frac14}(\tau)\mathrm{Ai}(-\hbar^{-\frac23}\tau)(1+\hbar a_0(-\tau;\alpha,\hbar))\\ &= \hbar^{\frac12}(x^2Q_0(x,\alpha))^{-\frac14}e^{-\frac{2}{3\hbar}(-\tau)^{\frac32}}(1+\hbar\tilde a_0(-\tau;\alpha,\hbar))\end{aligned} \tag{5.23}$$

Here $\tilde a_0(-\tau;\alpha,\hbar)$ satisfies $|\tilde a_0(-\tau;\alpha,\hbar)| \lesssim |\tau|^{-\frac32}$ and $\left|\partial_\tau^\ell \tilde a_0(-\tau;\alpha,\hbar)\right| \lesssim |\tau|^{-\frac32-\ell}$ for all $\tau \lesssim -\hbar^{\frac23}$. The representation (5.22) holds for $\tau \ge C\hbar^{\frac23}$, and (5.23) holds for $\tau \le -C\hbar^{\frac23}$ with some absolute constant C. Here τ is the global variable defined by (4.72). The third line of (5.22) is only valid if $\tau \ge C$ since then also $x = \hbar\xi^{\frac12}R \gg 1$ and $\rho'(\hbar\xi^{\frac12}R;\alpha) = O(x^{-2})$, see Lemma 4.11. In the second line of (5.23) we absorbed the error from the asymptotic expansion of $\mathrm{Ai}(-\hbar^{-\frac23}\tau)$ into the multiplicative correction $(1+\hbar\tilde a_0(-\tau;\alpha,\hbar))$. To pass to the second line we also used (4.73), which means that we can use this second line only sufficiently far away from the unique zero of Q_0 which is the turning point x_t. Thus, we need, $0 < x < \frac12 x_t$. On the other hand, if $\frac12 x_t \le x \le x_t$, then $x \simeq 1$, $q \simeq 1$.

Without loss of generality, in the rest of the proof we assume $\xi \le \eta$. We also first assume small $\hbar$. To estimate (5.21) we consider two regimes: (**Case A**) $\xi \simeq \eta$ and (**Case B**) $\xi \ll \eta$.

We start with **Case A:** $\xi \simeq \eta$. We break the integral in (5.21) into three separate regions, namely $x = \xi^{\frac12}\hbar R \ge C_1 \gg 1$ (case A_3), followed by $x \simeq 1$ (case A_2), and finally $0 < x \ll 1$ (case A_1).

With $\chi(\xi^{\frac12}\hbar R)$ a smooth cutoff to the region in **case A$_3$**, and with $\alpha = \hbar\xi^{\frac12}, \beta = \hbar\eta^{\frac12}$, and $\tau, \tilde\tau$ defined by (4.72) relative to ξ and η, respectively,

$$\begin{aligned}F_3(\xi,\eta;\hbar) &:= \int_0^\infty W_n^+(R)\phi(R,\xi;\hbar)\phi(R,\eta;\hbar)\chi(\xi^{\frac12}\hbar R)R\,dR \\ &= 4\pi^{-1}\int_0^\infty W_n^+(R)(R\xi^{\frac12})^{-\frac12}(1+\rho'(\hbar\xi^{\frac12}R;\alpha))^{-\frac12}\mathrm{Re}\Big(a(\xi)e^{i\xi^{\frac12}R}e^{\frac{i}{\hbar}[-y(\alpha)+\rho(\hbar\xi^{\frac12}R;\alpha)]}(1+\hbar\tilde a_1(-\tau;\alpha))\Big)\\ &\qquad (R\eta^{\frac12})^{-\frac12}(1+\rho'(\hbar\eta^{\frac12}R;\beta))^{-\frac12}\mathrm{Re}\Big(a(\eta)e^{i\eta^{\frac12}R}e^{\frac{i}{\hbar}[-y(\beta)+\rho(\hbar\eta^{\frac12}R;\beta)]}(1+\hbar\tilde a_1(-\tilde\tau;\beta))\Big)\chi(\xi^{\frac12}\hbar R)R\,dR\end{aligned} \tag{5.24}$$

$$= 4\pi^{-1}(\hbar\xi)^{-1}\zeta^{-\frac{1}{2}} \int_0^\infty W_n^+(x/\alpha)(1+\rho'(x;\alpha))^{-\frac{1}{2}}\mathrm{Re}\Big(a(\xi)e^{\frac{i}{\hbar}[x-y(\alpha)+\rho(x;\alpha)]}(1+\hbar O(x^{-1}))\Big)$$
$$(1+\rho'(x\zeta;\beta))^{-\frac{1}{2}}\mathrm{Re}\Big(a(\eta)e^{\frac{i}{\hbar}[x\zeta-y(\beta)+\rho(x\zeta;\beta)]}(1+\hbar O(x^{-1}))\Big)\chi(x)\,dx \tag{5.25}$$

where $\zeta = (\eta/\xi)^{\frac{1}{2}} \simeq 1$. The third equality sign follows by a change of variables and the behavior of a_1 under differentiation in τ, and thus in x. The notation $O(x^{-1})$ refers to symbol-type behavior under differentiation. Then, on the one hand, placing absolute values inside the integrals yields

$$|F_3(\xi,\eta;\hbar)| \lesssim \hbar^{-1}\xi^{-\frac{1}{2}} \int_{\xi^{\frac{1}{2}}\hbar R\gtrsim 1} \langle R\rangle^{-4}\,dR \lesssim (\hbar\xi^{\frac{1}{2}})^{-1}\min(1,(\hbar\xi^{\frac{1}{2}})^3) \tag{5.26}$$

On the other hand, integrating by parts in (5.25) yields

$$|F_3(\xi,\eta;\hbar)| \lesssim (\hbar\xi^{\frac{1}{2}})^{-1}|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|^{-1}\min(1,(\hbar\xi^{\frac{1}{2}})^4) \tag{5.27}$$

Indeed, the oscillatory terms in (5.25) are of the form $e^{\pm\frac{i}{\hbar}\Phi_+(x,\alpha;\hbar,\zeta)}$ and $e^{\pm\frac{i}{\hbar}\Phi_-(x,\alpha;\hbar,\zeta)}$ where

$$\Phi_\pm(x,\alpha;\hbar,\zeta) = x(\zeta\pm 1) \pm \rho(x;\alpha) + \rho(x\zeta;\alpha\zeta) \tag{5.28}$$

We will only discuss the destructive interference given by Φ_- as Φ_+ contributes less. One has

$$\begin{aligned}\partial_x\Phi_-(x,\alpha;\hbar) &= (\zeta-1)\{1+\int_0^1 \big[(\rho_x+y(s)\rho_{xx})(y(s),d(s))) + d(s)\rho_{x\alpha}(y(s),d(s))\big]\,ds\}\\ &= (\zeta-1)[1+O(x^{-2})]\end{aligned} \tag{5.29}$$

with $y(s) = x(1-s+s\zeta), d(s) = \alpha(1-s+s\zeta)$, where the final $O(x^{-2})$ is a consequence of Lemma 4.13. Using that

$$(i\hbar^{-1}\partial_x\Phi_-(x,\alpha;\hbar,\zeta))^{-1}\partial_x e^{\frac{i}{\hbar}\Phi_-(x,\alpha;\hbar,\zeta)} = e^{\frac{i}{\hbar}\Phi_-(x,\alpha;\hbar,\zeta)}$$

one integration by parts in (5.25) yields

$$\begin{aligned}|F_3(\xi,\eta;\hbar)| &\lesssim (\hbar\xi)^{-1}|1-\zeta|^{-1}\int_1^\infty \Big(\langle x/\alpha\rangle^{-6}x\alpha^{-2} + \langle x/\alpha\rangle^{-4}x^{-2}\Big)\,dx\\ &\lesssim (\hbar\xi)^{-1}|1-\zeta|^{-1}\min(1,(\hbar\xi^{\frac{1}{2}})^4),\end{aligned}$$

which is the same as (5.27). After $\ell \geq 1$ integrations by parts we obtain

$$\begin{aligned}|F_3(\xi,\eta;\hbar)| &\lesssim (\hbar\xi^{\frac{1}{2}})^{-2}\hbar^\ell|1-\zeta|^{-\ell}\min(1,(\hbar\xi^{\frac{1}{2}})^4)\\ &\lesssim (\hbar\xi^{\frac{1}{2}})^{\ell-2}|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|^{-\ell}\min(1,(\hbar\xi^{\frac{1}{2}})^4)\end{aligned} \tag{5.30}$$

Combining (5.26) with (5.30) (with $\ell = 5$) one obtains with $\alpha = \hbar\xi^{\frac{1}{2}}$,

$$|F_3(\xi,\eta;\hbar)| \lesssim \alpha^{-1}\min(1,\alpha^3)\min\big(1,\alpha^5|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|^{-5}\big) \tag{5.31}$$

We now turn to **case $\mathbf{A_2}$**, i.e., the second integral for which $x \simeq 1$, i.e., $-C \le \tau \le C$ with some arbitrary but fixed constant C. Then, in view of (5.23) and (5.22), with a smooth cutoff $\chi(x)$ to $x \simeq 1$,

$$\begin{aligned} F_2(\xi,\eta;\hbar) &:= \int_0^\infty W_n^+(R)\phi(R,\xi;\hbar)\phi(R,\eta;\hbar)\chi(x)R\,dR \\ &= \int_0^\infty W_n^+(R)\chi(\alpha R)R\,\hbar^{\frac13}x^{-\frac12}q^{-\frac14}(\tau)\Big[\mathrm{Ai}(-\hbar^{-\frac23}\tau)(1+\hbar a_0(-\tau;\alpha,\hbar))\chi_{[\tau\lesssim\hbar^{\frac23}]}+ \\ &\quad + 2\mathrm{Re}\,\big(a(\xi)(\mathrm{Ai}(-\hbar^{-\frac23}\tau)-i\mathrm{Bi}(-\hbar^{-\frac23}\tau))(1+\hbar\overline{a_1(-\tau;\alpha)})\big)\chi_{[\tau\gtrsim\hbar^{\frac23}]}\Big] \\ &\quad \hbar^{\frac13}\tilde{x}^{-\frac12}\tilde{q}^{-\frac14}(\tilde\tau)\Big[\mathrm{Ai}(-\hbar^{-\frac23}\tilde\tau)(1+\hbar a_0(-\tilde\tau;\beta,\hbar))\chi_{[\tilde\tau\lesssim\hbar^{\frac23}]}+ \\ &\quad + 2\mathrm{Re}\,\big(a(\eta)(\mathrm{Ai}(-\hbar^{-\frac23}\tilde\tau)-i\mathrm{Bi}(-\hbar^{-\frac23}\tilde\tau))(1+\hbar\overline{a_1(-\tilde\tau;\beta)})\big)\chi_{[\tilde\tau\gtrsim\hbar^{\frac23}]}\Big]\,dR \end{aligned} \tag{5.32}$$

Here $\chi_{[\tau\lesssim\hbar^{\frac23}]}$ and $\chi_{[\tau\gtrsim\hbar^{\frac23}]}$ are smooth cutoffs to the indicated regions and a partition of unity, i.e.,

$$\chi_{[\tau\lesssim\hbar^{\frac23}]}+\chi_{[\tau\gtrsim\hbar^{\frac23}]}=1.$$

In (5.32), we are passing beyond the turning point in the non-oscillatory fundamental solution by an amount $\simeq \hbar^{\frac23}$, cf. Remark 4.33. As usual, $\tilde\tau$ and $\beta=\zeta\alpha$ play the role of τ and α, respectively, with η in place of ξ. To handle case A_2, we analyze $\tilde\tau$ as a function of τ. By Lemmas 4.10 and 4.11 we have

$$\tau=(x-x_t(\alpha))\Phi(x;\alpha), \qquad \tilde\tau=(\zeta x-x_t(\zeta\alpha))\Phi(\zeta x;\zeta\alpha)$$

where $|\partial_\alpha\Phi(x,\alpha)|+|\partial_\alpha x_t(\alpha)|\lesssim\hbar\langle\alpha\rangle^{-3}$. In particular, $\tau=0$ corresponds to (with $x_t=x_t(\alpha)$)

$$\begin{aligned} \tilde\tau_0 &:= (\zeta x_t(\alpha)-x_t(\zeta\alpha))\Phi(\zeta x_t;\zeta\alpha) \\ &= (\zeta-1)[x_t-\alpha\int_0^1 x_t'(s\zeta\alpha+(1-s)\alpha)\,ds]\Phi(\zeta x_t;\zeta\alpha)=(\zeta-1)[1+O(\hbar)], \end{aligned} \tag{5.33}$$

see (4.70), and

$$\tilde\tau=\zeta\tau\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(x;\alpha)}+\tilde\tau_0\cdot\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(\zeta x_t;\zeta\alpha)} \tag{5.34}$$

Since $\zeta\ge1$, we have $\tilde\tau_0\ge0$. By Lemma 4.11, we have

$$\Phi(\zeta x;\zeta\alpha)\simeq1, \quad \Phi(x;\alpha)\simeq1$$

and thus

$$\tilde\tau\simeq\tau+\zeta-1 \tag{5.35}$$

uniformly in the parameters. It follows that, see (4.73),

$$\begin{aligned} \frac{\partial\tilde\tau}{\partial\tau} &= \zeta\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(x;\alpha)}+\zeta\tau\left(\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(x;\alpha)}\right)_x q^{-\frac12}+\tilde\tau_0\left(\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(\zeta x_t;\zeta\alpha)}\right)_x q^{-\frac12} \\ \frac23\partial_\tau\left(\tilde\tau^{\frac32}-\tau^{\frac32}\right) &= \tau^{\frac12}\left(-1+\frac{\partial\tilde\tau}{\partial\tau}\right)+\frac{\tilde\tau-\tau}{\tau^{\frac12}+\tilde\tau^{\frac12}}\frac{\partial\tilde\tau}{\partial\tau} \end{aligned} \tag{5.36}$$

We claim that the functions $\mathcal{R}, \mathcal{S}$ defined by

$$\begin{aligned}\zeta\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(x;\alpha)} - 1 &=: (\zeta-1)\mathcal{R}(\zeta, x;\alpha,\hbar)\\ \left(\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(x;\alpha)}\right)_x &=: (\zeta-1)\mathcal{S}(\zeta, x;\alpha,\hbar)\\ \left(\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(\zeta x_t;\zeta\alpha)}\right)_x &=: \mathcal{T}(\zeta, x;\alpha,\hbar)\end{aligned} \tag{5.37}$$

satisfy uniformly in this regime of parameters

$$|\mathcal{R}(\zeta, x;\alpha,\hbar)| + |\mathcal{S}(\zeta, x;\alpha,\hbar)| + |\mathcal{T}(\zeta, x;\alpha,\hbar)| \lesssim 1 \tag{5.38}$$

To estimate $\mathcal{R}$, we write

$$(\zeta-1)\Phi(\zeta x;\zeta\alpha) + \Phi(\zeta x;\zeta\alpha) - \Phi(x;\alpha) = (\zeta-1)\mathcal{R}(\zeta, x;\alpha,\hbar)\Phi(x;\alpha).$$

The difference on the left-hand side is

$$\Phi(\zeta x;\zeta\alpha) - \Phi(x;\alpha) = (\zeta-1)\int_0^1 (x(\partial_x\Phi) + \alpha(\partial_\alpha\Phi))\,(s\zeta x + (1-s)x; s\zeta\alpha + (1-s)\alpha)\,ds \tag{5.39}$$

By (4.77), $|\alpha\partial_\alpha\Phi| \lesssim \hbar$ and $|\partial_x\Phi| \lesssim 1$, whence $|\mathcal{R}(\zeta, x;\alpha,\hbar)| \lesssim 1$. For $\mathcal{S}(\zeta, x;\alpha,\hbar)$ we consider

$$\begin{aligned}\left(\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(x;\alpha)}\right)_x = \left(\frac{\Phi(\zeta x;\zeta\alpha)-\Phi(x;\alpha)}{\Phi(x;\alpha)}\right)_x &= -\frac{\Phi(\zeta x;\zeta\alpha)-\Phi(x;\alpha)}{\Phi(x;\alpha)^2}\cdot\partial_x\Phi(x;\alpha) + \frac{(\Phi(\zeta x;\zeta\alpha)-\Phi(x;\alpha))_x}{\Phi(x;\alpha)}\\ &=:(\zeta-1)(I+II).\end{aligned}$$

$|\partial_x\Phi(x;\alpha)| \lesssim 1$ and (5.39) imply $|I| \lesssim 1$. Furthermore,

$$\begin{aligned}&\partial_x(\Phi(\zeta x;\zeta\alpha) - \Phi(x;\alpha))\\ =&(\zeta-1)\int_0^1 \Big(\big((\zeta-1)x^2(\partial_x^2\Phi) + (\zeta-1)x\alpha(\partial^2_{x\alpha}\Phi) + \partial_x\Phi\big)(s\zeta x + (1-s)x; s\zeta\alpha + (1-s)\alpha)\Big)\,ds.\end{aligned} \tag{5.40}$$

Bounding the integrand by Lemma 4.11 implies $|II| \lesssim 1$, whence $|\mathcal{S}(\zeta, x;\alpha,\hbar)| \lesssim 1$. The estimate on $\mathcal{T}(\zeta, x;\alpha,\hbar)$ follows from

$$\mathcal{T}(\zeta, x;\alpha,\hbar) = \left(\frac{\Phi(\zeta x;\zeta\alpha)}{\Phi(\zeta x_t;\zeta\alpha)}\right)_x = \frac{\zeta\partial_x\Phi(\zeta x;\zeta\alpha)}{\Phi(\zeta x_t;\zeta\alpha)}.$$

and (5.38) holds. Returning to (5.36), we conclude that

$$\begin{aligned}\frac{\partial\tilde\tau}{\partial\tau} - 1 &= (\zeta-1)\big[\mathcal{R}(\zeta, x;\alpha,\hbar) + \zeta\tau\mathcal{S}(\zeta, x;\alpha,\hbar)q^{-\frac12} + (1+O(\hbar))\mathcal{T}(\zeta, x;\alpha,\hbar)q^{-\frac12}\big]\\ \tilde\tau - \tau &= \int_0^\tau \left[\frac{\partial\tilde\tau}{\partial\tau} - 1\right] dv + \tilde\tau_0\\ &= (\zeta-1)[1 + O(\tau) + O(\hbar)],\end{aligned} \tag{5.41}$$

where we invoked (5.33) to pass to the third line. By (5.36) and (5.35)

$$\partial_\tau(\tilde\tau^{\frac32} - \tau^{\frac32}) \simeq (\zeta-1)(\tau+\tilde\tau_0)^{-\frac12} \simeq (\zeta-1)(\tau+\zeta-1)^{-\frac12} \tag{5.42}$$

uniformly in $0 \leq \tau \ll 1$, $\alpha > 0$ and small $\hbar$ (recall that we are in the range $1 \leq \zeta \lesssim 1$). We will also require the following bound on the second derivatives

$$\left|\partial_\tau^2\left(\tilde{\tau}^{\frac{3}{2}} - \tau^{\frac{3}{2}}\right)\right| \lesssim (\zeta-1)\tau^{-\frac{1}{2}}(\tau+\zeta-1)^{-1} \tag{5.43}$$

for the same regime of parameters. We write the left-hand side here in the form

$$\frac{4}{3}\partial_\tau^2\left(\tilde{\tau}^{\frac{3}{2}} - \tau^{\frac{3}{2}}\right) = 2\tilde{\tau}^{\frac{1}{2}}\frac{\partial^2\tilde{\tau}}{\partial\tau^2} + \tilde{\tau}^{-\frac{1}{2}} - \tau^{-\frac{1}{2}} + \tilde{\tau}^{-\frac{1}{2}}\left(\left(\frac{\partial\tilde{\tau}}{\partial\tau}\right)^2 - 1\right) \tag{5.44}$$

The second term on the right-hand side is

$$\begin{aligned}\tilde{\tau}^{-\frac{1}{2}} - \tau^{-\frac{1}{2}} &= \frac{\tau-\tilde{\tau}}{(\tilde{\tau}\tau)^{\frac{1}{2}}(\tilde{\tau}^{\frac{1}{2}}+\tau^{\frac{1}{2}})} \\ &= O\left((\zeta-1)\tau^{-\frac{1}{2}}(\tau+\zeta-1)^{-1}\right)\end{aligned} \tag{5.45}$$

and the third

$$\tilde{\tau}^{-\frac{1}{2}}\left(\left(\frac{\partial\tilde{\tau}}{\partial\tau}\right)^2 - 1\right) = O\left((\zeta-1)(\tau+\zeta-1)^{-\frac{1}{2}}\right) \tag{5.46}$$

Taking a derivative of (5.41) yields

$$\frac{\partial^2\tilde{\tau}}{\partial\tau^2} = (\zeta-1)\partial_\tau\left[\mathcal{R}(\zeta,x;\alpha,\hbar) + \zeta\tau\mathcal{S}(\zeta,x;\alpha,\hbar)q^{-\frac{1}{2}} + (1+O(\hbar))\mathcal{T}(\zeta,x;\alpha,\hbar)q^{-\frac{1}{2}}\right] = O(\zeta-1)$$

The final bound here follows from the expressions for $\mathcal{R}$ etc. which we obtained above, see the calculations leading from (5.38) to (5.41). In conclusion, (5.43) holds. We now break up (5.32) into two pieces, the first being over $\tau \lesssim \hbar^{\frac{2}{3}}$ (here ℓ stands for *left*):

$$\begin{aligned}F_{2\ell}(\xi,\eta;\hbar) = \int_0^\infty & W_n^+(R)\hbar^{\frac{1}{3}}x^{-\frac{1}{2}}q^{-\frac{1}{4}}(\tau)\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)(1+\hbar a_0(-\tau;\alpha,\hbar))\chi_{[\tau\lesssim\hbar^{\frac{2}{3}}]} \\ & \hbar^{\frac{1}{3}}\tilde{x}^{-\frac{1}{2}}\tilde{q}^{-\frac{1}{4}}(\tilde{\tau})\Big[\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau})(1+\hbar a_0(-\tilde{\tau};\beta,\hbar))\chi_{[\tilde{\tau}\lesssim\hbar^{\frac{2}{3}}]} + \\ & + 2\mathrm{Re}\,(a(\eta)(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau}) - i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tilde{\tau}))(1+\hbar\overline{a_1(-\tilde{\tau};\beta)}))\chi_{[\tilde{\tau}\gtrsim\hbar^{\frac{2}{3}}]}\Big]\chi(\alpha R)R\,dR\end{aligned} \tag{5.47}$$

This is estimated using only the size of the integrand:

$$|F_{2\ell}(\xi,\eta;\hbar)| \lesssim \hbar^{-\frac{1}{3}}\alpha^{-2}\langle\alpha^{-1}\rangle^{-4}\int_{-\infty}^\infty \chi_{[-C\leq\tau\lesssim\hbar^{\frac{2}{3}}]}\left\langle\hbar^{-\frac{2}{3}}\tau\right\rangle^{-\frac{1}{4}}\exp\left(-\frac{2}{3}\hbar^{-1}|\tau|^{\frac{3}{2}}\right)\left\langle\hbar^{-\frac{2}{3}}\tilde{\tau}\right\rangle^{-\frac{1}{4}}\,d\tau \tag{5.48}$$

We do not need to use the rapid decay of $\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau})$ for negative $\tilde{\tau}$ since it does not improve the upper bound. If $-C \leq \tau \leq -c$ with some small constant $c > 0$, then we obtain a gain of $O(\hbar^\infty)$ from the exponential. If $-c \leq \tau \lesssim \hbar^{\frac{2}{3}}$, then we use (5.41) to write $\tilde{\tau} = \tau + (\zeta-1)(1+o(1))$. The dominant contribution to (5.48) therefore comes from the interval $|\tau| \simeq \hbar^{\frac{2}{3}}$ whence

$$|F_{2\ell}(\xi,\eta;\hbar)| \lesssim \alpha^2\langle\alpha\rangle^{-4}\min(\hbar^{\frac{1}{3}},\hbar^{\frac{1}{4}}\Lambda^{\frac{1}{4}}), \tag{5.49}$$

where $\Lambda := \hbar(\zeta-1)^{-1} = \frac{\hbar\xi^{\frac12}}{\eta^{\frac12}-\xi^{\frac12}}$. By (5.41), $\tau \gtrsim \hbar^{\frac23}$ implies the same for $\tilde\tau$. Hence we can write $F_{2r} := F_2 - F_{2\ell}$ in the form

$$F_{2r}(\xi,\eta;\hbar) = 4\hbar^{\frac23}\int_0^\infty W_n^+(R)x^{-\frac12}q^{-\frac14}(\tau)\tilde x^{-\frac12}\tilde q^{-\frac14}(\tilde\tau)\mathrm{Re}\,\big(a(\xi)(\mathrm{Ai}(-\hbar^{-\frac23}\tau) - i\mathrm{Bi}(-\hbar^{-\frac23}\tau))(1+\hbar\overline{a_1(-\tau;\alpha)})\big)$$
$$\mathrm{Re}\,\big(a(\eta)(\mathrm{Ai}(-\hbar^{-\frac23}\tilde\tau) - i\mathrm{Bi}(-\hbar^{-\frac23}\tilde\tau))(1+\hbar\overline{a_1(-\tilde\tau;\beta)})\big)\chi_{[\tau\gtrsim\hbar^{\frac23}]}\chi(x)R\,dR$$

Integrating by parts using (5.42) and (5.43) requires that $\hbar^{\frac23} \lesssim \tau \ll 1$. Thus, we split F_{2r} further by means of a partition of unity:

$$F_{2r} = F_{2r1} + F_{2r2}.$$

Here F_{2r1} covers the integration over $\hbar^{\frac23} \lesssim \tau \ll 1$, and F_{2r2} deals with $\tau \simeq 1$. Changing variables via $dR = \alpha^{-1}q^{-\frac12}\,d\tau$ yields

$$F_{2r1}(\xi,\eta;\hbar) := 4\hbar^{\frac23}\alpha^{-2}\int_{-\infty}^\infty W_n^+(R)\chi(x)\mathrm{Re}\,\big(a(\xi)(\mathrm{Ai}(-\hbar^{-\frac23}\tau) - i\mathrm{Bi}(-\hbar^{-\frac23}\tau))(1+\hbar\overline{a_1(-\tau;\alpha)})\big)$$
$$x^{\frac12}q^{-\frac34}(\tau)\tilde x^{-\frac12}\tilde q^{-\frac14}(\tilde\tau)\mathrm{Re}\,\big(a(\eta)(\mathrm{Ai}(-\hbar^{-\frac23}\tilde\tau) - i\mathrm{Bi}(-\hbar^{-\frac23}\tilde\tau))(1+\hbar\overline{a_1(-\tilde\tau;\beta)})\big)\chi_{[1\gg\tau\gtrsim\hbar^{\frac23}]}\,d\tau$$

The dominant contributions are made by the resonant terms due to phases exhibiting opposite signs. Without loss of generality it suffices to bound

$$\begin{aligned} F^{+-}_{2r1}(\xi,\eta;\hbar) &:= \hbar\alpha^{-2}\int_{-\infty}^\infty W_n^+(R)a(\xi)\overline{a(\eta)}e^{\frac{2i}{3\hbar}(\tilde\tau^{\frac32}-\tau^{\frac32})}(1+\hbar\overline{\tilde a_1(-\tau;\alpha)})) \\ &\qquad x^{\frac12}q^{-\frac34}(\tau)\tilde x^{-\frac12}\tilde q^{-\frac14}(\tilde\tau)\tau^{-\frac14}\tilde\tau^{-\frac14}(1+\hbar\tilde a_1(-\tilde\tau;\beta)))\chi_{[1\gg\tau\gtrsim\hbar^{\frac23}]}\,d\tau \\ &= \frac32 i\hbar^2\alpha^{-2}a(\xi)\overline{a(\eta)}\int_{-\infty}^\infty e^{\frac{2i}{3\hbar}(\tilde\tau^{\frac32}-\tau^{\frac32})}\partial_\tau\Big\{[\partial_\tau(\tilde\tau^{\frac32}-\tau^{\frac32})]^{-1}\Big[W_n^+(R)(1+\hbar\overline{\tilde a_1(-\tau;\alpha)}) \\ &\qquad x^{\frac12}q^{-\frac34}(\tau)\tilde x^{-\frac12}\tilde q^{-\frac14}(\tilde\tau)\tau^{-\frac14}\tilde\tau^{-\frac14}(1+\hbar\tilde a_1(-\tilde\tau;\beta))\chi_{[1\gg\tau\gtrsim\hbar^{\frac23}]}\Big]\Big\}\,d\tau \end{aligned} \tag{5.50}$$

The second expression, which was obtained by integration by parts, is only useful if ξ and η are not too close. We estimate the first term on the right-hand side of (5.50) by passing absolute values inside yielding

$$|F^{+-}_{2r1}(\xi,\eta;\hbar)| \lesssim \alpha^2\langle\alpha\rangle^{-4} \tag{5.51}$$

To obtain the refined bounds in terms of Λ provided $\Lambda \ll 1$, we further split the interval $\hbar^{\frac23} \ll \tau \ll 1$ according to whether $\tau^{\frac12} \lesssim \Lambda$ or $\tau^{\frac12} \gtrsim \Lambda$. In other words, using a smooth partition of unity as before, we split one more time

$$F^{+-}_{2r1}(\xi,\eta;\hbar) = F^{+-}_{2r11}(\xi,\eta;\hbar) + F^{+-}_{2r12}(\xi,\eta;\hbar)$$

The first term $F^{+-}_{2r11}(\xi,\eta;\hbar)$ corresponds to $\hbar^{\frac23} \ll \tau \lesssim \Lambda^2$, and we bound it by placing absolute values inside and using that $\tilde\tau^{-\frac14} \le \tau^{-\frac14}$, which yields

$$\Big|F^{+-}_{2r11}(\xi,\eta;\hbar)\Big| \lesssim \alpha^2\langle\alpha\rangle^{-4}\Lambda, \quad \text{for} \quad \Lambda \ll 1. \tag{5.52}$$

The second term F^{+-}_{2r12} lives on $\Lambda^2 \lesssim \tau \ll 1$, and we integrate by parts as in the second line of (5.50) but with $\chi_{[1\gg\tau\gg\hbar^{\frac{2}{3}}]}$ replaced by $\chi_{[1\gg\tau\gtrsim\Lambda^2]}$. By (5.42) and (5.43) we have

$$\frac{|\partial_\tau^2(\tilde{\tau}^{\frac{3}{2}} - \tau^{\frac{3}{2}})|}{|\partial_\tau(\tilde{\tau}^{\frac{3}{2}} - \tau^{\frac{3}{2}})|^2} \lesssim (\zeta-1)^{-1}\tau^{-\frac{1}{2}}$$

Placing absolute values inside the integral in the second expression in (5.50) produces the bound

$$\begin{aligned}\left|F^{+-}_{2r12}(\xi,\eta;\hbar)\right| \lesssim \Lambda\alpha^2\langle\alpha\rangle^{-4}\Big\{ &\int_{-\infty}^{\infty} \tau^{-\frac{3}{4}}(\tau+\zeta-1)^{-\frac{1}{4}}\chi_{[1\gg\tau\gg\Lambda^2]}\,d\tau \\ &+ \int_{-\infty}^{\infty}(\tau+\zeta-1)^{\frac{1}{4}}(\hbar\tau^{-\frac{3}{4}} + \tau^{-\frac{5}{4}})\chi_{[1\gg\tau\gg\Lambda^2]}\,d\tau\Big\}.\end{aligned}$$

On the one hand,

$$\int \tau^{-\frac{3}{4}}(\tau+\zeta-1)^{-\frac{1}{4}}\chi_{[1\gg\tau\gg\Lambda^2]}\,d\tau \lesssim \int \tau^{-1}\chi_{[1\gg\tau\gg\Lambda^2]}\,d\tau \lesssim |\log\Lambda|.$$

Note that without the condition $\tau \gg \Lambda^2$ we would have obtained $|\log(\zeta-1)|$ resulting in a loss of $\log\hbar$ which is inadmissible. On the other hand,

$$\int(\tau+\zeta-1)^{\frac{1}{4}}\tau^{-\frac{5}{4}}\chi_{[1\gg\tau\gg\Lambda^2]}\,d\tau \lesssim \int(\tau^{-1} + (\zeta-1)^{\frac{1}{4}}\tau^{-\frac{5}{4}})\chi_{[1\gg\tau\gg\Lambda^2]}\,d\tau \lesssim |\log\Lambda| + \hbar^{\frac{1}{4}}\Lambda^{-\frac{3}{4}},$$

and

$$\int_{-\infty}^{\infty}(\tau+\zeta-1)^{\frac{1}{4}}\hbar\tau^{-\frac{3}{4}}\chi_{[1\gg\tau\gg\Lambda^2]}\,d\tau \lesssim \hbar$$

In combination with (5.51) these estimates imply

$$\left|F^{+-}_{2r12}(\xi,\eta;\hbar)\right| \lesssim \alpha^2\langle\alpha\rangle^{-4}\Lambda\left[\hbar + |\log\Lambda| + \hbar^{\frac{1}{4}}\Lambda^{-\frac{3}{4}}\right] \lesssim \alpha^2\langle\alpha\rangle^{-4}\Lambda^{\frac{1}{4}}$$

and in summary,

$$|F_{2r1}(\xi,\eta;\hbar)| \lesssim \alpha^2\langle\alpha\rangle^{-4}\min(1,\Lambda^{\frac{1}{4}}). \tag{5.53}$$

Next we turn to F_{2r2} which covers the integration over $\tau \simeq 1$. We will proceed as in Case A_3 above. However, since x is no longer large, we need to add the following observation concerning the lower bound on the derivative of the phase in (5.29). By (4.83) and (4.84),

$$y\rho_y = -\frac{1}{y+\sqrt{y^2-1}} + O(\hbar), \quad (y\rho_y)_y = \frac{1}{(y+\sqrt{y^2-1})\sqrt{y^2-1}} + O(\hbar)$$

Note that the derivative on the right-hand side is large if y is close to 1, but it is positive. On the other hand, by Lemma 4.13, $\alpha\rho_{\alpha y} = O(\hbar)$. This shows that (5.29) continues to hold in the form $\partial_x\Phi_- \simeq \zeta - 1$. The same analysis we used above to bound (5.25) thus still goes through whence

$$|F_{2r2}(\xi,\eta;\hbar)| \lesssim \alpha^{-1}\min(1,\alpha^3)\min(1,\Lambda^5) \tag{5.54}$$

This completes the discussion for **case $\mathbf{A_2}$**.

It remains to deal with **case A_1**, namely, $0 < x \ll 1$. Lemmas 4.17, 4.20, 4.22 and 4.24 imply that, for sufficiently small $\hbar$,

$$(-\tau)^{\frac{3}{2}} \simeq -\log x$$

uniformly in the parameters. By the definition of q and (4.90), for $\tau \ll -1$ (i.e. $0 < x \leq \frac{1}{2}x_t$), we have

$$q = -\frac{Q_0}{\tau} \simeq \frac{1}{x^2\tau}.$$

This together with the asymptotic profile for the Airy function gives

$$\left|\hbar^{\frac{1}{3}} x^{-\frac{1}{2}} q^{-\frac{1}{4}}(\tau)\mathrm{Ai}\left(-\hbar^{-\frac{2}{3}}\tau\right)(1+\hbar a_0(-\tau,\alpha;\hbar))\right| \lesssim \hbar^{\frac{1}{2}} e^{-\frac{2}{3}\hbar^{-1}(-\tau)^{\frac{3}{2}}} \lesssim \hbar^{\frac{1}{2}} \cdot x^{c\hbar^{-1}} \tag{5.55}$$

Since $\tilde{x} = x\zeta \simeq x$, we have $\tilde{\tau} \ll -1$ and

$$\left|\hbar^{\frac{1}{3}} \tilde{x}^{-\frac{1}{2}} q^{-\frac{1}{4}}(\tilde{\tau})\mathrm{Ai}\left(-\hbar^{-\frac{2}{3}}\tilde{\tau}\right)(1+\hbar a_0(-\tilde{\tau},\beta;\hbar))\right| \lesssim \hbar^{\frac{1}{2}} \cdot x^{c\hbar^{-1}}$$

Therefore, with $p = c\hbar^{-1}$ and a fixed small constant $c > 0$,

$$\begin{aligned}
&\Big|\int_0^\infty \chi_{[0<x\ll 1]} \cdot W_n^+(R)\hbar^{\frac{1}{3}} x^{-\frac{1}{2}} q^{-\frac{1}{4}}(\tau)\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)(1+\hbar a_0(-\tau,\alpha;\hbar)) \\
&\cdot \hbar^{\frac{1}{3}} \tilde{x}^{-\frac{1}{2}} q^{-\frac{1}{4}}(\tilde{\tau})\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau})(1+\hbar a_0(-\tilde{\tau},\beta;\hbar))R\,dR\Big| \\
&\lesssim \alpha^{-2}\int_0^{\frac{1}{2}} \langle x/\alpha\rangle^{-4} x^p\,dx = \alpha^{-1+p}\int_0^{(2\alpha)^{-1}} \langle y\rangle^{-4} y^p\,dy \\
&\lesssim \hbar\alpha^2\langle\alpha\rangle^{-4} \lesssim \alpha^2\langle\alpha\rangle^{-4}\min(1,\Lambda),
\end{aligned} \tag{5.56}$$

since $\Lambda \gtrsim \hbar$. Adding this estimate to the sum of the prior cases, i.e., (5.31), (5.49), (5.53), (5.54), establishes the bound stated in (5.3). The bound in the last line of (5.56) is good enough for our purposes, but crude. In fact, we gain an exponential factor of the form $2^{-p} = e^{-c\hbar^{-1}}$.

Now we turn to **Case B** where $\xi \ll \eta$. As before, we break the integral in (5.21) into three separate regions, namely $x \gg 1$ (case B_3), followed by $x \simeq 1$ (case B_2), and finally $0 < x \ll 1$ (case B_1). Beginning with $\mathbf{B_3}$, we need to estimate the expression $F_3(\xi,\eta;\hbar)$ defined by (5.25). The bound (5.26) is replaced by

$$|F_3(\xi,\eta;\hbar)| \lesssim \zeta^{-\frac{1}{2}}\alpha^{-1}\min(1,\alpha^3)$$

In Case B, we have $\zeta - 1 \simeq \zeta = (\eta/\xi)^{\frac{1}{2}}$ which implies that the phases in (5.28) satisfy

$$|\partial_x\Phi_\pm(x,\alpha;\hbar,\zeta)| \simeq \zeta$$

uniformly in the parameters. Integrating by parts as in the calculation leading up to (5.30) yields

$$|F_3(\xi,\eta;\hbar)| \leq C_\ell\,\zeta^{-\frac{1}{2}}(\hbar/\zeta)^\ell\alpha^2\langle\alpha\rangle^{-4} \tag{5.57}$$

for any $\ell \geq 1$. In **Case B$_2$**, we have $x \simeq 1$, i.e., $|\tau| \lesssim 1$ as well as $\tilde{x} \gg 1$ and $\tilde{\tau} \gg 1$. Thus, (5.32) now takes the form

$$\begin{aligned}
F_2(\xi, \eta; \hbar) &:= \int_0^\infty W_n^+(R)\phi(R, \xi; \hbar)\phi(R, \eta; \hbar)\chi(x)R\, dR \\
&= 2\alpha^{-2} \int_0^\infty W_n^+(R)\chi(\alpha R)\hbar^{\frac{1}{3}} x^{\frac{1}{2}} q^{-\frac{1}{4}}(\tau)\Big[\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)(1 + \hbar a_0(-\tau; \alpha, \hbar))\chi_{[-1\lesssim\tau\lesssim\hbar^{\frac{2}{3}}]} + \\
&\quad + 2\mathrm{Re}\,(a(\xi)(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau) - i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tau))(1 + \hbar\overline{a_1(-\tau; \alpha)}))\chi_{[1\gtrsim\tau\gtrsim\hbar^{\frac{2}{3}}]}\Big] \\
&\qquad \hbar^{\frac{1}{3}} \tilde{x}^{-\frac{1}{2}} \tilde{q}^{-\frac{1}{4}}(\tilde{\tau})\mathrm{Re}\,(a(\eta)(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau}) - i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tilde{\tau}))(1 + \hbar\overline{a_1(-\tilde{\tau}; \beta)}))\, dx \\
&:= F_{2\ell}(\xi, \eta; \hbar) + F_{2r}(\xi, \eta; \hbar)
\end{aligned} \tag{5.58}$$

where the summands in the last line correspond to the respective summands inside the brackets. Below we will simplify (5.58) using that $x^{\frac{1}{2}} \tilde{x}^{-\frac{1}{2}} = \zeta^{-\frac{1}{2}}$, see (5.63), (5.64). The factors $a(\xi), a(\eta)$ are uniformly bounded and do not affect the bounds. By Lemma 4.13 we have

$$\Psi = \Psi(x; \alpha, \zeta, \hbar) := \frac{2}{3}\tilde{\tau}^{\frac{3}{2}} = \tilde{x} - y(\beta; \hbar) + \rho(\tilde{x}, \beta; \hbar) = \zeta x - y(\beta; \hbar) + \rho(\zeta x, \beta; \hbar) \simeq \zeta x \tag{5.59}$$

and the oscillatory Airy functions are as follows, see (4.124):

$$\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau}) \mp i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tilde{\tau}) = c_\pm(\hbar^{-\frac{2}{3}}\tilde{\tau})^{-\frac{1}{4}} \cdot e^{\pm\frac{i}{\hbar}\Psi} \cdot (1 + b(\hbar^{-\frac{2}{3}}\tilde{\tau})).$$

By (4.122), $b(\hbar^{-\frac{2}{3}}\tilde{\tau}) = \hbar\tilde{b}(\tilde{x}, \beta; \hbar)$ with $\partial_{\tilde{x}}^k\tilde{b}(\tilde{x}, \beta; \hbar) = O(\tilde{x}^{-k-1})$. We can thus absorb the factor $(1 + b(\hbar^{-\frac{2}{3}}\tilde{\tau}))$ into the factor $(1 + \hbar\overline{a_1(\tilde{\tau}; \beta)})$ above. We shall carry out this step without further mention, also for the non-oscillatory Airy function. Integration by parts in (5.58) is carried out by means of the identity

$$\begin{aligned}
\mathcal{L}\, e^{\pm\frac{i}{\hbar}\Psi} &= \pm e^{\pm\frac{i}{\hbar}\Psi} \\
\mathcal{L} &:= \hbar\big[i\,\partial_x\Psi(x; \alpha, \zeta, \hbar)\big]^{-1}\partial_x
\end{aligned} \tag{5.60}$$

Since $\tilde{x} \gg 1$, the derivative of the phase satisfies

$$\begin{aligned}
\partial_x\Psi &= \zeta(1 + \rho_{\tilde{x}}(\tilde{x}, \beta; \hbar)) = \zeta(1 + O(\tilde{x}^{-2})) \simeq \zeta \\
\partial_x^k\Psi &= \zeta^k O(\tilde{x}^{-1-k}) \simeq_k \zeta^{-1}, \quad k \geq 2
\end{aligned} \tag{5.61}$$

where the final $\simeq$ holds due to $\tilde{x} \simeq \zeta$. From (4.73),

$$\hbar^{\frac{1}{3}} \tilde{x}^{-\frac{1}{2}} q^{-\frac{1}{4}}(\tilde{\tau})(\hbar^{-\frac{2}{3}}\tilde{\tau})^{-\frac{1}{4}} = \hbar^{\frac{1}{2}} \tilde{x}^{-\frac{1}{2}}(-Q_0(\tilde{x}, \beta; \hbar))^{-\frac{1}{4}},$$

with Corollary 4.9 providing the bounds

$$-Q_0(\tilde{x}, \alpha; \hbar) \simeq 1, \quad \left|\partial_{\tilde{x}}^k Q_0(\tilde{x}, \alpha; \hbar)\right| \lesssim \tilde{x}^{-k-2} \simeq \zeta^{-k-2}, \ k \geq 1. \tag{5.62}$$

We break up $F_{2\ell} = F_{2\ell 0} + F_{2\ell 1}$ further, by means of a smooth partition of unity adapted to $|x - x_t(\alpha, \hbar)| \lesssim \hbar^{\frac{2}{3}}$, respectively $-1 \lesssim x - x_t \lesssim -\hbar^{\frac{2}{3}}$. Note that by Lemma 4.11 we may interchange τ with $x - x_t(\alpha)$ in these conditions. Thus, up to a constant multiplicative factor which we ignore,

$$F_{2\ell 0}(\xi,\eta;\hbar) = \hbar^{\frac{5}{6}}\alpha^{-2}a(\eta)\zeta^{-\frac12}\int_0^\infty e^{\frac{i}{\hbar}\Psi}\,(\mathcal{L}^*)^m\Big[W_n^+(x/\alpha)q^{-\frac14}(\tau)\mathrm{Ai}(-\hbar^{-\frac23}\tau)(1+\hbar a_0(-\tau;\alpha,\hbar)) \\ (-Q_0(\tilde{x},\beta;\hbar))^{-\frac14}(1+\hbar\overline{\tilde{a}_1(-\tilde{\tau};\beta)})\chi_{[|x-x_t(\alpha)|\lesssim\hbar^{\frac23}]}\Big]\,dx + \mathrm{cc} \tag{5.63}$$

where "cc" stands for *complex conjugate*. Similarly,

$$F_{2\ell 1}(\xi,\eta;\hbar) = \hbar\alpha^{-2}a(\eta)\zeta^{-\frac12}\int_0^\infty e^{\frac{i}{\hbar}\Psi}\,(\mathcal{L}^*)^m\Big[W_n^+(x/\alpha)q^{-\frac14}(\tau)(-\tau)^{-\frac14}e^{-\frac{2}{3\hbar}(-\tau)^{\frac32}}(1+\hbar\tilde{a}_0(-\tau;\alpha,\hbar)) \\ (-Q_0(\tilde{x},\beta;\hbar))^{-\frac14}(1+\hbar\overline{\tilde{a}_1(-\tilde{\tau};\beta)})\chi_{[-1\lesssim x-x_t(\alpha)\lesssim-\hbar^{\frac23}]}\Big]\,dx + \mathrm{cc}. \tag{5.64}$$

We now analyze the contribution of each factor as it arises by integrating by parts in $F_{2\ell 0}$, see (5.58). First, it follows by induction that

$$(\mathcal{L}^*)^m := -\hbar^m\Big(\partial_x[i\,\partial_x\Psi(x;\alpha,\zeta,\hbar)]^{-1}\Big)^m \\ = \hbar^m\sum \mathrm{coeff.}\frac{\partial_x^{j_1+1}\Psi\partial_x^{j_2+1}\Psi\cdot\ldots\cdot\partial_x^{j_\ell+1}\Psi}{(\partial_x\Psi)^{m+\ell}}\partial_x^k \tag{5.65}$$

where the sum runs over those terms which obey $j_1+j_2+\ldots+j_\ell+k=m$, $0\le\ell\le m$, $\min(j_1,j_2,\ldots,j_\ell)\ge 1$. The coefficients are some absolute constants. In view of (5.61), the factors in the sum are of size

$$\frac{\partial_x^{j_1+1}\Psi\partial_x^{j_2+1}\Psi\cdot\ldots\cdot\partial_x^{j_\ell+1}\Psi}{(\partial_x\Psi)^{m+\ell}} \simeq \frac{\zeta^{-(j_1+\ldots+j_\ell)-\ell}}{\zeta^{m+\ell}} \simeq \zeta^{k-2(m+\ell)} \lesssim \zeta^{-m-3\ell} \lesssim \zeta^{-m} \tag{5.66}$$

since $\ell+k\le m$. Thus, we gain $(\hbar/\zeta)^m$ in (5.63) and (5.63), and it remains to estimate the effect of ∂_x^k, $0\le k\le m$, in these integrals. We now analyze ∂_x^k derivatives of each of the factors in (5.63) for all $k\ge 1$, and uniformly in the parameters such as $x\simeq 1$:

- $\alpha^{-2}|\partial_x^k(W_n^+(x/\alpha))|\le C_k\hbar^{-1}\alpha^2\langle\alpha\rangle^{-4}$, as one checks by differentiation of (5.20).
- By Lemma 4.13, $|\partial_x^k q^{-\frac14}(\tau)|\le C_k$.
- $|\partial_x^k(\mathrm{Ai}(-\hbar^{-\frac23}\tau)\chi_{[|x-x_t(\alpha)|\lesssim\hbar^{\frac23}]})|\le C_k\hbar^{-\frac{2k}{3}}$, again using Lemma 4.13.
- By Lemmas 4.30, 4.31, $|\partial_x^k(1+\hbar a_0(-\tau;\alpha,\hbar))|\le C_k\,\hbar^{\frac{4-2k}{3}}$ for $|\tau|\lesssim\hbar^{\frac23}$.
- By (4.122), $|\partial_x^k(1+\hbar\tilde{a}_0(-\tau;\alpha,\hbar))|\le C_k\,\hbar|\tau|^{-\frac32-k}$ for $\hbar^{\frac23}\ll|\tau|\lesssim 1$.
- By (5.62) and $\frac{\partial\tilde{x}}{\partial x}=\zeta$, one has $|\partial_x^k(-Q_0(\tilde{x},\beta;\hbar))^{-\frac14}|\le C_k\,\zeta^{-2}$.
- By Lemma 4.29, see (4.127), and $\frac{\partial\tilde{\tau}}{\partial\tilde{x}}=(-Q_0(\tilde{x})/\tilde{\tau})^{\frac12}$, one checks that $|\partial_x^k(1+\hbar\tilde{a}_1(-\tilde{\tau};\beta,\hbar))|\le C_k\,\hbar\zeta^{-1}$.

We remark that $\zeta=-\tau$ in Lemmas 4.29, 4.30, 4.31, and elsewhere in that section has nothing to do with $\zeta=(\eta/\xi)^{\frac12}$ as it appears here. In summary, these bounds imply that for any $m\ge 0$,

$$|F_{2\ell 0}(\xi,\eta;\hbar)|\le C_m\,\alpha^2\langle\alpha\rangle^{-4}\,(\hbar/\zeta)^{\frac12}(\hbar^{\frac13}/\zeta)^m \tag{5.67}$$

The factor $\hbar^{\frac{1}{2}}$ arises as $\hbar^{\frac{5}{6}}\hbar^{-1}\hbar^{\frac{2}{3}}$, the latter being the length of the integration interval. This same estimate also holds for $F_{2\ell 1}(\xi,\eta;\hbar)$ as can be seen by a dyadic decomposition $-\tau\simeq 2^j\hbar^{\frac{2}{3}}$. Indeed, one then has

$$\hbar(-\tau)^{-\frac{1}{4}}\exp\left(-\frac{2}{3\hbar}(-\tau)^{\frac{3}{2}}\right)\lesssim \hbar^{\frac{5}{6}}2^{-\frac{j}{4}}e^{-\frac{2}{3}2^{3j/2}}$$

which is rapidly decaying in j. We leave the remaining details to the reader. Thus, $F_{2\ell}$ also satisfies the bound (5.67).

The analysis of $F_{2r}(\xi,\eta;\hbar)$ is similar. Indeed, up to uniformly bounded factors, this term is the sum of the following four integrals:

$$\begin{aligned}F^{\pm}_{\pm} := \alpha^{-2}\zeta^{-\frac{1}{2}}\hbar\int_0^\infty W_n^+(x/\alpha)(\tau q(\tau))^{-\frac{1}{4}}(-Q_0(\tilde{x};\beta,\hbar))^{-\frac{1}{4}}\exp\left(\frac{2i}{3\hbar}(\pm\tau^{\frac{3}{2}}\pm\tilde{\tau}^{\frac{3}{2}})\right)\chi_{[\hbar^{\frac{2}{3}}\lesssim x-x_t(\alpha)\lesssim 1]}\\ \cdot\left(1+\hbar\tilde{a}_1^{\pm}(-\tau;\alpha,\hbar)\right)\left(1+\hbar\tilde{a}_1^{\pm}(-\tilde{\tau};\beta,\hbar)\right)\,dx\end{aligned}$$

We write the complex exponential as $e^{\frac{i}{\hbar}\Phi^{\pm}_{\pm}}$ with phases

$$\Phi^{\pm}_{\pm}(x;\alpha,\zeta,\hbar)=\frac{2}{3}\left(\pm\tau^{\frac{3}{2}}\pm\tilde{\tau}^{\frac{3}{2}}\right)$$

and $a_1^+ := a_1,\quad a_1^- := \overline{a_1},\quad \tilde{a}_1^+ := \tilde{a}_1,\quad \tilde{a}_1^- := \overline{\tilde{a}_1}$.

It suffices to consider one choice of signs here due to the fact that $\tilde{\tau}\gg 1$ and $0<\tau\lesssim 1$, and we write $\Phi=\Phi^+_+$. Then

$$\begin{aligned}&\partial_x\Phi=\tau^{\frac{1}{2}}q^{\frac{1}{2}}+\partial_x\Psi\simeq\zeta\\&\partial_x^2\Phi=\frac{1}{2}\tau^{-\frac{1}{2}}q+\frac{1}{2}\tau^{\frac{1}{2}}\partial_\tau q+\partial_x^2\Psi\simeq\tau^{-\frac{1}{2}}\\&|\partial_x^k\Phi|\simeq_k\tau^{\frac{3}{2}-k},\quad k\geq 2\end{aligned}\tag{5.68}$$

see (5.61). We redefine $\mathcal{L}$ as in (5.60) but with Φ in place of Ψ. In analogy with (5.64) we now have

$$\begin{aligned}F^+_+ = \hbar\alpha^{-2}\zeta^{-\frac{1}{2}}\int_0^\infty e^{\frac{i}{\hbar}\Psi}(\mathcal{L}^*)^m[W_n^+(x/\alpha)(\tau q(\tau))^{-\frac{1}{4}}(-Q_0(\tilde{x};\beta,\hbar))^{-\frac{1}{4}}\chi_{[\hbar^{\frac{2}{3}}\lesssim x-x_t(\alpha)\lesssim 1]}\\ \cdot(1+\hbar\tilde{a}_1(-\tau;\alpha,\hbar))\,(1+\hbar\tilde{a}_1(-\tilde{\tau};\beta,\hbar))\,]\,dx\end{aligned}$$

where $(\mathcal{L}^*)^m$ is given by (5.65) but with Φ in place of Ψ. In view of the preceding bullet list, and with $m\geq 2$ and k,ℓ and the sum as in (5.65), we have

$$\begin{aligned}|F_{2r}(\xi,\eta;\hbar)| &\lesssim \hbar^{m+1}\alpha^{-2}\zeta^{-\frac{1}{2}}\int_0^\infty\sum\frac{\prod_{i=1}^{\ell}\tau^{\frac{1}{2}-j_i}}{\zeta^{m+\ell}}\Big|\partial_x^k[W_n^+(x/\alpha)(-\tau q(\tau)Q_0(\tilde{x};\beta))^{-\frac{1}{4}}\chi_{[\hbar^{\frac{2}{3}}\lesssim x-x_t(\alpha)\lesssim 1]}\\&\qquad(1+\hbar\tilde{a}_1(-\tau;\alpha,\hbar))\,(1+\hbar\tilde{a}_1(-\tilde{\tau};\beta,\hbar))\,]\Big|\,dx\\&\lesssim \hbar^m\alpha^2\langle\alpha\rangle^{-4}\sum\zeta^{-\frac{1}{2}-m-\ell}\int_{\hbar^{\frac{2}{3}}}^1\tau^{\frac{\ell}{2}+k-m}\left(\tau^{-\frac{1}{4}-k}+\hbar\tau^{-\frac{3}{2}-k}\right)d\tau\\&\lesssim \alpha^2\langle\alpha\rangle^{-4}\sum\hbar^m\zeta^{-\frac{1}{2}-m-\ell}(\hbar^{\frac{2}{3}})^{\frac{\ell}{2}-m+\frac{3}{4}}\lesssim\alpha^2\langle\alpha\rangle^{-4}(\hbar/\zeta)^{\frac{1}{2}}\sum_{\ell\geq 0}(\hbar^{\frac{1}{3}}/\zeta)^{m+\ell}\\&\lesssim \alpha^2\langle\alpha\rangle^{-4}(\hbar/\zeta)^{\frac{1}{2}}(\hbar^{\frac{1}{3}}/\zeta)^m\end{aligned}$$

To pass to the third line we used that $\frac{\ell}{2} - m - \frac{1}{4} < -1$ due to $\ell \le m$ and $m \ge 2$. The final estimate exactly matches (5.67) whence

$$|F_2(\xi,\eta;\hbar)| \le C_m\, \alpha^2 \langle\alpha\rangle^{-4}\, (\hbar/\zeta)^{\frac{1}{2}} (\hbar^{\frac{1}{3}}/\zeta)^m, \quad \forall\, m \ge 0 \tag{5.69}$$

It remains to analyze the contribution of Case B_1. Let $\chi(x)$ be a smooth cutoff which equals 1 on $0 < x \le \frac{1}{4}$ and is supported on $0 < x \le \frac{1}{2}$. We split the integral

$$\begin{aligned} F_1(\xi,\eta;\hbar) &:= \int_0^\infty W_n^+(R)\phi(R,\xi;\hbar)\phi(R,\eta;\hbar)\chi(x)R\, dR = \alpha^{-2}\int_0^\infty W_n^+(x/\alpha)\phi(R,\xi;\hbar)\phi(R,\eta;\hbar)\chi(x)x\, dx \\ &= \sum_{j=1}^3 \alpha^{-2}\int_0^\infty W_n^+(x/\alpha)\hbar^{\frac{1}{3}}x^{\frac{1}{2}}q(\tau;\hbar)^{-\frac{1}{4}}\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)(1+\hbar a_0(-\tau;\alpha,\hbar))\phi(R,\eta;\hbar)\chi_j(\tilde\tau;\hbar)\, dx \\ &= (F_{11}+F_{12}+F_{13})(\xi,\eta;\hbar) \end{aligned} \tag{5.70}$$

where $\sum_{j=1}^3 \chi_j(\tilde\tau;\hbar) = \chi(x)$ is a smooth partition of unity corresponding to $\tilde\tau \lesssim -1$, $|\tilde\tau| \lesssim 1$, $\tilde\tau \gtrsim 1$, respectively. Then, on the one hand, by (4.112) and with $N = (2\hbar)^{-1}$,

$$\begin{aligned} |F_{11}(\xi,\eta;\hbar)| &\lesssim \alpha^{-2}\zeta^{-\frac{1}{2}}\int_0^\infty \langle x/\alpha\rangle^{-4}\,(Q_0(x;\alpha,\hbar)Q_0(\tilde x;\beta,\hbar))^{-\frac{1}{4}}\, e^{-\frac{2}{3\hbar}((-\tau)^{\frac{3}{2}}+(-\tilde\tau)^{\frac{3}{2}})} \\ &\qquad |(1+\hbar\tilde a_0(-\tau;\alpha,\hbar))(1+\hbar\tilde a_0(-\tilde\tau;\beta,\hbar))|\chi_1(\tilde\tau;\hbar)\, dx \\ &\lesssim \alpha^{-2}\zeta^{-\frac{1}{2}}\int_0^{\frac{1}{2\zeta}} \langle x/\alpha\rangle^{-4}(x\tilde x)^{\frac{1}{2}}(x\tilde x)^N\, dx \lesssim \alpha^{-2}\zeta^N\int_0^{\frac{1}{2\zeta}}\langle x/\alpha\rangle^{-4}x^{2N+1}\, dx \\ &\lesssim \hbar\alpha^{-2}\zeta^{-N-2}\min(1,(\alpha\zeta)^4) \\ &\lesssim \hbar\alpha^2\langle\alpha\rangle^{-4}\zeta^{-N+2} \end{aligned} \tag{5.71}$$

and, on the other hand,

$$\begin{aligned} F_{13}(\xi,\eta;\hbar) &= 2\alpha^{-2}\int_0^\infty W_n^+(x/\alpha)\hbar^{\frac{1}{3}}x^{\frac{1}{2}}q(\tau;\hbar)^{-\frac{1}{4}}\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)(1+\hbar a_0(-\tau;\alpha,\hbar)) \\ &\qquad \hbar^{\frac{1}{3}}\tilde x^{-\frac{1}{2}}\tilde q(\tilde\tau;\hbar)^{-\frac{1}{4}}\mathrm{Re}\Big(a(\eta)(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde\tau) - i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tilde\tau))(1+\hbar\overline{a_1}(-\tilde\tau;\alpha,\hbar))\Big)\chi_3(\tilde\tau;\hbar)\, dx \\ &= 2\hbar\,\alpha^{-2}\zeta^{-\frac{1}{2}}\int_0^\infty W_n^+(x/\alpha)\,(Q_0(x;\alpha,\hbar))^{-\frac{1}{4}}\,(-Q_0(\tilde x;\beta,\hbar))^{-\frac{1}{4}}\, e^{-\frac{2}{3\hbar}(-\tau)^{\frac{3}{2}}}(1+\hbar\tilde a_0(-\tau;\alpha,\hbar)) \\ &\qquad \mathrm{Re}\Big(a(\eta)e^{\frac{i}{\hbar}\Psi}(1+\hbar\overline{\tilde a_1}(-\tilde\tau;\beta,\hbar))\Big)\chi_{[\zeta^{-1}\ll x\ll 1]}\, dx \end{aligned} \tag{5.72}$$

where we wrote the smooth cutoff function $\chi_3(\tilde\tau;\hbar) = \chi_{[\zeta^{-1}\ll x\ll 1]}$, and Ψ is defined in (5.59). Integrating by parts by means of $\mathcal{L}$ defined in (5.60) we obtain

$$\begin{aligned} F_{13}(\xi,\eta;\hbar) &= \hbar\alpha^{-2}a(\eta)\zeta^{-\frac{1}{2}}\int_0^\infty e^{\frac{i}{\hbar}\Psi}\,(\mathcal{L}^*)^m\Big[W_n^+(x/\alpha)\,(Q_0(x;\alpha,\hbar))^{-\frac{1}{4}}\,(-Q_0(\tilde x;\beta,\hbar))^{-\frac{1}{4}}\, e^{-\frac{2}{3\hbar}(-\tau)^{\frac{3}{2}}} \\ &\qquad (1+\hbar\tilde a_0(-\tau;\alpha,\hbar))(1+\hbar\overline{\tilde a_1}(-\tilde\tau;\beta,\hbar))\chi_{[\zeta^{-1}\ll x\ll 1]}\Big]\, dx + \mathrm{cc.}, \end{aligned} \tag{5.73}$$

cf. (5.64). In view of (5.65) and (5.66),

$$\begin{aligned}|F_{13}(\xi,\eta;\hbar)| \lesssim \hbar\alpha^{-2}\zeta^{-\frac{1}{2}} \sum_{\ell+k\le m} \zeta^{k-2(m+\ell)} \int_0^\infty \Big|\partial_x^k\Big\{W_n^+(x/\alpha)\,(Q_0(x;\alpha,\hbar))^{-\frac{1}{4}}\,(-Q_0(\tilde{x};\beta,\hbar))^{-\frac{1}{4}}\,e^{-\frac{2}{3\hbar}(-\tau)^{\frac{3}{2}}} \\ \big(\chi_{[x\lesssim\alpha]}+\chi_{[x\gtrsim\alpha]}\big)\,(1+\hbar\tilde{a}_0(-\tau;\alpha,\hbar))(1+\hbar\overline{\tilde{a}_1}(-\tilde{\tau};\beta,\hbar))\chi_{[\zeta^{-1}\ll x\ll 1]}\Big\}\Big|\,dx\end{aligned} \tag{5.74}$$

Here $\chi_{[x\lesssim\alpha]}$ and $\chi_{[x\gtrsim\alpha]}$ are smooth cutoffs.

We now analyze ∂_x^k derivatives of each of the factors in (5.74) for all $k \ge 1$, and uniformly in the parameters such as $x \ll 1$:

- By differentiation of (5.20), $\alpha^{-2}\left|\partial_x^k\left(W_n^+(x/\alpha)\right)\right| \le C_k\hbar^{-1}\alpha^{-2-k}$ if $x \lesssim \alpha$ and $\alpha^{-2}\left|\partial_x^k\left(W_n^+(x/\alpha)\right)\right| \le C_k\hbar^{-1}\alpha^2x^{-4-k}$ if $x \gg \alpha$.
- $\left|\partial_x^k|Q_0(x;\alpha,\hbar)|^{-\frac{1}{4}}\right| \le C_kx^{\frac{1}{2}-k}$ by the definition of $Q_0(x;\alpha,\hbar)$ and the fact that $0 < x \ll 1$
- $\left|\partial_x^k\left(-Q_0(\tilde{x};\beta,\hbar)\right)^{-\frac{1}{4}}\right| \le C_k\zeta^{-2}$ by (5.62)
- $\left|\partial_x^k\left(\chi_{x\le\alpha}e^{-\frac{2}{3\hbar}(-\tau)^{\frac{3}{2}}}\right)\right| \le C_kx^{N-k}$ by Lemma 4.17 and the second equation in (4.112)
- $\left|\partial_x^k\left(\chi_{x\ge\alpha}e^{-\frac{2}{3\hbar}(-\tau)^{\frac{3}{2}}}\right)\right| \le C_kx^{N-k}$ by the first equation in (4.112)
- $\left|\partial_x^k(1+\hbar\tilde{a}_1(-\tilde{\tau};\beta,\hbar))\right| \le C_k\hbar\zeta^{-1}$ by Lemma 4.29.
- By Lemma 4.30 $\left|\partial_x^k(1+\hbar\tilde{a}_0(-\tau;\alpha,\hbar))\right| \le C_k\hbar(-\tau)^{-3}x^{-k}$ for $\alpha \gtrsim 1$
- By Lemma 4.31 $\left|\partial_x^k(1+\hbar\tilde{a}_0(-\tau;\alpha,\hbar))\right| \le C_k\hbar x^{-k}$ for $0 < \alpha \ll 1$.

To obtain the last two estimates in the above list, besides Lemma 4.30 and Lemma 4.31, we also use the estimate $\left|\frac{\partial(-\tau)}{\partial x}\right| \lesssim (-\tau)^{-\frac{1}{2}}$.

This bullet list implies that for a constant $c \in (0,1)$,

$$|F_{13}(\xi,\eta;\hbar)| \lesssim \hbar\alpha^2\langle\alpha\rangle^{-4}\hbar^{-1}\zeta^{-\frac{1}{2}}\zeta^{-m}\int_{\zeta^{-1}}^1 x^{c\hbar^{-1}}\,dx \lesssim \hbar\alpha^2\langle\alpha\rangle^{-4}\zeta^{-\frac{1}{2}-m}. \tag{5.75}$$

Finally we consider F_{12}. Similar to (5.72), we have

$$\begin{aligned}F_{12}(\xi,\eta;\hbar) = 2\alpha^{-2}\int_0^\infty W_n^+(x/\alpha)\hbar^{\frac{1}{3}}x^{\frac{1}{2}}q(\tau;\hbar)^{-\frac{1}{4}}\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)(1+\hbar a_0(-\tau;\alpha,\hbar)) \\ \hbar^{\frac{1}{3}}\tilde{x}^{-\frac{1}{2}}\tilde{q}(\tilde{\tau};\hbar)^{-\frac{1}{4}}\mathrm{Re}\Big(a(\eta)(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau})-i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tilde{\tau}))(1+\hbar\overline{a_1}(-\tilde{\tau};\alpha,\hbar))\Big)\chi_2(\tilde{\tau};\hbar)\,dx\end{aligned} \tag{5.76}$$

We simply bound the oscillatory Airy function in absolute value by

$$\left|\tilde{x}^{-\frac{1}{2}}\tilde{q}(\tilde{\tau};\hbar)^{-\frac{1}{4}}\mathrm{Re}\Big(a(\eta)(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau})-i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tilde{\tau}))(1+\hbar\overline{a_1}(-\tilde{\tau};\alpha,\hbar))\Big)\right| \lesssim 1.$$

Then similar as for F_{11}, we have

$$\begin{aligned}|F_{12}(\xi,\eta;\hbar)| &\lesssim \alpha^{-2}\int_0^{\frac{1}{\zeta}}\langle x/\alpha\rangle^{-4}\,|Q_0(x;\alpha,\hbar)|^{-\frac{1}{4}}\,e^{-\frac{2}{3\hbar}(-\tau)^{\frac{3}{2}}}\,|1+\hbar\tilde{a}_0(-\tau;\alpha,\hbar)|\,\chi_2(\tilde{\tau};\hbar)\,dx \\ &\lesssim \alpha^{-2}\int_0^{\frac{1}{\zeta}}\langle x/\alpha\rangle^{-4}x^{N+\frac{1}{2}}\,dx \\ &\lesssim \hbar\alpha^2\langle\alpha\rangle^{-4}\zeta^{-N+1}.\end{aligned} \tag{5.77}$$

So the estimates (5.71),(5.75) and (5.77) finally conclude the proof of (5.3) and (5.4).

We now turn to bounding the derivatives in the variables $\xi^{\frac{1}{2}}$, resp., $\eta^{\frac{1}{2}}$, and we consider $k_1 = 1, k_2 = 0$ in (5.6). We need to verify that differentiating once in $\xi^{\frac{1}{2}}$ results in the same bound that we obtained above for the undifferentiated case, but with a loss of α^{-1} if $0 < \alpha < 1$. We will instead differentiate in α, resp., β using that $\partial_{\xi^{\frac{1}{2}}} = \hbar\partial_\alpha$, $\partial_{\eta^{\frac{1}{2}}} = \hbar\partial_\beta$. Beginning with Case A_3, rewrite (5.25) in the form

$$
\begin{aligned}
F_3(\xi,\eta;\hbar) &= 4\pi^{-1}\hbar\alpha^{-2}\zeta^{-\frac{1}{2}}\int_0^\infty W_n^+(x/\alpha)(1+\rho'(x;\alpha))^{-\frac{1}{2}}\mathrm{Re}\,\Big(a(\xi)e^{\frac{i}{\hbar}[x-y(\alpha)+\rho(x;\alpha)]}(1+\hbar O(x^{-1}))\Big)\\
&\qquad (1+\rho'(x\zeta;\beta))^{-\frac{1}{2}}\mathrm{Re}\,\Big(a(\eta)e^{\frac{i}{\hbar}[x\zeta-y(\beta)+\rho(x\zeta;\beta)]}(1+\hbar O(x^{-1}))\Big)\,\chi(x)\,dx \qquad (5.78)\\
&= 4\pi^{-1}\hbar\alpha^{-2}\zeta^{-\frac{1}{2}}a(\xi)\overline{a(\eta)}\int_0^\infty e^{\frac{i}{\hbar}x(1-\zeta)}W_n^+(x/\alpha)(1+\rho'(x;\alpha))^{-\frac{1}{2}}(1+\rho'(x\zeta;\beta))^{-\frac{1}{2}}\\
&\qquad e^{\frac{i}{\hbar}[y(\beta)-y(\alpha)+\rho(x;\alpha)-\rho(x\zeta;\beta)]}(1+\hbar O(x^{-1}))\chi(x)\,dx + \mathrm{cc} \qquad (5.79)\\
&\quad + 4\pi^{-1}\hbar\alpha^{-2}\zeta^{-\frac{1}{2}}a(\xi)a(\eta)\int_0^\infty e^{\frac{i}{\hbar}x(1+\zeta)}W_n^+(x/\alpha)(1+\rho'(x;\alpha))^{-\frac{1}{2}}(1+\rho'(x\zeta;\beta))^{-\frac{1}{2}}\\
&\qquad e^{\frac{i}{\hbar}[-y(\beta)-y(\alpha)+\rho(x;\alpha)+\rho(x\zeta;\beta)]}(1+\hbar O(x^{-1}))\chi(x)\,dx + \mathrm{cc} \qquad (5.80)
\end{aligned}
$$

where “cc” stands for *complex conjugate*. As noted above, the $O(x^{-1})$ has symbol behavior under differentiation in x, and each derivative in α brings out a factor of α^{-1}. The latter follows from (4.127). We only treat the term (5.79), since (5.80) satisfies better bounds. By (4.154) and (4.163), we have $\partial_\alpha a(\xi) = O(\alpha^{-1})$ uniformly in all parameters (while $a(\xi) = O(1)$). Writing $\zeta = \beta/\alpha$ implies $\zeta_\alpha = -\zeta/\alpha$ and $\zeta_\beta = \zeta/\beta$. Therefore, in view of Lemma 4.13, differentiating

$$
\alpha^{-2}\zeta^{-\frac{1}{2}}a(\xi)\overline{a(\eta)}(1+\rho'(x;\alpha))^{-\frac{1}{2}}(1+\rho'(x\zeta;\beta))^{-\frac{1}{2}}e^{\frac{i}{\hbar}[y(\beta)-y(\alpha)+\rho(x;\alpha)-\rho(x\zeta;\beta)]}(1+\hbar O(x^{-1}))
$$

in α produces a sum of terms of similar form, but multiplied with extra factors which are bounded by α^{-1} or even by $\hbar(1+\alpha)^{-1}$ (or $\hbar(1+\alpha)^{-3}$ as for $y'(\alpha)$). On the other hand,

$$
\partial_\alpha\left(W_n^+(x/\alpha)\right) = -\alpha^{-1}(W_n^+)'(x/\alpha)x/\alpha = \hbar^{-1}\alpha^{-1}O(\langle x/\alpha\rangle^{-4})
$$

which differs from $W_n^+(x/\alpha) = \hbar^{-1}O(\langle x/\alpha\rangle^{-4})$ only by the α^{-1} factor. Note that all these terms are better by a factor of $\hbar$ than what we need since we are actually taking a derivative $\hbar\partial_\alpha$. The most important contribution results from

$$
\hbar\partial_\alpha\, e^{\frac{i}{\hbar}x(1-\zeta)} = i\zeta\frac{x}{\alpha}e^{\frac{i}{\hbar}x(1-\zeta)}
$$

Since we are in the regime $\zeta \simeq 1$, the only effect here is to multiply the potential $W_n^+(x/\alpha)$ by one factor of $\frac{x}{\alpha}$. Consider first $\alpha > 1$. Due to the fact that in our estimation of F_3 above the dominant contribution in the integrals $\int_1^\infty \ldots dx$ came from $x \simeq \alpha$, we see that we arrive at the same estimate as above by analogous arguments. The only essential feature here is that $W_n^+(x/\alpha)x/\alpha$ remains integrable at $x = \infty$ since it decays like x^{-3} (this changes if we were to take three derivatives). On the other hand, if $0 < \alpha \lesssim 1$, then we lose a factor of α^{-1} since the dominant contribution to the integral is derived from the region $x \simeq 1$. In summary, the upshot from this discussion is that Case A_3 satisfies the claimed estimate.

Turning to Case A_2, we rewrite (5.32) in the form

$$
\begin{aligned}
F_2(\xi,\eta;\hbar) = \hbar^{\frac{2}{3}}\alpha^{-2}\zeta^{-\frac{1}{2}}\int_0^\infty & W_n^+(x/\alpha)q^{-\frac{1}{4}}(\tau)\tilde{q}^{-\frac{1}{4}}(\tilde{\tau})\Big[\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau)(1+\hbar a_0(-\tau;\alpha,\hbar))\chi_{[\tau\lesssim\hbar^{\frac{2}{3}}]}+ \\
&+ 2\mathrm{Re}\,(a(\xi)(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tau) - i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tau))(1+\hbar\overline{a_1(-\tau;\alpha)}))\chi_{[\tau\gtrsim\hbar^{\frac{2}{3}}]}\Big] \\
&\Big[\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau})(1+\hbar a_0(-\tilde{\tau};\beta,\hbar))\chi_{[\tilde{\tau}\lesssim\hbar^{\frac{2}{3}}]}+ \\
&+ 2\mathrm{Re}\,(a(\eta)(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\tilde{\tau}) - i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\tilde{\tau}))(1+\hbar\overline{a_1(-\tilde{\tau};\beta)}))\chi_{[\tilde{\tau}\gtrsim\hbar^{\frac{2}{3}}]}\Big]\chi(x)\,dx
\end{aligned}
\tag{5.81}
$$

where the integration runs over $x \simeq 1$. We differentiate this expression by $\partial_{\xi^{\frac{1}{2}}} = \hbar\partial_\alpha$ using Lemma 4.11 in the regime $x \simeq 1$. Viewed as functions of x and α, the derivative $\partial_\alpha\tau$ gains a factor of $\hbar(1+\alpha)^{-3}$, uniformly in this regime, while

$$\partial_\alpha\tilde{\tau} = \partial_\alpha\tau(x\zeta,\beta) = -(\partial_1\tau)(x\zeta,\beta)\zeta x/\alpha = O(\alpha^{-1})$$

By Lemma 4.11 and the chain rule, both $\partial_\alpha q(\tau(x,\alpha),\alpha)$ and $\partial_\alpha\tilde{q}(\tilde{\tau}) = \partial_\alpha q(\tau(x\zeta,\beta),\beta)$ are bounded by $O(\alpha^{-1})$. The potential term is treated the same as in Case A_3 above, as are $\alpha^{-2}\zeta^{-\frac{1}{2}}$ and $a(\xi), a(\eta)$. We now consider the case when the ∂_α derivative falls on a_j, $j = 0,1$. Note that both ∂_α and a_j come with a factor of $\hbar$, so in total we gain a factor of $\hbar^2$. First (suppressing $\hbar$ from the notation), by the preceding bound on τ_α and Lemmas 4.30 and 4.31 (with $\zeta = -\tau$ in those lemmas having nothing to do with the ζ used in this proof),

$$\partial_\alpha a_0(-\tau(x,\alpha);\alpha) = -\tau_\alpha\,\partial_1 a_0(-\tau,\alpha) + \partial_\alpha a_0(-\tau,\alpha) = O(\alpha^{-1}), \quad -1 \lesssim \tau \lesssim \hbar^{\frac{2}{3}}$$

Here we used that $\partial_1 a_0(-\tau,\alpha) = O(\hbar^{-\frac{1}{3}})$, see (4.136) and (4.143), which is harmless due to the gains of factors of $\hbar$. We remark that the main argument above for Case A_2 only needs the *size of the integrand* in the regime of τ in which a_0 is relevant, so these bounds suffice (without any further differentiation, in contrast to the oscillatory a_1 regime we integrate by parts once in τ as independent variable). Next,

$$\partial_\alpha a_0(-\tilde{\tau};\beta) = \partial_\alpha\, a_0(-\tau(x\zeta,\beta);\beta) = x\zeta\alpha^{-1}\partial_1\tau(x\zeta,\beta)\,\partial_1 a_0(-\tilde{\tau},\beta) = O(\alpha^{-1}\hbar^{-\frac{1}{3}}), \quad \frac{1}{2} \le x \le x_t + \hbar^{\frac{2}{3}}$$

By (4.127)

$$\partial_\alpha a_1(-\tau(x,\alpha),\alpha) = -\tau_\alpha\,\partial_1 a_1(-\tau,\alpha) + \partial_2 a_1(-\tau,\alpha) = O(\langle\alpha\rangle^{-1})$$

Differentiating this once in x (which is the same as differentiating in τ up to a factor of $q^{\frac{1}{2}} \simeq 1$), leads to a loss of a factor of $\hbar^{-\frac{1}{3}}$ which is harmless given the aforementioned $\hbar^2$ whence $\partial_x\partial_\alpha a_1(-\tau(x,\alpha),\alpha) = O(\hbar^{-\frac{1}{3}}\langle\alpha\rangle^{-1})$. A more delicate term is $\hbar a_1(-\tilde{\tau},\beta) = \hbar a_1(-\tau(\zeta x,\beta),\beta)$ since it is involved in one integration by parts relative to τ, see (5.50). On the one hand,

$$\hbar\partial_\alpha\hbar a_1(-\tilde{\tau},\beta) = \hbar^2(\partial_1 a_1)(-\tilde{\tau},\beta)(\partial_1\tau)(x\zeta,\beta)\zeta x/\alpha = O\left(\hbar^2\tilde{\tau}^{-\frac{1}{2}}\alpha^{-1}\right) = O\left(\hbar^2\hbar^{-\frac{1}{3}}\alpha^{-1}\right), \quad \tilde{\tau} \gtrsim \hbar^{\frac{2}{3}} \tag{5.82}$$

and on the other, by (4.127), another derivative in x (or τ) loses a further factor of $\hbar^{-\frac{2}{3}}$. In conclusion, those terms in which the ∂_α derivative *does not fall on the Airy functions* are treated in the exact same fashion as in Case A_2 above, with the net effect of a factor of $\hbar\alpha^{-1}$ for all $\alpha > 0$ which multiplies the original estimate.

From (4.124), for all $x \geq 1$,

$$\begin{aligned}(\mathrm{Ai}(-x) + i\mathrm{Bi}(-x))' &= c\, x^{\frac12}(1 + b_1(x))\,(\mathrm{Ai}(-x) + i\mathrm{Bi}(-x)) \\ b_1^{(k)}(x) &= O(x^{-\frac32-k}), \quad k \geq 0\end{aligned} \tag{5.83}$$

whence

$$\hbar\partial_\alpha\left(\mathrm{Ai}\left(-\hbar^{-\frac23}\tau\right) - i\mathrm{Bi}\left(-\hbar^{-\frac23}\tau\right)\right) = c\,\tau^{\frac12}\tau_\alpha\left(1 + \hbar O\left(\tau^{-\frac32}\right)\right)\left(\mathrm{Ai}\left(-\hbar^{-\frac23}\tau\right) - i\mathrm{Bi}\left(-\hbar^{-\frac23}\tau\right)\right) \tag{5.84}$$

for $\tau \geq \hbar^{\frac23}$, with $|\tau_\alpha| \lesssim \hbar(1+\alpha)^{-3}$. Similarly,

$$\begin{aligned}\mathrm{Ai}'(x) &= c\, x^{\frac12}(1 + b_2(x))\mathrm{Ai}(x) \\ b_2^{(k)}(x) &= O(x^{-\frac32-k}), \quad k \geq 0\end{aligned} \tag{5.85}$$

leading to the same type of gain of the $\hbar(1+\alpha)^{-3}$ factor. Using that $\hbar\partial_\alpha\tilde\tau = O(\hbar\alpha^{-1})$ from above, we conclude that $\hbar\partial_\alpha$ acting on the second bracket in (5.81) yields the oscillatory term

$$\hbar\partial_\alpha\left(\mathrm{Ai}\left(-\hbar^{-\frac23}\tilde\tau\right) - i\mathrm{Bi}\left(-\hbar^{-\frac23}\tilde\tau\right)\right) = c\,\tilde\tau^{\frac12}O\left(\alpha^{-1}\right)\left(1 + \hbar O\left(\tilde\tau^{-\frac32}\right)\right)\left(\mathrm{Ai}\left(-\hbar^{-\frac23}\tilde\tau\right) - i\mathrm{Bi}\left(-\hbar^{-\frac23}\tilde\tau\right)\right) \tag{5.86}$$

and analogously for the decaying Airy function. This shows that by repeating the exact same arguments as in the undifferentiated Case A_2, see in particular (5.50), we obtain the desired bound.

It remains to consider Case A_1. Using that $q = -\frac{Q_0}{\tau}$, we rewrite the relevant integral in this case, see (5.56), in the form

$$\begin{aligned}&\int_0^{x_0/\alpha} W_n^+(R)\hbar^{\frac13}x^{-\frac12}q^{-\frac14}(\tau)\mathrm{Ai}(-\hbar^{-\frac23}\tau)(1+\hbar a_0(-\tau,\alpha;\hbar))\hbar^{\frac13}\tilde x^{-\frac12}q^{-\frac14}(\tilde\tau)\mathrm{Ai}(-\hbar^{-\frac23}\tilde\tau)(1+\hbar a_0(-\tilde\tau,\beta;\hbar))R\,dR \\ &= \alpha^{-2}\hbar\zeta^{-\frac12}\int_0^{x_0} W_n^+(x/\alpha)(Q_0(x,\alpha)Q_0(\zeta x,\beta))^{-\frac14}e^{-\frac{2}{3\hbar}[(-\tau)^{\frac32}+(-\tilde\tau)^{\frac32}]}(1+\hbar a_0(-\tau,\alpha;\hbar))(1+\hbar a_0(-\tilde\tau,\beta;\hbar))\,dx\end{aligned}$$

with some $0 < x_0 \ll 1$. The terms arising if ∂_α hits $\alpha^{-2}\zeta^{-\frac12}W_n^+(x/\alpha)$ are the same as above. Next,

$$\partial_\alpha(Q_0(x,\alpha)Q_0(\zeta x,\beta))^{-\frac14} = -\frac14(Q_0(x,\alpha)Q_0(\zeta x,\beta))^{-\frac14}\left[\frac{\partial_\alpha Q_0(x,\alpha)}{Q_0(x,\alpha)} - \frac{\partial_1 Q_0(\zeta x,\beta)}{Q_0(\zeta x,\beta)}x\zeta/\alpha\right]$$

By (4.69) the term in brackets is $O(\alpha^{-1})$. Here we also used the fact that if $0 < x \ll 1, 0 < \zeta x \ll 1$, then $\frac{1}{Q_0(x,\alpha)} \simeq x^2$, $\frac{1}{Q_0(\zeta x,\beta)} \simeq \zeta^2x^2$. The dominant contribution is (recall $\tau = \tau(x,\alpha;\hbar)$ and $\tilde\tau = \tau(x\zeta,\beta;\hbar)$)

$$\hbar\partial_\alpha e^{-\frac{2}{3\hbar}[(-\tau)^{\frac32}+(-\tilde\tau)^{\frac32}]} = e^{-\frac{2}{3\hbar}[(-\tau)^{\frac32}+(-\tilde\tau)^{\frac32}]}\left((-\tau)^{\frac12}\tau_\alpha - (-\tilde\tau)^{\frac12}\partial_1\tau(x\zeta,\beta)x\zeta/\alpha\right) \tag{5.87}$$

By (4.112), $\tau_\alpha = O((-\tau)^{-\frac12}\alpha^{-1})$ and $\partial_1\tau(x\zeta,\beta) = O(x^{-1}(-\tilde\tau)^{-\frac12})$ whence

$$\hbar\partial_\alpha e^{-\frac{2}{3\hbar}[(-\tau)^{\frac32}+(-\tilde\tau)^{\frac32}]} = O(\alpha^{-1})e^{-\frac{2}{3\hbar}[(-\tau)^{\frac32}+(-\tilde\tau)^{\frac32}]} \tag{5.88}$$

as desired. Finally, by (4.136) and (4.143),

$$\begin{aligned}\partial_\alpha a_0(-\tau(x,\alpha);\alpha) &= -\tau_\alpha\,\partial_1 a_0(-\tau,\alpha) + \partial_\alpha a_0(-\tau,\alpha) \\ &= O((-\tau)^{-\frac12}\alpha^{-1})(-\tau)^{\frac12} + O(\alpha^{-1}) = O(\alpha^{-1}),\end{aligned}$$

and

$$\partial_\alpha a_0(-\tilde\tau;\beta) = \partial_\alpha a_0(-\tau(x\zeta,\beta);\beta) = x\zeta\alpha^{-1}\partial_1\tau(x\zeta,\beta)\,\partial_1 a_0(-\tilde\tau,\beta)$$
$$= x\zeta\alpha^{-1}O((-\tilde\tau)^{-\frac12}x^{-1})O((-\tilde\tau)^{\frac12}) = O(\alpha^{-1})$$

uniformly in the regime $0 < x \ll 1$ (which corresponds to $\tau \lesssim -1$).

Case B, which means $\eta \gg \xi$ or $\zeta \gg 1$, now proceeds entirely analogously. For example, (5.57) needs to be replaced by

$$|\partial_{\xi^{\frac12}} F_3(\xi,\eta;\hbar)| \le C_\ell\,\zeta^{-\frac12}(\hbar/\zeta)^\ell \alpha\langle\alpha\rangle^{-3} \tag{5.89}$$

for any $\ell \ge 1$, reflecting a loss of a factor of $\max(1,\alpha^{-1})$. This can be seen by applying $\hbar\partial_\alpha$ to (5.79) and (5.80), followed by the exact same integration by parts which yield (5.25). The analysis of B_2 and B_1 proceed in the same way and we skip the details. The other derivatives $\partial^{k_1}_{\xi^{\frac12}}\partial^{k_1}_{\eta^{\frac12}}$ which can appear in the regime $k_1 + k_2 \le 2$ are also treated in the same fashion, leading to the estimates claimed in the proposition.

We now turn to the expressions

$$\left(\partial_{\xi^{\frac12}} + \partial_{\eta^{\frac12}}\right)^k F(\xi,\eta;\hbar) = \hbar^k\left(\partial_\alpha + \partial_\beta\right)^k F(\xi,\eta;\hbar)$$

assuming $\xi \simeq \eta$ or $\zeta \simeq 1$. Thus it suffices to apply the operator $\hbar\left(\partial_\alpha + \partial_\beta\right)$ and its powers to Case A above. We begin with Case A_3, see (5.79) and (5.80). The key identities for (5.79) are

$$\begin{aligned} -i\hbar\partial_x e^{\frac{i}{\hbar}x(1-\zeta)} &= (1-\zeta)e^{\frac{i}{\hbar}x(1-\zeta)} \\ \hbar(\partial_\alpha+\partial_\beta)e^{\frac{i}{\hbar}x(1-\zeta)} &= -i\frac{x}{\alpha}(1-\zeta)e^{\frac{i}{\hbar}x(1-\zeta)} = -\hbar\frac{x}{\alpha}\partial_x e^{\frac{i}{\hbar}x(1-\zeta)} \end{aligned} \tag{5.90}$$

and for (5.80) they take the form

$$\begin{aligned} -i\hbar\partial_x e^{\frac{i}{\hbar}x(1+\zeta)} &= (1+\zeta)e^{\frac{i}{\hbar}x(1+\zeta)} \\ \hbar(\partial_\alpha+\partial_\beta)e^{\frac{i}{\hbar}x(1+\zeta)} &= i\frac{x}{\alpha}(1-\zeta)e^{\frac{i}{\hbar}x(1+\zeta)} = \hbar\frac{x}{\alpha}\frac{1-\zeta}{1+\zeta}\partial_x e^{\frac{i}{\hbar}x(1+\zeta)}. \end{aligned} \tag{5.91}$$

Thus, applying (5.90) to (5.79), respectively (5.91) to (5.80), yields, upon integration by parts in x,

$$\hbar\left(\partial_\alpha+\partial_\beta\right)F_3(\xi,\eta;\hbar) = 4\pi^{-1}\hbar\alpha^{-2}\zeta^{-\frac12}a(\xi)\overline{a(\eta)}\int_0^\infty e^{\frac{i}{\hbar}x(1-\zeta)}\hbar\partial_x\Big[W_n^+(x/\alpha)x/\alpha\,(1+\rho'(x;\alpha))^{-\frac12}\,(1+\rho'(x\zeta;\beta))^{-\frac12}$$
$$e^{\frac{i}{\hbar}[y(\beta)-y(\alpha)+\rho(x;\alpha)-\rho(x\zeta;\beta)]}\left(1+\hbar O(x^{-1})\right)\chi(x)\Big]\,dx + \mathrm{cc} \tag{5.92}$$
$$+\,4\pi^{-1}\hbar\alpha^{-2}\zeta^{-\frac12}a(\xi)a(\eta)\frac{\zeta-1}{1+\zeta}\int_0^\infty e^{\frac{i}{\hbar}x(1+\zeta)}\hbar\partial_x\Big[W_n^+(x/\alpha)x/\alpha(1+\rho'(x;\alpha))^{-\frac12}(1+\rho'(x\zeta;\beta))^{-\frac12}$$
$$e^{\frac{i}{\hbar}[-y(\beta)-y(\alpha)+\rho(x;\alpha)+\rho(x\zeta;\beta)]}\left(1+\hbar O(x^{-1})\right)\chi(x)\Big]\,dx + \mathrm{cc} + O\left(\hbar\alpha^{-1}\Gamma\right) \tag{5.93}$$

The final O-term here is a result of those expressions in which the $\hbar(\partial_\alpha+\partial_\beta)$ derivatives fall on the non-oscillatory terms, and they are treated as above. It is essential here that we obtain $O(\hbar\alpha^{-1}\Gamma)$ rather than $O(\hbar\max(1,\alpha^{-1})\Gamma)$ as in the case of $\hbar\partial_\alpha$ and $\hbar\partial_\beta$, see (5.6). As explained in the paragraph preceding (5.81), the reason for the absence of a gain of α^{-1} for $\alpha > 1$ lies only with the extra x-factor resulting from ∂_α hitting the complex exponential. This term is absent in the $O(\cdot)$ in (5.93). Placing absolute values in the integrals in (5.92) and (5.93), and using Lemma 4.13 as before, yields a bound of $O(\hbar\alpha^{-2}) = O(\xi^{-\frac{1}{2}}\Gamma)$. Iterating this process leads to (5.7).

For Case A_2, we return to the integral (5.81). As we proved in the paragraph preceding (5.83), if either $\hbar\partial_\alpha$ or, by symmetry, $\hbar\partial_\beta$, fall on any term other than the Airy functions in the integral in (5.81), then we obtain a net factor of $\hbar\alpha^{-1}$ as required for (5.7) (recall $\alpha \simeq \beta$). Furthermore, if $(\hbar\partial_\alpha)^k$, respectively, $\left(\hbar\partial_\beta\right)^k$ with $k \geq 2$ hits any term in (5.81) other than the Airy functions and a_0, a_1, then by Lemmas 4.10, 4.11, 4.13, we obtain the desired $(\hbar\alpha^{-1})^k$ factor. For the case of $a_j(-\tau,\alpha) = a_j(-\tau(x,\alpha),\alpha)$, with $j = 0, 1$, we have from (4.127), (4.136), and (4.143),

$$\begin{aligned}\hbar\partial_\alpha\, \hbar a_j(-\tau,\alpha) &= -\hbar^2\partial_\alpha\tau(x,\alpha)(\partial_1 a_j)(-\tau,\alpha) + \hbar^2(\partial_2 a_j)(-\tau,\alpha)\\ &= O\left(\hbar^3\langle\alpha\rangle^{-3}|\tau|^{-\frac{1}{2}}\right) + O\left(\hbar^2\alpha^{-1}\right) = O\left(\hbar^2\alpha^{-1}\right)\end{aligned}$$

for $\hbar^{\frac{2}{3}} \lesssim |\tau| \lesssim 1$ (and as the reader can check also for $|\tau| \leq \hbar^{\frac{2}{3}}$). The higher order derivatives $(\hbar\partial_\alpha)^\ell\, \hbar a_j(-\tau,\alpha)$ are linear combinations of terms of the form

$$\hbar^{\ell+1}\,\partial_\alpha^{k_1}\tau\cdot\ldots\cdot\partial_\alpha^{k_n}\tau\,\partial_1^n\partial_2^m a_j(-\tau,\alpha), \qquad k_1+\ldots+k_n+m=\ell,\ k_i\geq 1,\ n\geq 0 \tag{5.94}$$

By Lemmas 4.10 and 4.11, $\partial_\alpha^{k_i}\tau = O(\hbar(1+\alpha)^{-k_i-2})$, whereas by the aforementioned bounds on a_j and their derivatives, $\partial_1^n\partial_2^m a_j(-\tau,\alpha) = O\left(|\tau|^{\frac{1}{2}-n}\alpha^{-m}\right)$. In conclusion, the term in (5.94) is bounded by, for $\hbar^{\frac{2}{3}} \leq |\tau| \lesssim 1$,

$$\lesssim \hbar^{2\ell+1-m}(1+\alpha)^{-2n-(\ell-m)}|\tau|^{\frac{1}{2}-n}\alpha^{-m} \lesssim \hbar^{\ell+1}\cdot\hbar^{\ell-m}(1+\alpha)^{-\ell+m}|\tau|^{\frac{1}{2}-n}\alpha^{-m} \lesssim \hbar^{\ell+1}\alpha^{-\ell}$$

using that $m+n \leq \ell$, and the same bounds also applies to $|\tilde\tau| \lesssim \hbar^{\frac{2}{3}}$. While we have so far not made use of the sum of the derivatives, higher derivatives of the more delicate $a_j(-\tilde\tau,\beta) = a_j\,(-\tau(x\zeta,\beta),\beta)$ do require the sum $\partial_\alpha + \partial_\beta$, cf. (5.82). The point being that $(\hbar\partial_\alpha)^\ell\, \hbar a_j(-\tilde\tau,\beta)$ generates expressions of the form

$$\hbar^{\ell+1}\partial_1^\ell a_j(-\tilde\tau,\beta)(\partial_\alpha\tilde\tau)^\ell = O\left(\hbar^{\ell+1}|\tilde\tau|^{\frac{1}{2}-\ell}\alpha^{-\ell}\right)$$

which violate our desired bounds if $\tilde\tau \simeq \hbar^{\frac{2}{3}}$ and ℓ becomes large. Note that there are other contributions from the expression for $(\hbar\partial_\alpha)^\ell\, \hbar a_j(-\tilde\tau,\beta)$, which, however, contributes less powers in $|\tilde\tau|^{-1}$. Now

$$\begin{aligned}\hbar(\partial_\alpha+\partial_\beta)\hbar a_j(-\tilde\tau,\beta) &= \hbar^2 x\zeta\beta^{-1}(\zeta-1)(\partial_1 a_j)(-\tilde\tau,\beta)\partial_1\tau(x\zeta,\beta)\\ &\quad - \hbar^2(\partial_1 a_j)(-\tilde\tau,\beta)(\partial_\beta\tau)(x\zeta,\beta) + \hbar^2\partial_2 a_j(-\tilde\tau,\beta)\end{aligned} \tag{5.95}$$

The final term is $O\left(\hbar^2\alpha^{-1}\right)$, which is better by a factor of $\hbar$ than what we need. The derivative $\partial_1 a_j$ loses $\left(|\tilde\tau| + \hbar^{\frac{2}{3}}\right)^{-\frac{1}{2}}$. In the second term this loss is compensated for by $(\partial_\beta\tau)(x\zeta,\beta) = O\left(\hbar(1+\alpha)^{-3}\right)$. For the integration by parts in τ, we need the following estimates (by Lemma 4.11, Lemma 4.29 and (5.46)):

$$\frac{\partial\tilde\tau}{\partial\tau} \simeq 1, \quad \frac{\partial x}{\partial\tau} \simeq 1, \quad \left|\partial_1^2 a_1(-\tilde\tau,\beta)\right| = O\left(\hbar^{-1}\right), \quad |\partial_1\partial_2 a_1(-\tilde\tau,\beta)| = O\left(\hbar^{-\frac{1}{3}}\beta^{-1}\right).$$

The powers in $\hbar^{-1}$ above are compensated by the extra power in $\hbar$. On the other hand, by (5.35) we have $\tilde\tau \geq \zeta - 1$ if $\tau \geq 0$. Thus, in the first term of (5.95) with $j = 1$ one has $(\zeta-1)(\partial_1 a_1)(-\tilde\tau,\beta) = O\left(\tilde\tau^{\frac{1}{2}}\right)$ (note that a_1 in (5.81) only involves $\tau \geq 0$). In the subsequent integration by parts in τ, the operator ∂_τ hits the first term on the RHS of (5.95). Again by Lemma 4.11, Lemma 4.29 and (5.46), the most troublesome contribution is

$$\hbar^2 x\zeta\beta^{-1}(\zeta-1)(\partial_1^2 a_1)(-\tilde\tau,\beta)\partial_1\tau(x\zeta,\beta)\cdot\frac{\partial\tilde\tau}{\partial\tau}$$

, which is bounded in absolute value by

$$\lesssim \hbar^2\beta^{-1}(\zeta-1)|\tilde\tau|^{\frac12-2}\lesssim \hbar^{\frac53}\beta^{-1},$$

which is better than what we need. This argument for $j=1$ can be iterated: the term $\tilde\tau^{\frac12-k}$ arising in $(\hbar(\partial_\alpha+\partial_\beta))^m\hbar a_j(-\tilde\tau,\beta)$ from (4.127) are compensated for by $(\zeta-1)^k$. The details are as in (5.94) above, and we skip them. The end result is that this particular expression derived from (5.81) gives $O(\hbar^m\alpha^{-m})$ (the integration by parts in τ, see (5.50), generates a loss of at most $\hbar^{-\frac23}$ which is absorbed by the extra $\hbar$ factor at our disposal). Here we also used the estimate on the higher order derivatives: $\left|\frac{\partial^m\tilde\tau}{\partial\tau^m}\right|\lesssim 1$, which is seen by Lemma 4.11, (5.37).

For $j=0$, i.e., $a_0(-\tilde\tau,\beta)=a_0(-\tau(x\zeta,\beta),\beta)$, the mechanism is slightly different. It is only the first term of (5.95) that requires a different argument. Using (5.35) once again, we distinguish three cases: (i) if $\tau\geq -c(\zeta-1)$ with some absolute constant $0<c\ll 1$, then $\hbar^{\frac23}\gtrsim\tilde\tau\gtrsim\zeta-1$ and the same argument as for a_1 applies. (ii) if $\tau\leq -C(\zeta-1)$ with some absolute constant $C\gg 1$, then $\tilde\tau\simeq\tau\leq -C(\zeta-1)$ whence $|\tilde\tau|\gtrsim\zeta-1$ and we can argue again as for a_1. (iii) if $-\tau\simeq\zeta-1$, then we use the factor $\mathrm{Ai}\left(-\hbar^{-\frac23}\tau\right)$ from (5.81) to bound

$$\left|\mathrm{Ai}\left(-\hbar^{-\frac23}\tau\right)(\zeta-1)\,(\partial_1 a_0)\,(-\tilde\tau,\beta)\partial_1\tau(x\zeta,\beta)\right|\lesssim(\zeta-1)\,\hbar^{-\frac13}e^{-\frac{c}{\hbar}|\zeta-1|^{\frac32}}\lesssim 1$$

uniformly in the parameters, which is sufficient (we can ignore the $-\frac14$ power preceding the exponential bound of Ai). For the higher derivatives we encounter terms of the form

$$\begin{aligned}\left|\mathrm{Ai}\left(-\hbar^{-\frac23}\tau\right)\hbar^{m+1}\beta^{-m}(\zeta-1)^m\left(\partial_1^m a_j\right)(-\tilde\tau,\beta)\right|&\lesssim\hbar^{m+1}\alpha^{-m}(\zeta-1)^m(|\tilde\tau|+\hbar^{\frac23})^{\frac12-m}e^{-\frac{c}{\hbar}|\zeta-1|^{\frac32}}\\&\lesssim\hbar^{m+1}\alpha^{-m}(\zeta-1)^m\hbar^{-\frac23 m}e^{-\frac{c}{\hbar}|\zeta-1|^{\frac32}}\lesssim\hbar^{m+1}\alpha^{-m}\end{aligned}$$

It remains to consider the derivatives of the Airy functions, which are governed by (5.84) and (5.86). In view of these equations, we lose a factor of $\hbar$ by differentiating $\mathrm{Ai}\left(-\hbar^{\frac23}\tau\right)$ and the other Airy function. However, if the argument is τ and not $\tilde\tau$, then we regain this factor in τ_α, resp., τ_β which are each bounded by $\hbar(1+\alpha)^{-3}$. This process can be repeated and we gain the desired $(\hbar/\alpha)^m$ *without* having to exploit the trace type derivative $\partial_\alpha+\partial_\beta$. In contrast, we do not gain $\hbar$ in (5.86) and instead invoke the following:

$$\begin{aligned}\hbar\left(\partial_\alpha+\partial_\beta\right)\left(\mathrm{Ai}\left(-\hbar^{-\frac23}\tilde\tau\right)-i\mathrm{Bi}\left(-\hbar^{-\frac23}\tilde\tau\right)\right)&=c\,\tilde\tau^{\frac12}\left(\tilde\tau_\alpha+\tilde\tau_\beta\right)\left(1+\hbar O(\tilde\tau^{-\frac32})\right)\left(\mathrm{Ai}\left(-\hbar^{-\frac23}\tilde\tau\right)-i\mathrm{Bi}\left(-\hbar^{-\frac23}\tilde\tau\right)\right)\\\tilde\tau_\alpha+\tilde\tau_\beta&=(\partial_1\tau)(x\zeta,\beta)x\zeta\beta^{-1}(1-\zeta)+(\partial_2\tau)(x\zeta,\beta)\end{aligned}\tag{5.96}$$

The second term is $\hbar/\alpha$ and again regains the essential $\hbar$ factor, while the first one is of size $(\zeta-1)/\alpha$. By (5.42) and (5.35), and assuming $\tau\geq -c(\zeta-1)$ for simplicity as in case (i) in the preceding paragraph,

$$\left|\partial_\tau\left(\tilde\tau^{\frac32}-\tau^{\frac32}\right)\right|\simeq(\zeta-1)\tilde\tau^{-\frac12}$$

Integrating by parts as in (5.50) in τ therefore gains a factor of $\hbar\tilde\tau/\alpha$ (note the $\tilde\tau^{\frac12}$ in (5.96)). As we iterate this process, we accumulate a factor of $(\hbar/\alpha)^m\tilde\tau^m$, albeit at the expense of ∂_τ^m hitting all terms in the integrand other than the complex exponential, cf. (5.50). If the operator $\hbar(\partial_\alpha+\partial_\beta)$ hits the coefficient of the oscillatory Airy function in (5.96), we again obtain a bound like $\hbar\tilde{\tau}/\alpha$. In fact we have

$$\hbar(\partial_\alpha+\partial_\beta)\left(\tilde{\tau}^{\frac{1}{2}}\right)\cdot\left(\tilde{\tau}_\alpha+\tilde{\tau}_\beta\right)\simeq\hbar\tilde{\tau}^{-\frac{1}{2}}\,((\partial_1\tau)(x\zeta,\beta))^2\,x^2\zeta^2\beta^{-2}(1-\zeta)^2.$$

Then upon integration by parts twice, we obtain a bound as (in view of $\hbar^{\frac{2}{3}}\lesssim\tilde{\tau}$, since we are in the oscillatory regime)

$$\hbar^3\,((\partial_1\tau)(x\zeta,\beta))^2\,x^2\zeta^2\beta^{-2}\tilde{\tau}\lesssim\hbar^2\tilde{\tau}^2/\beta^{-2}.$$

The case when the derivative $\hbar(\partial_\alpha+\partial_\beta)$ hits the factor $\tilde{\tau}_\alpha+\tilde{\tau}_\beta$ is handled similarly, and it is in fact lower order.

Note that in order to obtain the factor involving $\Lambda=\hbar(\zeta-1)^{-1}$ which is necessary for the Γ bound (5.3), we then need to carry out one more integration by parts exactly as in (5.50). If the ∂_τ^m hits $\tilde{\tau}^{-\frac{1}{4}}\tilde{a}_1(-\tilde{\tau},\beta)$ in (5.50), then we lose $\tilde{\tau}^{-m}$ which is exactly compensated for by the $\tilde{\tau}^m$ that we gained. However, if ∂_τ^m hits $\tau^{-\frac{1}{4}}\tilde{a}_1(-\tau,\alpha)$, then we lose τ^{-m}. Recall our conditions $|\xi-\eta|\leq\hbar^{\frac{2}{3}}\xi$ (which we have not used before) and $\xi\simeq\eta$ for (5.7). They imply that $0\leq\zeta-1\lesssim\hbar^{\frac{2}{3}}$, which ensures that $\tau\simeq\tilde{\tau}$, since we can assume that $\tau\geq\hbar^{\frac{2}{3}}$ (If $|\tau|\lesssim\hbar^{\frac{2}{3}}$, then we don't even need the integration by parts argument).

Finally, Case A_1 is easy due to the exponential gain inherent in the regime $0<x\ll1$, see the comment following (5.56). We can therefore pass any number of $\hbar\partial_\alpha$, resp. $\hbar\partial_\beta$ derivatives into the integral in (5.56) without any concern about losing powers of $\hbar$. The are swallowed by the gain of $\exp(-c\hbar^{-1})$ which we have at our disposal.

Now we turn to the case when $2\leq n\leq N_0$ for some fixed N_0. We use $F_n(\xi,\eta)$ to denote the off-diagonal kernel (5.21):

$$F_n(\xi,\eta):=\int_0^\infty W_n^+(R)\phi_n(R,\xi)\phi_n(R,\eta)\,dR. \tag{5.97}$$

Here $\phi_n(R,\cdot)=w_n(\cdot)\phi(R,\cdot)$ is as in Proposition 4.37 and was constructed with respect to the measure dR, which appears in (5.97). We start with the bound on $F_n(\xi,\eta)$ itself. When $0\leq R\xi^{\frac{1}{2}}\leq R\eta^{\frac{1}{2}}\lesssim1$, we have, by (4.178),

$$\begin{aligned}\left|\int_0^\infty W_n^+(R)\chi_{R\xi^{\frac{1}{2}}\lesssim\frac{1}{2}}\chi_{R\eta^{\frac{1}{2}}\lesssim\frac{1}{2}}\phi_n(R,\xi)\phi_n(R,\eta)\,dR\right|&\lesssim\xi^{-\frac{1}{4}}\eta^{-\frac{1}{4}}\cdot\xi^{\frac{n}{2}-\frac{1}{4}}\eta^{\frac{n}{2}-\frac{1}{4}}\int_0^{\eta^{-\frac{1}{2}}}\frac{R^{2n-1}}{(1+R^2)^2}dR\\&\lesssim\begin{cases}\eta^{-1}\cdot\left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^{n-1}, & \text{if}\quad\eta\gtrsim1,\\ \eta\cdot|\log\eta|\cdot\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}, & \text{if}\quad\eta\ll1.\end{cases}\end{aligned}\tag{5.98}$$

Next we turn to the case $0 < R\xi^{\frac{1}{2}} \lesssim 1 \ll R\eta^{\frac{1}{2}}$. We have, by (4.178) and (4.179) (Here we only look at $e^{iR\xi^{\frac{1}{2}}}(1+g_+(R,\xi))$, and its conjugate can be handled similarly),

$$\begin{aligned}
&\int_0^\infty W_n^+(R)\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\chi_{R\eta^{\frac{1}{2}}\gg 1}\phi_n(R,\xi)\phi_n(R,\eta)\,dR \\
&= \int_0^\infty W_n^+(R)\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_n(R,\xi)\cdot\chi_{R\eta^{\frac{1}{2}}\gg 1}\eta^{-\frac{1}{4}}\frac{a(\eta)}{|a(\eta)|}e^{iR\eta^{\frac{1}{2}}}(1+g_+(R,\eta))\,dR \\
&= \int_0^\infty W_n^+(R)\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_n(R,\xi)\cdot\chi_{R\eta^{\frac{1}{2}}\gg 1}\eta^{-\frac{1}{4}}\frac{a(\eta)}{|a(\eta)|}\cdot\frac{1}{i\eta^{\frac{1}{2}}}\partial_R\left(e^{iR\eta^{\frac{1}{2}}}\right)(1+g_+(R,\eta))\,dR
\end{aligned} \tag{5.99}$$

By Lemma 4.36 and (4.178), upon a differentiation in R on the factor $W_n^+(R)\phi_n(R,\xi)(1+g_+(R,\eta))$, we obtain a factor of $\xi^{\frac{1}{2}}$, compared to the undifferentiated one. Therefore after integration by parts, we obtain

$$\left|\int_0^\infty W_n^+(R)\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\chi_{R\eta^{\frac{1}{2}}\gg 1}\phi_n(R,\xi)\phi_n(R,\eta)\,dR\right| \lesssim \frac{\min\left\{1,\eta^{\frac{3}{2}}\right\}}{\xi^{\frac{1}{4}}\eta^{\frac{1}{4}}}\cdot\left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^N, \quad \text{for} \quad 0 < N \le n. \tag{5.100}$$

Now we consider the case when $R\xi^{\frac{1}{2}} \gg 1$ and $R\eta^{\frac{1}{2}} \gg 1$. This is similar to the previous case, because when ∂_R hits $W_n^+(R)\phi_n(R,\xi)(1+g_+(R,\eta))$, we again obtain an extra factor of $\xi^{\frac{1}{2}}$. Therefore we have

$$\left|\int_0^\infty W_n^+(R)\chi_{R\xi^{\frac{1}{2}}\gg 1}\chi_{R\eta^{\frac{1}{2}}\gg 1}\phi_n(R,\xi)\phi_n(R,\eta)\,dR\right| \lesssim \frac{\min\{1,\xi^{\frac{3}{2}}\}}{\xi^{\frac{1}{4}}\eta^{\frac{1}{4}}}\cdot\left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^N, \quad \text{for any} \quad N > 0. \tag{5.101}$$

Now we refine the estimates when $R\xi^{\frac{1}{2}} \ll 1$ and $\eta \ge 1$ to obtain the rapid decay factor $\left(\frac{\langle\xi\rangle}{\eta}\right)^N$. We again start with the expression

$$F_n(\xi,\eta) := \int_0^\infty W_n^+(R)\phi_n(R,\xi)\phi_n(R,\eta)R\,dR.$$

Here $\phi_n(R,\xi)$ is the Fourier basis with respect to the measure $R\,dR$, and $W_n^+(R)$ is a linear combination of $\frac{1}{(1+R^2)^k}$ where $k \ge 2$. Since now $2 \le n \le N_0$, the n-dependence of $W_n^+(R)$ is not a concern here. Therefore without loss of generality, we consider

$$\int_0^\infty \frac{1}{(1+R^2)^k}\phi_n(R,\xi)\phi_n(R,\eta)R\,dR. \tag{5.102}$$

A direct calculation gives

$$\begin{aligned}\eta\int_0^\infty \frac{1}{(1+R^2)^k}\phi_n(R,\xi)\phi_n(R,\eta)R\,dR =& -\int_0^\infty \frac{1}{(1+R^2)^k}\phi_n(R,\xi)H_n^+\left(\phi_n(R,\eta)\right)R\,dR\\ =&\xi\int_0^\infty \frac{1}{(1+R^2)^k}\phi_n(R,\xi)\phi_n(R,\eta)R\,dR\\ &+2\int_0^\infty \partial_R\left(\frac{1}{(1+R^2)^k}\right)\partial_R\left(\phi_n(R,\xi)\right)\phi_n(R,\eta)R\,dR\\ &+\int_0^\infty R^{-1}\partial_R\left(\frac{1}{(1+R^2)^k}\right)\phi_n(R,\xi)\phi_n(R,\eta)R\,dR\\ &-\int_0^\infty \partial_R^2\left(\frac{1}{(1+R^2)^k}\right)\phi_n(R,\xi)\phi_n(R,\eta)R\,dR.\end{aligned}\tag{5.103}$$

Since $\partial_R^2\left(\frac{1}{(1+R^2)^k}\right)$ and $R^{-1}\partial_R\left(\frac{1}{(1+R^2)^k}\right)$ are again linear combinations of the functions of the form $\frac{1}{(1+R^2)^k}$, the third and the fourth terms on the RHS of (5.103) can be the same way as (5.102). So we focus on the second term on the RHS of (5.103). With $\phi_0(R), \phi_j(R)$, $j \geq 1$ given by Lemma 4.35 and $w_n(\xi)$ given by Proposition 4.37, we have, for $R\xi^{\frac12} \ll 1$,

$$\phi_n(R,\xi) = \phi_0(R)w_n(\xi)\left[1+\sum_{j\geq 1}\phi_j(R)\left(R^2\xi\right)\right].$$

It follows that, for a constant C_n depending on n,

$$\chi_{R\xi^{\frac12}\ll 1}\partial_R\left(\phi_n(R,\xi)\right) = C_nR^{-1}\phi_n(R,\xi)+R\phi_n(R,\xi)\cdot\left[\sum_{j\geq 0}\psi_j(R)\left(R^2\xi\right)^j\right]$$

for certain coefficient functions $\psi_j(R)$ satisfying analogous bounds as the $\phi_j(R)$. In fact, if $R\xi^{\frac12} \ll 1$ is sufficiently small, the reciprocal of $\left[1+\sum_{j\geq 1}\phi_j(R)(R^2\xi)^j\right]$ can be also expanded into a power series in $R^2\xi$. So we conclude that, for some constant $C_{k,n}$ depending on k, n, and some constant D_k depending on k,

$$\begin{aligned}&\int_0^\infty \chi_{R\xi^{\frac12}\ll 1}\partial_R\left(\frac{1}{(1+R^2)^k}\right)\partial_R\left(\phi_n(R,\xi)\right)\phi_n(R,\eta)R\,dR\\ =&\int_0^\infty \chi_{R\xi^{\frac12}\ll 1}C_{k,n}\frac{1}{(1+R^2)^{k+1}}\phi_n(R,\xi)\phi_n(R,\eta)R\,dR\\ &+\int_0^\infty \chi_{R\xi^{\frac12}\ll 1}D_k\frac{1}{(1+R^2)^k}\phi_n(R,\xi)\left[\sum_{j\geq 0}\psi_j(R)\left(R^2\xi\right)^j\right]\phi_n(R,\eta)R\,dR.\end{aligned}\tag{5.104}$$

Using the fact that $\psi_j(R)$ has rapid decay in j and $\psi_j(R)$ is a function of R^2, the terms on the RHS of (5.104) can be handled the same way as (5.102).

The refined derivative estimates follow in the same way, using the relations:

$$H_n^+\left(\partial_\xi^k\phi_n(R,\xi)\right) = -\xi\partial_\xi^k\phi_n(R,\xi)-k\partial_\xi^{k-1}\phi_n(R,\xi), \quad \text{for} \quad k\geq 1.\tag{5.105}$$

Now we turn to the derivatives of $F_n(\xi,\eta)$, if $R\xi^{\frac{1}{2}} \lesssim 1$ and $R\eta^{\frac{1}{2}} \lesssim 1$, then by (4.178), upon a differentiation in $\xi^{\frac{1}{2}}$ (or $\eta^{\frac{1}{2}}$), we obtain an extra factor of $\xi^{-\frac{1}{2}}$ (or $\eta^{-\frac{1}{2}}$). This gives all the derivative estimates in this regime. If $R\xi^{\frac{1}{2}} \lesssim 1$ and $R\eta^{\frac{1}{2}} \gg 1$, we use the fact

$$R\partial_R\left(e^{iR\eta^{\frac{1}{2}}}\right) = \eta^{\frac{1}{2}}\partial_{\eta^{\frac{1}{2}}}\left(e^{iR\eta^{\frac{1}{2}}}\right), \quad \Rightarrow \quad \partial_{\eta^{\frac{1}{2}}}\left(e^{iR\eta^{\frac{1}{2}}}\right) = R\eta^{-\frac{1}{2}}\partial_R\left(e^{iR\eta^{\frac{1}{2}}}\right), \tag{5.106}$$

and integration by parts to obtain the derivative estimates in this regime. Finally we look at the more delicate case $1 \ll R\xi^{\frac{1}{2}} \le R\eta^{\frac{1}{2}}$. For the pure $\partial_{\eta^{\frac{1}{2}}}$-derivatives of $F_n(\xi,\eta)$, we again use (5.106) and integration by parts to obtain

$$\left|\partial^k_{\eta^{\frac{1}{2}}} F_n(\xi,\eta)\right| \lesssim \max\{\xi^{-\frac{k}{2}}, 1\}\cdot\Gamma_n\cdot\frac{\xi^{\frac{k}{2}}}{\eta^{\frac{k}{2}}}, \quad \text{for} \quad k=1,2 \quad \text{and} \quad 1 \ll R\xi^{\frac{1}{2}} \le R\eta^{\frac{1}{2}}. \tag{5.107}$$

Here Γ_n is the bound for the undifferentiated $F_n(\xi,\eta)$ in the same regime. For the derivatives in $\xi^{\frac{1}{2}}$. We distinguish the discussion according to the order of the derivative in $\xi^{\frac{1}{2}}$. We first consider the bound for $\partial^k_{\xi^{\frac{1}{2}}} F_n(\xi,\eta), k = 1,2$. It is straightforward to see that for $k = 1,2$,

$$\left|\partial^k_{\xi^{\frac{1}{2}}} F_n(\xi,\eta)\right| \lesssim \xi^{-\frac{k}{2}}\cdot\Gamma_n \quad \text{for} \quad \xi \ll 1, \quad \text{and} \quad \left|\partial^k_{\xi^{\frac{1}{2}}} F_n(\xi,\eta)\right| \lesssim \Gamma_n \quad \text{for} \quad \xi \gtrsim 1. \tag{5.108}$$

$\partial_{\xi^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}} F_n(\xi,\eta)$ is even more delicate. Without loss of generality, we consider the integral:

$$\int_0^\infty W_n^+(R)\chi_{R\xi^{\frac{1}{2}}\gg 1}\chi_{R\eta^{\frac{1}{2}}\gg 1}\xi^{-\frac{1}{4}}\eta^{-\frac{1}{4}}\partial_{\xi^{\frac{1}{2}}}\left(e^{iR\xi^{\frac{1}{2}}}(1+g_+(R,\xi))\right)\cdot\partial_{\eta^{\frac{1}{2}}}\left(e^{iR\eta^{\frac{1}{2}}}(1+g_+(R,\eta))\right)dR, \tag{5.109}$$

and we only focus on the contribution when the derivatives $\partial_{\xi^{\frac{1}{2}}}, \partial_{\eta^{\frac{1}{2}}}$ hit the oscillatory factors. Therefore the integral in consideration is given by (omitting the constant coefficients)

$$\begin{aligned}
&\left|\partial_{\xi^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}} F_n(\xi,\eta)\right| \\
\simeq&\xi^{-\frac{1}{4}}\eta^{-\frac{1}{4}}\left|\int_0^\infty W_n^+(R)\chi_{R\xi^{\frac{1}{2}}\gg 1}\chi_{R\eta^{\frac{1}{2}}\gg 1}R\cdot e^{iR\xi^{\frac{1}{2}}}(1+g_+(R,\xi))\cdot\partial_{\eta^{\frac{1}{2}}}\left(e^{iR\eta^{\frac{1}{2}}}\right)\cdot(1+g_+(R,\eta))\,dR\right| \\
=&\xi^{-\frac{1}{4}}\eta^{-\frac{1}{4}}\left|\int_0^\infty W_n^+(R)\chi_{R\xi^{\frac{1}{2}}\gg 1}\chi_{R\eta^{\frac{1}{2}}\gg 1}R\cdot e^{iR\xi^{\frac{1}{2}}}(1+g_+(R,\xi))\cdot\frac{R}{\eta^{\frac{1}{2}}}\partial_R\left(e^{iR\eta^{\frac{1}{2}}}\right)\cdot(1+g_+(R,\eta))\,dR\right| \\
\lesssim&\max\{\xi^{-1},1\}\cdot\Gamma_n\cdot\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}.
\end{aligned} \tag{5.110}$$

using integration by parts. At the end we consider the third order derivatives in the regime $1 \ll R\xi^{\frac{1}{2}} \le R\eta^{\frac{1}{2}}$. In this case we will encounter a logarithmic divergence any way, so we bound the contributions from all the possible third order derivatives by the absolute value of (Again here we only consider the phase $e^{iR(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}$, and the contribution from $e^{iR(\xi^{\frac12}+\eta^{\frac12})}$ can be treated similarly)

$$\begin{aligned}&\xi^{-\frac14}\eta^{-\frac14}\left(\int_{\xi^{-\frac12}}^{|\xi^{\frac12}-\eta^{\frac12}|^{-1}} R^{-1}\,dR+\int_{|\xi^{\frac12}-\eta^{\frac12}|^{-1}}^{\infty} R^{-1}e^{iR(\eta^{\frac12}-\xi^{\frac12})}\,dR\right)\\ \lesssim&\xi^{-\frac14}\eta^{-\frac14}\left(1+\left|\log\left(\xi^{\frac12}\left|\xi^{\frac12}-\eta^{\frac12}\right|^{-1}\right)\right|\right)\end{aligned}\tag{5.111}$$

□

5.1.2. *Negative frequency.* Next we turn to the case when $n\le -2$. Let us define the operator $\mathcal{K}_{\widetilde{\hbar}}$ as

$$\widehat{R\partial_R u}=-2\xi\partial_\xi\widehat{u}+\mathcal{K}_{\widetilde{\hbar}}\widehat{u}.\tag{5.112}$$

Here the Fourier transform is taken with respect to the Fourier basis $\phi(R,\xi;\widetilde{\hbar})=R^{-\frac12}\phi_*(R,\xi;\widetilde{\hbar})$ in $L^2(R\,dR)$ with $\phi_*(R,\xi;\widetilde{\hbar})$ being the Fourier basis in $L^2(dR)$ (See Propositions 4.39 and 4.45). Let $\rho_{-n}(d\xi)$ be the spectral measure associated with $\phi_*(R,\xi;\widetilde{\hbar})$, where $\widetilde{\hbar}=\frac{1}{1-n}$. As for the positive frequencies, $\rho_{-n}(d\xi)$ is also the spectral measure of $\phi(R,\xi;\widetilde{\hbar})$.

Similar as Proposition 5.1, we have

Proposition 5.2. *For any $-1\le\widetilde{\hbar}<0$ the operator $\mathcal{K}_{\widetilde{\hbar}}$ is given by*

$$\mathcal{K}_{\widetilde{\hbar}}f(\xi)=-\left(2f(\eta)+\frac{\eta\left(\frac{d\rho_{-n}}{d\eta}(\eta)\right)'}{\frac{d\rho_{-n}(\eta)}{d\eta}}f(\eta)\right)\delta(\eta-\xi)+\left(\mathcal{K}_{\widetilde{\hbar}}^{(0)}f\right)(\xi)$$

where the off-diagonal part $\mathcal{K}_{\widetilde{\hbar}}^{(0)}$ has a kernel $K_0(\xi,\eta;\widetilde{\hbar})$ given by

$$K_0(\xi,\eta;\widetilde{\hbar})=\frac{\frac{d\rho_{-n}(\eta)}{d\eta}}{\xi-\eta}F(\xi,\eta;\widetilde{\hbar})\tag{5.113}$$

and the symmetric function $F(\xi,\eta;\widetilde{\hbar})$ satisfies (for any $0\le k\le k_0$ and sufficiently small $\hbar=\hbar(k_0)$, where k_0 is arbitrary but fixed, and $\xi\le\eta$)

$$|F(\xi,\eta;\widetilde{\hbar})|\lesssim\left(\widetilde{\hbar}\xi^{\frac12}\right)^{-1}\min\left\{1,\left(\widetilde{\hbar}\xi^{\frac12}\right)^3\right\}\cdot G:=\Gamma_{\widetilde{\hbar}}.\tag{5.114}$$

with

$$G:=\begin{cases}\min\left\{1,\left(\widetilde{\hbar}\xi^{\frac12}\right)^{\frac14}\left|\eta^{\frac12}-\xi^{\frac12}\right|^{-\frac14}\right\}, & \textit{for}\quad \left|\frac{\eta^{\frac12}}{\xi^{\frac12}}-1\right|\lesssim 1,\\ \hbar\left(\widetilde{\hbar}\xi^{\frac12}\right)^{-1}\min\left\{1,\widetilde{\hbar}\xi^{\frac12}\right\}\cdot\left(\frac{\xi^{\frac12}}{\eta^{\frac12}}\right)^k, & \textit{for}\quad \left|\frac{\eta^{\frac12}}{\xi^{\frac12}}-1\right|\gg 1.\end{cases}\tag{5.115}$$

For $\widetilde{\hbar}\gtrsim 1$, we have, for any $N>0$

$$|F(\xi,\eta;\widetilde{\hbar})|\lesssim\Gamma_n:=\begin{cases}\eta|\log\eta|\cdot\frac{\xi^{\frac12}}{\eta^{\frac12}}, & \textit{for}\quad \eta\ll 1,\\ \max\{\xi^{-\frac12}\eta^{-\frac12},\eta^{-\frac14}\xi^{-\frac14}\}\cdot\left(\frac{\xi^{\frac12}}{\eta^{\frac12}}\right)^n\cdot\left(\frac{\langle\xi\rangle}{\eta}\right)^N, & \textit{for}\quad \eta\gtrsim 1\end{cases}.\tag{5.116}$$

For the derivatives of $F(\xi,\eta;\widetilde{\hbar})$, we have, for $k_1+k_2\le 2$

$$\left|\partial^{k_1}_{\xi^{\frac12}}\partial^{k_2}_{\eta^{\frac12}}F(\xi,\eta;\widetilde{\hbar})\right|\lesssim \max\left\{1,\left(\widetilde{\hbar}\xi^{\frac12}\right)^{-k_1}\right\}\cdot\max\left\{1,\left(\widetilde{\hbar}\eta^{\frac12}\right)^{-k_2}\right\}\cdot\Gamma \tag{5.117}$$

Here Γ is $\Gamma_{\widetilde{\hbar}}$ if $\hbar\ll 1$, and Γ_n if $\widetilde{\hbar}\gtrsim 1$. The following estimate holds for trace-type derivatives for $\widetilde{\hbar}\ll 1$:

$$\left|\left(\partial_{\xi^{\frac12}}+\partial_{\eta^{\frac12}}\right)^k F(\xi,\eta;\widetilde{\hbar})\right|\le C_k\,\xi^{-\frac{k}{2}}\cdot\Gamma,\quad \text{if}\quad \xi\simeq\eta,\ |\eta-\xi|\le\widetilde{\hbar}^{\frac23}\xi \tag{5.118}$$

for all $k\ge 0$.

Proof. Again we will often drop $\widetilde{\hbar}$ from the notation. The proof is identical to that of Proposition 5.1 except that for $2\le n\le N_0$ we use Proposition 4.39, since the Airy functions are only used to construct $\phi_{-n}(R,\xi)$ for $n\ge N_0$. A similar calculation as in Proposition 5.1 gives

$$\begin{aligned}\mathcal{K}f(\eta)=&\left\langle\int_0^\infty f(\xi)[R\partial_R-2\xi\partial_\xi]\phi(R,\xi)\rho(\xi)\,d\xi,\phi(R,\eta)\right\rangle_{L^2_{R\,dR}}\\&-2\left(1+\frac{\eta\rho'(\eta)}{\rho(\eta)}\right)f(\eta).\end{aligned}$$

To extract any $\delta(\xi-\eta)$ from the first line above we recall from Proposition 4.39,

$$\phi(R,\xi)=2\xi^{-\frac14}R^{-\frac12}\mathrm{Re}\left(a_-(\xi)e^{iR\xi^{\frac12}}\left(1+g^-(R,\xi)\right)\right)$$

with

$$\begin{aligned}&g^-(R,\xi)=\frac{1}{R\xi^{\frac12}}\left(C\,i+O\left(\frac{1}{1+R^2}\right)\right)+O\left(R^{-2}\xi^{-1}\right),\\&\left|(R\partial_R)^k g^-(R,\xi)\right|\le c_k(R\xi^{\frac12})^{-1},\quad\text{and}\quad\left|(\xi\partial_\xi)^k g^-(R,\xi)\right|\le c_k(R\xi^{\frac12})^{-1}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\left(R\partial_R-2\xi\partial_\xi\right)\phi(R,\xi)=&-4\xi^{-\frac14}R^{-\frac12}\mathrm{Re}\left(\xi a_-'(\xi)e^{iR\xi^{\frac12}}\left(1+g^-(R,\xi)\right)\right)\\&+2\xi^{-\frac14}R^{-\frac12}\mathrm{Re}\left(a_-(\xi)e^{iR\xi^{\frac12}}\left(R\partial_R-2\xi\partial_\xi\right)g^-(R,\xi)\right).\end{aligned}$$

According to the profile of $g^-(R,\xi)$, we have

$$\left|\left(R\partial_R-2\xi\partial_\xi\right)g^-(R,\xi)\right|\lesssim\frac{1}{R\xi^{\frac12}}O\left(\frac{R^2}{(1+R^2)^2}\right).$$

Using that $\mathrm{Re}\,z\mathrm{Re}\,w=\frac12\left(\mathrm{Re}\,(zw)+\mathrm{Re}\,(z\overline{w})\right)$ we infer that the δ measure on the diagonal in the integral

$$\lim_{A\to\infty}\int_0^A\left(R\partial_R-2\xi\partial_\xi\right)\phi(R,\xi)\phi(R,\eta)R\,dR$$

comes from the expression

$$(\xi\eta)^{-\frac{1}{4}} \lim_{L\to\infty} \mathrm{Re} \int_0^\infty -4\left(\xi a_-'(\xi)\overline{a_-(\eta)}e^{iR(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}\left(1+g^-(R,\xi)\right)\left(\overline{1+g^-(R,\eta)}\right)\right)\chi_1(R)\chi_2(R/L)dR$$
$$+ (\xi\eta)^{-\frac{1}{4}} \lim_{L\to\infty} \mathrm{Re} \int_0^\infty 2e^{iR(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}a_-(\xi)\overline{a_-(\eta)}\left(R\partial_R - 2\xi\partial_\xi\right)g^-(R,\xi)\cdot\overline{(1+g^-(R,\eta))}\cdot\chi_1(R)\chi_2(R/L)\,dR$$

Here χ_1 is a smooth cutoff which equals 1 near ∞ and which vanishes near 0, and $\chi_2 = 1-\chi_1$. According to the estimate on $\left(R\partial_R - 2\xi\partial_\xi\right)g^-(R,\xi)$ and the profile of g^-, the δ measure comes from

$$\begin{aligned}
&(\xi\eta)^{-\frac{1}{4}} \lim_{L\to\infty} \mathrm{Re} \int_0^\infty -4\xi a_-'(\xi)\overline{a_-(\eta)}e^{iR(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}\chi_1(R)\chi_2(R/L)dR\\
=& -4\pi(\xi\eta)^{-\frac{1}{4}}\mathrm{Re}\left(\xi a_-'(\xi)\overline{a_-(\eta)}\right)\delta(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})\\
=& -8\pi\xi\mathrm{Re}\left(a_-'(\xi)\overline{a_-(\xi)}\right)\delta(\xi-\eta) = -4\pi\xi\frac{d}{d\xi}\left(|a_-(\xi)|^2\right) = -\xi\frac{d}{d\xi}\left(\rho(\xi)^{-1}\right)\delta(\xi-\eta).
\end{aligned}$$

This together with the calculation in (5.19) gives the desired result. □

5.2. **Transference operator at angular momentum** $n = 1$**.** The operator $\mathcal{K}^1$ is defined as

$$\widehat{R\partial_R u} = -2\xi\partial_\xi\widehat{u} + \mathcal{K}^1\widehat{u}. \tag{5.119}$$

In this section, the Fourier transform is taken with respect to the Fourier basis $\tilde{\phi}(R,\xi) = R^{-\frac{1}{2}}\phi(R,\xi)$ in $L^2(RdR)$ where $\phi(R,\xi)$ is introduced in (4.21). Let $\rho_1(d\xi)$ be the spectral measure associated with $\phi(R,\xi)$. Using an argument similar to the $n \geq 2$ case, we obtain the Plancherel identity

$$\|f\|^2_{L^2_{RdR}} = \left\|\langle f, R^{-\frac{1}{2}}\phi\rangle_{L^2_{RdR}}\right\|^2_{L^2_{\rho_1(d\xi)}},$$

and it follows that $\rho_1(d\xi)$ is also the spectral measure for $\tilde{\phi}(R,\xi)$. The main result of this section is:

Proposition 5.3. *The operator $\mathcal{K}^1$ is given by*

$$\left(\mathcal{K}^1 f\right)(\xi) = -\left(2f(\eta) + \frac{\eta\rho_1'(\eta)}{\rho_1(\eta)}f(\eta)\right)\delta(\eta-\xi).$$

In other words, the off-diagonal part of $\mathcal{K}^1$ vanishes. Here for simplicity we write the spectral measure density $\frac{d\rho_1(\xi)}{d\xi}$ as $\rho_1(\xi)$, and recall from (4.32) *that $\rho_1(\xi) = \frac{2\xi}{\pi^2}$ for $\xi > 0$.*

Proof. Similar to the proof of Proposition 5.1, for a function $f \in C_0^\infty((0,\infty))$, we define $u(R) = \int_0^\infty \tilde{\phi}(R,\xi)f(\xi)\rho_1(\xi)d\xi$. A computation similar to (5.8) gives

$$\begin{aligned}
\widehat{R\partial_R u}(\xi) =& \left\langle \int_0^\infty [R\partial_R - 2\eta\partial_\eta]\tilde{\phi}(R,\eta)f(\eta)\rho_1(\eta)d\eta, \tilde{\phi}(R,\xi)\right\rangle_{L^2(RdR)}\\
&- 2f(\xi) - 2\frac{\xi\rho_1'(\xi)}{\rho_1(\xi)}f(\xi) - 2\xi f'(\xi).
\end{aligned} \tag{5.120}$$

Here we again used the representation (5.9) for Dirac measure. It follows that

$$\begin{aligned}\left(\mathcal{K}^1 f\right)(\eta) =& \left\langle \int_0^\infty f(\xi)[R\partial_R - 2\xi\partial_\xi]\tilde{\phi}(R,\xi)\rho_1(\xi)d\xi, \tilde{\phi}(R,\eta)\right\rangle_{L^2_{RdR}} \\ &- 2\left(1+\frac{\eta\rho_1'(\eta)}{\rho_1(\eta)}\right)f(\eta).\end{aligned} \tag{5.121}$$

To extract any $\delta(\xi-\eta)$-contribution from the RdR-integral above, we recall from (4.29), (4.31) and (4.32) (Here we adopt the notations from these equations):

$$\begin{aligned}\tilde{\phi}(R,\xi) =& 2R^{-\frac{1}{2}}\mathrm{Re}\ (a(\xi)\psi(R,\xi)) \\ =& 2\mathrm{Re}\left(-e^{-\frac{\pi i}{4}}\frac{\pi}{4i\xi^{\frac{3}{4}}}R^{-\frac{1}{2}}e^{iR\xi^{\frac{1}{2}}}\left(\sqrt{\frac{2}{\pi}}+\frac{a_1}{R\xi^{\frac{1}{2}}}+O((R^2\xi)^{-1})\right)\right).\end{aligned} \tag{5.122}$$

Here $O((R^2\xi)^{-1})$ obeys the "symbol-type" behavior upon differentiation. In view of (5.15), applying the differential operator $R\partial_R - 2\xi\partial_\xi$ to (5.122) yields

$$\begin{aligned}(R\partial_R - 2\xi\partial_\xi)\tilde{\phi}(R,\xi) =& -2\mathrm{Re}\left(e^{-\frac{\pi i}{4}}\frac{3\pi}{8i\xi^{\frac{3}{4}}}R^{-\frac{1}{2}}e^{iR\xi^{\frac{1}{2}}}\left(\sqrt{\frac{2}{\pi}}+\frac{a_1}{R\xi^{\frac{1}{2}}}+O((R^2\xi)^{-1})\right)\right) \\ &+\mathrm{Re}\left(e^{-\frac{\pi i}{4}}\frac{\pi}{4i\xi^{\frac{3}{4}}}R^{-\frac{1}{2}}e^{iR\xi^{\frac{1}{2}}}\left(\sqrt{\frac{2}{\pi}}+\frac{a_1}{R\xi^{\frac{1}{2}}}+O((R^2\xi)^{-1})\right)\right) \\ =& -\mathrm{Re}\left(e^{-\frac{\pi i}{4}}\frac{\pi}{2i\xi^{\frac{3}{4}}}R^{-\frac{1}{2}}e^{iR\xi^{\frac{1}{2}}}\left(\sqrt{\frac{2}{\pi}}+\frac{a_1}{R\xi^{\frac{1}{2}}}+O((R^2\xi)^{-1})\right)\right)\end{aligned} \tag{5.123}$$

Now we again use the fact $\mathrm{Re}\, z \mathrm{Re}\, w = \frac{1}{2}(\mathrm{Re}\,(zw) + \mathrm{Re}\,(z\overline{w}))$ to find the δ measure on the diagonal in the integral

$$\lim_{A\to\infty}\int_0^A [R\partial_R - 2\xi\partial_\xi]\tilde{\phi}(R,\xi)\tilde{\phi}(R,\eta)RdR.$$

The contribution from the δ measure is given by

$$\lim_{L\to\infty}\frac{\pi^2}{8}\cdot\frac{2}{\pi}\xi^{-\frac{3}{4}}\eta^{-\frac{3}{4}}\mathrm{Re}\int_0^\infty e^{iR(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}\chi_1(R)\chi_2(R/L)dR := I(\xi,\eta;1). \tag{5.124}$$

Here the cutoff functions χ_1, χ_2 are the same as in (5.16). The contribution from the profile $O((R^2\xi)^{-1})$ in (5.122)-(5.123) is integrable in R, therefore is not part of the δ measure. The contribution from $\frac{a_1}{R\xi^{\frac{1}{2}}}$ in (5.122)-(5.123) gives an integral of the form (5.18), therefore is not part of the δ measure either. Then a standard calculation gives

$$I(\xi,\eta;1) = \frac{\pi^2}{4}\xi^{-\frac{3}{4}}\eta^{-\frac{3}{4}}\delta(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}) = \frac{\pi^2}{2}\xi^{-\frac{1}{4}}\eta^{-\frac{3}{4}}\delta(\xi-\eta),$$

which in term gives the diagonal contribution

$$\left\langle \int_0^\infty f(\xi)[R\partial_R - 2\xi\partial_\xi]\tilde{\phi}(R,\xi)\rho_1(\xi)d\xi, \tilde{\phi}(R,\eta)\right\rangle_{L^2_{RdR}}$$

$$= \left\langle \frac{\pi^2}{2} \xi^{-\frac{1}{4}} \eta^{-\frac{3}{4}} \delta(\xi - \eta), f(\xi)\rho_1(\xi) \right\rangle_{d\xi} = f(\eta) = \frac{\eta \rho_1'(\eta)}{\rho_1(\eta)} f(\eta).$$

This gives the desired representation for the diagonal part of $\mathcal{K}^1$. For the off-diagonal part, we proceed similarly as in Proposition 5.1 to obtain that the kernel of the off-diagonal part of $\mathcal{K}^1$ is given by

$$F(\xi, \eta; 1) = \int_0^\infty W_1(R)\phi(R, \xi)\phi(R, \eta) R dR$$

where

$$W_1(R) := [H_1^+, R\partial_R] - 2H_1^+.$$

But a direct computation shows that $[H_1^+, R\partial_R] - 2H_1^+ = 0$. Therefore the off-diagonal part vanishes and the proof is completed. □

5.3. **Transference operator at angular momentum** $n = -1$. The operator $\mathcal{K}_{-1}$ is defined as

$$\widehat{R\partial_R u} = -2\xi \partial_\xi \widehat{u} + \mathcal{K}_{-1} \widehat{u}. \tag{5.125}$$

In this section, the Fourier transform is taken with respect to the Fourier basis $\phi_{-1}(R, \xi) = \xi^{-1} \mathcal{D}_- \left(R^{-\frac{1}{2}} \Phi^{-1}(R, \xi) \right)$ where $\Phi^{-1}(R, \xi)$ is constructed in Proposition 4.3. Let $\rho_{-1}(d\xi)$ be the spectral measure associated with $\Phi^{-1}(R, \xi)$. Let $f \in L^2(RdR)$ as in Lemma 4.6 and recall from (4.61),

$$\mathcal{D}_- f(R) = \int_0^\infty x_{-1}(\xi) \phi_{-1}(R, \xi) \widetilde{\rho}_{-1}(d\xi), \quad x_{-1}(\xi) = \langle \mathcal{D}_- f, \phi_{-1}(R, \xi) \rangle_{L^2_{RdR}},$$

where $\widetilde{\rho}_{-1}(d\xi) = \xi \rho_{-1}(d\xi)$. Moreover, Proposition 4.7 gives the Plancherel identity:

$$\|\mathcal{D}_- f\|^2_{L^2(RdR)} = \|\langle \mathcal{D}_- f, \phi_{-1}(R, \xi) \rangle\|^2_{L^2(\widetilde{\rho}_{-1})}.$$

The main result of this section is

Proposition 5.4. *The operator $\mathcal{K}_{-1}$ is given by*

$$\mathcal{K}_{-1} f(\xi) = -\left(2f(\eta) + \frac{\eta \widetilde{\rho}'_{-1}(\eta)}{\widetilde{\rho}_{-1}(\eta)} f(\eta) \right) \delta(\eta - \xi) + \mathcal{K}^0_{-1} f(\xi)$$

where the off-diagonal part $\mathcal{K}^0_{-1}$ has a kernel $K_0(\xi, \eta; -1)$ given by

$$K_0(\xi, \eta; -1) = \frac{\widetilde{\rho}_{-1}(\eta)}{\xi - \eta} F_{-1}(\xi, \eta) \tag{5.126}$$

where the symmetric function $F_{-1}(\xi, \eta)$ satisfies (for $\xi \le \eta$, and any $N \in \mathbb{N}$)

$$\begin{aligned} |F_{-1}(\xi, \eta)| \lesssim & \left(\frac{\langle \xi \rangle}{\langle \eta \rangle} \right)^N \left(\langle \eta \rangle^{-1} \cdot \frac{\min\{\xi^{\frac{1}{2}}, \xi^{-2}\}}{\eta^{\frac{1}{2}}} + \langle \eta \rangle^{-1} \min\{1, \eta^{-\frac{3}{4}}\} \right) \\ & + \langle \xi \rangle^{-1} \langle \eta \rangle^{-1} \eta^{-\frac{3}{4}} \cdot \min\{\xi^{\frac{3}{4}}, \xi^{-\frac{3}{4}}\} \left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}} \right)^N := \Gamma_{-1}. \end{aligned} \tag{5.127}$$

For the derivatives of $F_{-1}(\xi,\eta)$, we have

$$\begin{aligned}&\left|\partial_{\xi^{\frac12}}F_{-1}(\xi,\eta)\right|\lesssim\left(1+\xi^{-\frac12}\right)\cdot\Gamma_{-1},\quad\left|\partial_{\eta^{\frac12}}F_{-1}(\xi,\eta)\right|\lesssim\frac{\Gamma_{-1}}{\eta^{\frac12}}\left(1+\xi^{\frac12}\right),\\&\left|\partial^2_{\xi^{\frac12}}F_{-1}(\xi,\eta)\right|\lesssim\left(1+\xi^{-1}\right)\cdot\Gamma_{-1},\quad\left|\partial^2_{\eta^{\frac12}}F_{-1}(\xi,\eta)\right|\lesssim\frac{\Gamma_{-1}}{\eta}\cdot(1+\xi),\\&\left|\partial_{\xi^{\frac12}}\partial_{\eta^{\frac12}}F_{-1}(\xi,\eta)\right|\lesssim\frac{\Gamma_{-1}}{\eta^{\frac12}}\left(\xi^{\frac12}+\xi^{-\frac12}\right).\end{aligned}\tag{5.128}$$

Proof. Similar to the proof of Proposition 5.1 and Proposition 5.3, for a function $f\in C_0^\infty((0,\infty))$, we define $u(R)=\int_0^\infty\phi_{-1}(R,\xi)f(\xi)\rho_{-1}(\xi)d\xi$. A computation similar to (5.8) and (5.10) gives

$$\begin{aligned}(\mathcal{K}_{-1}f)(\eta)=&\left\langle\int_0^\infty f(\xi)[R\partial_R-2\xi\partial_\xi]\phi_{-1}(R,\xi)\widetilde{\rho}_{-1}(\xi)d\xi,\phi_{-1}(R,\eta)\right\rangle_{L^2_{RdR}}\\&-2\left(1+\frac{\eta\widetilde{\rho}'_{-1}(\eta)}{\widetilde{\rho}_{-1}(\eta)}\right)f(\eta).\end{aligned}\tag{5.129}$$

To extract the δ measure, we recall from Proposition 4.4, Proposition 4.5 and (4.17):

$$\begin{aligned}\phi_{-1}(R,\xi)=&\xi^{-1}\mathcal{D}_-\left(2R^{-\frac12}\mathrm{Re}\left(a_{-1}(\xi)\Psi^+_{-1}(R,\xi)\right)\right)\\=&2\xi^{-1}\mathcal{D}_-\left(\mathrm{Re}\left(R^{-\frac12}a_{-1}(\xi)\xi^{-\frac14}e^{iR\xi^{\frac12}}\left(1-\frac{i}{8R\xi^{\frac12}}+\frac{1}{R\xi^{\frac12}}O\left(\frac{1}{1+R^2}\right)+O\left(\frac{1}{R^2\xi}\right)\right)\right)\right)\\=&2\mathrm{Re}\left(R^{-\frac12}i\xi^{-\frac34}a_{-1}(\xi)e^{iR\xi^{\frac12}}\left(1+O\left(\frac{1}{R\xi^{\frac12}}\right)+O\left(\frac{1}{R^2}\right)\right)\right)\end{aligned}\tag{5.130}$$

Again the profiles $O\left(\frac{1}{R\xi^{\frac12}}\right)$ and $O\left(\frac{1}{R^2}\right)$ obey "symbol-type" behavior upon differentiation. Applying the differential operator $R\partial_R-2\xi\partial_\xi$ to (5.130) yields

$$\begin{aligned}&(R\partial_R-2\xi\partial_\xi)\phi_{-1}(R,\xi)\\=&-R^{-\frac12}\mathrm{Re}\left(a_{-1}(\xi)i\xi^{-\frac34}e^{iR\xi^{\frac12}}\left(1+O\left(\frac{1}{R\xi^{\frac12}}\right)+O\left(\frac{1}{R^2}\right)\right)\right)\\&+\mathrm{Re}\left(R^{-\frac12}\left(3i\xi^{-\frac34}a_{-1}(\xi)-4i\xi^{\frac14}a'_{-1}(\xi)\right)e^{iR\xi^{\frac12}}\left(1+O\left(\frac{1}{R\xi^{\frac12}}\right)+O\left(\frac{1}{R^2}\right)\right)\right)\\=&\mathrm{Re}\left(R^{-\frac12}\left(2i\xi^{-\frac34}a_{-1}(\xi)-4i\xi^{\frac14}a'_{-1}(\xi)\right)e^{iR\xi^{\frac12}}\left(1+O\left(\frac{1}{R\xi^{\frac12}}\right)+O\left(\frac{1}{R^2}\right)\right)\right)\end{aligned}\tag{5.131}$$

We again use the fact $\mathrm{Re}\,z\mathrm{Re}\,w=\frac12(\mathrm{Re}\,(zw)+\mathrm{Re}\,(z\overline{w})$ to find the δ measure on the diagonal in the integral

$$\lim_{A\to\infty}\int_0^A[R\partial_R-2\xi\partial_\xi]\phi_{-1}(R,\xi)\phi_{-1}(R,\eta)RdR.$$

As in the proof for Proposition 5.1 and Proposition 5.4, the profiles $O\left(\frac{1}{R\xi^{\frac{1}{2}}}\right)$ and $O\left(\frac{1}{R^2}\right)$ in the parenthesis Re (...) in (5.130) and (5.131) do not contribute to the δ measure. Therefore the contribution in consideration is given by

$$\lim_{L\to\infty}\int_0^\infty \mathrm{Re}\left(\left(\left(2\xi^{-\frac{3}{4}}a_{-1}(\xi)-4\xi^{\frac{1}{4}}a_{-1}'(\xi)\right)\eta^{-\frac{3}{4}}\overline{a_{-1}(\eta)}\right)e^{iR(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}\right)\chi_1(R)\chi_2(R/L)dR := I(\xi,\eta;-1) \quad (5.132)$$

where χ_1, χ_2 is again the same as in (5.16). Now a routine calculation gives

$$\begin{aligned}I(\xi,\eta;-1) =&\pi\mathrm{Re}\left(\left(2\xi^{-\frac{3}{4}}a_{-1}(\xi)-4\xi^{\frac{1}{4}}a_{-1}'(\xi)\right)\eta^{-\frac{3}{4}}\overline{a_{-1}(\eta)}\right)\delta(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})\\ =&2\xi^{\frac{1}{2}}\pi\mathrm{Re}\left(\left(2\xi^{-\frac{3}{4}}a_{-1}(\xi)-4\xi^{\frac{1}{4}}a_{-1}'(\xi)\right)\eta^{-\frac{3}{4}}\overline{a_{-1}(\eta)}\right)\delta(\xi-\eta),\end{aligned}$$

which in turn gives the diagonal contribution

$$\begin{aligned}&\left\langle\int_0^\infty f(\xi)[R\partial_R-2\xi\partial_\xi]\phi_{-1}(R,\xi)\widetilde{\rho}_{-1}(\xi)d\xi,\phi_{-1}(R,\eta)\right\rangle_{L^2_{RdR}}\\ =&\left\langle 2\xi^{\frac{1}{2}}\pi\mathrm{Re}\left(\left(2\xi^{-\frac{3}{4}}a_{-1}(\xi)-4\xi^{\frac{1}{4}}a_{-1}'(\xi)\right)\eta^{-\frac{3}{4}}\overline{a_{-1}(\eta)}\right)\delta(\xi-\eta), f(\xi)\widetilde{\rho}_{-1}(\xi)\right\rangle_{d\xi}\\ =&2\pi\eta^{\frac{1}{2}}\mathrm{Re}\left(2\eta^{-\frac{3}{2}}|a_{-1}(\eta)|^2-4\eta^{-\frac{1}{2}}a_{-1}'(\eta)\overline{a_{-1}(\eta)}\right)f(\eta)\widetilde{\rho}_{-1}(\eta)\\ =&\left(4\pi\eta^{-1}|a_{-1}(\eta)|^2-4\pi\left(|a_{-1}(\eta)|^2\right)'\right)f(\eta)\widetilde{\rho}_{-1}(\eta)\\ =&\left(\frac{1}{\eta\rho_{-1}(\eta)}+\frac{\rho_{-1}'(\eta)}{(\rho_{-1}(\eta))^2}\right)f(\eta)\widetilde{\rho}_{-1}(\eta)=\frac{\eta\widetilde{\rho}_{-1}'(\eta)}{\widetilde{\rho}_{-1}(\eta)}f(\eta).\end{aligned}$$

This gives the desired result on the diagonal part of $\mathcal{K}_{-1}$. Here we used the relation $\rho_{-1}(\xi)=\frac{1}{4\pi}|a_{-1}(\xi)|^{-2}$ given in Proposition 4.5.

For the off-diagonal part $\mathcal{K}_{-1}^0$, a routing calculation as in the proof of Proposition 5.1 gives the expression for $F_{-1}(\xi,\eta)$:

$$F_{-1}(\xi,\eta)=\int_0^\infty W_{-1}(R)\phi_{-1}(R,\xi),\phi_{-1}(R,\eta)R\,dR,\quad W_{-1}(R)=\left[\tilde{H}_1^-,R\partial_R\right]-2\tilde{H}_1^-=\frac{16}{(1+R^2)^2}.$$

By Proposition 4.3 and $\mathcal{D}_-\left(R^{-\frac{1}{2}}\Phi_0^{-1}(R)\right)=0$, we have, for $R^2\xi\lesssim 1$

$$|\phi_{-1}(R,\xi)|\lesssim R^3\langle R^2\rangle^{-1}\lesssim\xi^{-\frac{1}{2}}\langle\xi\rangle^{-1},\quad |\partial_R\phi_{-1}(R,\xi)|\lesssim R^2\langle R^2\rangle^{-1}\lesssim\langle\xi\rangle^{-1}. \quad (5.133)$$

For $R^2\xi\gg 1$, we have, by Propositions 4.4, 4.5 and (4.60),

$$\begin{aligned}\phi_{-1}(R,\xi)=&\xi^{-1}\mathcal{D}_-\left(2\mathrm{Re}\left(a_{-1}(\xi)R^{-\frac{1}{2}}\xi^{-\frac{1}{4}}e^{iR\xi^{\frac{1}{2}}}\sigma_{-1}(R\xi^{\frac{1}{2}},R)\right)\right)\\ \simeq&\xi^{-\frac{5}{4}}\langle\xi\rangle^{-1}\mathcal{D}_-\left(\mathrm{Re}\left(R^{-\frac{1}{2}}e^{iR\xi^{\frac{1}{2}}}\left(1+O\left(R^{-1}\xi^{-\frac{1}{2}}\right)\right)\right)\right)\\ \simeq&\xi^{-\frac{3}{4}}\langle\xi\rangle^{-1}\mathrm{Re}\left(R^{-\frac{1}{2}}e^{iR\xi^{\frac{1}{2}}}\left(1+O\left(R^{-1}\xi^{-\frac{1}{2}}\right)\right)\right)\end{aligned} \quad (5.134)$$

Without loss of generality, we again assume $\xi \leq \eta$ and set $\eta^{\frac{1}{2}} = \zeta\xi^{\frac{1}{2}}$. We discuss the following three regimes: $R\eta^{\frac{1}{2}} \geq R\xi^{\frac{1}{2}} \gg 1$, $R\eta^{\frac{1}{2}} \gg 1 \gtrsim R\xi^{\frac{1}{2}}$ and $1 \gtrsim R\eta^{\frac{1}{2}} \geq R\xi^{\frac{1}{2}}$. We start with first case where both $\phi_{-1}(R,\xi)$ and $\phi_{-1}(R,\eta)$ are in the oscillatory regime. In this case we consider the integral

$$\begin{aligned} F^a_{-1}(\xi,\eta) &:= \int_0^\infty \chi_{R\eta^{\frac{1}{2}}\gg 1}\cdot\chi_{R\xi^{\frac{1}{2}}\gg 1}\cdot W_{-1}(R)\phi_{-1}(R,\xi)\phi_{-1}(R,\eta)R\,dR \\ &= \int_0^\infty \chi_{R\eta^{\frac{1}{2}}\gg 1}\chi_{R\xi^{\frac{1}{2}}\gg 1}W_{-1}(R)\xi^{-\frac{3}{4}}\langle\xi\rangle^{-1}\eta^{-\frac{3}{4}}\langle\eta\rangle^{-1} \\ &\quad\cdot \mathrm{Re}\left(e^{iR\xi^{\frac{1}{2}}}\left(1+O\left(R^{-1}\xi^{-\frac{1}{2}}\right)\right)\right)\mathrm{Re}\left(e^{iR\eta^{\frac{1}{2}}}\left(1+O\left(R^{-1}\eta^{-\frac{1}{2}}\right)\right)\right)dR. \end{aligned} \tag{5.135}$$

If we simply bound the oscillatory factors $e^{iR\xi^{\frac{1}{2}}}$ and $e^{iR\eta^{\frac{1}{2}}}$ in absolute value, then we have (taking into account the integral of $W_-(R)$)

$$\left|F^a_{-1}(\xi,\eta)\right| \lesssim \langle\xi\rangle^{-1}\langle\eta\rangle^{-1}\eta^{-\frac{3}{4}}\cdot\min\{\xi^{\frac{3}{4}},\xi^{-\frac{3}{4}}\}. \tag{5.136}$$

To gain any factor of $\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}$, we need to use integration by parts. We consider an integral of the following form. The other cases can be handled similarly:

$$\begin{aligned} &\int_0^\infty \chi_{R^2\eta\gg 1}\chi_{R^2\xi\gg 1}W_{-1}(R)\xi^{-\frac{3}{4}}\langle\xi\rangle^{-1}\eta^{-\frac{3}{4}}\langle\eta\rangle^{-1}\cdot e^{iR\xi^{\frac{1}{2}}}\left(1+O\left(R^{-1}\xi^{-\frac{1}{2}}\right)\right) \\ &\qquad\cdot\frac{\partial_R(e^{iR\eta^{\frac{1}{2}}})}{i\eta^{\frac{1}{2}}}\left(1+O\left(R^{-1}\eta^{-\frac{1}{2}}\right)\right)dR. \end{aligned} \tag{5.137}$$

Upon integration by parts, no matter ∂_R hits $W_-(R)$ or $O(R^{-1}\xi^{-\frac{1}{2}})$, we gain a factor of $R^{-1} \lesssim \xi^{\frac{1}{2}}$. Therefore $F^a_{-1}(\xi,\eta)$ can be also bounded by

$$|F^a_{-1}(\xi,\eta)| \lesssim \langle\xi\rangle^{-1}\langle\eta\rangle^{-1}\eta^{-\frac{3}{4}}\cdot\min\{\xi^{\frac{3}{4}},\xi^{-\frac{3}{4}}\}\left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^N, \quad \text{for any} \quad N>0. \tag{5.138}$$

Next we turn to the regime $R\eta^{\frac{1}{2}} \gg 1 \gtrsim R\xi^{\frac{1}{2}}$, in which we consider the integral

$$\begin{aligned} F^b_{-1}(\xi,\eta) &:= \int_0^\infty \chi_{R^2\eta\gg 1}\chi_{R^2\xi\lesssim 1}W_{-1}(R)\phi_{-1}(R,\xi)\phi_{-1}(R,\eta)R\,dR \\ &= \int_0^\infty \chi_{R^2\eta\gg 1}\chi_{R^2\xi\lesssim 1}W_{-1}(R)\eta^{-\frac{3}{4}}\langle\eta\rangle^{-1}R^{\frac{1}{2}} \\ &\quad\cdot\phi_{-1}(R,\xi)\mathrm{Re}\left(e^{iR\eta^{\frac{1}{2}}}\left(1+O\left(R^{-1}\eta^{-\frac{1}{2}}\right)\right)\right)dR. \end{aligned} \tag{5.139}$$

By (5.133) this integral is bounded by

$$|F^b_{-1}(\xi,\eta)| \lesssim \langle\eta\rangle^{-1}\min\{1,\eta^{-\frac{3}{4}}\}. \tag{5.140}$$

Finally we consider the regime $0 < R\xi^{\frac{1}{2}} \leq R\eta^{\frac{1}{2}} \lesssim 1$. The integral in consideration now is given by

$$F^{c}_{-1}(\xi,\eta) := \int_0^\infty \chi_{R^2\eta\lesssim 1}\chi_{R^2\xi\lesssim 1} W_{-1}(R)\phi_{-1}(R,\xi)\phi_{-1}(R,\eta)R\,dR. \tag{5.141}$$

In view of (5.133), this can be bounded by

$$\left|F^{c}_{-1}(\xi,\eta)\right| \lesssim \left|\int_0^\infty \chi_{R^2\eta\lesssim 1}\chi_{R^2\xi\lesssim 1} W_{-1}(R)R^3\langle R^2\rangle^{-1}\eta^{-\frac{1}{2}}\langle\eta\rangle^{-1}R\,dR\right| \lesssim \langle\eta\rangle^{-1}\cdot\frac{\min\{\xi^{\frac{1}{2}},\xi^{-2}\}}{\eta^{\frac{1}{2}}}. \tag{5.142}$$

To obtain the rapid decay factor $\left(\frac{\langle\xi\rangle}{\langle\eta\rangle}\right)^N$ for any $N > 0$, we consider

$$\eta\int_0^\infty \chi_{R\xi^{\frac{1}{2}}\ll 1} W_{-1}(R)\phi_{-1}(R,\xi)\phi_{-1}(R,\eta)R\,dR = -\int_0^\infty \chi_{R\xi^{\frac{1}{2}}\ll 1} W_{-1}(R)\phi_{-1}(R,\xi)H^{+}_{-1}\left(\phi_{-1}(R,\eta)\right)R\,dR,$$

and use an argument similar to the proof of Proposition 5.1.

Next we turn to the derivatives of $F_{-1}(\xi,\eta)$. By Proposition 4.3 and (4.60), if $R\xi^{\frac{1}{2}} \lesssim 1$, then upon a differentiation in $\xi^{\frac{1}{2}}$ for $\phi_{-1}(R,\xi)$, we gain an extra factor of $\xi^{-\frac{1}{2}}$. This of course also holds for $\phi_{-1}(R,\eta)$ when $R\eta^{\frac{1}{2}} \lesssim 1$. Therefore the desired estimates hold for $R\xi^{\frac{1}{2}} \lesssim 1$ and $R\eta^{\frac{1}{2}} \lesssim 1$. If $R\eta^{\frac{1}{2}} \gg 1$, we use Propositions 4.4 and 4.5, as well as (4.60), to obtain that compared to the undifferentiated basis, we gain a factor of R upon a differentiation in $\eta^{\frac{1}{2}}$. In the estimates for the derivatives of $F_{-1}(\xi,\eta)$, this extra factor of R is absorbed into the function $W_{-1}(R)$. Therefore if $\eta \ll 1$, we simply obtain a factor of $\eta^{-\frac{1}{2}}$. If $\eta \gtrsim 1$ and $R\xi^{\frac{1}{2}} \lesssim 1$, we use (5.106)and integration by parts in R to obtain the desired estimates. The case when $1 \ll R\xi^{\frac{1}{2}} \leq R\eta^{\frac{1}{2}}$ is more delicate. First, for $\partial_{\eta^{\frac{1}{2}}}F_{-1}(\xi,\eta)$ and $\partial^2_{\eta^{\frac{1}{2}}}F_{-1}(\xi,\eta)$, we again use (5.106) and the above integration by parts argument to obtain:

$$\left|\partial^k_{\eta^{\frac{1}{2}}}F_{-1}(\xi,\eta)\right| \lesssim \max\{1,\xi^{-k}\}\cdot\Gamma_{-1}\cdot\left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^k, \quad \text{for} \quad k = 1,2, \quad \text{and} \quad 1 \ll R\xi^{\frac{1}{2}} \leq R\eta^{\frac{1}{2}}. \tag{5.143}$$

Here Γ_{-1} is the bound for the undifferentiated $F_{-1}(\xi,\eta)$ in the same regime. For the derivatives in $\xi^{\frac{1}{2}}$, we again distinguish the cases when $\xi \ll 1$ and $\xi \gtrsim 1$. If $\xi \ll 1$, we simply obtain a factor of $\xi^{-\frac{1}{2}}$ upon a differentiation in $\xi^{\frac{1}{2}}$. If $\xi \gtrsim 1$, then we obtain the same bound as the undifferentiated one. Therefore we have

$$\begin{aligned}&\left|\partial_{\xi^{\frac{1}{2}}}F_{-1}(\xi,\eta)\right| \lesssim \Gamma_{-1}, \quad \left|\partial^2_{\xi^{\frac{1}{2}}}F_{-1}(\xi,\eta)\right| \lesssim \Gamma_{-1},\\ &\left|\partial_{\xi^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}}F_{-1}(\xi,\eta)\right| \lesssim \max\{1,\xi^{-1}\}\cdot\Gamma_{-1}\cdot\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}} \quad \text{for} \quad 1 \ll R\xi^{\frac{1}{2}} \leq R\eta^{\frac{1}{2}}.\end{aligned} \tag{5.144}$$

This completes the proof. □

5.4. **Transference operator at angular momentum $n = 0$.** The operator $\mathcal{K}$ is defined as

$$\widehat{R\partial_R u} = -2\xi\partial_\xi\widehat{u} + \mathcal{K}\widehat{u}. \tag{5.145}$$

In this section, the Fourier transform is taken with respect to the Fourier basis (See (4.1) in [16])

$$\phi_0(R,\xi) := \xi^{-1}\mathcal{D}_0\left(R^{-\frac{1}{2}}\phi_{KST}(R,\xi)\right) \tag{5.146}$$

Here $\phi_{KST}(R,\xi)$ (See [18]) is the Fourier basis associated to the operator

$$\mathcal{L}_{KST} := -\partial_R^2 + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2}$$

with respect to the measure dR. The operator $\mathcal{D}_0$ (See [16]) is given by

$$\mathcal{D}_0 := \partial_R + \frac{1}{R} - \frac{2}{R(R^2+1)}.$$

As in [16], we have

Proposition 5.5. *The operator $\mathcal{K}$ is given by, with $\widetilde{\rho}_0(\xi)$ being the spectral measure density associated to the Fourier basis $\phi_0(R,\xi)$,*

$$\mathcal{K}f(\xi) = -2f(\xi) - \frac{\widetilde{\rho}_0'(\xi)\xi}{\widetilde{\rho}_0(\xi)}\cdot f(\xi) + \mathcal{K}_0 f(\xi)$$

where the off-diagonal operator $\mathcal{K}_0$ is an integral operator with kernel

$$\frac{\widetilde{\rho}_0(\eta)F_0(\xi,\eta)}{\xi-\eta}.$$

The symmetric C^2 function $F_0(\xi,\eta)$ satisfies the bounds

$$\begin{aligned}|F_0(\xi,\eta)| \lesssim \min\{\xi^{\frac{3}{2}},1\}\xi^{-\frac{1}{4}}\log(1+\xi^{-1})\eta^{-\frac{1}{4}}\log(1+\eta^{-1})\cdot\left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^N \\ +\frac{\eta\left(\log(1+\eta^{-1})\right)^2}{\langle\eta\rangle^{N+\frac{1}{4}}} + \langle\eta\rangle^{-3}\left(\frac{\langle\xi\rangle}{\langle\eta\rangle}\right)^N := \Gamma_0.\end{aligned} \tag{5.147}$$

For the derivatives of $F_0(\xi,\eta)$, we have

$$\begin{aligned}&\left|\partial_{\xi^{\frac{1}{2}}}F_0(\xi,\eta)\right| \lesssim \left(1+\xi^{-\frac{1}{2}}\right)\cdot\Gamma_0, \quad \left|\partial_{\eta^{\frac{1}{2}}}F_0(\xi,\eta)\right| \lesssim \frac{\Gamma_0}{\eta^{\frac{1}{2}}}\left(1+\xi^{\frac{1}{2}}\right),\\ &\left|\partial^2_{\xi^{\frac{1}{2}}}F_0(\xi,\eta)\right| \lesssim \left(1+\xi^{-1}\right)\cdot\Gamma_0, \quad \left|\partial^2_{\eta^{\frac{1}{2}}}F_0(\xi,\eta)\right| \lesssim \frac{\Gamma_0}{\eta}\cdot(1+\xi),\\ &\left|\partial_{\xi^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}}F_0(\xi,\eta)\right| \lesssim \frac{\Gamma_0}{\eta^{\frac{1}{2}}}\left(\xi^{\frac{1}{2}}+\xi^{-\frac{1}{2}}\right).\end{aligned} \tag{5.148}$$

Proof. The diagonal part of the operator $\mathcal{K}$ was calculated in [16]. Here we strive to derive a more precise bound on the off-diagonal kernel $F_0(\xi,\eta)$. A routing calculation gives (See (4.17) in [16])

$$F_0(\xi,\eta) := \int_0^\infty W_0(R)\phi_0(R,\xi)\phi_0(R,\eta)R\,dR, \quad W_0(R) := \frac{16}{(1+R^2)^2} - \frac{32}{(1+R^2)^3}. \tag{5.149}$$

Again in view of the bounds derived in [16], we have

$$|\phi_0(R,\xi)| \lesssim \log(1+R^2) \quad \text{for} \quad R^2\xi \lesssim 1. \tag{5.150}$$

and

$$\phi_0(R,\xi) = R^{-\frac{1}{2}}\xi^{-\frac{1}{4}}\log\left(1+\xi^{-1}\right)\mathrm{Re}\left(e^{iR\xi^{\frac{1}{2}}}\left(1+O\left(R^{-1}\xi^{-\frac{1}{2}}\right)\right)\right) \quad \text{for} \quad R\xi^{\frac{1}{2}} \gg 1. \tag{5.151}$$

Again we split into the following three regimes: $R\eta^{\frac{1}{2}} \geq R\xi^{\frac{1}{2}} \gg 1, R\eta^{\frac{1}{2}} \gg 1 \gtrsim R\xi^{\frac{1}{2}}$ and $1 \gtrsim R\eta^{\frac{1}{2}} \geq R\xi^{\frac{1}{2}}$. We start with the case when both $\phi_0(R,\xi)$ and $\phi_0(R,\eta)$ are in the oscillatory regime:

$$\begin{aligned} F_0^a(\xi,\eta) &:= \int_0^\infty \chi_{R\eta^{\frac{1}{2}}\gg 1}\chi_{R\xi^{\frac{1}{2}}\gg 1} W_0(R)\phi_0(R,\xi)\phi_0(R,\eta)R\,dR \\ &= \int_0^\infty \chi_{R\eta^{\frac{1}{2}}\gg 1}\chi_{R\xi^{\frac{1}{2}}\gg 1} W_0(R)\xi^{-\frac{1}{4}}\log\left(1+\xi^{-1}\right)\eta^{-\frac{1}{4}}\log\left(1+\eta^{-1}\right) \\ &\quad \cdot \mathrm{Re}\left(e^{iR\xi^{\frac{1}{2}}}\left(1+O\left(R^{-1}\xi^{-\frac{1}{2}}\right)\right)\right)\mathrm{Re}\left(e^{iR\eta^{\frac{1}{2}}}\left(1+O\left(R^{-1}\eta^{-\frac{1}{2}}\right)\right)\right)dR. \end{aligned} \tag{5.152}$$

If we simply bound the oscillatory factors in absolute value, then we have

$$\left|F_0^a(\xi,\eta)\right| \lesssim \min\{\xi^{\frac{3}{2}},1\}\cdot\xi^{-\frac{1}{4}}\log\left(1+\xi^{-1}\right)\eta^{-\frac{1}{4}}\log\left(1+\eta^{-1}\right)$$

Writing $e^{iR\eta^{\frac{1}{2}}} = \frac{1}{i\eta^{\frac{1}{2}}}\partial_R\left(e^{iR\eta^{\frac{1}{2}}}\right)$, we can do integration by parts. No matter ∂_R hits $W_0(R)$ or $O(R^{-1}\xi^{-\frac{1}{2}})$, we gain a factor of $R^{-1} \lesssim \xi^{\frac{1}{2}}$. Therefore $F_0^a(\xi,\eta)$ can be bounded by

$$\left|F_0^a(\xi,\eta)\right| \lesssim \min\{\xi^{\frac{3}{2}},1\}\cdot\xi^{-\frac{1}{4}}\log\left(1+\xi^{-1}\right)\eta^{-\frac{1}{4}}\log\left(1+\eta^{-1}\right)\cdot\left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^N, \quad \text{for any} \quad N>0. \tag{5.153}$$

Next we turn to the regime $R\eta^{\frac{1}{2}} \gg 1 \gtrsim R\xi^{\frac{1}{2}}$:

$$\begin{aligned} F_0^b(\xi,\eta) &:= \int_0^\infty \chi_{R\eta^{\frac{1}{2}}\gg 1}\chi_{R\xi^{\frac{1}{2}}\lesssim 1} W_0(R)\phi_0(R,\xi)\phi_0(R,\eta)R\,dR \\ &= \int_0^\infty \chi_{R\eta^{\frac{1}{2}}\gg 1}\chi_{R\xi^{\frac{1}{2}}\lesssim 1} W_0(R)R^{\frac{1}{2}}\log(1+R^2)\eta^{-\frac{1}{4}}\log(1+\eta^{-1})\mathrm{Re}\left(e^{iR\eta^{\frac{1}{2}}}\left(1+O\left(R^{-1}\eta^{-\frac{1}{2}}\right)\right)\right)dR, \end{aligned} \tag{5.154}$$

which is bounded by, upon integration by parts

$$\left|F_0^b(\xi,\eta)\right| \lesssim \frac{\eta\left(\log\left(1+\eta^{-1}\right)\right)^2}{\langle\eta\rangle^{\frac{1}{4}}}\langle\eta\rangle^{-N}, \quad \text{for any} \quad N>0. \tag{5.155}$$

Finally we consider the regime $0 < R\xi^{\frac{1}{2}} \leq R\eta^{\frac{1}{2}} \lesssim 1$, and we have

$$\left|F_0^c(\xi,\eta)\right| \lesssim \left|\int_0^\infty \chi_{R^2\eta\lesssim 1}\chi_{R^2\xi\lesssim 1}W_0(R)\phi_0(R,\xi)\phi_0(R,\eta)R\,dR\right| \lesssim \langle\eta\rangle^{-3}. \tag{5.156}$$

To gain the rapid decay $\left(\frac{\langle\xi\rangle}{\langle\eta\rangle}\right)^N$ for any $N>0$, we consider

$$\eta\int_0^\infty \chi_{R\xi^{\frac{1}{2}}\ll 1}W_0(R)\phi_0(R,\xi)\phi_0(R,\eta)R\,dR = -\int_0^\infty \chi_{R\xi^{\frac{1}{2}}\ll 1}W_0(R)\phi_0(R,\xi)H_0\left(\phi_0(R,\eta)\right)R\,dR,$$

and then a routing argument gives the desired estimate.

The estimates on the derivatives of $F_0(\xi,\eta)$ are derived similarly as those for $F_{-1}(\xi,\eta)$. □

6. Estimates for smooth sources for $n \geq 2$

6.1. **Precise equations-Revisit.** In this section, we derive the precise equations for each angular mode and their counterpart on Fourier side with respect to the R-variable. We start by taking the Fourier transform for the equation (3.59) with respect to the angular variable. To this end, we (formally) expand $\mathfrak{N}(\varphi_\ell), \ell = 1,2$ (recall the notation in (3.60)) into Fourier series

$$\mathfrak{N}(\varphi_\ell)(t,R,\theta) = \sum_n \widehat{\mathfrak{N}}(\varphi_\ell)(n,t,R)e^{in\theta}, \quad \widehat{\mathfrak{N}}(\varphi_\ell)(n,t,R) := \int_0^{2\pi} \mathfrak{N}(\varphi_\ell)(t,R,\theta)e^{-in\theta}\frac{d\theta}{2\pi}, \quad n\in\mathbb{Z}, \quad \ell = 1,2 \tag{6.1}$$

Then we have the Fourier counterpart of the equation (3.59), with $\varepsilon_\pm(\tau,R,\theta) = \varphi_1(\tau,R,\theta) \mp i\varphi_2(\tau,R,\theta)$:

$$-\left(\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 + \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\right)\varepsilon_\pm(n) + H_n^\pm\varepsilon_\pm(n) = F_\pm(n), \tag{6.2}$$

where the source term admits a decomposition into the following pieces:

$$\begin{aligned} F_\pm(n) &= \lambda^{-2}N_\pm(n) - 2\frac{\sin[2Q+\epsilon]\sin\epsilon}{R^2}\varepsilon_\pm(n) - 4\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}i\varepsilon_{\pm,\theta}(n) \\ &\pm i\left(\frac{2\partial_R\epsilon}{1+R^2} + (\partial_R\epsilon)^2 - \frac{\lambda_\tau}{\lambda}\frac{8R}{1+R^2}\left(\partial_\tau\epsilon + \frac{\lambda_\tau}{\lambda}R\partial_R\epsilon\right) - \left(\partial_\tau\epsilon + \frac{\lambda_\tau}{\lambda}R\partial_R\epsilon\right)^2\right)\varphi_2(n) \\ &\mp \frac{(1+\nu)^2}{\nu^2\tau^2}\cdot\frac{4R^2}{1+2R^2+R^4}i\varphi_2(n), \end{aligned} \tag{6.3}$$

and where the first component of $F_\pm(n)$ is the angular momentum n projection (which we shall denote by Π_n) of $\lambda^{-2}N(\varphi_1) \mp i\lambda^{-2}N(\varphi_2)$:

$$\begin{aligned} \lambda^{-2}N_\pm(n) &= \Pi_n(\mathcal{P}\varepsilon_\pm) \\ &+ \Pi_n\left(\frac{2}{\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\left[U_R\sum_{j=1}^2\varphi_j\varphi_{j,R} - \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)U\sum_{j=1}^2\varphi_j\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\varphi_j\right]\right) \\ &\mp \Pi_n\left(\frac{2\sin U}{R^2\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\sum_{j=1}^2\varphi_j\varphi_{j,\theta}\right) \\ \mathcal{P} &= \left|\Psi_t\right|^2 - \left|\Phi_t\right|^2 - \left|\nabla\Psi\right|^2 + \left|\nabla\Phi\right|^2, \end{aligned} \tag{6.4}$$

and where the fine structure of the last term $\mathcal{P}$ is given in (3.49).

Now we derive the equations in (τ, ξ)-variable for $|n| \geq 2$. We first introduce the following notations:

$$\begin{aligned}\overline{x}^{\hbar}(\tau,\xi) :=& \mathcal{F}^{\hbar}(\varepsilon_+(\tau,R,n)) := \int_0^\infty \phi(R,\xi;\hbar)\varepsilon_+(\tau,R,n)R\,dR,\\ \overline{x}^{\widetilde{\hbar}}(\tau,\xi) :=& \mathcal{F}^{\widetilde{\hbar}}(\varepsilon_-(\tau,R,-n)) := \int_0^\infty \phi(R,\xi;\widetilde{\hbar})\varepsilon_-(\tau,R,-n)R\,dR.\end{aligned} \tag{6.5}$$

Here $\phi(R,\xi;\hbar) := R^{-\frac{1}{2}}\phi_n(R,\xi)$ where $\phi_n(R,\xi)$ are introduced in Proposition 4.32 for $n \gg 1$, and $\phi(R,\xi;\hbar)$: $R^{-\frac{1}{2}}w_n(\xi)\phi(R,\xi)$ where $w_n(\xi)\phi(R,\xi)$ are introduced in Proposition 4.37 for $2 \leq n \leq N_0$. Similarly, $\phi(R,\xi;\widetilde{\hbar}) := R^{-\frac{1}{2}}\phi_{-n}(R,\xi)$ where $\phi_{-n}(R,\xi)$ are introduced in Proposition 4.45 for $n \gg 1$, and $\phi(R,\xi;\widetilde{\hbar}) := R^{-\frac{1}{2}}\phi_{-n}(R,\xi)$ where $\phi_{-n}(R,\xi)$ are introduced in Proposition 4.39 for $2 \leq n \leq N_0$.

Applying the Fourier transform $\mathcal{F}^{\hbar}$ to both sides of the "+"-component of (3.61), we obtain

$$-\left(\partial_\tau - 2\frac{\lambda'(\tau)}{\lambda(\tau)}\xi\partial_\xi + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_\hbar\right)^2\overline{x}^{\hbar} - \frac{\lambda'(\tau)}{\lambda(\tau)}\left(\partial_\tau - 2\frac{\lambda'(\tau)}{\lambda(\tau)}\xi\partial_\xi + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_\hbar\right)\overline{x}^{\hbar} - \xi\overline{x}^{\hbar} = \mathcal{F}^{\hbar}(F_+). \tag{6.6}$$

By the definition of $\mathcal{K}_\hbar^{(0)}$, the above equation becomes

$$\begin{aligned}&-\left(\mathcal{D}_\tau^2 + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{D}_\tau + \xi\right)\overline{x}^{\hbar}\\ =&\mathcal{F}^{\hbar}(\mathfrak{R}_+(\varepsilon)(n))\\ &+2\frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_\hbar^{(0)}\mathcal{D}_\tau\overline{x}^{\hbar} + \left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)'\mathcal{K}_\hbar^{(0)}\overline{x}^{\hbar} + \frac{\lambda'(\tau)}{\lambda(\tau)}\left[\mathcal{D}_\tau, \mathcal{K}_\hbar^{(0)}\right]\overline{x}^{\hbar} + \left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)^2\left(\left(\mathcal{K}_\hbar^{(0)}\right)^2 + \mathcal{K}_\hbar^{(0)}\right)\overline{x}^{\hbar}\\ =:&\mathcal{F}^{\hbar}(\mathfrak{R}_+(\varepsilon)(n)) + \mathcal{R}_\hbar(\overline{x}^{\hbar}, \mathcal{D}_\tau\overline{x}^{\hbar}).\end{aligned} \tag{6.7}$$

Here the operator $\mathcal{D}_\tau$ is defined as

$$\mathcal{D}_\tau := \partial_\tau - 2\frac{\lambda'(\tau)}{\lambda(\tau)}\xi\partial_\xi - \frac{\lambda'(\tau)}{\lambda(\tau)}\frac{\rho_n'(\xi)\xi}{\rho_n(\xi)} - 2\frac{\lambda'(\tau)}{\lambda(\tau)}. \tag{6.8}$$

For the second order linear operator appearing in the equation (6.7), we have, by direct calculation (see [17]),

Lemma 6.1. *Let $\mathcal{D}_\tau$ be as in* (6.8)*. If $\overline{x}(\tau,\xi)$ satisfies*

$$\left(\mathcal{D}_\tau^2 + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{D}_\tau + \xi\right)\overline{x}(\tau,\xi) = 0; \quad \overline{x}(\tau_0,\xi) = \overline{x}_0(\xi), \quad \mathcal{D}_\tau\overline{x}(\tau_0,\xi) = \overline{x}_1(\xi), \tag{6.9}$$

then $\overline{x}(\tau,\xi)$ is given by

$$\begin{aligned}\overline{x}(\tau,\xi) =& \frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}\cos\left(\lambda(\tau)\xi^{\frac{1}{2}}\int_{\tau_0}^{\tau}\lambda(u)^{-1}du\right)\overline{x}_0\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right)\\ &+\xi^{-\frac{1}{2}}\frac{\lambda(\tau)}{\lambda(\tau_0)}\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}\sin\left(\lambda(\tau)\xi^{\frac{1}{2}}\int_{\tau_0}^{\tau}\lambda(u)^{-1}du\right)\overline{x}_1\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right).\end{aligned} \tag{6.10}$$

If $\overline{x}(\tau,\xi)$ satisfies

$$\left(\mathcal{D}_\tau^2 + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{D}_\tau + \xi\right)\overline{x}(\tau,\xi) = f(\tau,\xi); \quad \overline{x}(\tau_0,\xi) = \mathcal{D}_\tau\overline{x}(\tau_0,\xi) = 0, \tag{6.11}$$

then

$$\overline{x}(\tau,\xi) = \xi^{-\frac{1}{2}} \int_{\tau_0}^{\tau} \frac{\lambda(\tau)}{\lambda(\sigma)} \frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)} \sin\left(\lambda(\tau)\xi^{\frac{1}{2}} \int_\sigma^\tau \lambda(u)^{-1}du\right) f\left(\sigma, \frac{\lambda(\tau)^2}{\lambda(\sigma)^2}\xi\right) d\sigma \tag{6.12}$$

6.2. **Spaces for smooth sources.** In this section we strive to prove the estimates on the "regular" part of $\overline{x}^\hbar(\tau,\xi)$. Here "regular" means in $H^{5+}(\mathbb{R}^2)$. We first introduce the following space:

$$\|f(\xi)\|_{S_0^\hbar} := \|(\hbar^2\xi)^{1-\frac{\delta}{2}}\langle\hbar^2\xi\rangle^{\frac{3}{2}+\delta} f(\xi)\|_{L^2_{d\xi}}. \tag{6.13}$$

Here $\delta > 0$ is a sufficiently small constant.

We have the following decay estimate for (6.10) in $\|\cdot\|_{S_0^\hbar}$:

Proposition 6.2. *Let $\overline{x}(\tau,\xi)$ be as in* (6.10) *with $\overline{x}_0(\xi)$ and $\overline{x}_1(\xi)$ satisfying*

$$\|\overline{x}_0\|_{S_0^\hbar} < \infty, \quad \|\overline{x}_1\|_{S_1^\hbar} < \infty, \quad \textit{where} \quad \|\cdot\|_{S_1^\hbar} := \|\xi^{-\frac{1}{2}}\cdot\|_{S_0^\hbar}.$$

Then we have, for some constant $c > 0$,

$$\|\overline{x}(\tau,\cdot)\|_{S_0^\hbar} \lesssim \left(\frac{\tau}{\tau_0}\right)^{-3+c\nu}\left(\|\overline{x}_0\|_{S_0^\hbar} + \|\overline{x}_1\|_{S_1^\hbar}\right),$$

where the implicit constant is independent of $\hbar$.

Proof. Without loss of generality, we may assume that $\overline{x}_1 = 0$. A direct calculation gives

$$\begin{aligned}\|(\hbar^2\xi)^{1-\frac{\delta}{2}}\langle\hbar^2\xi\rangle^{\frac{3}{2}+\delta}\overline{x}(\tau,\xi)\|_{L^2_{d\xi}} &\lesssim \left(\frac{\lambda(\tau)}{\lambda(\tau_0)}\right)^{-2+\delta}\left\|\left(\hbar^2\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right)^{1-\frac{\delta}{2}}\left\langle\hbar^2\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right\rangle^{\frac{3}{2}+\delta}\overline{x}(\tau,\xi)\right\|_{L^2_{d\xi}} \\ &\lesssim \left(\frac{\lambda(\tau)}{\lambda(\tau_0)}\right)^{-1+\delta}\|\overline{x}_0\|_{S_0^\hbar}.\end{aligned}$$

For any $\nu \in \left(0,\frac{1}{2}\right)$, we simply choose $\delta > 0$ sufficiently small such that $\delta \lesssim \nu^2$ to obtain the desired result. □

We also have the following estimate for (6.12) in $\|\cdot\|_{S_0^\hbar}$:

Proposition 6.3. *Let $\overline{x}(\tau,\xi)$ be as in* (6.12) *with f satisfying, for some constant $c > 0$,*

$$\left(\frac{\sigma}{\tau_0}\right)^{3-c\nu}\cdot\sigma\,\|f(\sigma,\cdot)\|_{S_1^\hbar} < \infty, \quad \textit{for} \quad \sigma\in[\tau_0,\tau].$$

Then for the same constant $c > 0$, we have

$$\left(\frac{\tau}{\tau_0}\right)^{3-c\nu}\cdot\|\overline{x}(\tau,\cdot)\|_{S_0^\hbar} \lesssim \sup_{\tau_0\le\sigma\le\tau}\left(\frac{\sigma}{\tau_0}\right)^{3-c\nu}\cdot\sigma\,\|f(\sigma,\cdot)\|_{S_1^\hbar}.$$

Here the implicit constant depends only on ν and c.

Proof. A direct calculation gives, for $\delta \ll \nu^2$ and some $c' \ll 1$,

$$\left(\frac{\tau}{\tau_0}\right)^{3-c\nu} \cdot \|\overline{x}(\tau,\cdot)\|_{S_0^\hbar} \lesssim \left(\frac{\tau}{\tau_0}\right)^{3-c\nu} \cdot \int_{\tau_0}^{\tau} \frac{\lambda(\tau)}{\lambda(\sigma)} \left\| \frac{(\hbar^2\xi)^{1-\frac{\delta}{2}}}{\xi^{\frac{1}{2}}} \langle \hbar^2\xi\rangle^{\frac{3}{2}+\delta} \cdot f\left(\sigma, \frac{\lambda(\tau)^2}{\lambda(\sigma)^2}\xi\right)\right\|_{L^2_{d\xi}} d\sigma$$

$$\lesssim \left(\frac{\tau}{\tau_0}\right)^{3-c\nu} \cdot \int_{\tau_0}^{\tau} \left(\frac{\lambda(\tau)}{\lambda(\sigma)}\right)^{-1+\delta} \|f(\sigma,\cdot)\|_{S_1^\hbar}\, d\sigma$$

$$\lesssim \left(\frac{\tau}{\tau_0}\right)^{3-c\nu} \cdot \int_{\tau_0}^{\tau} \left(\frac{\sigma}{\tau}\right)^{3-c\nu} \cdot \left(\frac{\sigma}{\tau}\right)^{c'\nu} \sigma^{-1} \cdot \sigma\|f(\sigma,\cdot)\|_{S_1^\hbar}\, d\sigma$$

$$\lesssim \sup_{\tau_0\le\sigma\le\tau} \left(\frac{\sigma}{\tau_0}\right)^{3-c\nu} \cdot \sigma\, \|f(\sigma,\cdot)\|_{S_1^\hbar},$$

as claimed. □

Based on the proof of the above proposition, we have the following:

Corollary 6.4. *Let $\overline{x}(\tau,\xi)$ be as in* (6.12) *with f satisfying, for some constant $c > 0$,*

$$\left(\frac{\sigma}{\tau_0}\right)^{3-c\nu} \cdot \sigma^2\, \|f(\sigma,\cdot)\|_{S_1^\hbar} < \infty, \quad \text{for} \quad \sigma \in [\tau_0,\tau].$$

Then for the same constant $c > 0$, we have

$$\left(\frac{\tau}{\tau_0}\right)^{3-c\nu} \cdot \|\overline{x}(\tau,\cdot)\|_{S_0^\hbar} \ll_{\tau_0} \sup_{\tau_0\le\sigma\le\tau} \left(\frac{\sigma}{\tau_0}\right)^{3-c\nu} \cdot \sigma^2\, \|f(\sigma,\cdot)\|_{S_1^\hbar}.$$

We also have the following estimate on the operator $\mathcal{K}_\hbar^{(0)}$ acting on the space $S_0^\hbar$ and $S_1^\hbar$:

Proposition 6.5. *Let $\mathcal{K}_\hbar^{(0)}$ be as in Proposition 5.1. We have the bounds:*

$$\left\|\mathcal{K}_\hbar^{(0)} f\right\|_{S_0^\hbar} \lesssim \|f\|_{S_0^\hbar}, \quad \left\|\mathcal{K}_\hbar^{(0)} f\right\|_{S_1^\hbar} \lesssim \|f\|_{S_1^\hbar}, \quad \left\|\mathcal{K}_\hbar^{(0)} f\right\|_{S_1^\hbar} \lesssim \|f\|_{S_0^\hbar}$$

uniformly in $\hbar$.

Proof. The idea is to reduce the operator to a simple Hilbert operator, by exploiting the differentiability properties of $F(\xi,\eta;\hbar)$. For simplicity, we denote $F(\xi,\eta;\hbar)$ by $F(\xi,\eta)$. We first prove the bounds for $\|\cdot\|_{S_0^\hbar} \to \|\cdot\|_{S_0^\hbar}$ and $\|\cdot\|_{S_1^\hbar} \to \|\cdot\|_{S_1^\hbar}$:

- $\underline{\xi \simeq \eta}$. We reformulate the norms in terms of the variable $\xi^{\frac{1}{2}}$, using the fact $\|f\|_{L^2_{d\xi}} \simeq \left\|\xi^{\frac{1}{4}} f\right\|_{L^2_{d\xi^{\frac{1}{2}}}}$. So the integral can be rewritten as

$$\int_0^\infty \chi_{\xi\simeq\eta} \frac{F(\xi,\eta)\rho_n(\eta)}{\xi-\eta} f(\eta)\, d\eta = 2\int_0^\infty \chi_{\xi^{\frac{1}{2}}\simeq\eta^{\frac{1}{2}}} \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}} \cdot \frac{F(\xi,\eta)\rho_n(\eta)}{\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}} f(\eta)\, d\eta^{\frac{1}{2}}.$$

  If $|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}| \ge 1$, we have, by Proposition 5.1,

$$\left|\chi_{\xi\simeq\eta} \frac{(\hbar^2\xi)^{1-\frac{\delta}{2}}\langle\hbar^2\xi\rangle^{\frac{3}{2}+\delta}}{(\hbar^2\eta)^{1-\frac{\delta}{2}}\langle\hbar^2\eta\rangle^{\frac{3}{2}+\delta}} \cdot \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}} \cdot \frac{F(\xi,\eta)}{\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}}\right| \lesssim \left|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\right|^{-\frac{5}{4}},$$

which is in $L^1_{d\eta^{\frac{1}{2}}}$. This gives

$$\left\|\int_0^\infty \chi_{\xi\simeq\eta, |\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|\geq 1} \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}} \cdot \frac{F(\xi,\eta)\rho_n(\eta)}{\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}} f(\eta) d\eta^{\frac{1}{2}}\right\|_{S_0^\hbar} \lesssim \|f\|_{S_0^\hbar}.$$

When $|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}| < 1$, we write

$$F(\xi,\eta) = F(\xi,\xi) + \partial_{\eta^{\frac{1}{2}}} F \cdot O\left(|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|\right).$$

Since Proposition 5.1 gives that $\left|\partial_{\eta^{\frac{1}{2}}} F\right| \lesssim 1$ uniformly in $\hbar$, the contribution from the error $\partial_{\eta^{\frac{1}{2}}} F \cdot O\left(|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|\right)$ again leads to a $L^1_{d\eta^{\frac{1}{2}}}$-kernel, which can be handled as for the case $|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}| \geq 1$. The contribution from the main term $F(\xi,\xi)$ is handled via the boundedness of $F(\xi,\eta)$ and the estimate

$$\left\|\int_0^\infty \chi_{\xi^{\frac{1}{2}}\simeq\eta^{\frac{1}{2}}, |\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}|<1} \frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}} \cdot \frac{f(\eta)}{\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}} d\eta^{\frac{1}{2}}\right\|_{L^2_{d\xi^{\frac{1}{2}}}} \lesssim \|f\|_{L^2_{d\eta^{\frac{1}{2}}}}.$$

The bound for $\|\cdot\|_{S_1^\hbar} \to \|\cdot\|_{S_1^\hbar}$ is handled similarly using the fact $\xi \simeq \eta$.

- $\underline{\xi \gg \eta}$. We split the integral into the cases $\hbar^2\xi \geq 1$ and $\hbar^2\xi \leq 1$. For the latter we have

$$\begin{aligned}&\left\|\chi_{\hbar^2\xi\leq 1}(\hbar^2\xi)^{1-\frac{\delta}{2}} \int_0^\infty \frac{F(\xi,\eta)\rho_n(\eta)}{(\xi-\eta)(\hbar^2\eta)^{1-\frac{\delta}{2}}} (\hbar^2\eta)^{1-\frac{\delta}{2}} f(\eta) d\eta\right\|_{L^2_{d\xi}} \\ \lesssim \hbar &\left\|\chi_{\hbar^2\xi\leq 1} \frac{(\hbar^2\xi)^{1-\frac{\delta}{2}}}{\hbar^2\xi}\right\|_{L^2_{\hbar^2 d\xi}} \cdot \int_0^\infty \chi_{\eta\ll\xi}(\hbar^2\eta)^{-\frac{1}{2}+\frac{\delta}{2}} \cdot (\hbar^2\eta)^{1-\frac{\delta}{2}} f(\eta) d\eta.\end{aligned}$$

Here we have used the bound $|F(\xi,\eta)| \lesssim \hbar\eta^{\frac{1}{2}}$ for $\hbar\eta^{\frac{1}{2}} \leq 1$. The desired estimate follows from the bound

$$\left|\int_0^\infty \chi_{\eta\ll\xi}(\hbar^2\eta)^{-\frac{1}{2}+\frac{\delta}{2}} \cdot (\hbar^2\eta)^{1-\frac{\delta}{2}} f(\eta) d\eta\right| \lesssim \left\|\chi_{\eta\ll\xi}(\hbar^2\eta)^{-\frac{1}{2}+\frac{\delta}{2}}\right\|_{L^2_{d\eta}} \cdot \|f\|_{S_0^\hbar} \lesssim \hbar^{-1}\|f\|_{S_0^\hbar}.$$

The estimate for $\|\cdot\|_{S_1^\hbar} \to \|\cdot\|_{S_1^\hbar}$ follows in the same way using the fact $\xi \geq \eta$.

For the case $\hbar^2\xi \geq 1$, we have, by Proposition 5.1, that for $\hbar^2\eta \leq 1$

$$\begin{aligned}&\left\|\chi_{\hbar^2\xi\geq 1}(\hbar^2\xi)^{\frac{5}{2}+\frac{\delta}{2}} \int_0^\infty \frac{F(\xi,\eta)\rho_n(\eta)}{(\xi-\eta)(\hbar^2\eta)^{1-\frac{\delta}{2}}} (\hbar^2\eta)^{1-\frac{\delta}{2}} f(\eta) d\eta\right\|_{L^2_{d\xi}} \\ \lesssim \hbar &\left\|\chi_{\hbar^2\xi\geq 1} \frac{(\hbar^2\xi)^{\frac{5}{2}+\frac{\delta}{2}}}{(\hbar^2\xi)^{\frac{7}{2}}}\right\|_{L^2_{\hbar^2 d\xi}} \cdot \int_0^\infty \chi_{\eta\ll\xi, \hbar^2\eta\leq 1}(\hbar^2\eta)^{-\frac{1}{2}+\frac{\delta}{2}} \cdot (\hbar^2\eta)^{1-\frac{\delta}{2}} f(\eta) d\eta,\end{aligned}$$

which is bounded similarly as for the case $\hbar^2\xi \leq 1$.

If $\hbar^2\eta \geq 1$, Proposition 5.1 implies

$$\left|\frac{F(\xi,\eta)}{\xi-\eta}\right| \lesssim \hbar^2(\hbar^2\eta)^{\frac{3}{4}}(\hbar^2\xi)^{-\frac{9}{4}} \cdot (\hbar^2\eta)^{\frac{5}{4}}(\hbar^2\xi)^{-\frac{5}{4}} \leq \hbar^2(\hbar^2\eta)^2(\hbar^2\xi)^{-\frac{7}{2}}$$

Therefore we have the bound

$$\left\|\chi_{\hbar^2\xi\geq 1}(\hbar^2\xi)^{\frac{5}{2}+\frac{\delta}{2}}\int_0^\infty \frac{F(\xi,\eta)\rho_n(\eta)}{(\xi-\eta)(\hbar^2\eta)^{\frac{5}{2}+\frac{\delta}{2}}}(\hbar^2\eta)^{\frac{5}{2}+\frac{\delta}{2}}f(\eta)d\eta\right\|_{L^2_{d\xi}}$$

$$\lesssim\hbar\left\|\chi_{\hbar^2\xi\geq 1}\frac{(\hbar^2\xi)^{\frac{5}{2}+\frac{\delta}{2}}}{(\hbar^2\xi)^{\frac{7}{2}}}\right\|_{L^2_{\hbar^2 d\xi}} \cdot \int_0^\infty \chi_{\eta\ll\xi,\hbar^2\eta\geq 1}(\hbar^2\eta)^{-\frac{1}{2}-\frac{\delta}{2}} \cdot (\hbar^2\eta)^{\frac{5}{2}+\frac{\delta}{2}}f(\eta)d\eta,$$

which again is bounded in the similar way. The estimate for $\|\cdot\|_{S_1^\hbar} \to \|\cdot\|_{S_1^\hbar}$ is handled similarly using the fact $\xi \geq \eta$.

- $\underline{\xi \ll \eta}$. We start with the case when $\hbar^2\xi \geq 1$. By Proposition 5.1

$$\left|\frac{F(\xi,\eta)}{\eta-\xi}\right| \lesssim \hbar^2(\hbar^2\xi)^{\frac{3}{4}}(\hbar^2\eta)^{-\frac{9}{4}} \cdot (\hbar^2\xi)^{\frac{5}{4}}(\hbar^2\eta)^{-\frac{5}{4}} \leq \hbar^2(\hbar^2\xi)^{-\frac{7}{2}}(\hbar^2\eta)^2,$$

which means that this case can be handled in the same way as the case when $\hbar^2\xi \gg \hbar^2\eta \geq 1$. If $\hbar^2\xi \leq 1$ and $\hbar^2\eta \geq 1$, we have

$$\left|\frac{F(\xi,\eta)}{\eta-\xi}\right| \lesssim \hbar^2(\hbar^2\xi)^{\frac{1}{2}}(\hbar^2\eta)^{-\frac{9}{4}},$$

which is surely enough to bound the integral

$$\left\|\chi_{\hbar^2\xi\leq 1,\hbar^2\eta\geq 1}(\hbar^2\xi)^{1-\frac{\delta}{2}}\int_0^\infty \frac{F(\xi,\eta)\rho_n(\eta)}{(\xi-\eta)(\hbar^2\eta)^{\frac{5}{2}+\frac{\delta}{2}}}(\hbar^2\eta)^{\frac{5}{2}+\frac{\delta}{2}}f(\eta)d\eta\right\|_{L^2_{d\xi}}$$

Finally for the case $\hbar^2\xi \ll \hbar^2\eta \leq 1$, we have

$$\left|\frac{F(\xi,\eta)}{\xi-\eta}\right| \lesssim \hbar^2(\hbar^2\xi)^{\frac{1}{2}}(\hbar^2\eta)^{-\frac{1}{2}}|\log\hbar^2\eta| \lesssim \hbar^2(\hbar^2\xi)^{-1}(\hbar^2\eta)|\log\hbar^2\eta|,$$

which is the desired estimate.

For the bound for $\|\cdot\|_{S_1^\hbar} \to \|\cdot\|_{S_1^\hbar}$, we proceed as follows. For $\hbar \ll 1$, since we can have as many powers of $\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}$ as we wish, the argument is the same as that for $\|\cdot\|_{S_0^\hbar} \to \|\cdot\|_{S_0^\hbar}$. For $\hbar \simeq 1$, if $1 \leq \hbar^2\xi \ll \hbar^2\eta$, then the argument is the same as for $\hbar \ll 1$. For the remaining two cases, we use the bound:

$$\left|\frac{F(\xi,\eta)}{\xi-\eta}\right| \lesssim \hbar^2(\hbar^2\xi)^{-\frac{1}{2}}(\hbar^2\eta)^{-\frac{5}{4}}, \quad \text{for} \quad \hbar^2\xi \leq 1 \leq \hbar^2\eta,$$

$$\left|\frac{F(\xi,\eta)}{\xi-\eta}\right| \lesssim \hbar^2(\hbar^2\xi)^{-\frac{1}{2}}(\hbar^2\eta)^{\frac{1}{2}}|\log\hbar^2\eta|, \quad \text{for} \quad \hbar^2\xi \ll \hbar^2\eta \leq 1.$$

The bound for $\|\cdot\|_{S_0^\hbar} \to \|\cdot\|_{S_1^\hbar}$ can be handled as follows: First, if $\hbar^2\xi \geq 1$, the desired bound follows from the bound for $\|\cdot\|_{S_0^\hbar} \to \|\cdot\|_{S_0^\hbar}$. If $\hbar^2\xi \leq 1$, the argument is similar to that of deriving the bound for $\|\cdot\|_{S_0^\hbar} \to \|\cdot\|_{S_0^\hbar}$, and here we only give an outline:

- When $\xi \simeq \eta$ and $|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}| < 1$, for $\hbar \gtrsim 1$ we have $\partial_{\eta^{\frac{1}{2}}} F(\xi,\eta) \simeq \xi^{\frac{1}{2}}|\log\eta|$. Here $O(|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}|)$ cancels with singular denominator and $L^1_{d\eta^{\frac{1}{2}}}$-integrability of $|\log\eta|$ gives the desired bound.
- When $\hbar^2\xi \leq 1$ and $\hbar^2\eta \geq 1$, the same bound
$$\left|\frac{F(\xi,\eta)}{\eta-\xi}\right| \lesssim \hbar^2(\hbar^2\xi)^{\frac{1}{2}}(\hbar^2\eta)^{-\frac{9}{4}}$$
suffices.
- When $\hbar^2\xi \ll \hbar^2\eta \leq 1$, we have
$$\left|\frac{F(\xi,\eta)}{\xi-\eta}\right| \lesssim \hbar^2(\hbar^2\xi)^{\frac{1}{2}}(\hbar^2\eta)^{-\frac{1}{2}}|\log\hbar^2\eta| \lesssim \hbar^2(\hbar^2\xi)^{\frac{1}{2}}(\hbar^2\xi)^{-\frac{3}{2}+\delta}(\hbar^2\eta)^{1-\delta}|\log\hbar^2\eta|,$$
which suffices.

□

6.3. **Linear smooth sources.** Now we start to estimate the contribution of the smooth part of $\overline{x}^\hbar(\tau,\xi)$ to the RHS of the equation (6.7). We start with the contribution to $\mathcal{R}_\hbar(\overline{x}^\hbar, \mathcal{D}_\tau\overline{x}^\hbar)$:

Proposition 6.6. *Let S be any of the expressions in $\mathcal{R}(\overline{x}^\hbar, \mathcal{D}_\tau\overline{x}^\hbar)$, and we define*
$$\widetilde{x}^\hbar(\tau,\xi) := \xi^{-\frac{1}{2}}\int_{\tau_0}^{\tau}\frac{\lambda(\tau)}{\lambda(\sigma)}\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}\sin\left(\lambda(\tau)\xi^{\frac{1}{2}}\int_\sigma^\tau\lambda(u)^{-1}du\right)S\left(\sigma,\frac{\lambda(\tau)^2}{\lambda(\sigma)^2}\xi\right)d\sigma.$$
Then we have, for $S = 2\frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_\hbar^{(0)}\overline{x}^\hbar$,
$$\left(\frac{\tau}{\tau_0}\right)^{3-c\nu}\left(\left\|\widetilde{x}^\hbar(\tau,\cdot)\right\|_{S_0^\hbar} + \left\|\mathcal{D}_\tau\widetilde{x}^\hbar(\tau,\cdot)\right\|_{S_1^\hbar}\right) \lesssim \left(\frac{\tau}{\tau_0}\right)^{3-c\nu}\|S\|_{S_1^\hbar}.$$
If S is one of the terms $\left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)'\mathcal{K}_\hbar^{(0)}\overline{x}^\hbar$, $\left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)^2\mathcal{K}_\hbar^{(0)}\overline{x}^\hbar$, $\left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)'\left(\mathcal{K}_\hbar^{(0)}\right)^2\overline{x}^\hbar$ or $\frac{\lambda'(\tau)}{\lambda(\tau)}\left[\mathcal{D}_\tau,\mathcal{K}_\hbar^{(0)}\right]\overline{x}^\hbar$, then we have
$$\left(\frac{\tau}{\tau_0}\right)^{3-c\nu}\left(\left\|\widetilde{x}^\hbar(\tau,\cdot)\right\|_{S_0^\hbar} + \left\|\mathcal{D}_\tau\widetilde{x}^\hbar(\tau,\cdot)\right\|_{S_1^\hbar}\right) \ll \left(\frac{\tau}{\tau_0}\right)^{3-c\nu}\|S\|_{S_0^\hbar}.$$

Proof. Except the commutator $\frac{\lambda'(\tau)}{\lambda(\tau)}\left[\mathcal{D}_\tau,\mathcal{K}_\hbar^{(0)}\right]\overline{x}^\hbar$, the estimates for the terms in $\mathcal{R}_\hbar(\overline{x}^\hbar, \mathcal{D}_\tau\overline{x}^\hbar)$ follow directly from Proposition 6.3 and Proposition 6.5. Now we consider the commutator term
$$\frac{\lambda'(\tau)}{\lambda(\tau)}\left[\mathcal{D}_\tau,\mathcal{K}_\hbar^{(0)}\right]\overline{x}^\hbar = -2\left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)^2\left[\xi\partial_\xi,\mathcal{K}_\hbar^{(0)}\right]\overline{x}^\hbar - \left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)^2\left[\frac{\rho_n'(\xi)\xi}{\rho_n(\xi)},\mathcal{K}_\hbar^{(0)}\right]\overline{x}^\hbar.$$

Since $\left|\frac{\rho_n'(\xi)\xi}{\rho_n(\xi)}\right| \simeq 1$, the second commutator on the RHS above can be handled similarly as the other terms in $\mathcal{R}_\hbar(\overline{x}^\hbar, \mathcal{D}_\tau\overline{x}^\hbar)$. We focus on the first commutator. In the following calculation, we omit the factor $\left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)^2$ and write $2\xi\partial_\xi = \xi^{\frac{1}{2}}\partial_{\xi^{\frac{1}{2}}}$. For a function $f \in C_c^\infty(\mathbb{R}_+)$, we consider

$$\left[\xi^{\frac{1}{2}}\partial_{\xi^{\frac{1}{2}}}, \mathcal{K}_\hbar^{(0)}\right] f(\xi) = \int_0^\infty \xi^{\frac{1}{2}}\partial_{\xi^{\frac{1}{2}}}\left(\frac{F(\xi,\eta)}{\xi-\eta}\right)\rho_n(\eta)f(\eta)\,d\eta - \int_0^\infty \frac{F(\xi,\eta)}{\xi-\eta}\rho_n(\eta)\eta^{\frac{1}{2}}\partial_{\eta^{\frac{1}{2}}}f(\eta)d\eta := I + II.$$

For II we use integration by parts to obtain

$$\begin{aligned}
II =& 2\int_0^\infty \partial_{\eta^{\frac{1}{2}}}\left(\frac{F(\xi,\eta)\eta}{\xi-\eta}\rho_n(\eta)\right)d\eta^{\frac{1}{2}} \\
=& -2\int_0^\infty \partial_{\eta^{\frac{1}{2}}}\left(F(\xi,\eta)\rho_n(\eta)\right)f(\eta)d\eta^{\frac{1}{2}} + 2\xi\int_0^\infty \partial_{\eta^{\frac{1}{2}}}\left(\frac{F(\xi,\eta)}{\xi-\eta}\rho_n(\eta)\right)f(\eta)d\eta^{\frac{1}{2}} \\
=& -\int_0^\infty \frac{1}{\eta^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}}\left(\frac{F(\xi,\eta)\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\rho_n(\eta)\right)f(\eta)d\eta - \int_0^\infty \frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}}\left(\frac{F(\xi,\eta)}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\rho_n(\eta)\right)f(\eta)\,d\eta \\
&+ \int_0^\infty \frac{\xi}{\eta^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}}\left(\frac{F(\xi,\eta)\rho_n(\eta)}{(\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}})(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}\right)d\eta := II_1 + II_2 + II_3.
\end{aligned}$$

We write II_2 as

$$II_2 = -\int_0^\infty \frac{\xi}{\eta^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}}\left(\frac{F(\xi,\eta)\rho_n(\eta)}{(\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}})(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}\right)f(\eta)d\eta + \int_0^\infty \frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}}\left(\frac{F(\xi,\eta)\eta^{\frac{1}{2}}\rho_n(\eta)}{(\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}})(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}\right)f(\eta)d\eta.$$

Therefore

$$\begin{aligned}
I + II_2 + II_3 &= \int_0^\infty \frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\left(\partial_{\xi^{\frac{1}{2}}} + \partial_{\eta^{\frac{1}{2}}}\right)\left(\frac{F(\xi,\eta)\eta^{\frac{1}{2}}\rho_n(\eta)}{(\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}})(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}\right)f(\eta)d\eta \\
&= \int_0^\infty \frac{1}{\xi-\eta}\cdot\frac{\xi^{\frac{1}{2}}(\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}})}{\eta^{\frac{1}{2}}}\left(\partial_{\xi^{\frac{1}{2}}} + \partial_{\eta^{\frac{1}{2}}}\right)\left(\frac{F(\xi,\eta)\eta^{\frac{1}{2}}\rho_n(\eta)}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right)f(\eta)d\eta.
\end{aligned}$$

So we have

$$\left[\xi^{\frac{1}{2}}\partial_{\xi^{\frac{1}{2}}}, \mathcal{K}_\hbar^{(0)}\right] f(\xi) = \int_0^\infty \frac{H_1(\xi,\eta)}{\xi-\eta}f(\eta)d\eta + \int_0^\infty H_2(\xi,\eta)f(\eta)d\eta,$$

where

$$H_1(\xi,\eta) := \frac{\xi^{\frac{1}{2}}(\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}})}{\eta^{\frac{1}{2}}}\left(\partial_{\xi^{\frac{1}{2}}} + \partial_{\eta^{\frac{1}{2}}}\right)\left(\frac{F(\xi,\eta)\eta^{\frac{1}{2}}\rho_n(\eta)}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right), \quad H_2(\xi,\eta) := -\frac{1}{\eta^{\frac{1}{2}}}\partial_{\eta^{\frac{1}{2}}}\left(\frac{F(\xi,\eta)\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\rho_n(\eta)\right).$$

Let us first look at the contribution from $H_2(\xi,\eta)$, whose principal contribution is given by $-\frac{\partial_{\eta^{\frac{1}{2}}}F(\xi,\eta)}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\rho_n(\eta)$. Here we try to mimic the proof of Proposition 6.5:

- $\underline{\xi \simeq \eta}$. After change of the variable $\eta \to \eta^{\frac{1}{2}}$, we consider

$$\left|\chi_{\xi\simeq\eta}\hbar\frac{(\hbar^2\xi)^{\frac{1}{2}-\frac{\delta}{2}}\langle\hbar^2\xi\rangle^{\frac{3}{2}+\delta}}{(\hbar^2\eta)^{1-\frac{\delta}{2}}\langle\hbar^2\eta\rangle^{\frac{3}{2}+\delta}}\cdot\frac{\eta^{\frac{1}{2}}\partial_{\eta^{\frac{1}{2}}}F(\xi,\eta)}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right|. \tag{6.14}$$

  If $\hbar^2\eta \gtrsim 1$, Proposition 5.1 shows that the above absolute value is bounded by $\hbar\cdot(\hbar\eta^{\frac{1}{2}})^{-2}$, which is in $L^1_{d\eta^{\frac{1}{2}}}$. If $\hbar^2\eta \ll 1$, Proposition 5.1 shows that the above absolute value is bounded by $\hbar\log(\hbar^2\eta)$, which is again in $L^1_{d\eta^{\frac{1}{2}}}$.
- $\underline{\xi \ll \eta}$. This can be treated in the same way as the case $\xi \simeq \eta$.
- $\underline{\xi \gg \eta}$. If $\hbar^2\eta \geq 1$, Proposition 5.1 shows that (6.14)is bounded by

$$\hbar\cdot(\hbar^2\xi)^{2+\frac{\delta}{2}}\cdot(\hbar^2\xi)^{-\frac{5}{4}}\cdot(\hbar^2\eta)^{-\frac{5}{2}-\frac{\delta}{2}}\cdot(\hbar^2\eta)^{\frac{3}{4}}\cdot(\hbar^2\xi)^{-\frac{1}{2}}\cdot\left(\frac{\hbar^2\eta}{\hbar^2\xi}\right)^N \leq \hbar(\hbar^2\eta)^{-\frac{3}{2}-\frac{\delta}{2}},$$

  which is in $L^1_{d\eta^{\frac{1}{2}}}$. If $\hbar^2\eta \leq 1$ and $\hbar^2\xi \geq 1$, then (6.14) is bounded by

$$\hbar\cdot(\hbar^2\xi)^{\frac{5}{2}+\frac{\delta}{2}}\cdot(\hbar^2\eta)^{-1+\frac{\delta}{2}}\cdot(\hbar^2\eta)^{-\frac{1}{2}}\cdot(\hbar^2\xi)^{-\frac{1}{4}}\frac{\hbar^2\eta}{\hbar^2\xi}\cdot(\hbar^2\xi)^{-N} \leq (\hbar^2\eta)^{-\frac{1}{2}+\frac{\delta}{2}},$$

  which is again in $L^1_{d\eta}$. Finally if $\hbar^2\eta \ll \hbar^2\xi \leq 1$, Proposition 3.1 shows that (6.14) is bounded by

$$\hbar\cdot(\hbar^2\xi)^{\frac{1}{2}-\frac{\delta}{2}}\cdot(\hbar^2\eta)^{-1+\frac{\delta}{2}}\cdot(\hbar^2\eta)^{\frac{1}{2}}|\log\hbar^2\eta| \leq \hbar|\log\hbar^2\eta|,$$

  which is again in $L^1_{d\eta^{\frac{1}{2}}}$. This completes the estimate for $H_2(\xi,\eta)$.

Next we turn to the estimate for $\frac{H_1(\xi,\eta)}{\xi-\eta}$. Systematically $H_1(\xi,\eta)$ is a linear combination of the following terms:

$$\xi^{\frac{1}{2}}\partial_{\xi^{\frac{1}{2}}}F(\xi^{\frac{1}{2}},\eta^{\frac{1}{2}})\rho_n(\eta),\quad \xi^{\frac{1}{2}}\partial_{\eta^{\frac{1}{2}}}F(\xi^{\frac{1}{2}},\eta^{\frac{1}{2}})\rho_n(\eta),\quad \frac{\xi^{\frac{1}{2}}}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}}F(\xi,\eta)\rho_n(\eta),\quad \frac{\hbar\xi^{\frac{1}{2}}}{\hbar\eta^{\frac{1}{2}}}F(\xi,\eta)\rho_n(\eta) \tag{6.15}$$

The contribution from the third term in (6.15) can be treated exactly the same as $\mathcal{K}_\hbar^{(0)}$ itself. For the fourth term in (6.15), $\hbar\xi^{\frac{1}{2}}$ in numerator cancels with the $\xi^{-\frac{1}{2}}$ in the definition of $S_1^\hbar$, giving an extra $\hbar$. In the denominator, if $\hbar\eta^{\frac{1}{2}} \leq 1$, then we use the small factors of $\hbar\eta^{\frac{1}{2}}$ in $F(\xi,\eta)$ to absorb it. Therefore the argument is similar as the one treating the $S_0^\hbar \to S_1^\hbar$ bound for $\mathcal{K}_\hbar^{(0)}$. The second term in (6.15) is treated in the same way, except that $\xi^{\frac{1}{2}}$ cancels exactly with the $\xi^{-\frac{1}{2}}$ in the definition of $S_1^\hbar$, without an extra $\hbar$. This approach also applies to the first term in (6.15). Therefore we obtain the following estimate on $\left[\mathcal{D}_\tau,\mathcal{K}_\hbar^{(0)}\right]$:

$$\left\|\left[\mathcal{D}_\tau,\mathcal{K}_\hbar^{(0)}\right]f\right\|_{S_1^\hbar} \lesssim \|f\|_{S_0^\hbar},$$

which completes the proof. □

6.4. **Nonlinear smooth sources.** Observing the expressions (3.53) and (3.54) for $N(\varphi_1)$ and $N(\varphi_2)$, there are two types of nonlinearities in $N(\varphi_1)$ and $N(\varphi_2)$: One is a quadratic expression of $\varphi_i, i = 1, 2$ and their derivatives multiplied by a spatial weight depending on the background solution, the other is $\varphi_i, i = 1, 2$ multiplied by a quadratic expression of the derivatives of $\varphi_i, i = 1.2$. Since in this section we only handle smooth sources and the background solution has limited smoothness, now we focus on the second type of nonlinearity. More precisely, in this section we derive multilinear estimates for products of regular functions, i.e. those whose Fourier transforms are in $S_0^\hbar$, where the $\hbar$ may differ amongst the factors. The key point here is that in these estimates, we do not lose in the smallest $\hbar$, i.e. the estimates are *uniform with respect to the smallest $\hbar$ present*. At the most basic level we need to bound the expressions

$$\partial_R\phi_1 \cdot \partial_R\phi_2, \quad \left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\phi_1 \cdot \left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\phi_2,$$

We start with the following basic

Proposition 6.7. *Let*

$$f(R) := \int_0^\infty \phi_n(R;\xi)\overline{x}(\xi)\rho_n(\xi)\,d\xi,$$

we have the bound, for $n \ll k$,

$$\left\|\partial_R^k f(R)\right\|_{L^2_{R\,dR}} \lesssim_k \left\|\xi^{\frac{k}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}},$$

the implied constant being uniform in $\hbar$.

Proof. We cover the case $n \gg 1$, the remaining case $n \lesssim 1$ being simpler since no attention needs to be paid to the $\hbar$-dependence, and can be handled similarly by Proposition 4.37. Moreover, the cases $k \geq 2$ being similar to the one of $k = 1$, we treat this latter one in detail. We do this by explicit use of the asymptotics of $\phi(R;\xi,\hbar)$, decomposing the frequency regime into a small frequency, turning point, and oscillatory regime:

Contribution of the small frequency regime. Here we consider the norm of the expression

$$\int_0^\infty \chi_{R\xi^{\frac{1}{2}}\hbar<\frac{x_t}{2}}\partial_R\phi_n(R;\xi)x(\xi)\rho_n(\xi)\,d\xi,$$

where the cutoff smoothly localizes to the indicated region. Then using the representation from Proposition 4.32, we easily infer the bound

$$\left|\chi_{R\xi^{\frac{1}{2}}\hbar<\frac{x_t}{2}}\partial_R\phi_n(R,\xi)\right| \lesssim R^{-1}\cdot\hbar^{n-\frac{1}{2}}\xi^{\frac{n-1}{2}}R^{n-1} \leq \hbar^{\frac{1}{2}}\cdot\hbar\xi^{\frac{1}{2}}\cdot\left(\hbar\xi^{\frac{1}{2}}R\right)^{n-2}$$

Hence subdividing this frequency region into dyadic sub-intervals, we infer

$$\begin{aligned}
&\Big\|\int_0^\infty \chi_{R\xi^{\frac12}\hbar<\frac{x_t}{2}}\partial_R\phi_n(R;\xi)x(\xi)\rho_n(\xi)\,d\xi\Big\|_{L^2_{R\,dR}}\\
&\lesssim \sum_{\mu<\frac{x_t}{2}}\mu^{n-2}\Big\|\int_0^\infty \chi_{R\xi^{\frac12}\hbar\simeq\mu}\hbar^{\frac32}\xi^{\frac12}\cdot\overline{x}(\xi)\rho_n(\xi)d\xi\Big\|_{L^2_{R\,dR}}\\
&\lesssim \sum_{\mu<\frac{x_t}{2}}\mu^{n-2}\Big(\sum_\lambda\Big\|\int_0^\infty \chi_{R\xi^{\frac12}\hbar\simeq\mu}\hbar^{\frac32}\xi^{\frac12}\cdot\chi_{\xi^{\frac12}\simeq\lambda}\overline{x}(\xi)\rho_n(\xi)d\xi\Big\|_{L^2_{R\,dR}}\Big)^{\frac12}\\
&\lesssim \sum_{\mu<\frac{x_t}{2}}\mu^{n-2}\Big(\sum_\lambda\Big(\hbar^{\frac32}\frac{\mu}{\hbar\lambda}\Big)^2\cdot\lambda^2\Big\|\chi_{\xi^{\frac12}\simeq\lambda}\xi^{\frac12}\overline{x}(\xi)\Big\|^2_{L^2_{d\xi}}\Big)^{\frac12}\\
&\lesssim\hbar^{\frac12}\Big\|\xi^{\frac12}\overline{x}(\xi)\Big\|_{L^2_{d\xi}}\sum_{\mu<\frac{x_t}{2}}\mu^{n-1}\lesssim\hbar^{\frac12}\Big\|\xi^{\frac12}\overline{x}(\xi)\Big\|_{L^2_{d\xi}}.
\end{aligned}$$

We have used Hölder's inequality to obtain the second line from the bottom.

Contribution of the turning point. Here we include the smooth cutoff $\chi_{R\xi^{\frac12}\hbar\in[\frac{x_t}{2},2x_t]}$. By means of a sharp cutoff, we split this into two portions

$$\begin{aligned}
\int_0^\infty \chi_{R\xi^{\frac12}\hbar\in[\frac{x_t}{2},2x_t]}\partial_R\phi_n(R;\xi)\overline{x}(\xi)\rho_n(\xi)\,d\xi &= \int_0^\infty \chi_{R\xi^{\frac12}\hbar\in[\frac{x_t}{2},x_t]}\partial_R\phi_n(R;\xi)\overline{x}(\xi)\rho_n(\xi)\,d\xi\\
&\quad+\int_0^\infty \chi_{R\xi^{\frac12}\hbar\in[x_t,2x_t]}\partial_R\phi_n(R;\xi)\overline{x}(\xi)\rho_n(\xi)\,d\xi
\end{aligned}$$

Correspondingly, we get a contribution from the non-oscillatory regime to the left of the turning point, and a contribution of the oscillatory regime to the right.

Non-oscillatory regime $R\xi^{\frac12}\hbar\in\big[\frac{x_t}{2},x_t\big]$. Invoking again Proposition 4.32 and the estimates for τ in the regime $R\xi^{\frac12}\hbar\in[\frac{x_t}{2},x_t]$ which come from Lemma 4.11, we infer in this regime the bound

$$\Big|\chi_{R\xi^{\frac12}\hbar\in[\frac{x_t}{2},x_t]}\partial_R\phi_n(R;\xi)\Big|\lesssim\hbar^{\frac23}\xi^{\frac12}.$$

This time we have to be more careful to avoid any $\hbar$-losses since we no longer get the very rapid exponential decay in $\hbar^{-1}$ for free, compared to the preceding case. Thus for $\hbar\ll1$ we have to use finer asymptotics of Airy functions. Setting $x_t-x=\hbar^{\frac23}\mu$ with $\lambda\geq1$, $x=R\xi^{\frac12}\hbar$, we have

$$\Big|\chi_{R\xi^{\frac12}\hbar\in[\frac{x_t}{2},x_t]}\partial_R\phi_n(R;\xi)\Big|\lesssim\hbar^{\frac23}\xi^{\frac12}\mu^{-\frac14}e^{-\frac23\mu^{\frac32}}.$$

Now we divide the ξ integral into one where $x_t-x\in\big[0,\hbar^{\frac23}\big]$ and its complement. For the contribution of the first of these, we observe that for fixed R, and using the fact that

$$\chi_{R\xi^{\frac12}\hbar\in[\frac{x_t}{2},2x_t]}\partial_\xi(x-x_t)\simeq\chi_{R\xi^{\frac12}\hbar\in[\frac{x_t}{2},2x_t]}\cdot\xi^{-1}$$

thanks to Lemma 4.10, we have that $x_t - x \in \left[0, \hbar^{\frac{2}{3}}\right]$ implies that ξ is confined to an interval I_R of length $\simeq \frac{\hbar^{\frac{2}{3}}}{(R\hbar)^2}$. So if $R \simeq \lambda$, then

$$\left|\int_0^\infty \chi_{x_t - x\in[0,\hbar^{\frac{2}{3}}]} \partial_R\phi_n(R;\xi)\overline{x}(\xi)\rho_n(\xi)\, d\xi\right| \lesssim \hbar^{\frac{2}{3}} \cdot |I_R|^{\frac{1}{2}} \cdot \left\|\xi^{\frac{1}{2}}\overline{x}\right\|_{L^2(\xi\simeq\lambda^{-2}\hbar^{-2})} \lesssim \frac{1}{\lambda} \cdot \left\|\xi^{\frac{1}{2}}\overline{x}\right\|_{L^2(\xi\simeq\lambda^{-2}\hbar^{-2})}.$$

Here we have used Hölder's inequality for the first inequality. We conclude, using orthogonality, that

$$\begin{aligned}
&\left\|\int_0^\infty \chi_{x_t - x\in\left[0,\hbar^{\frac{2}{3}}\right]} \partial_R\phi_n(R;\xi)\overline{x}(\xi)\rho_n(\xi)\, d\xi\right\|_{L^2_{R\,dR}} \\
&\lesssim \left(\sum_\lambda \left\|\int_0^\infty \chi_{x_t - x\in[0,\hbar^{\frac{2}{3}}]} \partial_R\phi_n(R;\xi)\overline{x}(\xi)\rho_n(\xi)\, d\xi\right\|^2_{L^2_{R\,dR}(R\simeq\lambda)}\right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_\lambda \left\|\xi^{\frac{1}{2}}\overline{x}\right\|^2_{L^2(\xi\sim\lambda^{-2}\hbar^{-2})}\right)^{\frac{1}{2}} \lesssim \left\|\xi^{\frac{1}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}}.
\end{aligned}$$

In the regime $x_t - x \in \left[\hbar^{\frac{2}{3}}, \frac{x_t}{2}\right]$, we write $x_t - x \simeq \mu\hbar^{\frac{2}{3}}$ for dyadic $\mu \geq 1$. Then fixing such μ the corresponding interval I_R for ξ is of length $\simeq \frac{\mu\hbar^{\frac{2}{3}}}{(R\hbar)^2}$, and we gain $\mu^{-\frac{1}{4}}e^{-\frac{3}{2}\mu^{\frac{3}{2}}}$ from the decay of the Airy function. Proceeding as before one gets an analogous bound with (super)exponential decay in μ which can be summed up over dyadic $\mu \geq 1$. We omit the similar details.

Oscillatory regime $R\xi^{\frac{1}{2}}\hbar \in [x_t, 2x_t]$. We start by observing that the bound in the transition regime $x - x_t \in \left[0, \hbar^{\frac{2}{3}}\right]$ is handled exactly like the one for $x_t - x \in \left[0, \hbar^{\frac{2}{3}}\right]$. There is, however, a difference in the regime $x_t - x \in \left[\hbar^{\frac{2}{3}}, x_t\right]$, since we no longer get the rapid decay in terms of the re-scaled variable $\hbar^{-\frac{2}{3}}(x - x_t)$, and so we have to argue more carefully, exploiting oscillation. Defining the kernel function

$$G(\xi,\eta) := \int_0^\infty \chi_{x\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]} \partial_R\phi_n(R;\xi) \cdot \phi_n(R;\eta)\, R\, dR,$$

we strive for an estimate like

$$\left\|\int_0^\infty G(\xi,\eta)\overline{x}(\eta)\, d\eta\right\|_{L^2_{d\xi}} \lesssim \left\|\eta^{\frac{1}{2}}\overline{x}(\eta)\right\|_{L^2_{d\eta}}.$$

We start with the most difficult situation when the second factor $\phi_n(R;\eta)$ is also in the weakly oscillatory regime to the right of the turning point, i.e. the expression

$$G_1(\xi,\eta) := \int_0^\infty \chi_{x\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]}\chi_{\tilde{x}\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]} \partial_R\phi_n(R;\xi) \cdot \phi_n(R;\eta)\, R\, dR,$$

where we recall the notation $x = R\xi^{\frac{1}{2}}\hbar$, $\tilde{x} = R\eta^{\frac{1}{2}}\hbar$. Then by Proposition 4.32 we have

$$\phi_n(R;\xi) = c\hbar^{\frac{1}{3}}R^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}q^{-\frac{1}{4}}(\tau) \cdot \mathrm{Re}\left[a(\xi) \cdot \left(\mathrm{Ai}(\hbar^{-\frac{2}{3}}\tau) - i\mathrm{Bi}(\hbar^{-\frac{2}{3}}\tau)\right)\left(1 + \hbar\overline{a_1(\tau;\alpha,\hbar)}\right)\right]$$

$$\phi_n(R;\eta) = c\hbar^{\frac{1}{3}}R^{-\frac{1}{2}}\beta^{-\frac{1}{2}}q^{-\frac{1}{4}}(\tilde{\tau}) \cdot \mathrm{Re}\left[a(\eta) \cdot \left(\mathrm{Ai}(\hbar^{-\frac{2}{3}}\tilde{\tau}) - i\mathrm{Bi}(\hbar^{-\frac{2}{3}}\tilde{\tau})\right)\left(1 + \hbar\overline{a_1(\tilde{\tau};\beta,\hbar)}\right)\right].$$

Using the standard asymptotics of Airy functions in the oscillatory regime, and only keeping track of the worst term with destructive resonance of the phases and ∂_R falling on the oscillatory phase, we obtain the oscillatory integrals (omitting unnecessary constants)

$$G_{1,\pm}(\xi,\eta) := \xi^{\frac{1}{2}} \int_0^\infty \chi_{x\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]}\chi_{\tilde{x}\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]}(\xi\eta)^{-\frac{1}{4}} a(\xi)\overline{a(\eta)}\tau^{\frac{1}{4}} \frac{e^{\pm i\frac{2}{3}\hbar^{-1}\left[\tau^{\frac{3}{2}}-\tilde{\tau}^{\frac{3}{2}}\right]}}{\tilde{\tau}^{\frac{1}{4}}} \cdot (1 + b(\tau,\tilde{\tau},\alpha,\beta,\hbar))\,dR, \tag{6.16}$$

where the function $b(\dots)$ has symbol behavior with respect to all its variables. In order to obtain (6.16) we have used Lemma 4.10 and Lemma 4.11. We split the discussion according to $\widetilde{\tau} \gtrsim \tau$ and $\widetilde{\tau} \ll \tau$. By the presence of the factor $\frac{\tau^{\frac{1}{4}}}{\tilde{\tau}^{\frac{1}{4}}}$, the second case is more difficult, which we discuss first. Let us introduce $\tilde{\tau} := \hbar^{\frac{2}{3}} y^{\frac{2}{3}}$, whence in our domain $y \gtrsim 1$, and recalling $\xi \simeq \eta$ in our region of integration, we find (by Lemma 4.11)

$$\frac{\partial\tau}{\partial y} = \frac{\partial\tau}{\partial R}\cdot\frac{\partial R}{\partial\tilde{\tau}}\cdot\frac{\partial\tilde{\tau}}{\partial y} \sim y^{-\frac{1}{3}}\hbar^{\frac{2}{3}},$$

and so (note that $\tau^{\frac{1}{2}}\hbar^{-\frac{1}{3}}y^{-\frac{1}{3}} \gg 1$)

$$\chi_{\tau\gg\tilde{\tau}}\partial_y\left[e^{\pm i\frac{2}{3}\hbar^{-1}\left[\tau^{\frac{3}{2}}-\tilde{\tau}^{\frac{3}{2}}\right]}\right] \simeq \tau^{\frac{1}{2}}\hbar^{-\frac{1}{3}}y^{-\frac{1}{3}}\cdot e^{\pm i\frac{2}{3}\hbar^{-1}\left[\tau^{\frac{3}{2}}-\tilde{\tau}^{\frac{3}{2}}\right]}.$$

Thus we may replace $\chi_{\tau\gg\tilde{\tau}}e^{\pm i\frac{2}{3}\hbar^{-1}\left[\tau^{\frac{3}{2}}-\tilde{\tau}^{\frac{3}{2}}\right]}$ by $\chi_{\tau\gg\tilde{\tau}}\frac{\tilde{\tau}^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}\partial_y\left[e^{\pm i\frac{2}{3}\hbar^{-1}\left[\tau^{\frac{3}{2}}-\tilde{\tau}^{\frac{3}{2}}\right]}\right]$, and performing integration by parts after passing to the variable y (which does not cost anything since it produces additional factors $y^{-1} \lesssim 1$) and then reverting to the variable R, we have effectively replaced $\chi_{\tau\gg\tilde{\tau}}\frac{\tau^{\frac{1}{4}}}{\tilde{\tau}^{\frac{1}{4}}}$ by $\chi_{\tau\gg\tilde{\tau}}\left[\frac{\tau^{\frac{1}{4}}}{\tilde{\tau}^{\frac{1}{4}}}\right]^{-1}$.
Returning to (6.16), we may henceforth assume that $\tilde{\tau} \gtrsim \tau$. Now introduce the new variable

$$\omega := \frac{2}{3}\hbar^{-1}\left[\tau^{\frac{3}{2}} - \tilde{\tau}^{\frac{3}{2}}\right].$$

Observe that

$$\frac{\partial\omega}{\partial R} = \xi^{\frac{1}{2}}\frac{\partial\tau}{\partial x}\tau^{\frac{1}{2}} - \eta^{\frac{1}{2}}\frac{\partial\tilde{\tau}}{\partial\tilde{x}}\tilde{\tau}^{\frac{1}{2}}.$$

By Lemma 4.11

$$\frac{\partial\tau}{\partial x} = \Phi(x;\alpha,\tau) + \tau\cdot\frac{\partial_x\Phi(x;\alpha,\tau)}{\Phi(x;\alpha,\tau)} \simeq 1$$

for τ sufficiently small. Therefore we have

$$\tau^{\frac{1}{2}} - \tilde{\tau}^{\frac{1}{2}} = \frac{\tau-\tilde{\tau}}{\tau^{\frac{1}{2}}+\tilde{\tau}^{\frac{1}{2}}} \simeq \frac{\left[\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right]\cdot R\hbar}{\tau^{\frac{1}{2}}+\tilde{\tau}^{\frac{1}{2}}} \simeq \frac{\left[\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right]}{\xi^{\frac{1}{2}}(\tau^{\frac{1}{2}}+\tilde{\tau}^{\frac{1}{2}})}.$$

On the other hand, we have

$$\left|\frac{\partial}{\partial\xi^{\frac{1}{2}}}\left(\xi^{\frac{1}{2}}\frac{\partial\tau}{\partial x}\right)\right| = \left|\frac{\partial\tau}{\partial x} + \xi^{\frac{1}{2}}\frac{\partial^2\tau}{\partial x^2}\cdot\hbar R\right| \lesssim 1.$$

Therefore we have

$$\frac{\partial\omega}{\partial R} = \left(\xi^{\frac{1}{2}}\frac{\partial\tau}{\partial x} - \eta^{\frac{1}{2}}\frac{\partial\tilde{\tau}}{\partial\tilde{x}}\right)\cdot\tau^{\frac{1}{2}} + \left(\tau^{\frac{1}{2}} - \tilde{\tau}^{\frac{1}{2}}\right)\cdot\eta^{\frac{1}{2}}\frac{\partial\tilde{\tau}}{\partial\tilde{x}} \simeq \frac{\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}}{\tau^{\frac{1}{2}} + \tilde{\tau}^{\frac{1}{2}}},$$

since

$$\left|\left(\xi^{\frac{1}{2}}\frac{\partial\tau}{\partial x} - \eta^{\frac{1}{2}}\frac{\partial\tilde{\tau}}{\partial\tilde{x}}\right)\cdot\tau^{\frac{1}{2}}\right| \lesssim \left|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\right| \ll \left|\frac{\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}}{\tau^{\frac{1}{2}} + \tilde{\tau}^{\frac{1}{2}}}\right|$$

at last if $\tau + \tilde{\tau} \ll 1$, which we may as well assume(one can replace the bound $2x_t$ of the x-interval by any bound of the form $(1+c)x_t$ for our purposes). The integral under consideration can then be re-formulated as

$$\begin{aligned}&\xi^{\frac{1}{2}}\cdot(\xi\eta)^{-\frac{1}{4}}\cdot\int_0^\infty \chi_{x\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]}\chi_{\tilde{x}\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]}\chi_{\tau\lesssim\tilde{\tau}}\cdot e^{i\omega}\Psi(\omega;\alpha,\beta,\hbar)\frac{dR}{d\omega}\,d\omega\\ &= \frac{\xi^{\frac{1}{2}}\cdot(\xi\eta)^{-\frac{1}{4}}}{\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}}\cdot\int_0^\infty \chi_{x\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]}\chi_{\tilde{x}\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]}\chi_{\tau\lesssim\tilde{\tau}}\cdot e^{i\omega}\tilde{\Psi}(\omega;\alpha,\beta,\hbar)\,d\omega,\end{aligned}$$

where $\tilde{\Psi}(\omega;\alpha,\beta,\hbar)$ has symbol behavior with respect to all derivatives, and is uniformly bounded. In fact, observe that

$$\frac{\partial\tilde{\Psi}(\omega;\alpha,\beta,\hbar)}{\partial\omega} = \frac{\partial\tilde{\Psi}}{\partial R}\cdot\frac{\partial R}{\partial\omega}$$

and we have

$$\omega \simeq \hbar^{-1}\left(\tau^{\frac{1}{2}} - \tilde{\tau}^{\frac{1}{2}}\right)\cdot(\tau+\tilde{\tau}) \simeq \hbar^{-1}\frac{\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}}\cdot\left(\tau^{\frac{1}{2}} + \tilde{\tau}^{\frac{1}{2}}\right) \simeq R\cdot\left(\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\right)\cdot\left(\tau^{\frac{1}{2}} + \tilde{\tau}^{\frac{1}{2}}\right),$$

which implies

$$\left|\frac{\partial\tilde{\Psi}(\omega;\alpha,\beta,\hbar)}{\partial\omega}\right| \lesssim R^{-1}\cdot\frac{\tau^{\frac{1}{2}} + \tilde{\tau}^{\frac{1}{2}}}{\left|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\right|} \lesssim \left|\omega^{-1}\right|.$$

We have used here the easily verified symbol behavior of $\tilde{\Psi}$ with respect to R. The higher order derivatives are treated analogously. Finally we write

$$\int_0^\infty \chi_{x\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]}\chi_{\tilde{x}\in\left[x_t+\hbar^{\frac{2}{3}},2x_t\right]}\chi_{\tau\lesssim\tilde{\tau}}\cdot e^{i\omega}\tilde{\Psi}(\omega;\alpha,\beta,\hbar)\,d\omega =: H(\xi,\eta),$$

where we have $|H(\xi,\eta)| \lesssim 1$ and symbol behavior with respect to ξ,η. But (keeping in mind that $\xi \simeq \eta$ on the support of H) it is easy to then verify the bound

$$\left\|\int_0^\infty \frac{\xi^{\frac{1}{2}}\chi_{\xi\simeq\eta}H(\xi,\eta)\cdot(\xi\eta)^{-\frac{1}{4}}}{\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}}\cdot x(\eta)\,d\eta\right\|_{L^2_{d\xi}} \lesssim \left\|\eta^{\frac{1}{2}}x\right\|_{L^2_{d\eta}},$$

as desired.

Oscillatory regime: $R\xi^{\frac{1}{2}}\hbar > 2x_t$. Here, in analogy to the analysis of the transference operator, we have to understand the distribution valued kernel

$$\lim_{A\to\infty}\int_0^A \partial_R\phi_n(R;\xi)\phi_n(R;\eta,\hbar)R\,dR =: F(\xi,\eta),$$

and show that it induces an operator obeying the asserted bound. For this we again use the oscillatory expansion

$$\phi_n(R;\xi) = 2\pi^{-\frac{1}{2}}\left(R\xi^{\frac{1}{2}}\right)^{-\frac{1}{2}}(\tau q(x;\alpha,\hbar))^{-\frac{1}{4}}\,\mathrm{Re}\left(a(\xi)e^{i\frac{2}{3\hbar}\tau^{\frac{3}{2}}}(1+\hbar a_1(\tau,\alpha,\hbar))\right),$$

where the description of the variable τ in terms of $x,\alpha,\hbar$ is furnished by Lemma 4.13. We only deal with the most delicate case when $\xi\simeq\eta$ and the resonant case where the phases cancel. The combined phases arising will then be of the form, by Lemma 4.13,

$$e^{\pm i\left[R\left(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right)-\hbar^{-1}y(\alpha;\hbar)+\hbar^{-1}y(\beta;\hbar)-\hbar^{-1}\rho(x;\alpha,\hbar)+\hbar^{-1}\rho(\tilde{x};\beta,\hbar)\right]}.$$

When the operator ∂_R hits the non-phase factor $(1+\hbar a_1(\tau,\alpha,\hbar))$, using Lemma 4.29 we obtain an extra factor

$$\lesssim \hbar x^{-\frac{1}{3}}R^{-1}$$

and hence the corresponding term can be integrated over the region $R\xi^{\frac{1}{2}}\hbar\gtrsim 1$ to give a contribution to $\chi_{\xi\simeq\eta}F(\xi,\eta)$ of size $\lesssim\xi^{-\frac{1}{2}}$. By Minkowski inequality and the fact $\xi\simeq\eta$, such a bound for $\chi_{\xi\simeq\eta}F(\xi,\eta)$ gives

$$\left\|\int_0^\infty \chi_{\xi\simeq\eta}F(\xi,\eta)x(\eta)\,d\eta\right\|_{L^2_{d\xi}}\lesssim\left\|\eta^{\frac{1}{2}}x(\eta)\right\|_{L^2_{d\eta}}.$$

We henceforth only consider the contribution arising when ∂_R hits the phase associated with ξ. This then results in the term (coming purely from the phase, and omitting constant coefficients)

$$\begin{aligned}&\mathrm{Re}\left(a(\xi)\overline{a(\eta)}\right)\xi^{\frac{1}{2}}\cdot[1-\rho_x(x;\alpha,\hbar)]\\&\cdot\sin\left[R(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})-\hbar^{-1}y(\alpha;\hbar)+\hbar^{-1}y(\beta;\hbar)-\hbar^{-1}\rho(x;\alpha,\hbar)+\hbar^{-1}\rho(\tilde{x};\beta,\hbar)\right]\\&+\xi^{\frac{1}{2}}\cdot[1-\rho_x(x;\alpha,\hbar)]\,\mathrm{Im}\left(a(\eta)\overline{a(\xi)}\right)\\&\cdot\cos\left[R(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})-\hbar^{-1}y(\alpha;\hbar)+\hbar^{-1}y(\beta;\hbar)-\hbar^{-1}\rho(x;\alpha,\hbar)+\hbar^{-1}\rho(\tilde{x};\beta,\hbar)\right].\end{aligned}\tag{6.17}$$

Using the bounds on $a(\cdot)$ and its derivatives, we have

$$\chi_{\xi\simeq\eta}\left(a(\eta)\overline{a(\xi)}-a(\xi)\overline{a(\eta)}\right)=\chi_{\xi\simeq\eta}\left((a(\eta)-a(\xi))\,\overline{a(\xi)}+a(\xi)\left(\overline{a(\xi)}-\overline{a(\eta)}\right)\right)=\frac{(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})}{\xi^{\frac{1}{2}}}\cdot b(\xi,\eta),$$

where b itself obeys symbol behavior. Writing the phase as $\left(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right)\cdot\tilde{R}$, $\tilde{R}=\tilde{R}(R;\alpha,\beta,\hbar)$, we have in our regime $\hbar\xi^{\frac{1}{2}}R\simeq\hbar\eta^{\frac{1}{2}}R>(1+c)x_t$, $c>0$ that

$$\frac{d\tilde{R}}{dR}\simeq 1$$

uniformly in all parameters. The contribution of the second term in (6.17) can then be handled by integration by parts: omitting constant terms and proceeding schematically, we reduce this to

$$\frac{b(\xi,\eta)}{\xi^{\frac{1}{2}}}\int_0^\infty[1-\rho_x(x;\alpha,\hbar)]\cdot(1+\hbar a_1(\tau,\alpha,\hbar))(1+\hbar a_1(\tilde{\tau},\beta,\hbar))\cdot\partial_{\tilde{R}}\left(\sin\left(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right)\right)\cdot\left(\frac{d\tilde{R}}{dR}\right)^{-1}d\tilde{R}.$$

Carrying out the integration by parts again leads to a contribution to $\chi_{\xi\sim\eta}F(\xi,\eta)$ of size $\lesssim \xi^{-\frac{1}{2}}$.
We can thus reduce things to the first, principal term in (6.17), which is the following contribution to $F(\xi,\eta)$:

$$\frac{\mathrm{Re}\left(a(\xi)\overline{a(\eta)}\right)}{(\xi\eta)^{\frac{1}{4}}}\xi^{\frac{1}{2}}$$
$$\cdot \lim_{A\to\infty}\int_0^A \chi_{R\xi^{\frac{1}{2}}\hbar>2x_t}\left[1-\rho_x(x;\alpha,\hbar)\right]\cdot\frac{(1+\hbar a_1(\tau,\alpha,\hbar))}{(q(x;\alpha,\hbar)\tau)^{\frac{1}{4}}}\frac{(1+\hbar a_1(\tilde{\tau},\beta,\hbar))}{(q(\tilde{x};\beta,\hbar)\tilde{\tau})^{\frac{1}{4}}}\cdot\sin\left[\tilde{R}\left(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right)\right]\cdot\left(\frac{d\tilde{R}}{dR}\right)^{-1}d\tilde{R} \tag{6.18}$$

Writing

$$\chi_{R\xi^{\frac{1}{2}}\hbar>2x_t}[1-\rho_x(x;\alpha,\hbar)]\cdot\frac{(1+\hbar\tilde{a}_1(\tau,\alpha,\hbar))}{(q(x;\alpha,\hbar)\tau)^{\frac{1}{4}}}\frac{(1+\hbar\tilde{a}_1(\tilde{\tau},\beta,\hbar))}{(q(\tilde{x};\beta,\hbar)\tilde{\tau})^{\frac{1}{4}}}\cdot\left(\frac{d\tilde{R}}{dR}\right)^{-1}=\Phi(\tilde{R};\alpha,\beta,\hbar)$$

and

$$\Phi(\tilde{R};\alpha,\beta,\hbar)=1-\chi_{R\xi^{\frac{1}{2}}\hbar<2x_t}+\Psi(\tilde{R};\alpha,\beta,\hbar),$$

with $\left|\Psi(\tilde{R};\alpha,\beta,\hbar)\right|\lesssim x^{-1}$ and symbol behavior with respect to its arguments. Then we have

$$\chi_{\xi\simeq\eta}\frac{a(\xi)\overline{a(\eta)}}{(\xi\eta)^{\frac{1}{4}}}\xi^{\frac{1}{2}}\cdot\lim_{A\to\infty}\int_0^A\sin\left[\tilde{R}(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})\right]d\tilde{R}=c\chi_{\xi\simeq\eta}\frac{a(\xi)\overline{a(\eta)}}{(\xi\eta)^{\frac{1}{4}}}\xi^{\frac{1}{2}}\cdot\left(\frac{1}{\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}}\right)_{P.V.}$$

and acts like a Hilbert transform like operator which is easily seen to satisfy the desired bound. Furthermore we can write

$$\lim_{A\to\infty}\int_0^A\chi_{R\xi^{\frac{1}{2}}\hbar<2x_t}\sin\left[\tilde{R}(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})\right]d\tilde{R}=c\left(\left(\frac{1}{\cdot}\right)_{P.V.}*m\right)(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})$$

for a function m of bounded L^1-norm (independent of $\hbar$), and hence this contribution can be bounded analogously to the preceding one.
Finally, for the operator

$$\chi_{\xi\simeq\eta}\frac{a(\xi)\overline{a(\eta)}}{(\xi\eta)^{\frac{1}{4}}}\xi^{\frac{1}{2}}\cdot\lim_{A\to\infty}\int_0^A\Psi(\tilde{R};\alpha,\beta,\hbar)\sin\left[\tilde{R}(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})\right]d\tilde{R}$$
$$=\chi_{\xi\simeq\eta}\frac{a(\xi)\overline{a(\eta)}}{(\xi\eta)^{\frac{1}{4}}}\xi^{\frac{1}{2}}\cdot\sum_{\lambda\gtrsim 2x_t}\lim_{A\to\infty}\int_0^A\chi_{x\simeq\lambda}\Psi(\tilde{R};\alpha,\beta,\hbar)\sin\left[\tilde{R}(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})\right]d\tilde{R},$$

where the sum is over dyadic λ, we use that

$$\lim_{A\to\infty}\int_0^A\chi_{x\simeq\lambda}\Psi(\tilde{R};\alpha,\beta,\hbar)\sin\left[\tilde{R}(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})\right]d\tilde{R}=c\left(\left(\frac{1}{\cdot}\right)_{P.V.}*m_\lambda\right)(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}})$$

where $\|m_\lambda\|_{L^1}\lesssim\lambda^{-1}$, and we can bound this contribution as before and sum over all dyadic $\lambda\gtrsim x_t$. □

In a similar vein we have the following:

Proposition 6.8. *Assuming, with $\hbar = \frac{1}{n+1}$, $|n| \geq 2$,*

$$f(R) = \int_0^\infty \phi_n(R;\xi)\overline{x}(\xi)\rho_n(\xi)\,d\xi,$$

we have for $k+\ell \ll n$ the bound

$$\left\|\left(\frac{n}{R}\right)^k \partial_R^\ell f(R)\right\|_{L^2_{R\,dR}} \lesssim_{k,\ell} \left\|\xi^{\frac{k+\ell}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}},$$

the implied constant being uniform in $\hbar$.

Proof. This is similar to the preceding proposition. One exploits that for $x = R\xi^{\frac{1}{2}}\hbar \gtrsim 1$, we have

$$\left(\frac{n}{R}\right)^k \sim \left(\frac{1}{R\hbar}\right)^k \lesssim \xi^{\frac{k}{2}},$$

while if $x = R\xi^{\frac{1}{2}}\hbar < \frac{x_t}{2}$, we have

$$\left|\left(\frac{n}{R}\right)^k \phi_n(R;\xi)\right| \lesssim \hbar^{\frac{1}{2}}\left(R\xi^{\frac{1}{2}}\hbar\right)^{n-1-k}\cdot\xi^{\frac{k}{2}}.$$

We omit the rest similar details. □

The following lemma gives the L^∞-bound in terms of the $S_0^\hbar$-norm:

Lemma 6.9. *Let*

$$f(R) = \int_0^\infty \phi_n(R;\xi)\overline{x}(\xi)\rho_n(\xi)\,d\xi.$$

Then for $k \geq 1$ we can estimate

$$\left\|\partial_R^k f\right\|_{L^\infty_{dR}} \lesssim \hbar^{-\frac{3}{2}-\delta}\left\|(\xi\hbar^2)^{1-\frac{\delta}{2}}\langle\xi\hbar^2\rangle^\delta\xi^{\frac{k-1}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}}.$$

In particular we have (with $\hbar = \frac{1}{n+1}$)

$$\|\partial_R f\|_{L^\infty_{dR}} \lesssim \hbar^{-\frac{3}{2}-\delta}\|\overline{x}\|_{S_0^\hbar}.$$

Finally, we also have, for $k+\ell \geq 1$,

$$\left\|\left(\frac{n}{R}\right)^\ell \partial_R^k f\right\|_{L^\infty_{dR}} \lesssim \hbar^{-2-\delta}\left\|(\xi\hbar^2)^{1-\frac{\delta}{2}}\langle\xi\hbar^2\rangle^\delta\xi^{\frac{k+\ell-1}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}}.$$

Proof. We treat the case $k = 1$. The higher order derivatives can be handled similarly. From the proof of Proposition 6.7, we have

$$|\partial_R\phi_n(R,\xi)| \lesssim \hbar^{\frac{1}{2}}\xi^{\frac{1}{2}}.$$

Then we have

$$\|\partial_R f\|_{L^\infty_{dR}} \lesssim \hbar^{\frac{1}{2}}\int_0^\infty \xi^{\frac{1}{2}}\,|\overline{x}(\xi)|\,d\xi$$

$$\lesssim \hbar^{\frac{1}{2}}\left(\int_0^1 \left|\xi^{1-\frac{\delta}{2}}\overline{x}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \cdot \left(\int_0^1 \xi^{-1+\delta}\, d\xi\right)^{\frac{1}{2}}$$
$$+ \hbar^{\frac{1}{2}}\left(\int_1^\infty \left|\xi^{1+\frac{\delta}{2}}\overline{x}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \cdot \left(\int_1^\infty \xi^{-1-\delta}\, d\xi\right)^{\frac{1}{2}}$$
$$\lesssim \hbar^{-\frac{3}{2}-\delta}\left\|(\xi\hbar^2)^{1-\frac{\delta}{2}}\langle\xi\hbar^2\rangle^{\delta}\overline{x}(\xi)\right\|_{L^2_{d\xi}} \lesssim \hbar^{-\frac{3}{2}-\delta}\left\|\overline{x}\right\|_{S_0^{\hbar}}.$$

□

We now state the first main proposition on multilinear estimates. The assumption here is that we have the following Fourier representations

$$\phi_j(R) = \int_0^\infty \phi_{n_j}(R;\xi)\overline{x}_j(\xi)\rho_{n_j}(\xi)\, d\xi, \quad j = 1,\, 2, \quad \overline{x}_j(\xi) \in S_0^{\hbar_j}. \tag{6.19}$$

Proposition 6.10. *Assume that $\hbar_1 \lesssim \hbar_2 \ll 1$, and either $\hbar_3 \simeq \hbar_1$ or $\hbar_3 \gg \hbar_1, |\hbar_1| \simeq |\hbar_2|$. Then we have the bound*

$$\left\|\langle\phi_{n_3}(R;\xi),\, \partial_R\phi_1\cdot\partial_R\phi_2\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}} \lesssim \hbar_2^{-2}\prod_{j=1,2}\left\|\overline{x}_j\right\|_{S_0^{\hbar_j}}.$$

Proof. We treat the case $\hbar_1 \simeq \hbar_3$ in detail. The second case is similar and simpler, in the sense that this case happens when $\hbar_1$ and $\hbar_2$ have opposite signs and n_1, n_2 almost cancel each other. Therefore there is extra decay in $|\hbar_1|$ and $|\hbar_2|$ in the product $\prod_{j=1,2}\left\|\overline{x}_j\right\|_{S_0^{\hbar_j}}$. We distinguish between the case when the output frequency ξ is less than the maximal frequency of the factors, and the case when it is larger. Call the output frequency ξ and those of the factors ϕ_j ξ_j, $j = 1, 2$.

(1): $\xi < \max\{\xi_1, \xi_2\}$. Due to the asymmetry of the situation, we split this further into two sub-cases.

(1.a): $\xi < \xi_1$. We write this contribution as

$$\sum_{\mu<\lambda}\chi_{\xi\simeq\mu}\langle\phi_{n_3}(R;\xi),\, \partial_R\phi_{1,\lambda}\cdot\partial_R\phi_2\rangle_{L^2_{R\,dR}}$$

where μ, λ range over dyadic numbers, and we use the notation

$$\phi_{1,\lambda} = \int_0^\infty \chi_{\xi_1\simeq\lambda}\phi_{n_1}(R;\xi_1)\overline{x}_j(\xi_1)\rho_{n_1}(\xi_1)\, d\xi_1$$

By Plancherel's theorem for the distorted Fourier transform we have (recall $\hbar_3 \simeq \hbar_1$)

$$\left\|\chi_{\xi\simeq\mu}\langle\phi_{n_3}(R;\xi),\, \partial_R\phi_{1,\lambda}\cdot\partial_R\phi_2\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}$$
$$\lesssim \hbar_1(\mu\hbar_1^2)^{\frac{1}{2}-\frac{\delta}{2}}\langle\mu\hbar_1^2\rangle^{\delta+\frac{3}{2}}\left\|\langle\phi_{n_3}(R;\xi),\, \partial_R\phi_{1,\lambda}\cdot\partial_R\phi_2\rangle_{L^2_{R\,dR}}\right\|_{L^2_{d\xi}}$$
$$\simeq \hbar_1(\mu\hbar_1^2)^{\frac{1}{2}-\frac{\delta}{2}}\langle\mu\hbar_1^2\rangle^{\delta+\frac{3}{2}}\left\|\langle\phi_{n_3}(R;\xi),\, \partial_R\phi_{1,\lambda}\cdot\partial_R\phi_2\rangle_{L^2_{R\,dR}}\right\|_{L^2_{\rho_{n_3}(\xi)d\xi}}$$
$$\lesssim \hbar_1(\mu\hbar_1^2)^{\frac{1}{2}-\frac{\delta}{2}}\langle\mu\hbar_1^2\rangle^{\delta+\frac{3}{2}}\left\|\partial_R\phi_{1,\lambda}\cdot\partial_R\phi_2\right\|_{L^2_{R\,dR}}.$$

The last $L^2_{R\,dR}$-norm of the product can be estimated using Proposition 6.8 and Lemma 6.9:

$$\left\|\partial_R\phi_{1,\lambda}\cdot\partial_R\phi_2\right\|_{L^2_{R\,dR}} \lesssim \left\|\partial_R\phi_{1,\lambda}\right\|_{L^2_{R\,dR}}\cdot\|\partial_R\phi_2\|_{L^\infty_{R\,dR}}$$
$$\lesssim \lambda^{\frac{1}{2}}\,\|\overline{x}_1\|_{L^2_{d\xi_1}(\xi_1\simeq\lambda)}\cdot\hbar_2^{-\frac{3}{2}-\delta}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}}\,.$$

Combining the two preceding bounds, we infer that

$$\left\|\chi_{\xi\simeq\mu}\,\langle\phi(R;\xi,\hbar_3),\,\partial_R\phi_{1,\lambda}\cdot\partial_R\phi_2\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}$$
$$\lesssim \hbar_2^{-\frac{3}{2}-\delta}\lambda^{\frac{1}{2}}\hbar_1(\mu\hbar_1^2)^{\frac{1}{2}-\frac{\delta}{2}}\langle\mu\hbar_1^2\rangle^{\delta+\frac{3}{2}}\,\|\overline{x}_1\|_{L^2_{d\xi_1}(\xi_1\simeq\lambda)}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}}$$
$$\lesssim \hbar_2^{-\frac{3}{2}-\delta}\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\lambda)}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}}\,.$$

Finally exploiting orthogonality as well as the Cauchy-Schwarz inequality, we infer that

$$\left\|\sum_{\mu<\lambda}\chi_{\xi\simeq\mu}\,\langle\phi_{n_3}(R;\xi),\,\partial_R\phi_{1,\lambda}\cdot\partial_R\phi_2\rangle_{L^2_{R\,dR}}\right\|^2_{S_1^{\hbar_3}}$$
$$\lesssim \sum_{\mu}\left\|\sum_{\lambda>\mu}\chi_{\xi\simeq\mu}\,\langle\phi_{n_3}(R;\xi),\,\partial_R\phi_{1,\lambda}\cdot\partial_R\phi_2\rangle_{L^2_{R\,dR}}\right\|^2_{S_1^{\hbar_3}}$$
$$\lesssim \hbar_2^{-3-2\delta}\sum_{\mu<\lambda}\left(\frac{\mu}{\lambda}\right)^{1-\delta}\cdot\|\overline{x}_1\|^2_{S_0^{\hbar_1}(\xi_1\simeq\lambda)}\cdot\|\overline{x}_2\|^2_{S_0^{\hbar_2}}$$
$$\lesssim \hbar_2^{-3-2\delta}\,\|\overline{x}_1\|^2_{S_0^{\hbar_1}}\cdot\|\overline{x}_2\|^2_{S_0^{\hbar_2}}\,,$$

which is desired.

(1.b): $\xi_1\le\xi<\xi_2$. Write this term as

$$\sum_{\mu<\lambda}\chi_{\xi\simeq\mu}\langle\phi_{n_3}(R;\xi),\,\partial_R\phi_{1,\le\mu}\cdot\partial_R\phi_{2,\lambda}\rangle_{L^2_{R\,dR}}.$$

This time we use Lemma 6.9 to bound the factor $\partial_R\phi_{2,\lambda}$: as in the preceding case, it suffices to bound

$$\hbar_1(\mu\hbar_1^2)^{\frac{1}{2}-\frac{\delta}{2}}\langle\mu\hbar_1^2\rangle^{\delta+\frac{3}{2}}\cdot\left\|\partial_R\phi_{1,\le\mu}\cdot\partial_R\phi_{2,\lambda}\right\|_{L^2_{R\,dR}}$$
$$\lesssim \hbar_1(\mu\hbar_1^2)^{\frac{1}{2}-\frac{\delta}{2}}\langle\mu\hbar_1^2\rangle^{\delta+\frac{3}{2}}\left\|\partial_R\phi_{1,\le\mu}\right\|_{L^\infty_{R\,dR}}\cdot\left\|\partial_R\phi_{2,\lambda}\right\|_{L^2_{R\,dR}}$$
$$\lesssim \hbar_2^{-2+\delta}\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\hbar_1^{\frac{1}{2}-2\delta}\cdot(\lambda\hbar_2^2)^{1-\frac{\delta}{2}}\langle\lambda\hbar_2^2\rangle^{\delta+\frac{3}{2}}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\le\mu)}\cdot\|\overline{x}_2\|_{L^2_{d\xi}(\xi_2\simeq\lambda)}$$
$$\sim \hbar_2^{-2+\delta}\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\hbar_1^{\frac{1}{2}-2\delta}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\le\mu)}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi_2\sim\lambda)}\,.$$

From this point the estimate can be completed just as in the preceding case.

We next deal with the case where the output frequency ξ dominates the input frequencies:

(2): $\xi \geq \max\{\xi_1, \xi_2\}$. Here we perform integration by parts in order to gain in terms of $\frac{\max\{\xi_1,\xi_2\}}{\xi}$. We can do this rather carelessly in the present situation since we are assuming that all angular parameters $\hbar_j$, $j = 1, 2$, are very small, which ensures sufficient vanishing at the origin $R = 0$. We have to exploit that $\phi_{n_3}(R;\xi)$ is a generalized eigenfunction of $H^{\pm}_{n_3}$, $\hbar_3 = \frac{1}{n_3+1}$ which we denote as H_{n_3} for simplicity. Then we have

$$-H_{n_3}\phi_{n_3}(R;\xi) = \xi\phi_{n_3}(R;\xi)$$

We apply H_{n_3} three times and integrate by parts, resulting schematically in

$$\begin{aligned}
&\xi^3 \left\langle \phi_{n_3}(R;\xi),\, \partial_R\phi_{1,<\xi} \cdot \partial_R\phi_{2,<\xi} \right\rangle_{L^2_{R\,dR}} \\
&= \sum_{i+j+k=6} \left\langle \phi_{n_3}(R;\xi),\, \left(\frac{n_3}{R}\right)^i \partial_R^{1+j}\phi_{1,<\xi} \cdot \partial_R^{1+k}\phi_{2,<\xi} \right\rangle_{L^2_{R\,dR}},
\end{aligned}$$

where we have only included the most singular terms, the remaining ones being much simpler to treat. Importantly note that we combine the factors $\frac{n_3}{R}$ with $\phi_{1,<\xi}$, in order not to lose inverse powers of $\hbar_1 \simeq \hbar_3$. In order to proceed we dyadically localize the output frequency $\xi \simeq \lambda$ and the inner factors to dyadic frequencies μ_1, μ_2 respectively. Invoking orthogonality, we infer

$$\begin{aligned}
&\left\| \left\langle \phi_{n_3}(R;\xi),\, \partial_R\phi_{1,<\xi} \cdot \partial_R\phi_{2,<\xi} \right\rangle_{L^2_{R\,dR}} \right\|^2_{S_1^{\hbar_3}} \\
&\simeq \sum_{\lambda} \left\| \left\langle \phi_{n_3}(R;\xi),\, \partial_R\phi_{1,<\xi} \cdot \partial_R\phi_{2,<\xi} \right\rangle_{L^2_{R\,dR}} \right\|^2_{S_1^{\hbar_3}(\xi\simeq\lambda)} \\
&\lesssim \sum_{i+j+k=6} \sum_{\lambda} \lambda^{-6} \left\| \left\langle \phi_{n_3}(R;\xi),\, \left(\frac{n_3}{R}\right)^i \partial_R^{1+j}\phi_{1,<\xi} \cdot \partial_R^{1+k}\phi_{2,<\xi} \right\rangle_{L^2_{R\,dR}} \right\|^2_{S_1^{\hbar_3}(\xi\simeq\lambda)} \\
&\lesssim \sum_{i+j+k=6} \sum_{\lambda} \lambda^{-6} \left\| \sum_{\mu_1,\mu_2<\lambda} \left\langle \phi_{n_3}(R;\xi),\, \left(\frac{n_3}{R}\right)^i \partial_R^{1+j}\phi_{1,\mu_1} \cdot \partial_R^{1+k}\phi_{2,\mu_2} \right\rangle_{L^2_{R\,dR}} \right\|^2_{S_1^{\hbar_3}(\xi\simeq\lambda)}.
\end{aligned}$$

Using Plancherel's theorem for the distorted Fourier transform we obtain, for $\mu_2 \leq \mu_1$,

$$\begin{aligned}
&\lambda^{-3} \left\| \sum_{\mu_1,\mu_2<\lambda} \left\langle \phi_{n_3}(R;\xi),\, \left(\frac{n_3}{R}\right)^i \partial_R^{1+j}\phi_{1,\mu_1} \cdot \partial_R^{1+k}\phi_{2,\mu_2} \right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}} \\
&\lesssim \frac{(\hbar_3^2\lambda)^{1-\frac{\delta}{2}} \langle \hbar_3^2\lambda \rangle^{\delta+\frac{3}{2}}}{\lambda^{\frac{7}{2}}} \sum_{\mu_{1,2}<\lambda} \left\| \left(\frac{n_3}{R}\right)^i \partial_R^{1+j}\phi_{1,\mu_1} \right\|_{L^2_{R\,dR}} \cdot \left\| \partial_R^{1+k}\phi_{2,\mu_2} \right\|_{L^\infty_{R\,dR}}.
\end{aligned} \tag{6.20}$$

By Proposition 6.8 as well as Lemma 6.9, we bound the preceding by

$$\hbar_2^{-\frac{3}{2}-\delta}\frac{(\hbar_3^2\lambda)^{1-\frac{\delta}{2}}\langle\hbar_3^2\lambda\rangle^{\delta+\frac{3}{2}}}{\lambda^{\frac{7}{2}}}\sum_{\mu_{1,2}<\lambda}\mu_1^{\frac{1+i+j}{2}}\|\overline{x}_1\|_{L^2_{d\xi}(\xi_1\simeq\mu_1)}\cdot\mu_2^{\frac{k}{2}}\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi_2\simeq\mu_2)}$$

$$\lesssim\hbar_2^{-\frac{3}{2}-\delta}\sum_{\mu_{1,2}<\lambda}\frac{(\hbar_3^2\lambda)^{1-\frac{\delta}{2}}\langle\hbar_3^2\lambda\rangle^{\delta+\frac{3}{2}}}{(\hbar_1^2\mu_1)^{1-\frac{\delta}{2}}\langle\hbar_1^2\mu_1\rangle^{\delta+\frac{3}{2}}}\cdot\left(\frac{\mu_1}{\lambda}\right)^{\frac{1+i+j}{2}}\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\cdot\left(\frac{\mu_2}{\lambda}\right)^{\frac{k}{2}}\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\mu_2)}.$$

Finally, observe that

$$\frac{(\hbar_3^2\lambda)^{1-\frac{\delta}{2}}\langle\hbar_3^2\lambda\rangle^{\delta+\frac{3}{2}}}{(\hbar_1^2\mu_1)^{1-\frac{\delta}{2}}\langle\hbar_1^2\mu_1\rangle^{\delta+\frac{3}{2}}}\cdot\left(\frac{\mu_1}{\lambda}\right)^{\frac{1+i+j}{2}}\cdot\left(\frac{\mu_2}{\lambda}\right)^{\frac{k}{2}}\lesssim\prod_{j=1,2}\left(\frac{\mu_j}{\lambda}\right)^{\frac{1}{4}}.$$

We conclude by using the Cauchy-Schwarz inequality to infer that

$$\hbar_2^{-\frac{3}{2}-\delta}\left(\sum_{\lambda}\left[\sum_{\mu_{1,2}<\lambda}\prod_{j=1,2}\left(\frac{\mu_j}{\lambda}\right)^{\frac{1}{4}}\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu_1)}\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\mu_2)}\right]^2\right)^{\frac{1}{2}}$$
$$\lesssim\hbar_2^{-\frac{3}{2}-\delta}\|\overline{x}_1\|_{S_0^{\hbar_1}}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}},$$

as desired. If $\mu_1\leq\mu_2$, then we bound (6.20) as

$$\frac{(\hbar_3^2\lambda)^{1-\frac{\delta}{2}}\langle\hbar_3^2\lambda\rangle^{\delta+\frac{3}{2}}}{\lambda^{\frac{7}{2}}}\sum_{\mu_{1,2}<\lambda}\left\|\left(\frac{n_3}{R}\right)^i\partial_R^{1+j}\phi_{1,\mu_1}\right\|_{L^\infty_{R\,dR}}\cdot\left\|\partial_R^{1+k}\phi_{2,\mu_2}\right\|_{L^2_{R\,dR}}.$$

By Proposition 6.7 and Lemma 6.9 we bound this as

$$\hbar_1^{-\frac{3}{2}-\delta}\frac{(\hbar_3^2\lambda)^{1-\frac{\delta}{2}}\langle\hbar_3^2\lambda\rangle^{\delta+\frac{3}{2}}}{\lambda^{\frac{7}{2}}}\sum_{\mu_{1,2}<\lambda}\mu_1^{\frac{i+j}{2}}\|\overline{x}_1\|_{S_1^{\hbar_1}(\xi_1\simeq\mu_1)}\cdot\mu_2^{\frac{k+1}{2}}\|\overline{x}_2\|_{L^2_{d\xi}(\xi_2\simeq\mu_2)}$$

$$\lesssim\hbar_3^{\frac{1}{2}}\hbar_2^{-2+\delta}\sum_{\mu_{1,2}<\lambda}\frac{\lambda^{1-\frac{\delta}{2}}\langle\hbar_3^2\lambda\rangle^{\delta+\frac{3}{2}}}{\mu_2^{1-\frac{\delta}{2}}\langle\hbar_2^2\mu_2\rangle^{\delta+\frac{3}{2}}}\cdot\left(\frac{\mu_1}{\lambda}\right)^{\frac{i+j}{2}}\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\cdot\left(\frac{\mu_2}{\lambda}\right)^{\frac{1+k}{2}}\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\mu_2)}.$$

The rest argument is similar to the case when $\mu_2\leq\mu_1$. □

We next aim to derive an analogue of the preceding proposition but with all angular frequencies $\hbar_j\gtrsim 1$, again assuming the compatibility conditions on the $\hbar_j$ as in the proposition. Thus in that situation we no longer worry about losses in the $\hbar_j^{-1}$, but instead we need to worry about the last step in the proof, where we transferred the operator H_n from the left side of the inner product to the right.

6.5. **The good spaces for the exceptional angular momenta $n=0,\pm1$.** The next estimate will be formulated for arbitrary inputs, also those for angular momenta $n=0,\pm1$. Thus we need to introduce a norm for these as well.

$n = -1$. Recall from (4.61)

$$\mathcal{D}_- f(R) = \int_0^\infty y(\xi)\phi_-(R,\xi)\tilde{\rho}_{-1}(\xi)d\xi, \quad y(\xi) := \langle \mathcal{D}_- f(R),\, \phi_-(R,\xi)\rangle_{L^2_{R\,dR}}.$$

To pass back from this formula to the underlying function f, write it as

$$f(R) = c_- \cdot \phi_{-1}(R) + \phi_{-1}(R)\cdot \int_0^R (\phi_{-1}(s))^{-1}\,\mathcal{D}_- f(s)\,ds, \quad \phi_{-1}(R) := \frac{R^2}{1+R^2}. \tag{6.21}$$

Recall the asymptotics of the spectral measure

$$\tilde{\rho}_{-1}(\xi) \sim \xi\langle\xi\rangle^2.$$

In the sequel, it will also be important to recall the asymptotic bounds for the Fourier basis $\phi_{-1}(R,\xi)$, which gives $\left|\phi_{-1}(R,\xi)\right| \lesssim R^3/\langle R\rangle^2$, whence linear growth toward $R = +\infty$. Then we introduce the norm

$$\|y(\xi)\|_{S_0^-} := \left\|\xi^{1-\frac{\delta}{2}}\langle\xi\rangle^{\frac{5}{2}+\delta}y(\xi)\right\|_{L^2_{d\xi}}. \tag{6.22}$$

Then we shall describe the function $f(R)$ in terms of the pair (c_-, y). For later reference we also introduce the space S_1^- such that $\|\cdot\|_{S_1^-} := \|\xi^{-\frac{1}{2}}\cdot\|_{S_0^-}$.

$n = 0$. We recall from [16] the representation

$$\mathcal{D}f(R) = \int_0^\infty y(\xi)\phi_0(R,\xi)\tilde{\rho}_0(\xi)d\xi, \quad y(\xi) := \langle \mathcal{D}f(R),\, \phi_0(R,\xi)\rangle_{L^2_{R\,dR}}.$$

Here we use the notation $\phi_0(R,\xi) := \phi(R,\xi)$ where the latter Fourier basis is described in [16], and the spectral measure $\tilde{\rho}_0$ corresponds to $\tilde{\rho}(\xi)$ in loc. cit.. Moreover, denoting by $\phi_0(R) := \frac{R}{1+R^2}$ the resonance at zero frequency, we have the representation

$$f(R) = c_0\cdot\phi_0(R) + \phi_0(R)\cdot\int_0^R \phi_0^{-1}(s)\mathcal{D}f(s)\,ds, \quad \phi_0(R) := \frac{R}{1+R^2}. \tag{6.23}$$

Here we recall the spectral asymptotics

$$\tilde{\rho}_0(\xi) \sim \langle\log\xi\rangle^{-2}, \quad \xi \lesssim 1,$$
$$\tilde{\rho}_0(\xi) \sim \xi^2, \quad \xi \gg 1.$$

Then we introduce the norm

$$\|y(\xi)\|_{S_0^0} := \left\|\left(\frac{\langle\log\langle\xi\rangle\rangle}{\langle\log\xi\rangle}\right)^{1+\delta}\xi^{\frac{1}{2}}\langle\xi\rangle^{\frac{5}{2}+\frac{\delta}{2}}y(\xi)\right\|_{L^2_{d\xi}}. \tag{6.24}$$

Remark 6.11. *This norm is similar to the one use in [16], except it reflects H^{4+} regularity (for the derivative $\mathcal{D}f$) rather than H^{3+}-regularity as in [16]; moreover, the logarithmic weight is slightly altered here in order for this norm to exactly match the regularity for the other n at high frequencies.*

$n = 1$. Recall from Lemma 4.2 the representation

$$\mathcal{D}_+ f(R) = \frac{2}{\pi^2}\int_0^\infty y(\xi)\phi_1(R,\xi)\xi d\xi, \quad y(\xi) := \langle \mathcal{D}_+ f(R),\, \phi_1(R,\xi)\rangle_{L^2_{R\,dR}}.$$

where we set $\phi_1(R,\xi)$ is identical to $\phi(R,z)$ in (4.21) with $z = \xi$. In the following we will denote by $\widetilde{\rho}_1(\xi) := \frac{2}{\pi^2}\xi$ the spectral measure associated to $\phi_1(R,\xi)$. Then we have the representation

$$f(R) = c_1 \cdot \phi_1(R) + \phi_1(R) \cdot \int_0^R \phi_1(s)^{-1}\mathcal{D}_+ f(s)\,ds, \quad \phi_1(R) := \frac{1}{1+R^2}. \tag{6.25}$$

Then we introduce the norm

$$\|y(\xi)\|_{S_0^+} := \left\| \xi^{1-\frac{\delta}{2}} \langle\xi\rangle^{\frac{3}{2}+\delta} y(\xi) \right\|_{L^2_{d\xi}}. \tag{6.26}$$

We then have the following basic analogous of Proposition 6.7 and Lemma 6.9:

Proposition 6.12. *Let*

$$\begin{aligned} f_{-1}(R) &= c_{-1} \cdot \phi_{-1}(R) + \phi_{-1}(R) \cdot \int_0^R \phi_{-1}(s)^{-1}\mathcal{D}_- f_{-1}(s)\,ds \\ f_0(R) &= c_0 \cdot \phi_0(R) + \phi_0(R) \cdot \int_0^R \phi_0(s)^{-1}\mathcal{D} f_0(s)\,ds \\ f_1(R) &= c_1 \cdot \phi_1(R) + \phi_1(R) \cdot \int_0^R \phi_1(s)^{-1}\mathcal{D}_+ f_1(s)\,ds \end{aligned}$$

Moreover, set

$$y_{\mp 1}(\xi) = \langle \mathcal{D}_\mp f_{\mp 1}(R),\, \phi_{\mp 1}(R,\xi)\rangle_{L^2_{R\,dR}}, \quad y_0(\xi) := \langle \mathcal{D} f_0(R),\, \phi_0(R,\xi)\rangle_{L^2_{R\,dR}}$$

Then for $k \geq 1$ we have the derivative bounds (for $\tau \gg 1$)

$$\begin{aligned} \left\|\partial_R^k f_{-1}(R)\right\|_{L^2_{R\,dR}} &\lesssim |c_{-1}| + \left\| \xi^{\frac{1}{2}-\frac{\delta}{2}} \langle\xi\rangle^{\frac{1}{2}+\delta} (\tilde{\rho}_{-1}(\xi))^{\frac{1}{2}}\, y_{-1}(\xi) \right\|_{L^2_{d\xi}} \\ &\quad + \left\| \xi^{\frac{k-1}{2}} (\tilde{\rho}_{-1}(\xi))^{\frac{1}{2}}\, y_{-1}(\xi) \right\|_{L^2_{d\xi}}, \\ \left\|\partial_R^k f_{+1}(R)\right\|_{L^2_{R\,dR}(R\lesssim\tau)} &\lesssim |c_1| + \tau^{\max\{2-k,0\}(1-\delta)} \left\| \xi^{\frac{1}{2}-\frac{\delta}{2}} \langle\xi\rangle^{\delta} (\tilde{\rho}_1(\xi))^{\frac{1}{2}}\, y_1(\xi) \right\|_{L^2_{d\xi}} \\ &\quad + \left\| \xi^{\frac{k-1}{2}} (\tilde{\rho}_1(\xi))^{\frac{1}{2}}\, y_1(\xi) \right\|_{L^2_{d\xi}}, \\ \left\|\partial_R^k f_0(R)\right\|_{L^2_{R\,dR}} &\lesssim |c_0| + \tau^{(1-\delta)\max\{2-k,0\}} \left\| \xi^{\frac{1}{2}-\frac{\delta}{2}} \langle\xi\rangle^{\delta} (\tilde{\rho}_0(\xi))^{\frac{1}{2}}\, y_0(\xi) \right\|_{L^2_{d\xi}} \\ &\quad + \left\| \xi^{\frac{k-1}{2}} (\tilde{\rho}_0(\xi))^{\frac{1}{2}}\, y_0(\xi) \right\|_{L^2_{d\xi}} \end{aligned} \tag{6.27}$$

and in terms of the norms $S_0^{\pm}, S_0^0$, we have the bounds

$$\begin{aligned} \|\partial_R f_{\pm 1}(R)\|_{L^2_{R\,dR}(R\lesssim\tau)} &\lesssim |c_{\pm 1}| + \tau^{1-\delta} \|y_{\pm 1}\|_{S_0^{\pm}}, \\ \|\partial_R f_0(R)\|_{L^2_{R\,dR}(R\lesssim\tau)} &\lesssim |c_0| + \tau \|y_0\|_{S_0^0}, \\ \left\|\partial_R^2 f_0(R)\right\|_{L^2_{R\,dR}(R\lesssim\tau)} &\lesssim |c_0| + \sqrt{|\log\tau|}\, \|y_0\|_{S_0^0}. \end{aligned} \tag{6.28}$$

Moreover, for $k \geq 1$ we have the L^∞-bounds

$$\left\|\partial_R^k f_{-1}(R)\right\|_{L^\infty_{R\,dR}} \lesssim |c_{-1}| + \left\|\xi^{\frac{k}{2}-\frac{\delta}{2}}\langle\xi\rangle^{\delta}\tilde{\rho}_{-1}(\xi)y_j(\xi)\right\|_{L^2_{d\xi}}$$

and in particular we have

$$\|\partial_R f_{-1}(R)\|_{L^\infty_{R\,dR}} \lesssim |c_{-1}| + \|y_{-1}\|_{S_0^-}.$$

Similarly, we have

$$\left\|\partial_R^k f_1(R)\right\|_{L^\infty_{R\,dR}} \lesssim |c_1| + \left\|\xi^{\frac{k}{2}-\frac{\delta}{2}}\langle\xi\rangle^{\delta}\tilde{\rho}_1(\xi)y_1(\xi)\right\|_{L^2_{d\xi}}$$

and in particular we have

$$\|\partial_R f_1(R)\|_{L^\infty_{R\,dR}} \lesssim |c_1| + \|y_1\|_{S_0^+}.$$

Finally we have

$$\left\|\partial_R^k f_0(R)\right\|_{L^\infty_{R\,dR}} \lesssim |c_0| + \left\|\xi^{\frac{k}{2}}\langle\xi\rangle^{\frac{\delta}{2}}\left(\frac{\langle\log\xi\rangle}{\langle\log\langle\xi\rangle\rangle}\right)^{1-\delta}\tilde{\rho}_0(\xi)y_0(\xi)\right\|_{L^2_{d\xi}},$$

and in particular we have

$$\|\partial_R f_0(R)\|_{L^\infty_{R\,dR}} \lesssim |c_0| + \|y_0\|_{S_0^0}.$$

Proof. We start by proving (6.27).

First estimate of (6.27). To begin with, we observe that

$$\left\|c_{-1}\partial_R^k\phi_{-1}(R)\right\|_{L^2_{R\,dR}} \lesssim |c_{-1}|, \quad k \geq 1.$$

Next, write

$$\partial_R^k\left(\phi_{-1}(R)\cdot\int_0^R \phi_{-1}(s)^{-1}\mathcal{D}_- f_{-1}(s)\,ds\right) = \sum_{i+j=k,\, j\geq 1} C_{i,j}\partial_R^i\left(\phi_{-1}(R)\right)\partial_R^{j-1}\left(\phi_{-1}(R)^{-1}\mathcal{D}_- f_{-1}(R)\right)$$
$$+\,\partial_R^k\left(\phi_{-1}(R)\right)\cdot\int_0^R \phi_{-1}(s)^{-1}\mathcal{D}_- f_{-1}(s)\,ds$$

Here we can quickly dispose of the second term on the right by observing that

$$\left\|\langle R\rangle\cdot\partial_R^k\left(\phi_-(R)\right)\right\|_{L^2_{R\,dR}} < \infty,\ k \geq 1,$$

as well as the bounds (valid for all positive s)

$$\left|\phi_{-1}(s)^{-1}\phi_{-1}(s,\xi)\right| \lesssim \min\left\{s,\, s^{-2}(1+s^2)\langle\xi\rangle^{-1}\xi^{-\frac{1}{2}}(s\xi^{\frac{1}{2}})^{-\frac{1}{2}}\right\}$$

Then write

$$\int_0^R \phi_{-1}(s)^{-1}\mathcal{D}_- f_{-1}(s)\,ds$$
$$= \int_0^{\min\{1,R\}} \phi_{-1}(s)^{-1}\int_0^1 \phi_{-1}(s,\xi)y_{-1}(\xi)\tilde{\rho}_{-1}(\xi)\,d\xi\,ds$$

$$
\begin{aligned}
&+\int_0^{\min\{1,R\}} \phi_{-1}(s)^{-1} \int_1^{s^{-2}} \phi_{-1}(s,\xi) y_{-1}(\xi)\tilde{\rho}_{-1}(\xi)\, d\xi\, ds \\
&+\int_0^{\min\{1,R\}} \phi_{-1}(s)^{-1} \int_{s^{-2}}^{\infty} \phi_{-1}(s,\xi) y_{-1}(\xi)\tilde{\rho}_{-1}(\xi)\, d\xi\, ds \\
&+\int_{\min\{1,R\}}^{R} \phi_{-1}(s)^{-1}\mathcal{D}_{-} f_{-1}(s)\, ds \\
&=\sum_{j=1}^{4} S_j.
\end{aligned}
$$

We easily obtain

$$
|S_1| \lesssim \int_0^1 \int_0^1 s\xi\, |y_{-1}(\xi)|\, d\xi\, ds \lesssim \left\|\xi^{\frac{3}{2}-\delta} y_{-1}(\xi)\right\|_{L^2_{d\xi}(\xi<1)},
$$

which is better than the bound needed for (6.27). For $S_{2,3}$, we change the order of integration:

$$
|S_2| \lesssim \int_1^{\infty} \int_0^{\min\{R,\xi^{-\frac{1}{2}}\}} s\big|y_{-1}(\xi)\big|\tilde{\rho}_{-1}(\xi)\, ds\, d\xi \lesssim \int_1^{\infty} \xi^2\, |y_{-1}(\xi)|\, d\xi \lesssim \left\|\xi^{\frac{5}{2}+\frac{\delta}{2}} y(\xi)\right\|_{L^2_{d\xi}(\xi>1)}
$$

in accordance with the bound needed for (6.27), and similarly

$$
\begin{aligned}
|S_3| &\lesssim \int_1^{\infty} \left(\int_{\xi^{-\frac{1}{2}}}^{R} s^{-2}(1+s^2)\langle\xi\rangle^{-1}\xi^{-\frac{1}{2}}(s\xi^{\frac{1}{2}})^{-\frac{1}{2}}\, ds\right) |y_{-1}(\xi)|\, \tilde{\rho}_{-1}(\xi)\, d\xi \\
&\lesssim \langle R\rangle \cdot \int_1^{\infty} \xi^2\, |y_{-1}(\xi)|\, d\xi \lesssim \langle R\rangle \cdot \left\|\xi^{\frac{5}{2}+\frac{\delta}{2}} y_{-1}(\xi)\right\|_{L^2_{d\xi}(\xi>1)},
\end{aligned}
$$

and so the contribution of the this term to the full expression can be estimated by

$$
\left\|\partial_R^k(\phi_{-1}(R))\cdot S_3\right\|_{L^2_{R\,dR}} \lesssim \left\|\langle R\rangle\cdot\partial_R^k(\phi_{-}(R))\right\|_{L^2_{R\,dR}} \cdot \left\|\langle R\rangle^{-1} S_3\right\|_{L^\infty_{R\,dR}} \lesssim \left\|\xi^{\frac{5}{2}+\frac{\delta}{2}} y_{-1}(\xi)\right\|_{L^2_{d\xi}(\xi>1)},
$$

which is consistent with (6.27). Finally, for the last term, we may assume $R > 1$, of course, and then

$$
\begin{aligned}
|S_4| &\lesssim \int_1^R \int_0^1 \chi_{s\xi^{\frac{1}{2}}\lesssim 1}\, s\xi^{\frac{1}{2}}\cdot\xi^{\frac{1}{2}}\big|y_{-1}(\xi)\big|\, d\xi\, ds \\
&\quad+\int_1^R \int_0^1 \chi_{s\xi^{\frac{1}{2}}\gtrsim 1}\,\xi^{-\frac{1}{2}}(s\xi^{\frac{1}{2}})^{-\frac{1}{2}}\big|y_{-1}(\xi)\big|\,\xi d\xi\, ds \\
&\quad+\int_1^R \int_1^{\infty} \langle\xi\rangle^{-1}\xi^{-\frac{1}{2}}(s\xi^{\frac{1}{2}})^{-\frac{1}{2}}\big|y_{-1}(\xi)\big|\xi^3\, d\xi\, ds,
\end{aligned}
$$

and all of these are seen to be bounded by

$$
\lesssim \langle R\rangle \left\|\xi^{\frac{1}{2}-\frac{\delta}{2}}\langle\xi\rangle^{\frac{1}{2}+\delta}\,(\tilde{\rho}_{-1}(\xi))^{\frac{1}{2}}\, y_{-1}(\xi)\right\|_{L^2_{d\xi}},
$$

which then leads to the desired bound as see before. As far as the remaining term

$$\sum_{i+j=k,\, j\geq 1} C_{i,j}\partial_R^i\left(\phi_{-1}(R)\right)\partial_R^{j-1}\left(\phi_{-1}(R)^{-1}\mathcal{D}_- f_{-1}(R)\right),\quad k\geq 1,$$

is concerned, using arguments analogous to the ones for the proof of Proposition 6.7, we have for $i+j\leq k$ that

$$\left\|\partial_R^i\left(\phi_{-1}(R)\right)\partial_R^{j-1}\left(\phi_{-1}(R)^{-1}\mathcal{D}_- f_{-1}(R)\right)\right\|_{L^2_{R\,dR}}\lesssim\left\|\xi^{\frac{k-1}{2}}\left(\tilde\rho_{-1}(\xi)\right)^{\frac12}y_{-1}(\xi)\right\|_{L^2_{d\xi}}.$$

In fact, the term $\phi_-^{-1}(R)$ being singular at the origin costs two powers of $\xi^{\frac12}$ for $\xi\gg 1$, but the factor $\partial_R^i(\phi_-(R))$ contributes at least $\max\{2-i,0\}$ powers of R, whence we have to pay

$$j+1-\max\{2-i,0\}=k-i+1-\max\{2-i,0\}\leq k-1$$

many powers of $\xi^{\frac12}$ in the high-frequency regime. In the low frequency regime $\xi\lesssim 1$ derivatives translate into gains of R^{-1} which in turn translate into additional factors $\xi^{\frac12}$. This completes the proof of the first inequality in (6.27).

Second estimate of (6.27). As in the preceding case the contribution of the root mode $c_1\phi_1(R)$ is trivial, and so we now treat the contribution of the integral expression. As before write

$$\partial_R^k\left(\phi_1(R)\cdot\int_0^R\phi_1(s)^{-1}\mathcal{D}_+ f_1(s)\,ds\right)=\sum_{i+j=k,\, j\geq 1} C_{i,j}\partial_R^i\left(\phi_1(R)\right)\partial_R^{j-1}\left(\phi_{+1}(R)^{-1}\mathcal{D}_+ f_1(R)\right)$$
$$+\,\partial_R^k\left(\phi_1(R)\right)\cdot\int_0^R\phi_1(s)^{-1}\mathcal{D}_+ f_1(s)\,ds$$

The loss of a factor τ for $k=1$ in the estimate is due to the case $k=1$ when the integral does not get hit by ∂_R. Considering this case first, we get the bound

$$\begin{aligned}&\left|\partial_R\left(\phi_1(R)\right)\cdot\int_0^R\phi_1(s)^{-1}\mathcal{D}_+ f_1(s)\,ds\right|\\&\lesssim\left|\partial_R\left(\phi_1(R)\right)\right|\cdot\int_0^1\int_0^{\min\{R,\xi^{-\frac12}\}}(1+s^2)s\,ds\,|y_1(\xi)|\,\xi d\xi\\&+\left|\partial_R\left(\phi_1(R)\right)\right|\cdot\int_0^1\int_{\xi^{-\frac12}}^R(1+s^2)\cdot\xi^{-\frac12}(s\xi^{\frac12})^{-\frac12}\,ds\,|y_1(\xi)|\,\xi d\xi\\&+\left|\partial_R\left(\phi_1(R)\right)\right|\cdot\int_1^\infty\int_0^R(1+s^2)\min\{s,\,\xi^{-\frac12}(s\xi^{\frac12})^{-\frac12}\}\,ds\cdot|y_1(\xi)|\,\xi d\xi\end{aligned}$$

Each of these expressions can be easily bounded by

$$\lesssim\langle R\rangle^{-\delta}\left\|\xi^{\frac12-\frac{\delta}{2}}\langle\xi\rangle^{\delta}\left(\tilde\rho_1(\xi)\right)^{\frac12}y_1(\xi)\right\|_{L^2_{d\xi}}.$$

Correspondingly the $L^2_{R\,dR}$-norm over the region $R\lesssim\tau$ is bounded by the second expression on the right of (6.27) in the case $k=1$. Observe that expression we obtain in the $k=1$ case when ∂_R falls on the integral is simply $\mathcal{D}_+ f_1(R)$, and its $L^2_{R\,dR}$ norm is simply bounded by $\left\|(\tilde{\rho}_1(\xi))^{\frac{1}{2}}\, y_1(\xi)\right\|_{L^2_{d\xi}}$ by Plancherel for the distorted Fourier transform, again compatible with the second bound in (6.27).

Next consider the case $k = 2$, again assuming that both derivatives fall on the first factor $\phi_1(R)$. For $R \gg 1$, we have

$$\left|\partial_R^2\,(\phi_1(R))\cdot\int_0^R \phi_1(s)^{-1}\mathcal{D}_+ f_1(s)\,ds\right| \lesssim \langle R\rangle^{-1-\delta}\left\|\xi^{\frac{1}{2}-\frac{\delta}{2}}\langle\xi\rangle^{\delta}\,(\tilde{\rho}_1(\xi))^{\frac{1}{2}}\,y_1(\xi)\right\|_{L^2_{d\xi}},$$

whose L^2_{RdR}-norm is bounded by the second expression on the right of (6.27) in the case $k = 2$. The remaining terms arising in the $k = 2$ case are

$$\partial_R\,(\phi_1(R))\cdot[\phi_1(R)]^{-1}\mathcal{D}_+ f_1(R),\quad \phi_1(R)\cdot\partial_R\left([\phi_1(R)]^{-1}\mathcal{D}_+ f_1(R)\right).$$

In order to estimate them, we use the Plancherel's theorem for the distorted Fourier transform, as well as the bounds

$$\left|\partial_R\,(\phi_1(R))\,[\phi_1(R)]^{-1}\right| + \left|\phi_1(R)\cdot\partial_R\left([\phi_{+1}(R)]^{-1}\right)\right| \lesssim 1,$$
$$\|\partial_R\mathcal{D}_+ f_1(R)\|_{L^2_{R\,dR}} \lesssim \left\|\xi^{\frac{1}{2}}\,(\tilde{\rho}_1(\xi))^{\frac{1}{2}}\,y_1(\xi)\right\|_{L^2_{d\xi}},$$

which imply the desired estimate in accordance with the second inequality in (6.27). The case of derivatives of degree $k \geq 3$ is straightforward due to the fact that now

$$\partial_R^k\,(\phi_1(R))\cdot\langle R\rangle\cdot[\phi_1(R)]^{-1} \in L^2_{R\,dR}.$$

We omit the straightforward details.

When $n = 0$, the argument is similar to that of $n = 1$, and most delicate step is to estimate the contribution from

$$\begin{aligned}
&\left|\partial_R\,(\phi_0(R))\cdot\int_0^R \phi_0(s)^{-1}\mathcal{D} f_0(s)\,ds\right| \\
&\lesssim |\partial_R\,(\phi_0(R))|\cdot\int_0^1\int_0^{\min\{R,\xi^{-\frac{1}{2}}\}}\left(s+s^{-1}\right)s^2\,ds\,|y_0(\xi)|\,\langle\log\xi\rangle^{-2}d\xi \\
&\quad+ |\partial_R\,(\phi_0(R))|\cdot\int_0^1\int_{\xi^{-\frac{1}{2}}}^R\left(s+s^{-1}\right)\cdot(s\xi^{\frac{1}{2}})^{-\frac{1}{2}}\langle\log\xi\rangle\,ds\,|y_0(\xi)|\,\langle\log\xi\rangle^{-2}d\xi \\
&\quad+ |\partial_R\,(\phi_0(R))|\cdot\int_1^\infty\int_0^R\left(s+s^{-1}\right)\min\{s^2,(s\xi^{\frac{1}{2}})^{-\frac{1}{2}}\xi^{-1}\}\,ds\cdot|y_0(\xi)|\,\langle\xi\rangle^2 d\xi.
\end{aligned}$$

Similar as the $n = 1$ case, each of these expressions is bounded by

$$\lesssim \langle R\rangle^{-\delta}\left\|\xi^{\frac{1}{2}-\frac{\delta}{2}}\langle\xi\rangle^{\delta}\,(\widetilde{\rho_0}(\xi))^{\frac{1}{2}}\,y_0(\xi)\right\|_{L^2_{d\xi}}.$$

The rest argument is identical to the $n = 1$ case, and we omit the details.

Now we turn to the second group of estimates (6.28). Based on the estimates in (6.27), the proof for (6.28) is more straightforward and here we only give an outline of its proof. For instance we look at $\partial_R f_1$:

$$\partial_R f_1(R) = c_1 \partial_R \phi_1(R) + \mathcal{D}_+ f_1(R) + (\partial_R \phi_1(R)) \cdot \int_0^R (\phi_1(s))^{-1} \mathcal{D}_+ f_1(s)\, ds$$

The first and the last term on the RHS above can be estimated exactly the same way as for the proof of (6.27), in view of the fact that the $\|\cdot\|_{S_0^+}$-norm controls the norm $\left\|\xi^{\frac{1}{2}-\frac{\delta}{2}}\langle\xi\rangle^{\delta}\,(\widetilde{\rho}_1(\xi))^{\frac{1}{2}}\,\cdot\right\|_{L^2_{d\xi}}$. For the second term, we first estimate the pointwise bound for $\mathcal{D}_+ f_1(R)$, which decays for $R \gg 1$, and take the L^2_{RdR}-norm over the regime $R \lesssim \tau$. More precisely, we have (assuming $R \gg 1$, and omitting the constant coefficients)

$$\mathcal{D}_+ f_1(R) = \int_0^\infty \phi_1(R,\xi) y_1(\xi) \xi\, d\xi = \left(\int_0^{R^{-2}} + \int_{R^{-2}}^1 + \int_1^\infty \right) \phi_1(R,\xi) y_1(\xi) \xi\, d\xi.$$

For the integral $\int_0^{R^{-2}}$ ... we have

$$\left| \int_0^{R^{-2}} \phi_1(R,\xi) y_1(\xi) \xi\, d\xi \right| \lesssim \int_0^{R^{-2}} R\xi |y_1(\xi)|\, d\xi \lesssim \left(\int_0^{R^{-2}} \xi^{-1+\delta}\, d\xi \right)^{\frac{1}{2}} \|y_1\|_{S_0^+} \lesssim R^{-\delta} \|y_1\|_{S_0^+}.$$

For the integral $\int_{R^{-2}}^1$ ... we have

$$\left| \int_0^{R^{-2}} \phi_1(R,\xi) y_1(\xi) \xi\, d\xi \right| \lesssim R^{-\frac{1}{2}} \int_{R^{-2}}^1 \xi^{\frac{1}{4}} |y_1(\xi)|\, d\xi \lesssim R^{-\frac{1}{2}} \left(\int_{R^{-2}}^1 \xi^{-\frac{3}{2}+\delta}\, d\xi \right)^{\frac{1}{2}} \|y_1\|_{S_0^+} \lesssim R^{-\delta} \|y_1\|_{S_0^+}.$$

Finally for the integral $\int_1^\infty$ ... we have

$$\left| \int_1^{\infty} \phi_1(R,\xi) y_1(\xi) \xi\, d\xi \right| \lesssim R^{-\frac{1}{2}} \int_1^\infty \xi^{\frac{1}{4}} |y_1(\xi)|\, d\xi \lesssim R^{-\frac{1}{2}} \|y_1\|_{S_0^+}.$$

Therefore the estimate on $\|\partial_R f_1\|_{L^2_{RdR}(R \lesssim \tau)}$ follows. The other estimates in (6.28) follow in the same way.

Finally the L^∞-estimates on $\partial_R^k f_j$ can be proved directly using the Fourier representations for $\mathcal{D}_- f_{-1}, \mathcal{D} f_0$ and $\mathcal{D}_+ f_1$. We omit the details here. □

In the sequel, we shall need certain weighted versions of inequalities in the preceding proposition which allow to mostly avoid the losses in τ, which may be understood as a low-frequency issue:

Lemma 6.13. *We have the following weighted and paradifferentiated derivative bounds, where y_j denotes the Fourier transform in analogy to the preceding proposition, and where we assume $c_j = 0$:*

$$\left(\sum_{\lambda \lesssim 1} \lambda^{1-\delta} \left\| \partial_R f_{j,[\lambda,1]} \right\|^2_{L^2_{RdR}} \right)^{\frac{1}{2}} \lesssim \left\| y_j \right\|_{S_0^j}, \quad j = \pm 1,$$

$$\left(\sum_{\lambda \lesssim 1} \lambda^{1-\delta} \left\| \partial_R f_{0,[\lambda,1]} \right\|^2_{L^2_{RdR}(R \lesssim \tau)} \right)^{\frac{1}{2}} \lesssim \tau^{\delta} \left\| y_j \right\|_{S_0^0}, \quad \tau \gtrsim 1.$$

Here λ ranges over dyadic frequencies. The same bound obtains if $[\lambda, 1]$ is replaced by $[\lambda, a)$ where $a \in [1, \infty]$. Here we use the notations $S_0^1 := S_0^+, S_0^{-1} := S_0^-$.

Proof. We observe that the difference between the cases $j = \pm 1$ and $j = 0$ arises due to the different weights in the small frequency region in the norms $\|\cdot\|_{S_0^j}$. We treat the cases $j = \mp 1$, the case $j = 0$ being similar.
$j = -1$. Write as before

$$\partial_R f_{-1,[\lambda,1]} = \partial_R \phi_{-1}(R) \cdot \int_0^R [\phi_{-1}(s)]^{-1} \cdot \mathcal{D}_- f_{-1,[\lambda,1]}(s)\, ds + \mathcal{D}_- f_{-1,[\lambda,1]}(R)$$

Using Plancherel's theorem for the distorted Fourier transform we infer for dyadic $\mu \in [\lambda, 1]$

$$\begin{aligned}\lambda^{\frac{1}{2}-\frac{\delta}{2}} \left\|\mathcal{D}_- f_{-1,\mu}(R)\right\|_{L^2_{R\,dR}} &\lesssim \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \mu^{\frac{1}{2}-\frac{\delta}{2}} \left\|(\tilde{\rho}_{-1}(\xi))^{\frac{1}{2}} y_{-1}(\xi)\right\|_{L^2_{d\xi}(\xi\simeq\mu)} \\ &\lesssim \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \|y_{-1}(\xi)\|_{S_0^{-1}(\xi\simeq\mu)}\,.\end{aligned}$$

It follows that

$$\begin{aligned}\left(\sum_{\lambda\lesssim 1} \lambda^{1-\delta} \left\|\mathcal{D}_- f_{-1,[\lambda,1]}\right\|^2_{L^2_{R\,dR}}\right)^{\frac{1}{2}} &\lesssim \left(\sum_{\lambda\lesssim 1} \sum_{\lambda\lesssim\mu\lesssim 1} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \left\|y_{-1}(\xi)\right\|^2_{S_0^{-1}(\xi\simeq\mu)}\right)^{\frac{1}{2}} \\ &\lesssim \|y_{-1}(\xi)\|_{S_0^{-1}}\,.\end{aligned}$$

Next consider the integral term above.

$$\partial_R \phi_{-1}(R) \cdot \int_0^R [\phi_{-1}(s)]^{-1} \cdot \mathcal{D}_{-1} f_{-1,[\lambda,1]}(s)\, ds.$$

Following from the proof of the preceding proposition, we immediately obtain

$$\left\|\partial_R \phi_{-1}(R) \cdot \int_0^R [\phi_{-1}(s)]^{-1} \cdot \mathcal{D}_- f_{-1,[\lambda,1]}(s)\, ds\right\|_{L^2_{RdR}} \lesssim \|y_{-1}\|_{S_0^{-1}}\,,$$

and the desired weighted bound follows.
$j = +1$. Write

$$\partial_R f_{1,[\lambda,1]} = \partial_R \phi_1(R) \cdot \int_0^R [\phi_1(s)]^{-1} \cdot \mathcal{D}_+ f_{1,[\lambda,1]}(s)\, ds + \mathcal{D}_+ f_{1,[\lambda,1]}(R)$$

The contribution of the second term is treated exactly as in the preceding case. However, the contribution of the first term is more complicated. Our main goal here is to control the growth in R of its L^2_{RdR}-norm. Therefore without loss of generality, we assume that $s \in [1, R]$. Fix dyadic $\mu \in [\lambda, 1]$ and consider the localized term(where $y_{1,\mu}(\xi) = \chi_{\xi\simeq\mu} y_1(\xi)$)

$$\begin{aligned}&\partial_R \phi_1(R) \cdot \int_0^R [\phi_1(s)]^{-1} \cdot \mathcal{D}_+ f_{1,\mu}(s)\, ds \\ =&\partial_R \phi_1(R) \cdot \int_0^R [\phi_1(s)]^{-1} \cdot \int_0^\infty \chi_{s\xi^{\frac{1}{2}}\lesssim 1} \phi_1(s,\xi) y_{1,\mu}(\xi) \tilde{\rho}_1(\xi)\, d\xi\, ds\end{aligned}$$

$$+\partial_R\phi_1(R)\cdot\int_0^R[\phi_1(s)]^{-1}\cdot\int_0^\infty\chi_{s\xi^{\frac12}\gtrsim1}\phi_1(s,\xi)y_{1,\mu}(\xi)\tilde\rho_1(\xi)\,d\xi\,ds$$

Then we bound the contribution of the first term on the right as follows, where we first localize R to dyadic scale $R\simeq\rho$ (Recall that since $s\geq1$, we have $\xi\leq1$ in this case.):

$$\lambda^{\frac12-\frac\delta2}\left\|\partial_R\phi_1(R)\cdot\int_0^R[\phi_1(s)]^{-1}\cdot\int_0^\infty\chi_{s\xi^{\frac12}\lesssim1}\phi_1(s,\xi)y_{1,\mu}(\xi)\tilde\rho_1(\xi)\,d\xi\,ds\right\|_{L^2_{R\,dR}(R\simeq\rho)}$$
$$\lesssim\left(\frac{\lambda}{\mu}\right)^{\frac12-\frac\delta2}\cdot\left(\sum_{\kappa\lesssim\min\{\mu^{-\frac12},\rho\}}\left(\frac{\kappa}{\rho}\right)^2\cdot\left(\mu^{\frac12}\kappa\right)^2\right)\cdot\left\|y_{1,\mu}\right\|_{S_0^1(\xi\simeq\mu)}$$

where κ runs over dyadic scales for $s\in[1,R]$, and we have

$$\sum_{\kappa\lesssim\min\{\mu^{-\frac12},\rho\}}\left(\frac{\kappa}{\rho}\right)^2\cdot\left(\mu^{\frac12}\kappa\right)^2\lesssim\frac{\mu^{\frac12}\rho}{\langle\mu^{\frac12}\rho\rangle^2}.$$

We conclude that

$$\left(\sum_{\lambda\lesssim1}\lambda^{1-\delta}\left\|\partial_R\phi_1(R)\cdot\int_0^R[\phi_1(s)]^{-1}\cdot\int_0^\infty\chi_{s\xi^{\frac12}\lesssim1}\phi_1(s,\xi)y_{1,[\lambda,1]}(\xi)\tilde\rho_1(\xi)\,d\xi\,ds\right\|^2_{L^2_{R\,dR}}\right)^{\frac12}$$
$$\lesssim\left(\sum_{\lambda\lesssim1}\sum_\rho\left[\sum_{\mu\in[\lambda,1]}\left(\frac{\lambda}{\mu}\right)^{\frac12-\frac\delta2}\cdot\frac{\mu^{\frac12}\rho}{\langle\mu^{\frac12}\rho\rangle^2}\cdot\left\|y_{1,\mu}\right\|_{S_0^+(\xi\simeq\mu)}\right]^2\right)^{\frac12}$$
$$\lesssim\left(\sum_{\lambda\lesssim1}\sum_\rho\sum_{\mu\in[\lambda,1]}\left(\frac{\lambda}{\mu}\right)^{\frac12-\frac\delta2}\cdot\frac{\mu^{\frac12}\rho}{\langle\mu^{\frac12}\rho\rangle^2}\cdot\left\|y_{1,\mu}\right\|^2_{S_0^+(\xi\simeq\mu)}\right)^{\frac12}$$
$$\lesssim\|y_1\|_{S_0^+},$$

which corresponds to the desired bound. As for the oscillatory term above where the integration is restricted to $s\xi^{\frac12}\gtrsim1$, we have to perform integration by parts (for instance, three times) with respect to s, which leads to the analogous bound

$$\lambda^{\frac12-\frac\delta2}\left\|\partial_R\phi_1(R)\cdot\int_0^R[\phi_1(s)]^{-1}\cdot\int_0^\infty\chi_{s\xi^{\frac12}\gtrsim1}\phi_1(s,\xi)y_{1,\mu}(\xi)\tilde\rho_1(\xi)\,d\xi\,ds\right\|_{L^2_{R\,dR}(R\simeq\rho)}$$
$$\lesssim\left(\frac{\lambda}{\mu}\right)^{\frac12-\frac\delta2}\cdot\left(\sum_{\mu^{-\frac12}\lesssim\kappa\lesssim\rho}\left(\frac{\kappa}{\rho}\right)^2\cdot\left(\mu^{\frac12}\kappa\right)^{-2}\right)\cdot\left\|y_{1,\mu}\right\|_{S_0^+(\xi\simeq\mu)},$$

and the estimate is concluded from here in the same way as before.

We omit the simple modifications of the preceding proof to also handle the generalised bounds involving $\partial_R f_{j,[\lambda,a)}$.

□

6.6. **More general bilinear estimates.** We can now formulate the following proposition, in which we still restrict the output to 'non-exceptional angular momenta', i.e. $|n_3| \geq 2$.

Proposition 6.14. *Let $|n_1| \gg 1$, $|n_2| \leq |n_1|$, where n_2 is allowed to take any integer value. Also, assume that $|n_3| \geq 2$ and that either $n_3 \simeq n_1$ or else $|n_3| \ll |n_1|$ as well as $n_1 \simeq -n_2$. For $|n| \geq 2$, we say that $\phi(R)$ is an angular momentum n function provided*

$$\phi(R) = \int_0^\infty \phi_n(R;\xi)\overline{x}(\xi)\tilde{\rho}_n(\xi)\,d\xi,$$

in which case we set

$$\|\phi\|_{\tilde{S}^{(n)}} := \|\overline{x}(\xi)\|_{S_0^{\hbar}}.$$

For $n = 0, \pm 1$ we say that $\phi(R)$ is an angular momentum n function provided

$$\phi(R) = c_n\phi_n(R) + \phi_n(R)\cdot\int_0^R [\phi_n(s)]^{-1}\,\tilde{\mathcal{D}}\phi(s)\,ds, \quad \tilde{\mathcal{D}}\phi(R) = \int_0^\infty x(\xi)\phi_n(R,\xi)\tilde{\rho}_n(\xi)\,d\xi,$$

where $\tilde{\mathcal{D}} = \mathcal{D}_-, \mathcal{D}, \mathcal{D}_+$ according to n as in the preceding. Then set

$$\|\phi\|_{\tilde{S}^{(n)}} := |c_n| + \|\overline{x}(\xi)\|_{S_0^{(n)}}, \quad \text{where} \quad S_0^{(0)} := S_0^0, \quad S_0^{(\pm 1)} = S_0^{\pm}.$$

With the conventions and under the assumptions on the n_j stated at the beginning, and assuming $\tau \gg 1$, we have the following bound

$$\Big\|\langle\phi_{n_3}(R;\xi), \chi_{R\lesssim\tau}\partial_R\phi_1\cdot\partial_R\phi_2\rangle_{L^2_{R\,dR}}\Big\|_{S_1^{\hbar_3}} \lesssim \tau^{\frac{\delta}{2}}\langle n_2\rangle^2\prod_{j=1}^2\big\|\phi_j\big\|_{\tilde{S}^{(n_j)}} \tag{6.29}$$

where we assume that the factors ϕ_j are angular momentum n_j functions, $j = 1, 2$.
Next, assume that $|n_1| \lesssim 1$, while the other assumptions on n_j stated above are still valid. Then if $\phi_{1,2}$ are angular momentum n_j functions with finite $\|\cdot\|_{\tilde{S}^{(n_j)}}$-norm, the product

$$\partial_R\phi_1\cdot\partial_R\phi_2$$

admits a third order Taylor expansion at $R = 0$ of the form

$$P_3(R) := \sum_{\ell=0}^3 \gamma_\ell R^\ell,$$

where we have the bound

$$\sum_{\ell=0}^3 |\gamma_\ell| \lesssim \prod_{j=1,2}\big\|\phi_j\big\|_{\tilde{S}^{(n_j)}}. \tag{6.30}$$

Furthermore, we have the bound

$$\Big\|\langle\phi_{n_3}(R;\xi), \chi_{R\lesssim\tau}\partial_R\phi_1\cdot\partial_R\phi_2 - \chi_{R\lesssim 1}P_3(R)\rangle_{L^2_{R\,dR}}\Big\|_{S_1^{(\hbar_3)}} \lesssim \tau^{\frac{\delta}{2}}\prod_{j=1}^2\big\|\phi_j\big\|_{\tilde{S}^{(n_j)}} \tag{6.31}$$

The last inequality remains correct if we subtract from P_3 those terms $\gamma_l R^l$ with $l \geq |n_3 - 1|, l \equiv n_3 - 1$(mod 2), provided $|n_3 - 1| \leq 3$.

Proof. Proof of (6.29):
In order to prove (6.29), in light of Proposition 6.10, it suffices to deal with the case when $n_2 = 0, \pm 1$, and so $n_3 \simeq n_1$. Then we repeat the proof of Proposition 6.10. We split the proof into the same cases, and use the same terminology. Frequencies for the second factor ϕ_2 correspond to the frequencies of the Fourier integral.

(1.a): $\xi < \xi_1$. This proceeds exactly as in the earlier proof by taking advantage of the L^∞-bounds coming from Proposition 6.12. In particular, the contribution of the resonant/root part in ϕ_2 leads to an admissible contribution, and we can omit it from now on.

(1.b): $\xi_2 \geq \xi \geq \xi_1$. This we can write as in the earlier proof as

$$\begin{aligned}&\sum_{\mu} \chi_{\xi\simeq\mu} \left\langle \phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu} \cdot \partial_R\phi_{2,[\mu,\infty]} \right\rangle_{L^2_{R\,dR}} \\ &= \sum_{\mu<\lambda} \chi_{\xi\simeq\mu} \left\langle \phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu} \cdot [c_n\partial_R\phi_n(R) + \partial_R\phi_{2,\lambda}] \right\rangle_{L^2_{R\,dR}}.\end{aligned}$$

where we set

$$\phi_{2,\lambda}(R) = \phi_j(R)\cdot\int_0^R \left[\phi_j(s)\right]^{-1}\, \tilde{\mathcal{D}}\phi_{2,\lambda}(s)\,ds,\ \tilde{\mathcal{D}}\phi_{2,\lambda}(R) = \int_0^\infty \chi_{\xi\simeq\lambda}\overline{x}_2(\xi)\phi_j(R,\xi)\tilde{\rho}_j(\xi)\,d\xi, \quad j = 0, \pm 1.$$

We distinguish between small output frequencies $\mu \lesssim 1$ and large ones.

Small output frequency $\mu \lesssim 1$. The contribution of the root/resonant part is straightforward to handle by means of Plancherel's Theorem and Lemma 6.9:

$$\begin{aligned}&\left\| \sum_{\mu\lesssim 1} \chi_{\xi\simeq\mu} \langle \phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot c_n\partial_R\phi_n(R)\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}} \\ &\lesssim |c_n| \cdot \sum_{\mu\lesssim 1} \hbar_3^{2-\delta}\mu^{\frac{1}{2}-\frac{\delta}{2}} \left\|\partial_R\phi_{1,\leq\mu}\right\|_{L^\infty_{R\,dR}} \cdot \|\partial_R\phi_n(R)\|_{L^2_{R\,dR}} \\ &\lesssim |c_n| \cdot \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \lesssim \prod_{j=1,2} \left\|\phi_j\right\|_{\tilde{S}^{(n_j)}}.\end{aligned}$$

Next, denoting $\tilde{\phi}_2 = \phi_2 - c_n\phi_n(R)$ and taking advantage of Lemma 6.13 as well as Lemma 6.9 as well as orthogonality, we have

$$\begin{aligned}&\left\| \sum_{\mu\lesssim 1} \chi_{\xi\simeq\mu} \left\langle \phi(R;\xi,\hbar_3),\, \chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\tilde{\phi}_{2,[\mu,\infty]} \right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}} \\ &\lesssim \left(\sum_{\mu\lesssim 1} \hbar_3^{4-\delta}\mu^{1-\delta} \left\|\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\tilde{\phi}_{2,[\mu,\infty]}\right\|^2_{L^2_{R\,dR}} \right)^{\frac{1}{2}}\end{aligned}$$

$$\lesssim \left(\sum_{\mu\lesssim 1}\hbar_3^{4-\delta}\mu^{1-\delta}\left\|\partial_R\phi_{1,\leq\mu}\right\|^2_{L^\infty_{R\,dR}}\cdot\left\|\partial_R\tilde{\phi}_{2,[\mu,\infty]}\right\|^2_{L^2_{R\,dR}(R\lesssim\tau)}\right)^{\frac{1}{2}}$$
$$\lesssim \hbar_1^{\frac{1}{2}}\tau^{\delta}\cdot\|\phi_1\|_{\tilde{S}^{(n_1)}}\cdot\|\phi_2\|_{\tilde{S}^{(n_2)}}$$

Large output frequency $\mu \gg 1$. Here in principle we intend to absorb the frequency weight $\mu^{\frac{1}{2}-\frac{\delta}{2}}\langle\mu\hbar_1^2\rangle^{\delta+\frac{3}{2}}$ into the high frequency factor, which requires some care, due to the particular structure of $\partial_R\phi_2$. We quickly dispose of the contribution of the resonant/root part via integration by parts: assuming $\phi_2(R) = c_n\cdot\phi_n(R)$, $n = 0, \pm 1$,

$$c_n\chi_{\xi\simeq\mu}\left\langle\phi_{n_3}(R;\xi),\ \chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\phi_n(R)\right\rangle_{L^2_{R\,dR}}$$
$$= c_n\mu^{-3}\cdot\chi_{\xi\simeq\mu}\left\langle\phi_{n_3}(R;\xi),\ H^3_{n_3}\left(\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\phi_n(R)\right)\right\rangle_{L^2_{R\,dR}},$$

where boundary terms at $R = 0$ play no role because $|n_3| \gg 1$ by assumption. Localize the term $\phi_{1,\leq\mu}$ to dyadic frequency $\simeq \lambda \lesssim \mu$ and write

$$H^3_{n_3}\left(\chi_{R\lesssim\tau}\partial_R\phi_{1,\lambda}\cdot\partial_R\phi_n(R)\right) = \sum_{i+j+k=6}\left(\frac{n_3}{R}\right)^i\partial_R^{j+1}(\phi_{1,\lambda})\cdot\partial_R^k(\chi_{R\lesssim\tau}\partial_R\phi_n(R)).$$

Then use Proposition 6.8, which gives (letting $\overline{x}_1$ be the Fourier variable for ϕ_1)

$$\left\|\left(\frac{n_3}{R}\right)^i\partial_R^{j+1}\left(\phi_{1,\lambda}\right)\cdot\partial_R^k\left(\chi_{R\lesssim\tau}\partial_R\phi_n(R)\right)\right\|_{L^2_{R\,dR}} \lesssim \left(\lambda^{\frac{1}{2}}+\lambda^{\frac{7}{2}}\right)\|\overline{x}_1\|_{L^2_{d\xi}},\quad i+j\leq 6,$$

and so we infer

$$|c_n|\left\|\chi_{\xi\simeq\mu}\langle\phi_{n_3}(R;\xi),\ \chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\phi_n(R)\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}$$
$$\lesssim |c_n|\mu^{2+\frac{\delta}{2}}\cdot\mu^{-3}\cdot\left(\lambda^{\frac{1}{2}}+\lambda^{\frac{7}{2}}\right)\|\overline{x}_1\|_{L^2_{d\xi}(\xi_1\simeq\lambda)}$$
$$\lesssim |c_n|\cdot\frac{\lambda}{\mu}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\lambda)}.$$

as long as we restrict $\lambda \gtrsim 1$. It follows that

$$|c_n|\left\|\sum_{\mu\gtrsim 1}\chi_{\xi\simeq\mu}\left\langle\phi_{n_3}(R;\xi),\ \chi_{R\lesssim\tau}\partial_R\phi_{1,1\leq\cdot\leq\mu}\cdot\partial_R\phi_n(R)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}$$
$$\lesssim |c_n|\left(\sum_{\mu\gtrsim 1}\left\|\chi_{\xi\simeq\mu}\left\langle\phi_{n_3}(R;\xi),\ \chi_{R\lesssim\tau}\partial_R\phi_{1,1\leq\cdot\leq\mu}\cdot\partial_R\phi_n(R)\right\rangle_{L^2_{R\,dR}}\right\|^2_{S_1^{\hbar_3}}\right)^{\frac{1}{2}}$$
$$\lesssim |c_n|\left(\sum_{\mu\gtrsim 1}\left[\sum_{1\lesssim\lambda\lesssim\mu}\frac{\lambda}{\mu}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\lambda)}\right]^2\right)^{\frac{1}{2}}$$
$$\lesssim |c_n|\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}} \leq \|\phi_1\|_{\tilde{S}_0^{(n_1)}}\cdot\|\phi_2\|_{\tilde{S}_0^{(n_2)}}.$$

The case when ϕ_1 is at frequency $\lesssim 1$ is easily handled by placing this factor into $L^\infty_{R\,dR}$, we omit the details.

This reduces things to the contribution of the integral term, where we shall again have to perform some integration by parts in the inner product for very large output frequencies $\gtrsim \hbar_3^{-2}$. Assume henceforth that ϕ_2 is given by the expression stated at the beginning of case *(1.b)*.

(i): intermediate output frequencies $1 \lesssim \mu \lesssim \hbar_3^{-2}$. Here we have

$$\begin{aligned}&\left\|\chi_{\xi\simeq\mu}\langle\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\phi_{2,[\mu,\infty]}\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\ &\lesssim \hbar_3^{2-\delta}\mu^{\frac12-\frac{\delta}{2}}\sum_{\lambda\geq\mu}\left\|\partial_R\phi_{1,\leq\mu}\right\|_{L^\infty_{R\,dR}}\cdot\left\|\partial_R\phi_{2,\lambda}\right\|_{L^2_{R\,dR}}.\end{aligned}$$

where we can expand

$$\begin{aligned}\partial_R\phi_{2,\lambda} &= \partial_R\left(\phi_n(R)\right)\cdot\int_0^R\left[\phi_n(s)\right]^{-1}\mathcal{D}_n\phi_{2,\lambda}(s)\,ds\\ &\quad+\mathcal{D}_n\phi_{2,\lambda}(R),\quad n=0,\pm1.\end{aligned}$$

Using Plancherel's theorem for the distorted Fourier transform and the fact that $\lambda\geq\mu\gtrsim 1$, we obtain (with x_2 denoting the Fourier transform of $D_n\phi_2$)

$$\mu^{\frac12-\frac{\delta}{2}}\left\|\mathcal{D}_n\phi_{2,\lambda}(R)\right\|_{L^2_{R\,dR}}\lesssim\left(\frac{\mu}{\lambda}\right)^{\frac12-\frac{\delta}{2}}\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\lambda)}.$$

To bound the contribution of the integral term to $\partial_R\phi_{2,\lambda}$, we observe that in all cases $n=0,\pm1$ we have (under the hypothesis $\lambda\gtrsim\mu\gtrsim 1$)

$$\begin{aligned}&\mu^{\frac12-\frac{\delta}{2}}\left\|\partial_R\left(\phi_n(R)\right)\cdot\int_0^R\left[\phi_n(s)\right]^{-1}\mathcal{D}_n\phi_{2,\lambda}(s)\,ds\right\|_{L^2_{R\,dR}}\\ &\lesssim\left(\frac{\mu}{\lambda}\right)^{\frac12-\frac{\delta}{2}}\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\lambda)}.\end{aligned}$$

In fact here we can restrict to the regime where $s\gg 1$, since the other regime $s\lesssim 1$ is much more straightforward. So in this case we have $s\lambda^{\frac12}\gg 1$ and we are in the oscillatory regime. As an example, for $n=0$, we first localize R to dyadic scale $R\simeq\rho$ and then perform integration by parts to obtain

$$\begin{aligned}&\mu^{\frac12-\frac{\delta}{2}}\left\|\partial_R\phi_0(R)\cdot\int_1^R\chi_{s\xi^{\frac12}\gtrsim1}\chi_{\xi\simeq\lambda}\phi_0(s,\xi)\overline{x}_2(\xi)\widetilde{\rho}_0(\xi)d\xi ds\right\|_{L^2_{RdR}(R\simeq\rho)}\\ &\lesssim\left(\frac{\mu}{\lambda}\right)^{\frac12-\frac{\delta}{2}}\cdot\lambda^{-1-\delta}\left(\sum_{\lambda^{-\frac12}\lesssim1\lesssim\kappa\lesssim\rho}\left(\frac{\kappa}{\rho}\right)\cdot\left(\lambda^{\frac12}\kappa\right)^{-1}\right)\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\lambda)}\\ &\lesssim\left(\frac{\mu}{\lambda}\right)^{\frac12-\frac{\delta}{2}}\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\lambda)}.\end{aligned}$$

If we combine these bounds with the usual L^∞-bound

$$\hbar_3^{2-\delta}\big\|\partial_R\phi_{1,\leq\mu}\big\|_{L^\infty_{R\,dR}} \lesssim \big\|\phi_1\big\|_{\tilde{S}_0^{(n_1)}}$$

and to exploit orthogonality and the Cauchy-Schwarz inequality as usual to deduce the desired bound.

(ii): large output frequencies $\mu \gg \hbar_3^{-2}$. Here the weight used for the norm $\|\cdot\|_{S_1^{\hbar_3}}$ becomes

$$\hbar_3\left(\mu\hbar_3^2\right)^{\frac{1}{2}-\frac{\delta}{2}}\left\langle\mu\hbar_3^2\right\rangle^{\delta+\frac{3}{2}} \simeq \hbar_3\left(\mu\hbar_3^2\right)^{2+\frac{\delta}{2}}$$

We may again assume that

$$\partial_R\phi_{2,[\mu,\infty]} = \partial_R\left(\phi_n(R)\right)\cdot\int_0^R\left[\phi_n(s)\right]^{-1}\mathcal{D}_n\phi_{2,[\mu,\infty]}(s)\,ds + \mathcal{D}_n\phi_{2,[\mu,\infty]}(R), \quad n = 0, \pm 1.$$

Substituting the second term on the right for $\partial_R\phi_{2,[\mu,\infty]}$ leads to term that can be bounded directly via the Plancherel's theorem and Lemma 6.9:

$$\begin{aligned}
&\left\|\chi_{\xi\simeq\mu}\left\langle\phi_{n_3}(R;\xi),\,\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\mathcal{D}_n\phi_{2,[\mu,\infty]}(R)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\
&\lesssim \hbar_3(\mu\hbar_3^2)^{2+\frac{\delta}{2}}\cdot\left\|\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\right\|_{L^\infty_{R\,dR}}\cdot\left\|\mathcal{D}_n\phi_{2,[\mu,\infty]}(R)\right\|_{L^2_{R\,dR}}\\
&\lesssim \sum_{\lambda\geq\mu}\left(\frac{\mu}{\lambda}\right)^{2+\frac{\delta}{2}}\|\phi_1\|_{\tilde{S}_0^{(n_1)}}\cdot\|\phi_2\|_{\tilde{S}_0^{(n_2)}(\xi\simeq\lambda)}\,.
\end{aligned}$$

As for the contrition of the integral term to $\partial_R\phi_{2,[\mu,\infty]}$ we perform integration by parts as needed: Setting now

$$\partial_R\phi_{2,[\mu,\infty]} = \partial_R\left(\phi_n(R)\right)\cdot\int_0^R\left[\phi_n(s)\right]^{-1}\mathcal{D}_n\phi_{2,[\mu,\infty]}(s)\,ds,$$

we write

$$\begin{aligned}
&\chi_{\xi\simeq\mu}\left\langle\phi_{n_3}(R;\xi),\,\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\phi_{2,[\mu,\infty]}\right\rangle_{L^2_{R\,dR}}\\
&= \mu^{-1}\chi_{\xi\simeq\mu}\left\langle\phi_{n_3}(R;\xi),\,H_{n_3}\left[\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\phi_{2,[\mu,\infty]}\right]\right\rangle_{L^2_{R\,dR}}\\
&= \mu^{-1}\chi_{\xi\simeq\mu}\left\langle\phi_{n_3}(R;\xi),\,H_{n_3}\left[\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\left(\phi_n(R)\right)\right]\cdot\left(\int_0^R\left[\phi_n(s)\right]^{-1}\mathcal{D}_n\phi_{2,[\mu,\infty]}(s)\,ds\right)\right\rangle_{L^2_{R\,dR}}\\
&+ \mu^{-1}\chi_{\xi\simeq\mu}\left\langle\phi_{n_3}(R;\xi),\,\partial_R\left(\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\phi_n(R)\left[\phi_n(R)\right]^{-1}\cdot\mathcal{D}_n\phi_{2,[\mu,\infty]}(R)\right)\right\rangle_{L^2_{R\,dR}}
\end{aligned} \tag{6.32}$$

The second term on the right can be bounded directly by using Plancherel's Theorem for the cases $n = 0, \pm 1$, as well as Leibinz' rule to expand things out more. Specifically, we have

$$\begin{aligned}
&\left\|\partial_R\left(\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\left(\phi_n(R)\right)\left[\phi_n(R)\right]^{-1}\right)\cdot\mathcal{D}_n\phi_{2,[\mu,\infty]}(R)\right\|_{L^2_{R\,dR}}\\
&\lesssim \left\|\partial_R\left(\chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\left(\phi_n(R)\right)\left[\phi_n(R)\right]^{-1}\right)\right\|_{L^\infty_{R\,dR}}\cdot\left\|\mathcal{D}_n\phi_{2,[\mu,\infty]}(R)\right\|_{L^2_{R\,dR}}
\end{aligned}$$

$$\lesssim \hbar_1^{-4-\delta} \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \cdot \left(\sum_{\lambda \geq \mu} \lambda^{-2-\frac{\delta}{2}} \left\| \phi_{2,\lambda} \right\|_{\tilde{S}_0^{(n_2)}(\xi \simeq \lambda)} \right),$$

where we have also taken advantage of Lemma 6.9. Further, we have

$$\begin{aligned}
&\left\| \chi_{R \lesssim \tau} \partial_R \phi_{1,\leq\mu} \cdot \partial_R \left(\phi_n(R)\right) [\phi_n(R)]^{-1} \cdot \partial_R \left(\mathcal{D}_n \phi_{2,[\mu,\infty]}(R) \right) \right\|_{L^2_{R\,dR}} \\
&\lesssim \left\| \chi_{R \lesssim \tau} \partial_R \phi_{1,\leq\mu} \cdot \partial_R \left(\phi_n(R)\right) [\phi_n(R)]^{-1} \right\|_{L^\infty_{R\,dR}} \cdot \left\| \partial_R \left(\mathcal{D}_n \phi_{2,[\mu,\infty]}(R) \right) \right\|_{L^2_{R\,dR}} \\
&\lesssim \hbar_1^{-4-\delta} \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \cdot \left(\sum_{\lambda \geq \mu} \lambda^{-\frac{3}{2}-\frac{\delta}{2}} \left\| \phi_{2,\lambda} \right\|_{\tilde{S}_0^{(n_2)}(\xi \simeq \lambda)} \right)
\end{aligned}$$

Keeping in mind orthogonality, we can the estimate the contribution of the second term on the right in (6.32) to the $\|\cdot\|_{S_1^{\hbar_3}}$-norm by

$$\begin{aligned}
&\left(\sum_{\mu \gtrsim \hbar_3^{-2}} \left[\hbar_3 (\mu \hbar_3^2)^{2+\frac{\delta}{2}} \cdot \mu^{-1} \hbar_1^{-4-\delta} \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \cdot \left(\sum_{\lambda \geq \mu} \lambda^{-\frac{3}{2}-\frac{\delta}{2}} \left\| \phi_{2,\lambda} \right\|_{\tilde{S}_0^{(n_2)}(\xi \sim \lambda)} \right) \right]^2 \right)^{\frac{1}{2}} \\
&\lesssim \hbar_3^{\frac{1}{2}} \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \cdot \left(\sum_{\mu \gtrsim \hbar_3^{-2}} \sum_{\lambda \geq \mu} \left(\frac{\mu}{\lambda} \right)^{\frac{1}{2}} \left\| \phi_{2,\lambda} \right\|^2_{\tilde{S}_0^{(n_2)}(\xi \simeq \lambda)} \right)^{\frac{1}{2}} \\
&\lesssim \hbar_3^{\frac{1}{2}} \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \cdot \|\phi_2\|_{\tilde{S}_0^{(n_2)}},
\end{aligned}$$

which is better than the bound we are striving to establish.

As for the contribution of the first term in (6.32), this requires another application of integration by parts. In fact, we can write

$$\begin{aligned}
&\mu^{-1} \chi_{\xi \simeq \mu} \left\langle \phi_{n_3}(R;\xi),\ H_{n_3} \left[\chi_{R \lesssim \tau} \partial_R \phi_{1,\leq\mu} \cdot \partial_R \left(\phi_n(R)\right) \right] \cdot \left(\int_0^R [\phi_n(s)]^{-1}\, \mathcal{D}_n \phi_{2,[\mu,\infty]}(s)\, ds \right) \right\rangle_{L^2_{R\,dR}} \\
&= \mu^{-2} \chi_{\xi \simeq \mu} \left\langle \phi_{n_3}(R;\xi),\ H^2_{n_3} \left[\chi_{R \lesssim \tau} \partial_R \phi_{1,\leq\mu} \cdot \partial_R \left(\phi_n(R)\right) \right] \cdot \left(\int_0^R [\phi_n(s)]^{-1}\, \mathcal{D}_n \phi_{2,[\mu,\infty]}(s)\, ds \right) \right\rangle_{L^2_{R\,dR}} \\
&+ \mu^{-2} \chi_{\xi \simeq \mu} \left\langle \phi_{n_3}(R;\xi),\ \partial_R \left(H_{n_3} \left[\chi_{R \lesssim \tau} \partial_R \phi_{1,\leq\mu} \cdot \partial_R \left(\phi_n(R)\right) \right] [\phi_n(R)]^{-1} \cdot \mathcal{D}_n \phi_{2,[\mu,\infty]}(R) \right) \right\rangle_{L^2_{R\,dR}}.
\end{aligned}$$

Here both terms on the right can be bounded directly. The second term is analogous (with the same exponent $\hbar_1^{-4-\delta}$) to the second term in (6.32) and hence omitted, while for the first term we use (with $\alpha_{-1} = 1, \alpha_0 = 2, \alpha_1 = 3$).

$$\left\| \langle R \rangle^{\alpha_n} H^2_{n_3} \left[\chi_{R \lesssim \tau} \partial_R \phi_{1,\leq\mu} \cdot \partial_R \left(\phi_n(R)\right) \right] \right\|_{L^2_{R\,dR}} \lesssim \hbar_1^{-5-\delta} \|\phi_1\|_{\tilde{S}_0^{(n_1)}},$$

$$\left\| \langle R\rangle^{-\alpha_n} \int_0^R [\phi_n(s)]^{-1}\, \mathcal{D}_n \phi_{2,[\mu,\infty]}(s)\, ds \right\|_{L^\infty_{R\,dR}} \lesssim \mu^{-1} \|\phi_2\|_{\tilde{S}_0^{(n_2)}},$$

where for the first bound we again take advantage of Proposition 6.7 and Lemma 6.9. These bounds imply

$$\begin{aligned} &\left\| \mu^{-2} \chi_{\xi\simeq\mu} \left\langle \phi_{n_3}(R;\xi),\, H_{n_3}^2 \left[\chi_{R\lesssim\tau} \partial_R \phi_{1,\leq\mu} \cdot \partial_R \left(\phi_n(R)\right)\right] \cdot \left(\int_0^R [\phi_n(s)]^{-1}\, \mathcal{D}_n \phi_{2,[\mu,\infty]}(s)\, ds \right) \right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}} \\ &\lesssim \hbar_3^{5+\delta} \mu^{2+\frac{\delta}{2}} \\ &\cdot \left\| \mu^{-2} \chi_{\xi\simeq\mu} \left\langle \phi_{n_3}(R;\xi),\, H_{n_3}^2 \left[\chi_{R\lesssim\tau} \partial_R \phi_{1,\leq\mu} \cdot \partial_R \left(\phi_n(R)\right)\right] \cdot \left(\int_0^R [\phi_n(s)]^{-1}\, \mathcal{D}_n \phi_{2,[\mu,\infty]}(s)\, ds \right) \right\rangle_{L^2_{R\,dR}} \right\|_{L^2_{d\xi}} \\ &\lesssim \hbar_3^{\delta} \mu^{-(1-\frac{\delta}{2})} \cdot \left\|\phi_1\right\|_{\tilde{S}_0^{(n_1)}} \cdot \left\|\phi_2\right\|_{\tilde{S}_0^{(n_2)}}, \end{aligned}$$

and this can be summed over dyadic $\mu \gtrsim \hbar_3^{-2}$. This finally concludes the estimate for case *(1.b)*.

(2): $\xi \geq \max\{\xi_1, \xi_2\}$. This is again completely analogous to *(2)* in the proof of Proposition 6.10, taking advantage of Proposition 6.12 as well as repeated integrations by parts as in the preceding case, and hence omitted here.

Proof of (6.31). We start with the case when the output frequency ξ dominates the two input frequencies $\xi_{1,2}$, which corresponded to the case *(2)* in the preceding argument.

In order to define P_3, we formally expand the product $\chi_{R\lesssim\tau}\partial_R\phi_1 \cdot \partial_R\phi_2$ in a Taylor series around $R = 0$ and stop at order three terms. For this we expand (observe that we do not include a cutoff here)

$$\phi_1(R) = c \int_0^\infty \left(R\xi^{\frac{1}{2}}\right)^{|n_1|\mp1} \left(1 + \sum_{j\geq1} \phi_j(R^2) \cdot \left(R^2\xi\right)^j\right) \overline{x}_1(\xi)\tilde{\rho}_{n_1}(\xi)\, d\xi,$$

provided ϕ_1 is an angular momentum n_1 function with $|n_1| \geq 2$, while if $|n_1| = 1, 0$, we use an analogous formula at the level of $\mathcal{D}_j\phi_1(R)$, $j = 0, \pm1$, which gets then inserted into the formula giving $\phi_1(R)$ in terms of $\left(c_j,\, \mathcal{D}_j\phi_1(R)\right)$. Keeping track only of terms up to order R^3, we easily infer the bound (6.30). Now write the bad term corresponding to the case *(2)* from before as

$$\begin{aligned} &\sum_\lambda \chi_{\xi\simeq\lambda} \left\langle \phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\partial_R\phi_{1,\ll\lambda} \cdot \partial_R\phi_{2,\ll\lambda} - \chi_{R\lesssim1}P_3(R)\right\rangle_{L^2_{R\,dR}} \\ &= \sum_\lambda \chi_{\xi\simeq\lambda} \left\langle \chi_{R\xi^{\frac{1}{2}}\gtrsim1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\partial_R\phi_{1,\ll\lambda} \cdot \partial_R\phi_{2,\ll\lambda} - \chi_{R\lesssim1}P_3(R)\right\rangle_{L^2_{R\,dR}} \\ &+ \sum_\lambda \chi_{\xi\simeq\lambda} \left\langle \chi_{R\xi^{\frac{1}{2}}\lesssim1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\partial_R\phi_{1,\ll\lambda} \cdot \partial_R\phi_{2,\ll\lambda} - \chi_{R\lesssim1}P_3(R)\right\rangle_{L^2_{R\,dR}} \end{aligned} \tag{6.33}$$

Observe that we no longer keep careful track of the $\hbar_3$-dependence, since $n_3 = O(1)$ now, and similarly for all other angular momenta.

To deal with the first term on the right of (6.33), observe that we can replace the cutoff $\chi_{R\lesssim\tau}$ by $\chi_{R\lesssim1}$, since including a cutoff $\chi_{1\lesssim R\lesssim\tau}$ the resulting term can be handled like *(2)* in the proof of Proposition 6.10

via integration by parts without incurring problematic boundary terms. Assuming for simplicity that both factors $\phi_{1,2}$ are angular momentum n_j functions with $|n_j| \geq 2,\ j = 1, 2$, we can then schematically write

$$\chi_{R\lesssim 1}\partial_R\phi_{1,\ll\lambda}\cdot\partial_R\phi_{2,\ll\lambda} - \chi_{R\lesssim 1}P_3(R) = \sum_{j=1,2} X_j - \chi_{R\lesssim 1}\tilde{P}_3(R)$$

$$X_1 = \chi_{R\lesssim 1}\sum_{i+j\geq 4}\left(\int_0^\infty \chi_{R\xi_1^{\frac{1}{2}}\lesssim 1}(R\xi_1^{\frac{1}{2}})^i\left(1+\sum_{k\geq 1}\phi_k(R)(R\xi_1^{\frac{1}{2}})^k\right)\xi_1^{\frac{1}{2}}\chi_{\xi_1\ll\lambda}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right)$$
$$\cdot\left(\int_0^\infty \chi_{R\xi_2^{\frac{1}{2}}\lesssim 1}(R\xi_2^{\frac{1}{2}})^j\left(1+\sum_{l\geq 1}\phi_l(R)(R\xi_2^{\frac{1}{2}})^l\right)\xi_2^{\frac{1}{2}}\chi_{\xi_2\ll\lambda}\overline{x}_2(\xi_2)\tilde{\rho}_{n_2}(\xi_2)\,d\xi_2\right)$$

$$X_2 = \chi_{R\lesssim 1}\left(\int_0^\infty \chi_{R\xi_1^{\frac{1}{2}}\gtrsim 1}\partial_R\phi_{n_1}(R;\xi_1)\chi_{\xi_1\ll\lambda}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right)$$
$$\cdot\left(\int_0^\infty \partial_R\phi_{n_2}(R;\xi_2)\chi_{\xi_2\ll\lambda}\overline{x}_2(\xi_2)\tilde{\rho}_{n_2}(\xi_2)\,d\xi_2\right)$$
$$+\chi_{R\lesssim 1}\left(\int_0^\infty \chi_{R\xi_1^{\frac{1}{2}}\lesssim 1}\partial_R\phi_{n_1}(R;\xi_1)\chi_{\xi_1\ll\lambda}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right)$$
$$\cdot\left(\int_0^\infty \chi_{R\xi_2^{\frac{1}{2}}\gtrsim 1}\partial_R\phi_{n_2}(R;\xi_2)\chi_{\xi_2\ll\lambda}\overline{x}_2(\xi_2)\tilde{\rho}^{n_2}(\xi_2)\,d\xi_2\right)$$

where the functions $\phi_{l,k}(R)$ in the definition of X_1 satisfy bounds stated in Lemma 2.34. Furthermore, we can write schematically

$$\tilde{P}_3(R) = \sum_{i+j\leq 3} C_{i,j}\left(\int_0^\infty \chi_{R\xi_1^{\frac{1}{2}}\gtrsim 1}\left(R\xi_1^{\frac{1}{2}}\right)^i\xi_1^{\frac{1}{2}}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right)\cdot\left(\int_0^\infty\left(R\xi_2^{\frac{1}{2}}\right)^j\xi_2^{\frac{1}{2}}\overline{x}_2(\xi_2)\tilde{\rho}_{n_2}(\xi_2)\,d\xi_2\right)$$
$$+\sum_{i+j\leq 3} C_{i,j}\left(\int_0^\infty \chi_{R\xi_1^{\frac{1}{2}}\lesssim 1}\left(R\xi_1^{\frac{1}{2}}\right)^i\xi_1^{\frac{1}{2}}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right)\cdot\left(\int_0^\infty \chi_{R\xi_2^{\frac{1}{2}}\gtrsim 1}\left(R\xi_2^{\frac{1}{2}}\right)^j\xi_2^{\frac{1}{2}}\overline{x}_2(\xi_2)\tilde{\rho}_{n_2}(\xi_2)\,d\xi_2\right)$$
$$+\dots,$$

and where the terms denoted by ... refer to similar products for $R\xi_1^{\frac{1}{2}} \lesssim 1, R\xi_2^{\frac{1}{2}} \lesssim 1$ where at least one cutoff $\chi_{\xi_{1,2}\gtrsim\lambda}$ is included into one of the factors.
Then we handle their contribution to the first term on the right of (6.33) as follows: for the contribution of X_1, i.e. the term

$$\left\|\sum_\lambda \chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\chi_{R\simeq\kappa}\phi_{n_3}(R;\xi),\ X_1\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}},$$

localize R to dyadic size $\kappa \gtrsim \lambda^{-\frac{1}{2}}$. Exploiting the oscillatory nature of $\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi)$ from Lemma 4.36, Proposition 4.37 and their analogues for negative n, and performing integration by parts with respect to R, we gain $\left(\frac{1}{\lambda^{\frac{1}{2}}\kappa}\right)^N$. Then localizing the frequencies $\xi_{1,2}$ to dyadic values $\mu_{1,2} \ll \lambda$, and calling $X_{1,\mu_{1,2}}$ the corresponding contribution to X_1, we have

$$\begin{aligned}&\Big\|\chi_{\xi\simeq\lambda}\Big\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\chi_{R\simeq\kappa}\phi_{n_3}(R;\xi),\,X_{1,\mu_{1,2}}\Big\rangle_{L^2_{R\,dR}}\Big\|_{S_1^{\hbar_3}}\\&\lesssim\lambda^{\frac{1}{2}-\frac{\delta}{2}}\langle\lambda\rangle^{\frac{3}{2}+\delta}\Big\|\chi_{\xi\simeq\lambda}\Big\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\chi_{R\simeq\kappa}\phi_{n_3}(R;\xi),\,X_{1,\mu_{1,2}}\Big\rangle_{L^2_{R\,dR}}\Big\|_{L^2_{d\xi}}\\&\lesssim\lambda^{1-\frac{\delta}{2}}\langle\lambda\rangle^{\frac{3}{2}+\delta}\Big\|\chi_{\xi\simeq\lambda}\Big\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\chi_{R\sim\kappa}\phi(R;\xi,\hbar_3),\,X_{1,\mu_{1,2}}\Big\rangle_{L^2_{R\,dR}}\Big\|_{L^\infty_{d\xi}},\end{aligned}$$

and further for fixed $\xi\simeq\lambda$ we have, by Lemma 4.36 and the profile of X_1,

$$\begin{aligned}&\Big|\Big\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\chi_{R\simeq\kappa}\phi_{n_3}(R;\xi),\,X_{1,\mu_{1,2}}\Big\rangle_{L^2_{R\,dR}}\Big|\\&\lesssim\frac{\kappa^6}{\big(\lambda^{\frac{1}{2}}\kappa\big)^N}\cdot\left(\mu_1^{\frac{1}{2}-\frac{\delta}{2}}\frac{\mu_2^{\frac{\delta}{2}}}{\langle\mu_2\rangle^{\frac{3}{2}+\delta}}+\mu_2^{\frac{1}{2}-\frac{\delta}{2}}\frac{\mu_1^{\frac{\delta}{2}}}{\langle\mu_1\rangle^{\frac{3}{2}+\delta}}\right)\prod_{j=1,2}\|\overline{x}_j\|_{S_0^{\hbar_j}(\xi_j\simeq\mu_j)}\end{aligned}$$

Summing over $\kappa\gtrsim\lambda^{-\frac{1}{2}}$ we obtain

$$\begin{aligned}&\sum_{\kappa\gtrsim\lambda}\frac{\kappa^6}{\big(\lambda^{\frac{1}{2}}\kappa\big)^N}\left(\mu_1^{\frac{1}{2}-\frac{\delta}{2}}\frac{\mu_2^{\frac{\delta}{2}}}{\langle\mu_2\rangle^{\frac{3}{2}+\delta}}+\mu_2^{\frac{1}{2}-\frac{\delta}{2}}\frac{\mu_1^{\frac{\delta}{2}}}{\langle\mu_1\rangle^{\frac{3}{2}+\delta}}\right)\cdot\prod_{j=1,2}\|x_j\|_{S_0^{\hbar_j}(\xi_j\simeq\mu_j)}\\&\lesssim\frac{\mu_1^{\frac{1}{2}-\frac{\delta}{2}}\frac{\mu_2^{\frac{\delta}{2}}}{\langle\mu_2\rangle^{\frac{3}{2}+\delta}}+\mu_2^{\frac{1}{2}-\frac{\delta}{2}}\frac{\mu_1^{\frac{\delta}{2}}}{\langle\mu_1\rangle^{\frac{3}{2}+\delta}}}{\lambda^3}\prod_{j=1,2}\|\overline{x}_j\|_{S_0^{\hbar_j}(\xi_j\simeq\mu_j)}.\end{aligned}$$

Combining with the inequality further above we obtain

$$\Big\|\chi_{\xi\simeq\lambda}\Big\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\,X_{1,\mu_{1,2}}\Big\rangle_{L^2_{R\,dR}}\Big\|_{S_1^{\hbar_3}}\lesssim\frac{\mu_1^{\frac{1}{2}-\frac{\delta}{2}}\frac{\mu_2^{\frac{\delta}{2}}}{\langle\mu_2\rangle^{\delta}}+\mu_2^{\frac{1}{2}-\frac{\delta}{2}}\frac{\mu_1^{\frac{\delta}{2}}}{\langle\mu_1\rangle^{\delta}}}{\lambda^{\frac{1}{2}-\frac{\delta}{2}}}\prod_{j=1,2}\|x_j\|_{S_0^{\hbar_j}(\xi_j\sim\mu_j)}$$

provided $\lambda\gtrsim 1$. Finally, exploiting orthogonality and Cauchy-Schwarz inequality, we infer that

$$\Big\|\Big\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\,X_{1,\mu_{1,2}}\Big\rangle_{L^2_{R\,dR}}\Big\|_{S_1^{\hbar_3}(\xi\gtrsim 1)}\lesssim\left(\sum_{\lambda}\Big\|\chi_{\xi\simeq\lambda}\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\,X_{1,\mu_{1,2}}\rangle_{L^2_{R\,dR}}\Big\|^2_{S_1^{\hbar_3}}\right)^{\frac{1}{2}}\lesssim\prod_{j=1}^{2}\|\phi_j\|_{\tilde{S}^{(n_j)}}.$$

The estimate for low output frequencies $\xi\lesssim 1$ is much easier due to the restriction on R in the definition of X_1 and omitted here.

The contribution of the term X_2 to the first term on the right hand side in (6.33) is handled similarly, except that now one has to use the oscillatory expansion for $\chi_{R\xi_1^{\frac{1}{2}}\gtrsim 1}\phi_{n_1}(R;\xi_1)$ and the other oscillatory terms and combine the phases with the one of $\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi)$ before performing integration by parts. We omit the

similar details.
In order to complete the estimate for the first term on the right hand side of (6.33), it thus suffices to deal with the contribution of $\chi_{R\lesssim 1}\tilde{P}_3(R)$. We observe right away that the restrictions $R\xi^{\frac{1}{2}} \gtrsim 1$, $R \lesssim 1$ imply $\xi \gtrsim 1$. Consider the first term in the definition of $\tilde{P}_3(R)$, and call this $\tilde{P}_3^{(1)}(R)$. Further localise the frequencies of the two factors to dyadic size $\mu_{1,2}$, respectively, resulting in $\tilde{P}^{(1)}_{3,\mu_{1,2}}(R)$. Then we bound for $\kappa \gtrsim \mu_1^{-\frac{1}{2}}$

$$\left|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\simeq\kappa}\chi_{R\lesssim 1}\tilde{P}^{(1)}_{3,\mu_{1,2}}(R)\right\rangle_{L^2_{R\,dR}}\right|$$
$$\lesssim \sum_{i+j\leq 3}\frac{\kappa^{2+i+j}}{(\lambda^{\frac{1}{2}}\kappa)^N}\cdot\left|\int_0^\infty \chi_{\xi_1\simeq\mu_1}\xi_1^{\frac{1+i}{2}}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right|\cdot\left|\int_0^\infty \chi_{\xi_2\simeq\mu_2}\xi_2^{\frac{1+j}{2}}\overline{x}_2(\xi_2)\tilde{\rho}^{n_2}(\xi_2)\,d\xi_2\right|.$$

The integrals are easily bounded by

$$\left|\int_0^\infty \chi_{\xi_1\simeq\mu_1}\xi_1^{\frac{1+i}{2}}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right| \lesssim \frac{\mu_1^{\frac{\delta}{2}}}{\langle\mu_1\rangle^\delta}\cdot\langle\mu_1\rangle^{-\frac{3}{2}+\frac{i}{2}}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\sim\mu_1)},$$
$$\left|\int_0^\infty \chi_{\xi_2\simeq\mu_1}\xi_2^{\frac{1+j}{2}}\overline{x}_2(\xi_2)\tilde{\rho}^{n_2}(\xi_2)\,d\xi_2\right| \lesssim \frac{\mu_2^{\frac{\delta}{2}}}{\langle\mu_2\rangle^\delta}\cdot\langle\mu_2\rangle^{-\frac{3}{2}+\frac{j}{2}}\cdot\|\overline{x}_2\|_{S_0^{\hbar_1}(\xi_2\sim\mu_2)}.$$

We conclude that

$$\sum_{\kappa\gtrsim\lambda^{-\frac{1}{2}}}\left|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\simeq\kappa}\chi_{R\lesssim 1}\tilde{P}^{(1)}_{3,\mu_{1,2}}(R)\right\rangle_{L^2_{R\,dR}}\right|$$
$$\lesssim \lambda^{-1-\frac{3}{2}}\cdot\left(\frac{\mu_1}{\lambda}\right)^{N'}\cdot\frac{\mu_1^{\frac{\delta}{2}}}{\langle\mu_1\rangle^\delta}\cdot\langle\mu_1\rangle^{-\frac{3}{2}+\frac{i}{2}}\frac{\mu_2^{\frac{\delta}{2}}}{\langle\mu_2\rangle^\delta}\cdot\langle\mu_2\rangle^{-\frac{3}{2}+\frac{j}{2}}\cdot\prod_{k=1,2}\|\overline{x}_k\|_{S_0^{\hbar_k}(\xi_k\simeq\mu_k)}$$

Finally, we infer

$$\left\|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1}\tilde{P}^{(1)}_{3,\mu_{1,2}}(R)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}$$
$$\lesssim \lambda^{\frac{1}{2}-\frac{\delta}{2}}\langle\lambda\rangle^{\frac{3}{2}+\delta}\left\|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1}\tilde{P}^{(1)}_{3,\mu_{1,2}}(R)\right\rangle_{L^2_{R\,dR}}\right\|_{L^2_{d\xi}}$$
$$\lesssim \lambda^{1-\frac{\delta}{2}}\langle\lambda\rangle^{\frac{3}{2}+\delta}\left\|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1}\tilde{P}^{(1)}_{3,\mu_{1,2}}(R)\right\rangle_{L^2_{R\,dR}}\right\|_{L^\infty_{d\xi}}$$

whence using the preceding bound we get

$$\left\|\chi_{\xi\simeq\lambda}\langle\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1}\tilde{P}^{(1)}_{3,\mu_{1,2}}(R)\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}$$

$$\lesssim \left(\frac{\mu_1}{\lambda}\right)^{N'} \cdot \frac{(\lambda\mu_1)^{\frac{\delta}{2}}}{\langle\mu_1\rangle^{\delta}} \cdot \frac{\mu_2^{\frac{\delta}{2}}}{\langle\mu_2\rangle^{\delta}} \cdot \prod_{k=1,2} \|\overline{x}_k\|_{S_0^{\hbar_k}(\xi_k\simeq\mu_k)}.$$

But then exploiting orthogonality and the Cauchy-Schwarz inequality as usual we infer (here $\tilde{P}^{(1)}_{3,\ll\lambda,\ll\lambda}(R)$ is the term arising from $\tilde{P}^{(1)}_3$ after restricting both factors to frequency $\ll \lambda$)

$$\begin{aligned}
&\left\| \sum_{\lambda} \left\langle \chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac12}\gtrsim 1}\phi(R;\xi,\hbar_3),\, \chi_{R\lesssim 1}\tilde{P}^{(1)}_{3,\ll\lambda,\ll\lambda}(R)\right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{(\hbar_3)}}\\
&\lesssim \left(\sum_{\lambda} \left[\sum_{\mu_{1,2}\ll\lambda} \left\| \left\langle \chi_{R\xi^{\frac12}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1}\tilde{P}^{(1)}_{3,\mu_{1,2}}(R)\right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}(\xi\simeq\lambda)} \right]^2 \right)^{\frac12}\\
&\lesssim \left(\sum_{\lambda} \sum_{\mu_{1,2}\ll\lambda} \left(\frac{\mu_1}{\lambda}\right)^{N'} \cdot \frac{(\lambda\mu_1)^{\frac{\delta}{2}}}{\langle\mu_1\rangle^{\delta}} \cdot \frac{\mu_2^{\frac{\delta}{2}}}{\langle\mu_2\rangle^{\delta}} \cdot \prod_{k=1,2} \|\overline{x}_k\|^2_{S_0^{\hbar_k}(\xi_k\simeq\mu_k)} \right)^{\frac12}\\
&\lesssim \prod_{k=1,2} \|\overline{x}_k\|_{S_0^{\hbar_k}}
\end{aligned}$$

This still leaves the contribution of

$$\chi_{R\lesssim 1}[\tilde{P}^{(1)}_3(R) - \tilde{P}^{(1)}_{3,\ll\lambda,\ll\lambda}(R)]$$

to be bounded, which will be done like below, when treating the contribution of the final term constituting $\tilde{P}_3(R)$. The second term in the decomposition of $\tilde{P}_3(R)$ is handled analogously to the first, and so we only need to bound the contribution of the last term, $\tilde{P}^{(3)}_3(R)$ to finish the bound for the first term on the right of (6.33). This contribution is then given by[2]

$$\begin{aligned}
\sum_{i+j\le 3} C_{i,j} \sum_{\lambda} \Big\langle \chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac12}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1} &\left(\int_0^\infty \left(R\xi_1^{\frac12}\right)^i \chi_{\xi_1\gtrsim\lambda}\xi_1^{\frac12}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1 \right)\\
&\cdot \left(\int_0^\infty \left(R\xi_2^{\frac12}\right)^j \xi_2^{\frac12}\overline{x}_2(\xi_2)\tilde{\rho}_{n_2}(\xi_2)\,d\xi_2 \right) \Big\rangle_{L^2_{R\,dR}}.
\end{aligned}$$

Then we use the integral bounds (for $i + j \le 3$)

$$\begin{aligned}
\left| \int_0^\infty \xi_1^{\frac{i+1}{2}} \chi_{\xi_1\gtrsim\lambda}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1 \right| &\lesssim \sum_{\mu\gtrsim\lambda} \mu^{-\frac32-\frac{\delta}{2}}\mu^{\frac{i}{2}} \cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)},\\
\left| \int_0^\infty \xi_2^{\frac{j+1}{2}} \overline{x}_2(\xi_2)\tilde{\rho}_{n_1}(\xi_2)\,d\xi_2 \right| &\lesssim \|\overline{x}_2\|_{S_0^{\hbar_2}},
\end{aligned}$$

[2]In addition there is a similar term with the roles of x_1, x_2 interchanged.

and so localizing R to $\kappa \lesssim 1$, $\kappa \gtrsim \lambda^{-\frac{1}{2}}$, we infer after repeated integrations by parts (denoting by $A^{i,j}_{\lambda}(R)$ the product of the two integrals in the long expression above)

$$\begin{aligned}&\left|\langle \chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\simeq\kappa}A^{i,j}_{\lambda}(R)\rangle_{L^2_{R\,dR}}\right|\\ &\lesssim \frac{\kappa^{2+i+j}}{(\lambda^{\frac{1}{2}}\kappa)^N}\cdot\left(\sum_{\mu\gtrsim\lambda}\mu^{-\frac{3}{2}-\frac{\delta}{2}}\mu^{\frac{i}{2}}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)}\right)\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}},\end{aligned}$$

whence summing over κ

$$\begin{aligned}&\sum_{\lambda^{-\frac{1}{2}}\lesssim\kappa\lesssim 1}\left|\left\langle \chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\simeq\kappa}A^{i,j}_{\lambda}(R)\right\rangle_{L^2_{R\,dR}}\right|\\ &\lesssim \lambda^{-1-\frac{3}{2}}\cdot\left(\sum_{\mu\gtrsim\lambda}\mu^{-\frac{3}{2}-\frac{\delta}{2}}\mu^{\frac{i}{2}}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)}\right)\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}}.\end{aligned}$$

Finally, we infer

$$\begin{aligned}&\left\|\sum_{i+j\le 3}C_{i,j}\sum_{\lambda}\langle \chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1}A^{i,j}_{\lambda}(R)\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\ &\lesssim \sum_{i+j\le 3}\left(\sum_{\lambda\gtrsim 1}\left\|\langle \chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1}A^{i,j}_{\lambda}(R)\rangle_{L^2_{R\,dR}}\right\|^2_{S_1^{\hbar_3}}\right)^{\frac{1}{2}}\\ &\lesssim \sum_{i+j\le 3}\left(\sum_{\lambda\gtrsim 1}\left[\lambda^{1-\frac{\delta}{2}}\langle\lambda\rangle^{\frac{3}{2}+\delta}\left\|\left\langle \chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1}A^{i,j}_{\lambda}(R)\right\rangle_{L^2_{R\,dR}}\right\|_{L^\infty_{d\xi}}\right]^2\right)^{\frac{1}{2}},\end{aligned}$$

and the preceding expression can be bounded by

$$\begin{aligned}&\sum_{i+j\le 3}\left(\sum_{\lambda\gtrsim 1}\left[\lambda^{1-\frac{\delta}{2}}\langle\lambda\rangle^{\frac{3}{2}+\delta}\left\|\left\langle \chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac{1}{2}}\gtrsim 1}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim 1}A^{i,j}_{\lambda}(R)\right\rangle_{L^2_{R\,dR}}\right\|_{L^\infty_{d\xi}}\right]^2\right)^{\frac{1}{2}}\\ &\lesssim \left(\sum_{\lambda\gtrsim 1}\left[\sum_{\mu\gtrsim\lambda}\left(\frac{\lambda}{\mu}\right)^{\delta}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)}\right]^2\right)^{\frac{1}{2}}\|\overline{x}_2\|_{S_0^{\hbar_2}}\lesssim\prod_{j=1,2}\left\|\overline{x}_j\right\|_{S_0^{\hbar_j}}.\end{aligned}$$

This finally concludes the bound for the first term on the right hand side of (6.33), provided that both factors $\phi_{1,2}$ are angular momentum n_j-functions with $|n_j| \ge 2$. The case $|n_j| < 2$ for at least one j is handled similarly and omitted.

As for the second term on the right hand side of (6.33), here we have to take advantage of the high degree of vanishing of the expression in the inner product at $R = 0$, as well as the shortness of the interval of integration. To begin with we can reduce $\chi_{R\lesssim\tau}\partial_R\phi_{1,\ll\lambda}\cdot\partial_R\phi_{2,\ll\lambda}$ to $\chi_{R\lesssim 1}\partial_R\phi_{1,\ll\lambda}\cdot\partial_R\phi_{2,\ll\lambda}$. In light of the cutoff $\chi_{R\xi^{\frac{1}{2}}\lesssim 1}$ in the second term in (6.33), this further localization is automatic if $\xi\gtrsim 1$. Thus consider now

$$\sum_{\lambda\lesssim 1}\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi),\,\chi_{1\lesssim R\lesssim\tau}\partial_R\phi_{1,\ll\lambda}\cdot\partial_R\phi_{2,\ll\lambda}\right\rangle_{L^2_{R\,dR}}$$

We shall again assume for simplicity that both factors $\phi_{1,2}$ are angular momentum n_j functions with $|n_j|\geq 2$, the remaining cases being dealt with similarly. A straightforward sharpening of Lemma 6.9 furnishes the bounds

$$\left|\partial_R\phi_{1,\mu_1}\right|\lesssim\mu_1^{\frac{\delta}{2}}\,\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)},\quad\left|\partial_R\phi_{2,\ll\lambda}\right|\lesssim\|\overline{x}_2\|_{S_0^{\hbar_2}},$$

and so we infer

$$\begin{aligned}&\left\|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi),\,\chi_{1\lesssim R\lesssim\tau}\partial_R\phi_{1,\ll\lambda}\cdot\partial_R\phi_{2,\ll\lambda}\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\&\lesssim\sum_{\mu_1\ll\lambda}\lambda^{1-\frac{\delta}{2}}\cdot\lambda^{-1}\cdot\left\|\partial_R\phi_{1,\mu_1}\cdot\partial_R\phi_{2,\ll\lambda}\right\|_{L^\infty_{R\,dR}}\\&\lesssim\sum_{\mu_1\ll\lambda}\left(\frac{\mu}{\lambda}\right)^{\frac{\delta}{2}}\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}}.\end{aligned}$$

Square-summing over λ yields the desired bound upon applying Cauchy-Schwarz and orthogonality.
It follows that its suffices to bound

$$\left\|\sum_{\lambda}\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi),\,\chi_{R\lesssim 1}\left[\partial_R\phi_{1,\ll\lambda}\cdot\partial_R\phi_{2,\ll\lambda}-P_3(R)\right]\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}.$$

Observe that in the inner product the factors $\partial_R\phi_{j,\ll\lambda}$ are automatically in the non-oscillatory regime due to the restrictions on their frequencies and the cutoff $\chi_{R\xi^{\frac{1}{2}}\lesssim 1}$. Thus we can write

$$\chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\chi_{R\lesssim 1}\left[\partial_R\phi_{1,\ll\lambda}\cdot\partial_R\phi_{2,\ll\lambda}-P_3(R)\right]=X_3+\tilde{P}_4,$$

where we can write schematically

$$\begin{aligned}X_3=\chi_{R\lesssim 1}\chi_{\xi\simeq\lambda}\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\sum_{i+j\geq 4}&\left(\int_0^\infty(R\xi_1^{\frac{1}{2}})^i\left(1+\sum_{k\geq 1}\phi_k(R)(R\xi_1^{\frac{1}{2}})^k\right)\xi_1^{\frac{1}{2}}\chi_{\xi_1\ll\lambda}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right)\\&\cdot\left(\int_0^\infty(R\xi_2^{\frac{1}{2}})^j\left(1+\sum_{l\geq 1}\phi_l(R)(R\xi_2^{\frac{1}{2}})^l\right)\xi_2^{\frac{1}{2}}\chi_{\xi_2\ll\lambda}\overline{x}_2(\xi_2)\tilde{\rho}_{n_2}(\xi_2)\,d\xi_2\right)\end{aligned}$$

as well as

$$\tilde{P}_4=\sum_{i+j\leq 3}C_{i,j}\left(\int_0^\infty\chi_{\xi_1\gtrsim\lambda}\left(R\xi_1^{\frac{1}{2}}\right)^i\xi_1^{\frac{1}{2}}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right)\cdot\left(\int_0^\infty\left(R\xi_2^{\frac{1}{2}}\right)^j\xi_2^{\frac{1}{2}}\overline{x}_2(\xi_2)\tilde{\rho}_{n_2}(\xi_2)\,d\xi_2\right)$$

$$+\sum_{i+j\leq 3} C_{i,j}\left(\int_0^\infty \chi_{\xi_1\lesssim\lambda}\left(R\xi_1^{\frac{1}{2}}\right)^i \xi_1^{\frac{1}{2}}\overline{x}_1(\xi_1)\tilde{\rho}_{n_1}(\xi_1)\,d\xi_1\right)\cdot\left(\int_0^\infty \chi_{\xi_2\gtrsim\lambda}\left(R\xi_2^{\frac{1}{2}}\right)^j \xi_2^{\frac{1}{2}}\overline{x}_2(\xi_2)\tilde{\rho}_{n_2}(\xi_2)\,d\xi_2\right).$$

We bound the contributions of $X_3, \tilde{P}_4$ as follows:
Contribution of X_3: First, the low frequencies $\lambda \lesssim 1$ are handled as follows. Localizing the variable $\overline{x}_1(\xi_1)$ to dyadic frequency $\mu_1 \ll \lambda$, and calling the resulting expression X_{3,μ_1}, we get

$$\begin{aligned}\left|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi),\,\chi_{R\lesssim 1}X_{3,\mu_1}\right\rangle_{L^2_{R\,dR}}\right| &\lesssim \lambda^{-1}\cdot\left\|\chi_{R\lesssim 1}X_{3,\mu_1}\right\|_{L^\infty_{R\,dR}}\\ &\lesssim \lambda^{-1}\mu_1^{\frac{\delta}{2}}\,\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}}\,.\end{aligned}$$

From here we conclude that

$$\begin{aligned}&\left\|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi),\,\chi_{R\lesssim 1}X_{3,\mu_1}\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\ &\lesssim \lambda^{1-\frac{\delta}{2}}\left\|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi),\,\chi_{R\lesssim 1}X_{3,\mu_1}\right\rangle_{L^2_{R\,dR}}\right\|_{L^\infty_{d\xi}}\\ &\lesssim \left(\frac{\mu_1}{\lambda}\right)^{\frac{\delta}{2}}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}}\,.\end{aligned}$$

Square-summing over λ and invoking the Cauchy-Schwarz inequality as usual gives the desired bound. For high frequencies $\lambda \gtrsim 1$, we take advantage of the high degree of vanishing at $R = 0$ of X_3: specifically, localising the frequencies of the factors to $\mu_{1,2} \ll \lambda$ as well as fixing i, j for simplicity and calling the resulting expression $X_{3,\mu_{1,2}}^{(i,j)}$, we get the poitnwise bound

$$\begin{aligned}&\left|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi),\,\chi_{R\lesssim 1}X_{3,\mu_{1,2}}^{(i,j)}\right\rangle_{L^2_{R\,dR}}\right|\\ &\lesssim \lambda^{-1-\frac{i+j}{2}}\cdot\mu_1^{\max\{\frac{i-3}{2}-\frac{\delta}{2},0\}}\cdot\mu_2^{\max\{\frac{j-3}{2}-\frac{\delta}{2},0\}}\cdot\prod_{k=1,2}\|\overline{x}_k\|_{S_0^{\hbar_k}(\xi_k\simeq\mu_k)}\end{aligned}$$

whence

$$\begin{aligned}&\left\|\chi_{\xi\simeq\lambda}\left\langle\chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi),\,\chi_{R\lesssim 1}X_{3,\mu_{1,2}}^{(i,j)}\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\ &\lesssim \lambda^{\frac{3}{2}+\frac{\delta}{2}-\frac{i+j}{2}}\cdot\mu_1^{\max\{\frac{i-3}{2}-\frac{\delta}{2},0\}}\cdot\mu_2^{\max\{\frac{j-3}{2}-\frac{\delta}{2},0\}}\cdot\prod_{k=1,2}\|\overline{x}_k\|_{S_0^{\hbar_k}(\xi_k\simeq\mu_k)}\,,\end{aligned}$$

where we keep the condition $i+j \geq 4$ in mind. Square-summing over $\lambda \gtrsim 1$ and exploiting simple orthogonal arguments, the desired bound easily follows.
As for the contribution of $\tilde{P}_4$, where we only treat the first line, the second being more of the same, we get smallness from the large frequencies inside at least one of the factors in this expression. Using that $i+j \leq 3$, and fixing i, j as well as the frequency $\xi_1 \simeq \mu_1$ in the first factor, which results in $\tilde{P}_{4,\mu_1}^{(i,j)}$, we can bound, for

large frequencies $\lambda \gtrsim 1$,

$$\left|\chi_{\xi\simeq\lambda}\left\langle \chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi), \tilde{P}^{(i,j)}_{4,\mu_1}\right\rangle_{L^2_{RdR}}\right| \lesssim \lambda^{-1-\frac{i}{2}}\cdot \mu_1^{-(\frac{3}{2}-\frac{i}{2}+\frac{\delta}{2})}\cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\|\overline{x}_2\|_{S_0^{\hbar_2}}.$$

This in turn implies that

$$\left\|\chi_{\xi\simeq\lambda}\left\langle \chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi), \tilde{P}^{(i,j)}_{4,\mu_1}\right\rangle_{L^2_{RdR}}\right\|_{S_1^{\hbar_3}} \lesssim \left(\frac{\lambda}{\mu_1}\right)^{(\frac{3}{2}-\frac{i}{2}+\frac{\delta}{2})}\cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\|\overline{x}_2\|_{S_0^{\hbar_2}}.$$

But from here, keeping in mind that $i \leq 3$, we infer that

$$\left(\sum_{\lambda\gtrsim 1}\left\|\sum_{\mu_1\gtrsim\lambda}\chi_{\xi\simeq\lambda}\left\langle \chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi), \tilde{P}^{(i,j)}_{4,\mu_1}\right\rangle_{L^2_{RdR}}\right\|_{S_1^{\hbar_3}}^2\right)^{\frac{1}{2}} \lesssim \prod_{k=1,2}\|\overline{x}_k\|_{S_0^{\hbar_k}}.$$

For small frequencies $\lambda \lesssim 1$, we distinguish two cases $\mu_1 \lesssim 1$ and $\mu_1 \gtrsim 1$. For the first case, we have

$$\left|\chi_{\xi\simeq\lambda}\left\langle \chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi), \tilde{P}^{(i,j)}_{4,\mu_1}\right\rangle_{L^2_{RdR}}\right| \lesssim \mu_1^{\frac{\delta}{2}}\cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\|\overline{x}_2\|_{S_0^{\hbar_2}},$$

which implies

$$\left\|\chi_{\xi\simeq\lambda}\left\langle \chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi), \tilde{P}^{(i,j)}_{4,\mu_1}\right\rangle_{L^2_{RdR}}\right\|_{S_1^{\hbar_3}} \lesssim \lambda^{1-\frac{\delta}{2}}\mu_1^{\frac{\delta}{2}}\cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\|\overline{x}_2\|_{S_0^{\hbar_2}}.$$

Then a routing argument gives the desired result. For the second case where $\mu_1 \gtrsim 1$, we have

$$\left|\chi_{\xi\simeq\lambda}\left\langle \chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi), \tilde{P}^{(i,j)}_{4,\mu_1}\right\rangle_{L^2_{RdR}}\right| \lesssim \mu_1^{-(\frac{3}{2}-\frac{i}{2}+\frac{\delta}{2})}\cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\|\overline{x}_2\|_{S_0^{\hbar_2}},$$

which implies

$$\left\|\chi_{\xi\simeq\lambda}\left\langle \chi_{R\xi^{\frac{1}{2}}\lesssim 1}\phi_{n_3}(R;\xi), \tilde{P}^{(i,j)}_{4,\mu_1}\right\rangle_{L^2_{RdR}}\right\|_{S_1^{\hbar_3}} \lesssim \lambda^{1-\frac{\delta}{2}}\mu_1^{-(\frac{3}{2}-\frac{i}{2}+\frac{\delta}{2})}\cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi_1\simeq\mu_1)}\|\overline{x}_2\|_{S_0^{\hbar_2}}.$$

Again a routing argument gives the desired result.

Now we turn to the case when $\xi_1 \leq \xi \leq \xi_2$. Reviewing the argument handling the case $|n_3| \gg 1$, there are two places where we used the integration by parts regarding the operator H_{n_3}. The first place is when we handle the contribution from

$$\chi_{\xi\simeq\mu}\left\langle \phi_{n_3}(R,\xi), \chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\phi_n(R)\right\rangle_{L^2_{RdR}}.$$

Note that if the vanishing order at $R = 0$ of $\partial_R\phi_{1,\leq\mu}$ is greater or equal to 4, then we can still perform the integration by parts argument. If the vanishing order is less or equal to 3, then we can follow the argument estimating the contribution from

$$\chi_{\xi\simeq\mu}\left\langle \phi_{n_3}(R,\xi), \chi_{R\lesssim 1}\tilde{P}_3(R)\right\rangle_{L^2_{RdR}}.$$

The other place where we used the integration by parts argument is when estimate the contribution from

$$\chi_{\xi\simeq\mu}\left\langle \phi_{n_3}(R,\xi), \chi_{R\lesssim\tau}\partial_R\phi_{1,\leq\mu}\cdot\partial_R\phi_{2,[\mu,\infty]}\right\rangle_{L^2_{RdR}},$$

where, with $n = 0, \pm 1$,

$$\partial_R \phi_{2,[\mu,\infty]} = \partial_R \left(\phi_n(R)\right) \cdot \int_0^R \left[\phi_n(s)\right]^{-1} \mathcal{D}_n \phi_{2,[\mu,\infty]}(s)\, ds.$$

Note that the vanishing order at $R = 0$ in the expression $\chi_{\xi \simeq \mu} \left\langle \phi_{n_3}(R,\xi), \chi_{R \lesssim 1} \tilde{P}_3(R) \right\rangle_{L^2_{RdR}}$ is at least of $O(R^{\frac{9}{2}})$, which is enough for integration by parts regarding H_{n_3} twice, as for the $|n_3| \gg 1$ case. This completes the proof of the proposition. □

In order to bound the quadratic "null-forms" arising in the nonlinearity, we also need to deal with the terms involving temporal derivatives, which in terms of the (τ, R)-coordinates involve the operator $\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R$. In order to understand the effect of this operator on the exceptional low angular momentum terms, it is useful to determine the effect of this operator on the expressions in (6.21), (6.23), (6.25). Letting $n = 0, \pm 1$, we get (now the function ϕ also depends on τ)

$$\begin{aligned}
&\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R\right)\left(c_n(\tau)\phi_n(R) + \phi_n(R) \cdot \int_0^R \left[\phi_n(s)\right]^{-1} \mathcal{D}_n \phi(\tau, s)\, ds\right) \\
&= c_n'(\tau)\phi_n(R) + c_n(\tau)\frac{\lambda_\tau}{\lambda} R\partial_R \phi_n(R) + \frac{\lambda_\tau}{\lambda}\left(R\partial_R \phi_n(R)\right) \cdot \int_0^R \left[\phi_n(s)\right]^{-1} \mathcal{D}_n \phi(\tau, s)\, ds \\
&+ \frac{\lambda_\tau}{\lambda} R \mathcal{D}_n \phi(\tau, R) + \phi_n(R) \cdot \int_0^R \left[\phi_n(s)\right]^{-1} \partial_\tau \left(\mathcal{D}_n \phi(\tau, s)\right)\, ds.
\end{aligned} \tag{6.34}$$

Here in light of the explicit algebraic nature of $\phi_n(R)$, the first three terms at the end are essentially like the terms in the original formula for $\phi(\tau, R)$, except that they come with an extra factor $\tau^{-1} \sim \frac{\lambda_\tau}{\lambda}$. As for the last term, write it as

$$\begin{aligned}
\phi_n(R) \cdot \int_0^R \left[\phi_n(s)\right]^{-1} \partial_\tau \left(\mathcal{D}_n \phi(\tau, s)\right)\, ds &= \phi_n(R) \cdot \int_0^R \left[\phi_n(s)\right]^{-1} \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} s \partial_s\right)\left(\mathcal{D}_n \phi(\tau, s)\right)\, ds \\
&+ \frac{\lambda_\tau}{\lambda}\phi_n(R) \cdot \int_0^R \partial_s \left(s \left[\phi_n(s)\right]^{-1}\right)\left(\mathcal{D}_n \phi(\tau, s)\right)\, ds \\
&- \frac{\lambda_\tau}{\lambda} R \mathcal{D}_n \phi(\tau, R),
\end{aligned} \tag{6.35}$$

where the last term cancels against the fourth term in the earlier identity. The middle term is completely analogous to the term

$$\phi_n(R) \cdot \int_0^R \left[\phi_n(s)\right]^{-1} \mathcal{D}_n \phi(\tau, s)\, ds,$$

The first term on the right is more delicate, and will be handled by formulating it on the Fourier side. Specifically, writing

$$\mathcal{D}_n \phi(\tau, R) = \int_0^\infty \phi_n(R, \xi)\overline{x}(\tau, \xi)\tilde{\rho}_n(\xi)\, d\xi,$$

we express the effect of the operator $\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R$ by

$$\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R\right)\mathcal{D}_n \phi(\tau, R) = \int_0^\infty \phi_n(R, \xi)\mathcal{D}_\tau^{(n)}\overline{x}(\tau, \xi)\tilde{\rho}_n(\xi)\, d\xi + \int_0^\infty \phi_n(R, \xi)\mathcal{K}_n^{(0)}\overline{x}(\tau, \xi)\tilde{\rho}_n(\xi)\, d\xi, \tag{6.36}$$

where $\mathcal{K}_n^{(0)}$ is the non-diagonal part of the transference operator, which vanishes when $n = 1$. Moreover, the dilation type operator $\mathcal{D}_\tau^{(n)}$ is given by the formula

$$\mathcal{D}_\tau^{(n)} = \partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi - \frac{\lambda_\tau}{\lambda}\frac{(\tilde{\rho}_n(\xi))'\,\xi}{\tilde{\rho}_n(\xi)} - \frac{\lambda_\tau}{\lambda}.$$

Next, we consider the effect of $\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R$ on large angular momentum functions, where things are more direct. In fact, writing

$$\phi(\tau, R) = \int_0^\infty \phi_n(R;\xi)\overline{x}(\tau,\xi)\tilde{\rho}_n(\xi)\,d\xi,$$

we obtain

$$\begin{aligned}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\Big)\phi(\tau,R) &= \int_0^\infty \phi_n(R;\xi)\mathcal{D}_\tau^{(\hbar)}\overline{x}(\tau,\xi)\tilde{\rho}_n(\xi)\,d\xi \\ &\quad + \int_0^\infty \phi_n(R;\xi)\mathcal{K}_\hbar^{(0)}\overline{x}(\tau,\xi)\tilde{\rho}_n(\xi)\,d\xi,\end{aligned} \tag{6.37}$$

where the dilation type operator is given by the formula

$$\mathcal{D}_\tau^{(\hbar)} = \partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi - \frac{\lambda_\tau}{\lambda}\frac{(\tilde{\rho}_n(\xi))'\,\xi}{\tilde{\rho}_n(\xi)} - 2\frac{\lambda_\tau}{\lambda}.$$

We shall then use bounds on $\overline{x}(\tau,\xi)$, $\mathcal{D}_\tau^{(\hbar)}\overline{x}(\tau,\xi)$ or else $\mathcal{D}_\tau^{(n)}\overline{x}(\tau,\xi)$ in order to control the bilinear terms involving time derivatives. We shall require an analogue of the L^∞-bounds in Proposition 6.12 without derivatives and Fourier coefficients that are better behaved in the low frequency regime:

Lemma 6.15. *Let the functions f_j, $j = \pm 1, 0$ be as in the statement of Proposition 6.12. Then we have the bounds*

$$\big\|f_j\big\|_{L^\infty_{R\,dR}} \lesssim \big|c_j\big| + \big\|y_j\big\|_{S_1^{(j)}}, \quad S_1^{(j)} = \xi^{\frac12}S_0^{(j)}.$$

Here $S_0^{(1)} := S_0^+, S_0^{(0)} := S_0^0, S_0^{(-j)} := S_0^-$.

Proof. We consider the integral expression contributing to f_j. Thus set now

$$\begin{aligned}f_j(R) &= \phi_j(R)\cdot\int_0^R\big[\phi_j(s)\big]^{-1}\cdot\Big(\int_0^\infty \phi_j(s,\xi)y_j(\xi)\tilde{\rho}_j(\xi)\,d\xi\Big)\,ds \\ &= \phi_j(R)\cdot\int_0^R\big[\phi_j(s)\big]^{-1}\cdot\Big(\int_0^\infty \chi_{s\xi^{\frac12}\lesssim 1}\phi_j(s,\xi)y_j(\xi)\tilde{\rho}_j(\xi)\,d\xi\Big)\,ds \\ &\quad + \phi_j(R)\cdot\int_0^R\big[\phi_j(s)\big]^{-1}\cdot\Big(\int_0^\infty \chi_{s\xi^{\frac12}\gtrsim 1}\phi_j(s,\xi)y_j(\xi)\tilde{\rho}_j(\xi)\,d\xi\Big)\,ds.\end{aligned}$$

Consider for example the case $j = 1$, the others being similar. Then recall the bound

$$|\phi_1(s,\xi)| \lesssim \min\{s, \xi^{-\frac12}(s\xi^{\frac12})^{-\frac12}\},$$

and so we get the bound

$$\Big|\chi_{s\xi^{\frac12}\lesssim 1}\phi_1(s,\xi)\tilde{\rho}_1(\xi)\Big| \lesssim s^{-1}\cdot(s\xi^{\frac12})$$

We conclude that

$$\left|\phi_1(R)\cdot\int_0^R[\phi_1(s)]^{-1}\cdot\left(\int_0^\infty \chi_{s\xi^{\frac12}\lesssim 1}\phi_1(s,\xi)y_1(\xi)\tilde\rho_1(\xi)\,d\xi\right)ds\right|$$
$$\lesssim \int_0^\infty \langle\xi\rangle^{\frac12}|y_1(\xi)|\,d\xi \lesssim \|y_1\|_{S_1^{(1)}}.$$

The second integral expression is bounded similarly, by using the oscillatory nature of $\chi_{s\xi^{\frac12}\gtrsim 1}\phi_j(s,\xi)$ and performing integration by parts. □

We also have the following estimates for large frequencies:

Lemma 6.16. *Let $\lambda\geq 1$ and $f_{j,\geq\lambda}$, $j=0,\pm1$ be defined as*

$$f_{j,\geq\lambda}(R) := \phi_j(R)\cdot\int_0^R\left[\phi_j(s)\right]^{-1}\cdot\left(\int_\lambda^\infty \phi_j(s,\xi)y_j(\xi)\tilde\rho_j(\xi)\,d\xi\right)ds. \tag{6.38}$$

Then we have the following L^∞-estimate

$$\left\|f_{j,\geq\lambda}\right\|_{L^\infty_{RdR}}\lesssim \lambda^{-\frac12}\left\|y_j\right\|_{S_1^{(j)}},\ \left\|\partial_R f_{j,\geq\lambda}\right\|_{L^2_{RdR}}\lesssim\sum_{\mu\geq\lambda}\mu^{-1-\frac{\delta}{2}}\left\|y_j(\xi)\right\|_{S_1^{(j)}(\xi\simeq\mu)}.$$

and the following L^2-estimates

$$\left\|\partial_R f_{j,\geq\lambda}\right\|_{L^2_{RdR}}\lesssim\sum_{\mu\geq\lambda}\mu^{-1-\frac{\delta}{2}}\left\|y_j(\xi)\right\|_{S_1^{(j)}(\xi\simeq\mu)},\quad \textit{for}\quad j=0,1,$$

$$\left\|\partial_R f_{-1,\geq\lambda}\right\|_{L^2_{RdR}(R\lesssim\lambda^{-\frac12})}\lesssim\lambda^{-1-\frac{\delta}{2}}\|y_{-1}(\xi)\|_{S_1^{(-1)}}.$$

Proof. The proof is similar to that of Lemma 6.15 and we omit the details. □

Proposition 6.17. *Let $|n_1|\gg 1$, $|n_2|\leq|n_1|$, where n_2 is allowed to take any integer value. Also, assume that $|n_3|\geq 2$ and that either $n_3\simeq n_1$ or else $|n_3|\ll|n_1|$ as well as $n_1\simeq -n_2$. For $|n|\geq 2$, assuming that $\phi(R)$ is an angular momentum n function with*

$$\phi(\tau,R)=\int_0^\infty \phi_n(R;\xi)\overline{x}(\tau,\xi)\tilde\rho_n(\xi)\,d\xi,\quad \hbar=\frac{1}{n+1},$$

we set (recalling $S_1^\hbar=\xi^{\frac12}S_0^\hbar$)

$$\|\phi(\tau,\cdot)\|_{\tilde S_1^{(n)}} := \|\mathcal{D}_\tau\overline{x}(\tau,\xi)\|_{S_1^\hbar}.$$

For $n=0,\pm1$ assuming that $\phi(\tau,R)$ is an angular momentum n function provided

$$\phi(\tau,R)=c_n(\tau)\phi_n(R)+\phi_n(R)\cdot\int_0^R[\phi_n(s)]^{-1}\,\tilde{\mathcal{D}}\phi(\tau,s)\,ds,\quad \tilde{\mathcal{D}}\phi(\tau,R)=\int_0^\infty\overline{x}(\tau,\xi)\phi_n(R,\xi)\tilde\rho_n(\xi)\,d\xi,$$

where $\tilde{\mathcal{D}}=\mathcal{D}_-,\mathcal{D},\mathcal{D}_+$ according to n as in the preceding. Then set

$$\|\phi(\tau,\cdot)\|_{\tilde S_1^{(n)}} := \left|c_n'(\tau)\right| + \|\mathcal{D}_\tau\overline{x}(\tau,\xi)\|_{S_1^{(n)}}.$$

With the conventions and under the assumptions on the n_j stated at the beginning, and assuming $\tau \gg 1$, we have the following bound

$$\left\| \left\langle \phi_{n_3}(R;\xi), \chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_1 \cdot \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_2 \right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}} \lesssim \tau^\delta \langle n_2\rangle^2 \prod_{j=1}^{2} \left\|\phi_j\right\|_{\tilde{S}_0^{(n_j)}\cap\tilde{S}_1^{(n_j)}} \tag{6.39}$$

where we assume that the factors ϕ_j are angular momentum n_j functions, $j = 1, 2$.
Next, assume that $|n_1| \lesssim 1$, while the other assumptions on n_j stated above are still valid. Then if $\phi_{1,2}$ are angular momentum n_j functions with finite $\|\cdot\|_{\cap_{k=0,1}\tilde{S}_k^{(n_j)}}$-norm, the product

$$\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_1 \cdot \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_2$$

admits a third order Taylor expansion at $R = 0$ of the form

$$P_3(R) := \sum_{l=0}^{3} \gamma_l R^l,$$

where we have the bound

$$\sum_{l=0}^{3} |\gamma_l| \lesssim \prod_{j=1,2} \left\|\phi_j\right\|_{\tilde{S}_0^{(n_j)}\cap\tilde{S}_1^{(n_j)}} . \tag{6.40}$$

Furthermore, we have the bound

$$\left\| \left\langle \phi_{n_3}(R;\xi), \chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_1 \cdot \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_2 - \chi_{R\lesssim 1}P_3(R) \right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}} \lesssim \tau^\delta \prod_{j=1}^{2} \left\|\phi_j\right\|_{\tilde{S}_0^{(n_j)}\cap\tilde{S}_1^{(n_j)}} \tag{6.41}$$

The last inequality remains correct if we subtract from P_3 those terms $\gamma_l R^l$ with $l \geq |n_3 - 1|, l \equiv n_3 - 1$(mod 2), provided $|n_3 - 1| \leq 3$.

Proof. First inequality. Start with the case when $|n_2| \geq 2$, i.e. we can use the representation (6.37). Observe that since $\mathcal{K}_\hbar^0$ maps $S_0^\hbar$ into $S_1^\hbar$, and we assume that $\mathcal{D}_\tau\overline{x} \in S_1^\hbar$, we can treat this case exactly as in the proof of Proposition 6.10, observing that there we always deal with the product $\xi^{\frac{1}{2}}\overline{x}(\tau,\xi) \in S_1^\hbar$ for the Fourier transform. Thus to conclude the proof of (6.39), it suffices to deal with the case $|n_2| \leq 1$, where we have to take advantage of (6.34), (6.35) as well as (6.37). In the following, we omit the contributions coming directly from the resonance/root part, i.e. the first, second term on the right hand side in (6.34), since their contribution is straightforward to bound. Throughout we have $n_3 \simeq n_1$ due to the assumptions on the angular momenta for the first inequality.

Contribution of the third term on the right hand side of (6.34) *and the second term on the right hand side in* (6.35). For all intents and purposes, we may assume here that

$$\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_2(R) = \frac{\lambda_\tau}{\lambda}\phi_n(R) \cdot \int_0^R [\phi_n(s)]^{-1}\, \mathcal{D}_n\phi_2(s)\, ds, \quad n = 0, \pm 1,$$

where $\mathcal{D}_n\phi_2$ admits the representation

$$\mathcal{D}_n\phi_2(R) = \int_0^\infty \phi_n(R,\xi)\overline{x}(\xi)\tilde{\rho}_n(\xi)\, d\xi.$$

On the other hand, $\phi_1(\tau, R)$ is assumed to admit a representation as at the beginning of the proposition, corresponding to large angular momentum, and we then have (6.37). We can then essentially repeat the proof of (6.29) in Prop. 6.14 to deal with this case.

This reduces things to the case when we substitute the two terms on the right of (6.36) for $\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} s\partial_s\right)\mathcal{D}_n\phi_2$, which in turn gets substituted into (6.35). On the other hand, we still use the representation (6.37) for $\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_1$. We follow the same steps as for the proof of (6.29). As before we call ξ the output frequency, and $\xi_{1,2}$ the frequencies of the factors (more precisely, in case of $\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_2$, the frequency of $\mathcal{D}_n\phi_2$).

(1.a) $\xi \le \xi_1$. We take advantage of Lemma 6.15, which under our current hypotheses on $\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_2$ give

$$\begin{aligned}\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_2\right\|_{L^\infty_{R\,dR}} &\lesssim \left\|\mathcal{D}_\tau^{(n)}\overline{x}(\tau,\xi)\right\|_{S_1^{(n)}} + \left\|\mathcal{K}_n^{(0)}\overline{x}\right\|_{S_1^{(n)}}\\ &\lesssim \left\|\mathcal{D}_\tau^{(n)}\overline{x}(\tau,\xi)\right\|_{S_1^{(n)}} + \|\overline{x}\|_{S_0^{(n)}}\\ &\lesssim \|\phi_2\|_{\tilde{S}_0^{(n)}\cap\tilde{S}_1^{(n)}}.\end{aligned}$$

We need to bound the expression

$$\begin{aligned}&\left\|\sum_\lambda \chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi), \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_{1\,\ge\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_2\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\ &\lesssim \left(\sum_\lambda \left\|\left\langle\phi_{n_3}(R;\xi), \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_{1\,\ge\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_2\right\rangle_{L^2_{R\,dR}}\right\|^2_{S_1^{(\hbar_3)}(\xi\sim\lambda)}\right)^{\frac12}\\ &\lesssim \hbar_3^{2-\delta}\left(\sum_\lambda \lambda^{1-\delta}\langle\lambda\hbar_3^2\rangle^{3+2\delta}\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_{1\,\ge\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_2\right\|^2_{L^2_{R\,dR}}\right)^{\frac12},\end{aligned}$$

where we have exploited orthogonality and the Plancherel's theorem for the distorted Fourier transform. Recalling (6.37) we get (letting $\overline{x}_1$ the Fourier transform at angular momentum n_1 of ϕ_1)

$$\begin{aligned}\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_{1\,\ge\lambda}\right\|_{L^2_{R\,dR}} &\lesssim \sum_{\mu\ge\lambda}\left[\left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{L^2_{d\xi}} + \left\|\mathcal{K}_{\hbar_1}^{(0)}\overline{x}_1\right\|_{L^2_{d\xi}}\right]\\ &\lesssim \sum_{\mu\ge\lambda}\hbar_1^{-1}(\mu\hbar_1^2)^{-\frac12+\frac{\delta}{2}}\langle\mu\hbar_1^2\rangle^{-\frac32-\delta}\cdot\left[\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)} + \left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{S_1^{\hbar_1}(\xi\simeq\mu)}\right].\end{aligned}$$

Recalling our hypothesis that $n_1 \simeq n_3$ and using Holder's inequality and also recalling the above L^∞ bound for the second factor, we then have

$$\hbar_3^{2-\delta}\lambda^{\frac12-\frac{\delta}{2}}\langle\lambda\hbar_3^2\rangle^{\frac32+\delta}\cdot\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_{1\,\ge\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_2\right\|_{L^2_{R\,dR}}$$

$$\lesssim \|\phi_2\|_{\tilde{S}_0^{(n)}\cap\tilde{S}_1^{(n)}} \sum_{\mu\geq\lambda} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \left[\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)} + \left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{S_1^{\hbar_1}(\xi\simeq\mu)}\right].$$

Using the Cauchy-Schwarz inequality as well as orthogonality, it follows that

$$\begin{aligned}
&\hbar_3^{2-\delta}\left(\sum_\lambda \lambda^{1-\delta}\langle\lambda\hbar_3^2\rangle^{3+2\delta}\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1\geq\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_2\right\|^2_{L^2_{R\,dR}}\right)^{\frac{1}{2}}\\
&\lesssim \|\phi_2\|_{\tilde{S}_0^{(n)}\cap\tilde{S}_1^{(n)}}\left(\sum_\lambda\left[\sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\left[\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)} + \left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{S_1^{\hbar_1}(\xi\simeq\mu)}\right]\right]^2\right)^{\frac{1}{2}}\\
&\lesssim \|\phi_2\|_{\tilde{S}_0^{(n)}\cap\tilde{S}_1^{(n)}}\left(\sum_\lambda\sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\left[\|\overline{x}_1\|_{S_1^{\hbar_1}(\xi\simeq\mu)} + \left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{S_1^{\hbar_1}(\xi\simeq\mu)}\right]^2\right)^{\frac{1}{2}}\\
&\lesssim \prod_{j=1,2}\left\|\phi_j\right\|_{\tilde{S}_0^{(n_j)}\cap\tilde{S}_1^{(n_j)}}.
\end{aligned}$$

(1.b) $\xi_1 < \xi \leq \xi_2$. We distinguish between different frequency ranges for the output frequency ξ.

Small output frequency $\xi \lesssim 1$. Localizing to dyadic $\xi \simeq \lambda \lesssim 1$, consider the expression

$$\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}},$$

The subscript in $\phi_{2,\geq\lambda}$ refers to the frequency variable occurring in the representation (6.36), which in turn gets substituted into (6.35), (6.34). For this small frequency regime, this localization does not play an important role, however. We decompose

$$\begin{aligned}
&\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}}\\
&= \chi_{\xi\simeq\lambda}\left\langle\chi_{\hbar_3R\lambda^{\frac{1}{2}}\leq\frac{1}{2}}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}}\\
&+ \chi_{\xi\simeq\lambda}\left\langle\chi_{\hbar_3R\lambda^{\frac{1}{2}}\geq\frac{1}{2}}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}}
\end{aligned}\tag{6.42}$$

By Proposition 4.32, the Fourier basis $\phi_{n_3}(R,\xi)$ satisfies the bound $|\phi_{n_3}(R,\xi)| \lesssim \left(\frac{1}{2}\right)^{c\hbar_3^{-1}}$ for some absolute constant $c > 0$. This together with a slightly sharpened version of Lemma 6.9 gives the following estimate for $\lambda \lesssim 1$ and an absolute constant $c > 0$

$$\left\|\int_0^\infty \chi_{\hbar R\lambda^{\frac{1}{2}}\leq\frac{1}{2}}\phi_n(R;\xi)\chi_{\xi\lesssim\lambda}\overline{x}(\xi)\tilde{\rho}_n(\xi)\,d\xi\right\|_{L^\infty_{R\,dR}} \lesssim \hbar^{-2+\delta}\lambda^{\frac{\delta}{2}}\cdot\left(\frac{1}{2}\right)^{c\hbar^{-1}}\cdot\|\overline{x}\|_{S_1^{\hbar}}.$$

Since $\hbar_1 \simeq \hbar_3$, we conclude, using Plancherel's theorem for the distorted Fourier transform as well as Holder's inequality that

$$\begin{aligned}
&\left\|\chi_{\xi\simeq\lambda}\left\langle\chi_{\hbar_3 R\lambda^{\frac{1}{2}}\le\frac{1}{2}}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\ge\lambda}\right\rangle_{L^2_{RdR}}\right\|_{S_1^{\hbar_3}}\\
&\lesssim \hbar_3^{2-\delta}\lambda^{\frac{1}{2}-\frac{\delta}{2}}\cdot\left\|\chi_{R\lesssim\min\{\hbar_3^{-1}\lambda^{-\frac{1}{2}},\tau\}}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\ge\lambda}\right\|_{L^2_{RdR}}\\
&\lesssim \hbar_3^{2-\delta}\lambda^{\frac{1}{2}-\frac{\delta}{2}}\cdot\hbar_3^{-1+\delta}\lambda^{-\frac{1}{2}+\frac{\delta}{2}}\tau^{\delta}\cdot\left\|\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\right\|_{L^\infty_{RdR}}\cdot\left\|\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\ge\lambda}\right\|_{L^\infty_{RdR}}.
\end{aligned}$$

Taking advantage of the preceding sharpened L^∞-bound, the fact that $\hbar_1\sim\hbar_3$, as well as Lemma 6.15, and also keeping in mind the representation (6.37), we can bound the preceding by

$$\begin{aligned}
&\hbar_3^{2-\delta}\lambda^{\frac{1}{2}-\frac{\delta}{2}}\cdot\lambda^{-\frac{1}{2}+\frac{\delta}{2}}\hbar_3^{-1+\delta}\tau^{\delta}\cdot\left\|\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\right\|_{L^\infty_{RdR}}\cdot\left\|\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\ge\lambda}\right\|_{L^\infty_{RdR}}\\
&\lesssim \tau^{\delta}\cdot\lambda^{\frac{\delta}{2}}\cdot\|\phi_1\|_{\tilde S_1^{(n_1)}\cap\tilde S_0^{(n_1)}}\cdot\|\phi_2\|_{\tilde S_1^{(n_2)}\cap\tilde S_0^{(n_2)}}.
\end{aligned}$$

This can then be summed over dyadic $\lambda\lesssim 1$ to yield the desired bound.
As for the second term on the right in (6.42), we perform integration by parts in the inner product, replacing it schematically by

$$\lambda^{-\frac{1}{2}}\chi_{\xi\simeq\lambda}\left\langle\chi_{\hbar_3 R\lambda^{\frac{1}{2}}\ge\frac{1}{2}}\phi_{n_3}(R;\xi),\,\partial_R\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\ge\lambda}\right]\right\rangle_{L^2_{RdR}}$$

To bound this, we recall our assumptions on the factors $\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_j$, $j=1,2$, and use the following L^2-bounds

$$\begin{aligned}
\left\|\partial_R\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\right\|_{L^2_{RdR}} &\lesssim \left\|\xi^{\frac{1}{2}}\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{L^2_{d\xi}(\xi<\lambda)}+\left\|\xi^{\frac{1}{2}}\mathcal{K}_{\hbar_1}^{(0)}\overline{x}_1\right\|_{L^2_{d\xi}(\xi<\lambda)}\\
&\lesssim \hbar_1^{-2+\delta}\lambda^{\frac{\delta}{2}}\,\|\phi_1\|_{\tilde S_1^{(\hbar_1)}\cap\tilde S_0^{(\hbar_1)}}\\
\left\|\partial_R\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\ge\lambda}\right\|_{L^2_{RdR}(R\lesssim\tau)} &\lesssim \|\phi_2\|_{\tilde S_1^{(n_2)}\cap\tilde S_0^{(n_2)}}.
\end{aligned}$$

In fact, to get the last bound for $n=0,\pm1$, write

$$\begin{aligned}
\partial_R\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\ge\lambda} &= \partial_R\phi_n(R)\cdot\int_0^R[\phi_n(s)]^{-1}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}s\partial_s\right)\mathcal{D}_n\phi_{2,\ge\lambda}(\tau,s)\,ds\\
&\quad+\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\mathcal{D}_n\phi_{2,\ge\lambda}(\tau,R)
\end{aligned}$$

and also recall the Fourier representation (6.36), which in light of the Plancherel's theorem for the distorted Fourier transform implies the desired bound for the second term on the right. As for the first term on the right, one can argue similarly to the proof of Lemma 6.15 to get the desired bound. If we add the easily verified bounds (see Lemma 6.15)

$$\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_{2,\geq\lambda}\right\|_{L^\infty_{R\,dR}} \lesssim \|\phi_2\|_{\tilde{S}_1^{(n_2)}\cap\tilde{S}_0^{(n_2)}},$$
$$\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_{1,<\lambda}\right\|_{L^\infty_{R\,dR}} \lesssim \hbar_1^{-2+\delta}\lambda^{\frac{\delta}{2}}\|\phi_1\|_{\tilde{S}_1^{(n_1)}\cap\tilde{S}_0^{(n_1)}}$$

and using Holder's inequality and the Leibniz product rule, we infer that (recall the restriction $\lambda \lesssim 1$)

$$\begin{aligned}
&\left\|\chi_{\xi\simeq\lambda}\left\langle \chi_{\hbar_3 R\lambda^{\frac12}\geq\frac12}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\
&\lesssim \hbar_3^{2-\delta}\lambda^{\frac12-\frac{\delta}{2}}\cdot\left\|\chi_{\xi\simeq\lambda}\left\langle \chi_{\hbar_3 R\lambda^{\frac12}\geq\frac12}\phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}}\right\|_{L^2_{d\xi}}\\
&\lesssim \hbar_3^{2-\delta}\tau^\delta\cdot\left\|\partial_R\left[\chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right]\right\|_{L^2_{R\,dR}}\\
&\lesssim \tau^\delta\lambda^{\frac{\delta}{2}}\cdot\prod_{j=1,2}\left\|\phi_j\right\|_{\tilde{S}_0^{(n_j)}\cap\tilde{S}_1^{(n_j)}},
\end{aligned}$$

which can also be summed over dyadic $\lambda \lesssim 1$, giving the desired bound. This concludes the case *(1.b)* in the small output regime.

Intermediate output frequency $\hbar_3^{-2} \gtrsim \xi \gtrsim 1$. This case can be handled similarly as the preceding case since the weight in the norm $\|\cdot\|_{S_1^{\hbar_3}}$ is the same as in the small frequency regime. However, since now the frequency $\lambda \geq 1$, we must proceed differently when we sum over λ. In the non-oscillatory regime $\hbar_3 R\lambda^{\frac12} \leq \frac12$ we use the bound

$$\left\|\chi_{\hbar_1 R\lambda^{\frac12}\leq\frac12}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,\leq\lambda}\right\|_{L^\infty_{RdR}} \lesssim \hbar_1^{-2+\delta}\lambda^{\frac{\delta}{2}}\cdot\left(\frac12\right)^{c\hbar_1^{-1}}\|\phi_1\|_{\tilde{S}_1^{(n_1)}\cap\tilde{S}_0^{(n_1)}}. \tag{6.43}$$

The rapid decaying factor $\left(\frac12\right)^{c\hbar_1^{-1}}$ absorbs the growth from summing over $1 \leq \lambda \lesssim \hbar_3^{-2}$, using the fact $\hbar_1 \simeq \hbar_3$. For the oscillatory regime $\hbar_3 R\lambda^{\frac12} \geq \frac12$, in addition to the rapid decaying estimate (6.43), we also need to use the following refined estimate for $\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}$:

$$\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\|_{L^\infty_{RdR}} \lesssim \lambda^{-\frac12}\|\phi_2\|_{\tilde{S}_1^{(n_2)}\cap\tilde{S}_0^{(n_2)}},$$

which is obtained by Lemma 6.16. The decaying factor $\lambda^{-\frac12}$ absorbs the growth from summing over $1 \leq \lambda \lesssim \hbar_3^{-2}$.

Large output frequency $\xi \gtrsim \hbar_3^{-2}$. This is accomplished by integration by parts: schematically we have

$$\begin{aligned}
&\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}}\\
&=\xi^{-1}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,H_{n_3}\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\cdot\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right]\right\rangle_{L^2_{R\,dR}}\\
&=\xi^{-1}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,H_{n_3}\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\right]\cdot\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}}\\
&+\sum_{i+j=2,i\leq1}\xi^{-1}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\left(\partial_R+\frac{1}{R}\right)^i\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\right]\cdot\partial_R^j\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}}
\end{aligned}\tag{6.44}$$

Taking advantage of Proposition 6.8 as well as Lemma 6.15, we can bound the first term on the right:

$$\begin{aligned}
&\left\|\xi^{-1}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,H_{n_3}\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\right]\cdot\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\
&\lesssim\hbar_3^{2-\delta}\lambda^{-\frac12-\frac{\delta}{2}}\left\langle\lambda\hbar_3^2\right\rangle^{\delta+\frac32}\cdot\left\|H_{n_3}\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\right]\right\|_{L^2_{R\,dR}}\cdot\left\|\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\|_{L^\infty_{R\,dR}}\\
&\lesssim\lambda^{-\frac12}\prod_{j=1,2}\left\|\tilde\phi_j\right\|_{\tilde S_0^{(n_j)}\cap\tilde S_1^{(n_j)}},
\end{aligned}$$

where we have taken advantage of the bounds (with $\overline{x}_j$ denoting the Fourier transforms as explained in Prop. 6.14)

$$\begin{aligned}
\hbar_3^{2-\delta}\lambda^{-\frac12-\frac{\delta}{2}}\left\langle\lambda\hbar_3^2\right\rangle^{\delta+\frac32}\cdot\left\|H_{n_3}\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\right]\right\|_{L^2_{R\,dR}}&\lesssim\left[\left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{S_1^{\hbar_1}}+\left\|\mathcal{K}_{\hbar_1}^{(0)}\overline{x}_1\right\|_{S_1^{\hbar_1}}\right]\\
&\lesssim\left\|\tilde\phi_1\right\|_{\tilde S_0^{(n_1)}\cap\tilde S_1^{(n_1)}}
\end{aligned}$$

$$\left\|\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\|_{L^\infty_{R\,dR}}\lesssim\lambda^{-\frac12}\cdot\left[\left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_2\right\|_{S_1^{(n_1)}}+\left\|\mathcal{K}_{(n_1)}^{(0)}\overline{x}_1\right\|_{S_1^{(n_1)}}\right]\lesssim\lambda^{-\frac12}\cdot\left\|\tilde\phi_2\right\|_{\tilde S_0^{(n_2}\cap\tilde S_1^{(n_2)}}.$$

In particular, summing over dyadic $\lambda \gtrsim \hbar_3^{-2}$ furnishes a bound for the first of the last two terms in (6.44) of the desired form.
As for the final term in (6.44), we get (under our current hypothesis on $\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_2$) the schematic (only keeping the term $\mathcal{D}_\tau^{(n_2)}\overline{x}_2$) decompositions

$$\begin{aligned}
\partial_R\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}&=\partial_R\phi_{n_2}(R)\cdot\left(\int_0^R\left[\phi_{n_2}(s)\right]^{-1}\int_0^\infty\chi_{\xi\geq\lambda}\phi_{n_2}(s,\xi)\mathcal{D}_\tau^{(n_2)}\overline{x}_2(\tau,\xi)\tilde\rho_{n_2}(\xi)\,d\xi\right)ds\\
&\quad+\int_0^\infty\chi_{\xi\geq\lambda}\phi_{n_2}(R,\xi)\mathcal{D}_\tau^{(n_2)}\overline{x}_2(\tau,\xi)\tilde\rho_{n_2}(\xi)\,d\xi,
\end{aligned}$$

$$\partial_R^2\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}=\partial_R^2\phi_{n_2}(R)\cdot\left(\int_0^R\left[\phi_{n_2}(s)\right]^{-1}\int_0^\infty\chi_{\xi\geq\lambda}\phi_{n_2}(s,\xi)\mathcal{D}_\tau^{(n_2)}\overline{x}_2(\tau,\xi)\tilde\rho_{n_2}(\xi)\,d\xi\right)ds$$

$$+\,\partial_R\phi_{n_2}(R)\cdot[\phi_{n_2}(R)]^{-1}\cdot\left(\int_0^\infty \chi_{\xi\geq\lambda}\phi_{n_2}(R,\xi)\mathcal{D}_\tau^{(n_2)}\overline{x}_2(\tau,\xi)\tilde{\rho}_{n_2}(\xi)\,d\xi\right)$$
$$+\,\partial_R\left(\int_0^\infty \chi_{\xi\geq\lambda}\phi_{n_2}(R,\xi)\mathcal{D}_\tau^{(n_2)}\overline{x}_2(\tau,\xi)\tilde{\rho}_{n_2}(\xi)\,d\xi\right).$$

For $j=0,1$ we directly use Lemma 6.16 to obtain for $\lambda>1$

$$\left\|\partial_R\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,\geq\lambda}\right\|_{L^2_{R\,dR}}\lesssim\sum_{\mu\geq\lambda}\mu^{-1-\frac{\delta}{2}}\left[\left\|\mathcal{D}_\tau^{(n_2)}\overline{x}_2(\tau,\xi)\right\|_{S_1^{(n_2)}(\xi\simeq\mu)}+\left\|\mathcal{K}_{n_2}^{(0)}\overline{x}_2\right\|_{S_1^{(n_2)}(\xi\simeq\mu)}\right].$$

For $j=-1$, Lemma 6.16 gives the desired estimate for $R\lesssim\lambda^{-\frac{1}{2}}$. For $R\lambda^{\frac{1}{2}}\gtrsim1$ we will treat differently at the end.

Furthermore, for the high-angular momentum term, in light of Lemma 6.9 we have the L^∞-bound (for $i\leq1$)

$$\left\|\left(\partial_R+\frac{1}{R}\right)^i\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1<\lambda}\right]\right\|_{L^\infty_{R\,dR}}\lesssim\hbar_1^{-\frac{5}{2}-\delta}\cdot\left[\left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{S_1^{\hbar_1}}+\left\|\mathcal{K}_{\hbar_1}^{(0)}\overline{x}_1\right\|_{S_1^{\hbar_1}}\right]$$

Combining these bounds leads to the estimate (except for $j=-1$ with $R\lambda^{\frac{1}{2}}\gtrsim1$)

$$\begin{aligned}
&\|(\text{last term of }(6.44))\|_{S_1^{\hbar_3}(\xi\gtrsim\hbar_3^{-2})}\\
&\lesssim\hbar_3^{5+\frac{\delta}{2}}\cdot\left(\sum_{\lambda\gtrsim\hbar_3^{-2}}\lambda^{-2}\lambda^{2+\delta}\,\|(\text{last term of }(6.44))\|^2_{L^2_{d\xi}}\right)^{\frac{1}{2}}\\
&\lesssim\hbar_1\cdot\left[\left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{S_1^{(\hbar_1)}}+\left\|\mathcal{K}_{\hbar_1}^{(0)}\overline{x}_1\right\|_{S_1^{\hbar_1}}\right]\\
&\qquad\cdot\left(\sum_{\lambda\gtrsim\hbar_3^{-2}}\lambda^{-2}\left(\sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{1+\frac{\delta}{2}}\left[\left\|\mathcal{D}_\tau^{(n_2)}\overline{x}_2(\tau,\xi)\right\|_{S_1^{(n_2)}(\xi\simeq\mu)}+\left\|\mathcal{K}_{n_2}^{(0)}\overline{x}_2\right\|_{S_1^{(n_2)}(\xi\simeq\mu)}\right]\right)^2\right)^{\frac{1}{2}}
\end{aligned}$$

For $j=-1$ and $R\lambda^{\frac{1}{2}}\gtrsim1$, we simply write $\lambda^{-2}=\lambda^{-1}\cdot\lambda^{-1}$ and use one of the two λ^{-1} to make up the discrepancy of the decay in μ. Applying the Cauchy-Schwarz inequality and exploiting orthogonality allows to bound the preceding by

$$\lesssim\hbar_1\cdot\prod_{j=1,2}\left\|\phi_j\right\|_{\tilde{S}_1^{(n_j)}\cap\tilde{S}_0^{(n_j)}}$$

This concludes the case of large output frequencies for the case *(1.b)*.

(2): $\xi\geq\max\{\xi_1,\xi_2\}$. Output frequency dominates both input frequencies. We proceed in analogy to case *(2)* in the proof Proposition 6.10. The case of output frequency $\xi\lesssim1$ here is handled exactly like the small output frequency case in *(1.b)* before. We shall henceforth restrict to output frequency $\xi\gtrsim1$. We shall again exploit multi-fold integration by parts in order to shift derivatives around. Precisely, we write, always

keeping the assumed underlying fine structure of $\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\phi_{2,<\lambda}$ in mind

$$
\begin{aligned}
&\chi_{\xi\simeq\lambda}\left\langle \phi_{n_3}(R;\xi),\, \chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,<\lambda}\right\rangle_{L^2_{R\,dR}} \\
&= \xi^{-3}\chi_{\xi\sim\lambda}\left\langle \phi(R;\xi,\hbar_3),\, H^3_{n_3}\left[\chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,<\lambda}\right]\right\rangle_{L^2_{R\,dR}} \\
&= \sum_{i+j=6} C_{i,j}\xi^{-3}\chi_{\xi\simeq\lambda}\left\langle \phi_{n_3}(R;\xi),\, \partial_R^i\left(\frac{n_3}{R}\right)^j\left[\chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\right]\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,<\lambda}\right\rangle_{L^2_{R\,dR}} \\
&+ \sum_{\substack{l+i+j=6\\ l\geq 1}} \xi^{-3}\chi_{\xi\simeq\lambda}\left\langle \phi_{n_3}(R;\xi),\, \partial_R^i\left(\frac{n_3}{R}\right)^j\left[\chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\right]\cdot\partial_R^l\left(\phi_n(R)\right)\cdot\int_0^R\left[\phi_n(s)\right]^{-1}f_{<\lambda}(s)\,ds\right\rangle_{L^2_{R\,dR}} \\
&+ \sum_{\substack{l+i+j+k=6\\ l\geq 1}} \xi^{-3}\chi_{\xi\simeq\lambda}\left\langle \phi_{n_3}(R;\xi),\, \partial_R^i\left(\frac{n_3}{R}\right)^j\left[\chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\right]\cdot\partial_R^{l-1}\left[\partial_R^k\left(\phi_n(R)\right)\left[\phi_n(R)\right]^{-1}\cdot f_{<\lambda}(R)\right]\right\rangle_{L^2_{R\,dR}},
\end{aligned}
\tag{6.45}
$$

where we have introduced the quantity

$$
f_{<\lambda}(R) := \int_0^\infty \phi_n(R,\xi)\left(\mathcal{D}_\tau^{(n_2)}\overline{x}_2(\tau,\xi) + \mathcal{K}^{(0)}_{\hbar_2}\overline{x}_2(\tau,\xi)\right)\tilde{\rho}_{n_2}(\xi)\,d\xi.
$$

Then we estimate the last three terms as follows: for the first term at the end in (6.45), we use Lemma 6.15 as well Proposition 6.8 to bound the first, respectively the second term below

$$
\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,<\lambda}\right\|_{L^\infty_{R\,dR}},\ \partial_R^i\left(\frac{n_3}{R}\right)^j\left[\chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\right].
$$

Invoking Plancherel's theorem for the distorted Fourier transform, this leads to the bound

$$
\begin{aligned}
&\left\|\xi^{-3}\chi_{\xi\simeq\lambda}\left\langle \phi_{n_3}(R;\xi),\, \partial_R^i\left(\frac{n_3}{R}\right)^j\left[\chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\right]\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,<\lambda}\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}} \\
&\lesssim \frac{\hbar_3^{2-\delta}\lambda^{\frac12-\frac{\delta}{2}}\langle\lambda\hbar_3^2\rangle^{\delta+\frac32}}{\lambda^3}\cdot\left\|\partial_R^i\left(\frac{n_3}{R}\right)^j\left[\chi_{R\lesssim\tau}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\right]\right\|_{L^2_{R\,dR}}\cdot\left\|\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{2,<\lambda}\right\|_{L^\infty_{R\,dR}} \\
&\lesssim \frac{\hbar_3^{2-\delta}\lambda^{\frac12-\frac{\delta}{2}}\left\langle\lambda\hbar_3^2\right\rangle^{\delta+\frac32}}{\lambda^3}\cdot\left(\sum_{\mu<\lambda}\left\|\xi^3\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{L^2_{d\xi}(\xi\simeq\mu)} + \left\|\xi^3\mathcal{K}^{(0)}_{\hbar_1}\overline{x}_1\right\|_{L^2_{d\xi}(\xi\simeq\mu)}\right)\cdot\|\phi_2\|_{\tilde{S}_0^{(n_2)}\cap\tilde{S}_1^{(n_2)}}.
\end{aligned}
$$

Here we can bound the product of the first two expressions by (recalling the hypothesis $\hbar_1 \simeq \hbar_3$)

$$
\begin{aligned}
&\frac{\hbar_3^{2-\delta}\lambda^{\frac12-\frac{\delta}{2}}\langle\lambda\hbar_3^2\rangle^{\delta+\frac32}}{\lambda^3}\cdot\left(\sum_{\mu<\lambda}\left\|\xi^3\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{L^2_{d\xi}(\xi\simeq\mu)} + \left\|\xi^3\mathcal{K}^{(0)}_{\hbar_1}\overline{x}_1\right\|_{L^2_{d\xi}(\xi\simeq\mu)}\right) \\
&\lesssim \sum_{\mu<\lambda}\left(\frac{\mu}{\lambda}\right)^{\frac12-\frac{\delta}{2}}\cdot\left[\left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{S_1^{\hbar_1}(\xi\simeq\mu)} + \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\sim\mu)}\right].
\end{aligned}
\tag{6.46}
$$

Substituting this bound into the preceding one, we easily infer that

$$\begin{aligned}&\Big(\sum_{\lambda\gtrsim 1}\|(\text{First term of }(6.45))\|_{S_1^{\hbar_3}}^2\Big)^{\frac{1}{2}}\\&\lesssim \|\phi_2\|_{\tilde{S}_0^{(n_2)}\cap\tilde{S}_1^{(n_2)}}\cdot\Bigg(\sum_{\lambda\gtrsim 1}\Bigg[\sum_{\mu<\lambda}\Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\Big[\Big\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\Big\|_{S_1^{\hbar_1}(\xi\simeq\mu)}+\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)}\Big]^2\Bigg]\Bigg)^{\frac{1}{2}}\\&\lesssim\prod_{j=1,2}\big\|\phi_j\big\|_{\tilde{S}_1^{(n_j)}\cap\tilde{S}_0^{(n_j)}},\end{aligned}$$

which is as desired, and concludes the bound for the first term in (6.45). For the second term, Lemma 6.15 implies

$$\Big\|\partial_R^l(\phi_{n_2}(R))\cdot\int_0^R[\phi_{n_2}(s)]^{-1}f_{<\lambda}(s)\,ds\Big\|_{L^\infty_{RdR}}\lesssim\Big\|\mathcal{D}_\tau^{(n_2)}\overline{x}_1\Big\|_{S_1^{(n_2)}}+\Big\|\mathcal{K}_{\hbar_2}^{(0)}\overline{x}_1\Big\|_{S_1^{(n_2)}},\quad l\geq 0,$$

As in the previous case, with $i+j\leq 6$, we have

$$\frac{\hbar_3^{2-\delta}\lambda^{\frac{1}{2}-\frac{\delta}{2}}\langle\lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}}}{\lambda^3}\Big\|\partial_R^i\Big(\frac{n_3}{R}\Big)^j\Big[\chi_{R\lesssim\tau}(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)\phi_{1,<\lambda}\Big]\Big\|_{L^2_{RdR}}$$

can be bounded by (6.46). Applying Plancherel's theorem as well as Holder's inequality suitably we then infer

$$\begin{aligned}&\Big(\sum_{\lambda\gtrsim 1}\|(\text{Second term of }(6.45))\|_{S_1^{\hbar_3}}^2\Big)^{\frac{1}{2}}\\&\lesssim \|\phi_2\|_{\tilde{S}_0^{(n_2)}\cap\tilde{S}_1^{(n_2)}}\cdot\Bigg(\sum_{\lambda\gtrsim 1}\Bigg[\sum_{\mu<\lambda}\Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\Big[\Big\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\Big\|_{S_1^{\hbar_1}(\xi\simeq\mu)}+\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)}\Big]^2\Bigg]\Bigg)^{\frac{1}{2}},\end{aligned}$$

which implies the desired bound.
The final term in (6.45) is bounded by applying Leibniz' rule to the last term. Considering the case $n=-1$, the cases $n=+1,0$ being handled similarly, we write

$$\begin{aligned}&\partial_R^{l-1}\Big[\partial_R^k(\phi_n(R))[\phi_n(R)]^{-1}\cdot f_{<\lambda}(R)\Big]\\&=\sum_{l_1+l_2+l_3=l-1}C_{l_{1,2,3}}\partial_R^{k+l_1}(\phi_n(R))\cdot\partial_R^{l_2}\Big([\phi_n(R)]^{-1}\Big)\partial_R^{l_3}(f_{<\lambda}(R))\\&=\sum_{l_1+l_2+l_3=l-1}g_{k,l_{1,2}}(R)\cdot\partial_R^{l_3}(f_{<\lambda}(R)),\end{aligned}$$

where the function $g_{k,l_{1,2}}(R)$ satisfies the bound

$$\big|g_{k,l_{1,2}}(R)\big|\lesssim\max\{R^{-2-l_2}\cdot R^{\max\{2-k-l_1,0\}},\langle R\rangle^{-(k+l_1+l_2)}\},$$

and so we get the bound

$$\left|\partial_R^i\left(\frac{n_3}{R}\right)^j\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\right]\cdot\partial_R^{l-1}\left[\partial_R^k\left(\phi_n(R)\right)[\phi_n(R)]^{-1}\cdot f_{<\lambda}(R)\right]\right|$$
$$\lesssim\sum_{l\leq 5-i-j-k}\left|R^{-(5-i-j-l)}\partial_R^i\left(\frac{n_3}{R}\right)^j\left[\chi_{R\lesssim\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\phi_{1,<\lambda}\right]\right|\cdot\left|\partial_R^l\left(f_{<\lambda}(R)\right)\right|.$$

Taking advantage of Prop. 6.8, Lemma 6.9 as well as Prop. 6.12 and Lemma 6.9 and placing the factors

$$R^{-(5-i-j-l)}\partial_R^i\left(\frac{n_3}{R}\right)^j[\dots],\quad \partial_R^l\left(f_{<\lambda}(R)\right)$$

into $L^\infty_{R\,dR}, L^2_{R\,dR}$ or the other way round according to whether $l\geq 3$, $l<3$, we infer the bound

$$\left\|\left\langle\xi^{-3}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\partial_R^i\left(\frac{n_3}{R}\right)^j\left[\chi_{R\lesssim\tau}(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R)\phi_{1,<\lambda}\right]\cdot\partial_R^{l-1}\left[\partial_R^k\left(\phi_n(R)\right)[\phi_n(R)]^{-1}\cdot f_{<\lambda}(R)\right]\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}$$
$$\lesssim\sum_{\mu_{1,2}<\lambda}\prod_{j=1,2}\min\{\mu_j,1\}^{\frac{\delta}{2}}\cdot\left(\frac{\max\{\mu_{1,2}\}}{\lambda}\right)\cdot\left[\left\|\mathcal{D}_\tau^{(\hbar_1)}\overline{x}_1\right\|_{S_1^{\hbar_1}(\xi\simeq\mu_1)}+\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu_1)}\right]$$
$$\cdot\left(\left\|\mathcal{D}_\tau^{(n_2)}\overline{x}_2\right\|_{S_1^{(n_2)}(\xi\simeq\mu_2)}+\left\|\mathcal{K}_{\hbar_2}^{(0)}\overline{x}_2\right\|_{S_1^{(n_2)}(\xi\simeq\mu_2)}\right)$$

The desired bound results from here in the usual fashion by square summing over λ and exploiting Cauchy-Schwarz as well as orthogonality. This concludes the proof of (6.39).
The proof of the remaining inequalities (6.40), (6.41) proceeds in analogy to the proof of (6.30), (6.31). This concludes the proof of Proposition 6.17. □

In order to deal with the higher order nonlinear source terms we need to pass from the above basic estimates to estimates for higher order terms. For this we have the following

Proposition 6.18. *Assume that $F(R)$ is a function on $[0,\infty)$ admitting a third order Taylor development $\sum_{j=0}^3\gamma_jR^j$ at $R=0$ and such that*

$$\left\|F(R)-\chi_{R\lesssim 1}\sum_{j=0}^3\gamma_jR^j\right\|_{\tilde{S}_1^{(n_1)}}+\sum_j\left|\gamma_j\right|=:\Lambda_1<\infty$$

for some $|n_1|\geq 2$, and where $\|\cdot\|_{\tilde{S}_1^{(n_1)}}$ is defined like $\|\cdot\|_{\tilde{S}_0^{(n_1)}}$ in Proposition 6.14, except that $\|\cdot\|_{S_0^{\hbar}}$ there is replaced by $\|\cdot\|_{S_1^{\hbar}}$. Also, assume that all $\gamma_j=0$ provided $|n_1|\geq K$. Next assume that ϕ_2 is an angular momentum n_2 function (for arbitrary integral n_2) with

$$\|\phi_2\|_{\tilde{S}_0^{(n_2)}}=:\Lambda_2<\infty.$$

Then the function $F\phi_2$ admits a Taylor expansion $P_3=\sum_{j=0}^3\tilde{\gamma}_jR^j$ of order three at $R=0$ with

$$\sum_j\left|\tilde{\gamma}_j\right|\lesssim\Lambda_1\cdot\Lambda_2,$$

and such that, with $|n_3| \geq 2$ and either (i) $n_1 \simeq n_3$ and $|n_2| \lesssim |n_1|$, or (ii) $|n_3| \ll |n_1|$ and $n_1 \simeq -n_2$, or (iii) $|n_3| \gg |n_1|$ and $n_3 \simeq n_2$, we have

$$\left\|\chi_{R\lesssim\tau}\left[F(R)\phi_2(R) - \chi_{R\lesssim 1}\sum_{j=0}^{3}\tilde{\gamma}_j R^j\right]\right\|_{S_1^{\hbar_3}} \lesssim \langle\chi_{|n_2|<K}|n_2|\rangle^{3+\delta}\langle\min\{n_1, n_2\}\rangle^2\tau^{1+\delta}\prod_{k=1,2}\Lambda_k,\ \tau \gg 1. \tag{6.47}$$

The last inequality remains correct if we subtract from P_3 those terms $\tilde{\gamma}_l R^l$ with $l \geq |n_3 - 1|, l \equiv n_3 - 1(\text{mod } 2)$, provided $|n_3 - 1| \leq 3$. Note the particular choice $n_3 = n_3(n_1, n_2)$ where $n_3(n_1, n_2) = n_1 + n_2$ if $|n_1 + n_2| \geq 2$ and $n_3(n_1, n_2) = 2$ otherwise, satisfies (i) - (iii).

Proof. This is in fact completely analogous to the one of Prop. 6.14, observing that we implicitly exploited there that $\partial_R\phi_j \in \tilde{S}_1^{(n_j)}$. Then if $|n_2| \gg 1$, we shall set $\tilde{\gamma}_j = 0$ for all j and we decompose

$$\begin{aligned}\chi_{R\lesssim\tau}F(R)\phi_2(R) &= \left(\chi_{R\lesssim\tau}F(R) - \chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_2(R)\\ &+ \left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_2(R).\end{aligned}$$

Here the first term on the right is bounded precisely like in the preceding argument, taking advantage of the last bound of Lemma 6.9(which causes the loss of a factor τ in the estimate). For the second term, one expands out

$$\phi_2(R) = \int_0^\infty \phi_{n_2}(R;\xi_2)\overline{x}_2(\xi_2)\tilde{\rho}_{n_2}(\xi_2)\,d\xi_2$$

and divides into two cases depending on the relation of the output frequency ξ to ξ_2. Thus write

$$\begin{aligned}&\left\langle\phi_{n_3}(R;\xi), \left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_2(R)\right\rangle_{L^2_{R\,dR}}\\ &= \sum_\lambda \chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi), \left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_{2,\ll\lambda}(R)\right\rangle_{L^2_{R\,dR}}\\ &+ \sum_\lambda \chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi), \left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_{2,\gtrsim\lambda}(R)\right\rangle_{L^2_{R\,dR}}\end{aligned} \tag{6.48}$$

To deal with the second term on the right, we further localize ϕ_2 to frequency $\mu_2 \gtrsim \lambda$, which then gives for fixed λ

$$\left\|\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi), \left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_{2,\mu_2}(R)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}$$

$$\lesssim \lambda^{-\frac{1}{2}}(\lambda\hbar_3^2)^{1-\frac{\delta}{2}}\langle\lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}} \cdot \left\|\left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)R^{-1}\phi_{2,\mu_2}(R)\right\|_{L^2_{R\,dR}},$$

where we have inserted the extra factor R^{-1} on purpose, since Proposition 6.8 gives

$$\left\|\left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)R^{-1}\phi_{2,\mu_2}(R)\right\|_{L^2_{R\,dR}} \lesssim \left\|\xi_2^{\frac{1}{2}}\overline{x}_2(\xi_2)\right\|_{L^2(\xi_2\simeq\mu_2)},$$

and so we infer

$$\begin{aligned}
&\left\|\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi), \left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_{2,\mu_2}(R)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\
&\lesssim \hbar_3\mu_2^{\frac{1}{2}}\cdot\left(\lambda\hbar_3^2\right)^{\frac{1}{2}-\frac{\delta}{2}}\left\langle\lambda\hbar_3^2\right\rangle^{\delta+\frac{3}{2}}\cdot\|\overline{x}_2(\xi_2)\|_{L^2(\xi_2\simeq\mu_2)}\\
&\lesssim [\chi_{|n_2|<K}|n_2|]^{3+\delta}\min\{n_1,n_2\}^2\left(\frac{\lambda}{\mu_2}\right)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}(\xi_2\simeq\mu_2)},
\end{aligned}$$

where we exploited our assumption that if $|n_2| \gg |n_3|$, then $K > |n_1| \simeq |n_2|$, since if $|n_1| \geq K$ the coefficients γ_j all vanish . Square summing over λ and exploiting Cauchy-Schwarz and orthogonality as usual leads to the desired bound. Bounding the first term in (6.48) involves integrating by parts in a manner similar to the one in the proof of the previous proposition, we omit the details.
There remains the case when all involved angular momenta n_j are of small size, $|n_j| \lesssim 1,\ j = 1, 2, 3$. There we use a slightly refined decomposition

$$\begin{aligned}
\chi_{R\lesssim\tau}F(R)\phi_2(R) &= \left(\chi_{R\lesssim\tau}F(R) - \chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_2(0)\\
&\quad+\left(\chi_{R\lesssim\tau}F(R) - \chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)[\phi_2(R) - \phi_2(0)]\\
&\quad+\left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_2(R).
\end{aligned}$$

Then proceeding as in the proof of the preceding proposition one shows that the first two terms on the right satisfy the desired bound

$$\left\|\left(\chi_{R\lesssim\tau}F(R) - \chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_2(0)\right\|_{S_1^{\hbar_3}} \lesssim \cdot\prod_{k=1,2}\Lambda_k$$

$$\left\|\left(\chi_{R\lesssim\tau}F(R)-\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)[\phi_2(R)-\phi_2(0)]\right\|_{S_1^{\hbar_3}}\lesssim \tau^{1+\frac{\delta}{2}}\cdot\prod_{k=1,2}\Lambda_k.$$

Finally, as for the remaining term, it is straightforward to check that it admits a degree Taylor expansion around $R=0$:

$$\sum_{j=1}^{3}\tilde{\gamma}_j R^j,\quad \sum_j|\tilde{\gamma}_j|\lesssim\prod_{k=1,2}\Lambda_k,$$

and such that

$$\left\|\left(\chi_{R\lesssim 1}\sum_{j\leq 3}\gamma_j R^j\right)\phi_2(R)-\chi_{R\lesssim 1}\sum_{j=1}^{3}\tilde{\gamma}_j R^j\right\|_{S_1^{\hbar_3}}\lesssim\prod_{k=1,2}\Lambda_k.$$

□

A consequence of the preceding proposition is the following multilinear estimate.

Proposition 6.19. *For an integer n define $n_{|\cdot|\geq 2}:=n$ if $|n|\geq 2$ and $n_{|\cdot|\geq 2}:=2$ otherwise. Assume that $\phi_1,\phi_2,\dots,\phi_k$ are angular momentum m_j functions, $j=1,2,\dots,k$, $m_j\in\mathbb{Z}$, where we make the same structural assumption on each of them as in Proposition 6.18. Further, assume that F is a angular momentum n_0 function with $|n_0|\geq 2$, admitting a third order Taylor development $\sum_{j=0}^{3}\gamma_j R^j$ at $R=0$ and such that*

$$\left\|F(R)-\chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j\right\|_{\tilde{S}_1^{(n_0)}}+\sum_j\left|\gamma_j\right|=:\Lambda_0<\infty.$$

Assume that all $\gamma_j=0$ if $|n_0|\geq C_0$. Then the function

$$F(R)\cdot\prod_{j=1}^{k}\phi_j(R)$$

admits a third order Taylor expansion $\sum_{j=0}^{3}\tilde{\gamma}_j R^j$ at $R=0$, where we have the coefficient bound

$$\sum_{j=0}^{3}\left|\tilde{\gamma}_j\right|\leq C_1^k\Lambda_0\cdot\prod_{j=1}^{k}\left\|\phi_j\right\|_{\tilde{S}_0^{(m_j)}}.$$

for a constant $C_1(C_0)$. Further, setting

$$n:=n_0+\sum_{j=1}^{k}m_j,$$

we have the bound

$$n_{|\cdot|\geq 2}^4\left\|\chi_{R\lesssim\tau}F(R)\cdot\prod_{j=1}^{n}\phi_j(R)-\chi_{R\lesssim 1}\sum_{j=0}^{3}\tilde{\gamma}_j R^j\right\|_{\tilde{S}_1^{(n_{|\cdot|\geq 2})}}$$

$$\leq C_2^k n_0^4 \Lambda_0 \cdot \prod_{j=1}^{k} \left(\tau^{1+\delta} \langle m_j \rangle^4 \left\| \phi_j \right\|_{\tilde{S}_0^{(m_j)}} \right).$$

Similar bounds obtain when the exponent 4 *is replaced by* $p \geq 4$.

Proof. By reordering the factors we may assume that $|m_1| \leq |m_2| \leq \ldots \leq |m_k|$, and either (i) $|n_0| \gtrsim \frac{|n_{|\cdot|\geq 2}|}{k}$ or else (ii) $|m_k| \gtrsim \frac{|n_{|\cdot|\geq 2}|}{k}$. Then we apply Prop. 6.18 consecutively to the sequence of products

$$F(R)\phi_1,\ F(R)\phi_1 \cdot \phi_2, \ldots, F(R) \cdot \prod_{j=1}^{k} \phi_j,$$

where the function $F(R)\phi_1 \cdot \ldots \phi_l$ will be interpreted as angular momentum $\left(n_0 + \sum_{j=1}^{l} m_j\right)_{|\cdot|\geq 2}$ function. Here we have to be careful not to lose on account of the factor

$$\chi_{|n_2|<K} |n_2|^{3+\delta}$$

in (6.47), where we note that the angular momenta of these products (and thus K) can in principle grow. However, if one of the factors ϕ_l is an angular momentum n function with $|n| \geq 5$, all products $F \cdot \prod_{j=1}^{l'} \phi_j, l' \geq l$ have trivial third order Taylor polynomial around $R = 0$, and so the term in the proof of Prop. 6.18 responsible for the preceding factor is not present. It follows that there is *at most one case* where one loses a factor $\chi_{|n_2|<K}|n_2|^{3+\delta}$, namely the first instance where ϕ_l is of absolute angular momentum ≥ 5, and we lose here at most a factor $Cl^{3+\delta}$. Such a factor can be absorbed into the factors C^l which automatically occur. In case (i) we inductively get the bounds

$$\left\| \chi_{R\lesssim\tau} F(R) \cdot \prod_{j=1}^{l} \phi_j - \chi_{R\lesssim 1} \sum_{j=1}^{3} \tilde{\gamma}^{(l)} R^j \right\|_{\tilde{S}_1^{\left((n_0+\sum_{j=1}^{l} m_j)_{|\cdot|\geq 2}\right)}} \leq C^l \Lambda_0 \cdot \prod_{j=1}^{l} \left(\tau^{1+\delta} \langle m_j \rangle^2 \left\| \phi_j \right\|_{\tilde{S}_0^{(m_j)}} \right) \tag{6.49}$$

where $\sum_{j=1}^{3} \tilde{\gamma}^{(l)} R^j$ is the Taylor polynomial of $\chi_{R\lesssim\tau} F(R) \cdot \prod_{j=1}^{l} \phi_j$ at $R = 0$, and the desired conclusion follows for $l = k$ by using assumption (i) and $k^4 C^k \leq C_2^k$ for suitable $C_2 = C_2(C)$.
In case (ii), use

$$\left| n_0 + \sum_{j=1}^{k-1} m_j \right| \leq |n_0| + (k-1)\,|m_{k-1}|\,,$$

and distinguish between $|n_0| < (k-1)\,|m_{k-1}|\,, |n_0| \geq (k-1)\,|m_{k-1}|$. in the former case, use (6.49) up to $l = k-1$ and further

$$\left\| \chi_{R\lesssim\tau} F(R) \cdot \prod_{j=1}^{k} \phi_j - \chi_{R\lesssim 1} \sum_{j=1}^{3} \tilde{\gamma}^{(k)} R^j \right\|_{\tilde{S}_1^{\left((n_0+\sum_{j=1}^{k} m_j)_{|\cdot|\geq 2}\right)}}$$

$$\leq \langle 2(k-1)\,|m_{k-1}| \rangle^2 C^k \Lambda_0 \cdot \prod_{j=1}^{k-1} \left(\tau^{1+\delta} \langle m_j \rangle^2 \left\| \phi_j \right\|_{\tilde{S}_0^{(m_j)}} \right) \cdot \left(\tau^{1+\delta} \left\| \phi_k \right\|_{\tilde{S}_0^{(m_k)}} \right)$$

$$\leq C_1^k \Lambda_0 \cdot \prod_{j=1}^{k-1} \left(\tau^{1+\delta} \langle m_j \rangle^4 \left\| \phi_j \right\|_{\tilde{S}_0^{(m_j)}} \right) \cdot \left(\tau^{1+\delta} \left\| \phi_k \right\|_{\tilde{S}_0^{(m_k)}} \right)$$

from which the desired bound follows due to assumption (ii). In the case $|n_0| \geq (k-1)|m_{k-1}|$ it suffices to replace the factor $\langle 2(k-1)|m_{k-1}|\rangle$ by $2|n_0|$. □

In a similar vein, we can bound products where each factor is a sum of different angular momentum components:

Corollary 6.20. *Assume that each ϕ_j can be written as a sum of angular momentum l functions $\phi_j^{(l)}$ satisfying the bound*

$$\tau^{1+\delta}\sum_{l\in\mathbb{Z}}\langle l\rangle^4\left\|\phi_j^{(l)}\right\|_{\tilde{S}_0^{(l)}} := \Lambda_j < \infty,\ j = 1, 2, \ldots, k.$$

Further, assume that F is a sum of angular momentum $l_{|\cdot|\geq 2}$-functions $F = \sum_{l\in\mathbb{Z}} F^{(l)}$ each of which has a third order Taylor polynomial $\sum_{j=1}^3 \gamma_j^{(l)}R^j$ at $R = 0$ and such that

$$\sum_{l\in\mathbb{Z}}\langle l\rangle^4\left[\left\|\chi_{R\lesssim\tau}F^{(l)} - \chi_{R\lesssim 1}\sum_{j=1}^3\gamma_j^{(l)}R^j\right\|_{\tilde{S}_1^{(l_{|\cdot|\geq 2})}} + \sum_{j=1}^3\left|\gamma_j^{(l)}\right|\right] := \Lambda_0 < \infty.$$

If we then set

$$\Psi_k^{(n)} := \sum_{l+\sum_{j=1}^k m_j = n} F^{(l)}\cdot\prod_{j=1}^k\phi_j^{(m_j)},$$

then each $\Psi_k^{(n)}$ admits a third order Taylor expansion $\sum_{j=0}^3\tilde{\gamma}_j^{(n)}R^j$ around $R = 0$ and we have, with C_3 a universal constant, the bounds

$$\sum_{n\in\mathbb{Z}}\langle n\rangle^4\left\|\chi_{R\lesssim\tau}\Psi_k^{(n)} - \chi_{R\lesssim 1}\sum_{j=0}^3\tilde{\gamma}_j^{(n)}R^j\right\|_{\tilde{S}_1^{(n_{|\cdot|\geq 2})}} \leq C_3^k\Lambda_0\prod_{j=1}^k\left(\tau^{1+\delta}\Lambda_j\right).$$

The same bound obtains if we subtract from the $\sum_{j=0}^3\tilde{\gamma}_j^{(n)}R^j$ those terms $\tilde{\gamma}_l^{(n)}R^l$ with $l \geq |n_{|\cdot|\geq 2} - 1|, l \equiv (n_{|\cdot|\geq 2} - 1)mod(2)$, provided $|n_{|\cdot|\geq 2} - 1| \leq 3$. The exponent 4 may be replaced by $p \geq 4$ throughout.

6.7. **Control over all source terms for angular momenta $|n| \geq 2$ and away from the light cone.** We use here the results of the preceding section to bound all the source terms arising in the equations for the angular momenta $|n| \geq 2$ in the interior of the light cone, away from the shock region. Dealing with the latter will require a different set of estimates exploiting the fine structure of the shock on the light cone. We will be using the equations (6.2)-(6.4). We observe that all terms on the right hand side of (6.3) with the exception of the first term are linear in $\varepsilon_{1,2}$, and in fact only depend on $\varepsilon_{1,2}(n)$. The fine structure of the correction $\epsilon = \epsilon(\tau, R)$ in turn is adopted from the works [10, 18], and we adopt the following function space essentially from the latter reference:

Definition 6.21. *Let $b_1 := \frac{\left(\log(1+R^2)\right)^2}{(t\lambda)^2}$, $b_2 = \frac{1}{(t\lambda)^2}$. Then we denote by*

$$S^m\left(R^k(\log R)^l\right)$$

the class of analytic functions $v : [0,\infty)\times[0,b_0]^2 \longrightarrow \mathbb{R}$, $b_0 > 0$ a fixed small positive number, such that

- *$v(R, b_1, b_2)$ vanishes to order m at $R = 0$, where we interpret b_1 as a function of R and t.*
- *v admits a convergent expansion at $R = \infty$*

$$v(R, b_1, b_2) = \sum_{\substack{0\leq j\leq l+i \\ i\geq 0}} c_{ij}(b_1, b_2)R^{k-i}(\log R)^j$$

and the functions c_{ij} are analytic on $[0, b_0]^2$.

We observe that this class of functions captures precisely the corrections ϵ which define Q in the region away from the light cone, say $r < \frac{t}{2}$, provided $m = 3, k = 1$. Specifically, we have

Theorem 6.22. *([10, 18]) The correction ϵ can be chosen in the form*

$$\chi_{r<\frac{t}{2}}\epsilon = \epsilon_1 + \epsilon_2$$

$$\epsilon_1 = \sum_{i=1}^{N} v_i,\ v_{2k-1} \in \frac{1}{(t\lambda)^{2k}}S^3\left(R(\log R)^{2k-1}\right),\ v_{2k} \in \frac{1}{(t\lambda)^{2k+2}}S^3\left(R^3(\log R)^{2k-1}\right)$$

where N can be chosen arbitrarily large, and such that ϵ_2 is a C^∞ function admitting a Taylor expansion around $R = 0$ in terms of odd powers of R, as well as the bounds

$$\left\|\nabla^l_{\tau,R}\epsilon_2\right\|_{L^\infty_{dR}} \lesssim_l \frac{1}{(t\lambda)^{N_1}},$$

for any $l \geq 0$, where $N_1 = N_1(N)$ grows linearly in N.

Using this structural ingredient, we can immediately bound all the linear terms in (6.3).

Proposition 6.23. *Let $|n| \geq 2$ and assume that ε is an angular momentum n function. Then denoting F_j, $j = 1, \dots, 4$ the final four terms in* (6.3)*, we have the bounds (with $\hbar = \frac{1}{|n|+1}$)*

$$\left\|\langle\phi(R;\xi,\hbar), \chi_{r<\frac{t}{2}}F_j\rangle_{L^2_{RdR}}\right\|_{S_1^\hbar} \lesssim \frac{1}{\tau^2}\sum_{\pm}\|\varepsilon_\pm(n)\|_{\tilde{S}_0^{(n)}},$$

where we recall that $\phi_2(n) = i\cdot(\epsilon_+(n) - \epsilon_-(n))$, $\tau = \int_t^\infty \lambda(s)\,ds$.

Proof. We give details for the estimate of the second term F_2, as the others can be handled similarly. To begin with, Theorem 6.22 implies that we have the bound

$$\left\|4\frac{\sin\left[Q + \frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\right\|_{L^2_{RdR}\cap L^\infty_{RdR}} \lesssim \frac{1}{(t\lambda)^2},\quad \left\|4\langle R\rangle\frac{\sin\left[Q + \frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\right\|_{L^\infty_{RdR}} \lesssim \frac{1}{(t\lambda)^2}$$

Also, we can obviously write $i\varepsilon_{\pm,\theta}(n) = -n\varepsilon_\pm(n)$. Then we localize the output frequency to dyadic size $\xi \simeq \lambda$ and split the source term into the sum of two pieces:

$$\chi_{r<\frac{t}{2}}F_2 = \chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q + \frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,\geq\lambda}(n) + \chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q + \frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,<\lambda}(n) \tag{6.50}$$

Correspondingly we have to bound two contributions:

(i) Contribution of first term in (6.50). This is the expression

$$\begin{aligned}&\chi_{\xi\simeq\lambda}\left\langle\phi_n(R;\xi),\quad \chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,\geq\lambda}(n)\right\rangle_{L^2_{R\,dR}}\\&=\sum_{\mu\geq\lambda}\chi_{\xi\simeq\lambda}\left\langle\phi_n(R;\xi),\quad \chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,\mu}(n)\right\rangle_{L^2_{R\,dR}}\end{aligned}$$

Using Plancherel's theorem for the distorted Fourier transform, the triangle inequality as well as the point wise bound above, we infer (recalling that $\lambda(t)\cdot t\sim\tau$)

$$\begin{aligned}&\left\|\chi_{\xi\simeq\lambda}\left\langle\phi_n(R;\xi),\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,\geq\lambda}(n)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^\hbar}\\&\lesssim\frac{(\lambda\hbar^2)^{1-\frac{\delta}{2}}\cdot\left\langle\lambda\hbar^2\right\rangle^{\delta+\frac32}}{\lambda^{\frac12}}\sum_{\mu\geq\lambda}\left\|\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,\mu}(n)\right\|_{L^2_{R\,dR}}\\&\lesssim\frac{1}{\tau^2}(\lambda\hbar^2)^{\frac12-\frac{\delta}{2}}\cdot\left\langle\lambda\hbar^2\right\rangle^{\delta+\frac32}\sum_{\mu\geq\lambda}\left\|\langle R\rangle^{-1}\varepsilon_{\pm,\mu}(n)\right\|_{L^2_{R\,dR}}\end{aligned}$$

The last term can be bounded by taking advantage of Proposition 6.8: letting $x(\xi)$ denote the distorted Fourier transform of $\varepsilon_\pm$ as angular momentum-n function, we have

$$\begin{aligned}&\frac{1}{\tau^2}(\lambda\hbar^2)^{\frac12-\frac{\delta}{2}}\cdot\langle\lambda\hbar^2\rangle^{\delta+\frac32}\sum_{\mu\geq\lambda}\left\|\langle R\rangle^{-1}\varepsilon_{\pm,\mu}(n)\right\|_{L^2_{R\,dR}}\\&\lesssim\frac{1}{\tau^2}(\lambda\hbar^2)^{\frac12-\frac{\delta}{2}}\cdot\langle\lambda\hbar^2\rangle^{\delta+\frac32}\sum_{\mu\geq\lambda}|n|^{-1}\left\|\xi^{\frac12}\overline{x}\right\|_{L^2_{d\xi}(\xi\simeq\mu)}\\&\lesssim\frac{1}{\tau^2}\sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{1-\frac{\delta}{2}}\cdot\|\overline{x}\|_{S_0^\hbar(\xi\simeq\mu)}.\end{aligned}$$

Finally, exploiting orthogonality, we infer the bound

$$\begin{aligned}&\left\|\left\langle\phi_n(R;\xi),\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,\geq\lambda}(n)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^\hbar}\\&\lesssim\left(\sum_{\lambda}\left\|\chi_{\xi\simeq\lambda}\left\langle\phi_n(R;\xi),\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,\geq\lambda}(n)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^\hbar}^2\right)^{\frac12}\\&\lesssim\tau^{-2}\left(\sum_{\lambda}\left[\sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{1-\frac{\delta}{2}}\cdot\|\overline{x}\|_{S_0^\hbar(\xi\simeq\mu)}\right]^2\right)^{\frac12}\end{aligned}$$

$$\lesssim \tau^{-2}\left(\sum_{\lambda}\sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{1-\frac{\delta}{2}}\cdot\|\overline{x}\|^2_{S_0^{\hbar}(\xi\simeq\mu)}\right)^{\frac{1}{2}}$$
$$\lesssim \tau^{-2}\,\|\overline{x}\|_{S_0^{\hbar}},$$

which confirms the estimate of the proposition for this contribution.

(ii) Contribution of second term in (6.50). This case is more delicate, and we have to distinguish between different output frequency regimes.

(ii.a) $\hbar^2\lambda < 1$. Here know that $\varepsilon_{\pm,<\lambda}(n)$ is in a low frequency regime, and we have to avoid losing factors $\hbar^{-1}$ when bounding this term. For this we take again advantage of Proposition 6.8 but use a multiplier $\langle R\rangle^{-2+}$ this time, exploiting the fact that

$$\left|\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\right| \lesssim \tau^{-2}\langle R\rangle^{-2}.$$

Then we obtain

$$\left\|\left\langle \chi_{\xi\simeq\lambda}\phi_n(R;\xi),\,\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,<\lambda}(n)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar}}$$
$$\lesssim \tau^{-2}(\lambda\hbar^2)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\left\|\langle R\rangle^{-2}\varepsilon_{\pm,<\lambda}(n)\right\|_{L^2_{R\,dR}}$$
$$\lesssim \tau^{-2}(\lambda\hbar^2)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\|\overline{x}\|_{S_0^{\hbar}(\xi<\lambda)}.$$

This can be l^1-summed over dyadic scales $\lambda < \hbar^{-2}$, giving the desired bound.

(ii.b) $\hbar^2\lambda \geq 1$. Here we perform integration by parts in order to absorb the outer weight into the expression. Importantly, observe that there will be no issues with boundary values at $R = 0$ since we use the same angular momentum for the output as well as for the factor $\varepsilon_{\pm,<\lambda}(n)$. Write

$$\chi_{\xi\simeq\lambda}\left\langle\phi_n(R;\xi),\,\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,<\lambda}(n)\right\rangle_{L^2_{R\,dR}}$$
$$= \xi^{-3}\chi_{\xi\simeq\lambda}\left\langle\phi_n(R;\xi),\,(H_n^{\pm})^3\left[\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,<\lambda}(n)\right]\right\rangle_{L^2_{R\,dR}}$$
$$= \sum_{i+j+k=6}C_{i,j,k}\xi^{-3}\chi_{\xi\simeq\lambda}\left\langle\phi_n(R;\xi),\,\partial_R^i\left[\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\right]\left(\frac{n}{R}\right)^j\partial_R^k\varepsilon_{\pm,<\lambda}(n)\right\rangle_{L^2_{R\,dR}}$$

To bound the expression on the right in the inner product, we use that

$$\langle R\rangle^{2-\delta}\left|\partial_R^i\left[\chi_{r<\frac{t}{2}}4n\frac{\sin[Q+\frac{\epsilon}{2}]\sin[\frac{\epsilon}{2}]}{R^2}\right]\right|\lesssim_{i,\delta}\frac{1}{\tau^2}$$

for any $i \geq 0$, and so we have the bound

$$\begin{aligned}&\left\|\partial_R^i\left[\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\right]\left(\frac{n}{R}\right)^j\partial_R^k\varepsilon_{\pm,<\lambda}(n)\right\|_{L^2_{RdR}}\\&\lesssim\tau^{-2}n\left\|\langle R\rangle^{-2+\frac{\delta}{2}}\left(\frac{n}{R}\right)^j\partial_R^k\varepsilon_{\pm,<\lambda}(n)\right\|_{L^2_{RdR}}\end{aligned}$$

It follows that

$$\begin{aligned}&\left\|\chi_{\xi\simeq\lambda}\left\langle\phi_n(R;\xi),\,\chi_{r<\frac{t}{2}}4n\frac{\sin\left[Q+\frac{\epsilon}{2}\right]\sin\left[\frac{\epsilon}{2}\right]}{R^2}\varepsilon_{\pm,<\lambda}(n)\right\rangle_{L^2_{RdR}}\right\|_{S_1^{\hbar}}\\&\lesssim\tau^{-2}\frac{(\lambda\hbar^2)^{\frac{1}{2}-\frac{\delta}{2}}\langle\lambda\hbar^2\rangle^{\delta+\frac{3}{2}}}{\lambda^3}\cdot\sum_{j+k\leq 6}\left\|\langle R\rangle^{-2+\delta}\left(\frac{n}{R}\right)^j\partial_R^k\varepsilon_{\pm,<\lambda}(n)\right\|_{L^2_{RdR}}\\&\lesssim\tau^{-2}\frac{(\lambda\hbar^2)^{\frac{1}{2}-\frac{\delta}{2}}\langle\lambda\hbar^2\rangle^{\delta+\frac{3}{2}}}{\lambda^3}\cdot\sum_{j+k\leq 6}\left\|\min\{(\xi\hbar^2)^{1-\frac{\delta}{2}},1\}\xi^{\frac{j+k}{2}}x(\xi)\right\|_{L^2_{d\xi}(\xi<\lambda)}\\&\lesssim\tau^{-2}\sum_{\mu<\lambda}\left(\frac{\mu}{\lambda}\right)^{1-\frac{\delta}{2}}\cdot\|\overline{x}(\xi)\|_{S_0^{\hbar}(\xi\simeq\mu)}\,.\end{aligned}$$

The desired estimate follows from this by square summing over dyadic scales $\lambda \geq \hbar^{-2}$ and applying the Cauchy-Schwarz inequality and orthogonality in the usual manner. □

The preceding proposition reveals that in order to control the right hand side in (6.3), it suffices to deal with the more difficult first term $\lambda^{-2}N_{\pm}(n)$, which contains all the nonlinear interactions. We commence by estimating the second and third term on the right hand side in (6.4). Recall that $U = Q + \epsilon$.

Proposition 6.24. *For n_1, n_2 arbitrary integers, let ϕ be an angular momentum n_1 function and ψ and angular momentum n_2 function in the sense of Prop. 6.14. Further, let $n_3, |n_3| \geq 2$, be an integer satisfying either (i) $n_1 \simeq n_3$ and $|n_2| \lesssim |n_1|$, or (ii) $|n_3| \ll |n_1|$ and $n_1 \simeq -n_2$, or (iii) $|n_3| \gg |n_1|$ and $n_3 \simeq n_2$. Then if $\min\{|n_1|, |n_2|\} \gg 1$, the functions*

$$F_1 := \chi_{R\ll\tau}U_R\cdot\phi\cdot\psi_R,\quad F_2=\chi_{R\ll\tau}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)U\cdot\phi\cdot\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\psi,\quad F_3=\chi_{R\ll\tau}\frac{2\sin U}{R^2}(n_1-n_2)\phi\cdot\overline{\psi}$$

satisfy the bound

$$\|F_l\|_{\tilde{S}_1^{(n_3)}}\lesssim\langle\min\{|n_1|,|n_2|\}\rangle^6\,\|\phi\|_{\tilde{S}_0^{(n_1)}}\cdot\|\psi\|_{\tilde{S}_0^{(n_2)}\cap\tilde{S}_1^{(n_2)}}\,.$$

If $|n_j| \lesssim 1$ *for all* j*, then the* F_l *admit third order Taylor expansions around* $R = 0$ *of the form* $P_3^{(l)} = \sum_{j=0}^{3} \gamma_j^{(l)} R^j$*,* $l = 1, 2, 3$*, with*

$$\sum_j \left|\gamma_j^{(l)}\right| \lesssim \|\phi\|_{\tilde{S}_0^{(n_1)}} \cdot \|\psi\|_{\tilde{S}_0^{(n_2)} \cap \tilde{S}_1^{(n_2)}},$$

and such that we have the bounds

$$\left\| F_l - \chi_{R\lesssim 1} \sum_{j=0}^{3} \gamma_j^{(l)} R^j \right\|_{\tilde{S}_1^{(n_3)}} \lesssim \|\phi\|_{\tilde{S}_0^{(n_1)}} \cdot \|\psi\|_{\tilde{S}_0^{(n_2)} \cap \tilde{S}_1^{(n_2)}}.$$

Remark 6.25. *The reason for the form of* F_3 *is the term*

$$\frac{2\sin U}{R^2} \sum_{j=1}^{2} \varphi_j \varphi_{j,\theta} = \frac{2\sin U}{R^2} (\varepsilon_+ \varepsilon_-)_\theta,$$

which occurs up to a small perturbative factor in (6.4) *and where we recall the relation* $\varepsilon_+ = \overline{\varepsilon_-}$*. Expanding out* $\varepsilon_\pm$ *in angular Fourier modes, we observe that there is a delicate cancellation in this term at* $R = 0$*, since the product* $e^{i(n+m)\theta}\varepsilon_+(n)\varepsilon_-(m)$ *does not vanish at* $R = 0$ *precisely in the case* $n = -m = 1$*, in which case the* θ*-derivative vanishes.*

Proof. We give details for the expressions $F_{1,3}$, the remaining F_2 being handled similarly.

(1): Bounding F_1. We distinguish between different situations involving the angular momentum and regular frequencies.

(1.1): $\max\{|n_j|\} \gg 1$. *Trivial Taylor polynomial for output.* Localizing the output frequency $\xi \simeq \lambda$ for dyadic λ, we need to estimate

$$\left\|\chi_{\xi\simeq\lambda} \langle \phi_{n_3}(R;\xi), \chi_{R\ll\tau} U_R \cdot \phi \cdot \psi_R \rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}},$$

where the output angular momentum n_3 is constrained by conditions (i) - (iii) in the statement of the proposition. Then decompose the expression as follows:

$$\begin{aligned}\chi_{\xi\simeq\lambda} \langle \phi_{n_3}(R;\xi), \chi_{R\lesssim\tau} U_R \phi \psi_R \rangle_{L^2_{R\,dR}} &= \chi_{\xi\simeq\lambda} \langle \phi_{n_3}(R;\xi), \chi_{R\ll\tau} U_R \phi \psi_{R,\geq\lambda} \rangle_{L^2_{R\,dR}} \\ &+ \chi_{\xi\simeq\lambda} \langle \phi_{n_3}(R;\xi), \chi_{R\ll\tau} U_R \phi_{\geq\lambda} \psi_{R,<\lambda} \rangle_{L^2_{R\,dR}} \\ &+ \chi_{\xi\simeq\lambda} \langle \phi_{n_3}(R;\xi), \chi_{R\ll\tau} U_R \phi_{<\lambda} \psi_{R,<\lambda} \rangle_{L^2_{R\,dR}}\end{aligned} \tag{6.51}$$

We use the simple bound $|U_R| \lesssim \langle R\rangle^{-2}$. Then the first term on the right is bounded by

$$\begin{aligned}&\left\|\chi_{\xi\simeq\lambda} \langle \phi_{n_3}(R;\xi), \chi_{R\ll\tau} U_R \phi \psi_{R,\geq\lambda} \rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}} \\ &\lesssim \hbar_3 (\lambda\hbar_3^2)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \langle \lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}} \cdot \left\|\chi_{R\ll\tau} U_R \phi \psi_{R,\geq\lambda}\right\|_{L^2_{R\,dR}}.\end{aligned}$$

Then in situations (iii) we have $|n_3| \gtrsim |n_2| \geq |n_1|$, and taking advantage of Lemma 6.9 and integrating $\partial_R \phi$ in R, we infer in this case the bound

$$\hbar_3 (\lambda\hbar_3^2)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \langle \lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}} \cdot \left\|\chi_{R\ll\tau} U_R \phi \psi_{R,\geq\lambda}\right\|_{L^2_{R\,dR}}$$

$$
\begin{aligned}
&\lesssim \|U_R\phi\|_{L^\infty_{R\,dR}} \cdot \hbar_3(\lambda\hbar_3^2)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \langle\lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}} \left\|\psi_{R,\geq\lambda}\right\|_{L^2_{R\,dR}} \\
&\lesssim |n_1|^{\frac{1}{2}+\delta} \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \cdot \sum_{\mu\geq\lambda} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\mu)},
\end{aligned}
$$

where $\overline{x}_2$ stands for the Fourier transform of ψ interpreted as angular momentum n_2-function, while in case (i), exploiting the bound

$$
\hbar_3 \|U_R\phi\|_{L^\infty_{R\,dR}} \lesssim \|\overline{x}_1\|_{S_0^{\hbar_1}},
$$

we get

$$
\begin{aligned}
&\hbar_3(\lambda\hbar_3^2)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \langle\lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}} \cdot \left\|\chi_{R\ll\tau} U_R\phi\psi_{R,\geq\lambda}\right\|_{L^2_{R\,dR}} \\
&\lesssim |n_2| \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \cdot \sum_{\mu\geq\lambda} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\mu)}.
\end{aligned}
$$

In the remaining case (ii), we have $|n_3| \ll |n_1| \simeq |n_2|$, and so we can bound

$$
\begin{aligned}
&\hbar_3(\lambda\hbar_3^2)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \langle\lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}} \cdot \left\|\chi_{R\ll\tau} U_R\phi\psi_{R,\geq\lambda}\right\|_{L^2_{R\,dR}} \\
&\lesssim \|U_R\phi\|_{L^\infty_{R\,dR}} \cdot \hbar_3(\lambda\hbar_3^2)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \langle\lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}} \left\|\psi_{R,\geq\lambda}\right\|_{L^2_{R\,dR}} \\
&\lesssim |n_1|^{\frac{1}{2}+\delta} \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \cdot |n_2|^{5+\delta} \sum_{\mu\geq\lambda} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\mu)} \\
&\lesssim |n_1|^6 \|\phi_1\|_{\tilde{S}_0^{(n_1)}} \cdot \sum_{\mu\geq\lambda} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \|\overline{x}_2\|_{S_0^{\hbar_2}(\xi\simeq\mu)}.
\end{aligned}
$$

The desired bound follows for this case by square-summing over dyadic λ and exploiting Cauchy-Schwarz and orthogonality.
In order to bound the second term in (6.51), use that according to Prop. 6.8 we have

$$
\left\|\langle R\rangle^{-1}\phi_{\geq\lambda}\right\|_{L^2_{R\,dR}} \lesssim |n_1|^{-1} \left\|\xi^{\frac{1}{2}}\overline{x}_1\right\|_{L^2_{d\xi}},
$$

and so in situations (i) we have the bound

$$
\hbar_3(\lambda\hbar_3^2)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \langle\lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}} \cdot \|U_R\phi_{\geq\lambda}\|_{L^2_{R\,dR}} \lesssim \hbar_3 \sum_{\mu\geq\lambda} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)},
$$

and so using $\hbar_3 \left\|\psi_{R,<\lambda}\right\|_{L^\infty_{R\,dR}} \lesssim |n_2|^{\frac{1}{2}+\delta} \|\overline{x}_2\|_{S_0^{\hbar_2}}$ according to Lemma 6.9, we get the bound

$$
\begin{aligned}
&\left\|\chi_{\xi\simeq\lambda} \langle\phi_{n_3}(R;\xi), \chi_{R\ll\tau} U_R\phi_{\geq\lambda}\psi_{R,<\lambda}\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}} \\
&\lesssim |n_2|^{\frac{1}{2}+\delta} \cdot \|\overline{x}_2\|_{S_0^{\hbar_2}} \cdot \sum_{\mu\geq\lambda} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)}.
\end{aligned}
$$

In situation (iii), we use (Lemma 6.9)

$$\hbar_3^{2-\delta} \left\|\psi_{R,<\lambda}\right\|_{L^\infty_{R\,dR}} \lesssim \|\overline{x}_2\|_{S_0^{\hbar_2}}$$

as well as (Prop. 6.8)

$$\lambda^{\frac{1}{2}-\frac{\delta}{2}} \cdot \langle\lambda\hbar_3^2\rangle^{\delta+\frac{3}{2}} \cdot \|U_R\phi_{\geq\lambda}\|_{L^2_{R\,dR}} \lesssim |n_1|^{1-\delta} \cdot \sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}} \cdot \|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)},$$

in order to infer the bound

$$\begin{aligned}&\left\|\chi_{\xi\sim\lambda}\left\langle\phi_{n_3}(R;\xi),\, \chi_{R\ll\tau}U_R\phi_{\geq\lambda}\psi_{R,<\lambda}\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_2}}\\ &\lesssim |n_1|^{1-\delta}\cdot\|\overline{x}_2\|_{S_0^{\hbar_2}}\cdot\sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)},\end{aligned}$$

while case (ii) is treated as for the first term in (6.51). The desired bound again follows for this case by square-summing over dyadic λ and exploiting Cauchy-Schwarz and orthogonality.
The last term in (6.51) is treated in the customary fashion by integration by parts (the fact that we assume high vanishing at the origin in case (1.1) preventing problems with boundary terms at $R = 0$), and we omit the details.

(1.2): $\max\{|n_j|\} \lesssim 1$. Here we have to take advantage of subtracting off the Taylor polynomial around $R = 0$. We shall treat the exceptional case when $n_j \in \{0, \pm1\}$, $j = 1, 2$. This means we assume that ϕ, ψ admit representations in terms of the root/resonant modes ϕ_n as detailed in the statement of Proposition 6.14:

$$\phi = c_1\cdot\phi_{n_1}(R) + \phi_{n_1}(R)\cdot\int_0^R \left[\phi_{n_1}(s)\right]^{-1}\mathcal{D}\phi(s)\,ds,\quad \psi = c_2\cdot\phi_{n_2}(R) + \phi_{n_2}(R)\cdot\int_0^R\left[\phi_{n_2}(s)\right]^{-1}\tilde{\mathcal{D}}\phi(s)\,ds$$

Also, observe that by assumption we set $n_3 = O(1)$. To begin with, the case when $\phi = c_1\cdot\phi_{n_1}(R)$, $\psi = c_2\cdot\phi_{n_2}(R)$, is easy to handle. In this case the third order Taylor polynomial $\sum_{j=0}^3\gamma_j R^j$ around $R = 0$ of $\chi_{R\ll\tau}U_R\phi\psi_R$ satisfies

$$\sum_j\left|\gamma_j\right| \lesssim \prod_{k=1,2}|c_k|\,.$$

We then need to verify the bound

$$\left\|\chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim1}\sum_{j=0}^3\gamma_j R^j\right\|_{S_1^{\hbar_3}} \lesssim \prod_{k=1,2}|c_k|\,,$$

which in light of the smoothness properties of the resonances/root modes reduces to verifying sufficient decay in R. In fact, we have

$$\left\|\left\langle\phi_{n_3}(R;\xi),\,\chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim1}\sum_{j=0}^3\gamma_j R^j\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}$$

$$\leq \left\| \left\langle \phi(R;\xi,\hbar_3),\, \chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j \right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}(\xi<\hbar_3^{-2})}$$
$$+ \left\| \left\langle \phi_{n_3}(R;\xi),\, \chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j \right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}(\xi\geq\hbar_3^{-2})},$$

and the first term on the right can be bounded by neglecting the weight defining $\|\cdot\|_{S_1^{\hbar_3}}$ and using Plancherel's theorem for the distorted Fourier transform:

$$\left\| \left\langle \phi_{n_3}(R;\xi),\, \chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j \right\rangle \right\|_{S_1^{\hbar_3}(\xi<\hbar_3^{-2})}$$
$$\lesssim \prod_{k=1,2}|c_k| \cdot \left\| \chi_{R\ll\tau}U_R\phi_{n_1}\phi_{n_2,R} - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j \right\|_{L^2_{R\,dR}} \lesssim \prod_{k=1,2}|c_k|\,.$$

For the second, large frequency contribution above, we use integration by parts: the fourth order vanishing of the term on the right in the inner product implies that

$$\left\langle \phi_{n_3}(R;\xi),\, \chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j \right\rangle_{L^2_{R\,dR}}$$
$$= \xi^{-2}\left\langle \phi_{n_3}(R;\xi),\, \left(H_{n_3}^{\pm}\right)^2 \left[\chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j\right] \right\rangle_{L^2_{R\,dR}}$$

Then we have the simple bound

$$\left| \left(H_{n_3}^{\pm}\right)^2 \left[\chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j\right] \right| \lesssim \frac{1}{\langle R\rangle^8}\cdot \prod_{k=1,2}|c_k|\,,$$

and using the asymptotics of $\phi_{n_3}(R;\xi)$ both in the oscillatory and non-oscillatory regime as well as simple integration by parts, we infer the bound

$$\left| \xi^{-2}\left\langle \phi_{n_3}(R;\xi),\, \left(H_{n_3}^{\pm}\right)^2 \left[\chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j\right] \right\rangle_{L^2_{R\,dR}} \right| \lesssim \xi^{-3}\langle\log\xi\rangle \cdot n_3^4 \prod_{k=1,2}|c_k|\,.$$

We conclude that

$$\left\| \left\langle \phi_{n_3}(R;\xi),\, \chi_{R\ll\tau}U_R\phi\psi_R - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j \right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}(\xi\geq\hbar_3^{-2})}$$

$$\lesssim \prod_{k=1,2} |c_k| \cdot \left\| \hbar_3 (\xi \hbar_3^2)^{\frac{1}{2}-\frac{\delta}{2}} \langle \xi \hbar_3^2 \rangle^{\delta+\frac{3}{2}} \cdot \xi^{-3} \langle \log \xi \rangle \right\|_{L^2_{d\xi}(\xi > \hbar_3^{-2})}$$
$$\lesssim \prod_{k=1,2} |c_k| .$$

Next we consider the mixed case, where, say,

$$\phi = \phi_{n_1}(R) \cdot \int_0^R [\phi_{n_1}(s)]^{-1} \mathcal{D}\phi(s)\, ds,$$

while $\psi = c_2 \phi_{n_2}(R)$ is a multiple of the root/resonant mode. By Proposition 6.14 (See (6.30)), the third order Taylor polynomial $\sum_{j=0}^3 \gamma_j R^j$ of $\chi_{R \ll \tau} U_R \phi \psi_R$ around $R = 0$ satisfies the bound (here $n \in \{0, \pm 1\}$, and $\overline{x}_1$ refers to the distorted Fourier coefficient at the level of $\mathcal{D}\phi$)

$$\sum_j \left| \gamma_j \right| \lesssim |c_2| \cdot \|\overline{x}_1\|_{S_0^{\hbar_1}} ,$$

The low frequency regime of the expression is again straightforward to estimate, since

$$\left\| \left\langle \phi_{n_3}(R; \xi),\, \chi_{R \ll \tau} U_R \phi \psi_R - \chi_{R \lesssim 1} \sum_{j=0}^3 \gamma_j R^j \right\rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar_3}(\xi < \hbar_3^{-2})} \lesssim \|\overline{x}_1\|_{S_0^{\hbar_1}} \cdot |c_2| ,$$

where we have used the fact that

$$\|U_R \phi\|_{L^2_{R\,dR}} \lesssim \left\| \langle R \rangle^{-(1-\delta)} \phi \right\|_{L^2_{R\,dR}} \lesssim \|\overline{x}_1\|_{S_0^{\hbar_1}} ,$$

recalling the proof of Prop. 6.12, as well as the bound $\left\| \chi_{R \lesssim 1} \sum_{j=0}^3 \gamma_j R^j \right\|_{L^2_{R\,dR}} \lesssim \|\overline{x}_1\|_{S_0^{\hbar_1}} \cdot |c_2|$. For the large frequency contribution, the interplay of the output frequency and the frequency of ϕ needs to be analyzed. Restricting the output frequency to dyadic scale $\lambda \geq \hbar_3^{-2}$, we have

$$\begin{aligned} &\chi_{\xi \simeq \lambda} \left\langle \phi_{n_3}(R; \xi),\, \chi_{R \ll \tau} U_R \phi \psi_R - \chi_{R \lesssim 1} \sum_{j=0}^3 \gamma_j R^j \right\rangle_{L^2_{R\,dR}} \\ &= \chi_{\xi \simeq \lambda} \left\langle \phi_{n_3}(R; \xi),\, \chi_{R \ll \tau} U_R \phi_{\geq \lambda} \psi_R \right\rangle_{L^2_{R\,dR}} \\ &+ \chi_{\xi \simeq \lambda} \left\langle \phi_{n_3}(R; \xi),\, \chi_{R \ll \tau} U_R \phi_{< \lambda} \psi_R - \chi_{R \lesssim 1} \sum_{j=0}^3 \gamma_j R^j \right\rangle_{L^2_{R\,dR}} \end{aligned} \tag{6.52}$$

To bound the first term on the right, we integrate by parts twice, using the fact that our choice for ϕ means that $\phi_{\geq \lambda}$ vanishes to order at least two at the origin. This gives

$$\begin{aligned} &\chi_{\xi \simeq \lambda} \left\langle \phi_{n_3}(R; \xi),\, \chi_{R \ll \tau} U_R \phi_{\geq \lambda} \psi_R \right\rangle_{L^2_{R\,dR}} \\ &= \lambda^{-1} \chi_{\xi \simeq \lambda} \left\langle \phi(R; \xi, \hbar_3),\, H^{\pm}_{n_3} \left[\chi_{R \ll \tau} U_R \phi_{\geq \lambda} \psi_R \right] \right\rangle_{L^2_{R\,dR}} \\ &= \lambda^{-1} \chi_{\xi \simeq \lambda} \left\langle \phi_{n_3}(R; \xi),\, H^{\pm}_{n_3} \left[\chi_{R \ll \tau} U_R \psi_R \right] \phi_{\geq \lambda} \right\rangle_{L^2_{R\,dR}} \end{aligned}$$

$$+\sum_{\substack{i+j=2\\ j\geq 1}} C_{i,j}\lambda^{-1}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\, \partial_R^i\left(\chi_{R\ll\tau}U_R\psi_R\right)\left(\partial_R^j\phi_n(R)\right)\left(\int_0^R[\phi_n(s)]^{-1}\mathcal{D}\phi_{\geq\lambda}(s)\,ds\right)\right\rangle_{L^2_{R\,dR}}$$
$$+\lambda^{-1}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\partial_R\left[\chi_{R\ll\tau}U_R\psi_R\mathcal{D}\phi_{\geq\lambda}(R)\right]\right\rangle_{L^2_{R\,dR}}$$
$$=:\sum_{l=1}^3 A_l.$$

To bound A_1, use a simple variation on Prop. 6.12 which is

$$\left\|\langle R\rangle^{-2}\phi_{\geq\lambda}\right\|_{L^\infty_{R\,dR}}\lesssim\sum_{\mu\geq\lambda}\left\|\xi\left[\tilde{\rho}_{n_1}(\xi)\right]^{\frac12}\overline{x}_1(\xi)\right\|_{L^2_{d\xi}(\xi\simeq\mu)},$$

and so we infer the bound (recall that the output frequency is restricted to $\xi\geq\hbar_3^{-2}$)

$$\|A_1\|_{S_1^{\hbar_3}}\lesssim\lambda^{1+\frac{\delta}{2}}\cdot\left\|\langle R\rangle^2H^{\pm}_{n_3}\left[\chi_{R\ll\tau}U_R\psi_R\right]\right\|_{L^2_{R\,dR}}\cdot\left\|\langle R\rangle^{-2}\phi_{\geq\lambda}\right\|_{L^\infty_{R\,dR}}$$
$$\lesssim|c_2|\sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{1+\frac{\delta}{2}}\cdot\|\overline{x}_1\|_{S_0^{(n_1)}(\xi\simeq\mu)}$$

The term A_2 leads to a similar bound, we omit the details. As for the last term A_3, expand it as follows

$$A_3=\lambda^{-1}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\partial_R\left[\chi_{R\ll\tau}U_R\psi_R\right]\mathcal{D}\phi_{\geq\lambda}(R)\right\rangle_{L^2_{R\,dR}}$$
$$+\lambda^{-1}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\chi_{R\ll\tau}U_R\psi_R\partial_R\left[\mathcal{D}\phi_{\geq\lambda}(R)\right]\right\rangle_{L^2_{R\,dR}}$$

Then use Prop. 6.7 to conclude that

$$\|\mathcal{D}\phi_{\geq\lambda}(R)\|_{L^2_{R\,dR}}\lesssim\|\overline{x}_1\|_{L^2_{d\xi}(\xi\geq\lambda)},\quad\left\|\partial_R\left[\mathcal{D}\phi_{\geq\lambda}(R)\right]\right\|_{L^2_{R\,dR}}\lesssim\left\|\xi^{\frac12}\overline{x}_1\right\|_{L^2_{d\xi}(\xi\geq\lambda)},$$

and so we can crudely bound

$$\|A_3\|_{S_1^{\hbar_3}}\lesssim\left\|\langle\partial_R\rangle\left[\chi_{R\ll\tau}U_R\psi_R\right]\right\|_{L^\infty_{R\,dR}}\cdot\lambda^{1+\frac{\delta}{2}}\cdot\left\|\xi^{\frac12}\overline{x}_1\right\|_{L^2_{d\xi}(\xi\geq\lambda)}$$
$$\lesssim|c_2|\cdot\sum_{\mu\geq\lambda}\left(\frac{\lambda}{\mu}\right)^{1+\frac{\delta}{2}}\|\overline{x}_1\|_{S_0^{\hbar_1}(\xi\simeq\mu)}$$

Combining the preceding bounds for A_j, $j=1,2,3$, and square-summing over λ as well as exploiting Cauchy-Schwarz and orthogonality gives the desired bound for the contribution of the first term on the right in (6.52).

Consider then the second term in (6.52), where we have to take advantage of a partial cancellation between the principal term and the truncated Taylor polynomial. To begin with, observe that in the case $n=-1$, the third order Taylor polynomial at $R=0$ is trivial, since

$$\left|\phi_{-1}(R)\cdot\int_0^R\left[\phi_{-1}(s)\right]^{-1}\mathcal{D}_{-1}\phi(s)\,ds\right|\lesssim R^{4+\delta}\|\overline{x}_1\|_{S_0^{(n_1)}}$$

Thus in this case a direct integration by parts argument works:

$$\begin{aligned}
&\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\chi_{R\ll\tau}U_R\phi_{<\lambda}\psi_R-\chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j\right\rangle_{L^2_{R\,dR}}\\
&=\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\chi_{R\ll\tau}U_R\phi_{<\lambda}\psi_R\right\rangle_{L^2_{R\,dR}}\\
&=\lambda^{-2}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\left(H^{\pm}_{n_3}\right)^2\left[\chi_{R\ll\tau}U_R\phi_{<\lambda}\psi_R\right]\right\rangle_{L^2_{R\,dR}}
\end{aligned}$$

Expand

$$\begin{aligned}
&\left(H^{\pm}_{n_3}\right)^2\left[\chi_{R\ll\tau}U_R\phi_{<\lambda}\psi_R\right]\\
&=\sum_{i+j=4}C_{i,j}\left(\frac{n_3}{R}\right)^i\partial_R^j\left(\chi_{R\ll\tau}U_R\psi_R\phi_{-1}(R)\right)\cdot\int_0^R\left[\phi_{-1}(s)\right]^{-1}\mathcal{D}_{-1}\phi_{<\lambda}(s)\,ds\\
&+\sum_{\substack{i+j+k=4\\k\geq 1}}C_{i,j,k}\left(\frac{n_3}{R}\right)^i\partial_R^j\left(\chi_{R\ll\tau}U_R\psi_R\phi_{-1}(R)\right)\cdot\partial_R^{k-1}\left(\left[\phi_{-1}(R)\right]^{-1}\mathcal{D}_{-1}\phi_{<\lambda}(R)\right),
\end{aligned}$$

and we have the crude bound (compare Prop. 6.12 and its proof)

$$\left|\langle R\rangle^{-1}\int_0^R\left[\phi_{-1}(s)\right]^{-1}\mathcal{D}_{-1}\phi_{<\lambda}(s)\,ds\right|\lesssim\|\overline{x}_1\|_{S_0^{(-1)}(\xi<\lambda)},$$

while for the second term we have the more sophisticated bound (under the constraint $i+j+k=4$)

$$\left|\left(\frac{n_3}{R}\right)^i\partial_R^j\left(\chi_{R\ll\tau}U_R\psi_R\phi_{-1}(R)\right)\partial_R^{k-1}\left(\left[\phi_{-1}(R)\right]^{-1}\mathcal{D}_{-1}\phi_{<\lambda}(R)\right)\right|\lesssim|c_2|\cdot\|\overline{x}_1\|_{S_0^{(-1)}(\xi<\lambda)}.$$

Back to the inner product $\langle\cdot\rangle_{L^2_{R\,dR}}$, we split this via a smooth cutoff into the regions $R\lambda^{\frac{1}{2}}<1$, $R\lambda^{\frac{1}{2}}\geq 1$, and perform further integration by parts as needed in the latter region(of course without generating boundary terms). This finally leads to the bound

$$\left|\lambda^{-2}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\left(H^{\pm}_{n_3}\right)^2\left[\chi_{R\ll\tau}U_R\phi_{<\lambda}\psi_R\right]\right\rangle_{L^2_{R\,dR}}\right|\lesssim\lambda^{-3}\cdot\|\overline{x}_1\|_{S_0^{(-1)}(\xi<\lambda)}\cdot|c_2|.$$

If we now apply the $\|\cdot\|_{S_1^{\hbar_3}}$-norm to this term and neglect the weight $\hbar_3\simeq 1$, we obtain

$$\begin{aligned}
&\left\|\lambda^{-2}\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\left(H^{\pm}_{n_3}\right)^2\left[\chi_{R\ll\tau}U_R\phi_{<\lambda}\psi_R\right]\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\
&\lesssim\lambda^{\frac{5}{2}+\frac{\delta}{2}}\cdot\lambda^{-3}\cdot\|\overline{x}_1\|_{S_0^{(-1)}(\xi<\lambda)}\cdot|c_2|
\end{aligned}$$

This can then even be l^1-summed over dyadic scales $\lambda\geq\hbar_3^{-2}$ to result in the desired bound.

Let us next consider the case $n_1 = 1$, where the Taylor polynomial of third order is not necessarily trivial. Then we write

$$\chi_{R\ll\tau}U_R\phi_{<\lambda}\psi_R - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j = \chi_{1\lesssim R\ll\tau}U_R\phi_{<\lambda}\psi_R + \chi_{R\lesssim 1}U_R\psi_R\phi_1(R)\cdot\int_0^R [\phi_1(s)]^{-1}\,\mathcal{D}_1\phi_{<\lambda}(s)\,ds - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j,$$

where throughout all cutoffs $\chi_{1\lesssim R\ll\tau}$ are chosen smoothly. Then the contribution of the first term on the right, which is

$$\chi_{\xi\simeq\lambda}\left\langle \phi_{n_3}(R;\xi),\, \chi_{1\lesssim R\ll\tau}U_R\phi_{<\lambda}\psi_R\right\rangle_{L^2_{R\,dR}},$$

can be handled in the customary fashion by shifting three copies of $H^{\pm}_{n_3}$ from left to right, since the term vanishes in a neighborhood of $R = 0$, and we omit the details. For the second term on the right, we note that here in fact $\sum_{j=0}^{3}\gamma_j R^j = \gamma_2 R^2$ or $\gamma_3 R^3$(depending on n_2), and we have

$$\begin{aligned}&\chi_{R\lesssim 1}U_R\psi_R\phi_1(R)\cdot\int_0^R [\phi_1(s)]^{-1}\,\mathcal{D}_1\phi_{<\lambda}(s)\,ds - \chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j R^j\\ &= \chi_{R\lesssim 1}U_R\psi_R\phi_1(R)\cdot\int_0^R [\phi_1(s)]^{-1}\,\widetilde{\mathcal{D}_1\phi_{<\lambda}}(s)\,ds + E,\end{aligned}\tag{6.53}$$

and where we define the expression $\widetilde{\mathcal{D}_1\phi_{<\lambda}}(s)$ as

$$\widetilde{\mathcal{D}_1\phi_{<\lambda}}(s) = \int_0^\infty \chi_{\xi<\lambda}\left[\phi_1(s,\xi) - \frac{\pi}{4}s\right]\overline{x}_1(\xi)\tilde{\rho}_1(\xi)\,d\xi,$$

and we recall that $\frac{\pi}{4}s$ is the linear part of $\phi(s,\xi)$ near $s = 0$. Then we claim that both terms in (6.53) lead to good contributions. In fact, for the first term, splitting the inner product $\langle\cdot\rangle_{L^2_{R\,dR}}$ smoothly into the ranges $R\lambda^{\frac{1}{2}} < 1, R\lambda^{\frac{1}{2}} > 1$ and performing integration by parts in the second regime as needed, and using the bound

$$\left|\chi_{R\lesssim 1}U_R\psi_R\phi_1(R)\cdot\int_0^R [\phi_1(s)]^{-1}\,\widetilde{\mathcal{D}_1\phi_{<\lambda}}(s)\,ds\right| \lesssim R^4\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}}$$

as well as simple analogues for its derivatives, we infer

$$\begin{aligned}&\left|\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\chi_{R\lesssim 1}U_R\psi_R\phi_1(R)\cdot\int_0^R [\phi_1(s)]^{-1}\,\widetilde{\mathcal{D}_1\phi_{<\lambda}}(s)\,ds\right\rangle_{L^2_{R\,dR}}\right|\\ &\lesssim \lambda^{-3}\cdot|c_2|\cdot\left\|x_1\right\|_{S_0^{\hbar_1}},\end{aligned}$$

which in combination with Hölder's inequality gives

$$\begin{aligned}&\left\|\chi_{\xi\simeq\lambda}\left\langle\phi_{n_3}(R;\xi),\,\chi_{R\lesssim 1}U_R\psi_R\phi_1(R)\cdot\int_0^R [\phi_1(s)]^{-1}\,\widetilde{\mathcal{D}_1\phi_{<\lambda}}(s)\,ds\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar_3}}\\ &\lesssim \lambda^{-\frac{1}{2}+\frac{\delta}{2}}\cdot|c_2|\cdot\|\overline{x}_1\|_{S_0^{\hbar_1}}.\end{aligned}$$

This bound can again be l^1-summed over dyadic scales $\lambda \geq \hbar_3^{-2}$. It remains to deal with the term E in (6.53). Assuming $n_2 = \pm 1$ to be concrete(the case $n_2 = 0$ being handled analogously), this term can be written as

$$\begin{aligned} E &= \chi_{R\lesssim 1} U_R \psi_R \phi_1(R) \cdot \int_0^R [\phi_1(s)]^{-1} \cdot \frac{\pi}{4} s\, ds \cdot \int_0^\infty \chi_{\xi<\lambda} \overline{x}_1(\xi) \tilde{\rho}_1(\xi)\, d\xi \\ &- \chi_{R\lesssim 1} U_R(0) R \psi_{RR}(0) \phi_1(0) \cdot \int_0^R \frac{\pi}{4} s\, ds \cdot \int_0^\infty \chi_{\xi<\lambda} \overline{x}_1(\xi) \tilde{\rho}_1(\xi)\, d\xi \\ &- \chi_{R\lesssim 1} U_R(0) R \psi_{RR}(0) \phi_1(0) \cdot \int_0^R \frac{\pi}{4} s\, ds \cdot \int_0^\infty \chi_{\xi\geq\lambda} \overline{x}_1(\xi) \tilde{\rho}_1(\xi)\, d\xi \end{aligned}$$

The difference of the first two terms is a C^∞ function on $[0, \infty)$ of size $O_{|c_2| \|\overline{x}_1\|_{S_0^{\hbar_1}}}\left(R^4\right)$ near the origin, and its contribution is handled analogously to the one of the first term on the right in (6.53). There remains the contribution of the last term, which is a smooth function of size

$$O_{|c_2| \cdot \|\overline{x}_1\|_{S_0^{\hbar_1}}}\left(\lambda^{-1-\frac{\delta}{2}} R^3\right).$$

Calling this last term E_3, we then conclude that

$$\left|\chi_{\xi\simeq\lambda} \left\langle \phi_{n_3}(R;\xi),\, E_3 \right\rangle_{L^2_{R\,dR}}\right| \lesssim |c_2| \cdot \|\overline{x}_1\|_{S_0^{\hbar_1}} \cdot \lambda^{-\frac{7}{2}-\frac{\delta}{2}},$$

from which the desired $\|\cdot\|_{S_1^{\hbar_3}}$ follows as usual via Hölder and summation over dyadic scales $\lambda \geq \hbar_3^{-2}$.

The case when $\phi = c_1 \phi_{n_1}(R)$ but $\psi = \phi_{n_2}(R) \cdot \int_0^R [\phi_{n_2}(s)]^{-1} \tilde{\mathcal{D}}\phi(s)\, ds$ is handled analogously, as is the case when both ϕ, ψ are of the latter form. This concludes case *(1.2)* and thereby case *(1)*, i.e. bounding the term F_1.

(2): Bounding F_3. This is largely analogous to the bound for F_1, since $\left|\frac{\sin U}{R}\right|$ has the same asymptotics as $R \to 0$ or $R \to \infty$ as $|U_R|$, and ψ_R gets replaced by $(n_1 - n_2)\frac{\psi}{R}$ or $(n_1 - n_2)\frac{\phi}{R}$. As remarked earlier after the statement of the proposition, the precise choice of the coefficient $n_1 - n_2$ ensures that the function is never singular at $R = 0$. □

Corollary 6.26. *Assume that $\phi = \sum_{n\in\mathbb{Z}} \phi^{(n)}$, where $\phi^{(n)}$ is an angular momentum n function, and similarly $\psi = \sum_{m\in\mathbb{Z}} \psi^{(m)}$, and we assume the bound*

$$\sum_n \langle n \rangle^{12} \left\|\phi^{(n)}\right\|_{\tilde{S}_0^{(n)} \cap \tilde{S}_1^{(n)}} = \Lambda_1 < \infty, \quad \sum_m \langle m \rangle^{12} \left\|\psi^{(m)}\right\|_{\tilde{S}_0^{(n)} \cap \tilde{S}_1^{(n)}} = \Lambda_2 < \infty.$$

Then if

$$F_l = F_l(\phi, \psi) = \sum_{n,m} F_l(\phi^{(n)}, \psi^{(m)}) = \sum_k \sum_{n+m=k} F_l(\phi^{(n)}, \psi^{(m)}) =: \sum_k F_l^{(k)}$$

where F_l is one of the bilinear expressions in Proposition 6.24, with n, m replacing n_1, n_2 if $l = 3$, then for each k the function $F_l^{(k)}$ admits a third order Taylor expansion

$$\sum_{j=0}^{3} \gamma_j^{(l,k)} R^j$$

around 0 *and we have*

$$\sum_{k\in\mathbb{Z}}\langle k\rangle^{12}\left[\left\|\chi_{R\lesssim\tau}F_l^{(k)}-\chi_{R\lesssim 1}\sum_{j=0}^{3}\gamma_j^{(l,k)}R^j\right\|_{\tilde{S}_1^{k_{|\cdot|\geq 2}}}+\sum_j\left|\gamma_j^{(l,k)}\right|\right]\lesssim\prod_{j=1,2}\Lambda_j.$$

We can now control the second and third term on the right in (6.4). For this we have to keep in mind that the angular decompositions and estimates will be at the level of the functions $\varepsilon_{\pm}$, which in turn determine the functions $\varphi_{1,2}$ by means of

$$\varphi_1=\frac{1}{2}\left[\varepsilon_++\varepsilon_-\right],\quad \varphi_2=\frac{1}{2i}\left[\varepsilon_--\varepsilon_+\right].$$

Then we decompose

$$\varepsilon_+=\sum_{n\in\mathbb{Z}}\varepsilon_+(n)e^{in\theta},\quad \varepsilon_-=\sum_{n\in\mathbb{Z}}\varepsilon_-(m)e^{im\theta},\tag{6.54}$$

where $\varepsilon_+(n)$ is an angular momentum n function in reference to H_n^+, while $\varepsilon_-(m)$ is an angular momentum m function in reference to H_m^-. We shall assume the bounds

$$\sum_{n\in\mathbb{Z}}\langle n\rangle^{12}\left\|\varepsilon_+(n)\right\|_{\tilde{S}_0^{(n)}\cap\tilde{S}_1^{(n)}}+\sum_{m\in\mathbb{Z}}\langle m\rangle^{12}\left\|\varepsilon_-(m)\right\|_{\tilde{S}_0^{(m)}\cap\tilde{S}_1^{(m)}}=:\Lambda<\infty.\tag{6.55}$$

Furthermore, we shall impose the smallness condition

$$\tau^{1+\delta}\Lambda\ll 1.\tag{6.56}$$

Proposition 6.27. *Denote by* $F_{1,2}$ *the functions*

$$F_1=\chi_{R\ll\tau}\frac{2}{\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\left[U_R\sum_{j=1}^{2}\varphi_j\varphi_{j,R}-\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)U\sum_{j=1}^{2}\varphi_j\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\varphi_j\right]$$

$$F_2=\frac{2\sin U}{R^2\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\sum_{j=1}^{2}\varphi_j\varphi_{j,\theta}$$

Then assuming (6.55), (6.56), *and setting*

$$F_j=\sum_n F_j^{(n)}e^{in\theta},\quad j=1,2.$$

we have the bound

$$\sum_{n\in\mathbb{Z}}\langle n\rangle^{12}\left[\left\|F_j^{(n)}-\chi_{R\lesssim 1}\sum_{l=0}^{3}\gamma_l^{(n,j)}R^l\right\|_{\tilde{S}_1^{(n_{|\cdot|\geq 2})}}+\sum_{l=0}^{3}\left|\gamma_l^{(n,j)}\right|\right]\lesssim\Lambda^2.$$

where $\sum_{l=0}^{3}\gamma_l^{(n,j)}R^l$ *is the third order Taylor development of* $F_j^{(n)}$ *at* $R=0$. *The same bound obtains if we subtract from the* $\sum_{l=0}^{3}\gamma_l^{(n,j)}R^l$ *those terms* $\gamma_l^{(n,j)}R^l$ *with* $l\geq|n_{|\cdot|\geq 2}-1|, l\equiv(n_{|\cdot|\geq 2}-1)mod(2)$, *provided* $|n_{|\cdot|\geq 2}-1|\leq 3$.

Proof. This is a consequence of Corollary 6.26 and Corollary 6.20 after expanding $\frac{2}{\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}$ in a power series. □

The problem of controlling the source terms (6.3), (6.4) in the regular regime away from the light cone is then reduced to bounding the term $\mathcal{P}\varepsilon_\pm$.

We now consider the last remaining term, for which we have to rely on the identities (3.48)-(3.49). Here we encounter a number of terms which appear singular at the origin $R = 0$, but these are of course spurious singularities that disappear when taking the algebraic fine structure into account. To formulate the necessary estimates allowing us to deal with this term, we first have to render these cancellations explicit. We shall sort these according to their degree.

(i) Linear terms singular at $R = 0$. These are given by

$$-2\,(1 + a\,(\Pi_{\Phi^\perp}\varphi))\,\frac{\sin U\cos U}{R^2}\varphi_1 - 2\frac{\sin U}{R^2}\,(1 + a\,(\Pi_{\Phi^\perp}\varphi))\,\varphi_{2,\theta}.$$

Since $1 - \cos U = O(R^2)$, we can replace this expression by the sum of a simpler one and a term regular at the origin:

$$\begin{aligned}&-2\,(1 + a\,(\Pi_{\Phi^\perp}\varphi))\,\frac{\sin U\cos U}{R^2}\varphi_1 - 2\frac{\sin U}{R^2}\,(1 + a\,(\Pi_{\Phi^\perp}\varphi))\,\varphi_{2,\theta}\\ &= -\frac{2\sin U}{R^2}\,(1 + a\,(\Pi_{\Phi^\perp}\varphi))\,[\varphi_1 + \varphi_{2,\theta}] + \frac{2\sin U}{R^2}(1-\cos U)\,(1 + a\,(\Pi_{\Phi^\perp}\varphi))\,\varphi_1\end{aligned}$$

Here the first, singular term can be written in terms of $\varepsilon_\pm$ as follows:

$$\begin{aligned}\frac{2\sin U}{R^2}\,(1 + a\,(\Pi_{\Phi^\perp}\varphi))\,[\varphi_1 + \varphi_{2,\theta}] &= \frac{\sin U}{R^2}\,(1 + a\,(\Pi_{\Phi^\perp}\varphi))\,[\varepsilon_+ + i\varepsilon_{+,\theta}]\\ &\quad+ \frac{\sin U}{R^2}\,(1 + a\,(\Pi_{\Phi^\perp}\varphi))\,[\varepsilon_- - i\varepsilon_{-,\theta}]\end{aligned}$$

Observe that the expression $[\varepsilon_+ + i\varepsilon_{+,\theta}]$ vanishes at angular momentum $n = 1$, which is precisely the value for which ε_+ does not vanish at the origin $R = 0$. Similarly $[\varepsilon_- - i\varepsilon_{-,\theta}]$ vanishes at angular momentum $n = -1$, which is precisely the value at which ε_- does not vanish at $R = 0$. It follows that both expressions on the right are in fact non-singular at $R = 0$.

(ii) Quadratic terms singular at $R = 0$. These are

$$-\frac{1}{R^2}\left[\varphi_{1,\theta}^2 + \varphi_{2,\theta}^2\right] - \frac{1}{R^2}\left[\varphi_2^2 + \cos^2 U\varphi_1^2\right] + \frac{2\cos U}{R^2}\,(\varphi_{1,\theta}\varphi_2 - \varphi_1\varphi_{2,\theta})$$

Again we can split this into a 'principal term' and two additional terms which are clearly non-singular at $R = 0$:

$$\begin{aligned}&-\frac{1}{R^2}\left[\varphi_{1,\theta}^2 + \varphi_{2,\theta}^2\right] - \frac{1}{R^2}\left[\varphi_2^2 + \varphi_1^2\right] + \frac{2}{R^2}\,(\varphi_{1,\theta}\varphi_2 - \varphi_1\varphi_{2,\theta})\\ &+ \frac{1}{R^2}(1-\cos^2 U)\varphi_1^2 + \frac{2(\cos U - 1)}{R^2}\,(\varphi_{1,\theta}\varphi_2 - \varphi_1\varphi_{2,\theta})\end{aligned}$$

Here the first term can be written as

$$\begin{aligned}
&-\frac{1}{R^2}\left[\varphi_{1,\theta}^2+\varphi_{2,\theta}^2\right]-\frac{1}{R^2}\left[\varphi_2^2+\varphi_1^2\right]+\frac{2}{R^2}\left(\varphi_{1,\theta}\varphi_2-\varphi_1\varphi_{2,\theta}\right)\\
&=-\frac{1}{R^2}\left(\varphi_1+\varphi_{2,\theta}\right)^2-\frac{1}{R^2}\left(\varphi_2-\varphi_{1,\theta}\right)^2
\end{aligned}$$

and since

$$\varphi_1+\varphi_{2,\theta}=\frac{\varepsilon_+ + i\varepsilon_{+,\theta}}{2}+\frac{\varepsilon_- - i\varepsilon_{-,\theta}}{2},\quad \varphi_2-\varphi_{1,\theta}=i\frac{\varepsilon_+ + i\varepsilon_{+,\theta}}{2}-i\frac{\varepsilon_- - i\varepsilon_{-,\theta}}{2},$$

we again verify that the preceding quadratic expression is non-singular at $R=0$.

(iii) Higher order singular terms at $R=0$:

$$-\frac{1}{R^2}\left[\left(a\left(\Pi_{\Phi^\perp}\varphi\right)\right)_\theta\right]^2+\frac{2\sin U}{R^2}\left(a\left(\Pi_{\Phi^\perp}\varphi\right)\right)_\theta\varphi_2$$

Recall that $a\left(\Pi_{\Phi^\perp}\varphi\right)=\sqrt{1-\left|\Pi_{\Phi^\perp}\varphi\right|^2}-1$, and so

$$\left(a\left(\Pi_{\Phi^\perp}\varphi\right)\right)_\theta=-\frac{\left(\left|\Pi_{\Phi^\perp}\varphi\right|^2\right)_\theta}{\sqrt{1-\left|\Pi_{\Phi^\perp}\varphi\right|^2}}.$$

Recalling Remark 6.25 we see that this expression vanishes at $R=0$, and hence the above terms are all non-singular at $R=0$.

Finally, control over the term $\mathcal{P}\varepsilon_\pm$ in the region away from the light cone will follow from

Proposition 6.28. *Assume* (6.54), (6.55), (6.56). *Define the functions* ((3.48)-(3.49))

$$\begin{aligned}
F_1=\chi_{R\ll\tau}\Bigg[&-\left(2a\left(\Pi_{\Phi^\perp}\varphi\right)+\left(a\left(\Pi_{\Phi^\perp}\varphi\right)\right)^2\right)\left(U_R^2-\left[\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)U\right]^2+\frac{\sin^2 U}{R^2}\right)\\
&+\left[\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\left(a\left(\Pi_{\Phi^\perp}\varphi\right)\right)\right]^2-\left[\partial_R\left(a\left(\Pi_{\Phi^\perp}\varphi\right)\right)\right]^2\\
&+\sum_{j=1,2}\left[\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\varphi_j\right]^2-\sum_{j=1,2}\left[\partial_R\varphi_j\right]^2\\
&+2\phi_1\left(\left(a\left(\Pi_{\Phi^\perp}\varphi\right)\right)_R U_R-\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\left(a\left(\Pi_{\Phi^\perp}\varphi\right)\right)\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)U\right)\\
&+\left(U_R^2-\left[\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)U\right]^2\right)\varphi_1^2\\
&+2\left(1+a\left(\Pi_{\Phi^\perp}\varphi\right)\right)\left(\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)U\cdot\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\varphi_1-U_R\cdot\varphi_{1,R}\right)\Bigg]\cdot\varepsilon_\pm,
\end{aligned}$$

$$\begin{aligned}
F_2=\chi_{R\ll\tau}\Bigg[&\frac{2\sin U}{R^2}\left(1-\cos U\right)\left(1+a\left(\Pi_{\Phi^\perp}\varphi\right)\right)\varphi_1\\
&+\frac{1}{R^2}\left(1-\cos^2 U\right)\varphi_1^2+\frac{2\left(\cos U-1\right)}{R^2}\left(\varphi_{1,\theta}\varphi_2-\varphi_1\varphi_{2,\theta}\right)\Bigg]\cdot\varepsilon_\pm,
\end{aligned}$$

$$F_3 = \chi_{R\ll\tau}\Bigg[\frac{\sin U}{R^2}\left(1+a\left(\Pi_{\Phi^\perp}\varphi\right)\right)[\varepsilon_+ + i\varepsilon_{+,\theta}] + \frac{\sin U}{R^2}\left(1+a\left(\Pi_{\Phi^\perp}\varphi\right)\right)[\varepsilon_- - i\varepsilon_{-,\theta}]\Bigg]\cdot\varepsilon_\pm,$$

$$F_4 = \chi_{R\ll\tau}\left[-\frac{1}{R^2}\left(\varphi_1+\varphi_{2,\theta}\right)^2 - \frac{1}{R^2}\left(\varphi_2-\varphi_{1,\theta}\right)^2\right]\cdot\varepsilon_\pm,$$

$$F_5 = \chi_{R\ll\tau}\left[\left(a\left(\Pi_{\Phi^\perp}\varphi\right)\right)_\theta\right]\cdot\varepsilon_\pm = -\frac{\left(|\Pi_{\Phi^\perp}\varphi|^2\right)_\theta}{\sqrt{1-|\Pi_{\Phi^\perp}\varphi|^2}}\cdot\varepsilon_\pm.$$

Then setting

$$F_j = \sum_n F_j^{(n)}e^{in\theta},\quad j=1,\dots,5,$$

we have the bound

$$\sum_{n\in\mathbb{Z}}\langle n\rangle^{12}\left[\left\|F_j^{(n)} - \chi_{R\lesssim 1}\sum_{l=0}^{3}\gamma_l^{(n,j)}R^l\right\|_{\tilde{S}_1^{(n_{|\cdot|\geq 2})}} + \sum_{l=0}^{3}\left|\gamma_l^{(n,j)}\right|\right] \lesssim \Lambda^2.$$

where $\sum_{l=0}^{3}\gamma_l^{(n,j)}R^l$ *is the third order Taylor development of* $F_j^{(n)}$ *at* $R=0$. *The same bound obtains if we subtract from the* $\sum_{l=0}^{3}\gamma_l^{(n,j)}R^l$ *those terms* $\gamma_l^{(n,j)}R^l$ *with* $l\geq|n_{|\cdot|\geq 2}-1|, l\equiv(n_{|\cdot|\geq 2}-1)mod(2)$, *provided* $|n_{|\cdot|\geq 2}-1|\leq 3$.

We observe that Proposition 6.23, Proposition 6.27 and Proposition 6.28 complete control over the source terms in the non-singular region for the $|n|\geq 2$ angular momentum modes.

7. Estimates for singular sources

7.1. **Description of the shock.** Here we briefly recall the function spaces from [10, 18] which in analogy to Definition 6.21 describe the correction ϵ used to build U close to the light cone:

Definition 1. *For* $i\in\mathbb{N}$, *let* $j(i)=i$ *if* ν *is irrational, respectively* $j(i)=2i^2$ *if* ν *is rational. Then*

- $\mathcal{Q}$ *is the algebra of continuous functions* $q:[0,1]\to\mathbb{R}$ *with the following properties:*
  (i) q *is analytic in* $[0,1)$ *with even expansion around* $a=0$.
  (ii) near $a=1$ *we have an absolutely convergent expansion of the form*

$$q(a) = q_0(a) + \sum_{i=1}^{\infty}(1-a)^{\beta(i)+\frac{1}{2}}\sum_{j=0}^{j(i)}q_{i,j}(a)\left(\log(1-a)\right)^j + \sum_{i=1}^{\infty}(1-a)^{\tilde\beta(i)+\frac{1}{2}}\sum_{j=0}^{j(i)}\tilde q_{i,j}(a)\left(\log(1-a)\right)^j$$

  with analytic coefficients $q_0, q_{i,j}$, *and* $\beta(i)=i\nu$, $\tilde\beta(i)=\nu i+\frac{1}{2}$.
- $\mathcal{Q}_n$ *is the algebra which is defined similarly, but also requiring* $q_{i,j}(1)=0$ *if* $i\geq 2n+1$.

- $Q_n^{(N)}$ *is the vector space of functions in* Q_n *for which* $q_{i,j}(a) = 0$ *for* $i \geq N$.

For later purposes, we also define the space of functions obtained by differentiating Q_n:

Definition 2. *Define* Q' *as in the preceding definition but replacing* $\beta(i)$ *by* $\beta'(i) := \beta(i) - 1$*, and similarly for* Q'_n.

Definition 3. *Pick t sufficiently small such that both* b_1, b_2 *(recall their definitions in Definition 6.21), when restricted to the light cone* $r \leq t$ *are of size at most* b_0.

- $S^m(R^k(\log R)^l, Q_n)$ *is the class of analytic functions* $v : [0,\infty) \times [0,1) \times [0, b_0]^2 \to \mathbb{R}$ *so that*
  (i) v is analytic as a function of R, b_1, b_2,
  $$v : [0,\infty) \times [0, b_0]^2 \to Q_n$$
  (ii) v vanishes to order m at $R = 0$.
  (iii) v admits a convergent expansion at $R = \infty$,
  $$v(R, \cdot, b_1, b_2) = \sum_{\substack{0 \leq j \leq l+i \\ i \geq 0}} c_{ij}(\cdot, b_1, b_2) R^{k-i} (\log R)^j$$
  where the coefficients $c_{ij} : [0, b_0]^2 \to Q_n$ *are analytic with respect to* $b_{1,2}$.
- $IS^m(R^k(\log R)^l, Q_n)$ *is the class of analytic functions w inside the cone* $r < t$ *which can be represented as*
  $$w(t,r) = v(R, a, b_1, b_2), \quad v \in S^m(R^k(\log R)^l, Q_n)$$
  and $t > 0$ *sufficiently small.*
- *Define* $IS^m(R^k(\log R)^l, Q_n^{(N)})$ *analogously by replacing* Q_n *by* $Q_n^{(N)}$.

Then the core content of [10, 18], customized for the application we have in mind, can be encapsulated by the following theorem:

Theorem 7.1. *Let* $\nu > 0$*. Then, given* $N > 0$*, there exist* $t_0 = t_0(\nu, N) > 0$*,* $M = M(\nu, N)$*,* $P = P(\nu)$ *and a co-rotational blow up solution* $U = Q(\lambda(t)r) + \epsilon(t,r)$*,* $\lambda(t) = t^{-1-\nu}$*, on* $(0, t_0] \times \mathbb{R}^2$*, and such that* $\epsilon(t,r)$ *admits the following fine structure:*

$$\epsilon = \epsilon_1 + \epsilon_2,$$

where we have[3] *(with* $R = \lambda(t)r$*,* $\tau = \int_t^\infty \lambda(s)\,ds$*)*

$$\|\epsilon_2(t,\cdot)\|_{H^{5+\delta}_{RdR}} \lesssim \frac{1}{\tau^N}, \quad \tau = \tau(t) \in [\tau_0, \infty), \quad \tau_0 = \int_{t_0}^\infty \lambda(s)\,ds,$$

for the 'small but unstructured term' ϵ_2*, while we have*

$$\epsilon_1 \in \sum_{k=1}^M \frac{1}{(t\lambda)^{2k}} IS^3\left(R(\log R)^{2k-1}, Q_{k-1}^{(P)}\right) + \sum_{k=1}^M \frac{1}{(t\lambda)^{2k+2}} IS^3\left(R^3(\log R)^{2k-1}, Q_k^{(P)}\right).$$

[3] The condition on ϵ_2 is chosen to conform with our requirements; one could impose any H^s_{RdR} condition here.

The additional restriction here replacing Q_k by $Q_k^{(P)}$ here is of course simply a reflection of the fact that those terms the expansion of a general function $q(a) \in Q_k$ with sufficiently large power of $(1-a)$ are automatically in $H^{5+\delta}_{R\,dR}$.

In order to fit the functions described in the preceding into our calculus set up on the Fourier side, we have to implement a concise translation of the algebraic structure on the physical side to the (distorted) Fourier side, for each angular momentum n. This is what we do in the next subsection. It is also here where we start to keep track of the *temporal decay* (as measured in terms of the variable τ), which will start to play an important role for the final iterative scheme, of course.

7.2. **Description of the shock on the distorted Fourier side I: prototypical expansions.** In order to handle the source terms near the light cone, we shall have to introduce a suitable algebra of functions which contains all the possible singular terms, is formulated on the distorted Fourier side, and flexible enough that it is preserved under certain para-differential operations, as well as the transference operators and the solution operator for the wave equation on the Fourier side. Due to the somewhat complicated nature of the function spaces concerned, we do this in a two-step fashion. First, we introduce a prototype which handles certain key aspects (it gives the correct translation from the physical singularity as detailed in the preceding subsection to the Fourier side at fixed times), but is still not adequate due to its incompatibility with the solution operator for the wave operator on the Fourier side. Once we have studied some of the key properties in this simplified setting, we can finally introduce the spaces that will overcome all hurdles. We already introduce aspects of the somewhat complicated temporal bounds for these function spaces, although strictly speaking this is not important yet at this stage. It will play a very prominent role later for the multilinear estimates, of course.

Definition 7.2. *We call a function $\overline{x}(\tau,\xi)$, $\xi \in [0,\infty)$, an prototype singular part at angular momentum $n, |n| \geq 2$, provided it allows an expansion (here all cutoffs are smooth localizers to the indicated regions)*

$$
\begin{aligned}
&\overline{x}(\tau,\xi)\\
&= \sum_{\pm}\sum_{k=1}^{N}\sum_{j=0}^{N_1}\chi_{\xi\geq 1}\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot a_{k,j}(\tau) + \sum_{\pm}\sum_{k=1}^{N}\sum_{j=0}^{N_1}\chi_{\xi\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i(\nu\tau\xi^{\frac12}+\hbar^{-1}\rho(x_\tau;\alpha,\hbar))}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot\tilde{a}_{k,j}(\tau)\\
&+ \sum_{\pm}\sum_{l=1}^{7}\sum_{k=1}^{N}\sum_{j=0}^{N_1}\chi_{\xi\geq 1}\hbar^{-1}\langle\hbar^2\xi\rangle^{-\frac{l}{4}}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot F_{l,k,j}(\tau,\xi)\\
&\qquad + \sum_{\pm}\sum_{l=1}^{7}\sum_{k=1}^{N}\sum_{j=0}^{N_1}\chi_{\xi\geq\hbar^{-2}}\hbar^{-1}\langle\hbar^2\xi\rangle^{-\frac{l}{4}}\frac{e^{\pm i(\nu\tau\xi^{\frac12}+\hbar^{-1}\rho(x_\tau;\alpha,\hbar))}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot\tilde{F}_{l,k,j}(\tau,\xi)
\end{aligned}
$$

Here $N_1 = N_1(N,\nu)$ is sufficiently large, $x_\tau = \xi^{\frac12}\hbar\cdot\nu\tau$ and ρ is as in Lemma 4.13, and the functions $F_{k,j}(\tau,\xi), F_{l,k,j}(\tau,\xi), \tilde{F}_{l,k,j}(\tau,\xi)$ have the following properties:

- *Pointwise bounds with (weak) temporal decay $\left|a_{k,j}(\tau)\right| + \left|\tilde{a}_{k,j}(\tau)\right| \lesssim (\log\tau)^{N_1-j}\cdot\tau^{-1-\nu}$, as well as $\left|F_{l,k,j}(\tau,\xi)\right| + \left|\tilde{F}_{l,k,j}(\tau,\xi)\right| \lesssim (\log\tau)^{N_1-j}\,\tau^{-1-\nu}$.*

- *Symbol type behavior: with* $k_1 \in \{0,1\}$, $k_2 \in \{0,1,\dots,5\}$

$$\left|\partial_\tau^{k_1} a_{k,j}(\tau)\right| + \left|\partial_\tau^{k_1} \tilde{a}_{k,j}(\tau)\right| \lesssim (\log\tau)^{N_1-j}\cdot \tau^{-1-\nu-k_1}$$

$$\left|\partial_\tau^{k_1}\partial_\xi^{k_2} F_{l,k,j}(\tau,\xi)\right| + \left|\partial_\tau^{k_1}\partial_\xi^{k_2} \tilde{F}_{l,k,j}(\tau,\xi)\right| \lesssim (\log\tau)^{N_1-j}\cdot \tau^{-1-\nu-k_1}\cdot \xi^{-k_2},$$

as well as the 'closure bounds'

$$\left\|\xi^{5+\delta-k_1}\partial_\tau^{k_1}\partial_\xi^{5-k_1} F_{l,k,j}(\tau,\xi)\right\|_{\dot{C}^\delta_\xi} + \left\|\xi^{5+\delta-k_1}\partial_\tau^{k_1}\partial_\xi^{5-k_1} \tilde{F}_{l,k,j}(\tau,\xi)\right\|_{\dot{C}^\delta_\xi} \lesssim (\log\tau)^{N_1-j}\cdot \tau^{-1-\nu-k_1},$$

where for some $\delta \in (0,1)$ *we set*

$$\|g\|_{\dot{C}^\delta} := \sup_{x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\delta}$$

We shall refer to the first two sums

$$\sum_\pm \sum_{k=1}^N \sum_{j=0}^{N_1} \hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot a_{k,j}(\tau), \quad \sum_\pm \sum_{k=1}^N \sum_{j=0}^{N_1} \chi_{\xi\geq 1}\hbar^{-1}\frac{e^{\pm i(\nu\tau\xi^{\frac12}+\hbar^{-1}\rho(x_\tau;\alpha,\hbar))}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot \tilde{a}_{k,j}(\tau)$$

as the principal singular part, and the second sum as the connecting singular part. This is because the second sum involves terms which are still too rough to be included into the space $S_0^\hbar$, *but still much less singular than the principal singular part. Here 'singular' refers of course to the physical side, and is measured in terms of decay of the Fourier transform with respect to* ξ *for large frequencies.*

We say that a function $f(\tau,R)$ *is a function at angular momentum n,* $|n|\geq 2$, *and prototypical singular part, provided it can be written as*

$$f(\tau,R) = \int_0^\infty \phi_n(R;\xi)\cdot \overline{x}(\tau,\xi)\cdot \rho_n(\xi)\,d\xi,$$

and $\overline{x} = \overline{x}_1 + \overline{x}_2$ *where* $\overline{x}_1 \in S_0^\hbar$ *for each* τ, *while* $\overline{x}_2$ *is a prototypical singular part at angular momentum n. Finally, we shall say that the principal singular part is of restricted type, provided we have the more specific structure*

$$a_{k,i}(\tau) = \sum_{l=i}^{N_1} c_{l,k,i}\cdot \tau^{-k(1+\nu)}\cdot(\log\tau)^{l-i} + b_{k,i}(\tau), \quad k\in\{1,2\}$$

for certain constants $c_{l,k,i}$, *where* $\left|b_{k,i}(\tau)\right| + \left|\tau\cdot b'_{k,i}(\tau)\right| \lesssim \tau^{-3-\nu}(\log\tau)^{N_1}$, *and similarly for* $\tilde{a}_{k,i}(\tau)$, *and for* $k\geq 3$ *we have*

$$\left|a_{k,i}(\tau)\right| + \left|\tau\cdot a'_{k,i}(\tau)\right| \lesssim \tau^{-3-\nu}(\log\tau)^{N_1}.$$

Remark 7.3. *It is important that the functions* $a_{k,i}(\tau), \tilde{a}_{k,i}(\tau)$ *do not depend on* ξ. *A similar property for the final admissible function space will play a key role when applying the modulation techniques for the exceptional modes* $n = 0, \pm 1$.

Remark 7.4. *The phase* $e^{\pm i(\nu\tau\xi^{\frac12}+\hbar^{-1}\rho(x_\tau;\alpha,\hbar))}$ *appears quite natural in light of the asymptotic properties of the angular momentum n Fourier basis in the oscillatory regime and near* $R=\nu\tau$. *However, the properties of the Duhamel parametrix will force a 'deformation' of the function* $\hbar^{-1}\rho(x_\tau;\alpha,\hbar)$ *toward smaller and smaller* *values, effectively making the phase converge toward $e^{\pm i\nu\tau\xi^{\frac{1}{2}}}$. Thus the 'true phase' that will appear in the admissible function space defined below will in fact be an interpolate between these two phases, involving a continuously varying family of phases. The fact that we can work with such in a certain sense 'unnatural phases' comes from an important monotonicity property of the function ρ with respect to the variable x, as well as the fact that we always restriction attention to the interior of the light cone $0 \le R < \nu\tau$.*

The preceding concept of function comes equipped with a natural concept of 'norm':

Definition 7.5. *Let $\overline{x}(\tau, \xi)$ be a prototype singular part at angular momentum $|n| \ge 2$. Then define*

$$
\begin{aligned}
\|\overline{x}\|_{proto} := & \sum_{k_1\in\{0,1\}} \sum_{k=1}^{N} \sum_{i=0}^{N_1} \left\| \tau^{1+\nu+k_1} (\log\tau)^{-N_1+i} \partial_\tau^{k_1} a_{k,i}(\tau) \right\|_{L^\infty([\tau_0,\infty))} \\
& + \sum_{k_1\in\{0,1\}} \sum_{l=1}^{7} \sum_{k=1}^{N} \sum_{i=0}^{N_1} \left\| \tau^{1+\nu+k_1} (\log\tau)^{-N_1+i} \partial_\tau^{k_1} \tilde{a}_{k,i}(\tau) \right\|_{L^\infty([\tau_0,\infty))} \\
& + \sum_{k_1\in\{0,1\}} \sum_{l=1}^{7} \sum_{k_2\in\{0,\dots,5\}} \left\| \tau^{1+\nu+k_1} (\log\tau)^{-N_1+i} \xi_2^{k} \partial_\tau^{k_1} \partial_\xi^{k_2} F_{l,k,i}(\tau,\xi) \right\|_{L^\infty_{\tau,\xi}([\tau_0,\infty)\times[0,\infty))} \\
& + \sum_{k_1\in\{0,1\}} \sum_{l=1}^{7} \sum_{k=1}^{N} \sum_{i=0}^{N_1} \sum_{k_2\in\{0,\dots,5\}} \left\| \tau^{1+\nu+k_1} (\log\tau)^{-N_1+i} \xi_2^{k} \partial_\tau^{k_1} \partial_\xi^{k_2} \tilde{F}_{l,k,i}(\tau,\xi) \right\|_{L^\infty_{\tau,\xi}([\tau_0,\infty)\times[0,\infty))} \\
& + \sum_{k_1\in\{0,1\}} \sum_{l=1}^{7} \sum_{k=1}^{N} \sum_{i=0}^{N_1} \left\| \sup_{\lambda>0} \left\| \tau^{1+\nu+k_1} (\log\tau)^{-N_1+i} \xi_2^{k} \partial_\tau^{k_1} \partial_\xi^{k_2} F_{l,k,i}(\tau,\xi) \right\|_{\dot{C}^{\delta_l}(\xi\simeq\lambda)} \right\|_{L^\infty_\tau([\tau_0,\infty))} \\
& + \sum_{k_1\in\{0,1\}} \sum_{l=1}^{7} \sum_{k=1}^{N} \sum_{i=0}^{N_1} \left\| \sup_{\lambda>0} \left\| \tau^{1+\nu+k_1} (\log\tau)^{-N_1+i} \xi_2^{k} \partial_\tau^{k_1} \partial_\xi^{k_2} \tilde{F}_{l,k,i}(\tau,\xi) \right\|_{\dot{C}^{\delta_l}(\xi\sim\lambda)} \right\|_{L^\infty_\tau([\tau_0,\infty))}
\end{aligned}
$$

In case the function $\overline{x}$ has principal part of restricted type, we also use the norm $\|\overline{x}\|_{proto(r)}$, where (keeping in mind the preceding definition) we replace the first expression on the right by

$$
\sum_{k_1\in\{0,1\}} \sum_{k=1}^{N} \sum_{i=0}^{N_1} \left\| \tau^{3+\nu+k_1} (\log\tau)^{-N_1+i} \partial_\tau^{k_1} b_{k,i}(\tau) \right\|_{L^\infty([\tau_0,\infty))} + \sum_{k=1}^{N} \sum_{i=0}^{N_1} \sum_{l=1}^{7} \left| c_{l,k,i} \right|.
$$

The first order of the day shall be to translate the information of the preceding definition to the physical side, i.e. identify a vector space of functions which correspond to the above Fourier description:

Lemma 7.6. *Assume that $\overline{x}$ is a prototypical singular part at angular momentum $n, |n| \ge 2$. Then the associated function*

$$
f(\tau, R) := \int_0^\infty \phi_n(R;\xi) \cdot \overline{x}(\tau,\xi) \cdot \rho_n(\xi)\, d\xi
$$

restricted to the light cone $R \le \nu\tau$, can be decomposed as

$$
f = f_1 + f_2 + f_3
$$

where $f_1 = f_1(\tau, R)$ is a C^∞-function supported in $\nu\tau - R \gtrsim 1$ and satisfying

$$\nabla_R^{k_2}\partial_\tau^{k_1} f_1(\tau,R) \lesssim \hbar^{-1}(\log\tau)^{N_1}\cdot\tau^{-\frac{3}{2}-\nu}|\nu\tau - R|^{-5} \quad k_1\in\{0,1\},\quad k_1+k_2\le 5,$$

while $f_2 = \sum_{l=1}^{8} f_{2l}$ where we have the explicit form

$$f_{2l}(\tau,R) = \chi_{|\nu\tau-R|\lesssim\hbar}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\frac{G_{k,l,i}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar^{-\frac{l+1}{2}}[\nu\tau-R]^{\frac{l}{2}+k\nu}(\log(\nu\tau-R))^i$$

Here the function $G_{k,i}(\tau,x)$ has symbol type behavior with respect to R, x, as follows:

$$\left|\partial_\tau^{k_1}\partial_x^{k_2}G_{k,l,i}(\tau,x)\right| \lesssim (\log\tau)^{N_1-i}\cdot\tau^{-1-\nu-k_1}x^{-k_2},\quad k_1\in\{0,1\},\quad k_2\in\{0,\dots,5\},\quad k_1+k_2\le 5$$

and we have the bound

$$\left\|x^{5+\delta-k_1}\partial_\tau^{k_1}\partial_x^{5-k_1}G_{k,l,i}(\tau,x)\right\|_{\dot{C}^\delta} \lesssim (\log\tau)^{N_1-i}\cdot\tau^{-1-\nu-k_1},\quad k_1\in\{0,1\}.$$

Fix the $+$-sign for the first oscillatory phase, meaning the $-$ sign for the second one.

Finally, the remaining function f_3 is C^∞ and supported in $|\nu\tau - R| \lesssim 1$ and satisfies

$$\|f_3\|_{\tilde{S}_0^{(n)}} \lesssim (\log\tau)^{N_1}\cdot\tau^{-1-\nu},\quad \|\partial_\tau f_3\|_{\tilde{S}_1^{(n)}} \lesssim (\log\tau)^{N_1}\cdot\tau^{-1-\nu}.$$

Moreover, we have the bounds

$$\left|\partial_R^k f_3\right| \lesssim (\log\tau)^{N_1}\tau^{-\frac{3}{2}-\nu}\cdot\hbar^{-1}\min\{(\nu\tau-R)^{-k},\hbar^{-k}\},$$
$$\left|\partial_R^k\partial_\tau f_3\right| \lesssim (\log\tau)^{N_1}\tau^{-\frac{3}{2}-\nu}\cdot\hbar^{-1}\min\{(\nu\tau-R)^{-k-1},\hbar^{-k-1}\},$$
$$0\le k\le 5.$$

Proof. Due to linearity, we can separately consider the contributions of the different parts constituting $\overline{x}$. We deal with the contribution of $\sum_\pm\sum_{k=1}^N\sum_{j=0}^{N_1}\chi_{\xi\ge1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot a_{k,j}(\tau)$, and later explain what changes are necessary to handle the terms in the second to the fourth sum constituting $\overline{x}(\tau,\xi)$, according to the definition. We split the Fourier integral into a number of regions reflecting the oscillatory/non-oscillatory nature of $\phi_n(R;\xi)$. Also we may as well omit the factors $a_{k,j}(\tau)$, modifying the required bounds accordingly. We split the integral for $f(\tau, R)$ into a number of regions.

(1) The rapidly decaying region: $R\xi^{\frac{1}{2}}\hbar < \frac{x_t}{2}$. Let $\psi(x)$ be a smooth function which is identically 1 for $x \le \frac{1}{2}$ and vanishes beyond $x = 1$. Then consider the integral

$$\int_0^\infty\psi\left(\frac{2x}{x_t}\right)\phi_n(R;\xi)\cdot\chi_{\xi\ge1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot\rho_n(\xi)\,d\xi,\quad x = R\xi^{\frac{1}{2}}\hbar.$$

Taking advantage of the asymptotics for $\phi_n(R;\xi)$ in the non-oscillatory regime away from the turning point x_t, this becomes

$$\int_0^\infty\psi\left(\frac{2x}{x_t}\right)\hbar^{\frac{1}{3}}x^{-\frac{1}{2}}q^{-\frac{1}{4}}(\zeta)\cdot\mathrm{Ai}\left(\hbar^{-\frac{2}{3}}\zeta\right)\cdot(1+\hbar a_0(-\zeta;\alpha,\hbar))\chi_{\xi\ge1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot\rho_n(\xi)\,d\xi$$

We claim that the preceding function is smooth with respect to R and τ, and that the function as well as its derivatives can be bounded by $\lesssim_N \tau^{-N} \lesssim |R-\nu\tau|^{-N}$ for any N. To see this, distinguish between the cases

$R \ll 1$ and $R \gtrsim 1$, and further between $\alpha = \xi^{\frac{1}{2}}\hbar \gtrsim 1$ and $\alpha \ll 1$, the latter requiring in principle a further decomposition, which we suppress. In case $\alpha \gtrsim 1$, we use Lemma 4.17 for the description of τ, and perform integration by parts with respect to $\xi^{\frac{1}{2}}$, where we also have to take advantage of Lemmas 4.30. For the case $\alpha \ll 1$, one uses either Lemma 4.24 for the case $R \gg 1$ or Lemma 4.22, Lemma 4.20 in case $R \lesssim 1$. Also, one uses Proposition 4.32 for the symbol behavior of the spectral measure. It follows from these sources that the worst case occurs when $\partial_{\xi^{\frac{1}{2}}}$ hits $\mathrm{Ai}\left(\hbar^{-\frac{2}{3}}\zeta\right)$, which in light of the Airy function asymptotics leads to a factor

$$\hbar^{-1} \cdot x^{-1} \cdot R\hbar = \frac{R}{x} = \frac{1}{\xi^{\frac{1}{2}}\hbar}.$$

Then performing N times integration by parts with respect to $\xi^{\frac{1}{2}}$, we get the bound

$$\left|\int_0^\infty \psi\left(\frac{2x}{x_t}\right)\hbar^{\frac{1}{3}}x^{-\frac{1}{2}}q^{-\frac{1}{4}}(\tau)\cdot \mathrm{Ai}\left(\hbar^{-\frac{2}{3}}\zeta\right)\cdot(1+\hbar a_0(-\zeta;\alpha,\hbar))\chi_{\xi\geq 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot\rho_n(\xi)\,d\xi\right|$$
$$\lesssim_N \tau^{-N}\int_0^\infty (\xi^{\frac{1}{2}}\hbar)^{-N}\cdot x^{\hbar^{-1}-3}\cdot\frac{\chi_{\xi\geq 1}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\,d\xi \lesssim \tau^{-N}$$

since we have $\sup_{0<\hbar\leq\frac{1}{3}}\hbar^{-N}x^{\hbar^{-1}-3} \lesssim_N 1$. Derivatives with respect to R, τ are handled similarly, as the additional factors $\xi^{\frac{1}{2}}$ can be easily absorbed.

(2) Region near the turning point, still non-oscillatory: $x_t + \hbar^{\frac{2}{3}} > R\xi^{\frac{1}{2}}\hbar \geq \frac{x_t}{2}$. Let $\varphi(x)$ be non-vanishing on $[-1,1]$ and zero on $[-2,2]^c$, chosen in such a way that

$$\sum_\lambda \varphi\left(\frac{x-\left[x_t-\lambda\hbar^{\frac{2}{3}}\right]}{\lambda\hbar^{\frac{2}{3}}}\right) = 1, \quad x\in(0, x_t+\hbar^{\frac{2}{3}}].$$

where λ ranges over $0\cup\{2^j\}_{j\geq 0}$. Then, in light of the fact that the above asymptotics for $\phi_n(R;\xi)$ are still valid in this regime, but with a different law for τ, we reduce to bounding

$$\sum_\lambda\int_0^\infty\left[1-\psi\left(\frac{2x}{x_t}\right)\right]\cdot\varphi\left(\frac{x-\left[x_t-\lambda\hbar^{\frac{2}{3}}\right]}{\lambda\hbar^{\frac{2}{3}}}\right)\cdot\Xi(R;\xi,\hbar)\chi_{\xi\geq 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\rho_n(\xi)\,d\xi, \quad x = R\xi^{\frac{1}{2}}\hbar.$$

where we set

$$\Xi(R;\xi,\hbar) = \hbar^{\frac{1}{3}}x^{-\frac{1}{2}}q^{-\frac{1}{4}}(\zeta)\cdot\mathrm{Ai}\left(\hbar^{-\frac{2}{3}}\right)\cdot(1+\hbar a_0(-\zeta;\alpha,\hbar)).$$

Also, ζ is described by Lemma 4.11. Then since

$$\hbar^{-\frac{2}{3}}\partial_{\xi^{\frac{1}{2}}}\zeta \simeq \hbar^{\frac{1}{3}}R, \quad R\xi^{\frac{1}{2}}\hbar\simeq 1$$

in the support of the integrand, we easily infer using integration by parts and taking into account the asymptotics of Airy functions, that

$$\left|\int_0^\infty\left[1-\psi\left(\frac{2x}{x_t}\right)\right]\cdot\varphi\left(\frac{x-\left[x_t-\lambda\hbar^{\frac{2}{3}}\right]}{\lambda\hbar^{\frac{2}{3}}}\right)\cdot\Xi(R;\xi,\hbar)\chi_{\xi\geq 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\rho_n(\xi)\,d\xi\right|$$

$$\lesssim_N \left(\hbar^{\frac{1}{3}}R\right)^N \cdot \tau^{-N}\cdot e^{-c\lambda^{\frac{3}{2}}} \lesssim \left(\frac{R}{\tau}\right)^N \cdot \left(\frac{1}{R\xi^{\frac{1}{2}}}\right)^{\frac{N}{3}} \cdot e^{-c\lambda^{\frac{3}{2}}} \lesssim \tau^{-\frac{N}{3}}\cdot \xi^{-\frac{N}{6}}\cdot e^{-c\lambda^{\frac{3}{2}}}.$$

Again derivatives with respect to R and τ are easily absorbed by the gain in $\xi^{-\frac{1}{2}}$, and the rapid exponential decay in λ allows for summation over dyadic λ. Here we also used the fact that inside the cone $r \le t$ we have $R \le \nu\tau < \tau$. The contributions from the derivatives on $a_0(-\zeta;\alpha,\hbar)$ are handled similarly by Lemma 4.30 and Lemma 4.31

(3) Region near the turning point, oscillatory regime: $x_t + \hbar^{\frac{2}{3}} \le R\xi^{\frac{1}{2}}\hbar < (1+\gamma)x_t, 0<\gamma \ll 1$. Here the asymptotics of the $\phi_n(R;\xi)$ are of the oscillatory kind, with a weaker decay. Proceeding as in the preceding case and letting ψ a smooth function which equals 1 on $[0,1]$ but vanishes beyond 2, we reduce to bounding (with λ again taking dyadic values)

$$\sum_{\lambda\ge 1}\int_0^\infty \psi\left(\frac{x}{(1+\gamma)x_t}\right)\cdot\varphi\left(\frac{x-[x_t+\lambda\hbar^{\frac{2}{3}}]}{\lambda\hbar^{\frac{2}{3}}}\right)\cdot\Xi(R;\xi,\hbar)\chi_{\xi\ge 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\rho_n(\xi)\,d\xi, \quad x = R\xi^{\frac{1}{2}}\hbar,$$

where we set this time

$$\Xi(R;\xi,\hbar) = \hbar^{\frac{1}{3}}x^{-\frac{1}{2}}q^{-\frac{1}{4}}(\zeta)\mathrm{Re}\left[2\,(a(\xi))\left(\mathrm{Ai}(-\hbar^{-\frac{2}{3}}\zeta) - i\mathrm{Bi}(-\hbar^{-\frac{2}{3}}\zeta)\right)(1+\hbar\overline{a_1(\zeta;\alpha)})\right]$$

recalling Proposition 4.32. Then using the oscillatory asymptotics of the complex Airy function, we get a phase $e^{\pm\frac{2}{3}i\hbar^{-1}\zeta^{\frac{3}{2}}}$, and we have (recall Lemma 4.11)

$$\begin{aligned}\partial_{\xi^{\frac{1}{2}}}\left[\hbar^{-1}\zeta^{\frac{3}{2}}\right] &= \frac{3}{2}\hbar^{-1}\zeta^{\frac{1}{2}}\cdot\left[R\hbar\cdot\Phi(x;\alpha,\hbar) + (1+O(x-x_t))\cdot O(\hbar(1+\alpha)^{-3}) + O(x-x_t)\cdot R\hbar\right]\\ &\simeq \zeta^{\frac{1}{2}}\cdot R,\end{aligned}$$

where we have used that $\left|O(\hbar(1+\alpha)^{-3})\right| \lesssim \hbar\cdot\left(\hbar\xi^{\frac{1}{2}}\right)^{-1} = O(\hbar R)$. In particular, for the combined phase $e^{\pm\frac{2}{3}i\hbar^{-1}\zeta^{\frac{3}{2}}\pm i\nu\tau\xi^{\frac{1}{2}}}$, we have the identity

$$\begin{aligned}e^{\pm\frac{2}{3}i\hbar^{-1}\zeta^{\frac{3}{2}}\pm i\nu\tau\xi^{\frac{1}{2}}} &= \frac{1}{\partial_{\xi^{\frac{1}{2}}}\left[\pm\frac{2}{3}i\hbar^{-1}\zeta^{\frac{3}{2}}\pm i\nu\tau\xi^{\frac{1}{2}}\right]}\cdot\partial_{\xi^{\frac{1}{2}}}\left(e^{\pm\frac{2}{3}i\hbar^{-1}\zeta^{\frac{3}{2}}\pm i\nu\tau\xi^{\frac{1}{2}}}\right)\\ &\simeq \frac{1}{\pm\nu\tau\pm\zeta^{\frac{1}{2}}R}\cdot\partial_{\xi^{\frac{1}{2}}}\left(e^{\pm\frac{2}{3}i\hbar^{-1}\zeta^{\frac{3}{2}}\pm i\nu\tau\xi^{\frac{1}{2}}}\right),\end{aligned}$$

and since $|\zeta| \ll 1$ on the support of the integrand, we have $\frac{1}{\pm\nu\tau\pm\zeta^{\frac{1}{2}}R} \simeq \frac{1}{\tau}$ for $R \le \nu\tau$. If we then again take advantage of the bound $\hbar^{-\frac{2}{3}}\partial_{\xi^{\frac{1}{2}}}\zeta \simeq \hbar^{\frac{1}{3}}R$ and perform repeated integration by parts (after combining the oscillatory phases), we obtain the gain (similar as in the previous regime)

$$\begin{aligned}&\left|\int_0^\infty \psi\left(\frac{x}{(1+\gamma)x_t}\right)\cdot\varphi\left(\frac{x-\left[x_t+\lambda\hbar^{\frac{2}{3}}\right]}{\lambda\hbar^{\frac{2}{3}}}\right)\cdot\Xi(R;\xi,\hbar)\chi_{\xi\ge 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\rho_n(\xi)\,d\xi\right|\\ &\lesssim \lambda^{-\frac{1}{4}}\cdot\tau^{-\frac{N}{3}}\cdot\xi^{-\frac{N}{6}},\end{aligned}$$

and this can again be summed over dyadic $\lambda \geq 1$. Derivatives are again handled similarly. Observe that up to this point all terms have been of the rapidly decaying type.

(4) Region away from the turning point and in oscillatory regime: $R\xi^{\frac{1}{2}}\hbar \geq (1+\gamma)\, x_t$. Here, recalling Lemma 4.13, we have to consider the integral

$$\int_0^\infty \left[1 - \psi\left(\frac{x}{(1+\gamma)x_t}\right)\right] \cdot \hbar^{\frac{1}{3}} x^{-\frac{1}{2}} q^{-\frac{1}{4}}(\zeta) \cdot \left(\hbar^{-\frac{2}{3}}\zeta\right)^{-\frac{1}{4}} \cdot e^{\pm i\hbar^{-1}\left[x - y(\alpha;\hbar) + \rho(x;\alpha,\hbar)\right]}$$
$$\cdot\, (1 + b(x;\alpha,\hbar))\, \chi_{\xi\geq 1} \frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}} (\log\xi)^j \rho_n(\xi)\, d\xi,$$

where $1 + b(x;\alpha,\hbar)$ accounts for the factor $1 + \hbar\overline{a_1(\zeta;\alpha)}$ as well as the corrections entailed by the Airy function asymptotics. Here we can combine the phases into

$$e^{i(\pm\nu\tau\pm R)\xi^{\frac{1}{2}}} \cdot e^{\pm i\hbar^{-1}[\rho(x;\alpha,\hbar) - y(\alpha;\hbar)]}$$

In the case of non-resonance, i.e. when the phases $e^{\pm i(\nu\tau+R)\xi^{\frac{1}{2}}}$ occur, we can prove rapid decay for the contribution as well as all its derivatives with respect to τ, whence also $|\nu\tau - R|$, within a dilate of the light cone. For this we also need to take into account the phase function $\hbar^{-1}\rho(x;\alpha,\hbar)$, for which we have, interpreting $x = R\hbar\xi^{\frac{1}{2}}$ as a function of R and ξ,

$$\partial_{\xi^{\frac{1}{2}}} \left(\hbar^{-1}\rho(x;\alpha,\hbar) - \hbar^{-1}y(\alpha,\hbar)\right) = -Rx^{-2}(1-2\hbar)^2 + O(\hbar),$$

where we note that the integration takes place over the region $x \geq (1+\gamma)x_t > 1$. We conclude that

$$\partial_{\xi^{\frac{1}{2}}} \left[(\nu\tau + R)\xi^{\frac{1}{2}} - \hbar^{-1}y(\alpha,\hbar) + \hbar^{-1}\rho(x;\alpha,\hbar)\right] \gtrsim \tau,$$

and the claim follows after repeated integration by parts with respect to $\xi^{\frac{1}{2}}$.

We may hence assume that there is destructive resonance, meaning the phases $e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}$ occur. Denoting for now

$$\left[1 - \psi\left(\frac{x}{(1+\gamma)x_t}\right)\right] \cdot \hbar^{\frac{1}{3}} x^{-\frac{1}{2}} q^{-\frac{1}{4}}(\zeta) \cdot \left(\hbar^{-\frac{2}{3}}\zeta\right)^{-\frac{1}{4}} \cdot (1 + b(x;\alpha,\hbar)) \cdot \chi_{\xi\geq 1} \frac{(\log\xi)^j \rho_n(\xi)}{\xi^{1+k\frac{\nu}{2}}} =: \Psi(x;\alpha,\hbar),$$

we reduce to bounding

$$\int_0^\infty e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}} \cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar) - y(\alpha;\hbar)\right]} \cdot \Psi(x;\alpha,\hbar)\, d\xi$$

We shall decompose this into a number of pieces with different structure. First, observe that if $\nu\tau - R \gg 1$, then the expression is a C^∞ function, which decays, in addition to all its derivatives, like $|\nu\tau - R|^{-N}\tau^{-\frac{1}{2}}$. This follows as before by considering the phase $(\nu\tau - R)\xi^{\frac{1}{2}} + \hbar^{-1}y(\alpha,\hbar) - \hbar^{-1}\rho(x;\alpha,\hbar)$ and using the bound

$$\left|\partial_{\xi^{\frac{1}{2}}} \left[(\nu\tau - R)\xi^{\frac{1}{2}} + \hbar^{-1}y(\alpha,\hbar) - \hbar^{-1}\rho(x;\alpha,\hbar)\right]\right| \gtrsim \nu\tau - R, \tag{7.1}$$

provided $\nu\tau - R \gg 1$, and in the regime $x > (1+\gamma)x_t > 1$. Note that to derive (7.1), we can use the fact that for $\hbar \ll 1$, $\partial_x\rho(x;\alpha,\hbar) \leq 0$
We next peel off a function in $S_0^{\hbar}$, by considering the intermediate region $\hbar \ll \nu\tau - R \lesssim 1$. Observe that

$$\left|\partial_R\left[(\nu\tau - R)\xi^{\frac{1}{2}} - \hbar^{-1}\rho(x;\alpha,\hbar)\right]\right| \lesssim \xi^{\frac{1}{2}} + \hbar^{-1}x^{-2}\cdot\hbar\xi^{\frac{1}{2}} \lesssim \xi^{\frac{1}{2}},$$
$$|\partial_R\Psi(x;\alpha,\hbar)| \lesssim R^{-1} \lesssim \tau^{-1},$$

under our assumptions on R, τ, and similarly for the j-th derivatives with the right hand sides replaced by the j-th powers. Furthermore, we have the estimate (7.1), where the right hand side can be replaced by $\hbar$ under the current assumption. Moreover, we have the now somewhat more delicate relation

$$\begin{aligned}\partial_{\xi^{\frac{1}{2}}}\left[(\nu\tau - R)\xi^{\frac{1}{2}} + \hbar^{-1}y(\alpha,\hbar) - \hbar^{-1}\rho(x;\alpha,\hbar)\right] &= (\nu\tau - R) + \hbar^{-1}\frac{\partial\alpha}{\partial\xi^{\frac{1}{2}}}\cdot\partial_\alpha y(\alpha,\hbar) - R\cdot\rho_x(x;\alpha,\hbar)\\ &\geq (\nu\tau - R) + \hbar^{-1}\frac{\partial\alpha}{\partial\xi^{\frac{1}{2}}}\cdot\partial_\alpha y(\alpha,\hbar)\\ &\gtrsim \nu\tau - R\end{aligned}$$

due to Lemma 4.13 and the assumption $\hbar \ll \nu\tau - R \lesssim 1$.

Again using integration by parts, we conclude that

$$\left\|\chi_{1\gtrsim\nu\tau-R\gg\hbar}\int_0^\infty e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\right\|_{\tilde{S}_0^{(n)}} \lesssim 1.$$

Here the large bound $(\nu\tau - R)^{-k} \ll \hbar^{-k}$ is compensated by the power of $\hbar$ in the definition of the norm $S_0^{\hbar}$. It follows that this contribution can be absorbed into f_3 (in fact, it is easily seen that we also have the point wise bounds required in the lemma), and so we reduce to controlling the remaining part.

Next we need to distinguish between different regions in terms of the functions $(\nu\tau - R)\xi^{\frac{1}{2}}, \hbar^{-1}\rho(x;\alpha,\hbar)$.

(4.i): $\hbar^{-1}\rho(x;\alpha,\hbar) \lesssim 1$. We enforce this condition by inclusion of a smooth cutoff $\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)$, and we shall expand

$$\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)e^{\mp i\hbar^{-1}\rho(x;\alpha,\hbar)} = \chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\sum_{l=0}^{\infty}\frac{1}{l!}\left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l.$$

Consider then a term (for $l \geq 0$)

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot\left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\ &= \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\lesssim 1}\left((\nu\tau-R)\xi^{\frac{1}{2}}\right)\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot\left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\ &+ \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\gtrsim 1}\left((\nu\tau-R)\xi^{\frac{1}{2}}\right)\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot\left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\cdot\Psi(x;\alpha,\hbar)\,d\xi.\end{aligned}$$

For the first term here on the right containing the cutoff $\chi_{\lesssim 1}\left((\nu\tau - R)\xi^{\frac{1}{2}}\right)$, we also expand the exponential $e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}$ into a Taylor series, which leads us to consider the expressions (for $l \geq 0, k \geq 0$)

$$\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\lesssim 1}\left((\nu\tau - R)\xi^{\frac{1}{2}}\right)\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\left[i(\nu\tau - R)\xi^{\frac{1}{2}}\right]^k \cdot \left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l e^{\pm i\hbar^{-1}y(\alpha;\hbar)} \cdot \Psi(x;\alpha,\hbar)\,d\xi.$$

If $k = 0$, writing

$$\begin{aligned}
&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\lesssim 1}\left((\nu\tau - R)\xi^{\frac{1}{2}}\right)\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l e^{\pm i\hbar^{-1}y(\alpha;\hbar)} \cdot \Psi(x;\alpha,\hbar)\,d\xi \\
&= \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l e^{\pm i\hbar^{-1}y(\alpha;\hbar)} \cdot \Psi(x;\alpha,\hbar)\,d\xi \\
&- \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\gtrsim 1}\left((\nu\tau - R)\xi^{\frac{1}{2}}\right)\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l e^{\pm i\hbar^{-1}y(\alpha;\hbar)} \cdot \Psi(x;\alpha,\hbar)\,d\xi.
\end{aligned}$$

and using the simple bounds

$$\left|\partial_R^k\left[\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l\right]\right| \lesssim_k C^l \cdot l^k \cdot \tilde{\chi}_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\frac{\left(\hbar\xi^{\frac{1}{2}}\right)^k}{x^k} \lesssim l^k \cdot \tau^{-k}$$

$$\left|\partial_R^k\left(\chi_{\nu\tau-R\lesssim\hbar}\right)\right| \lesssim \hbar^{-k},$$

we can easily place this term into $\tilde{S}_0^{(n)}$, with norm bound $\lesssim \tau^{-1-\nu}$, and the term is as f_3 in the lemma. For the second term we can write it as

$$\begin{aligned}
&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\gtrsim 1}\left((\nu\tau - R)\xi^{\frac{1}{2}}\right)\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l e^{\pm i\hbar^{-1}y(\alpha;\hbar)} \cdot \Psi(x;\alpha,\hbar)\,d\xi \\
&= (\nu\tau - R)^{\frac{1}{2}+k\nu}\left(\log(\nu\tau - R)\right)^j \cdot H(\tau, \nu\tau - R),
\end{aligned}$$

where the term $H(\tau, x)$ has the symbol behavior asserted in the lemma. This also applies to the terms with $k \geq 1$ (In fact here we do not write $\chi_{(\nu\tau-R)\xi^{\frac{1}{2}}\lesssim 1}$ as $1 - \chi_{(\nu\tau-R)\xi^{\frac{1}{2}}\gtrsim 1}$. Instead, we directly integrate the expression in the variable $\tilde{\xi}$, because now the power in $\tilde{\xi}$ from the $k \geq 1$ expansion kills one such a power in the denominator of the expression. Therefore we are able to integrate over the range $\tilde{\xi} \lesssim 1$.). On the other hand for the term with oscillatory phase $e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}$ in the regime $(\nu\tau - R)\xi^{\frac{1}{2}} \gtrsim 1$, pass to the new integration variable $\tilde{\xi} = (\nu\tau - R)\xi^{\frac{1}{2}}$, which also gives the structure

$$\begin{aligned}
&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\gtrsim 1}\left((\nu\tau - R)\xi^{\frac{1}{2}}\right)\chi_{\lesssim 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}} \cdot \left(\mp i\hbar^{-1}\rho(x;\alpha,\hbar)\right)^l e^{\pm i\hbar^{-1}y(\alpha;\hbar)} \cdot \Psi(x;\alpha,\hbar)\,d\xi \\
&= \chi_{\nu\tau-R\lesssim\hbar} \cdot (\nu\tau - R)^{\frac{1}{2}+k\nu} \cdot H(\tau, \nu\tau - R),
\end{aligned}$$

which concludes the case *(4.i)*.

(4.ii): $\hbar^{-1}\rho(x;\alpha,\hbar)\gg 1$, $(\nu\tau-R)\xi^{\frac{1}{2}}\ll\hbar^{-1}\rho(x;\alpha,\hbar)$. We effect the additional restriction by means of the following smooth cutoff

$$\chi_{\gg 1}\left(\frac{\hbar^{-1}\rho(x;\alpha,\hbar)}{(\nu\tau-R)\xi^{\frac{1}{2}}}\right)\cdot\chi_{\gg 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)=:\Lambda(x;\alpha,\hbar).$$

Observe that then thinking of $\frac{x}{\xi^{\frac{1}{2}}\hbar}=\nu\tau-(\nu\tau-R)$ as a function of $\tau,\nu\tau-R$,

$$\left|\frac{\partial\Lambda}{\partial\xi^{\frac{1}{2}}}\right|\lesssim\xi^{-\frac{1}{2}},\quad\left|\frac{\partial\Lambda}{\partial(\nu\tau-R)}\right|\lesssim(\nu\tau-R)^{-1},\quad\left|\frac{\partial\Lambda}{\partial\tau}\right|\lesssim\tau^{-1}.$$

We decompose

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\Lambda(x;\alpha,\hbar)e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&=\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\lesssim 1}((\nu\tau-R)\xi^{\frac{1}{2}})\Lambda(x;\alpha,\hbar)e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&+\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\gtrsim 1}((\nu\tau-R)\xi^{\frac{1}{2}})\Lambda(x;\alpha,\hbar)e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\end{aligned}$$

For the first term on the right, we further decompose it into

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\lesssim 1}((\nu\tau-R)\xi^{\frac{1}{2}})\Lambda(x;\alpha,\hbar)e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&=\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\lesssim 1}((\nu\tau-R)\xi^{\frac{1}{2}})\Lambda(x;\alpha,\hbar)\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&+\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\lesssim 1}((\nu\tau-R)\xi^{\frac{1}{2}})\Lambda(x;\alpha,\hbar)[e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}-1]\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&=\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\Lambda(x;\alpha,\hbar)\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&-\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\gtrsim 1}((\nu\tau-R)\xi^{\frac{1}{2}})\Lambda(x;\alpha,\hbar)\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&+\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\lesssim 1}((\nu\tau-R)\xi^{\frac{1}{2}})\Lambda(x;\alpha,\hbar)[e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}-1]\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\end{aligned}$$

Here the first of the last three terms can be reformulated as

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\gg 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&-\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\lesssim 1}\left(\frac{\hbar^{-1}\rho(x;\alpha,\hbar)}{(\nu\tau-R)\xi^{\frac{1}{2}}}\right)\chi_{\gg 1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi,\end{aligned}$$

of which the first term is easily seen to be in $\tilde{S}_0^{(n)}$ and to also satisfy the bounds required for f_3 in the lemma, while the second admits the representation

$$(\nu\tau - R)^{\frac{1}{2}+k\nu} \log(\nu\tau - R)^j \cdot \frac{H(\tau, \nu\tau - R)}{\tau^{\frac{1}{2}}}$$

where H has the desired symbol behavior. The latter conclusion also applies to the terms

$$\chi_{\nu\tau-R\lesssim\hbar} \int_0^\infty \chi_{\gtrsim 1}((\nu\tau - R)\xi^{\frac{1}{2}})\Lambda(x;\alpha,\hbar) \cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]} \cdot \Psi(x;\alpha,\hbar)\,d\xi$$

$$\chi_{\nu\tau-R\lesssim\hbar} \int_0^\infty \chi_{\lesssim 1}((\nu\tau - R)\xi^{\frac{1}{2}})\Lambda(x;\alpha,\hbar)[e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}} - 1] \cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]} \cdot \Psi(x;\alpha,\hbar)\,d\xi$$

In fact here we simply change the variable to $\tilde{\xi} := (\nu\tau - R)^2\xi$.

It remains to deal with the term

$$\chi_{\nu\tau-R\lesssim\hbar} \int_0^\infty \chi_{\gtrsim 1}\left((\nu\tau - R)\xi^{\frac{1}{2}}\right)\Lambda(x;\alpha,\hbar)e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}} \cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]} \cdot \Psi(x;\alpha,\hbar)\,d\xi.$$

Introduce the phase function

$$\tilde{\xi} := (\nu\tau - R)\xi^{\frac{1}{2}} - \hbar^{-1}\rho(x;\alpha,\hbar),$$

whence

$$\frac{\partial\tilde{\xi}}{\partial\xi^{\frac{1}{2}}} = (\nu\tau - R) - \hbar^{-1}\rho_x \cdot x\xi^{-\frac{1}{2}} - \hbar^{-1}\rho_\alpha \cdot \hbar,$$

and so since $\rho_x \cdot x \simeq -\rho$ on the support of the integrand and further $\hbar^{-1}\rho \gg 1$ there, as well as $|\rho_\alpha| \lesssim \hbar\alpha^{-1} \lesssim \xi^{-\frac{1}{2}}$, we infer

$$\left|\frac{\partial\tilde{\xi}}{\partial\xi^{\frac{1}{2}}}\right| \simeq \hbar^{-1}\rho\xi^{-\frac{1}{2}}$$

on the support of the integrand. Then we write the preceding integral as

$$\chi_{\nu\tau-R\lesssim\hbar} \int_0^\infty \chi_{\gtrsim 1}\left((\nu\tau - R)\xi^{\frac{1}{2}}\right)\Lambda(x;\alpha,\hbar)e^{\pm i\tilde{\xi}} \cdot e^{\pm i\hbar^{-1}y(\alpha;\hbar)]} \cdot \tilde{\Psi}(x;\alpha,\hbar)\,d\tilde{\xi}$$

where we set

$$\tilde{\Psi}(x;\alpha,\hbar) = 2\xi^{\frac{1}{2}}\frac{\partial\xi^{\frac{1}{2}}}{\partial\tilde{\xi}} \cdot \Psi(x;\alpha,\hbar).$$

Then we observe that

$$\left|\partial_{\tilde{\xi}}^l\tilde{\Psi}(x;\alpha,\hbar)\right| \lesssim \tau^{-\frac{1}{2}} \cdot \xi^{-\frac{1}{4}}(\log\xi)^j \cdot \left(\hbar^{-1}\rho\right)^{-l-1}, \quad \left|\partial_{\tilde{\xi}}^l\chi_{\gtrsim 1}\left((\nu\tau - R)\xi^{\frac{1}{2}}\right)\right| \lesssim \left(\hbar^{-1}\rho\right)^{-l},$$

$$\left|\partial_{\tilde{\xi}}^l\Lambda(x;\alpha,\hbar)\right| \lesssim \left(\hbar^{-1}\rho\right)^{-l}, \quad \left|\partial_{\tilde{\xi}}^l\left(e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\right)\right| \lesssim \left(\hbar^{-1}\rho\right)^{-l}.$$

Note that applying a derivative $\partial_{\nu\tau-R}$ to the original expression (formulated as integral with respect to ξ, and interpreted as function of $\nu\tau - R$ and τ) results at worst in a loss of

$$\lesssim \xi^{\frac{1}{2}} + \hbar^{-1}\rho \cdot \tau^{-1} \lesssim \left(\hbar^{-1}\rho\right) \cdot \left((\nu\tau - R)^{-1} + \tau^{-1}\right),$$

while applying ∂_τ results in at worst the loss $\hbar^{-1}\rho\cdot\tau^{-1}$. We conclude that after performing integration by parts in the variable $\tilde{\xi}$ and then reverting to the original integration variable ξ, we have

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{\gtrsim1}\left((\nu\tau-R)\xi^{\frac12}\right)\Lambda(x;\alpha,\hbar)e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&=(\nu\tau-R)^{\frac12+k\nu}\left(\log(\nu\tau-R)\right)^j\frac{H(\tau,\nu\tau-R)}{\tau^{\frac12}},\end{aligned}$$

where H has the symbol behavior asserted in the lemma. (Here the factor $(\nu\tau-R)^{\frac12+k\nu}$... with limited smoothness comes from the factor $\xi^{-\frac14}(\log\xi)^j$ and the cutoff. The rapid decay in $\hbar^{-1}\rho$ produced by integration by parts gives the integrability.)

(4.iii): $\hbar^{-1}\rho(x;\alpha,\hbar)\gg1$, $(\nu\tau-R)\xi^{\frac12}\gtrsim\hbar^{-1}\rho(x;\alpha,\hbar)$. Start by considering the term

$$\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\tilde{\Lambda}(x;\alpha,\hbar)\cdot e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\cdot\Psi(x;\alpha,\hbar)\,d\xi,$$

where we set

$$\tilde{\Lambda}(x;\alpha,\hbar):=\chi_{\lesssim1}\left(\frac{\hbar^{-1}\rho(x;\alpha,\hbar)}{(\nu\tau-R)\xi^{\frac12}}\right)\cdot\chi_{\gg1}\left(\hbar^{-1}\rho(x;\alpha,\hbar)\right)$$

This term can be seen to be of the explicit form

$$(\nu\tau-R)^{\frac12+k\nu}\left(\log(\nu\tau-R)\right)^j\cdot\frac{H(\tau,\nu\tau-R)}{\tau^{\frac12}}$$

by arguing as for the term

$$\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\Lambda(x;\alpha,\hbar)\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi$$

This leaves us with the term

$$\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\tilde{\Lambda}(x;\alpha,\hbar)\left[e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}\rho(x;\alpha,\hbar)}-1\right]e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\cdot\Psi(x;\alpha,\hbar)\,d\xi$$

Introduce the variable

$$\tilde{\xi}:=(\nu\tau-R)\xi^{\frac12}-\hbar^{-1}\rho(x;\alpha,\hbar),$$

and so (for suitable $c>0$)

$$\frac{\partial\tilde{\xi}}{\partial\xi^{\frac12}}\geq\nu\tau-R+c\hbar^{-1}x^{-1}\xi^{-\frac12}\geq\nu\tau-R,\quad\xi^{\frac12}=\left[\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)\right](\nu\tau-R)^{-1}.$$

Also, the cutoff $\tilde{\Lambda}(x;\alpha,\hbar)$ ensures that $(\nu\tau-R)\xi^{\frac12}=\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)\gtrsim\left|\tilde{\xi}\right|$ on the support of the integrand. Then, assuming $k\nu<\frac12$, we can write (with $c(\tau,\nu\tau-R)=\nu\tau-R-\hbar^{-1}\rho(\hbar R;\hbar,\hbar)$)

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\tilde{\Lambda}(x;\alpha,\hbar)\left[e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}\rho(x;\alpha,\hbar)}-1\right]e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\&=\sum_{j'+j''=j}C_{j',j''}\chi_{\nu\tau-R\lesssim\hbar}\cdot(\nu\tau-R)^{\frac32+\nu}\left[\log(\nu\tau-R)\right]^{j'}\cdot\end{aligned}$$

$$\int_{c(\tau,\nu\tau-R)}^{\infty} \frac{e^{\pm i\tilde{\xi}}-1}{[\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)]^{\frac{3}{2}+k\nu}} \left(\log\left(\tilde{\xi}+\hbar^{-1}\rho\right)\right)^{j''} \tilde{\Psi}(x;\alpha,\hbar)\,d\tilde{\xi},$$

where we put

$$\tilde{\Psi}(x;\alpha,\hbar) = \tilde{\Lambda}(x;\alpha,\hbar)e^{\pm i\hbar^{-1}y(\alpha;\hbar)} \cdot \Psi(x;\alpha,\hbar) \cdot \xi^{\frac{5}{4}+\frac{k\nu}{2}}(\log\xi)^{-j} \cdot \frac{\partial\xi^{\frac{1}{2}}}{\partial\tilde{\xi}}.$$

Then from the definitions we directly infer the bound

$$\left|\tilde{\Psi}(x;\alpha,\hbar)\right| \lesssim \tau^{-\frac{1}{2}} \cdot (\nu\tau-R)^{-1}$$

provided $\hbar \gtrsim \nu\tau - R > 0$, and putting

$$G(\tau,\nu\tau-R) = \tau^{\frac{1}{2}}(\nu\tau-R) \cdot \int_{c(\tau,\nu\tau-R)}^{\infty} \frac{e^{\pm i\tilde{\xi}}-1}{\left[\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)\right]^{\frac{3}{2}+k\nu}} \left(\log\left(\tilde{\xi}+\hbar^{-1}\rho\right)\right)^{j''} \tilde{\Psi}(x;\alpha,\hbar)\,d\tilde{\xi},$$

we easily infer the desired symbol type behavior asserted in the lemma. In fact, note that upon writing $x = \hbar\xi^{\frac{1}{2}}\,[\nu\tau - (\nu\tau - R)]$,

$$\partial_{(\nu\tau-R)}\left(\frac{1}{\left[\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)\right]^{\frac{3}{2}+k\nu}}\right) = -\left(\frac{3}{2}+k\nu\right) \cdot \frac{\hbar^{-1}\rho_x \cdot \hbar\xi^{\frac{1}{2}}}{\left[\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)\right]^{\frac{5}{2}+k\nu}},$$

and so (again using $\tau \simeq R$)

$$\begin{aligned}\left|\partial_{(\nu\tau-R)}\left(\frac{1}{\left[\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)\right]^{\frac{3}{2}+k\nu}}\right)\right| &\lesssim \frac{\hbar^{-1}\rho_x \cdot x\tau^{-1}}{\left[\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)\right]^{\frac{5}{2}+k\nu}} \\ &\lesssim \frac{\tau^{-1}}{\left[\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)\right]^{\frac{3}{2}+k\nu}}\end{aligned}$$

on the support of the integrand, which leads to better than symbol behavior in the case the derivative $\partial_{(\nu\tau-R)}$ falls on this term in the integral. However, this is not the case for the contribution when this derivative falls on the factor

$$\rho_n(\xi) = \rho_n(\xi(\tilde{\xi})),$$

for which we have to use Proposition 4.32. The case $k\nu \geq \frac{1}{2}$ is handled by subtracting further terms from $e^{\pm i\tilde{\xi}}$ to handle the term $\left[\tilde{\xi}+\hbar^{-1}\rho(x;\alpha,\hbar)\right]^{-(\frac{3}{2}+k\nu)}$ when $\tilde{\xi} \ll 1$.

It remains to explain how to handle the contributions of the second to fourth sum constituting $\overline{x}(\tau,\xi)$ in Definition 7.2. This is quite straightforward for the terms in the third sum. In fact, observe that if we restrict this sum to the region $\hbar^2\xi \lesssim 1$, then it contributes to a term of type f_3, i.e. of bounded $\tilde{S}_0^{(n)}$-norm (and bounded $\tilde{S}_1^{(n)}$-norm for its temporal derivative). This is a consequence of the fact that (for $k \geq 1$)

$$\begin{aligned}&\left\|\chi_{\hbar^{-2}\gtrsim\xi\geq 1}\hbar^{-1}\langle\hbar^2\xi\rangle^{-\frac{l}{4}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}\left(\log\xi\right)^i\cdot F_{l,k,i}(\tau,\xi)\right\|_{S_0^{\hbar}}\\ &\lesssim \left\|\chi_{\hbar^{-2}\gtrsim\xi\geq 1}\langle\hbar^2\xi\rangle^{-\frac{l}{4}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}+k\frac{\nu}{2}}}\left(\log\xi\right)^j\cdot F_{l,k,j}(\tau,\xi)\right\|_{L^2_{d\xi}} \lesssim 1.\end{aligned}$$

Then restricting the expressions to the high-frequency region $\xi \gtrsim \hbar^{-2}$, one may replace the factors $\langle\hbar^2\xi\rangle^{-\frac{l}{4}}$ by $(\hbar^2\xi)^{-\frac{l}{4}}$ for all intents and purposes, which upon following the steps in the preceding is seen to lead to additional factors of the form $\left(\hbar^{-1}[\nu\tau - R]\right)^{\frac{l-1}{2}}$ for the terms of explicit form. Moreover, the extra factors $F_{l,k,j}(\tau,\xi)$ are seen to not lead to any additional complications due to their symbol type bounds.

It remains to deal with the contribution of terms in the second and fourth sum, which differ more significantly due to the presence of the phase function $e^{\pm i\hbar^{-1}\rho(x_\tau;\alpha,\hbar)}$. This requires revisiting all terms where integration by parts was required. To do so we briefly revisit the cases *(1) - (4)* in the preceding, explaining the differences. For simplicity we shall set $l = j = 0$, the more general case being similar, and so we can unambiguously refer to these cases from before.

In cases *(1), (2)*, one replaces the term $e^{\pm i\nu\tau\xi^{\frac{1}{2}}}$ by $e^{\pm i(\nu\tau\xi^{\frac{1}{2}}+\hbar^{-1}\rho(x_\tau;\alpha,\hbar))}$. Keeping in mind that $\xi\hbar^2 \gtrsim 1$ for these terms, one has

$$\begin{aligned}\partial_{\xi^{\frac{1}{2}}}\left(\nu\tau\xi^{\frac{1}{2}} + \hbar^{-1}\rho(x_\tau;\alpha,\hbar)\right) &= \nu\tau + \hbar^{-1}\rho_x(x_\tau;\alpha,\hbar)\cdot\hbar\nu\tau + \hbar^{-1}\rho_\alpha(x_\tau;\alpha,\hbar)\cdot\hbar\\ &= \nu\tau + O(1),\end{aligned}$$

since we have $x_\tau \gtrsim \tau$ on the support of the expression, and so $\hbar^{-1}\rho_x(x_\tau;\alpha,\hbar)\cdot\hbar\nu\tau \lesssim \tau^{-1}$, while $|\rho_\alpha| \lesssim \hbar$. Thus the integration by parts with respect to $\xi^{\frac{1}{2}}$ works in the same way here.

Case *(3)* also basically doesn't change, since here we encounter the phase function

$$\pm\frac{2}{3}\hbar^{-1}\zeta^{\frac{3}{2}} \pm (\nu\tau\xi^{\frac{1}{2}} + \hbar^{-1}\rho(x_\tau;\alpha,\hbar)),$$

Here we have the same bound

$$\left|\partial_{\xi^{\frac{1}{2}}}\left(\pm\frac{2}{3}\hbar^{-1}\zeta^{\frac{3}{2}} \pm \left(\nu\tau\xi^{\frac{1}{2}} + \hbar^{-1}\rho(x_\tau;\alpha,\hbar)\right)\right)\right| \gtrsim \tau,$$

on account of the above bound for $\partial_{\xi^{\frac{1}{2}}}\left(\hbar^{-1}\rho(x_\tau;\alpha,\hbar)\right) = \hbar^{-1}\rho_x(x_\tau;\alpha,\hbar)\cdot\hbar\nu\tau$, and then the argument proceeds as before.

In case *(4)*, using the fact that $\nu\tau - R \gg 1$, we can again immediately reduce to the destructively resonant case involving the phase $(\nu\tau - R)\xi^{\frac{1}{2}}$ and the case close to the light cone $\nu\tau - R \lesssim 1$. Now to deal with the intermediate case $\hbar \ll \nu\tau - R \lesssim 1$, we encounter the phase

$$\left[(\nu\tau - R)\xi^{\frac{1}{2}} + \hbar^{-1}y(\alpha,\hbar) - \hbar^{-1}\rho(x;\alpha,\hbar) + \hbar^{-1}\rho(x_\tau;\alpha,\hbar)\right],$$

for which we have the relation

$$\begin{aligned}&\partial_{\xi^{\frac12}}\left[(\nu\tau-R)\xi^{\frac12}+\hbar^{-1}y(\alpha,\hbar)-\hbar^{-1}\rho(x;\alpha,\hbar)+\hbar^{-1}\rho(x_\tau;\alpha,\hbar)\right]\\&=(\nu\tau-R)+y_\alpha(\alpha,\hbar)-\hbar^{-1}\partial_{\xi^{\frac12}}\left(\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\cdot(x-x_\tau)\,ds\right),\end{aligned}$$

and we have the relation (for $\xi\gtrsim\hbar^{-2}$)

$$\begin{aligned}&\hbar^{-1}\partial_{\xi^{\frac12}}\left(\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\cdot(x-x_\tau)\,ds\right)\\&=(R-\nu\tau)\cdot\xi^{\frac12}\int_0^1\left[\rho_{xx}(sx+(1-s)x_\tau;\alpha,\hbar)\cdot(s\hbar R+(1-s)\hbar\nu\tau)+\hbar\rho_{x\alpha}(sx+(1-s)x_\tau;\alpha,\hbar)\right]\,ds\\&\qquad+(R-\nu\tau)\cdot\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\,ds\\&=(R-\nu\tau)\cdot O\left(\tau^{-2}\right).\end{aligned}$$

In light of $y_\alpha(\alpha,\hbar)=O(\hbar)$ because of Lemma 4.13, we conclude that

$$\left|\partial_{\xi^{\frac12}}\left[(\nu\tau-R)\xi^{\frac12}+\hbar^{-1}y(\alpha,\hbar)-\hbar^{-1}\rho(x;\alpha,\hbar)+\hbar^{-1}\rho(x_\tau;\alpha,\hbar)\right]\right|\gtrsim\nu\tau-R,$$

and one can again argue as in the earlier situation to handle the regime $\hbar\ll\nu\tau-R\lesssim1$.
Finally, consider the expression

$$\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)-y(\alpha;\hbar)]}\cdot\chi_{\xi\hbar^2\gtrsim1}\Psi(x;\alpha,\hbar)\,d\xi$$

where $\Psi(x;\alpha,\hbar)$ is again defined as in the earlier case *(4)*. Dealing with this is now in fact a bit simpler than in the earlier situation. Write the preceding as

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)-y(\alpha;\hbar)\right]}\cdot\chi_{\xi\hbar^2\gtrsim1}\Psi(x;\alpha,\hbar)\,d\xi\\&=\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{(\nu\tau-R)\xi^{\frac12}\lesssim1}e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]}e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\cdot\chi_{\xi\hbar^2\gtrsim1}\Psi(x;\alpha,\hbar)\,d\xi\\&+\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{(\nu\tau-R)\xi^{\frac12}\gtrsim1}e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]}e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\cdot\chi_{\xi\hbar^2\gtrsim1}\Psi(x;\alpha,\hbar)\,d\xi\\&=:X_1+X_2.\end{aligned}$$

In order to treat the first term, write it as (assuming $\frac12+k\nu<1$, the case $\frac12+k\nu\geq1$ being handled analogously by subtracting more terms of the Taylor expansion of e^z, $z=\pm i(\nu\tau-R)\xi^{\frac12}\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]$)

$$\begin{aligned}X_1=&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{(\nu\tau-R)\xi^{\frac12}\lesssim1}\left[e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)]}-1\right]\tilde\Psi(x;\alpha,\hbar)\,d\xi\\&+\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty\chi_{(\nu\tau-R)\xi^{\frac12}\lesssim1}\tilde\Psi(x;\alpha,\hbar)\,d\xi,\end{aligned}$$

and we set $\tilde{\Psi}(x;\alpha,\hbar) = e^{\pm i\hbar^{-1}y(\alpha;\hbar)}\cdot\chi_{\xi\hbar^2\gtrsim 1}\Psi(x;\alpha,\hbar)$. The second term on the right, call it X_{12}, is easily seen to be in $\tilde{S}_0^{(\hbar)}$ with the desired bound $\|X_{12}\|_{\tilde{S}_0^{(\hbar)}} + \|\partial_\tau X_{21}\|_{\tilde{S}_1^{(\hbar)}} \lesssim 1$. The first term, call it X_{11}, can be written as

$$\begin{aligned} X_{11} =&\chi_{\nu\tau-R\lesssim\hbar}(\nu\tau-R)\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac12}\lesssim 1}\frac{\left[e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]}-1\right]}{\left[\xi^{\frac12}(\nu\tau-R)\right]}\xi^{\frac12}\tilde{\Psi}(x;\alpha,\hbar)\,d\xi\\ =:&\chi_{\nu\tau-R\lesssim\hbar}(\nu\tau-R)^{\frac12+k\nu}\cdot\frac{H(\tau,\nu\tau-R)}{\tau^{\frac12}}. \end{aligned}$$

We claim that the function $H(\tau,x)$ satisfies the symbol bounds $\left|\partial_\tau^{k_1}\partial_x^{k_2}H(\tau,x)\right| \lesssim \tau^{-k_1}x^{-k_2}$. To begin with, note that the function

$$\frac{\left[e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)]}-1\right]}{\left[\xi^{\frac12}(\nu\tau-R)\right]}$$

is bounded on the support of the integrand, since

$$\left|\pm i(\nu\tau-R)\xi^{\frac12}\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]\right| \lesssim (\nu\tau-R)\xi^{\frac12}$$

there. As for the derivatives, write

$$\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right] = \int_0^1\rho_x\left(sx+(1-s)x_\tau;\alpha,\hbar\right)\,ds\cdot(R-\nu\tau)\xi^{\frac12},$$

whence (as usual interpreting x as function of τ and $\nu\tau-R$), we infer the relation

$$\begin{aligned} &\left|\partial_\tau\left[\frac{\left[e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)]}-1\right]}{[\xi^{\frac12}(\nu\tau-R)]}\right]\right|\\ &\leq\left|1+\int_0^1\rho_x\left(sx+(1-s)x_\tau;\alpha,\hbar\right)\,ds\right|\cdot\left|Z'\left((\nu\tau-R)\xi^{\frac12}\cdot(1+\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\,ds)\right)\right|\\ &\qquad\cdot\left|(\nu\tau-R)\xi^{\frac12}\cdot\partial_\tau\left(\int_0^1\rho_x\left(sx+(1-s)x_\tau;\alpha,\hbar\right)\,ds\right)\right|\\ &+\left|\partial_\tau(\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\,ds)\right|\cdot\left|Z((\nu\tau-R)\xi^{\frac12}\cdot(1+\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\,ds))\right|, \end{aligned}$$

where we set $Z(x) := \frac{e^{\pm ix}-1}{x}$. Using the crude bound

$$\left|\partial_\tau\left(\int_0^1\rho_x\left(sx+(1-s)x_\tau;\alpha,\hbar\right)\,ds\right)\right| \lesssim \tau^{-1},$$

we easily infer, also using the definition of $\tilde{\Psi}(x;\alpha,\hbar)$, that on the support of the integrand for the integral in the definition of X_{11}, whence in particular with $(\nu\tau-R)\xi^{\frac12}\lesssim 1$, $x\gtrsim 1$, we have

$$|\partial_\tau H(\tau,\nu\tau-R)| \lesssim \tau^{-1},$$

and analogously for higher derivatives with respect to τ. Derivatives with respect to $\nu\tau - R$ are handled analogously: here one may lose a factor $\xi^{\frac{1}{2}} \lesssim (\nu\tau - R)^{-1}$ on the support of the integrand.
It remains to deal with the term X_2, which is

$$X_2 = \chi_{\nu\tau-R\lesssim\hbar} \int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac{1}{2}}\gtrsim 1} e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}} \cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]} e^{\pm i\hbar^{-1}y(\alpha;\hbar)} \cdot \chi_{\xi\hbar^2\gtrsim 1}\Psi(x;\alpha,\hbar)\, d\xi.$$

We claim that this is also of the form $\chi_{\nu\tau-R\lesssim\hbar}(\nu\tau - R)^{\frac{1}{2}+k\nu} \cdot \frac{H(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}$, with H having the appropriate symbol behavior. For this, introduce the integration variable

$$\tilde{\xi} := (\nu\tau - R)\xi^{\frac{1}{2}} - \hbar^{-1}[\rho(x;\alpha,\hbar) - \rho(x_\tau;\alpha,\hbar)].$$

Then the preceding formula for the difference $\hbar^{-1}[\rho(x;\alpha,\hbar) - \rho(x_\tau;\alpha,\hbar)]$ and the fact that $sx + (1-s)x_\tau \simeq \tau$ on the support of the integrand for X_2 imply that

$$\tilde{\xi} \simeq (\nu\tau - R)\xi^{\frac{1}{2}}, \quad \frac{\partial\tilde{\xi}}{\partial\xi^{\frac{1}{2}}} \simeq \nu\tau - R$$

on the support of the integrand. We can then write

$$X_2 = \chi_{\nu\tau-R\lesssim\hbar}(\nu\tau - R)^{\frac{3}{2}+k\nu} \int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac{1}{2}}\gtrsim 1} \frac{e^{\mp i\tilde{\xi}}}{\left(\tilde{\xi} + \hbar^{-1}\left[\rho(x;\alpha,\hbar) - \rho(x_\tau;\alpha,\hbar)\right]\right)^{\frac{3}{2}+k\nu}} \cdot \frac{\partial\xi^{\frac{1}{2}}}{\partial\tilde{\xi}} \tilde{\tilde{\Psi}}\, d\tilde{\xi},$$

where we have set $\tilde{\tilde{\Psi}} := 2\xi^{\frac{5}{4}+\frac{k\nu}{2}} e^{\pm i\hbar^{-1}y(\alpha;\hbar)} \cdot \chi_{\xi\hbar^2\gtrsim 1}\Psi(x;\alpha,\hbar)$, and we recall that for simplicity of exposition we assume $l = j = 0$(recall the definition of Ψ). Then since $\left|\tilde{\tilde{\Psi}}(x;\alpha,\hbar)\right| \lesssim \tau^{-\frac{1}{2}}$ on the support of the integrand, writing $X_2 = \chi_{\nu\tau-R\lesssim\hbar}(\nu\tau - R)^{\frac{1}{2}+k\nu} \cdot \frac{H(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}$ we infer $|H(\tau, \nu\tau - R)| \lesssim 1$. Further using

$$\begin{aligned}\frac{\partial\tilde{\xi}}{\partial\xi^{\frac{1}{2}}} =&(\nu\tau - R) \cdot \left(1 + \int_0^1 \rho_x(sx + (1-s)x_\tau;\alpha,\hbar)\, ds + \int_0^1 \rho_{xx}(sx + (1-s)x_\tau;\alpha,\hbar) \cdot (sx + (1-s)x_\tau)\, ds\right)\\ &+ (\nu\tau - R) \cdot \hbar \int_0^1 \rho_{x\alpha}(sx + (1-s)x_\tau;\alpha,\hbar)\, ds,\end{aligned}$$

taking advantage of Lemma 4.13, we find

$$\left|\partial_\tau^{k_1}\partial_{\nu\tau-R}^{k_2}\left(\frac{\partial\xi^{\frac{1}{2}}}{\partial\tilde{\xi}}\right)\right| \lesssim \tau^{-k_1}(\nu\tau - R)^{-k_2-1},$$

and in conjunction with the definition of $\tilde{\tilde{\Psi}}$, the desired symbol behavior of H follows. □

The preceding lemma admits a counterpart translating a physical singular expansion into a Fourier expansion. Note that this is not a simple inverse, and adapted to the specific needs later on:

Lemma 7.7. *Assume that $f = f(\tau,R)$ is a function supported on $\frac{\nu\tau}{2} < R \leq \nu\tau$ which admits an expansion*

$$f = \sum_{j=1}^{3} f_j$$

where the f_1 is as in the statement of the preceding lemma, f_2 can be written as $f_2 = \sum_{l=0}^{7} f_{2l}$ while the f_{2l} vanish on the set $\nu\tau < R$ and are defined on $R < \nu\tau$ as in the preceding lemma except that $\hbar^{-\frac{l+1}{2}}$ is replaced by $\hbar^{-\frac{l}{2}-1}$, and f_3 is C^∞ with $\|f_3\|_{\tilde{S}_1^{(\hbar)}} \lesssim \tau^{-1-\nu}$ and satisfies the pointwise bounds

$$\left|\partial_R^k f_3(\tau, R)\right| \lesssim \tau^{-\frac{3}{2}-\nu} \cdot \hbar^{-1} \min\{|\nu\tau - R|^{-k}, \hbar^{-k}\}, \quad 0 \leq k \leq 5.$$

Then setting

$$\overline{x}(\tau, \xi) := \int_0^\infty \phi_n(R;\xi) \cdot f(\tau, R) \cdot \rho_n(\xi)\, d\xi,$$

we have $\overline{x} = \overline{x}_1 + \overline{x}_2$, where

$$\|\overline{x}_1\|_{S_1^\hbar} \lesssim \tau^{-1-\nu},$$

and furthermore we can write

$$\overline{x}_1(\tau, \xi) = \sum_{\pm} e^{\pm i\nu\tau\xi^{\frac{1}{2}}} \cdot g_\pm(\tau, \xi)$$

where $g_\pm(\tau, \xi)$ is C^∞ with respect to the second variable, and satisfies the bounds

$$\left|\partial_\xi^{k_2} g_\pm(\tau, \xi)\right| \lesssim \log\tau \cdot \tau^{-1-\nu} \cdot \hbar^{-1}\xi^{-\frac{1}{2}}\langle\hbar\xi^{\frac{1}{2}}\rangle^{-4}, \quad 0 \leq k_2 \leq 5.$$

Furthermore, $\overline{x}_2$ admits the symbol expansion (with ρ as in Lemma 4.13 and $x_\tau = \hbar\xi^{\frac{1}{2}} \cdot \nu\tau$)

$$\overline{x}_2(\tau, \xi) = \sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1} \chi_{\xi \geq \hbar^{-2}} \hbar^{-1}\langle\hbar^2\xi\rangle^{-\frac{l}{4}} \frac{e^{\pm i(\nu\tau\xi^{\frac{1}{2}} + \hbar^{-1}\rho(x_\tau;\alpha,\hbar))}}{\xi^{\frac{3}{4}+k\frac{\nu}{2}}} (\log\xi)^j \cdot F_{l,k,j}(\tau, \xi)$$

and we have the symbol type bounds (with $k_1 \in \{0, 1\}, k_2 \in \{0, \ldots, 5\}$, $k_1 + k_2 \leq 5$)

$$\left|\partial_\tau^{k_1}\partial_\xi^{k_2} F_{l,k,j}(\tau, \xi)\right| \lesssim \log\tau \cdot \tau^{-1-\nu-k_1} \cdot \xi^{-k_2},$$

as well as the closure bound

$$\left\|\xi^{5+\delta-k_1}\partial_\tau^{k_1}\partial_\xi^{k_2} F_{l,k,j}(\tau, \xi)\right\|_{\dot{C}^\delta} \lesssim \log\tau \cdot \tau^{-1-\nu-k_1}.$$

Proof.

$$\overline{x}(\tau, \xi) = \sum_{j=1}^{3} \overline{y}_j(\tau, \xi),$$

where we set

$$\overline{y}_j(\tau, \xi) = \int_0^\infty \phi_n(R;\xi) f_j(\tau, R)\, R\, dR.$$

Then it is immediate that $\left\|\overline{y}_3\right\|_{S_1^\hbar} \lesssim \tau^{-1-\nu}$, whence $\overline{y}_3$ can be incorporated into $\overline{x}_1$. The same holds for f_1 by using a similar argument and the cutoff $\nu\tau - R \gtrsim 1$.

It thus remains to deal with $f_2 = \sum_{l=0}^{7} f_{2l}$, where it suffices to set (with l, k, i as in the specified ranges above)

$$g(\tau, R) := \chi_{|\nu\tau - R| \lesssim \hbar} \frac{G_{k,l,i}(\tau, \nu\tau - R)}{\tau^{\frac{1}{2}}} \hbar^{-\frac{l}{2}-1}[\nu\tau - R]^{\frac{l}{2}+k\nu}(\log(\nu\tau - R))^i$$

We split the distorted Fourier transform of this term into several pieces. To begin with, write

$$\begin{aligned}&\left\langle \phi_n(R;\xi),\, \chi_{|\nu\tau-R|\lesssim\hbar}\frac{G_{k,l,j}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar^{-\frac{l}{2}-1}\left[\nu\tau-R\right]^{\frac{l}{2}+k\nu}(\log(\nu\tau-R))^j\right\rangle_{L^2_{R\,dR}}\\&=\chi_{\xi\hbar^2\lesssim1}\langle\ldots\rangle_{L^2_{R\,dR}}+\chi_{\xi\hbar^2\gtrsim1}\langle\ldots\rangle_{L^2_{R\,dR}}.\end{aligned}$$

Then the low-frequency term can again be included into $S_1^{\hbar}$:

$$\left\|\chi_{\xi\hbar^2\lesssim1}\langle\ldots\rangle_{L^2_{R\,dR}}\right\|_{S_1^{\hbar}}\lesssim\hbar\cdot\|g\|_{L^2_{R\,dR}}\lesssim\tau^{-1-\nu},$$

where we take advantage of the fact that

$$\left|\chi_{|\nu\tau-R|\lesssim\hbar}\hbar^{-\frac{l}{2}}\cdot|\nu\tau-R|^{\frac{l}{2}}\right|\lesssim1.$$

We can thus reduce to considering

$$\overline{x}_2(\tau,\xi):=\chi_{\xi\hbar^2\gtrsim1}\int_0^\infty\phi_n(R;\xi)\chi_{|\nu\tau-R|\lesssim\hbar}\frac{G_{k,l,j}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar^{-\frac{l}{2}-1}\left[\nu\tau-R\right]^{\frac{l}{2}+k\nu}(\log(\nu\tau-R))^j R\,dR$$

Observe that since $\xi^{\frac{1}{2}}\hbar R\simeq\xi^{\frac{1}{2}}\hbar\tau\gg1$ on the support of the above integrand, the function $\phi_n(R;\xi)$ is in the oscillatory regime, and we may replace this function by

$$\hbar^{\frac{1}{3}}x^{-\frac{1}{2}}\left(\hbar^{-\frac{2}{3}}\zeta\right)^{-\frac{1}{4}}q^{-\frac{1}{4}}(\zeta)\mathrm{Re}\left[a(\xi)e^{\pm i\left(R\xi^{\frac{1}{2}}-y(\alpha;\hbar)+\hbar^{-1}\rho(x;\alpha,\hbar)\right)}\cdot\left(1+\hbar\overline{\tilde{a}_1(\zeta;\alpha)}\right)\right],$$

where as usual we have incorporated the asymptotic correction terms for the oscillatory Airy functions into the term $1+\hbar\overline{\tilde{a}_1(\zeta;\alpha)}$. Write

$$\begin{aligned}\phi_n(R;\xi)&=\sum_{\pm}e^{\pm i\nu\tau\xi^{\frac{1}{2}}}\cdot\frac{e^{\mp i(\nu\tau-R)\xi^{\frac{1}{2}}\pm i\hbar^{-1}\rho(x;\alpha,\hbar)}}{(R\xi^{\frac{1}{2}})^{\frac{1}{2}}}\cdot\psi(x;\alpha,\hbar)\cdot(1+\hbar\overline{\tilde{a}_1(\zeta;\alpha)})\\&=:\sum_{\pm}e^{\pm i\nu\tau\xi^{\frac{1}{2}}}\cdot\phi_{\pm,\tau}(R;\xi,\hbar),\end{aligned}$$

where ψ has symbol behavior with respect to x,α. Then we can write

$$\begin{aligned}&\overline{x}_2(\tau,\xi)\\&=\sum_{\pm}\chi_{\xi\hbar^2\gtrsim1}e^{\pm i\nu\tau\xi^{\frac{1}{2}}}\cdot\int_0^\infty\phi_{\pm,\tau}(R;\xi,\hbar)\chi_{|\nu\tau-R|\lesssim\hbar}\frac{G_{k,l,j}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar^{-\frac{l}{2}-1}\left[\nu\tau-R\right]^{\frac{l}{2}+k\nu}(\log(\nu\tau-R))^j R\,dR\\&=\sum_{\pm}\chi_{\xi\hbar^2\gtrsim1}e^{\pm i\nu\tau\xi^{\frac{1}{2}}}\cdot\int_0^\infty\phi_{\pm,\tau}(R;\xi,\hbar)\chi_{|\nu\tau-R|\lesssim\xi^{-\frac{1}{2}}}\frac{G_{k,l,j}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar^{-\frac{l}{2}-1}\left[\nu\tau-R\right]^{\frac{l}{2}+k\nu}(\log(\nu\tau-R))^j R\,dR\\&+\sum_{\pm}\chi_{\xi\hbar^2\gtrsim1}e^{\pm i\nu\tau\xi^{\frac{1}{2}}}\cdot\int_0^\infty\phi_{\pm,\tau}(R;\xi,\hbar)\chi_{\xi^{-\frac{1}{2}}\lesssim|\nu\tau-R|\lesssim\hbar}\frac{G_{k,l,j}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar^{-\frac{l}{2}-1}\left[\nu\tau-R\right]^{\frac{l}{2}+k\nu}(\log(\nu\tau-R))^j R\,dR\\&=:\overline{x}_{21}+\overline{x}_{22}.\end{aligned}$$

To deal with $\overline{x}_{21}$, we simply write it as

$$\overline{x}_{21} = \sum_{\pm}\chi_{\xi\hbar^2\gtrsim 1}\hbar^{-1}\left\langle\xi\hbar^2\right\rangle^{-\frac{l}{4}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{3}{4}+\frac{k\nu}{2}}}(\log\xi)^j\cdot F^{(1),\pm}_{l,k,j}(\tau,\xi),$$

where we define

$$\begin{aligned}&F^{(1),\pm}_{l,k,j}(\tau,\xi)\\&=\tilde{\chi}_{\xi\hbar^2\gtrsim 1}\frac{\left\langle\xi\hbar^2\right\rangle^{\frac{l}{4}}}{(\xi\hbar^2)^{\frac{l}{4}}}\cdot\frac{\xi^{\frac{3}{4}}}{(\log\xi)^j}\\&\cdot\int_0^\infty\phi_{\pm,\tau}(R;\xi,\hbar)\chi_{|\nu\tau-R|\lesssim\xi^{-\frac{1}{2}}}\frac{G_{k,l,j}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\left[\xi^{\frac{1}{2}}(\nu\tau-R)\right]^{\frac{l}{2}+k\nu}(\log(\nu\tau-R))^j R\,dR\end{aligned}$$

Then it is straightforward to check that

$$\left|F^{(1),\pm}_{l,k,j}(\tau,\xi)\right|\lesssim\tau^{-1-\nu},$$

due to the fact that $\left|\phi_{\pm,\tau}(R;\xi,\hbar)\right|\lesssim\tau^{-\frac{1}{2}}\xi^{-\frac{1}{4}}$ on the support of the integrand, as well as the fact that the R-integration interval is of length $\lesssim\xi^{-\frac{1}{2}}$. We claim that $F^{(1),\pm}_{l,k,i}(\tau,\xi)$ also has the kind of symbol behavior asserted in the lemma. The main difficulty here comes from dealing with the phase $e^{\pm i\hbar^{-1}\rho(x;\alpha,\hbar)}$. The idea is to exploit the localization near the light cone expressed by the condition $(\nu\tau-R)\xi^{\frac{1}{2}}\lesssim 1$ to expand this phase into Taylor expansion around $R=\nu\tau$. In fact, setting $x_\tau := \xi^{\frac{1}{2}}\hbar\nu\tau$, we can write

$$\hbar^{-1}\rho(x;\alpha,\hbar)=\hbar^{-1}\rho(x_\tau;\alpha,\hbar)+\hbar^{-1}\sum_{j\geq 1}\frac{1}{j!}(\partial_x^j\rho)(x_\tau;\alpha,\hbar)\cdot(x-x_\tau)^j \tag{7.2}$$

It follows that we can write

$$\overline{x}_{21} = \sum_{\pm}\chi_{\xi\hbar^2\gtrsim 1}\hbar^{-1}\left\langle\xi\hbar^2\right\rangle^{-\frac{l}{4}}\frac{e^{\pm i\left[\nu\tau\xi^{\frac{1}{2}}+\hbar^{-1}\rho(x_\tau;\alpha,\hbar)\right]}}{\xi^{\frac{3}{4}+\frac{k\nu}{2}}}(\log\xi)^j\cdot\tilde{F}^{(1),\pm}_{l,k,j}(\tau,\xi),$$

where we have

$$\begin{aligned}&\tilde{F}^{(1),\pm}_{l,k,j}(\tau,\xi)\\&=\tilde{\chi}_{\xi\hbar^2\gtrsim 1}\frac{\left\langle\xi\hbar^2\right\rangle^{\frac{l}{4}}}{(\xi\hbar^2)^{\frac{l}{4}}}\cdot\frac{\xi^{\frac{3}{4}}}{(\log\xi)^j}\int_0^\infty\tilde{\phi}_{\pm,\tau}(R;\xi,\hbar)\chi_{|\nu\tau-R|\lesssim\xi^{-\frac{1}{2}}}\frac{G_{k,l,j}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\left[\xi^{\frac{1}{2}}(\nu\tau-R)\right]^{\frac{l}{2}+k\nu}(\log(\nu\tau-R))^j R\,dR,\end{aligned}$$

$$\tilde{\phi}_{\pm,\tau}(R;\xi,\hbar):=\frac{e^{\mp i(\nu\tau-R)\xi^{\frac{1}{2}}\pm i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]}}{\left(R\xi^{\frac{1}{2}}\right)^{\frac{1}{2}}}\cdot\psi(x;\alpha,\hbar)\cdot\left(1+\hbar\overline{\tilde{a}_1(\zeta;\alpha)}\right)$$

But then taking advantage of the localization of the integral as well as (7.2) it is straightforward to check that $\tilde{F}^{(1),\pm}_{l,k,i}(\tau,\xi)$ has the desired Symbol type behavior.

It remains to deal with the term $\overline{x}_{22}(\tau,\xi)$. Proceeding in analogy to $\overline{x}_{21}$, write this as

$$\overline{x}_{22} = \sum_{\pm}\chi_{\xi\hbar^2\gtrsim 1}\hbar^{-1}\left\langle\xi\hbar^2\right\rangle^{-\frac{l}{4}}\frac{e^{\pm i\left[\nu\tau\xi^{\frac{1}{2}}+\hbar^{-1}\rho(x_\tau;\alpha,\hbar)\right]}}{\xi^{\frac{3}{4}+\frac{k\nu}{2}}}(\log\xi)^j\cdot\tilde{F}^{(2),\pm}_{l,k,j}(\tau,\xi),$$

where we have

$$\tilde{F}^{(2),\pm}_{l,k,j}(\tau,\xi)$$
$$=\tilde{\chi}_{\xi\hbar^2\gtrsim 1}\frac{\left\langle\xi\hbar^2\right\rangle^{\frac{l}{4}}}{(\xi\hbar^2)^{\frac{l}{4}}}\cdot\frac{\xi^{\frac{3}{4}}}{(\log\xi)^j}\int_0^\infty\chi_{\xi^{-\frac{1}{2}}\lesssim|\nu\tau-R|\lesssim\hbar}\tilde{\phi}_{\pm,\tau}(R;\xi,\hbar)\frac{G_{k,l,j}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\left[\xi^{\frac{1}{2}}(\nu\tau-R)\right]^{\frac{l}{2}+k\nu}(\log(\nu\tau-R))^j R\,dR,$$

$$\tilde{\phi}_{\pm,\tau}(R;\xi,\hbar):=\frac{e^{\mp i(\nu\tau-R)\xi^{\frac{1}{2}}\pm i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]}}{\left(R\xi^{\frac{1}{2}}\right)^{\frac{1}{2}}}\cdot\psi(x;\alpha,\hbar)\cdot\left(1+\hbar\overline{\tilde{a}_1(\zeta;\alpha)}\right)$$

Then introduce the new variable

$$\tilde{R}:=(\nu\tau-R)\xi^{\frac{1}{2}}-\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right].$$

Then observe that on account of $x\gg 1$ on the support of the integrand, we have

$$\frac{\partial\tilde{R}}{\partial R}=-\xi^{\frac{1}{2}}-\hbar^{-1}\rho_x\cdot\hbar\xi^{\frac{1}{2}}\simeq-\xi^{\frac{1}{2}}.$$

Moreover, due to

$$\begin{aligned}\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]&=\hbar^{-1}\left(\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\,ds\right)\cdot(x-x_\tau)\\&=\left(\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\,ds\right)\cdot(R-\nu\tau)\xi^{\frac{1}{2}}\end{aligned}$$

and $|\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)|\lesssim\tau^{-2}$ on the region $|R-\nu\tau|<1,\ x\gtrsim 1$, we see that

$$\tilde{R}\simeq(\nu\tau-R)\xi^{\frac{1}{2}}$$

on the support of the integrand. Furthermore, interpreting $\nu\tau-R=(\nu\tau-R)(\tilde{R},\xi,\tau)$, we get

$$\begin{aligned}\left[(\nu\tau-R)\xi^{\frac{1}{2}}\right]_{,\xi^{\frac{1}{2}}}&=\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_\tau;\alpha,\hbar)\right]_{,\xi^{\frac{1}{2}}}\\&=\hbar^{-1}\cdot\left(\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\,ds\cdot(x-x_\tau)\right)_{,\xi^{\frac{1}{2}}}\\&=\xi^{\frac{1}{2}}\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\,ds\cdot(R-\nu\tau)_{,\xi^{\frac{1}{2}}}\\&+\left(\xi^{\frac{1}{2}}\int_0^1\rho_x(sx+(1-s)x_\tau;\alpha,\hbar)\,ds\right)_{,\xi^{\frac{1}{2}}}\cdot(R-\nu\tau).\end{aligned}$$

This relation easily implies that

$$\left|(R-\nu\tau)_{,\xi^{\frac{1}{2}}}\right| \lesssim \frac{\nu\tau - R}{\xi^{\frac{1}{2}}}$$

on the support of the integrand defining $\tilde{F}^{(2),\pm}_{l,k,j}(\tau,\xi)$. On the other hand, using the fact $\tilde{R} \simeq (\nu\tau - R)\xi^{\frac{1}{2}}$ we have

$$\left|(\nu\tau - R)_{,\tilde{R}}\right| \lesssim \xi^{-\frac{1}{2}} \simeq \frac{\nu\tau - R}{\tilde{R}}.$$

Then expressing the integral for $\tilde{F}^{(2),\pm}_{l,k,j}(\tau,\xi)$ in terms if $\tilde{R}$ and performing sufficiently many integrations by parts with respect to $\tilde{R}$, as well as exploiting the symbol behavior of $G_{k,l,j}(\tau,\nu\tau - R)$ and the preceding inequality, the desired symbol behavior for $\tilde{F}^{(2),\pm}_{l,k,j}(\tau,\xi)$ follows easily. □

7.3. **Description of the shock on the distorted Fourier side II: admissible expansions.** When formulating the precise asymptotic expansions we shall use for the description of the singular part on the Fourier side, we have to take into account the action of the solution of the wave equation at angular momentum n, $|n| \geq 2$, and formulated on the (distorted) Fourier side, given by the Duhamel formula

$$\int_{\tau_0}^{\tau} \frac{\lambda(\tau)}{\lambda(\sigma)} \cdot \left(\frac{\rho_n\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n(\xi)}\right)^{\frac{1}{2}} \cdot \frac{\sin\left[\lambda(\tau)\xi^{\frac{1}{2}}\int_\sigma^\tau \lambda^{-1}(u)\,du\right]}{\xi^{\frac{1}{2}}} \cdot F\left(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) d\sigma$$

If $F(\sigma,\xi)$ is a singular source term admitting an expansion as in the preceding subsection, then the re-scaled term $F\left(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)$ will come in general with a phase

$$e^{\pm i\left(\nu\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}} + \hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)\right)},$$

while the other oscillatory term $\sin\left[\lambda(\tau)\xi^{\frac{1}{2}}\int_\sigma^\tau \lambda^{-1}(u)\,du\right]$ will contribute

$$e^{\pm i\nu\left(\tau - \frac{\lambda(\tau)}{\lambda(\sigma)}\sigma\right)\cdot\xi^{\frac{1}{2}}}.$$

These phases will either enter 'destructive resonance', resulting in a singularity propagating precisely along the light cone characteristic $R = \nu\tau$, or else enter into a 'constructive resonance' resulting in a singularity propagating outside of the light cone. The latter situation, when translated to the interior of the light cone, will result in smoother terms, in principle analogous to the connecting singular terms from the preceding section, but with a more complicated algebraic structure, which we will have to keep track of. Precisely, in essence we will encounter integrals of the form (where as usual $\alpha = \hbar\xi^{\frac{1}{2}}$)

$$e^{\pm i\nu\tau\xi^{\frac{1}{2}}}\int_{\tau_0}^{\tau} e^{\pm i\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)} \cdot G\left(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) d\sigma$$

$$\int_{\tau_0}^{\tau} e^{\pm i\left(\left(\nu\tau - 2\frac{\lambda(\tau)}{\lambda(\sigma)}\nu\sigma\right)\xi^{\frac{1}{2}} - \hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)\right)} \cdot G\left(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) d\sigma,$$

where the two ±-signs in the first integral are synchronized, and where the factors $G\left(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)$ have suitable symbol behaviour. The key point concerning the second integral represents a function whose singularity (on the physical side) is located at $R = 2\frac{\lambda(\tau)}{\lambda(\sigma)}\nu\sigma - \nu\tau \geq \nu\tau$ due to $\sigma \leq \tau$, and the integration over σ ensures this function, upon restriction to the interior of the light cone, is of regularity H^{2+}.
Of course, the preceding integration expressions need to be re-inserted into suitable source terms and again subjected to the wave parametrix, resulting in more complex expressions in principle. However, a simple formalism is possible to capture all the terms that arise in this fashion, which is realized in the following definition:

Definition 7.8. *We call a function $\overline{x}(\tau, \xi)$, $\xi \in [0, \infty)$, an admissible singular part at angular momentum $n, |n| \geq 2$, provided it allows a representation*

$$\overline{x}(\tau, \xi) = \overline{x}_{in}(\tau, \xi) + \overline{x}_{out}(\tau, \xi) + \overline{x}_{prot}(\tau, \xi)$$

where $\overline{x}_{in}$ represents the ingoing part of the singularity, which constitutes the dominant part, while $\overline{x}_{out}$ represents the outgoing part, $\overline{x}_{prot}(\tau, \xi)$ is a prototypical singularity as in Definition 7.2 and the first two of these admit the following expansions:
We can write (in the following all cutoffs χ are smooth)

$$\begin{aligned}\overline{x}_{in}(\tau, \xi) = &\sum_{\pm}\sum_{k=1}^{N}\sum_{j=0}^{N_1}\chi_{\xi\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)}\cdot a_{k,j}^{(\pm)}(\tau, \sigma)\,d\sigma\\ &+\sum_{\pm}\sum_{l=1}^{7}\sum_{k=1}^{N}\sum_{j=0}^{N_1}\chi^{(l)}_{\xi\geq\hbar^{-2}}\hbar^{-1}\left\langle\hbar^2\xi\right\rangle^{-\frac{l}{4}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^j\\ &\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)}\cdot F_{l,k,j}^{(\pm)}\left(\tau, \sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,\end{aligned} \tag{7.3}$$

where the ±-signs in each expression on the right are synchronized, and we have the following bounds, where the δ_l are small positive numbers decreasing in l: :

$$\begin{aligned}&\left|a_{k,j}^{(\pm)}(\tau, \sigma)\right| + \tau^{k_1}\left|\partial_\tau^{k_1}a_{k,j}^{(\pm)}(\tau, \sigma)\right| \lesssim_{k_1} (\log\tau)^{N_1-j}\tau^{-1-\nu}\cdot\sigma^{-3},\\ &\left|\xi^{k_2}\partial_\tau^{\iota}\partial_\xi^{k_2}F_{l,k,j}^{(\pm)}(\tau, \sigma, \xi)\right| \lesssim_{k_1} (\log\tau)^{N_1-j}\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right], \quad 0\leq k_2\leq 5, \quad \iota\in\{0,1\}\\ &\left\|\xi^{5+\delta-k_1}\partial_\tau^{\iota}\partial_\xi^{5}F_{l,k,i}^{(\pm)}(\tau, \sigma, \xi)\right\|_{\dot{C}^{\delta}_{\xi}(\xi\simeq\mu)} \lesssim_{k_1} (\log\tau)^{N_1-j}\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\mu^{\frac{1}{2}}\right)\right], \quad \iota\in\{0,1\}\end{aligned}$$

where we set $\kappa(x) := \frac{x}{1+x^2}$ and μ can be any positive number. We shall again call the first sum on the right in (7.3) *the principal part of the ingoing singularity, while we call the second sum the connecting part of the ingoing singularity.*
For the outgoing part, we can split it into

$$\overline{x}_{out} = \overline{x}_{out,1} + \overline{x}_{out,2}$$

where the first term on the right represents the 'outgoing seed singularity' given by

$$\begin{aligned}\overline{x}_{out,1}(\tau,\xi) = \sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{j=0}^{N_1}\chi_{\xi\geq\hbar^{-2}}\hbar^{-1}\left\langle\hbar^2\xi\right\rangle^{-\frac{l}{4}}\frac{(\log\xi)^j}{\xi^{1+\frac{k\nu}{2}}}\\ \cdot\int_{\tau_0}^{\tau}e^{\pm i\left[\left(\nu\tau-2\frac{\lambda(\tau)}{\lambda(\sigma)}\nu\sigma\right)\xi^{\frac12}-\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)\right]}\cdot\tilde{F}^{\pm}_{l,k,j}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,\end{aligned} \tag{7.4}$$

and we have the bounds

$$\left|\xi^r\partial_\tau^\iota\partial_\xi^r\tilde{F}^{(\pm)}_{l,k,j}(\tau,\sigma,\xi)\right|\lesssim(\log\tau)^{N_1-j}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\xi^{\frac12}\right)\right],\quad 0\leq r\leq 5,\quad \iota\in\{0,1\}$$

$$\left\|\xi^{5+\delta}\partial_\tau^\iota\partial_\xi^5F^{(\pm)}_{l,k,j}(\tau,\sigma,\xi)\right\|_{\dot{C}^\delta_\xi(\xi\simeq\mu)}\lesssim(\log\tau)^{N_1-j}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\mu^{\frac12}\right)\right],\quad \iota\in\{0,1\}.$$

The second term $\overline{x}_{out,2}$, which is the 'outgoing perpetuated singularity' admits the description

$$\begin{aligned}\overline{x}_{out,2}(\tau,\xi) = \sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{j=0}^{N_1}\chi_{\xi\geq\hbar^{-2}}\hbar^{-1}\left\langle\hbar^2\xi\right\rangle^{-\frac{l}{4}}\frac{(\log\xi)^j}{\xi^{1+k\frac{\nu}{2}}}\\ \cdot\int_0^\infty\int_{\tau_0}^{\tau}e^{\pm i\left[\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x+\tau\right)\xi^{\frac12}+\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)\right]}\cdot G^{\pm}_{l,k,j}\left(\tau,\sigma,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma dx,\end{aligned} \tag{7.5}$$

where $F^{\pm}_{0,k,i}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)=b^{\pm}_{k,i}(\tau,\sigma)$, $G^{\pm}_{0,k,i}\left(\tau,\sigma,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)=c^{\pm}_{k,i}(\tau,\sigma,x)$, and we have the bounds

$$\left\|\xi^{k_2}\partial_\xi^{k_2}G^{(\pm)}_{l,k,i}(\tau,\sigma,x,\xi)\right\|_{L^1_x}\lesssim(\log\tau)^{N_1-i}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\xi^{\frac12}\right)\right],\quad 0\leq k_2\leq 5,$$

$$\left\|\left\|\xi^{5+\delta_l}\partial_\xi^5G^{(\pm)}_{l,k,i}(\tau,\sigma,x,\xi)\right\|_{\dot{C}^{\delta_l}_\xi(\xi\simeq\lambda)}\right\|_{L^1_x}\lesssim(\log\tau)^{N_1-i}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\lambda^{\frac12}\right)\right],$$

Moreover, we require[4] *that the same structure and bounds apply to $\xi^{-\frac12}\cdot\left(\partial_\tau-2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi\right)\overline{x}_{out,2}$ up to terms of the kinds $\overline{x}_{in}$, $\overline{x}_{out,1}$. Finally, we say that the principal singular part is of restricted type, provided for $k\in\{1,2\}$ we have*

$$a_{k,i}(\tau,\sigma)=\sum_{l=i}^{N_1}c^{(\pm)}_{l,k,i}(\sigma)\cdot\tau^{-k(1+\nu)}\cdot(\log\tau)^{l-i}+b^{(\pm)}_{k,i}(\tau,\sigma),$$

$$\left|c^{(\pm)}_{l,k,i}(\sigma)\right|\lesssim\sigma^{-3},\quad \left|b_{k,i}(\tau,\sigma)\right|+\tau\cdot\left|b'_{k,i}(\tau,\sigma)\right|\lesssim\tau^{-3-\nu}(\log\tau)^{N_1}\cdot\sigma^{-3},$$

while for $k\geq 3$ we have

$$\left|a^{(\pm)}_{k,i}(\tau,\sigma)\right|+\tau\left|\partial_\tau a^{(\pm)}_{k,i}(\tau,\sigma)\right|\lesssim(\log\tau)^{N_1-i}\,\tau^{-3-\nu}\cdot\sigma^{-3}.$$

The preceding space of functions comes equipped with a natural norm, given in the following

[4]This last somewhat delicate requirement is not needed for the previous term $x_{\text{out},1}$, since its structure implies this essentially as a corollary

Definition 7.9. *Assume that $\overline{x}(\tau,\xi)$ is an admissible singular part at angular momentum $n, |n| \geq 2$. Then we set*

$$
\begin{aligned}
&\|x\|_{adm} := \\
&\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\left\|\left[\left|a_{k,i}^{(\pm)}(\tau,\sigma)\right| + \tau\left|\partial_\tau a_{k,i}^{(\pm)}(\tau,\sigma)\right|\right]\cdot(\log\tau)^{-N_1+i}\,\tau^{1+\nu}\cdot\sigma^3\right\|_{L^\infty([\tau_0,\infty))L^\infty([\tau_0,\tau])} \\
&+\sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\sum_{\substack{0\leq k_2\leq 5\\ \iota\in\{0,1\}}}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^{k_2}\partial_\xi^{k_2}\partial_\tau^\iota F_{l,k,i}^{(\pm)}(\tau,\sigma,\xi)}{\left(\sigma^{-2}+\kappa(\hbar\xi^{\frac12})\right)}\right\|_{L_\xi^\infty([0,\infty))}\right\|_{L_\tau^\infty([\tau_0,\infty))L_\sigma^\infty([\tau_0,\tau])} \\
&+\sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\sum_{\substack{0\leq k_2\leq 5\\ \iota\in\{0,1\}}}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^{k_2}\partial_\xi^{k_2}\partial_\tau^\iota \tilde{F}_{l,k,i}^{(\pm)}(\tau,\sigma,\xi)}{\left(\sigma^{-2}+\kappa(\hbar\xi^{\frac12})\right)}\right\|_{L_\xi^\infty([0,\infty))}\right\|_{L_\tau^\infty([\tau_0,\infty)L_\sigma^\infty([\tau_0,\tau]} \\
&+\sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\sum_{\iota}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^5\partial_\xi^5\partial_\tau^\iota F_{l,k,i}^{(\pm)}(\tau,\sigma,\xi)}{\left(\sigma^{-2}+\kappa(\hbar\xi^{\frac12})\right)}\right\|_{\dot{C}_\xi^{\delta_*}}\right\|_{L_\tau^\infty([\tau_0,\infty))L_\sigma^\infty([\tau_0,\tau])} \\
&+\sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\sum_{\iota}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^5\partial_\xi^5\partial_\tau^\iota \tilde{F}_{l,k,i}^{(\pm)}(\tau,\sigma,\xi)}{\left(\sigma^{-2}+\kappa(\hbar\xi^{\frac12})\right)}\right\|_{\dot{C}_\xi^{\delta_*}}\right\|_{L_\tau^\infty([\tau_0,\infty))L_\sigma^\infty([\tau_0,\tau])} \\
&+\sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\sum_{0\leq k_2\leq 5}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^{k_2}\partial_\xi^{k_2}G_{l,k,i}^{(\pm)}(\tau,\sigma,x,\xi)}{\left(\sigma^{-2}+\kappa(\hbar\xi^{\frac12})\right)}\right\|_{L_\xi^\infty([0,\infty)}\right\|_{L_x^1([0,\infty))L_\tau^\infty(([\tau_0,\infty))L_\sigma^\infty(([\tau_0,\tau])} \\
&+\sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^5\partial_\xi^5\partial_\tau^\iota G_{l,k,i}^{(\pm)}(\tau,\sigma,x,\xi)}{\left(\sigma^{-2}+\kappa(\hbar\xi^{\frac12})\right)}\right\|_{\dot{C}_\xi^{\delta_*}}\right\|_{L_x^1L_\tau^\infty([\tau_0,\infty))L_\sigma^\infty([\tau_0,\tau])} \\
&+\ldots,
\end{aligned}
$$

where the term $\ldots$ stands for the additional terms of a similar form which get included when applying $(\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi)$ to the final term in the structure formula for $\overline{x}_{1,smooth}$.
If the principal singular part is of restricted type, we shall replace the first term

$$\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\left\|\left[\left|a_{k,i}^{(\pm)}(\tau,\sigma)\right| + \tau\left|\partial_\tau a_{k,i}^{(\pm)}(\tau,\sigma)\right|\right]\cdot(\log\tau)^{-N_1+i}\,\tau^{1+\nu}\cdot\sigma^3\right\|_{L^\infty([\tau_0,\infty))L^\infty([\tau_0,\tau])}$$

by

$$\sum_{\pm}\sum_{l=0}^{7}\sum_{k=1,2}\left\|\sigma^3\cdot c_{l,k,i}^{(\pm)}(\sigma)\right\|_{L^\infty([\tau_0,\infty))} + \sum_{\pm}\sum_{k=1,2}\sum_{i=0}^{N_1}\left\|b_{k,i}^{(\pm)}(\tau,\sigma)\cdot(\log\tau)^{-N_1+i}\,\tau^{3+\nu}\cdot\sigma^3\right\|_{L^\infty([\tau_0,\infty))L^\infty([\tau_0,\tau])}$$

$$+\sum_{\pm}\sum_{k=3}^{N}\sum_{i=0}^{N_1}\left\|\left[\left|a_{k,i}^{(\pm)}(\tau,\sigma)\right|+\tau\left|\partial_\tau a_{k,i}^{(\pm)}(\tau,\sigma)\right|\right]\cdot(\log\tau)^{-N_1+i}\,\tau^{3+\nu}\cdot\sigma^3\right\|_{L^\infty([\tau_0,\infty))L^\infty([\tau_0,\tau])}$$

and call the resulting norm $\|\cdot\|_{adm(r)}$.

In the rest of this subsection as well as the next one, we verify that this choice of function space is compatible with the key operations that will arise during the iterative scheme. The first order of the day shall be to recover an analogue of Lemma 7.6. The key point is that the next lemma has an almost identical conclusion, except for some subtle logarithmic loss with respect to time. We also carefully note that for now we omit conclusions about time derivatives, since the correct formulation of these will require the more complicated but natural operator $\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R$.

Lemma 7.10. *Assume that* $\overline{x}$ *is an admissible singular part at angular momentum* $n, |n| \geq 2$. *Then the associated function*

$$f(\tau,R) := \int_0^\infty \phi_n(R;\xi)\cdot\overline{x}(\tau,\xi)\cdot\rho_n(\xi)\,d\xi,$$

restricted to the light cone $R < \nu\tau$, *can be decomposed as*

$$f = f_1 + f_2 + f_3$$

where $f_1 = f_1(\tau,R)$ *is a* C^5*-function supported in* $\nu\tau - R \gtrsim 1$ *and satisfying*

$$\nabla_R^{k_2} f_1(\tau,R) \lesssim \hbar^{-1}(\log\tau)^{N_1+1}\cdot\tau^{-\frac{3}{2}-\nu}\,|\nu\tau-R|^{-5}, \quad 0\leq k_2\leq 5,$$

while $f_2 = \sum_{l=1}^{8} f_{2l}$ *where we have the explicit form*

$$f_{2l}(\tau,R) = \chi_{|\nu\tau-R|\lesssim\hbar}\sum_{k=1}^{N}\sum_{j=0}^{N_1}\frac{G_{k,l,j}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar^{-\frac{l+1}{2}}\,[\nu\tau-R]^{\frac{l}{2}+k\nu}\,(\log(\nu\tau-R))^j$$

Here the function $G_{k,j}(\tau,x)$ *has symbol type behavior with respect to* x, *as follows:*

$$\left|\partial_x^{k_2}G_{k,l,j}(\tau,x)\right| \lesssim (\log\tau)^{N_1-j+1}\cdot\tau^{-1-\nu}x^{-k_2}, \quad 0\leq k_2\leq 5$$

and we have the bound

$$\left\|x^{5+\delta}\partial_x^5 G_{k,l,j}(\tau,x)\right\|_{\dot{C}^\delta} \lesssim (\log\tau)^{N_1-j+1}\cdot\tau^{-1-\nu}.$$

Finally, the remaining function f_3 *is also* C^5 *and supported in* $|\nu\tau-R|\lesssim 1$ *and satisfies*

$$\|f_3\|_{\tilde{S}_0^{(\hbar)}} \lesssim (\log\tau)^{N_1+1}\cdot\tau^{-1-\nu}, \quad \|\partial_\tau f_3\|_{\tilde{S}_1^{(\hbar)}} \lesssim (\log\tau)^{N_1+1}\cdot\tau^{-1-\nu}.$$

Moreover, we have the bounds

$$\left|\partial_R^k f_3\right| \lesssim (\log\tau)^{N_1+1}\,\tau^{-\frac{3}{2}-\nu}\cdot\hbar^{-1}\min\{(\nu\tau-R)^{-k},\hbar^{-k}\}, \quad 0\leq k\leq 5.$$

Furthermore, the terms $\overline{x}_{out}$ *enjoy a smoothness gain visible on the physical side upon restriction to the interior of the light cone: defining*

$$g(\tau,R) := \int_0^\infty \phi_n(R;\xi)\cdot\overline{x}_{out}(\tau,\xi)\cdot\rho_n(\xi)\,d\xi,$$

we can write

$$g = g_1 + g_2 + g_3,$$

where $g_{1,3}$ have the same properties as $f_{1,3}$, while g_2 admits the representation

$$g_2 = \chi_{|\nu\tau - R|\lesssim\hbar} \sum_{l=3}^{8} \sum_{k=1}^{N} \sum_{j=0}^{N_1} \frac{H_{k,l,j}(\tau, \nu\tau - R)}{\tau^{\frac{1}{2}}} \hbar^{-\frac{l+1}{2}} [\nu\tau - R]^{\frac{l}{2}+k\nu} (\log(\nu\tau - R))^j$$

and we have the symbol type bounds

$$\left|\partial_x^{k_2} H_{k,l,j}(\tau, x)\right| \lesssim (\log\tau)^{N_1-j+1} \cdot \tau^{-1-\nu} x^{-k_2}, \quad k_2 \in \{0, \ldots, 3\},$$

as well as

$$\left\| x^{3+\delta} \partial_x^3 H_{k,l,j}(\tau, x) \right\|_{\dot{C}^\delta} \lesssim (\log\tau)^{N_1-j+1} \cdot \tau^{-1-\nu}.$$

The preceding two bounds hold also with derivative up to degree 5, provided we control the ξ-derivatives of the functions $F^{\pm}_{l,k,j}, G^{\pm}_{l,k,j}$ up to degree 7.

Proof. We shall use the crude bound $0 \leq \kappa\left(\hbar\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}\right) \leq 1$ for the auxiliary function appearing in the bounds in the preceding definition, since this fine structure will only play a role later on. We can then work at fixed time σ and deduce the final result by integration over σ. The proof follows exactly the one of Lemma 7.6, and we explain the minor differences arising.

The case of ingoing singularity, i.e. the contribution of (7.3). The argument is the same for the two integral expressions up to minor details. Fixing $\sigma \in [\tau_0, \tau]$, we encounter the oscillatory expression

$$e^{\pm i\left(\nu\tau + \hbar^{-1}\rho(x_{\sigma'};\alpha',\hbar)\right)\xi^{\frac{1}{2}}}, \quad \sigma' = \sigma \cdot \frac{\lambda(\tau)}{\lambda(\sigma)} \geq \tau, \quad \alpha' = \alpha \cdot \frac{\lambda(\tau)}{\lambda(\sigma)}.$$

Then going through the case *(1) - (4)* in the proof of Lemma 7.6, in cases *(1)*, *(2)* we replace $e^{\pm i\nu\tau\xi^{\frac{1}{2}}}$ by $e^{\pm i\left(\nu\tau+\hbar^{-1}\rho(x_{\sigma'};\alpha',\hbar)\right)\xi^{\frac{1}{2}}}$, then we use the bound

$$\begin{aligned}\partial_{\xi^{\frac{1}{2}}}\left(\left(\nu\tau\xi^{\frac{1}{2}} + \hbar^{-1}\rho(x_{\sigma'};\alpha',\hbar)\right)\xi^{\frac{1}{2}}\right) &= \nu\tau + \hbar^{-1}\rho_x(x_{\sigma'};\alpha',\hbar)\cdot\hbar\nu\sigma' + \hbar^{-1}\rho_\alpha(x_{\sigma'};\alpha',\hbar)\cdot\hbar \\ &= \nu\tau + O\left(\sigma'^{-1} + \hbar\right) \gtrsim \tau\end{aligned}$$

on the support of the expression which we recall is $\xi \gtrsim \hbar^{-2}$.

In case *(3)*, repeating the notation from the end of the proof of Lemma 7.6, we encounter the phase

$$\pm\frac{2}{3}\hbar^{-1}\zeta^{\frac{3}{2}} \pm \left(\nu\tau\xi^{\frac{1}{2}} + \hbar^{-1}\rho(x_{\sigma'};\alpha,\hbar)\right),$$

and thanks to $\sigma' \geq \tau$ and the condition $\xi \gtrsim \hbar^{-2}$ on the support, we again have

$$\left|\partial_{\xi^{\frac{1}{2}}}\left(\pm\frac{2}{3}\hbar^{-1}\zeta^{\frac{3}{2}} \pm \left(\nu\tau\xi^{\frac{1}{2}} + \hbar^{-1}\rho(x_{\sigma'};\alpha,\hbar)\right)\right)\right| \gtrsim \tau.$$

Case *(4)*, which we recall means $x \geq (1+\gamma)\,x_t$, is again the most complicated, and we argue as follows. Here in the regime $\nu\tau - R \gg 1$ and the most delicate case of resonant phases we encounter the phase function

$$\pm(\nu\tau - R)\xi^{\frac{1}{2}} \mp \hbar^{-1}\left[\rho(x;\alpha,\hbar) - \rho(x_{\sigma'};\alpha',\hbar) - y(\alpha;\hbar)\right]$$

$$
\begin{aligned}
&= \pm(\nu\tau - R)\xi^{\frac{1}{2}} \mp \hbar^{-1}\left[\rho(x;\alpha,\hbar) - \rho(x_\tau;\alpha,\hbar) - y(\alpha;\hbar)\right] \pm \hbar^{-1}\left[\rho(x_{\sigma'};\alpha',\hbar) - \rho(x_\tau;\alpha,\hbar)\right]\\
&=: \pm\Omega(\tau,R,\alpha,\hbar)
\end{aligned}
$$

and the following lower bound obtains:

$$
\begin{aligned}
\partial_{\xi^{\frac{1}{2}}}\Omega &\gtrsim \nu\tau - R + \hbar^{-1}\rho_x(x_{\sigma'};\alpha',\hbar)\cdot\hbar\nu\sigma' - \hbar^{-1}\rho_x(x_\tau;\alpha,\hbar)\cdot\hbar\nu\tau + O(\hbar)\\
&\gtrsim \nu\tau - R
\end{aligned}
$$

provided $\nu\tau - R \gg \hbar$, since we have the important positivity property

$$
\hbar^{-1}\rho_x(x_{\sigma'};\alpha',\hbar)\cdot\hbar\nu\sigma' - \hbar^{-1}\rho_x(x_\tau;\alpha,\hbar)\cdot\hbar\nu\tau \geq 0 \tag{7.6}
$$

on account of $\sigma' \geq \tau$, $\alpha' \geq \alpha$. This implies that the term (we re-use the notation from case *(4)* in the proof of Lemma 7.6) and where the temporally decaying factor $a^{\pm}_{k,j}(\tau,\sigma)$ according to Definition 7.8 has been included into $\Psi(x;\alpha,\hbar)$, and the cutoff $\chi_{\xi\geq 1}$ has been replaced by $\chi_{\xi\geq\hbar^{-2}}$

$$
\chi_{\nu\tau-R\gg 1}\int_0^\infty e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi
$$

can be included into $f_1(\tau,R)$, and similarly the term

$$
\chi_{1\gtrsim\nu\tau-R\gg\hbar}\int_0^\infty e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi
$$

can be incorporated into $f_3(\tau,R)$.

This reduces things to controlling the integral

$$
\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi
$$

Write this as

$$
\begin{aligned}
&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\
&= \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac{1}{2}}\lesssim 1}e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi \qquad (7.7)\\
&+ \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac{1}{2}}\gtrsim 1}e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi
\end{aligned}
$$

Split the first term on the right into the following

$$
\begin{aligned}
&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac{1}{2}}\lesssim 1}e^{\pm i(\nu\tau-R)\xi^{\frac{1}{2}}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\
&= \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac{1}{2}}\lesssim 1}\cdot e^{\mp i\hbar^{-1}\left[\rho(x_\tau;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\
&+ \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac{1}{2}}\lesssim 1}\left(e^{\pm i\left[(\nu\tau-R)\xi^{\frac{1}{2}}+\hbar^{-1}(\rho(x_\tau;\alpha,\hbar)-\rho(x;\alpha,\hbar))\right]}-1\right)\\
&\qquad\qquad\cdot e^{\mp i\hbar^{-1}\left[\rho(x_\tau;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\
&=: A_1 + A_2.
\end{aligned}
$$

Then the first term A_1 can be split into a regular term and a explicit term via

$$\begin{aligned}
&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac12}\lesssim 1}\cdot e^{\mp i\hbar^{-1}\left[\rho(x_\tau;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\
&=\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty e^{\mp i\hbar^{-1}\left[\rho(x_\tau;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi\\
&-\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac12}\gtrsim 1}\cdot e^{\mp i\hbar^{-1}\left[\rho(x_\tau;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi.
\end{aligned}$$

Here the first of the last two terms is of the required form for f_3, as the oscillatory phase is now independent of R, while the second term admits the representation required of terms of the form f_2 in the lemma. The term A_2 is also of the explicit form, by writing it as

$$\begin{aligned}
A_2 =&(\nu\tau-R)\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac12}\lesssim 1}\frac{\left(e^{\pm i\left[(\nu\tau-R)\xi^{\frac12}+\hbar^{-1}(\rho(x_\tau;\alpha,\hbar)-\rho(x;\alpha,\hbar))\right]}-1\right)}{(\nu\tau-R)\xi^{\frac12}}\\
&\qquad\cdot e^{\mp i\hbar^{-1}\left[\rho(x_\tau;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\xi^{\frac12}\Psi(x;\alpha,\hbar)\,d\xi,
\end{aligned}$$

and arguing as in the proof of Lemma 7.6.
It remains to deal with the integral

$$\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{(\nu\tau-R)\xi^{\frac12}\gtrsim 1}e^{\pm i(\nu\tau-R)\xi^{\frac12}}\cdot e^{\mp i\hbar^{-1}\left[\rho(x;\alpha,\hbar)-y(\alpha;\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]}\cdot\Psi(x;\alpha,\hbar)\,d\xi,$$

which we claim is of the explicit form, i.e. can be included into f_2. This can easily be seen by combining the term $e^{\pm i\hbar^{-1}y(\alpha;\hbar)}$ with $\Psi(x;\alpha,\hbar)$, and taking advantage of the inequality, in view of the positivity property (7.6),

$$\partial_{\xi^{\frac12}}\left[(\nu\tau-R)\xi^{\frac12}-\hbar^{-1}\left[\rho(x;\alpha,\hbar)-\rho(x_{\sigma'};\alpha',\hbar)\right]\right]\gtrsim \nu\tau-R,$$

and invoking integration by parts with respect to $\xi^{\frac12}$. This concludes the argument for the contribution of the terms (7.3).

The case of outgoing singularity, i.e. the contribution of (7.4)*,* (7.5)*.* Here the argument is very similar except that we have to replace the phases

$$e^{\pm i\left[(\nu\tau-R)\xi^{\frac12}+\hbar^{-1}\rho\left(x_{\sigma'};\alpha',\hbar\right)\right]}$$

by

$$e^{\pm i\left[\left(\nu\tau-2\frac{\lambda(\tau)}{\lambda(\sigma)}\nu\sigma+R\right)\xi^{\frac12}-\hbar^{-1}\rho\left(x_{\sigma'};\alpha',\hbar\right)\right]},\quad e^{\pm i\left[\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x+\tau\right)\xi^{\frac12}-R\xi^{\frac12}+\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)\right]}$$

for the contributions of (7.4), (7.5), respectively, and on account of (recall $\sigma\leq\tau$, $x\geq 0$)

$$\nu\tau-2\frac{\lambda(\tau)}{\lambda(\sigma)}\nu\sigma\leq-\nu\tau,\quad \nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x+\tau\right)-R\geq\nu\tau-R,$$

we can repeat the arguments from before since all functions will be restricted to the interior of the light cone, and hence we encounter functions of the type

$$(\nu\tau - R + y)^{\frac{l}{2}+k\nu} H(\tau, \nu\tau - R + y), \quad y \geq 0,$$

where H has symbol behavior with respect to the second variable. In fact we claim that the resulting explicit terms are smoother than the ones obtained for $\overline{x}_{\text{in}}$, as explained in the second part of the lemma. We briefly outline the arguments for the explicit singular terms generated by (7.5). In fact, consider a schematically written term(where we omit several factors which do not affect its smoothness but only its decay with respect to time)

$$\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\left[\nu\tau - R + \frac{\lambda(\tau)}{\lambda(\sigma)}x\right]^{\frac{1}{2}+k\nu}\left(\log\left(\nu\tau - R + \frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\right)^i H\left(\nu\tau - R + \frac{\lambda(\tau)}{\lambda(\sigma)}x\right)g(x)\,dx,$$

where g is bounded as well as in $L^1([0,\infty))$, $H(\cdot)$ is bounded and has symbol type behavior, and we have included an extra cutoff $\chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)$ since the contribution of (7.5) where $\frac{\lambda(\tau)}{\lambda(\sigma)}x \gtrsim 1$ is seen to lead to terms of type f_3. Then splitting

$$\begin{aligned}
&\chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\left[\nu\tau - R + \frac{\lambda(\tau)}{\lambda(\sigma)}x\right]^{\frac{1}{2}+k\nu}\left(\log\left(\nu\tau - R + \frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\right)^i H\left(\nu\tau - R + \frac{\lambda(\tau)}{\lambda(\sigma)}x\right)g(x)\,dx\\
&= \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\left[\frac{\lambda(\tau)}{\lambda(\sigma)}x\right]^{\frac{1}{2}+k\nu}\left(\log\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\right)^i H\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)g(x)\,dx\\
&+ \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\left[\nu\tau - R + \frac{\lambda(\tau)}{\lambda(\sigma)}x\right]^{\frac{1}{2}+k\nu}\left(\log\left(\nu\tau - R + \frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\right)^i H\left(\nu\tau - R + \frac{\lambda(\tau)}{\lambda(\sigma)}x\right)g(x)\,dx\\
&- \chi_{\nu\tau-R\lesssim\hbar}\int_0^\infty \chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\left[\frac{\lambda(\tau)}{\lambda(\sigma)}x\right]^{\frac{1}{2}+k\nu}\left(\log\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)\right)^i H\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)g(x)\,dx,
\end{aligned}$$

the first term on the right is again seen to lead to a contribution which can be absorbed into f_3 (upon also taking into account the currently omitted factors ensuring the right temporal decay). The second and the third (difference) term on the right can be written as

$$\chi_{\nu\tau-R\lesssim\hbar}(\nu\tau - R)\cdot\int_0^\infty\int_0^1 Z'\left(s(\nu\tau - R) + x\right)\cdot\chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)g(x)\,dsdx,$$

which of course involves a derivative falling onto the term $H(\cdot)$, which will then be responsible for the derivative loss which the 'upgrade of smoothness' for the contribution of $\overline{x}_{\text{out}}$ entails. Write the preceding integral as

$$\begin{aligned}
&\chi_{\nu\tau-R\lesssim\hbar}(\nu\tau - R)\cdot\int_0^\infty\int_0^1 Z'\left(s(\nu\tau - R) + x\right)\cdot\chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)g(x)\,dsdx\\
&= \chi_{\nu\tau-R\lesssim\hbar}\cdot(\nu\tau - R)\cdot\int_0^{\nu\tau-R}\int_0^1 Z'\left(s(\nu\tau - R) + x\right)\cdot\chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)g(x)\,dsdx\\
&+ \chi_{\nu\tau-R\lesssim\hbar}\cdot(\nu\tau - R)\cdot\int_{\nu\tau-R}^\infty\int_0^1 Z'\left(s(\nu\tau - R) + x\right)\cdot\chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right)g(x)\,dsdx
\end{aligned}$$

Observe that

$$Z'(z) = z^{k\nu-\frac{1}{2}}(\log z)^i \cdot \tilde{H}(z),$$

where z has properties like H in the region $z > 0$. It easily follows that the first integral on the right admits an explicit representation

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar} \cdot (\nu\tau - R) \cdot \int_0^{\nu\tau-R} \int_0^1 Z'\left(s(\nu\tau - R) + x\right) \cdot \chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right) g(x)\, ds dx \\ &= (\nu\tau - R)^{\frac{3}{2}+k\nu} \log\left(\nu\tau - R\right)^i \cdot H_1\left(\tau, \sigma, \nu\tau - R\right),\end{aligned}$$

where H_1 is bounded and has symbol type bound with respect to all its variables. For the second integral above over the range $x \in [\nu\tau - R, \infty)$, we use one more splitting of a similar kind:

$$Z'\left(s(\nu\tau - R) + x\right) = Z'(x) + s(\nu\tau - R) \cdot \int_0^1 Z''\left(s_1 s(\nu\tau - R) + x\right)\, ds_1,$$

which then yields

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar} \cdot (\nu\tau - R) \cdot \int_{\nu\tau-R}^{\infty} \int_0^1 Z'\left(s(\nu\tau - R) + x\right) \cdot \chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right) g(x)\, ds dx \\ &= \chi_{\nu\tau-R\lesssim\hbar} \cdot (\nu\tau - R) \cdot \left[\int_0^{\infty} \chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right) Z'(x) g(x)\, dx - \int_0^{\nu\tau-R} \chi_{\lesssim 1}\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x\right) Z'(x) g(x)\, dx\right] \\ &+ \chi_{\nu\tau-R\lesssim\hbar} \cdot (\nu\tau - R)^2 \cdot \int_{\nu\tau-R}^{\infty} \int_0^1 \int_0^1 s Z''\left(s_1 s(\nu\tau - R) + x\right) g(x)\, ds_1 ds dx\end{aligned}$$

Here, the first term on the right is the difference of a function which can be included into f_3 and a function of the explicit type, while for the second term on the right, due to the inequality

$$\left|Z''(z)\right| \lesssim z^{k\nu-\frac{3}{2}} \left|\log z\right|^i,$$

it will be of the explicit form

$$\begin{aligned}&\chi_{\nu\tau-R\lesssim\hbar} \cdot (\nu\tau - R)^2 \cdot \int_{\nu\tau-R}^{\infty} \int_0^1 \int_0^1 s Z''\left(s_1 s(\nu\tau - R) + x\right) g(x)\, ds_1 ds dx \\ &= (\nu\tau - R)^{\frac{3}{2}+k\nu} \log\left(\nu\tau - R\right)^i \cdot H_2(\tau, \sigma, \nu\tau - R)\end{aligned}$$

as long as $k\nu < \frac{1}{2}$. If $k\nu \geq \frac{1}{2}$, then we repeat the above process for finite times.
The contribution of the 'seed outgoing singular terms' (7.4) is handled similarly by replacing the role of x by σ. □

Remark 7.11. *We will only take advantage of the additional smoothness of the physical incarnation of the outgoing singular terms for the contribution of the seed outgoing part with $l = 0$, where the coefficients $F_{l,k,i}$ do not depend on the frequency variable, and hence no smoothness loss will be incurred.*

7.4. **Operations on functions with admissible singular part.** Our definition of admissible singular part is chosen to be flexible enough that important operations, such as frequency localisations as well as the transference operator, essentially preserve such functions.

7.4.1. *The effect of the transference operator.* Recall from Proposition 5.1 the precise form of the transference operator at angular momentum $n, |n| \geq 2$, and consisting of a diagonal and an off-diagonal part. Two key structural properties ensure that this preserves an admissible singular part: On the one hand, the spectral density $\rho_n(\xi)$ admits an asymptotic expansion toward $\xi = +\infty$ of Hankel type. On the other hand the off-diagonal part $\mathcal{K}_\hbar^{(0)}$ improves decay toward $\xi = +\infty$ and its kernel has further remarkable regularity properties uniformly in n. The latter property means that the off-diagonal transference operator will send the principal part of the ingoing singular term into its connecting part, which is less rigid structurally but decays better. The following lemma gives the basic bounds for the Hilbert transform concerning the weighted Hölder type norms we shall work with:

Lemma 7.12. *Introduce the norm (for $\delta \in (0,1)$)*

$$\|f\|_{wC^\delta} := \|xf(x)\|_{L^\infty([0,\infty))} + \sup_{\lambda>0} \lambda^{1+\delta} \|\chi_{x\simeq\lambda} f\|_{\dot{C}^\delta}$$

Assume that f is supported at $\eta \simeq \lambda \gtrsim \hbar^{-2}$, and consider the function

$$\left(\mathcal{H}^\pm_{twisted} f\right)(\xi;\tau,\sigma') := \int_0^\infty \frac{e^{\pm i\left[\nu\tau\eta^{\frac{1}{2}} + \hbar^{-1}\rho\left(x^{(\eta)}_{\sigma'};\beta',\hbar\right)\right]}}{(\xi-\eta)\langle\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\rangle^\gamma} f(\eta)\,d\eta, \quad \gamma > 0,$$

where we use the notation $\beta' = \hbar\eta^{\frac{1}{2}} \cdot \frac{\lambda(\tau)}{\lambda(\sigma)}$, $\quad x^{(\eta)}_{\sigma'} = \hbar\eta^{\frac{1}{2}} \cdot \nu\sigma \cdot \frac{\lambda(\tau)}{\lambda(\sigma)}$. Then for any $\delta' < \delta$ we have the relation

$$\left(\mathcal{H}^\pm_{twisted} f\right)(\xi;\tau,\sigma') = e^{\pm i\left[\nu\tau\xi^{\frac{1}{2}} + \hbar^{-1}\rho\left(x^{(\xi)}_{\sigma'};\alpha',\hbar\right)\right]} \cdot g_\pm(\xi;\tau,\sigma'), \quad \left\|\chi_{\xi\simeq\lambda} g(\xi;\tau,\sigma')\right\|_{wC^{\delta'}_\xi} \lesssim_{\delta',\delta,\gamma} \|f\|_{wC^\delta},$$

where $\|\cdot\|_{wC^{\delta'}_\xi}$ indicates that the norm is with respect to the variable ξ, we use the notations $\alpha' = \hbar\xi^{\frac{1}{2}} \cdot \frac{\lambda(\tau)}{\lambda(\sigma)}$ and $x^{(\xi)}_{\sigma'} = \hbar\xi^{\frac{1}{2}} \cdot \nu\sigma \cdot \frac{\lambda(\tau)}{\lambda(\sigma)}$, and the bound is uniform in $\hbar, \tau, \sigma'$.

Proof. By definition we have

$$g_\pm(\xi;\tau,\sigma') = \int_0^\infty \frac{e^{i\left[\nu\tau\left(\eta^{\frac{1}{2}} - \xi^{\frac{1}{2}}\right) + \hbar^{-1}\left[\rho(x^{(\eta)}_{\sigma'};\beta',\hbar) - \rho(x^{(\xi)}_{\sigma'};\alpha',\hbar)\right]\right]}}{(\xi-\eta)\langle\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\rangle^\gamma} f(\eta)\,d\eta$$

Introduce the variable $\tilde{\eta} = \nu\tau(\eta^{\frac{1}{2}} - \xi^{\frac{1}{2}})$, and write the phase as (we suppress the $\hbar$-dependence in the notation since $\hbar \ll 1$ is fixed)

$$\begin{aligned}
\Psi(\xi,\eta,\tau,\sigma') &:= \nu\tau(\eta^{\frac{1}{2}} - \xi^{\frac{1}{2}}) + \hbar^{-1}\left[\rho(x^{(\eta)}_{\sigma'};\beta',\hbar) - \rho(x^{(\xi)}_{\sigma'};\alpha',\hbar)\right] \\
&= \tilde{\eta} + \hbar^{-1}\int_0^1 \Bigg[\rho_x\left(sx^{(\eta)}_{\sigma'} + (1-s)x^{(\xi)}_{\sigma'}; s\beta' + (1-s)\alpha', \hbar\right) \cdot \hbar\frac{\sigma'}{\tau}\tilde{\eta} \\
&\qquad + \rho_\alpha\left(sx^{(\eta)}_{\sigma'} + (1-s)x^{(\xi)}_{\sigma'}; s\beta' + (1-s)\alpha', \hbar\right) \cdot \hbar\frac{\tilde{\eta}}{\nu\tau} \cdot \frac{\lambda(\tau)}{\lambda(\sigma)}\Bigg]\,ds
\end{aligned}$$

By the bounds on ρ_x and ρ_α in Lemma 4.13, if we pick $\xi \simeq \eta$, the phase is in effect bounded by $|\Psi(\xi,\eta,\tau,\sigma')| \lesssim \tilde{\eta}$ under those conditions. Write $f(x^2) = g(x)$, restricted to the positive real axis, and switch variables in the integral

$$g_{\pm}(\xi;\tau,\sigma') = \int_{-\infty}^{\infty} \frac{e^{i\Psi(\xi,\eta,\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{\tilde{\eta}}{\tau}\right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}} \cdot g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi^{\frac{1}{2}}\right) d\tilde{\eta},$$

where of course $\eta^{\frac{1}{2}} = \frac{\tilde{\eta}}{\nu\tau} + \xi^{\frac{1}{2}}$. In fact, restricting $\xi \simeq \lambda$, the integration limits may be set to be $\pm\lambda^{\frac{1}{2}}\tau$. We now prove the two bounds required for g:

The weighted L^{∞}-bound. Denote by $P_{>\kappa}$, $\kappa \in (0,\infty)$ a Littlewood-Paley type frequency cutoff to frequencies $\gtrsim \kappa$, and which can be realized by convolution with the function $\kappa\hat{\chi}(\eta\kappa)$, where χ is a smooth function supported on $(-\infty,-\frac{1}{2}] \cup [\frac{1}{2},\infty)$ and identically 1 on $[1,\infty)$. Then we have

$$P_{>\kappa}g(\eta) = \int_{-\infty}^{\infty} \kappa\hat{\chi}\left((\eta-\zeta)\kappa\right) g(\zeta)\, d\zeta = \int_{-\infty}^{\infty} \kappa\hat{\chi}\left((\eta-\zeta)\kappa\right)\left[g(\zeta)-g(\eta)\right]\, d\zeta, \tag{7.8}$$

(we used the fact that $\chi(\cdot)$ is supported away from the origin while the delta measure is supported at the origin) and setting $\kappa = \left(\frac{\tau^2}{\lambda}\right)^{\frac{1}{4}}$ and taking advantage of the rapid decay of the function $\hat{\chi}$ as well as the bound

$$|g(x_1)-g(x_2)| = \left|f(x_1^2)-f(x_2^2)\right| \lesssim \lambda^{-1-\delta}\,\|f\|_{wC^{\delta}} \cdot \left|x_1^2-x_2^2\right|^{\delta} \lesssim \lambda^{-1-\frac{\delta}{2}}\,\|f\|_{wC^{\delta}} \cdot |x_1-x_2|^{\delta}$$

provided $x_{1,2} \simeq \lambda^{\frac{1}{2}}$, we infer

$$\left|P_{>\left(\frac{\tau^2}{\lambda}\right)^{\frac{1}{4}}} g(\eta)\right| \lesssim \tau^{-\frac{\delta}{2}}\lambda^{-1-\frac{\delta}{4}} \cdot \|f\|_{wC^{\delta}},$$

and so we infer the bound

$$\left|\int_{-\infty}^{\infty} \psi_{|\tilde{\eta}|\gtrsim 1}\psi_{\eta\sim\lambda} \frac{e^{i\Psi(\xi,\eta,\tau,\sigma')}}{\tilde{\eta}\langle\frac{\tilde{\eta}}{\tau}\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}} \cdot P_{>\left(\frac{\tau^2}{\lambda}\right)^{\frac{1}{4}}} g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi^{\frac{1}{2}}\right) d\tilde{\eta}\right| \lesssim \lambda^{-1}\frac{\log(\lambda^{\frac{1}{2}}\tau)}{\tau^{\frac{\delta}{2}}\lambda^{\frac{\delta}{4}}} \cdot \|f\|_{wC^{\delta}},$$

where the extra localiser $\psi_{\eta\simeq\lambda}$ ensures the correct support of the integral after application of the frequency cutoff.

In order to handle the remaining integral with the additional restriction $|\tilde{\eta}| \gtrsim 1$ and involving $P_{\leq\left(\frac{\tau^2}{\lambda}\right)^{\frac{1}{4}}} g$, we perform integration by parts with respect to $\tilde{\eta}$, using

$$\left|\partial_{\tilde{\eta}}\Psi(\xi,\eta,\tau,\sigma')\right| \gtrsim 1,$$

as well as straightforward higher derivative bounds with respect to $\tilde{\eta}$. On the other hand, by the standard property of Littlewood-Paley cutoff, we have

$$\left|P_{\leq\left(\frac{\tau^2}{\lambda}\right)^{\frac{1}{4}}} g(\eta)\right| \lesssim \left(\frac{\lambda^{\frac{1}{4}}}{\tau^{\frac{1}{2}}}\right)^{\delta} \cdot \lambda^{-1-\frac{\delta}{2}}\|f\|_{wC^{\delta}}.$$

Then if the derivative falls on $P_{\leq\left(\frac{\tau^2}{\lambda}\right)^{\frac14}} g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi^{\frac12}\right)$, we gain a factor

$$\left(\frac{\tau^2}{\lambda}\right)^{\frac14}\cdot\tau^{-1},$$

which, together with the bound $\left(\frac{\lambda^{\frac14}}{\tau^{\frac12}}\right)^{\delta}\cdot\lambda^{-1-\frac{\delta}{2}}\|f\|_{wC^\delta}$, gives a bound

$$\left(\frac{\tau^{\frac12}}{\lambda^{\frac14}}\right)^{-1-\delta}\cdot\lambda^{-1}\|f\|_{wC^\delta}$$

This again beats the loss of $\log(\lambda^{\frac12}\tau)$. While if the derivative falls on $\tilde{\eta}^{-1}\left(\partial_{\tilde{\eta}}\Psi(\xi,\eta,\tau,\sigma')\right)^{-1}\cdot\left(\frac{\tilde{\eta}}{\nu\tau}+\xi^{\frac12}\right)$, we gain $\tilde{\eta}^{-1}$, whence forcing integrability of the expression.
This reduces things to bounding the integral

$$\begin{aligned}&\int_{-\infty}^{\infty}\psi_{|\tilde{\eta}|\lesssim 1}\frac{e^{i\Psi(\xi,\eta,\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{\tilde{\eta}}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac12}}{\xi^{\frac12}+\eta^{\frac12}}\cdot g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi^{\frac12}\right)d\tilde{\eta}\\ &=\sum_{\pm}(-1)^{\pm}\int_0^{\infty}\psi_{|\tilde{\eta}|\lesssim 1}\frac{e^{i\Psi(\xi,\eta(\pm\tilde{\eta},\xi,\tau),\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{\tilde{\eta}}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac12}(\pm\tilde{\eta},\xi,\tau)}{\xi^{\frac12}+\eta^{\frac12}(\pm\tilde{\eta},\xi,\tau)}\cdot g\left(\frac{\pm\tilde{\eta}}{\nu\tau}+\xi^{\frac12}\right)d\tilde{\eta},\end{aligned}$$

where we have made explicit the fact that η is a function of $\tilde{\eta},\xi,\tau$. But here the differencing easily leads to gains of factors $\lesssim\tilde{\eta}^{\delta}$, which counteracts the singular term $\tilde{\eta}^{-1}$, which easily concludes the weighted L^∞-bound. Here we also used the fact that the kernel $\tilde{\eta}$ is odd and the other factors in the integral except $g\left(\frac{\pm\tilde{\eta}}{\nu\tau}+\xi^{\frac12}\right)$ are smooth in $\tilde{\eta}$.

Hölder differencing bound. Fix two values $\xi_1,\xi_2\in\mathbb{R}_+,\xi_1\simeq\xi_2\simeq\lambda$, and set $l:=\left|\xi_1^{\frac12}-\xi_2^{\frac12}\right|$. As before set $\tilde{\eta}=\nu\tau\left(\eta^{\frac12}-\xi^{\frac12}\right)$, and decompose the integrals giving $\left(\mathcal{H}^{\pm}_{\text{twisted}}f\right)(\xi_j;\tau,\sigma'),\ j=1,2$, into (where for now $\xi=\xi_{1,2}$)

$$\begin{aligned}&\int_{-\infty}^{\infty}\psi_{|\tilde{\eta}|\lesssim 1}\frac{e^{i\Psi(\xi,\eta,\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac12}}{\xi^{\frac12}+\eta^{\frac12}}\cdot g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi^{\frac12}\right)d\tilde{\eta}\\ &+\int_{-\infty}^{\infty}\psi_{|\tilde{\eta}|\gtrsim 1}\frac{e^{i\Psi(\xi,\eta,\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac12}}{\xi^{\frac12}+\eta^{\frac12}}\cdot g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi^{\frac12}\right)d\tilde{\eta}\end{aligned}\tag{7.9}$$

First integral above, small $\tilde{\eta}$. Due the the bound $|\Psi(\xi,\eta,\tau,\sigma')|\lesssim|\tilde{\eta}|\lesssim 1$, we can split the exponential into

$$e^{i\Psi(\xi,\eta,\tau,\sigma')}=1+(e^{i\Psi(\xi,\eta,\tau,\sigma')}-1),\quad\left|(e^{i\Psi(\xi,\eta,\tau,\sigma')}-1)\right|\lesssim|\tilde{\eta}|,$$

and it is also easy to check that under our support conditions $\xi_1\simeq\xi_2\simeq\eta\simeq\lambda\gtrsim\hbar^{-2}$ we have

$$\left|\partial_\xi e^{i\Psi(\xi,\eta,\tau,\sigma')}\right|\lesssim\xi^{-1}|\tilde{\eta}|,$$

which easily furnishes the desired bound for the modified contribution with the exponential replaced by $e^{i\Psi(\xi,\eta,\tau,\sigma')}-1$:

$$\begin{aligned}\left|\sum_{j=1,2}(-1)^j\int_{-\infty}^{\infty}\psi_{|\tilde{\eta}|\lesssim 1}\frac{e^{i\Psi(\xi_j,\eta,\tau,\sigma')}-1}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\cdot g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)d\tilde{\eta}\right| &\lesssim \lambda^{-1-\frac{\delta}{2}}l^{\delta}\|f\|_{wC^{\delta}}\\ &\lesssim \lambda^{-1-\delta}|\xi_1-\xi_2|^{\delta}\|f\|_{wC^{\delta}}.\end{aligned}$$

This allows us to reduce the case of small $\tilde{\eta}$ to the following integral, where we may re-arrange things if we pick the cutoff ψ symmetrically around the origin:

$$\sum_{j=1,2}(-1)^j\int_{0}^{\infty}\psi_{|\tilde{\eta}|\lesssim 1}\frac{1}{\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\cdot\frac{g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)-g\left(\frac{-\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)}{\tilde{\eta}}\,d\tilde{\eta}.$$

We split the above integrals further by including smooth cutoffs $\chi_{|\tilde{\eta}|\lesssim\tau l},\chi_{|\tilde{\eta}|\gtrsim\tau l}$. Including the former, we take advantage of the bound

$$\left|g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)-g\left(\frac{-\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)\right|\lesssim\left|\frac{\tilde{\eta}}{\nu\tau}\right|^{\delta}\cdot\lambda^{-1-\frac{\delta}{2}}\cdot\|f\|_{wC^{\delta}},$$

which leads to the desired bound upon integration over $0\le\tilde{\eta}\lesssim\tau l$:

$$\begin{aligned}&\left|\int_{0}^{\infty}\psi_{|\tilde{\eta}|\lesssim\tau l}\frac{1}{\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\cdot\frac{g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)-g\left(\frac{-\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)}{\tilde{\eta}}\,d\tilde{\eta}\right|\\ &\lesssim\left(\frac{\tau l}{\tau}\right)^{\delta}\cdot\lambda^{-1-\frac{\delta}{2}}\cdot\|f\|_{wC^{\delta}}=l^{\delta}\cdot\lambda^{-1-\frac{\delta}{2}}\cdot\|f\|_{wC^{\delta}}.\end{aligned}$$

On the other hand, in the regime $|\tilde{\eta}|\gtrsim\tau l$, the difference structure will become important. Split

$$g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)=P_{<l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)+P_{\geq l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)$$

where the subscripts denote frequency cutoffs defined in analogy to (7.8). Then arguing as earlier, we get the bound

$$\left|P_{\geq l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)\right|\lesssim l^{\delta}\cdot\lambda^{-1-\frac{\delta}{2}}\cdot\|f\|_{wC^{\delta}},.$$

Moreover, using orthogonality (or Plancherel identity) and the fact that $P_{\geq l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)=P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}}\left(g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)\right)$ where $P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}}$ means the frequency cutoff applied to the following expression interpreted as a function of $\tilde{\eta}$, we infer

$$\int_{-\infty}^{\infty}\psi_{\tau l\lesssim|\tilde{\eta}|\lesssim 1}\frac{1}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\cdot P_{\geq l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)d\tilde{\eta}$$

$$= \int_{-\infty}^{\infty} P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}} \left(\psi_{\tau l \lesssim |\tilde{\eta}| \lesssim 1} \frac{1}{\tilde{\eta} \left\langle \frac{|\tilde{\eta}|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \right) \cdot P_{\geq l^{-1}} g\left(\frac{\tilde{\eta}}{\nu\tau} + \xi_j^{\frac{1}{2}} \right) d\tilde{\eta}$$

Then note that for any $k \geq 0$ we have

$$\left\| P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}} \left(\psi_{2^k \tau l \simeq |\tilde{\eta}|} \frac{1}{\tilde{\eta} \left\langle \frac{|\tilde{\eta}|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \right) \right\|_{L^1_{d\tilde{\eta}}} \lesssim 2^{-k}. \tag{7.10}$$

In fact we denote by

$$h(\zeta) := \psi_{2^k \tau l \simeq |\zeta|} \frac{1}{\zeta \left\langle \frac{|\zeta|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}(\zeta)}{\xi_j^{\frac{1}{2}} + \eta^{\frac{1}{2}}(\zeta)}.$$

By definition of the operator $P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}}$, we have

$$\left| \left(P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}} h \right)(\tilde{\eta}) \right| \lesssim \nu\tau l \cdot \left| \partial_{\tilde{\eta}} h(\tilde{\eta}) \right|.$$

On the other hand, we have

$$\left\| \partial_{\tilde{\eta}} h(\tilde{\eta}) \right\|_{L^1_{d\tilde{\eta}}} \lesssim 2^{-k} \tau^{-1} l^{-1}.$$

Therefore the estimate (7.10) follows. (7.10) in turn implies

$$\begin{aligned}
&\left\| P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}} \left(\psi_{\tau l \lesssim |\tilde{\eta}| \lesssim 1} \frac{1}{\tilde{\eta} \left\langle \frac{|\tilde{\eta}|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \right) \right\|_{L^1_{d\tilde{\eta}}} \\
&\lesssim \sum_{j \geq 0} \left\| P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}} \left(\phi_{2^k \tau l \simeq |\tilde{\eta}|} \frac{1}{\tilde{\eta} \langle \frac{|\tilde{\eta}|}{\tau} \rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \right) \right\|_{L^1_{d\tilde{\eta}}} \\
&\lesssim 1.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}} \left(\phi_{\tau l \lesssim |\tilde{\eta}| \lesssim 1} \frac{1}{\tilde{\eta} \left\langle \frac{|\tilde{\eta}|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \right) \cdot P_{\geq l^{-1}} g\left(\frac{\tilde{\eta}}{\nu\tau} + \xi_j^{\frac{1}{2}} \right) d\tilde{\eta} \right| \\
&\lesssim \left\| P^{(\tilde{\eta})}_{\geq l^{-1}(\nu\tau)^{-1}} \left(\psi_{\tau l \lesssim |\tilde{\eta}| \lesssim 1} \frac{1}{\tilde{\eta} \left\langle \frac{|\tilde{\eta}|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \right) \right\|_{L^1_{d\tilde{\eta}}} \cdot \left\| P_{\geq l^{-1}} g\left(\frac{\eta}{\nu\tau} + \xi_j^{\frac{1}{2}} \right) \right\|_{L^\infty_{d\tilde{\eta}}} \\
&\lesssim l^{\delta} \cdot \lambda^{-1-\frac{\delta}{2}} \cdot \|f\|_{wC^{\delta}},
\end{aligned}$$

as desired.

Next, to deal with the contribution of the low frequency part of g, namely $P_{<l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_j^{\frac{1}{2}}\right)$, we localize this further to frequency $2^j l^{-1}$, $j<0$, whence we get

$$\left|P_{2^j l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_1^{\frac{1}{2}}\right)-P_{2^j l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_2^{\frac{1}{2}}\right)\right|\lesssim 2^j\cdot 2^{-\delta j}l^{\delta}\cdot\lambda^{-1-\frac{\delta}{2}}\cdot\|f\|_{wC^{\delta}}.$$

Here we used the following fact about Hölder functions: Let P_j be the standard dyadic Littlewood-Paley projection and f be a Hölder function. Then we have

$$\left|P_jf(x+y)-P_jf(x)\right|\lesssim|y|\left\|\partial P_jf\right\|_{L^\infty}\lesssim[f]_\alpha 2^{j(1-\gamma)}.$$

Here $[\cdot]_\alpha$ is the Hölder difference norm for a Hölder function in C^α. For our use, recall that $[g]_\alpha\lesssim\lambda^{-1-\frac{\delta}{2}}\|f\|_{wC^\delta}$.

On the other hand, it is easily seen that

$$\left\|P^{(\tilde{\eta})}_{2^jl^{-1}(\nu\tau)^{-1}}\left(\phi_{\tau l\lesssim|\tilde{\eta}|\lesssim 1}\frac{1}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right)\right\|_{L^1_{d\tilde{\eta}}}\lesssim|j|$$

$$\left\|\sum_{r=1,2}(-1)^rP^{(\tilde{\eta})}_{<l^{-1}(\nu\tau)^{-1}}\left(\psi_{\tau l\lesssim|\tilde{\eta}|\lesssim 1}\frac{1}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right)\right\|_{L^1_{d\tilde{\eta}}}$$
$$\lesssim l|\log\tau l|\cdot\lambda^{-\frac{1}{2}}\lesssim l|\log l|\cdot\lambda^{-\frac{1}{2}}.$$

For the first estimate above, we simply write the cutoff $\psi_{\tau l\lesssim|\tilde{\eta}|\lesssim 1}=\psi_{\tau l\lesssim|\tilde{\eta}|\lesssim 2^{-j}\tau l}+\psi_{2^{-j}\tau l\lesssim|\tilde{\eta}|\lesssim 1}$. The contribution from $\psi_{\tau l\lesssim|\tilde{\eta}|\lesssim 2^{-j}\tau l}$ is bounded by $|j|$, upon integrating $|\tilde{\eta}|^{-1}$. While the contribution from $\psi_{2^{-j}\tau l\lesssim|\tilde{\eta}|\lesssim 1}$ is bounded by 1, using an earlier argument. For the second estimate above, we use the crude bound by ignoring the cutoff $P^{(\tilde{\eta})}_{<l^{-1}(\nu\tau)^{-1}}$, and using the Lagrange Mean Value Theorem for the function inside the parenthesis.

In total, the remaining contribution to the case $|\tilde{\eta}|\lesssim 1$ is then estimated as follows:

$$\left|\sum_{r=1,2}(-1)^r\int_{-\infty}^{\infty}\psi_{\tau l\lesssim|\tilde{\eta}|\lesssim 1}\frac{1}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\cdot P_{<l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_r^{\frac{1}{2}}\right)d\tilde{\eta}\right|$$
$$\lesssim\sum_{j<0}\left|\int_{-\infty}^{\infty}P^{(\tilde{\eta})}_{2^jl^{-1}(\nu\tau)^{-1}}\left(\psi_{\tau l\lesssim|\tilde{\eta}|\lesssim 1}\frac{1}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_1^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right)\cdot\sum_{r=1,2}(-1)^rP_{2^jl^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_r^{\frac{1}{2}}\right)d\tilde{\eta}\right|$$
$$+\left|\int_{-\infty}^{\infty}\sum_{r=1,2}(-1)^rP^{(\tilde{\eta})}_{<l^{-1}(\nu\tau)^{-1}}\left(\psi_{\tau l\lesssim|\tilde{\eta}|\lesssim 1}\frac{1}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right)\cdot P_{<l^{-1}}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_2^{\frac{1}{2}}\right)d\tilde{\eta}\right|$$
$$\lesssim\sum_{j<0}|j|2^{(1-\delta)j}\cdot l^{\delta}\cdot\lambda^{-1-\frac{\delta}{2}}\cdot\|f\|_{wC^\delta}+\chi_{l\ll 1}l^{1+\delta}|\log l|\cdot\lambda^{-\frac{3}{2}-\frac{\delta}{2}}\cdot\|f\|_{wC^\delta}$$
$$\lesssim l^{\delta}\cdot\lambda^{-1-\frac{\delta}{2}}\cdot\|f\|_{wC^\delta},$$

as desired.

Second integral on the right in (7.9)*, the contribution of large* $\tilde{\eta}$*.* Here we can no longer neglect the oscillatory term $e^{i\Psi(\xi,\eta,\tau,\sigma')}$, and in fact we shall exploit that

$$\Psi(\xi,\eta,\tau,\sigma') = \left(1 + O\left(\frac{1}{\tau}\right)\right)\cdot\tilde{\eta}.$$

By directly evaluating the Fourier transform, we infer for $r \geq 0$ that

$$\left\|P_{\ll 1}^{(\tilde{\eta})}\left(\psi_{|\tilde{\eta}|\simeq 2^r}\frac{e^{i\Psi(\xi,\eta,\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right)\right\|_{L^1_{d\tilde{\eta}}} = O_N\left(\frac{1}{2^{Nr}}\right)$$

for any $N > 0$, and in particular by summing over $r \geq 0$ we find that

$$\left\|P_{\ll 1}^{(\tilde{\eta})}\left(\psi_{|\tilde{\eta}|\gtrsim 1}\frac{e^{i\Psi(\xi,\eta,\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right)\right\|_{L^1_{d\tilde{\eta}}} \lesssim 1. \tag{7.11}$$

In fact, one can easily show that the Fourier transform of the function $P_{\ll 1}^{(\tilde{\eta})}\left(\psi_{|\tilde{\eta}|\simeq 2^r}\frac{e^{i\Psi(\xi,\eta,\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_j^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right)$ has a rapid decay in 2^{-r}, then the desired result follows by again taking its Fourier inverse transform.

Furthermore, we have the difference bound

$$\left\|\sum_{r=1,2}(-1)^r P_{\ll 1}^{(\tilde{\eta})}\left(\psi_{|\tilde{\eta}|\gtrsim 1}\frac{e^{i\Psi(\xi_r,\eta,\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\right)\right\|_{L^1_{d\tilde{\eta}}} \lesssim \lambda^{-\frac{1}{2}}\cdot l.$$

Now we break the second integral on the right in (7.9) into two contributions: restricting the function g to large frequency $> C^{-1}\tau$, we get the term

$$\sum_{r=1,2}(-1)^r\int_{-\infty}^{\infty}\psi_{|\tilde{\eta}|\gtrsim 1}\frac{e^{i\Psi(\xi_r,\eta,\tau,\sigma')}}{\tilde{\eta}\left\langle\frac{|\tilde{\eta}|}{\tau}\right\rangle^{\gamma}}\cdot\frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}}+\eta^{\frac{1}{2}}}\cdot P_{>C^{-1}\tau}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_r^{\frac{1}{2}}\right)d\tilde{\eta}$$

and using (7.8) we find

$$\left|\sum_{r=1,2}(-1)^r P_{>C^{-1}\tau}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_r^{\frac{1}{2}}\right)\right| \lesssim \min\left\{l^{\delta}\cdot\lambda^{-1-\frac{\delta}{2}},\, \lambda^{-1-\frac{\delta}{2}}\cdot\tau^{-\delta}\right\}\cdot\|f\|_{wC^{\delta}}$$

$$\sum_{r=1,2}\left|P_{>C^{-1}\tau}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_r^{\frac{1}{2}}\right)\right| \lesssim \lambda^{-1-\frac{\delta}{2}}\cdot\tau^{-\delta}\cdot\|f\|_{wC^{\delta}},$$

and in particular for any $\delta' < \delta$ we have

$$\left|\sum_{r=1,2}(-1)^r P_{>C^{-1}\tau}g\left(\frac{\tilde{\eta}}{\nu\tau}+\xi_r^{\frac{1}{2}}\right)\right| \lesssim_{\delta',\delta} l^{\delta'}\cdot\lambda^{-1-\frac{\delta'}{2}}\cdot(\lambda\tau)^{-\tilde{\delta}}\cdot\|f\|_{wC^{\delta}}$$

for suitable $\tilde{\delta} = \tilde{\delta}(\delta', \delta) > 0$. Since we also have

$$\left| \sum_{r=1,2} (-1)^r e^{i\Psi(\xi_r,\eta,\tau,\sigma')} \frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \right| \lesssim \lambda^{-\frac{1}{2}} \cdot l$$

in the domain $\xi_r \simeq \lambda$, $r = 1, 2$, we easily conclude that

$$\left| \sum_{r=1,2} (-1)^r \int_{-\infty}^{\infty} \psi_{|\tilde{\eta}|\gtrsim 1} \frac{e^{i\Psi(\xi_r,\eta,\tau,\sigma')}}{\tilde{\eta} \left\langle \frac{|\tilde{\eta}|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \cdot P_{>C^{-1}\tau} g\left(\frac{\tilde{\eta}}{\nu\tau} + \xi_r^{\frac{1}{2}} \right) d\tilde{\eta} \right|$$
$$\lesssim_{\delta',\delta} l^{\delta'} \cdot \lambda^{-1-\frac{\delta'}{2}} \cdot \|f\|_{wC^{\delta}}$$

It remains to deal with the contribution of the term arising when g is replaced by $P_{\leq C^{-1}\tau} g$. Using Plancherel's theorem, we can write this as

$$\sum_{r=1,2} (-1)^r \int_{-\infty}^{\infty} \psi_{|\tilde{\eta}|\gtrsim 1} \frac{e^{i\Psi(\xi_r,\eta,\tau,\sigma')}}{\tilde{\eta} \left\langle \frac{|\tilde{\eta}|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \cdot P_{\leq C^{-1}\tau} g\left(\frac{\tilde{\eta}}{\nu\tau} + \xi_r^{\frac{1}{2}} \right) d\tilde{\eta}$$
$$= \sum_{r=1,2} (-1)^r \int_{-\infty}^{\infty} P^{(\tilde{\eta})}_{\lesssim C^{-1}} \left(\psi_{|\tilde{\eta}|\gtrsim 1} \frac{e^{i\Psi(\xi_r,\eta,\tau,\sigma')}}{\tilde{\eta} \left\langle \frac{|\tilde{\eta}|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \right) \cdot P_{\leq C^{-1}\tau} g\left(\frac{\tilde{\eta}}{\nu\tau} + \xi_r^{\frac{1}{2}} \right) d\tilde{\eta}.$$

Here we can take advantage of (7.11) and the bound following it provided $C \gg 1$, which gives

$$\left| \sum_{r=1,2} (-1)^r \int_{-\infty}^{\infty} P^{(\tilde{\eta})}_{\lesssim C^{-1}} \left(\psi_{|\tilde{\eta}|\gtrsim 1} \frac{e^{i\Psi(\xi_r,\eta,\tau,\sigma')}}{\tilde{\eta} \left\langle \frac{|\tilde{\eta}|}{\tau} \right\rangle^{\gamma}} \cdot \frac{2\eta^{\frac{1}{2}}}{\xi_r^{\frac{1}{2}} + \eta^{\frac{1}{2}}} \right) \cdot P_{\leq C^{-1}\tau} g\left(\frac{\tilde{\eta}}{\nu\tau} + \xi_r^{\frac{1}{2}} \right) d\tilde{\eta} \right|$$
$$\lesssim \left(l^{\delta} \cdot \lambda^{-1-\frac{\delta}{2}} + \chi_{l \lesssim \lambda^{\frac{1}{2}}} \lambda^{-\frac{3}{2}} \cdot l \right) \cdot \|f\|_{wC^{\delta}}$$
$$\lesssim l^{\delta} \cdot \lambda^{-1-\frac{\delta}{2}} \cdot \|f\|_{wC^{\delta}}.$$

This concludes the proof. □

Remark 7.13. *The loss of Holder regularity is only due to the contribution of the large frequency part $P_{>C^{-1}\tau} g$ of g. In particular, if g is very smooth, then this contribution will be very small since $\tau \gg 1$.*

In a similar vein, we can improve the conclusion if the function $f(\eta)$ has added differentiability:

Lemma 7.14. *Assume that f is supported at $\eta \simeq \lambda \gtrsim \hbar^{-2}$, and consider the function*

$$\left(\mathcal{H}^{\pm}_{twisted} f \right)(\xi; \tau, \sigma') := \int_0^{\infty} \frac{e^{\pm i \left[\nu\tau\eta^{\frac{1}{2}} + \hbar^{-1} \rho\left(x^{(\eta)}_{\sigma'}; \beta', \hbar \right) \right]}}{(\xi - \eta) \left\langle \xi^{\frac{1}{2}} - \eta^{\frac{1}{2}} \right\rangle^{\gamma}} f(\eta)\, d\eta, \quad \gamma > 0,$$

where we use the notation $\beta' = \hbar\eta^{\frac{1}{2}} \cdot \frac{\lambda(\tau)}{\lambda(\sigma)}$, $x^{(\eta)}_{\sigma'} = \hbar\eta^{\frac{1}{2}} \cdot \nu\sigma \cdot \frac{\lambda(\tau)}{\lambda(\sigma)}$. Then for any $\delta' < \delta$ we have the relation

$$\left(\mathcal{H}^{\pm}_{twisted} f \right)(\xi; \tau, \sigma') = e^{\pm i \left[\nu\tau\xi^{\frac{1}{2}} + \hbar^{-1} \rho\left(x^{(\xi)}_{\sigma'}; \alpha', \hbar \right) \right]} \cdot g_{\pm}(\xi; \tau, \sigma'),$$
$$\sum_{0 \leq r \leq N} \sup_{\xi > 0} \left| \chi_{\xi\simeq\lambda} \xi^r \partial_\xi^r g(\xi; \tau, \sigma') \right| + \left\| \chi_{\xi\simeq\lambda} \xi^{N+\delta'} \partial_\xi^N g(\xi; \tau, \sigma') \right\|_{wC^{\delta'}_\xi}$$

$$\lesssim_{\delta',\delta,\gamma} \sum_{0\leq r\leq N} \sup_{\xi>0} \left|\chi_{\xi\simeq\lambda}\xi^r \partial_\xi^r f(\xi;\tau,\sigma')\right| + \left\|\chi_{\xi\simeq\lambda}\xi^{N+\delta}\partial_\xi^N f(\xi;\tau,\sigma')\right\|_{wC^\delta}.$$

The proof is analogous to the one of the preceding lemma.

Using the preceding lemmas, we can now analyze the effect of the transference operator at angular momentum $n, |n| \geq 2$ on admissible singular terms. For the following proposition, recall the terminology from Proposition 5.1.

Proposition 7.15. *Assume that $\overline{x}(\tau,\xi)$ is an admissible singular part at angular momentum $n, |n| \geq 2$, as in Definition 7.8. Then we have*

$$\left(\mathcal{K}_\hbar^{(0)}\overline{x}\right)(\tau,\xi) = \overline{x}_1(\tau,\xi) + \overline{x}_2(\tau,\xi) + \overline{x}_3(\tau,\xi)$$

where $\overline{x}_1$ is admissibly singular with vanishing principal ingoing part, while $\overline{x}_2$ satisfies

$$\|\overline{x}_2\|_{S_0^\hbar} \lesssim \tau^{-5}.$$

Furthermore $\overline{x}_3 = \sum_\pm e^{\pm i\nu\tau\xi^{\frac{1}{2}}} \cdot g_\pm(\tau,\xi)$ where $g_\pm(\tau,\xi)$ is C^∞ with respect to the second variable, and satisfies the bounds

$$\left|\partial_\xi^{k_2} g_\pm(\tau,\xi)\right| \lesssim \log\tau \cdot \tau^{-1-\nu} \cdot \hbar^{-1}\xi^{-\frac{1}{2}} \left\langle \hbar\xi^{\frac{1}{2}} \right\rangle^{-4}, \quad 0 \leq k_2 \leq 5.$$

As a corollary, we see that

$$\left(\mathcal{K}_\hbar\overline{x}\right)(\tau,\xi) = \overline{x}_1(\tau,\xi) + \overline{x}_2(\tau,\xi) + \overline{x}_3(\tau,\xi)$$

with $\overline{x}_1$ admissibly singular and $\overline{x}_2, \overline{x}_3$ as before.

Remark 7.16. *The reason for the presence of the term $\overline{x}_2$ is the fact that application of the transference operator always causes a small loss of differentiability of the coefficients $F_{l,k,i}^{(\pm)}(\tau,\sigma,\xi)$ with respect to the last variable, which means things don't close for the top order terms with l maximal. However, these terms get mapped into the better space $S_0^\hbar$, and moreover, the fact that the preceding lemma and Remark 7.13 identify the precise source for this loss of differentiability allow us to gain an additional temporal smallness for the terms $\overline{x}_2$. This added temporal decay will be crucial, since otherwise the fine structure would have to remain visible. Similarly, the term $\overline{x}_3$ is also in the good space $S_0^\hbar$ but has poor temporal decay (its presence is forced by very technical reasons, related to the cutoff in the definition of admissibly singular function), and so we have to retain enough spatial structure to be able to handle its contributions, when dealing with the modulation theory for the exceptional modes.*

Proof. We verify the conclusion for both $\overline{x}_{in}$ and $\overline{x}_{out}$ according to Definition 7.8.

Contribution of the principal ingoing singular part, i.e. the first term on the right in (7.3). We fix a sign $\pm$ as well as k, i and the time σ, as we can move the temporal integral to the beginning of the resulting expression after applying the transference operator. The transference operator shall then be given by the kernel

$$\frac{F(\xi,\eta;\hbar)\cdot\rho_n(\eta)}{\xi-\eta},$$

and we naturally divide the integral expressing the action of the transference operator on a function into three portions, corresponding to *(i) $\xi \ll \eta$, (ii) $\xi \simeq \eta$, (iii) $\xi \gg \eta$*. The middle term is then essentially handled by the preceding lemma, in conjunction to refined bounds on the kernel function F and the spectral density ρ_n.

(i): $\xi \ll \eta$. Since τ, σ are fixed, we may as well move the function $a_{k,i}^{(\pm)}(\tau,\sigma)$ to the outside of the η-integral. Then we need to show that the expression (where we use the same notation as in the proof of the preceding lemma)

$$\Xi_1^{(\pm)}(\tau,\xi) := \int_{\tau_0}^{\tau} a_{k,i}^{(\pm)}(\tau,\sigma)\cdot\int_0^\infty \chi_{\xi\ll\eta}\frac{F(\xi,\eta;\hbar)\cdot\rho_n(\eta)}{\xi-\eta}\cdot\chi_{\eta\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\left[\nu\tau\eta^{\frac{1}{2}}+\hbar^{-1}\rho\left(x_{\sigma'}^{(\eta)};\beta',\hbar\right)\right]}}{\eta^{1+\frac{k\nu}{2}}}(\log\eta)^i\,d\eta$$

is as asserted in the proposition. Write the preceding expression as

$$\begin{aligned}\Xi_1^{(\pm)}(\tau,\xi) &= \chi_{\xi\geq\hbar^{-2}}\int_{\tau_0}^{\tau} a_{k,i}^{(\pm)}(\tau,\sigma)e^{\pm i\left[\nu\tau\xi^{\frac{1}{2}}+\hbar^{-1}\rho(x_{\sigma'}^{(\xi)};\alpha',\hbar)\right]}\cdot\tilde{\Xi}_1^{(\pm)}(\tau,\sigma,\xi)\,d\sigma\\ &\quad+\tilde{\tilde{\Xi}}_1^{(\pm)}(\tau,\xi),\end{aligned}$$

where we have

$$\tilde{\Xi}_1^{(\pm)}(\tau,\sigma,\xi) = \int_0^\infty \chi_{\xi\ll\eta}\frac{F(\xi,\eta;\hbar)\cdot\rho_n(\eta)}{\xi-\eta}\cdot\chi_{\eta\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\Psi(\xi,\eta,\tau,\sigma')}}{\eta^{1+\frac{k\nu}{2}}}(\log\eta)^i\,d\eta$$

We claim that in the preceding the error term $\tilde{\tilde{\Xi}}_1^{(\pm)}(\tau,\xi)$ is of type $\overline{x}_2$ while we may write

$$a_{k,i}^{(\pm)}(\tau,\sigma)\cdot\tilde{\Xi}_1^{(\pm)}(\tau,\sigma,\xi) = \hbar^{-1}\left\langle\hbar^2\xi\right\rangle^{-\frac{1}{4}}\cdot\xi^{-1-\frac{k\nu}{2}}(\log\xi)^i\cdot F_{1,k,i}^{\pm}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right),$$

where the function $F_{1,k,i}^{\pm}$ satisfies the bounds required in Definition 7.8 and with $l=1$. Due to the scaling invariance of these bounds, it suffices to prove them for the function

$$Z_1^{\pm}(\tau,\sigma,\xi) := \tilde{\chi}_{\xi\geq\hbar^{-2}}a_{k,i}^{(\pm)}(\tau,\sigma)\cdot\tilde{\Xi}_1^{(\pm)}(\tau,\sigma,\xi)\cdot\hbar\left\langle\hbar^2\xi\right\rangle^{\frac{1}{4}}\cdot\xi^{1+\frac{k\nu}{2}}(\log\xi)^{-i}$$

where $\tilde{\chi}_{\xi\geq\hbar^{-2}}\chi_{\xi\geq\hbar^{-2}} = \chi_{\xi\geq\hbar^{-2}}$. Due to Proposition 5.1, which in particular gives the bound $|F(\xi,\eta;\hbar)| \lesssim \left\langle\hbar^2\xi\right\rangle^{-\frac{1}{2}}$, we infer the undifferentiated bound

$$\left|Z_1^{\pm}(\tau,\sigma,\xi)\right| \lesssim \tau^{-1-\nu}(\log\tau)^{N_1-i}\cdot\sigma^{-3}.$$

In order to get the derivative bounds, introduce the variable

$$\tilde{\tilde{\eta}} := \Psi(\xi,\eta,\tau,\sigma'),$$

whence in terms of the preceding variable $\tilde{\eta} = \nu\tau\left(\eta^{\frac{1}{2}}-\xi^{\frac{1}{2}}\right)$, we have in light of the proof of the preceding lemma as well as Lemma 4.13

$$\tilde{\tilde{\eta}} = \tilde{\eta}\cdot\left(1+\kappa\left(\tilde{\tilde{\eta}},\xi,\tau,\sigma'\right)\right),$$

where on the support of the integrand (i.e. $\eta\geq\hbar^{-2}$ and also $\xi\geq\hbar^{-2}$) we have

$$\left|\kappa\left(\tilde{\tilde{\eta}},\xi,\tau,\sigma'\right)\right| = O\left([\sigma'\tau]^{-1}\right)\ll 1,\quad \left|\partial_\xi^{k_1}\partial_\tau^{k_2}\kappa\left(\tilde{\tilde{\eta}},\xi,\tau,\sigma'\right)\right|\lesssim\tau^{-1-k_2}\sigma'^{-1}\xi^{-k_1},$$

and furthermore we get

$$\frac{\partial\tilde{\tilde{\eta}}}{\partial\tilde{\eta}} \simeq 1, \quad \left|\partial_\xi^{k_1}\partial_\tau^{k_2}\left(\left[\frac{\partial\tilde{\tilde{\eta}}}{\partial\tilde{\eta}}\right]^\iota\right)\right| \lesssim \tau^{-1-k_2}\sigma'^{-1}\xi^{-k_1}, \quad \iota = \pm 1.$$

Then write

$$F(\xi,\eta;\hbar) = \tilde{F}\left(\xi^{\frac{1}{2}},\eta^{\frac{1}{2}};\hbar\right) = \tilde{F}\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}} + \frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)$$

and interpret $\eta = \left(\xi^{\frac{1}{2}} + \frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)}\right)^2$ as function of $\xi,\tilde{\tilde{\eta}},\tau,\sigma'$. Thus we now have

$$\tilde{\Xi}_1^{(\pm)}(\tau,\sigma,\xi) = \int_0^\infty \chi_{\xi\ll\eta}\frac{\tilde{F}\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}} + \frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)}{\tilde{\tilde{\eta}}}\cdot e^{\pm i\tilde{\tilde{\eta}}}\cdot\zeta(\tilde{\tilde{\eta}},\xi,\tau,\sigma')\,d\tilde{\tilde{\eta}},$$

where we have set

$$\zeta(\tilde{\tilde{\eta}},\xi,\tau,\sigma') = \chi_{\eta\geq\hbar^{-2}}\hbar^{-1}\frac{(\log\eta)^i}{\eta^{1+\frac{k\nu}{2}}}\rho_n(\eta)\cdot\frac{2\eta^{\frac{1}{2}}}{\eta^{\frac{1}{2}}+\xi^{\frac{1}{2}}}\cdot\frac{\partial\tilde{\eta}}{\partial\tilde{\tilde{\eta}}}.$$

Using the bounds from above, it is then easy to verify that

$$\left|\partial_\xi^{k_1}\partial_\tau^{k_2}\left(\zeta(\tilde{\tilde{\eta}},\xi,\tau,\sigma')\cdot\hbar\left\langle\hbar^2\xi\right\rangle^{\frac{1}{4}}\cdot\xi^{1+\frac{k\nu}{2}}(\log\xi)^{-i}\right)\right| \lesssim \left\langle\hbar^2\xi\right\rangle^{\frac{1}{4}}\cdot\tau^{-1-k_2}\xi^{-k_1}$$

on the support of the integrand. Furthermore, observe the relation

$$\begin{aligned}\partial_{\xi^{\frac{1}{2}}}\left(\tilde{F}\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}} + \frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)\right) &= \left(\left(\partial_{\xi^{\frac{1}{2}}} + \partial_{\eta^{\frac{1}{2}}}\right)\tilde{F}\right)\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}} + \frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)\\ &\quad - \frac{\tilde{\tilde{\eta}}\cdot\partial_{\xi^{\frac{1}{2}}}\kappa}{\nu\tau(1+\kappa)^2}\cdot\left(\partial_{\eta^{\frac{1}{2}}}\tilde{F}\right)\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}} + \frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right),\end{aligned}$$

and we can bound each term on the right due to the refined derivative bounds for the kernel of the transference operator:

$$\begin{aligned}&\left|\left(\left(\partial_{\xi^{\frac{1}{2}}} + \partial_{\eta^{\frac{1}{2}}}\right)\tilde{F}\right)\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}} + \frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)\right| \lesssim \xi^{-\frac{1}{2}}\cdot\langle\hbar^2\xi\rangle^{-\frac{1}{2}},\\ &\left|\frac{\tilde{\tilde{\eta}}\cdot\partial_{\xi^{\frac{1}{2}}}\kappa}{\nu\tau(1+\kappa)^2}\cdot\left(\partial_{\eta^{\frac{1}{2}}}\tilde{F}\right)\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}} + \frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)\right| \lesssim \left|\partial_{\xi^{\frac{1}{2}}}\kappa\right|\cdot\left|\left(\left(\eta^{\frac{1}{2}} - \xi^{\frac{1}{2}}\right)\partial_{\eta^{\frac{1}{2}}}\tilde{F}\right)\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}} + \frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)\right|\\ &\qquad\qquad \lesssim (\tau\sigma')^{-1}\,\xi^{-\frac{1}{2}}\cdot\left\langle\hbar^2\xi\right\rangle^{-\frac{1}{2}}.\end{aligned}$$

In the first estimate above, the operator $\partial_{\xi^{\frac{1}{2}}} + \partial_{\eta^{\frac{1}{2}}}$ kills the destructive phase $e^{\pm iR\left(\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right)}$. For the non-destructive phase $e^{\pm iR\left(\xi^{\frac{1}{2}}+\eta^{\frac{1}{2}}\right)}$ we use integration by parts in the variable R to gain decay in $\xi^{\frac{1}{2}} + \eta^{\frac{1}{2}}$. The decay rate $\xi^{-\frac{1}{2}}$ is from the case when the derivative hits other factors in the presence of the destructive phase.

Analogous bounds are obtained for higher derivatives. For temporal derivatives, we find

$$\left|\partial_\tau\left(\tilde{F}\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}}+\frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)\right)\right|\lesssim\tau^{-2}(\sigma')^{-1}\cdot\left\langle\hbar^2\xi\right\rangle^{-\frac{1}{2}},$$

and analogously for higher derivatives, as well as for mixed derivatives.
The preceding considerations then easily imply the desired higher derivative bounds (here to arbitrary order)

$$\left|\partial_\xi^{k_1}\partial_\tau^{k_2}Z_1^{\pm}(\tau,\sigma,\xi)\right|\lesssim_{k_1,k_2}\tau^{-1-\nu-k_2}(\log\tau)^{N_1-i}\sigma^{-3}\cdot\xi^{-k_1}$$

The Hölder type bounds in Definition 7.8 are then a consequence of interpolation.
To conclude case *(i)*, we still need to deal with the error term $\tilde{\tilde{\Xi}}_1^{(\pm)}(\tau,\xi)$. We shall show that this can be included into $\overline{x}_2$. In fact, using integration by parts with respect to $\tilde{\tilde{\eta}}$, we infer the point wise bound

$$\left|\tilde{\Xi}_1^{(\pm)}(\tau,\sigma,\xi)\right|\lesssim_N\tau^{-N}\cdot\hbar^{-1}\left\langle\hbar^2\xi\right\rangle^{-\frac{1}{4}}\cdot\xi^{-1-\frac{k\nu}{2}}(\log\xi)^i.$$

In fact we have, by the fact $\eta\geq\hbar^{-2}$,

$$\eta\gg\xi\quad\Rightarrow\quad\tilde{\eta}\gtrsim\nu\tau\eta^{\frac{1}{2}}\gtrsim\tau\quad\Rightarrow\quad\tilde{\tilde{\eta}}\gtrsim\tau.$$

In the integral defining $\tilde{\Xi}_1^{(\pm)}(\tau,\sigma,\xi)$, when the $\partial_{\tilde{\tilde{\eta}}}$-derivative hits $\tilde{F}\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}}+\frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)$, the τ^{-1}-gain comes from the coefficient $\frac{1}{\nu\tau(1+\kappa)}$. The case when the $\partial_{\tilde{\tilde{\eta}}}$-derivative hits the factor $\zeta(\tilde{\tilde{\eta}},\xi,\tau,\sigma')$ can be handled similarly, using the explicit expression for $\kappa\left(\tilde{\tilde{\eta}},\xi,\tau,\sigma\right)$.
Therefore we have

$$\begin{aligned}&\left|\tilde{\tilde{\Xi}}_1^{(\pm)}(\tau,\xi)\right|\\&=\left|\chi_{\xi<\hbar^{-2}}\int_{\tau_0}^{\tau}a_{k,i}^{(\pm)}(\tau,\sigma)e^{\pm i\left[\nu\tau\xi^{\frac{1}{2}}+\hbar^{-1}\rho\left(x_{\sigma'}^{(\xi)};\alpha',\hbar\right)\right]}\cdot\tilde{\Xi}_1^{(\pm)}(\tau,\sigma,\xi)\,d\sigma\right|\\&\lesssim\chi_{\xi<\hbar^{-2}}\left(\int_{\tau_0}^{\tau}\tau^{-1-\nu}(\log\tau)^{N_1-i}\cdot\sigma^{-3}\,d\sigma\right)\cdot O_N\left(\tau^{-N}\right)\cdot\hbar^{-1}\left\langle\hbar^2\xi\right\rangle^{-\frac{1}{4}}\cdot\xi^{-1-\frac{k\nu}{2}}(\log\xi)^i,\end{aligned}$$

which in turn gives

$$\left\|\tilde{\tilde{\Xi}}_1^{(\pm)}(\tau,\xi)\right\|_{S_0^\hbar}\lesssim_N\tau^{-N},$$

stronger than what we need.

(ii): $\xi\simeq\eta$. Here we need to show that the expression

$$\Xi_2^{(\pm)}(\tau,\xi):=\int_{\tau_0}^{\tau}a_{k,i}^{(\pm)}(\tau,\sigma)\cdot\int_0^\infty\chi_{\xi\simeq\eta}\frac{F(\xi,\eta;\hbar)\cdot\rho_n(\eta)}{\xi-\eta}\cdot\chi_{\eta\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\left[\nu\tau\eta^{\frac{1}{2}}+\hbar^{-1}\rho\left(x_{\sigma'}^{(\eta)};\beta',\hbar\right)\right]}}{\eta^{1+\frac{k\nu}{2}}}(\log\eta)^i\,d\eta$$

is as asserted in the proposition. We shall again decompose the expression into two by including suitable cutoffs in front:

$$\Xi_2^{(\pm)}(\tau,\xi)=\chi_{\xi\geq\hbar^{-2}}\int_{\tau_0}^{\tau}a_{k,i}^{(\pm)}(\tau,\sigma)\cdot\int_0^\infty\chi_{\xi\simeq\eta}\frac{F(\xi,\eta;\hbar)\cdot\rho_n(\eta)}{\xi-\eta}\cdot\chi_{\eta\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\left[\nu\tau\eta^{\frac{1}{2}}+\hbar^{-1}\rho\left(x_{\sigma'}^{(\eta)};\beta',\hbar\right)\right]}}{\eta^{1+\frac{k\nu}{2}}}(\log\eta)^i\,d\eta$$

$$+ \tilde{\tilde{\Xi}}_2^{(\pm)}(\tau,\xi)$$
$$=: \chi_{\xi\geq\hbar^{-2}} \int_{\tau_0}^{\tau} a_{k,i}^{(\pm)}(\tau,\sigma) e^{\pm i\left[\nu\tau\xi^{\frac{1}{2}}+\hbar^{-1}\rho\left(x_{\sigma'}^{(\xi)};\alpha',\hbar\right)\right]} \cdot \tilde{\Xi}_2^{(\pm)}(\tau,\sigma,\xi)\,d\sigma + \tilde{\tilde{\Xi}}_2^{(\pm)}(\tau,\xi).$$

To proceed, we decompose $\tilde{\Xi}_2^{(\pm)}(\tau,\sigma,\xi)$ further into a 'weakly diagonal' and a 'strongly diagonal' part via inclusion of smooth cutoffs $\chi_{\left|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right|\geq 1}, \chi_{\left|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right|<1}$.

Weakly diagonal part. Set

$$\tilde{\Xi}_{21}^{(\pm)}(\tau,\sigma,\xi) := \int_0^\infty \chi_{\xi\simeq\eta}\cdot\chi_{\left|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right|\geq 1} \frac{F(\xi,\eta;\hbar)\cdot\rho_n(\eta)}{\xi-\eta}\cdot\chi_{\eta\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\Psi(\xi,\eta,\tau,\sigma')}}{\eta^{1+\frac{k\nu}{2}}}(\log\eta)^i\,d\eta$$

Here we proceed as in case *(i)* and, using the same notation we get

$$\tilde{\Xi}_{21}^{(\pm)}(\tau,\sigma,\xi) = \int_{-\infty}^{\infty} \chi_{\xi\simeq\eta}\chi_{\left|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right|\geq 1} \frac{\tilde{F}\left(\xi^{\frac{1}{2}},\xi^{\frac{1}{2}}+\frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)}{\tilde{\tilde{\eta}}}\cdot e^{\pm i\tilde{\tilde{\eta}}}\cdot\zeta(\tilde{\tilde{\eta}},\xi,\tau,\sigma')\,d\tilde{\tilde{\eta}},$$

where η is thought of as function of $\xi,\tilde{\tilde{\eta}},\tau,\sigma'$. Note that the restriction $\left|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right|\geq 1$ implies $\left|\tilde{\tilde{\eta}}\right|\gtrsim\tau$, and that we can write

$$\chi_{\left|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right|\geq 1} = \chi_{\left|\frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)}\right|\geq 1},$$

and we have

$$\left|\partial_\xi\left(\chi_{\left|\frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)}\right|\geq 1}\right)\right|\lesssim\xi^{-1}.$$

Taking advantage of the refined asymptotics of $F(\xi,\eta;\hbar)$ and its derivatives, as detailed in Proposition 5.1, we infer the bound

$$\left|\partial_\xi^{k_1}\partial_\tau^{k_2}\left(\tilde{\Xi}_{21}^{(\pm)}(\tau,\sigma,\xi)\cdot\hbar\left\langle\hbar^2\xi\right\rangle^{\frac{1}{4}}\cdot\xi^{1+\frac{k\nu}{2}}(\log\xi)^{-i}\right)\right|\lesssim\xi^{-k_1}\tau^{-k_2}.$$

In fact, integrability over $\left|\tilde{\tilde{\eta}}\right|\gtrsim\tau$ follows from the refined off-diagonal decay. The desired derivative bounds(of arbitrary degree) for

$$\hbar\left\langle\hbar^2\xi\right\rangle^{\frac{1}{4}}\cdot\xi^{1+\frac{k\nu}{2}}(\log\xi)^{-i}\cdot\int_{\tau_0}^{\tau} a_{k,i}^{(\pm)}(\tau,\sigma)\cdot\tilde{\Xi}_{21}^{(\pm)}(\tau,\sigma,\xi)\,d\sigma$$

as implied by the proposition and Definition 7.8 follow directly from this.

Strongly diagonal part. This is the term

$$\tilde{\Xi}_{22}^{(\pm)}(\tau,\sigma,\xi) := \int_0^\infty \chi_{\xi\simeq\eta}\cdot\chi_{\left|\xi^{\frac{1}{2}}-\eta^{\frac{1}{2}}\right|<1} \frac{F(\xi,\eta;\hbar)\cdot\rho_n(\eta)}{\xi-\eta}\cdot\chi_{\eta\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\Psi(\xi,\eta,\tau,\sigma')}}{\eta^{1+\frac{k\nu}{2}}}(\log\eta)^i\,d\eta$$

We split this into a term which can be handled directly by the preceding lemma, as well as a better error term. For this split $F(\xi, \eta; \hbar) = \tilde{F}\left(\xi^{\frac{1}{2}}, \eta^{\frac{1}{2}}; \hbar\right)$ as follows:

$$\tilde{F}\left(\xi^{\frac{1}{2}}, \eta^{\frac{1}{2}}; \hbar\right) = \tilde{F}\left(\xi^{\frac{1}{2}}, \xi^{\frac{1}{2}}; \hbar\right) + \int_0^1 \left(\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\right) \cdot \partial_{\eta^{\frac{1}{2}}} \tilde{F}\left(\xi^{\frac{1}{2}}, s\xi^{\frac{1}{2}} + (1-s)\eta^{\frac{1}{2}}; \hbar\right) ds,$$

and so we infer

$$\tilde{\Xi}_{22}^{(\pm)}(\tau, \sigma, \xi) = \tilde{\Xi}_{22}^{(\pm),a}(\tau, \sigma, \xi) + \tilde{\Xi}_{22}^{(\pm),b}(\tau, \sigma, \xi),$$

where we set

$$\tilde{\Xi}_{22}^{(\pm),a}(\tau, \sigma, \xi) = \tilde{F}\left(\xi^{\frac{1}{2}}, \xi^{\frac{1}{2}}; \hbar\right) \cdot \int_0^\infty \chi_{\xi \simeq \eta} \cdot \chi_{\left|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\right| < 1} \frac{\rho_n(\eta)}{\xi - \eta} \cdot \chi_{\eta \geq \hbar^{-2}} \hbar^{-1} \frac{e^{\pm i \Psi(\xi, \eta, \tau, \sigma')}}{\eta^{1 + \frac{k\nu}{2}}} (\log \eta)^i \, d\eta$$

$$\tilde{\Xi}_{22}^{(\pm),b}(\tau, \sigma, \xi) = \int_0^\infty \chi_{\xi \simeq \eta} \cdot \chi_{\left|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\right| < 1} \tilde{\tilde{F}}\left(\xi^{\frac{1}{2}}, \eta^{\frac{1}{2}}; \hbar\right) \rho_n(\eta) \cdot \chi_{\eta \geq \hbar^{-2}} \hbar^{-1} \frac{e^{\pm i \Psi(\xi, \eta, \tau, \sigma')}}{\eta^{1 + \frac{k\nu}{2}}} (\log \eta)^i \, d\eta,$$

where in the second integral we use the notation

$$\tilde{\tilde{F}}\left(\xi^{\frac{1}{2}}, \eta^{\frac{1}{2}}; \hbar\right) = \int_0^1 \partial_{\eta^{\frac{1}{2}}} \tilde{F}\left(\xi^{\frac{1}{2}}, s\xi^{\frac{1}{2}} + (1-s)\eta^{\frac{1}{2}}; \hbar\right) ds$$

Using the special derivative bounds for the transference kernel, as well as the preceding lemma, we then easily infer the better than required bounds

$$\left| \xi^{k_2} \partial_\tau^{k_1} \partial_\xi^{k_2} \left(a_{k,i}^{(\pm)}(\tau, \sigma) \tilde{\Xi}_{22}^{(\pm),a}(\tau, \sigma, \xi) \cdot \hbar \left\langle \hbar^2 \xi \right\rangle^{\frac{1}{4}} \cdot \xi^{1 + \frac{k\nu}{2}} (\log \xi)^{-i} \right) \right| \lesssim (\log \tau)^{N_1 - i} \tau^{-1-\nu-k_1} \cdot \sigma^{-3}$$

The same bounds obtain also for the second term $\tilde{\Xi}_{22}^{(\pm),b}(\tau, \sigma, \xi)$, where one can simply repeat the change of coordinates from case *(i)* and follow the same arguments as in that case.

It remains to deal with the contribution of the error term $\tilde{\tilde{\Xi}}_2^{(\pm)}(\tau, \xi)$. But this term is seen to be of type $\overline{x}_3$ as is easily seen by arranging the outer cutoff $\chi_{\xi < \hbar^{-2}}$. in such a way that the intersection of the supports of the two cutoffs $\chi_{\xi < \hbar^{-2}}, \chi_{\eta \geq \hbar^{-2}}$ is empty.

(iii): $\xi \gg \eta$. This can be handled analogously to case *(i)* due to the rapid off-diagonal gain $\left(\frac{\eta}{\xi}\right)^N$ for the transference kernel, and we omit the analogous details. Moreover, note that now there is no analogous error term as $\tilde{\tilde{\Xi}}_1^{(\pm)}(\tau, \xi)$.
This concludes the bounds for the contribution of the principal ingoing singular part, which we see gets transformed by the non-local part of the transference operator into a connecting ingoing singular part.

Contribution of the connecting ingoing singular part, i.e. the second term on the right in (7.3). Here one can essentially follow the same outline as in the preceding case, splitting into situations *(i) - (iii)*. Consider for example the case *(i)* here, which is given by

$$\int_{\tau_0}^{\tau}\int_0^\infty \chi_{\xi\ll\eta}\frac{F(\xi,\eta;\hbar)\cdot\rho_n(\eta)}{\xi-\eta}\cdot\chi_{\eta\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\left[\nu\tau\eta^{\frac12}+\hbar^{-1}\rho\left(x_{\sigma'}^{(\eta)};\beta',\hbar\right)\right]}}{\eta^{1+\frac{k\nu}{2}}}(\log\eta)^i\cdot F_{l,k,i}^{(\pm)}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\eta\right)d\eta d\sigma.$$

Using the same change of variables as in case *(i)* above, i.e. $\tilde{\tilde{\Psi}}(\xi,\eta,\tau,\sigma')$, we can write the preceding term as (abusing notation in order to emphasize the analogy)

$$\chi_{\xi\geq\hbar^{-2}}\int_{\tau_0}^{\tau}e^{\pm i\left[\nu\tau\xi^{\frac12}+\hbar^{-1}\rho\left(x_{\sigma'}^{(\xi)};\alpha',\hbar\right)\right]}\cdot\tilde{\tilde{\Xi}}_1^{(\pm)}(\tau,\sigma,\xi)\,d\sigma+\chi_{\xi<\hbar^{-2}}\int_{\tau_0}^{\tau}e^{\pm i\left[\nu\tau\xi^{\frac12}+\hbar^{-1}\rho\left(x_{\sigma'}^{(\xi)};\alpha',\hbar\right)\right]}\cdot\tilde{\tilde{\Xi}}_1^{(\pm)}(\tau,\sigma,\xi)\,d\sigma,$$

where this time we set

$$\tilde{\tilde{\Xi}}_1^{(\pm)}(\tau,\sigma,\xi)=\int_0^\infty\chi_{\xi\ll\eta}\frac{\tilde{F}\left(\xi^{\frac12},\xi^{\frac12}+\frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)};\hbar\right)}{\tilde{\tilde{\eta}}}\cdot e^{\pm i\tilde{\tilde{\eta}}}\cdot\zeta\left(\tilde{\tilde{\eta}},\xi,\tau,\sigma'\right)d\tilde{\tilde{\eta}}$$

and we define

$$\zeta\left(\tilde{\tilde{\eta}},\xi,\tau,\sigma'\right)=\chi_{\eta\geq\hbar^{-2}}\hbar^{-1}\frac{(\log\eta)^i}{\eta^{1+\frac{k\nu}{2}}}\rho_n(\eta)\cdot\frac{2\eta^{\frac12}}{\eta^{\frac12}+\xi^{\frac12}}\cdot\frac{\partial\tilde{\eta}}{\partial\tilde{\tilde{\eta}}}\cdot F_{l,k,i}^{(\pm)}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\left(\xi^{\frac12}+\frac{\tilde{\tilde{\eta}}}{\nu\tau(1+\kappa)}\right)^2\right),$$

and κ is defined as in Definition 7.8. This time we obtain the following bounds for ζ in the region $\xi\ll\eta$:

$$\begin{aligned}&\left|\partial_\xi^{k_1}\partial_\tau^{k_2}\left(\zeta\left(\tilde{\tilde{\eta}},\xi,\tau,\sigma'\right)\cdot\hbar\langle\hbar^2\xi\rangle^{\frac14}\cdot\xi^{1+\frac{k\nu}{2}}(\log\xi)^{-i}\right)\right|\\&\lesssim\langle\hbar^2\xi\rangle^{\frac14}\cdot\tau^{-1-\nu-k_2}(\log\tau)^{N_1-i}\cdot\sigma^{-1}\left[\sigma^{-2}+\kappa\left(\hbar\frac{\lambda(\tau)}{\lambda(\sigma)}\eta^{\frac12}\right)\right]\cdot\xi^{-k_1},\quad k_1+k_2\leq5,\end{aligned}$$

as well as the 'closing Holder bounds'

$$\begin{aligned}&\left\|\partial_\xi^{5-k_2}\partial_\tau^{k_2}\left(\zeta\left(\tilde{\tilde{\eta}},\xi,\tau,\sigma'\right)\cdot\hbar\left\langle\hbar^2\xi\right\rangle^{\frac14}\cdot\xi^{1+\frac{k\nu}{2}}(\log\xi)^{-i}\right)\right\|_{\dot{C}_\xi^\delta(\xi\simeq\lambda)}\\&\lesssim\lambda^{-(5-k_2+\delta)}\left\langle\hbar^2\lambda\right\rangle^{\frac14}\cdot\tau^{-1-\nu-k_2}(\log\tau)^{N_1-i}\cdot\sigma^{-1}\left[\sigma^{-2}+\kappa\left(\hbar\frac{\lambda(\tau)}{\lambda(\sigma)}\eta^{\frac12}\right)\right]\end{aligned}$$

Observe that the off-diagonal decay of the transference kernel $F(\xi,\eta;\hbar)=\tilde{F}\left(\xi^{\frac12},\eta^{\frac12};\hbar\right)$ then allows us to translate the gain $\kappa\left(\hbar\frac{\lambda(\tau)}{\lambda(\sigma)}\eta^{\frac12}\right)$ to $\kappa\left(\hbar\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac12}\right)$ because

$$\min\left\{\left(\frac{\xi}{\eta}\right)^N,\left(\frac{\eta}{\xi}\right)^N\right\}\cdot\kappa\left(\hbar\frac{\lambda(\tau)}{\lambda(\sigma)}\eta^{\frac12}\right)\lesssim\kappa\left(\hbar\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac12}\right)$$

and the rest of the argument proceeds as in the earlier case *(i)*. The required modifications for the cases *(ii)*, *(iii)* are similar.

Contribution of the outgoing singular part, i.e. the terms (7.4)*,* (7.5)*.* This can again be handled by small modifications of the preceding cases. Consider for example the contribution of an 'outgoing perpetuated singularity' (7.5). To handle it, we need to use a slightly modified version of Lemma 7.12, Lemma 7.14, where the oscillatory factor

$$e^{\pm i\nu\tau\eta^{\frac{1}{2}}}$$

gets replaced by

$$e^{\pm i\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x+\tau\right)\eta^{\frac{1}{2}}}, \quad x \geq 0.$$

This forces utilization of a different variable $\tilde{\eta}$ in the proof of these lemmas, namely

$$\tilde{\eta} = \nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x + \tau\right)\left(\eta^{\frac{1}{2}} - \xi^{\frac{1}{2}}\right),$$

while the remaining steps are again identical to the ones in the original proofs. Observe in particular that in Remark 7.13 the function $P_{>C^{-1}\tau}g$ is then replaced by

$$P_{>C^{-1}\cdot\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x+\tau\right)}g.$$

□

7.4.2. *The effect of derivatives on functions with admissibly singular distorted Fourier transform at angular momentum $n, |n| \geq 2$.* Recall that in the coordinates (τ, R), the derivatives arising in the non-linear source terms are of type $\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R \pm \partial_R$. Here we establish how these operators act on admissibly singular functions, taking advantage of the preceding subsection. This gets expressed in the form of a lemma analogous to Lemma 7.10:

Lemma 7.17. *Assume that $f(\tau, R)$ is an angular momentum $n, |n| \geq 2$ function, represented by*

$$f(\tau, R) = \int_0^\infty \phi_n(R, \xi) \cdot \overline{x}(\tau, \xi) \cdot \rho_n(\xi)\, d\xi,$$

with $\overline{x}(\tau, \xi)$ admissibly singular (at angular momentum n). Then for the action of the 'good derivative' $\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R$ we have the following representation on the physical side: restricting to $R < \nu\tau$ we can write

$$\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\right)f = f_1 + f_2 + f_3$$

where $f_1 = f_1(\tau, R)$ is a C^5-function supported in $\nu\tau - R \gtrsim 1$ and satisfying

$$\nabla_R^{k_2} f_1(\tau, R) \lesssim \hbar^{-1} (\log\tau)^{N_1+1} \cdot \tau^{-\frac{3}{2}-\nu} |\nu\tau - R|^{-5}, \quad 0 \leq k_2 \leq 5,$$

while $f_2 = \sum_{l=1}^{8} f_{2l}$ where we have the explicit form

$$f_{2l}(\tau, R) = \chi_{|\nu\tau-R|\lesssim\hbar} \sum_{k=1}^{N}\sum_{i=0}^{N_1} \frac{G_{k,l,i}(\tau, \nu\tau - R)}{\tau^{\frac{1}{2}}}\hbar^{-\frac{l+1}{2}} [\nu\tau - R]^{\frac{l}{2}+k\nu} (\log(\nu\tau - R))^i$$

Here the function $G_{k,i}(\tau, x)$ has symbol type behavior with respect to x, as follows:

$$\left|\partial_x^{k_2} G_{k,l,i}(\tau, x)\right| \lesssim (\log\tau)^{N_1-i+1} \cdot \tau^{-1-\nu} x^{-k_2}, \quad 0 \leq k_2 \leq 5$$

and we have the bound

$$\left\|x^{5+\delta}\partial_x^5 G_{k,l,i}(\tau,x)\right\|_{\dot{C}^\delta} \lesssim (\log\tau)^{N_1-i+1}\cdot\tau^{-1-\nu}.$$

Finally, the remaining function f_3 *is also* C^5 *and supported in* $|\nu\tau - R| \lesssim 1$ *and satisfies*

$$\|f_3\|_{S_0^\hbar} \lesssim (\log\tau)^{N_1+1}\cdot\tau^{-1-\nu}.$$

Moreover, we have the bounds

$$\left|\partial_R^k f_3\right| \lesssim (\log\tau)^{N_1+1}\,\tau^{-\frac{3}{2}-\nu}\cdot\hbar^{-1}\min\{(\nu\tau-R)^{-k},\hbar^{-k}\},\quad 0\le k\le 5.$$

The contribution of the terms $\overline{x}_{out}$ *enjoys a smoothness gain visible on the physical side upon restriction to the interior of the light cone: defining*

$$g(\tau,R) := \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\right)\int_0^\infty \phi_n(R;\xi)\cdot\overline{x}_{out}(\tau,\xi)\cdot\rho_n(\xi)\,d\xi,$$

we can write

$$g = g_1 + g_2 + g_3,$$

where $g_{1,3}$ *have the same properties as* $f_{1,3}$*, while* g_2 *admits the representation*

$$g_2 = \chi_{|\nu\tau-R|\lesssim\hbar}\sum_{l=3}^{8}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\frac{H_{k,l,i}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar^{-\frac{l+1}{2}}\,[\nu\tau-R]^{\frac{l}{2}+k\nu}\left(\log(\nu\tau-R)\right)^i$$

and we have the symbol type bounds

$$\left|\partial_x^{k_2}H_{k,l,i}(\tau,x)\right| \lesssim (\log\tau)^{N_1-i+1}\cdot\tau^{-1-\nu}x^{-k_2},\quad k_2\in\{0,\dots,3\},$$

as well as

$$\left\|x^{3+\delta}\partial_x^3 G_{k,l,i}(\tau,x)\right\|_{\dot{C}^\delta} \lesssim (\log\tau)^{N_1-i+1}\cdot\tau^{-1-\nu}.$$

The preceding two bounds hold also with derivatives up to degree 5*, provided we control the* ξ*-derivatives of the functions* $F^\pm_{l,k,i}, G^\pm_{l,k,i}$ *up to degree* 7*.*

For the action of the 'bad derivative' $\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R$*, we have analogous conclusion except that* k, k_2 *range up to* 4 *and the expression* $[\nu\tau-R]^{\frac{l}{2}+k\nu}$ *is replaced by*

$$[\nu\tau-R]^{\frac{l}{2}+k\nu-1},$$

and similarly at the end (concerning the improvement for the contribution of $\overline{x}_{out}$*)* k_2 *only ranges up to* 2*, respectively* 4 *for the conclusion at the very end.*

Proof. The idea is to decompose the derivative operator into

$$\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R \mp \partial_R = \partial_\tau + [(1+\nu)\mp 1]\,\partial_R + \left(1+\nu^{-1}\right)\frac{R-\nu\tau}{\tau}\partial_R$$

Fixing the $-$-sign first, this becomes

$$\partial_\tau + \nu\partial_R + \left(1+\nu^{-1}\right)\frac{R-\nu\tau}{\tau}\partial_R.$$

The operator $\partial_\tau + \nu\partial_R$ acts trivially on functions of the form $f(\nu\tau - R)$. Thus if we recall Lemma 7.10, the conclusion of the first part of the present lemma follows easily for the contribution arising by applying

$\partial_\tau + \nu\partial_R$, while the contribution arising upon applying $(1+\nu^{-1})\frac{R-\nu\tau}{\tau}\partial_R$ is handled by invoking the symbol behavior of the coefficients

$$G_{k,l,i}(\tau, \nu\tau - R)$$

with respect to the second variable. The case of the +-sign is handled similarly. □

Due to the technical difficulty of precisely translating the physical properties of functions to the Fourier properties, we give a more precise statement for the action of $\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R \mp \partial_R$ on the physical realisation of the principal ingoing part:

Lemma 7.18. *Assume that $f(\tau, R)$ is an angular momentum $n, |n| \geq 2$ function, represented by*

$$f(\tau, R) = \int_0^\infty \phi_n(R, \xi) \cdot \overline{x}(\tau, \xi) \cdot \rho_n(\xi)\, d\xi,$$

with $\overline{x}(\tau, \xi)$ given by the first term on the right in (7.3). *Then we have*

$$\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\right) f(\tau, R)|_{R<\nu\tau} = \int_0^\infty \phi_n(R, \xi) \cdot \overline{y}(\tau, \xi) \cdot \rho_n(\xi)\, d\xi$$

where $\overline{y} = \pm 2i\xi^{\frac{1}{2}} \cdot \overline{x} + \sum_{j=1}^2 \overline{y}_j$, with $\overline{y}_1$ an admissibly singular term, while $\overline{y}_2$ is in $S_0^\hbar$.

Proof. Write

$$\begin{aligned}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\right) f &= \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\right) f + 2\partial_R f \\ &= \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\right) f + 2\frac{R\partial_R f}{R} \\ &= \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\right) f + 2\frac{R\partial_R f}{\nu\tau} + 2\frac{\nu\tau - R}{\nu\tau} \cdot \partial_R f\end{aligned}$$

Then apply the preceding lemma to the first and third term, using

$$2\partial_R = \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\right) - \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\right)$$

to handle the third term, and further apply Lemma 7.7 to get the desired conclusion. It remains to determine the distorted Fourier transform of the middle term

$$2\frac{R\partial_R f}{\nu\tau},$$

for which we use (denoting by $\mathcal{F}$ the distorted Fourier transform at angular momentum n)

$$\begin{aligned}\mathcal{F}\left(2\frac{R\partial_R f}{\nu\tau}\right) &= -\frac{4}{\nu\tau}\left(\xi\partial_\xi\right)\mathcal{F}(f) + \frac{2}{\nu\tau}\mathcal{K}^{(\hbar)}\mathcal{F}(f) \\ &= -\frac{4}{\nu\tau}\left(\xi\partial_\xi\right)\overline{x} + \frac{2}{\nu\tau}\mathcal{K}^{(\hbar)}\overline{x}\end{aligned}$$

To handle the contribution of the last term on the right, use Proposition 7.15. It remains to deal with the term $-\frac{4}{\nu\tau}\left(\xi\partial_\xi\right)\overline{x}$. Referring to the principal ingoing term as in (7.3), we have

$$\frac{4}{\nu\tau}\left(\xi\partial_\xi\right)\left(\chi^{(l)}_{\xi\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^i\right) = \frac{4}{\nu\tau}\left(\xi\partial_\xi\right)\left(\chi^{(l)}_{\xi\geq\hbar^{-2}}\right)\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^i$$
$$+\chi^{(l)}_{\xi\geq\hbar^{-2}}\frac{4}{\nu\tau}\left(\xi\partial_\xi\right)\left(\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^i\right)$$

Here the second term on the right can be written as

$$\chi^{(l)}_{\xi\geq\hbar^{-2}}\frac{4}{\nu\tau}\left(\xi\partial_\xi\right)\left(\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^i\right) = \pm 2i\chi^{(l)}_{\xi\geq\hbar^{-2}}\cdot\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}+k\frac{\nu}{2}}}(\log\xi)^i + \text{error},$$

Here the first term on the right is responsible for the term $\pm 2i\xi^{\frac{1}{2}}\cdot\overline{x}$ in the lemma, while the term 'error' leads to a term of principal ingoing singular type. The remaining term

$$\frac{4}{\nu\tau}\left(\xi\partial_\xi\right)\left(\chi^{(l)}_{\xi\geq\hbar^{-2}}\right)\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^i$$

from above is easily seen to lead to a term in $S_0^{\hbar}$, since the above function is supported in the regime where $\hbar^2\xi\simeq 1$. □

7.5. **Multilinear estimates near the light cone with singular inputs at angular momentum $|n|\geq 2$.** In this subsection, we finally control the source terms near the light cone with singular inputs, which is particularly delicate for the null-form source terms. In a first stage, we shall assume that all factors (inputs) are angular momentum $|n|\geq 2$ functions, as the general case will be a rather straightforward extension of this case. However, at this stage we only consider the source terms at angular momentum $|n|\geq 2$, as the exceptional angular momenta $n=0,\pm 1$ will be treated in a separate section at the end.
We note that the null-form estimates are delicate, since a priori the regularity we are dealing with is only H^{1+}, whence we are at the limit of the strong local well-posedness regime. While the general theory requires the use of $H^{s,\delta}$ spaces in this setting, we can and have to take advantage of the very particular structure of our solutions involving the shock on the light cone, which turns out to be very naturally adapted to the null-form structure of the most singular source terms.

7.5.1. *Basic product estimates for angular momentum $|n|\geq 2$ functions with admissibly singular distorted Fourier transform.* Here we show that our concept of admissible singularity leads to good product estimates, and that these concepts are also compatible with forming paraproducts. These will arise naturally when proving the basic null-form estimates needed to handle the source terms.

Proposition 7.19. *Let n_j, $j=1,2,3$ obey the same conditions as in Prop. 6.14 . Assume that the functions $f_j(\tau,R)$, $j=1,2$ are angular momentum n_j, $|n_j|\geq 2$ functions admitting representations*

$$f_j(\tau,R) = \int_0^\infty \phi_{n_j}(R,\xi)\cdot\overline{x}_j(\tau,\xi)\cdot\rho_{n_j}(\xi)\,d\xi,\quad j=1,2,$$

where the distorted Fourier transforms $\overline{x}_j$, $j = 1, 2$ *each can be written as* $\overline{x}_j = \overline{y}_j + \overline{z}_j$ *with* $\overline{y}_j$ *admissibly singular (at angular momentum* n_j*) and* $\overline{z}_j \in S_0^{\hbar_j}$*. Then we have*

$$\prod_{j=1,2} f_j|_{R<\nu\tau} = g(\tau, R),$$

where, g admits an angular momentum n_3 *representation*

$$g(\tau, R) = \int_0^\infty \phi_{n_3}(R,\xi)\overline{x}_3(\tau,\xi)\rho_{n_3}(\xi)\,d\xi,$$

where $\overline{x}_3 = \overline{y}_3 + \overline{\overline{y}}_3 + \overline{z}_3$*, and where* $\overline{y}_3$ *is admissibly singular of principal ingoing type,* $\overline{\overline{y}}_3$ *is of prototypical singular type with vanishing principal part, and* $\overline{z}_3 \in S_0^{\hbar_3}$*.*

Proof. Assume that $\hbar_2 \gg \hbar_1$, say, whence the product is at angular momentum n_3 with $n_3 \simeq n_1$, the remaining case being handled similarly. In addition, first, assume that both $\overline{y}_{1,2}$ have vanishing principal ingoing part. Write

$$f_j(\tau, R) = \tilde{f}_j(\tau, R) + \tilde{\tilde{f}}_j(\tau, R),\ j = 1, 2,$$

where we define

$$\tilde{f}_j(\tau, R) = \int_0^\infty \phi_{n_j}(R,\xi)\cdot\overline{y}_j(\tau,\xi)\cdot\rho_{n_j}(\xi)\,d\xi,$$

i.e. corresponding to the singular part. Using Lemma 7.10, as well as Lemma 7.7, the conclusion follows readily for the product of the $\tilde{f}_j$, and a simple version of the basic product estimates such as Prop. 6.10 gives the conclusion for the product of the $\tilde{\tilde{f}}_j$. It remains to consider the mixed case, i.e. the products

$$\tilde{\tilde{f}}_1\cdot\tilde{f}_2,\ \tilde{f}_1\cdot\tilde{\tilde{f}}_2.$$

The product $\tilde{\tilde{f}}_1\cdot\tilde{f}_2$. To begin with, we can reduce $\tilde{f}_2$ to $\chi_{\nu\tau-R<\hbar_1}\tilde{f}_2$, by means of the following technical

Lemma 7.20. *We have*

$$\tilde{\tilde{f}}_1\cdot\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2 \in \tilde{S}_0^{(\hbar_3)},\quad \left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\left(\tilde{\tilde{f}}_1\cdot\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2\right) \in \tilde{S}_1^{(\hbar_3)}$$

Proof. (lemma) We sketch the argument, which is very similar to the one for Prop. 6.10: for the undifferentiated term, we need to bound

$$\left\|\left\langle\phi_{n_3}(R,\xi),\ \tilde{\tilde{f}}_1\cdot\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2\right\rangle_{L^2_{R\,dR}}\right\|_{S_0^{\hbar_3}}$$

Labeling ξ_1 the frequency in the angular momentum n_1 Fourier representation of $\tilde{\tilde{f}}_1$, due to $\hbar_3 \simeq \hbar_1$ the case $\xi_1 \geq \xi$ is easily handled by invoking Lemma 7.10 to bound the L^∞-norm of $\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2$ (for instance, the method to handle the case *(1)* in the proof of Proposition 6.10 can be directly applied here). It remains to deal with the case $\xi_1 < \xi$, where we have to perform integration by parts. For this, in the low frequency regime $\xi < 1$, write

$$\left\langle\phi_{n_3}(R,\xi),\ \tilde{\tilde{f}}_1\cdot\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2\right\rangle_{L^2_{R\,dR}} = \frac{1}{\xi}\left\langle H_{n_3}\phi_{n_3}(R,\xi),\ \tilde{\tilde{f}}_1\cdot\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2\right\rangle_{L^2_{R\,dR}}$$

Then the terms

$$\frac{1}{\xi}\left\langle\phi_{n_3}(R,\xi),\ \partial_R^2\left(\tilde{\tilde{f}}_1\right)\cdot\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2\right\rangle_{L^2_{R\,dR}},\quad \frac{1}{\xi}\left\langle\phi_{n_3}(R,\xi),\ \frac{n_3^2}{R^2}\left(\tilde{\tilde{f}}_1\right)\cdot\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2\right\rangle_{L^2_{R\,dR}}$$

are bounded by means of Prop. 6.8 (by again placing $\frac{n_3^2}{R^2}\tilde{\tilde{f}}_1, \partial_R^2\tilde{\tilde{f}}_1$ into L^2 and $\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2$ into L^∞). The term

$$\frac{1}{\xi}\left\langle\phi_{n_3}(R,\xi),\ \tilde{\tilde{f}}_1\cdot\partial_R^2\left(\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2\right)\right\rangle_{L^2_{R\,dR}}$$

is estimated by using the bound (see Lemma 6.9)

$$\left|\tilde{\tilde{f}}_1(\tau,R)\right|\lesssim\tau\hbar_1^{-\delta}\cdot\left\|\tilde{\tilde{f}}_1\right\|_{\tilde{S}_1^{(\hbar_1)}},\quad\text{for}\quad R\lesssim\tau,$$

in conjunction with (see Lemma 7.10)

$$\left\|\partial_R^2\left(\chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_2\right)\right\|_{L^2_{R\,dR}}\lesssim\hbar_1^{-1}\cdot\tau^{-1-\nu}\left(\log\tau\right)^C.$$

To deal with the high frequency regime $\xi>1$ further integration by parts are required, which can be handled analogously to the preceding. □

It remains to deal with $\tilde{\tilde{f}}_1\cdot\chi_{\nu\tau-R<\hbar_1}\tilde{f}_2$. Here it suffices to split

$$\tilde{\tilde{f}}_1(\tau,R)=\sum_{j=0}^{3}(\nu\tau-R)^j\cdot\frac{\tilde{\tilde{f}}_1^{(j)}(\tau,\nu\tau)}{j!}+\tilde{\tilde{g}}_1(\tau,R)=:P_3\tilde{\tilde{f}}_1(\tau,R)+\tilde{\tilde{g}}_1(\tau,R)$$

Also, we may assume that in accordance with Lemma 7.10 we have

$$\chi_{\nu\tau-R<\hbar_1}\tilde{f}_2(\tau,R)=\chi_{|\nu\tau-R|\lesssim\hbar_1}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\frac{G_{k,l,i}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar_2^{-\frac{l+1}{2}}\left[\nu\tau-R\right]^{\frac{l}{2}+k\nu}\left(\log\left(\nu\tau-R\right)\right)^i$$

with bounds as stated there for the coefficients $G_{k,l,i}(\tau,\nu\tau-R)$. We have the bounds

$$\begin{aligned}\left|\tilde{\tilde{f}}_1^{(j)}(\tau,\nu\tau)\right|&=\left|\partial_R^j\int_0^\infty\phi_{n_1}(R,\xi)\cdot\overline{z}_1(\tau,\xi)\rho_{n_1}(\xi)\,d\xi\right|_{R=\nu\tau}\\&\lesssim\tau\cdot\hbar_1^{-1-j}\cdot\|\overline{z}_1(\tau,\cdot)\|_{S_0^{\hbar_1}},\quad 0\leq j\leq 3,\end{aligned}$$

$$\left|\left(\partial_\tau+\frac{\lambda'}{\lambda}R\partial_R\right)\tilde{\tilde{f}}_1^{(j)}(\tau,\nu\tau)\right|\lesssim\hbar_1^{-1-j}\cdot\left(\|\overline{z}_1(\tau,\cdot)\|_{S_0^{\hbar_1}}+\|\mathcal{D}_\tau\overline{z}_1(\tau,\cdot)\|_{S_1^{\hbar_1}}\right),\quad 0\leq j\leq 3.$$

The second estimate above can be derived similarly as the first one. For the first estimate, we can actually avoid the growth in τ. In fact we write

$$\begin{aligned}\tilde{\tilde{f}}_1^{(j)}(\tau,\nu\tau)=&\chi_{0\leq\nu\tau-R\lesssim1}\tilde{\tilde{f}}_1^{(j)}(\nu\tau,R)|_{R=\nu\tau}=\int_{\nu\tau-1}^{\nu\tau}\partial_R\left(\chi_{0\leq\nu\tau-R\lesssim1}\tilde{\tilde{f}}_1^{(j)}(\nu\tau,R)\right)dR\\&=\int_{\nu\tau-1}^{\nu\tau}\partial_R\left(\chi_{0\leq\nu\tau-R\lesssim1}\right)\tilde{\tilde{f}}_1^{(j)}(\nu\tau,R)\,dR+\int_{\nu\tau-1}^{\nu\tau}\chi_{0\leq\nu\tau-R\lesssim1}\partial_R\tilde{\tilde{f}}_1(\tau,R)\,dR.\end{aligned}$$

This combined with Proposition 6.7 gives the desired estimate without the growth in τ.

Then the product of the pure polynomial of the singular part is given by

$$\begin{aligned}&\Big(P_3\tilde{\tilde{f}}_1(\tau,R)\Big)\cdot\chi_{\nu\tau-R<\hbar_1}\tilde{f}_2(\tau,R)\\&=\chi_{|\nu\tau-R|\lesssim\hbar_1}\sum_{j=0}^{3}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\frac{\tilde{G}_{k,l,j,i}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar_1^{-\frac{l+2j+1}{2}}\,[\nu\tau-R]^{\frac{l+2j}{2}+k\nu}\,(\log(\nu\tau-R))^i\,,\end{aligned}$$

where we set

$$\tilde{G}_{k,l,j,i}(\tau,\nu\tau-R)=\hbar_1^{\frac{l-1}{2}}\cdot\hbar_2^{-\frac{1+l}{2}}\cdot\tilde{\tilde{f}}_1^{(j)}(\tau,\nu\tau)\cdot G_{k,l,i}(\tau,\nu\tau-R)$$

and which is seen to satisfy the same bounds as the coefficients in $f_{2(l+2j)}$ in Lemma 7.10 up to a factor $\hbar_2^{-C}$. Next we consider the product

$$\tilde{\tilde{g}}_1\cdot\chi_{\nu\tau-R<\hbar_1}\tilde{f}_2$$

We claim that this function can be placed into $\tilde{S}_0^{(\hbar_1)}$, with time derivative in $\tilde{S}_1^{(\hbar_1)}$. To see this, we use Taylor's theorem to write

$$\tilde{\tilde{g}}_1(\tau,R)=\int_0^1\tilde{\tilde{f}}_1^{(4)}(\tau,\nu\tau+s(R-\nu\tau))\cdot(\nu\tau-R)^4\,ds,$$

Now we assume that of of $\overline{y}_j$, say $\overline{y}_1$, is of principal ingoing part. Then if $\overline{y}_2$ is also of principal ingoing type, write

$$\tilde{f}_j(\tau,R)=\chi_{\nu\tau-R\geq\hbar_j}\tilde{f}_j(\tau,R)+\chi_{\nu\tau-R<\hbar_j}\Big(\tilde{f}_j(\tau,R)-c_j(\tau)\Big)+\chi_{\nu\tau-R<\hbar_j}c_j(\tau),\quad c_j(\tau)=\tilde{f}_j(\tau,\nu\tau),\quad j=1,2,$$

where we let

$$\tilde{f}_j(\tau,R)=\int_0^\infty\phi_{n_j}(R,\xi)\cdot\overline{y}_j(\tau,\xi)\cdot\rho_{n_j}(\xi)\,d\xi.$$

Here $\chi_{\nu\tau-R\geq\hbar_j}\tilde{f}_j(\tau,R)\in\tilde{S}_0^{(\hbar_j)}$, and furthermore the preceding argument implies

$$\chi_{\nu\tau-R\geq\hbar_j}\tilde{f}_j(\tau,R)\cdot\chi_{\nu\tau-R<\hbar_k}\tilde{f}_k(\tau,R)\in\tilde{S}_0^{(\hbar_3)},\quad\{j,k\}=\{1,2\}.$$

It follows that to complete analysis of the product $\prod_{j=1,2}\tilde{f}_j(\tau,R)$, it suffices to consider

$$\begin{aligned}&\prod_{j=1,2}\Big[\chi_{\nu\tau-R<\hbar_j}\Big(\tilde{f}_j(\tau,R)-c_j(\tau)\Big)+\chi_{\nu\tau-R<\hbar_j}c_j(\tau)\Big]\\&=\prod_{j=1,2}\chi_{\nu\tau-R<\hbar_j}\Big(\tilde{f}_j(\tau,R)-c_j(\tau)\Big)\\&+\sum_{\{j,k\}=\{1,2\}}\chi_{\nu\tau-R<\hbar_j}\Big(\tilde{f}_j(\tau,R)-c_j(\tau)\Big)\cdot\chi_{\nu\tau-R<\hbar_k}c_k(\tau)\\&+\prod_{j=1,2}\chi_{\nu\tau-R<\hbar_j}c_j(\tau)\end{aligned}$$

Using Lemma 7.10 and Lemma 7.7, the first term on the right is seen to be of connecting singular type or smoother. For the second term on the right, considering the case $j=1,k=2$, say, we have

$$\chi_{\nu\tau-R<\hbar_1}\Big(\tilde{f}_1(\tau,R)-c_1(\tau)\Big)\cdot\chi_{\nu\tau-R<\hbar_2}c_2(\tau)$$

$$
\begin{aligned}
&= \chi_{\nu\tau-R<\hbar_1}\tilde{f}_1(\tau,R)\cdot\chi_{\nu\tau-R<\hbar_2}c_2(\tau) - \chi_{\nu\tau-R<\hbar_1}c_1(\tau)\cdot\chi_{\nu\tau-R<\hbar_2}c_2(\tau)\\
&= \tilde{f}_1(\tau,R)\cdot c_2(\tau)\\
&- \chi_{\nu\tau-R\geq\hbar_1}\tilde{f}_1(\tau,R)\cdot c_2(\tau)\\
&- \chi_{\nu\tau-R<\hbar_1}\tilde{f}_1(\tau,R)\cdot\chi_{\nu\tau-R\geq\hbar_2}c_2(\tau)\\
&- \chi_{\nu\tau-R<\hbar_1}c_1(\tau)\cdot\chi_{\nu\tau-R<\hbar_2}c_2(\tau).
\end{aligned}
$$

Here the last three terms are easily seen to be in $\tilde{S}_0^{(\hbar_3)}$ (Note that the third term on the RHS above actually vanishes, since $\hbar_2 \gg \hbar_1$), while for the first of the last four terms, this is clearly a function which is of principal ingoing type *when interpreted as angular momentum n_1 function*. However, this function needs to be interpreted as an angular momentum n_3 function, for which we need a 'translation device'. To begin with, using Lemma 7.10, we easily conclude that

$$\chi_{R<\frac{\nu\tau}{2}}\tilde{f}_1(\tau,R)\cdot c_2(\tau)\in\tilde{S}_0^{(\hbar_3)}.$$

It thus suffices to understand

$$\left\langle \phi_{n_3}(R,\xi),\, \chi_{R\geq\frac{\nu\tau}{2}}\tilde{f}_1(\tau,R)\cdot c_2(\tau)\right\rangle_{L^2_{R\,dR}}$$

For this we use the following lemma:

Lemma 7.21. *Let $f(R)$ be an angular momentum n_1 function, $|n_1|\geq 2$, let $|n_2|\geq 2$, and setting*

$$f(R)=\int_0^\infty \phi_{n_1}(R,\xi)\overline{x}(\xi)\rho_{n_1}(\xi)\,d\xi,$$

let the function $\mathcal{K}_\tau^{n_1,n_2}\overline{x}$ be defined by the relation

$$(\mathcal{K}_\tau^{n_1,n_2}\overline{x})(\eta) := \left\langle \phi_{n_2}(R,\eta),\,\chi_{R\geq\frac{\nu\tau}{2}}f(R)\right\rangle_{L^2_{R\,dR}}.$$

Then we have in analogy to the transference operator the distributional identity

$$\mathcal{K}_\tau^{n_1,n_2}(\xi,\eta)=\frac{\overline{a_{n_2}(\eta)}}{\overline{a_{n_1}(\xi)}}\delta(\xi-\eta)+\tilde{\mathcal{K}}_\tau^{n_1,n_2}(\xi,\eta),$$

where the operator $\tilde{\mathcal{K}}_\tau^{n_1,n_2}$ acts via integration against a kernel $\frac{F(\xi,\eta;\tau,n_1,n_2)\rho_{n_1}(\eta)}{\xi-\eta}$

$$\left(\tilde{\mathcal{K}}_\tau^{n_1,n_2}f\right)(\eta)=\int_0^\infty\frac{F(\xi,\eta;\tau,n_1,n_2)\rho_{n_1}(\xi)}{\xi-\eta}f(\xi)\,d\xi.$$

The kernel function $F(\xi,\eta;\tau,n_1,n_2)$ can be decomposed as

$$F(\xi,\eta;\tau,n_1,n_2)=F_P(\xi,\eta;\tau,n_1,n_2)+F_N(\xi,\eta;\tau,n_1,n_2)$$

where $F_P(\xi,\eta;\tau,n_1,n_2)$ satisfies (assuming $\xi\leq\eta$) for $k\leq k_0(n_1)$,

$$|F_P(\xi,\eta;\tau,n_1,n_2)|\lesssim P(|n_1-n_2|)\left(\hbar_1\xi^{\frac{1}{2}}\right)^{-1}\min\left\{1,\left(\hbar_1\xi^{\frac{1}{2}}\right)\right\}\cdot G, \tag{7.12}$$

with

$$G := \begin{cases} \min\left\{1, \left(\hbar_1 \xi^{\frac{1}{2}}\right)^{\frac{1}{4}} \left|\eta^{\frac{1}{2}} - \xi^{\frac{1}{2}}\right|^{-\frac{1}{4}}\right\}, & \text{for} \quad \left|\frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}} - 1\right| \lesssim 1, \\ \hbar_1 \left(\hbar_1 \xi^{\frac{1}{2}}\right)^{-1} \min\left\{1, \hbar_1 \xi^{\frac{1}{2}}\right\} \cdot \left(\frac{\xi^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}\right)^k \cdot \left(\frac{\langle \xi \rangle}{\eta}\right)^N, & \text{for} \quad \left|\frac{\eta^{\frac{1}{2}}}{\xi^{\frac{1}{2}}} - 1\right| \gg 1, \end{cases} \tag{7.13}$$

for arbitrary $N > 0$ and $\eta \geq 1$. Here $P(\cdot)$ is a polynomial. For the derivatives of $F_P(\xi, \eta; \tau, n_1, n_2)$, we have

$$\begin{aligned} &\left|\partial_{\xi^{\frac{1}{2}}} F_P(\xi, \eta; \tau, n_1, n_2)\right|, \quad \left|\partial_{\eta^{\frac{1}{2}}} F_P(\xi, \eta; \tau, n_1, n_2)\right| \\ \lesssim &\frac{P(|n_1 - n_2|)}{\left(\hbar_1 \xi^{\frac{1}{2}}\right)^{\frac{1}{2}} \left(\hbar_2 \eta^{\frac{1}{2}}\right)^{\frac{1}{2}}} \left(1 + \left|\log\left(\hbar_1 \xi^{\frac{1}{2}} \left|\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}\right|^{-1}\right)\right|\right) \quad \text{if} \quad \xi \simeq \eta, \end{aligned} \tag{7.14}$$

The operator corresponding to $F_N(\xi, \eta; n_1, n_2, \tau)$ has a stronger smoothing effect, which maps an admissibly singular function to a smooth function.

Proof. The proof is similar to that of Proposition 5.1, and here we only give an outline. According to its definition, the operator $(\mathcal{K}_\tau^{n_1,n_2} \overline{x})(\eta)$ is given by

$$(\mathcal{K}_\tau^{n_1,n_2} \overline{x})(\eta) = \int_0^\infty \left\langle \chi_{R \geq \frac{\nu\tau}{2}} \phi_{n_1}(R, \xi), \phi_{n_2}(R, \eta) \right\rangle_{L^2_{R\,dR}} \cdot \rho_{n_1}(\xi) f(\xi)\, d\xi.$$

In view of (5.14), we infer that the δ measure on the diagonal in the integral

$$\lim_{A \to \infty} \int_0^A \chi_{R \geq \frac{\nu\tau}{2}} \phi_{n_1}(R, \xi) \phi_{n_2}(R, \eta) R\, dR$$

comes from the expression

$$\begin{aligned} &2\pi^{-1} (\xi\eta)^{-\frac{1}{4}} \lim_{L \to \infty} \operatorname{Re} \int_0^\infty e^{i\left(\hbar_1^{-1} \Psi(R, \xi, \hbar_1) - \hbar_2^{-1} \Psi(R, \eta, \hbar_2)\right)} a_{n_1}(\xi) \left(1 + \hbar a_1(-\tau, \alpha)\right) \\ &\cdot \overline{a_{n_2}(\eta)} \left(1 + \hbar \overline{a_1(-\sigma; \beta)}\right) \chi_{R \geq \frac{\nu\tau}{2}} \chi_2(R/L)\, dR. \end{aligned}$$

Here

$$\begin{aligned} \alpha &:= \hbar_1 \xi^{\frac{1}{2}}, \quad \Psi(R, \xi, \hbar_1) := \hbar_1 \xi^{\frac{1}{2}} R - y(\alpha, \hbar_1) + \rho(\hbar_1 \xi^{\frac{1}{2}} R, \alpha, \hbar_1) \\ \beta &:= \hbar_2 \eta^{\frac{1}{2}}, \quad \Psi(R, \eta, \hbar_2) := \hbar_2 \eta^{\frac{1}{2}} R - y(\beta, \hbar_2) + \rho(\hbar_2 \eta^{\frac{1}{2}} R, \alpha, \hbar_2), \end{aligned}$$

and χ_2 is defined the same as in (5.16). The result for the diagonal part follows in a similar way as in the proof for Proposition 5.1.

Now we turn to the off-diagonal kernel $\tilde{\mathcal{K}}_\tau^{n_1,n_2}(\xi, \eta)$. A routing calculation similar to the one in Proposition 5.1, we have

$$\begin{aligned} \eta \left(\mathcal{K}_\tau^{n_1,n_2} f\right)(\eta) - \mathcal{K}_\tau^{n_1,n_2} \left(\xi f(\xi)\right)(\eta) = &- \int_0^\infty \left\langle \chi_{R \geq \frac{\nu\tau}{2}} \phi_{n_1}(R, \xi), H_{n_2}^+ \phi_{n_2}(R, \eta) \right\rangle_{L^2_{R\,dR}} \rho_{n_1}(\xi) f(\xi)\, d\xi \\ &+ \int_0^\infty \left\langle \chi_{R \geq \frac{\nu\tau}{2}} \phi_{n_1}(R, \xi), H_{n_1}^+ \phi_{n_2}(R, \eta) \right\rangle_{L^2_{R\,dR}} \rho_{n_1}(\xi) f(\xi)\, d\xi \\ &+ \int_0^\infty \left\langle \left(\partial_R^2 + R^{-1} \partial_R\right) \chi_{R \geq \frac{\nu\tau}{2}} \cdot \phi_{n_2}(R, \eta), \phi_{n_1}(R, \xi) \right\rangle_{L^2_{R\,dR}} \rho_{n_1}(\xi) f(\xi)\, d\xi \end{aligned}$$

$$+\int_0^\infty \left\langle \partial_R\chi_{R\geq\frac{\nu\tau}{2}}\partial_R\phi_{n_2}(R,\eta),\phi_{n_1}(R,\xi)\right\rangle_{L^2_{RdR}}\rho_{n_1}(\xi)f(\xi)\,d\xi.$$

Therefore the function $F(\xi,\eta;\tau,n_1,n_2)$ is given by

$$\begin{aligned}F(\xi,\eta;\tau,n_1,n_2) =& \left\langle\left(H^+_{n_1}-H^+_{n_2}\right)\phi_{n_2}(R,\eta),\chi_{R\geq\frac{\nu\tau}{2}}\phi_{n_1}(R,\xi)\right\rangle_{L^2_{RdR}}\\ &+\left\langle\left(\partial_R^2+R^{-1}\partial_R\right)\chi_{R\geq\frac{\nu\tau}{2}}\cdot\phi_{n_2}(R,\eta),\phi_{n_1}(R,\xi)\right\rangle_{L^2_{RdR}}\\ &+2\left\langle\partial_R\chi_{R\geq\frac{\nu\tau}{2}}\cdot\partial_R\phi_{n_2}(R,\eta),\phi_{n_1}(R,\xi)\right\rangle_{L^2_{RdR}}.\end{aligned}\tag{7.15}$$

We denote the first term on the RHS of (7.15) by $F_P(\xi,\eta;n_1,n_2,\tau)$ and the rest three terms by $F_N(\xi,\eta;n_1,n_2,\tau)$. Note that $F_P(\xi,\eta;n_1,n_2,\tau)$ has a similar structure as the function $F(\xi,\eta;\hbar)$ in Proposition 5.1, except that 1) the potential $H^+_{n_1}-H^+_{n_2}$ decays as $\langle R\rangle^{-2}$ only, 2) the coefficient of the potential is $n_1^2-n_2^2=(n_1-n_2)(n_1+n_2)$, and 3) the Fourier basis $\phi_{n_1}(R,\xi)$ and $\phi_{n_2}(R,\eta)$ are for different angular modes. If $\hbar_1\leq\hbar_2$ (the vice-versa being similar), then

$$\hbar_2=\hbar_1\cdot\frac{\hbar_2}{\hbar_1}=\hbar_1\cdot\frac{n_1+1}{n_2+1}=\hbar_1\cdot\left(1+\frac{n_1-n_2}{n_2+1}\right).$$

Therefore the estimates on $F_P(\xi,\eta;n_1,n_2,\tau)$ follows in a similar way as in Proposition 5.1. □

Using the preceding lemma, we infer that

$$\left\langle\phi_{n_3}(R,\xi),\chi_{R\geq\frac{\nu\tau}{2}}\tilde{f}_1(\tau,R)\cdot c_2(\tau)\right\rangle_{L^2_{RdR}}=c_2(\tau)\cdot\overline{y}_1(\tau)+\tilde{\overline{y}}_1(\tau,\xi),$$

where $\tilde{\overline{y}}_1(\tau,\xi)$ is of connecting singular type, and hence more regular. This completes the outline of the proof of the proposition. □

Applying the preceding proposition inductively, we can then also handle more general products:

Corollary 7.22. *Let ϕ_j, $j=1,2,\dots,k$, $k\geq 2$, be angular momentum n_j, $|n_j|\geq 2$ functions with admissibly singular distorted Fourier transform (in the angular momentum n_j-sense). Let the angular momenta m_l, $l=1,\dots,k-1$, be determined such that $|m_l|\geq 2$ and $\{m_2,n_1,n_2\}$ satisfy the conditions in of Prop. 6.14 (instead of satisfying the conditions $\{n_3,n_1,n_2\}$ in that proposition), and similarly for the triples $\{m_r,n_r,m_{r-1}\}$. Then the product*

$$\prod_{j=1}^k\phi_j$$

may be represented as an angular momentum m_{k-1}-function whose Fourier transform is a linear combination of an admissible function and a function in $S_0^{\tilde{\hbar}_{k-1}}$, where $\tilde{\hbar}_l:=\frac{1}{m_l+1}$.

7.5.2. *Fourier localization on admissibly singular functions.* One nice feature of our concept of admissibly singular functions is that except for the principal ingoing singular part, they are compatible with the application of Fourier localization operators

$$\chi_{\xi>\mu},\quad\chi_{\xi<\mu},\quad\chi_{\xi\simeq\mu}.$$

In fact, if $F^{(\pm)}_{l,k,i}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)$ is as in Definition 7.8, then the function

$$\chi_{\xi>\mu}F^{(\pm)}_{l,k,i}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) = \chi_{\frac{\lambda^2(\sigma)}{\lambda^2(\tau)}\cdot\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)>\mu}F^{(\pm)}_{l,k,i}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)$$
$$=: \tilde{F}^{(\pm)}_{l,k,i}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)$$

is easily seen to satisfy the same estimates as the function $F^{(\pm)}_{l,k,i}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)$, and similarly for the other Fourier cutoffs. Moreover, these cutoffs act boundedly on $S^{\hbar}_0, S^{\hbar}_1$ for trivial reasons.
In the sequel, if f is an angular momentum $n, |n| \geq 2$ function represented by

$$f(R) = \int_0^\infty \phi_n(R,\xi)x(\xi)\rho_n(\xi)\,d\xi,$$

then we define the Fourier localization operators $P_{<\mu}, P_\mu, P_{>\mu}$ by means of

$$P_{>\mu}f(R) = \int_0^\infty \phi_n(R,\xi)\chi_{\xi>\mu}x(\xi)\rho_n(\xi)\,d\xi,$$

and similarly for the other operators. Thus we commit abuse of notion here, in that these operators tacitly depend on the angular momentum n. In the sequel, it will always be understood what the underlying angular momentum is. However, the principal ingoing part will be modified into inadmissible form by application of a Fourier cutoff; nonetheless, we shall be able to show that certain paraproducts of admissibly singular inputs remain admissibly singular (up to smoother errors).

7.5.3. *Paraproduct estimates for angular momentum $|n| \geq 2$ functions with admissibly singular distorted Fourier transform.* Recall that expressions of the form $\sum_\mu P_{<\mu}fP_\mu g$ where μ ranges over dyadic frequencies. Here we show that this concept is compatible with our concept of admissible singularity:

Proposition 7.23. *Let n_j, $j = 1,2,3$ obey the same conditions as in Prop. 6.14 . Assume that the functions $f_j(\tau,R)$, $j = 1,2$ are angular momentum $n_j, |n_j| \geq 2$ functions admitting representations*

$$f_j(\tau,R) = \int_0^\infty \phi_{n_j}(R,\xi)\cdot\overline{x}_j(\tau,\xi)\cdot\rho_{n_j}(\xi)\,d\xi, \quad j = 1,2,$$

where the distorted Fourier transforms $\overline{x}_j$, $j = 1,2$ each can be written as $\overline{x}_j = \overline{y}_j + \overline{z}_j$ with $\overline{y}_j$ admissibly singular (at angular momentum n_j) and $\overline{z}_j \in S^{\hbar_j}_0$. Then we have (with $\mu = 2^N$ ranging over dyadic numbers)

$$\left(\sum_{1<\mu} P_{<\mu}f_1\cdot P_\mu f_2\right)|_{R<\nu\tau} = g(\tau,R),$$

where g admits an angular momentum n_3 representation

$$g(\tau,R) = \int_0^\infty \phi_{n_3}(R,\xi)\overline{x}_3(\tau,\xi)\rho_{n_3}(\xi)\,d\xi,$$

where $\overline{x}_3 = \overline{y}_3 + \tilde{\overline{y}}_3 + \overline{z}_3$, and where $\overline{y}_3$ is admissibly singular of principal ingoing type, $\tilde{\overline{y}}_3$ is of prototypical singular type with vanishing principal part, and $\overline{z}_3 \in S^{\hbar}_0$.

Proof. □

7.6. **Spaces for the source terms in the nonlinearity.** In analogy to the 'good spaces' $S_1^{\hbar}$ which are used to bound the smooth source terms, we shall introduce admissibly singular source terms, mimicking Definition 7.8:

Definition 7.24. *Let $F(\tau, R)$ be an angular momentum $n, |n| \geq 2$ function. Then we say that F is an admissibly singular source term, provided its distorted Fourier transform $\overline{y}(\tau, \xi)$ (at angular momentum n) has the property that*

$$\tau \cdot \xi^{-\frac{1}{2}} \cdot \overline{y}(\tau, \xi)$$

is admissibly singular in the sense of Definition 7.8, except all conditions involving ∂_τ are suppressed, and we have the slightly stronger bound (for $\tau \cdot \xi^{-\frac{1}{2}} \cdot \overline{y}$)

$$\left|a_{k,i}^{(\pm)}(\tau, \sigma)\right| \lesssim (\log \tau)^{N_1 - i}\, \tau^{-1-\nu-\delta} \cdot \sigma^{-3}, \quad 0 \leq k_1 \leq 6$$

for some $\delta > 0$, and in all subsequent estimates in Definition 7.8 there is either an extra gain $\tau^{-\delta}$ or an extra factor $\kappa\left(\hbar \frac{\lambda(\sigma)}{\lambda(\tau)} \xi^{\frac{1}{2}}\right)$, i. e. in the latter case

$$\left|\xi^{k_2} \partial_\xi^{k_2} F_{l,k,i}^{(\pm)}(\tau, \sigma, \xi)\right| \lesssim \kappa\left(\hbar \frac{\lambda(\sigma)}{\lambda(\tau)} \xi^{\frac{1}{2}}\right) \cdot (\log \tau)^{N_1 - i}\, \tau^{-1-\nu} \cdot \sigma^{-1} \cdot \left[\sigma^{-2} + \kappa(\hbar \xi^{\frac{1}{2}})\right], \quad 0 \leq k_2 \leq 5,$$

In particular, we have natural concepts of principally ingoing singular part and principal singular part of restricted type, for the admissibly singular source terms as well. Writing

$$\overline{y}(\tau, \xi) = \tau^{-1} \xi^{\frac{1}{2}} \cdot \overline{z}(\tau, \xi)$$

with $\overline{z}(\tau, \xi)$ admissibly singular in the sense of Definition 7.8, if the principal ingoing part of $\overline{z}$ vanishes and the sums $\sum_l \ldots$ occurring in the connecting incoming and outgoing parts are restricted to $l \geq l_1 \geq 1$, we say that $\overline{y}$ is an admissibly singular source term of level l_1.

7.7. **Null-form estimates near the light cone; only angular momenta $|n| \geq 2$ involved.** We shall now deal with the most delicate type of source term due to the derivatives it contains, which we cast here as a trilinear expression

$$\mathcal{N}_0(\phi_1, \phi_2, \phi_3) := \phi_1 \cdot \left[\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R\right) \phi_2 \cdot \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R\right) \phi_3 - \partial_R \phi_2 \cdot \partial_R \phi_3\right]$$

As hinted at in the title of this subsection, we shall assume for now that all factors as well as the expression itself will be at angular momentum $|n| \geq 2$, i. e. we shall use Fourier representations with respect to such angular momenta. This will simplify the presentation a bit, and the extension to general factors will be rather routine. Our way to deal with this expression will be to take advantage of Fourier localizations, as set up in the preceding subsection. We shall face one particular technical difficulty, which is intimately tied to our functional setup, and which will require a bit of detour to handle. To understand it, let us write the above term in terms of the 'good' derivative $\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R - \partial_R$ as well as the 'bad' derivative $\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R + \partial_R$:

$$\begin{aligned} \mathcal{N}_0(\phi_1, \phi_2, \phi_3) = {} & \phi_1 \cdot \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R - \partial_R\right) \phi_2 \cdot \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R + \partial_R\right) \phi_3 \\ & + \phi_1 \cdot \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R + \partial_R\right) \phi_2 \cdot \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R - \partial_R\right) \phi_3 \end{aligned}$$

Then the strategy is to show that if the factors have Fourier transform which is either admissibly singular or in $S_0^{\hbar}$(with $\hbar$ in accordance with their angular momentum), then the source term, upon application of the wave parametrix, leads again to such terms. This works for the most of the terms, except in the situation where a 'bad' derivative hits a singular term (i. e. with Fourier transform admissibly singular), while the 'good' derivative hits a term in $S_0^{\hbar}$. The problem is that these terms are still too rough to lead to a term in $S_1^{\hbar}$, but they also lack the precise fine structure that characterizes our admissibly singular terms(or more precisely, their distorted Fourier transform).
The trick to deal with this problem is to 'prepare' the wave equation we are trying to solve a bit, by modifying unknown variable (for us $\varepsilon_{\pm}(n)$) by subtracting a suitable correction term(whose distorted Fourier transform happens to be a linear combination of an admissibly singular and a $S_0^{(\hbar)}$-term) from it which modifies the equation in such a way that the 'troublesome' terms disappear, up to other troublesome terms of similar structure but which are smaller. Then the same method can be re-iterated to the remaining trouble some terms, until they eventually disappear. What makes this method work is the very special structure of the $\mathcal{N}_0$ null-structure. To begin with, we 'micro-localize' the expression as follows:

$$\begin{aligned}\mathcal{N}_0(\phi_1,\phi_2,\phi_3) &= \sum_{\lambda_2,\lambda_3}\mathcal{N}_0\left(P_{<\min\{\lambda_2,\lambda_3\}}\phi_1, P_{\lambda_2}\phi_2, P_{\lambda_3}\phi_3\right)\\ &\quad+\sum_{\lambda_2,\lambda_3}\mathcal{N}_0\left(P_{\geq\min\{\lambda_2,\lambda_3\}}\phi_1, P_{\lambda_2}\phi_2, P_{\lambda_3}\phi_3\right)\\ &=: \mathcal{N}_{01}(\phi_1,\phi_2,\phi_3)+\mathcal{N}_{02}(\phi_1,\phi_2,\phi_3).\end{aligned} \tag{7.16}$$

Of these it turns out that the second term $\mathcal{N}_{02}(\phi_1,\phi_2,\phi_3)$ is of the 'good kind', as exemplified by the following

Proposition 7.25. *Let $|n_j| \geq 2$, $j = 1,2,3$, and assume that $\{m_2, m_1\}$ satisfy $|m_j| \geq 2$ and the triples $\{m_2, m_1, n_1\}$, $\{m_1, n_2, n_3\}$ satisfy the conditions in of Prop. 6.14 (instead of $\{n_3, n_1, n_2\}$ there). Then the function*

$$\chi_{R\gtrsim\tau}\mathcal{N}_{02}(\phi_1,\phi_2,\phi_3)|_{R<\nu\tau} = g(\tau,R)|_{R<\nu\tau}$$

where $g(\tau,R)$ admits a angular momentum m_2 distorted Fourier representation

$$g(\tau,R) = \int_0^\infty \phi_{m_2}(R,\xi)\overline{y}(\tau,\xi)\rho_{m_2}(\xi)\,d\xi,$$

where $\overline{y} = \overline{y}_1 + \overline{y}_2$ and $\overline{y}_1$ is an admissibly singular source term at angular momentum m_2, while $\overline{y}_2 \in S_1^{\hbar_2}$, $\hbar_2 = \frac{1}{m_2+1}$.

Proof. For simplicity of notation, we shall indicate frequency localizations by subscripts, thus $P_\lambda\phi = \phi_\lambda$ etc. Expand the term out as

$$\begin{aligned}\mathcal{N}_{02}(\phi_1,\phi_2,\phi_3) &= \sum_{\lambda_3<\lambda_2}\phi_{1,\geq\lambda_3}\cdot\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R-\partial_R\Big)\phi_{2,\lambda_2}\cdot\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R+\partial_R\Big)\phi_{3,\lambda_3}\\ &\quad+\sum_{\lambda_3<\lambda_2}\phi_{1,\geq\lambda_3}\cdot\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R+\partial_R\Big)\phi_{2,\lambda_2}\cdot\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R-\partial_R\Big)\phi_{3,\lambda_3}\\ &\quad+\sum_{\lambda_3\geq\lambda_2}\phi_{1,\geq\lambda_2}\cdot\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R-\partial_R\Big)\phi_{2,\lambda_2}\cdot\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R+\partial_R\Big)\phi_{3,\lambda_3}\end{aligned}$$

$$+\sum_{\lambda_3\geq\lambda_2}\phi_{1,\geq\lambda_2}\cdot\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R+\partial_R\Big)\phi_{2,\lambda_2}\cdot\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R-\partial_R\Big)\phi_{3,\lambda_3}.$$

By symmetry it suffices to treat the first two terms on the right. Call them $\mathcal{N}_{02a}, \mathcal{N}_{02b}$ respectively.

The contribution of $\mathcal{N}_{02a}$. Observe that the 'bad' derivative falls on the low frequency term ϕ_{3,λ_3}. We shall derive the desired assertion by means of twofold application of suitable bilinear estimates. To begin with, we need a modification of the pure $S^{\hbar}_{j}$-based bilinear estimate Prop. 6.18:

Lemma 7.26. *Assume that $\phi_j,\ j=1,2$ are angular momentum n_j, $|n_j|\geq 2$ functions, with $\phi_1\in\tilde{S}_0^{(\hbar_1)}$, $\phi_2\in\tilde{S}_1^{(\hbar_2)}$. Let $\{m,n_1,n_2\}$, $|m|\geq 2$, be an admissible triple of angular momenta.Then*

$$\chi_{R<\nu\tau}\sum_{\lambda>0}P_{\geq\lambda}\phi_1\cdot P_\lambda\phi_2\in\langle\xi\rangle^{-\frac12}S_1^{\hbar},\quad \hbar=\frac{1}{m+1}$$

where the sum is over dyadic frequencies, and we have the bound

$$\Big\|\Big\langle\phi_m(R,\xi),\chi_{R<\nu\tau}\sum_{\lambda>0}P_{\geq\lambda}\phi_1\cdot P_\lambda\phi_2\Big\rangle_{L^2_{R\,dR}}\Big\|_{\langle\xi\rangle^{-\frac12}S_1^{\hbar}}\lesssim\tau\cdot\max\{\hbar_1,\hbar_2\}^{-2}\cdot\|\phi_1\|_{\tilde{S}_0^{(\hbar_1)}}\cdot\|\phi_2\|_{\tilde{S}_0^{(\hbar_2)}}.$$

The proof is analogous to the one of Prop. 6.18.

We now state the first main ingredient in the proof of the proposition:

Lemma 7.27. *Let ϕ_1,ϕ_3 be angular momentum $n_j,\ j=1,3$ functions (with $|n_j|\geq 2$) and with the property that their angular momentum n_j distorted Fourier transforms $\overline{y}_j(\tau,\xi)$ can be split into*

$$\overline{y}_j(\tau,\xi)=\tilde{\overline{y}}_j(\tau,\xi)+\tilde{\tilde{\overline{y}}}_j(\tau,\xi),$$

where $\tilde{\overline{y}}_j(\tau,\xi)\in S_0^{\hbar_j}$, $\mathcal{D}_\tau\tilde{\overline{y}}_j(\tau,\xi)\in S_1^{\hbar_j}$, and further $\tilde{\tilde{\overline{y}}}_j$ is admissibly singular. Then we can write

$$\chi_{R\gtrsim\tau}\phi_{1,\geq\lambda_3}\cdot\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R+\partial_R\Big)\phi_{3,\lambda_3}|_{R<\nu\tau}=g(\tau,R)|_{R<\nu\tau}$$

where, if $\{m,n_1,n_3\}$ with $|m|\geq 2$ is an admissible angular momentum triple in the sense of Prop. 6.14, we have an angular momentum m Fourier representation of $g(\tau,R)$

$$g(\tau,R)=\int_0^\infty\phi_m(R,\xi)\overline{z}(\tau,\xi)\rho_m(\xi)\,d\xi,\quad \hbar=\frac{1}{m+1}$$

where $\overline{z}$ is the sum of a function in $\overline{z}_1\in\langle\xi\rangle^{-\frac14-}S_1^{\hbar}$ and a function $\overline{z}_2$ which is admissibly singular at angular momentum m and level $l_1\geq 1$.

Proof. (lemma) We distinguish between different situations, depending on the nature of the factors.

(I): both factors of type $\tilde{\overline{y}}_j$. Here we write

$$\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R+\partial_R\Big)f(\tau,R)=\Big(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\Big)f+\partial_Rf.$$

Then the desired conclusion follows from Prop. 6.18, where the complications concerning the spatial origin $R = 0$ are avoided here since we localize things away from the origin.

(II): first factor of type $\tilde{\tilde{\tilde{y}}}_1$, *second factor of type* $\tilde{\tilde{y}}_3$. We may reduce the first factor to $\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}$, $\hbar_1 = \frac{1}{|n_1|+1}$, since else the assertion follows from the preceding case. Also, again using the preceding case and Lemma 7.10, we may assume that $\phi_{1,\geq\lambda_3}$ is like the middle term f_2 in that lemma. We write $\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R = D^+_\tau$, and decompose

$$D^+_\tau\phi_{3,\lambda_3} = [D^+_\tau\phi_{3,\lambda_3} - P_3\left(D^+_\tau\phi_{3,\lambda_3}\right)] + P_3\left(D^+_\tau\phi_{3,\lambda_3}\right),$$

where $P_3\left(D^+_\tau\phi_{3,\lambda_3}\right)$ denotes the third order Taylor polynomial of $D^+_\tau\phi_{3,\lambda_3}$ centered at $R = \nu\tau$. Then we claim that

$$\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}\cdot P_3\left(D^+_\tau\phi_{3,\lambda_3}\right)|_{R<\nu\tau}$$

coincides with the restriction there of an admissibly singular source term of level $l_1 \geq 2$ up to a function in $\tilde{S}^{(\hbar)}_1$, $\hbar = \frac{1}{|m|+1}$. To see this, assume, say, that $\hbar_3 \ll \hbar_1$. Then write

$$P_3\left(D^+_\tau\phi_{3,\lambda_3}\right) = \sum_{j=0}^{3}\frac{1}{j!}\left(D^+_\tau\phi_{3,\lambda_3}\right)^{(j)}(\tau,\nu\tau)(\nu\tau - R)^j,$$

where we have the coefficient bounds $\left|\left(D^+_\tau\phi_{3,\lambda_3}\right)^{(j)}(\tau,\nu\tau)\right| \lesssim \tau\hbar_3^{-\frac{3}{2}-j}\cdot\left\|D^+_\tau\phi_{3,\lambda_3}\right\|_{\tilde{S}^{(\hbar_3)}_1}$ Further, expand

$$\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3} = \chi_{|\nu\tau-R|\lesssim\hbar_3}\sum_{l=1}^{8}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\frac{G_{k,l,i}(\tau,\nu\tau-R)}{\tau^{\frac{1}{2}}}\hbar_1^{-\frac{l+1}{2}}\,[\nu\tau-R]^{\frac{l}{2}+k\nu}\left(\log(\nu\tau-R)\right)^i$$

Then the desired conclusion follows by multiplying the preceding expressions, reformulating things in the form of f_2 in Lemma 7.10 with respect to angular momentum n_3, and using Lemma 7.7. Observe that the resulting function is actually admissibly regular source of level $l_1 \geq 2$ (since it is one derivative more regular than a minimal regularity source), and this information lets us gain another power $\hbar_3$ to counteract the loss of $\hbar_3^{-\frac{3}{2}}$ to bound the coefficients of the Taylor polynomial. The situation $\hbar_3 \gtrsim \hbar_1$ is handled similarly.

Next, consider the product

$$\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}\cdot[D^+_\tau\phi_{3,\lambda_3} - P_3\left(D^+_\tau\phi_{3,\lambda_3}\right)].$$

We claim that this is in $\tilde{S}^{(\hbar)}_1$. For this let us again assume that $\hbar_3 \ll \hbar_1$, say, which implies the 'output angular momentum' $\hbar \simeq \hbar_3$. Then split the expression into a low and a high-frequency part:

$$\begin{aligned}
&\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}\cdot[D^+_\tau\phi_{3,\lambda_3} - P_3\left(D^+_\tau\phi_{3,\lambda_3}\right)]\\
&= P_{<\lambda_3}\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}\cdot[D^+_\tau\phi_{3,\lambda_3} - P_3\left(D^+_\tau\phi_{3,\lambda_3}\right)]\right)\\
&+ P_{\geq\lambda_3}\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}\cdot[D^+_\tau\phi_{3,\lambda_3} - P_3\left(D^+_\tau\phi_{3,\lambda_3}\right)]\right)\\
&=: A + B.
\end{aligned}$$

To bound A, use

$$\left\|P_{<\lambda_3}\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}\cdot[D^+_\tau\phi_{3,\lambda_3} - P_3\left(D^+_\tau\phi_{3,\lambda_3}\right)]\right)\right\|_{\tilde{S}^{(\hbar)}_1}$$

$$\lesssim \hbar \cdot \left(\sum_{\mu<\lambda_3} \left(\hbar^2\mu\right)^{1-\delta} \cdot \left\langle \hbar^2\mu \right\rangle^{3+2\delta} \cdot \left\| \chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3} \cdot D_\tau^+\phi_{3,\lambda_3} \right\|_{L^2_{R\,dR}}^2 \right)^{\frac{1}{2}}$$

$$+ \hbar \cdot \left(\sum_{\mu<\lambda_3} \left(\hbar^2\mu\right)^{1-\delta} \cdot \left\langle \hbar^2\mu \right\rangle^{3+2\delta} \cdot \left\| \chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3} \cdot P_3\left(D_\tau^+\phi_{3,\lambda_3}\right) \right\|_{L^2_{R\,dR}}^2 \right)^{\frac{1}{2}}$$

where μ ranges over dyadic scales. To bound the first term on the right, use the bound

$$\left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{L^2_{R\,dR}} \lesssim \hbar_3^{-1}\left(\hbar_3^2\lambda_3\right)^{-\frac{1}{2}+\frac{\delta}{2}}\left\langle \hbar_3^2\lambda_3 \right\rangle^{-\frac{3}{2}-\delta} \cdot \left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}}.$$

Further using the bound

$$\left\|\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}\right\|_{L^\infty_{R\,dR}} \lesssim \lambda_3^{-\frac{1}{4}} \cdot \tau^{-\frac{3}{2}-\nu}(\log\tau)^C$$

and taking advantage of Hölder's inequality, we bound the first sum by

$$\hbar \cdot \left(\sum_{\mu<\lambda_3} \left(\hbar^2\mu\right)^{1-\delta} \cdot \left\langle \hbar^2\mu \right\rangle^{3+2\delta} \cdot \left\| \chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3} \cdot D_\tau^+\phi_{3,\lambda_3} \right\|_{L^2_{R\,dR}}^2 \right)^{\frac{1}{2}}$$

$$\lesssim \lambda_3^{-\frac{1}{4}} \cdot \tau^{-\frac{3}{2}-\nu}(\log\tau)^C \cdot \left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}},$$

where we have also taken advantage of the fact that $\hbar \simeq \hbar_3$.

For the other square sum above involving the Taylor polynomial, we use the fact that the function $\phi_{1,\geq\lambda_3}$ decays rapidly beyond scale $\lambda_3^{-\frac{1}{2}}$ in $\nu\tau - R$, as follows by straightforward integration by parts. Thus we may for all intents and purposes restrict to the region $\nu\tau - R \lesssim \lambda_3^{-\frac{1}{2}}$, and there we have the bound

$$\begin{aligned}\left\|\chi_{\nu\tau-R<\lambda_3^{-\frac{1}{2}}} P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)\right\|_{L^2_{R\,dR}} &\lesssim \sum_{j=0}^{3}\left\|\chi_{\nu\tau-R<\lambda_3^{-\frac{1}{2}}}\left(D_\tau^+\phi_{3,\lambda_3}\right)^{(j)}(\tau,\nu\tau)\cdot(\nu\tau-R)^j\right\|_{L^2_{R\,dR}} \\ &\lesssim \lambda_3^{-\frac{1}{4}} \cdot \sum_{j=0}^{3}\tau^{\frac{1}{2}}\left|\left(D_\tau^+\phi_{3,\lambda_3}\right)^{(j)}(\tau,\nu\tau)\right| \cdot \lambda_3^{-\frac{j}{2}} \\ &\lesssim \left(\sum_{j=0}^{3}\lambda_3^{\frac{j}{2}}\cdot\lambda_3^{-\frac{j}{2}}\right)\cdot\hbar_3^{-1}\left(\hbar_3^2\lambda_3\right)^{-\frac{1}{2}+\frac{\delta}{2}}\left\langle\hbar_3^2\lambda_3\right\rangle^{-\frac{3}{2}-\delta}\cdot\left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}}.\end{aligned}$$

Here we have taken into account the factors $R^{-\frac{1}{2}}\xi^{\frac{1}{4}}$ from the Fourier basis.

Combining with the L^∞-bound for $\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}$ from before, we infer the bound for the second square sum

$$\hbar \cdot \left(\sum_{\mu<\lambda_3} \left(\hbar^2\mu\right)^{1-\delta} \cdot \left\langle \hbar^2\mu \right\rangle^{3+2\delta} \cdot \left\| \chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3} \cdot P_3\left(D_\tau^+\phi_{3,\lambda_3}\right) \right\|_{L^2_{R\,dR}}^2 \right)^{\frac{1}{2}}$$

$$\lesssim \lambda_3^{-\frac{1}{4}} \cdot \tau^{-\frac{3}{2}-\nu}(\log\tau)^C \cdot \left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}}.$$

This concludes the bound for A.

To get the desired bound for B, split it into

$$\begin{aligned}
&P_{\geq\lambda_3}\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\lambda_3}\cdot[D_\tau^+\phi_{3,\lambda_3}-P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)]\right)\\
&=\sum_{\mu\geq\lambda_3}P_\mu\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,[\lambda_3,\mu]}\cdot[D_\tau^+\phi_{3,\lambda_3}-P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)]\right)\\
&+\sum_{\mu\geq\lambda_3}P_\mu\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\mu}\cdot[D_\tau^+\phi_{3,\lambda_3}-P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)]\right)\\
&=: B_1+B_2.
\end{aligned}\tag{7.17}$$

To handle the term B_1 we perform integration by parts after a further subdivision to shift derivatives (inherent in the definition of the norm) from the outside to the inner lower frequency factors, as in the proof of Prop. 6.14. Specifically, write

$$\begin{aligned}
&\mathcal{F}(B_1)(\xi)\\
&=\sum_{\hbar^{-2}\geq\mu\geq\lambda_3}\chi_{\xi\simeq\mu}\left\langle\phi_m(R,\xi),\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,[\lambda_3,\mu]}\cdot[D_\tau^+\phi_{3,\lambda_3}-P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)]\right)\right\rangle_{L^2_{R\,dR}}\\
&+\sum_{\mu\geq\max\{\lambda_3,\hbar^{-2}\}}\xi^{-3}\chi_{\xi\simeq\mu}\left\langle\phi_m(R,\xi),H_m^3\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,[\lambda_3,\mu]}\cdot[D_\tau^+\phi_{3,\lambda_3}-P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)]\right)\right\rangle_{L^2_{R\,dR}}
\end{aligned}$$

The first term on the right is estimated by using a point wise bound on the inner product and using Holder's inequality to bound the $S_1^\hbar$-norm of the output. In fact, note that we have

$$\begin{aligned}
&\left\|\sum_{\hbar^{-2}\geq\mu\geq\lambda_3}\chi_{\xi\simeq\mu}\left\langle\phi_m(R,\xi),\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,[\lambda_3,\mu]}\cdot[D_\tau^+\phi_{3,\lambda_3}-P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)]\right)\right\rangle_{L^2_{R\,dR}}\right\|_{S_1^\hbar}\\
&\lesssim\sum_{\hbar^{-2}\geq\mu\geq\lambda_3}\hbar\cdot\left(\hbar^2\mu\right)^{\frac{1}{2}-\frac{\delta}{2}}\cdot\mu^{\frac{1}{2}}\cdot\left\|\langle\phi_m(R,\xi),\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,[\lambda_3,\mu]}\cdot[D_\tau^+\phi_{3,\lambda_3}-P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)]\right)\rangle_{L^2_{R\,dR}}\right\|_{L^\infty_{d\xi}}.
\end{aligned}$$

Then we schematically[5] estimate for $R\simeq\tau$

$$\begin{aligned}
\left|D_\tau^+\phi_{3,\lambda_3}(\tau,R)\right|&=\left|\int_0^\infty\mathcal{F}\left(D_\tau^+\phi_{3,\lambda_3}\right)(\xi)\cdot\phi_{n_3}(R,\xi)\cdot\rho_{n_3}(\xi)\,d\xi\right|\\
&\lesssim\int_0^\infty\left|\mathcal{F}\left(D_\tau^+\phi_{3,\lambda_3}\right)(\xi)\right|\cdot\frac{1}{\xi^{\frac{1}{4}}\tau^{\frac{1}{2}}}\cdot\min\{\tau\xi^{\frac{1}{2}}\hbar_3,1\}\,d\xi\\
&\lesssim\hbar_3^{\frac{1}{2}}\cdot\hbar_3^{-2}\left\|\mathcal{F}\left(D_\tau^+\phi_{3,\lambda_3}\right)\right\|_{S_1^{\hbar_3}},
\end{aligned}$$

In fact the second line on the RHS above can be written as

$$\hbar_3^{\frac{1}{2}}\int_0^\infty\left|\mathcal{F}\left(D_\tau^+\phi_{3,\lambda_3}\right)(\xi)\right|\cdot\frac{1}{\hbar_3^{\frac{1}{2}}\xi^{\frac{1}{4}}\tau^{\frac{1}{2}}}\cdot\min\{\tau\xi^{\frac{1}{2}}\hbar_3,1\}\,d\xi$$

[5]We omit the now routine details required to handle the turning point

$$\lesssim \hbar_3^{\frac{1}{2}} \cdot \hbar_3^{-2+\delta} \int_0^\infty \xi^{-1+\frac{\delta}{2}} \cdot \xi^{\frac{1}{2}} \cdot \left(\hbar_3^2 \xi\right)^{1-\frac{\delta}{2}} \cdot \xi^{-\frac{1}{2}} \left|\mathcal{F}\left(D_\tau^+ \phi_{3,\lambda_3}\right)(\xi)\right| d\xi.$$

Then the desired estimate follows using Hölder inequality and the fact $\xi \leq \hbar_2^{-2}$, which kills the extra power $\hbar_3^\delta$ upon integrating in ξ.

as well as

$$\left|\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}} P_3\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right| \lesssim \hbar_3^{\frac{1}{2}} \cdot \hbar_3^{-2} \left\|\mathcal{F}\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right\|_{S_1^{\hbar_3}},$$

since the factors $(\nu\tau-R)^j$ compensate for the frequency loss $\lambda_3^{\frac{j}{2}}$ from the coefficients in the Taylor polynomial due to the restrictions $\nu\tau - R < \min\{\hbar_1, \hbar_3\} = \hbar_3$ and $\hbar_3 \lesssim \lambda_3^{-\frac{1}{2}}$. Finally we can estimate (for $R \simeq \tau$)

$$\begin{aligned}
&\left\|\left\langle \phi_m(R,\xi), \left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}} \phi_{1,[\lambda_3,\mu]} \cdot \left[D_\tau^+ \phi_{3,\lambda_3} - P_3\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right]\right)\right\rangle_{L^2_{R\,dR}}\right\|_{L^\infty_{d\xi}} \\
&\lesssim \tau\hbar_3 \cdot \|\phi_m(R,\xi)\|_{L^\infty_{R\,dR}} \cdot \left\|\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}} \phi_{1,[\lambda_3,\mu]}\right\|_{L^\infty_{R\,dR}} \cdot \left\|D_\tau^+ \phi_{3,\lambda_3} - P_3\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right\|_{L^\infty_{R\,dR}},
\end{aligned}$$

which in the regime where $\nu\tau\xi^{\frac{1}{2}}\hbar$ is away from the turning point (at output angular momentum m) can be bounded by

$$\tau\hbar_3 \cdot \hbar^{\frac{1}{2}} \cdot \tau^{-\frac{3}{2}-\nu} (\log\tau)^C \cdot \hbar_3^{-\frac{3}{2}} \left\|\mathcal{F}\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right\|_{S_1^{\hbar_3}} \lesssim \tau^{-\frac{1}{2}-\nu} (\log\tau)^C \cdot \left\|\mathcal{F}\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right\|_{S_1^{\hbar_3}},$$

where we have taken advantage of the assumption that $\hbar \simeq \hbar_3$. If $\nu\tau\xi^{\frac{1}{2}}\hbar$ is near the turning point, one a priori only gains $\hbar^{\frac{1}{3}}$ for $|\phi_m(R,\xi)|$, and an extra power $\hbar^{\frac{1}{6}}$ needs to be gained from the shortness of the ξ-integration interval, in analogy to arguments in the proof of Prop. 6.7. In fact, recall from the proof for Proposition 6.7, ξ is confined in an interval is length $\simeq \frac{\hbar^{\frac{2}{3}}}{(R\hbar)^2}$. Then in using Hölder inequality to get the pointwise bound for $D_\tau^+ \phi_{3,\lambda_3}$, we encounter the following (I_R being the interval where ξ lies in)

$$\left(\int_{I_R} \xi^{-1+\delta}\, d\xi\right)^{\frac{1}{2}} \lesssim \hbar_3^{\frac{1}{3}} \cdot \hbar_3^{-\delta},$$

which together with the factor $\hbar_3^{\frac{1}{3}} \cdot \hbar_3^{-2+\delta}$ in front of the ξ-integral gives the desired extra factor $\hbar_3^{\frac{1}{3}}$ (more then needed.

To conclude the estimate for B_1, we still need to handle the case of large output frequencies $\mu \geq \max\{\hbar^{-2}, \lambda_3\}$, i. e. the sum

$$\sum_{\mu \geq \max\{\lambda_3, \hbar^{-2}\}} \xi^{-3} \chi_{\xi\simeq\mu} \left\langle \phi_m(R,\xi), H_m^3 \left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}} \phi_{1,[\lambda_3,\mu]} \cdot \left[D_\tau^+ \phi_{3,\lambda_3} - P_3\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right]\right)\right\rangle_{L^2_{R\,dR}}$$

Using the Leibniz rule, this can be written as linear combination of terms of the form

$$\sum_{\mu \geq \max\{\lambda_3, \hbar^{-2}\}} \xi^{-3} \chi_{\xi\simeq\mu} \left\langle \phi_m(R,\xi), \left(\frac{m^2}{R^2}\right)^l \left(\partial_R^i \chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}} \partial_R^j \phi_{1,[\lambda_3,\mu]} \cdot \partial_R^k \left[D_\tau^+ \phi_{3,\lambda_3} - P_3\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right]\right)\right\rangle_{L^2_{R\,dR}}$$

where $2l + i + j + k = 6$. We consider the extreme cases $2l = 6, j = 6, k = 6$, the other situations being handled similarly.
$2l = 6$. Observe that

$$\left\| \sum_{\mu \geq \max\{\lambda_3, \hbar^{-2}\}} \xi^{-3} \chi_{\xi \simeq \mu} \langle \dots \rangle_{L^2_{R\,dR}} \right\|_{S_1^{\hbar}} \leq \sum_{\mu \geq \max\{\lambda_3, \hbar^{-2}\}} \hbar^7 \cdot \left\| \left(\xi \hbar^2\right)^{-1+\frac{\delta}{2}} \langle \dots \rangle_{L^2_{R\,dR}} \right\|_{L^2_{d\xi}(\xi \simeq \mu)}$$

Using Planhcerel's theorem for the distorted Fourier transform at angular momentum m, we can then bound

$$\begin{aligned} &\hbar^7 \cdot \left\| \left(\xi \hbar^2\right)^{-1+\frac{\delta}{2}} \langle \dots \rangle_{L^2_{R\,dR}} \right\|_{L^2_{d\xi}(\xi \simeq \mu)} \\ &\lesssim \hbar^7 \cdot \left(\mu \hbar^2\right)^{-1+\frac{\delta}{2}} \cdot \left\| \frac{m^6}{R^6} \chi_{\nu\tau - R < \min\{\hbar_1, \hbar_3\}} \phi_{1,[\lambda_3,\mu]} \cdot \left[D_\tau^+ \phi_{3,\lambda_3} - P_3\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right] \right\|_{L^2_{R\,dR}}, \end{aligned}$$

and we bound the last term by using

$$\begin{aligned} \left\| \chi_{\nu\tau - R < \min\{\hbar_1, \hbar_3\}} D_\tau^+ \phi_{3,\lambda_3} \right\|_{L^2_{R\,dR}} &\lesssim \tau^{\frac{1}{2}} \cdot \hbar_3^{-1} \left\| D_\tau^+ \phi_{3,\lambda_3} \right\|_{\tilde{S}_1^{(\hbar_3)}}, \\ \left\| \chi_{\nu\tau - R < \min\{\hbar_1, \hbar_3\}} P_3\left(D_\tau^+ \phi_{3,\lambda_3}\right) \right\|_{L^2_{R\,dR}} &\lesssim \tau^{\frac{1}{2}} \hbar_3^{\frac{1}{2}} \cdot \hbar_3^{\frac{1}{2}} \cdot \hbar_3^{-2} \cdot \left\| D_\tau^+ \phi_{3,\lambda_3} \right\|_{\tilde{S}_1^{(\hbar_3)}}, \end{aligned}$$

where the second bound follows from Holder's inequality and the estimates

$$\left| \left(D_\tau^+ \phi_{3,\lambda_3}\right)^{(j)} (\tau, \nu\tau) \right| \lesssim \hbar_3^{\frac{1}{2}} \cdot \hbar_3^{-2-j} \cdot \left\| D_\tau^+ \phi_{3,\lambda_3} \right\|_{\tilde{S}_1^{(\hbar_3)}}, \quad 0 \leq j \leq 3.$$

Combining the preceding estimates with

$$\left\| \chi_{\nu\tau - R < \min\{\hbar_1, \hbar_3\}} \phi_{1,[\lambda_3,\mu]} \right\|_{L^\infty_{R\,dR}} \lesssim \lambda_3^{-\frac{1}{4}} \cdot \tau^{-\frac{1}{2}} \cdot \tau^{-1-\nu} (\log \tau)^C,$$

and further taking advantage of $\hbar \simeq \hbar_3$ by our assumption as well as $m \simeq \hbar^{-1}$, we infer

$$\hbar^7 \cdot \left\| \left(\xi \hbar^2\right)^{-1+\frac{\delta}{2}} \langle \dots \rangle_{L^2_{R\,dR}} \right\|_{L^2_{d\xi}(\xi \simeq \mu)} \lesssim \tau^{-\frac{13}{2}-\nu} (\log \tau)^C \cdot \left\| \left(D_\tau^+ \phi_{3,\lambda_3}\right) \right\|_{S_1^{\hbar_3}},$$

where we exploit that $R \simeq \tau$ on the support of the expression. Note that in the case $i = 6$, which is formally similar we only gain a factor $\tau^{-\frac{1}{2}-\nu} (\log \tau)^C$.

$j = 6$. Here we exploit the fact that the factor $D_\tau^+ \phi_{3,\lambda_3} - P_3\left(D_\tau^+ \phi_{3,\lambda_3}\right)$ compensates for the singularity[6] of $\partial_R^6 \phi_{1,[\lambda_3,\mu]}$. Precisely, write

$$\left[D_\tau^+ \phi_{3,\lambda_3} - P_3\left(D_\tau^+ \phi_{3,\lambda_3}\right)\right](\tau, R) = C\,(\nu\tau - R)^4 \cdot \int_0^1 \left(D_\tau^+ \phi_{3,\lambda_3}\right)^{(4)} (\tau, \nu\tau - s(\nu\tau - R))\, ds, \tag{7.18}$$

where using Prop. 6.7 as well as Holder's inequality and a simple change of variables we have

$$\left| \int_0^1 \left(D_\tau^+ \phi_{3,\lambda_3}\right)^{(4)} (\tau, \nu\tau - s(\nu\tau - R))\, ds \right| \lesssim \hbar_3^{-5} \cdot \left\| D_\tau^+ \phi_{3,\lambda_3} \right\|_{\tilde{S}_1^{(\hbar_3)}} \cdot (\nu\tau - R)^{-\frac{1}{2}}.$$

[6] Of course this function is actually C^∞ due to the frequency cutoff but we need bounds which are uniform in μ

(Note that here we used the fact $\hbar_3^{-\frac{1}{2}} \lesssim (\nu\tau - R)^{-\frac{1}{2}}$.) Furthermore using Lemma 7.10 we infer the bound ((by re-arranging the powers on $(\nu\tau - R)$ and μ))

$$\left|\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\partial_R^6\phi_{1,[\lambda_3,\mu]}\right| \lesssim \mu^{1-}\tau^{-\frac{3}{2}-\nu}(\log\tau)^C \cdot \chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}} \cdot (\nu\tau - R)^{-\frac{7}{2}+\nu-},$$

whence in total we get the bound

$$\left\|\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\partial_R^6\phi_{1,[\lambda_3,\mu]}\left[D_\tau^+\phi_{3,\lambda_3} - P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)\right]\right\|_{L^2_{RdR}} \lesssim \hbar_3^{\frac{1}{2}-5} \cdot \mu^{1-}\tau^{-1-\nu}(\log\tau)^C \cdot \left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}}.$$

Using Plancherel's theorem for the distorted Fourier transform, we infer

$$\begin{aligned}
&\left\|\sum_{\mu\geq\max\{\lambda_3,\hbar^{-2}\}}\xi^{-3}\chi_{\xi\simeq\mu}\left\langle\phi_m(R,\xi),\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\partial_R^6\phi_{1,[\lambda_3,\mu]}\cdot\left[D_\tau^+\phi_{3,\lambda_3} - P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)\right]\right)\right\rangle_{L^2_{RdR}}\right\|_{S_1^\hbar}\\
&\lesssim \sum_{\mu\geq\max\{\lambda_3,\hbar^{-2}\}}\hbar^{5+\frac{\delta}{2}}\mu^{-(1-)}\cdot\left\|\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\partial_R^6\phi_{1,[\lambda_3,\mu]}\cdot\left[D_\tau^+\phi_{3,\lambda_3} - P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)\right]\right\|_{L^2_{RdR}}\\
&\lesssim \sum_{\mu\geq\max\{\lambda_3,\hbar^{-2}\}}\mu^{-(0+)}\cdot\tau^{-1-\nu}(\log\tau)^C\cdot\left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}}.
\end{aligned}$$

$k = 6$. Here use the estimate

$$\left\|\partial_R^6\left[D_\tau^+\phi_{3,\lambda_3} - P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)\right]\right\|_{L^2_{RdR}} \lesssim \hbar_3^{-5-\delta}\cdot\lambda_3^{1-\frac{\delta}{2}}\cdot\left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}}.$$

Then using the simple point wise bound

$$\left\|\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,[\lambda_3,\mu]}\right\|_{L^\infty_{RdR}} \lesssim \tau^{-\frac{3}{2}-\nu}(\log\tau)^C,$$

we obtain by using the Plancherel's theorem for the distorted Fourier transform

$$\begin{aligned}
&\left\|\sum_{\mu\geq\max\{\lambda_3,\hbar^{-2}\}}\xi^{-3}\chi_{\xi\simeq\mu}\langle\phi_m(R,\xi),\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,[\lambda_3,\mu]}\cdot\partial_R^6\left[D_\tau^+\phi_{3,\lambda_3} - P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)\right]\right)\rangle_{L^2_{RdR}}\right\|_{S_1^\hbar}\\
&\lesssim \sum_{\mu\geq\max\{\lambda_3,\hbar^{-2}\}}\hbar^{5+\delta}\mu^{-(1-\frac{\delta}{2})}\cdot\left\|\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,[\lambda_3,\mu]}\right\|_{L^\infty_{RdR}}\cdot\left\|\partial_R^6\left[D_\tau^+\phi_{3,\lambda_3} - P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)\right]\right\|_{L^2_{RdR}}\\
&\lesssim \tau^{-\frac{3}{2}-\nu}(\log\tau)^C\cdot\left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}},
\end{aligned}$$

again exploiting that $\hbar \simeq \hbar_3$.
This concludes the bound for the term B_1, recalling (7.17).

We next turn to the term B_2 there, where we have to take advantage of the large frequency in the singular term. Precisely, we use that $\phi_{1,\geq\mu}$ decays rapidly beyond scale $\mu^{-\frac{1}{2}}$ with respect to $\nu\tau - R$. If we then again invoke (7.18), we find the bound

$$\left\|\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\mu}\cdot\left[D_\tau^+\phi_{3,\lambda_3} - P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)\right]\right\|_{L^2_{RdR}} \lesssim \hbar_3^{-5-\delta}\mu^{-2-\frac{\delta}{2}-\frac{1}{4}}\cdot\tau^{-1-\nu}(\log\tau)^C\cdot\left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}}.$$

Here following the proof for Lemma 7.10, we integrate by parts to gain the decay in μ^{-1}. In the meantime we also see powers in $(\nu\tau - R)^{-1}$. We can have up to $(\nu\tau - R)^{-4}$, to be compensated by the power $(\nu\tau - R)^4$

from the expression for $D_\tau^+\phi_{3,\lambda_3} - P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)$.

Then using the Plancherel's theorem for the distorted Fourier transform, and the fact that by assumption $\hbar \simeq \hbar_3$, we get

$$\begin{aligned}
&\left\|\sum_{\mu\geq\lambda_3} P_\mu\left(\chi_{\nu\tau-R<\min\{\hbar_1,\hbar_3\}}\phi_{1,\geq\mu}\cdot\left[D_\tau^+\phi_{3,\lambda_3} - P_3\left(D_\tau^+\phi_{3,\lambda_3}\right)\right]\right)\right\|_{\tilde{S}_1^{(\hbar)}} \\
&\lesssim \sum_{\mu\geq\lambda_3}\mu^{-\frac{1}{4}}\cdot\tau^{-1-\nu}(\log\tau)^C\cdot\left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}} \\
&\lesssim \lambda_3^{-\frac{1}{4}}\cdot\tau^{-1-\nu}(\log\tau)^C\cdot\left\|D_\tau^+\phi_{3,\lambda_3}\right\|_{\tilde{S}_1^{(\hbar_3)}}.
\end{aligned}$$

This completes the estimate for B and thereby finally for case (II) under the assumption $\hbar_3 \ll \hbar_1$. The remaining cases $\hbar_3 \simeq \hbar_1$, $\hbar_3 \gg \hbar_1$ are handled analogously.

(III): first factor of type $\tilde{\tilde{y}}_1$, second factor of type $\tilde{\tilde{\tilde{y}}}_3$. It is in this case where the particular paraproduct structure of the expression in the lemma becomes important. Write

$$\begin{aligned}
&\chi_{R\gtrsim\tau}\phi_{1,\geq\lambda_3}\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\right)\phi_{3,\lambda_3} \\
&= \chi_{R\gtrsim\tau}\phi_{1,\geq\lambda_3}(\tau,\nu\tau)\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\right)\phi_{3,\lambda_3} \\
&+ \chi_{R\gtrsim\tau}\left[\phi_{1,\geq\lambda_3}(\tau,R) - \phi_{1,\geq\lambda_3}(\tau,\nu\tau)\right]\cdot\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\right)\phi_{3,\lambda_3},
\end{aligned}$$

where in light of Lemma 7.10 we may assume that ϕ_{3,λ_3} admits the expansion of the middle term in that lemma. Then for the first term on the right, we argue exactly as in the proof of Prop. 7.23, to conclude that it is an admissibly singular source term of level $l_1 \geq 1$. The second term on the right is again in $\tilde{S}_1^{(\hbar)}$ by essentially the same argument as the one used at the end in the preceding case.

(IV): Both factors of type $\tilde{y}_j$, i. e. admissibly singular. This is handled by using Lemma 7.17, Lemma 7.10, and translating things back to the Fourier side via Lemma 7.7. It makes sense that if both factors are of singular type, then it is straightforward to see that their product is of a smoother singular type.

This concludes the proof of the lemma. □

In order to complete the proof of the proposition as far as the contribution of the term $\mathcal{N}_{02a}$ is concerned, we need one more bilinear lemma

Lemma 7.28. *Let $F(\tau,R)$ be an angular momentum $\{m,n_1,n_2\}$ be an admissible triple of momenta all of absolute value ≥ 2, and assume that $F = F_1 + F_2$ is an angular momentum n_1 function which can be written as the sum of an angular momentum n_1 function $F_1 \in \tilde{S}_1^{(\hbar_1)}$, while F_2 is an angular momentum n_1 admissibly singular source function of level $l_1 \geq 1$. Then if ϕ_2 is an angular momentum n_2 function as in Prop. 7.25,* *then*

$$\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\Big)\phi_{2,\lambda_2}\cdot F|_{R<\nu\tau} = g(\tau,R)|_{R<\nu\tau},$$

where (with $\hbar = \frac{1}{1+m}$*)*

$$g(\tau,R) = \int_0^\infty \phi_m(R,\xi)\cdot\overline{z}(\tau,\xi)\rho_m(\xi)\,d\xi$$

with $\overline{z} = \overline{z}_1 + \overline{z}_2$ *and* $\overline{z}_1$ *an admissibly singular source term at level* $l_1 \geq 1$, *while* $\overline{z}_2 \in S_1^{\hbar}$.

Proof. (lemma) This follows again by distinguishing between different cases, as in the proof of the preceding lemma. Note that the 'good derivative' $\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R$ sends admissibly singular functions into functions of the same regularity, albeit at the loss of regularity of some of the coefficient functions in the definition of admissibly singular functions. For this recall the proof of Lemma 7.17. The details are rather similar to the ones in the preceding proof. Note that if ϕ_2 is admissibly singular and $F \in \tilde{S}_1^{(\hbar_1)}$ and $\hbar_2 \ll \hbar_1$, then since $\big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\big)\phi_2$ will be of essentially the same regularity, we can absorb the $\hbar_2^{-\frac{3}{2}}$-loss coming from controlling F since we are allowed to lose a factor $\hbar_2^{-1}$ and we can absorb another factor $\hbar_2^{-1}$ since the resulting expression will be of level $l_1 \geq 2$. □

Combining the preceding two lemmas and summing over $\lambda_3 < \lambda_2$ completes the proof for the contribution of $\mathcal{N}_{02a}$. Observe that a conclusion of the preceding considerations is that $\mathcal{N}_{02a}$ is above minimal regularity (as expressed by the statement concerning the level $l_1 \geq 1$).

The contribution of $\mathcal{N}_{02b}$. This is handled in the same manner, breaking things into a number of sub-steps. To begin with, we re-arrange the terms as follows:

$$\begin{aligned}
&\sum_{\lambda_3<\lambda_2}\phi_{1,\geq\lambda_3}\cdot\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_{2,\lambda_2}\cdot\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\Big)\phi_{3,\lambda_3}\\
&= \Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_2\cdot\Big(\sum_{\lambda_3}\phi_{1,\geq\lambda_3}\cdot\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\Big)\phi_{3,\lambda_3}\Big) \qquad (7.19)\\
&- \sum_{\lambda_3\geq\lambda_2}\phi_{1,\geq\lambda_3}\cdot\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_{2,\lambda_2}\cdot\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\Big)\phi_{3,\lambda_3}
\end{aligned}$$

Here the second term on the right will again have better than minimum regularity (it will be seen to be at least admissibly singular source of level $l_1 \geq 1$), while the first term on the right may be of minimal regularity. First, we need the following lemma analogous to Lemma 7.27:

Lemma 7.29. *Let* ϕ_1, ϕ_3 *be angular momentum* n_j, $j = 1, 3$ *functions (with* $|n_j| \geq 2$*) and with the property that their angular momentum* n_j *distorted Fourier transforms* $\overline{y}_j(\tau,\xi)$ *can be split into*

$$\overline{y}_j(\tau,\xi) = \tilde{\overline{y}}_j(\tau,\xi) + \tilde{\tilde{\overline{y}}}_j(\tau,\xi),$$

where $\tilde{\overline{y}}_j(\tau,\xi) \in S_0^{\hbar_j}$, $\mathcal{D}_\tau\tilde{\overline{y}}_j(\tau,\xi) \in S_1^{\hbar_j}$, *and further* $\tilde{\tilde{\overline{y}}}_j$ *is admissibly singular. Then we can write*

$$\chi_{R\gtrsim\tau}\phi_{1,\geq\lambda_3}\cdot\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R - \partial_R\Big)\phi_{3,\lambda_3}|_{R<\nu\tau} = g(\tau,R)|_{R<\nu\tau}$$

where, if $\{m, n_1, n_3\}$ with $|m| \geq 2$ is an admissible angular momentum triple in the sense of Prop. 6.14, we have an angular momentum m Fourier representation of $g(\tau, R)$

$$g(\tau, R) = \int_0^\infty \phi_m(R,\xi) z(\tau,\xi)\rho_m(\xi)\, d\xi, \quad \hbar = \frac{1}{m+1}$$

where $\overline{z}$ is the sum of a function in $\overline{z}_1 \in \langle\xi\rangle^{-\frac{1}{4}-} S_1^\hbar$ and a function $\overline{z}_2$ which is source admissibly singular at angular momentum m at level $l_1 \geq 2$.

The proof of this proceeds in analogy to the one of Lemma 7.27 and we omit it here.

To use the preceding lemma, the following refined L^∞-type estimate shall be useful:

Lemma 7.30. *Assume that $f(R)$ is an angular momentum $n, |n| \geq 2$ function represented by*

$$f(R) = \int_0^\infty \phi_n(R,\xi)\cdot \overline{x}(\xi)\cdot \rho_n(\xi)\, d\xi.$$

Then we have the point wise bound (with $\hbar = \frac{1}{n+1}$)

$$|f(R)| \lesssim \hbar^{-1}\tau^{\frac{1}{2}} \cdot \|\overline{x}\|_{\langle\xi\rangle^{-\frac{1}{4}-} S_1^\hbar}, \quad R \lesssim \tau.$$

Proof. (outline) We first recall that away from the turning point we have the bound $|\phi_n(R,\xi)| \lesssim \hbar^{\frac{1}{2}}$. Write

$$\int_0^\infty \phi_n(R,\xi)\cdot\overline{x}(\xi)\cdot\rho_n(\xi)\,d\xi = \int_0^\infty \chi_{R\xi^{\frac{1}{2}}\hbar \ll 1}\phi_n(R,\xi)\cdot\overline{x}(\xi)\cdot\rho_n(\xi)\,d\xi + \int_0^\infty \chi_{R\xi^{\frac{1}{2}}\hbar \gtrsim 1}\phi_n(R,\xi)\cdot\overline{x}(\xi)\cdot\rho_n(\xi)\,d\xi.$$

Then we can bound the first integral on the right by

$$\begin{aligned}\left|\int_0^\infty \chi_{R\xi^{\frac{1}{2}}\hbar \ll 1}\phi_n(R,\xi)\cdot\overline{x}(\xi)\cdot\rho_n(\xi)\,d\xi\right| &\lesssim \int_0^1 \chi_{R\xi^{\frac{1}{2}}\hbar \ll 1}\,|\phi_n(R,\xi)|\,|\overline{x}(\xi)|\;d\xi \\ &\quad + \int_1^\infty \chi_{R\xi^{\frac{1}{2}}\hbar \ll 1}\,|\phi_n(R,\xi)|\,|\overline{x}(\xi)|\;d\xi \\ &\lesssim c^{\hbar^{-1}}\left\|\xi^{\frac{1}{2}-\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\xi<1)} + c^{\hbar^{-1}}\left\|\xi^{\frac{1}{2}+\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\xi\geq 1)}\end{aligned}$$

for some positive $c < 1$, and the last two terms are easily seen to be bounded by $\ll \|\overline{x}\|_{S_1^\hbar}$, which is better than what is needed. To control

$$\int_0^\infty \chi_{R\xi^{\frac{1}{2}}\hbar \gtrsim 1}\phi_n(R,\xi)\cdot\overline{x}(\xi)\cdot\rho_n(\xi)\,d\xi,$$

use that $\tau\xi^{\frac{1}{2}}\hbar \gtrsim R\xi^{\frac{1}{2}}\hbar \gtrsim 1$, whence $\left(\xi^{\frac{1}{2}}\hbar\right)^{-\frac{1}{2}} \lesssim \tau^{\frac{1}{2}}$, and so away from the turning point $R\xi^{\frac{1}{2}}\hbar = x_t(\hbar)$, which we excise by the cutoff $\chi_1(\xi, R)$, we find by means of the Cauchy Schwarz inequality

$$\left|\int_0^\infty \chi_1(\xi,R)\chi_{R\xi^{\frac{1}{2}}\hbar \gtrsim 1}\phi_n(R,\xi)\cdot\overline{x}(\xi)\cdot\rho_n(\xi)\,d\xi\right| \lesssim \hbar^{\frac{1}{2}}\left\|\chi_{\tau\xi^{\frac{1}{2}}\hbar\gtrsim 1}\xi^{\frac{1}{2}-\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\xi<1)}$$

$$+ \hbar^{\frac12-\delta}\left\|\chi_{\tau\xi^{\frac12}\hbar\gtrsim1}\xi^{\frac12-\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\hbar^{-2}\geq\xi\geq1)}$$
$$+ \hbar^{\frac12+\delta}\left\|\chi_{\tau\xi^{\frac12}\hbar\gtrsim1}\xi^{\frac12+\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\xi\geq\hbar^{-2})}$$

and furthermore we have

$$\hbar^{\frac12}\left\|\chi_{\tau\xi^{\frac12}\hbar\gtrsim1}\xi^{\frac12-\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\xi<1)} \lesssim \tau^{\frac12}\hbar\left\|\chi_{\tau\xi^{\frac12}\hbar\gtrsim1}\xi^{\frac34-\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\xi<1)}$$
$$\lesssim \tau^{\frac12}\hbar^{-1}\cdot\|\overline{x}\|_{\langle\xi\rangle^{-\frac14-}S_1^{\hbar}},$$

On the other hand for the large frequency contributions

$$\hbar^{\frac12-\delta}\left\|\chi_{\tau\xi^{\frac12}\hbar\gtrsim1}\xi^{\frac12-\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\hbar^{-2}\geq\xi\geq1)} \lesssim \tau^{\frac12}\hbar^{1-\delta}\left\|\chi_{\tau\xi^{\frac12}\hbar\gtrsim1}\xi^{\frac34-\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\hbar^{-2}\geq\xi\geq1)}$$
$$\lesssim \tau^{\frac12}\hbar^{-1}\cdot\|\overline{x}\|_{\langle\xi\rangle^{-\frac14}S_1^{\hbar}},$$

and similarly

$$\hbar^{\frac12+\delta}\left\|\chi_{\tau\xi^{\frac12}\hbar\gtrsim1}\xi^{\frac12+\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\xi\geq\hbar^{-2})} \lesssim \tau^{\frac12}\hbar^{1+\delta}\left\|\chi_{\tau\xi^{\frac12}\hbar\gtrsim1}\xi^{\frac34+\frac{\delta}{2}}\overline{x}(\xi)\right\|_{L^2_{d\xi}(\xi\geq\hbar^{-2})}$$
$$\lesssim \tau^{\frac12}\hbar^{-1}\cdot\|\overline{x}\|_{\langle\xi\rangle^{-\frac14}S_1^{\hbar}},$$

which is as desired. It remains to deal with the turning point, which follows by means of the usual refined asymsptotics of $\phi_n(R,\xi)$ for $x = R\xi^{\frac12}\hbar$ near $x_t(\alpha;\hbar)$. This again can be achieved using the fact that ξ lies in an interval of length $\simeq \frac{\hbar^{\frac23}}{(R\hbar)^2}$. □

To finish the argument for the first of the terms on the right of (7.19), we need the following analogue of Lemma 7.28:

Lemma 7.31. *Let $F(\tau,R)$ be an angular momentum $\{m,n_1,n_2\}$ be an admissible triple of momenta all of absolute value ≥ 2, and assume that $F = F_1 + F_2$ is an angular momentum n_1 function which can be written as the sum of an angular momentum n_1 function F_1 with $F_1 \in \langle\xi\rangle^{-\frac14} S_1^{\hbar_1}$, while F_2 is an angular momentum n_1 admissibly singular source function of level $l_1 \geq 2$. Then if ϕ_2 is an angular momentum n_2 function as in Prop. 7.25, then*

$$\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\right)\phi_2\cdot F|_{R<\nu\tau} = g(\tau,R)|_{R<\nu\tau},$$

where (with $\hbar = \frac{1}{1+m}$)

$$g(\tau,R) = \int_0^\infty \phi_m(R,\xi)\cdot\overline{z}(\tau,\xi)\rho_m(\xi)\,d\xi$$

with $\overline{z} = \overline{z}_1 + \overline{z}_2$ and $\overline{z}_1$ an admissibly singular source term, while $\overline{z}_2 \in S_1^{\hbar}$.

Proof. (sketch) This follows again by considering the various combinations of inputs allowed. The interaction of two $S_1^{\hbar_j}$-functions being routine by now, we assume that ϕ_2 is admissibly singular, and also $F_1 \in \langle\xi\rangle^{-\frac{1}{4}} S_1^{\hbar_1}$. Assuming $n_1 \gg n_2$, say, whence $\hbar = \frac{1}{|m|+1} \simeq \hbar_1$, we split

$$\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_2 \cdot F = \chi_{\nu\tau - R \gtrsim \hbar_1}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_2 \cdot F + \chi_{\nu\tau - R \lesssim \hbar_1}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_2 \cdot F$$

The first term is easy to handle since it is the product of two $S_1^{\hbar_j}$-functions. For the second term, we may assume that $\chi_{\nu\tau - R \lesssim \hbar_1}\big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\big)\phi_2$ is given in accordance with Lemma 7.17 for the 'connecting part' of the singularity, while we have to invoke Lemma 7.18 to describe the contribution of the principal incoming singular part. Now if ϕ_2 is of principal ingoing singular type, we decompose

$$\chi_{\nu\tau - R \lesssim \hbar_1}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_2 \cdot F = \chi_{\nu\tau - R \lesssim \hbar_1}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_2 \cdot F(\tau, \nu\tau) + \chi_{\nu\tau - R \lesssim \hbar_1}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_2 \cdot [F(\tau, R) - F(\tau, \nu\tau)]$$

Here the second term on the right is at least source admissibly singular at level $l_1 \geq 2$, and is handled as in the proof of Lemma 7.27. Observe that the fact that if $F \in \langle\xi\rangle^{-\frac{1}{4}-} S_1^{\hbar_1}$ implies that $F(\tau, R) - F(\tau, \nu\tau) = O_{\hbar_1}\big((\nu\tau - R)^{4+}\big)$, which suffices to then place the product into $\tilde{S}_1^{\hbar}$ by following the same argument as for Lemma 7.27.

As for the first term on the right, in accordance with the conclusion of Lemma 7.18 we may assume (i. e. this representation is valid on $R < \nu\tau$)

$$\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_2(\tau, R) = \int_0^\infty \phi_{n_2}(R, \xi) \cdot \xi^{\frac{1}{2}} \cdot \overline{x}(\tau, \xi)\rho_{n_2}(\xi)\, d\xi + \int_0^\infty \phi_{n_2}(R, \xi) \cdot \overline{y}(\tau, \xi)\tilde{\rho}^{(\hbar_2)}(\xi)\, d\xi,$$

where $\overline{x}$ is principal singular in the sense of Definition 7.24, while $\overline{y}(\tau, \xi)$ is admissibly singular (not in the source sense!) or in $S_0^{\hbar}$. It follows that

$$\chi_{\nu\tau - R \lesssim \hbar_1}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R + \partial_R\Big)\phi_2 \cdot F(\tau, \nu\tau)$$

agrees on the inner light cone $R < \nu\tau$ with the function

$$F(\tau, \nu\tau) \cdot \int_0^\infty \phi_{n_2}(R, \xi)\xi^{\frac{1}{2}} \cdot \overline{x}(\tau, \xi)\rho_{n_2}(\xi)\, d\xi$$

up to errors either of connection singular source type or in $\tilde{S}_1^{\hbar}$. The size control of the term follows by invoking Lemma 7.30 with $R = \nu\tau$. □

Combining Lemma 7.29, Lemma 7.31 easily implies the conclusion of the proposition for the first term on the right in (7.19), while for the second term there, one easily checks that the microlocalization forces it to be of at connecting admissibly singular source type of level $l_1 \geq 1$. This then completes the desired conclusion for $\mathcal{N}_{02b}$, and hence the proposition. □

Returning to (7.16), we have dealt with the second term on the right there, and now need a way to handle the first term, which we write as

$$\sum_{\lambda_2,\lambda_3} \mathcal{N}_0\left(P_{<\min\{\lambda_2,\lambda_3\}}\phi_1, P_{\lambda_2}\phi_2, P_{\lambda_3}\phi_3\right) = \sum_{\lambda_2<\lambda_3} \mathcal{N}_0\left(P_{<\lambda_2}\phi_1, P_{\lambda_2}\phi_2, P_{\lambda_3}\phi_3\right) + \sum_{\lambda_2\geq\lambda_3} \mathcal{N}_0\left(P_{<\lambda_3}\phi_1, P_{\lambda_2}\phi_2, P_{\lambda_3}\phi_3\right)$$

Introduce the wave type operator

$$\Box_n' := \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 - \partial_R^2 + \frac{n^2}{R^2}.$$

Then we can write the preceding terms in the following manner:

$$\begin{aligned}
\chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3} \mathcal{N}_0\left(P_{<\lambda_2}\phi_1, P_{\lambda_2}\phi_2, P_{\lambda_3}\phi_3\right) &= \Box_n'\left(\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2<\lambda_3}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\phi_{3,\lambda_3}\right)\right)\\
&\quad + \left(\sum_{k=1}^{3}\frac{n_k^2}{R^2} - \frac{n^2}{R^2}\right)\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2<\lambda_3}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\phi_{3,\lambda_3}\right)\\
&\quad - [\Box_n', \chi_{R\gtrsim\tau}]\left(\sum_{\lambda_2<\lambda_3}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\phi_{3,\lambda_3}\right)\\
&\quad - \chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3}\mathcal{N}_0\left(\phi_{3,\lambda_3}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\right)\\
&\quad - \chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3}\mathcal{N}_0\left(\phi_{2,\lambda_2}\phi_{1,<\lambda_2}\phi_{3,\lambda_3}\right)\\
&\quad - \chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3}\Box_{n_1}'\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\phi_{3,\lambda_3}\\
&\quad - \chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3}\phi_{1,<\lambda_2}\Box_{n_2}'\phi_{2,\lambda_2}\phi_{3,\lambda_3}\\
&\quad - \chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\Box_{n_3}'\phi_{3,\lambda_3}\\
&=: \sum_{j=1}^{8} E_j
\end{aligned} \tag{7.20}$$

We shall then treat each of the terms E_j separately. Some of these are good source terms and can be directly bounded, others can only be dealt with under the assumption that ϕ_j itself satisfy a wave equation (which is automatically fulfilled in an iterative scheme), one term will require further transformation(E_8), and one term will be used to modify the wave equation(E_1), thereby generating some additional but harmless source terms.

E_1: Recall that the wave equation at angular momentum n, $|n| \geq 2$, is given by

$$-\left(\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)^2 + \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\right)\varepsilon_\pm(n) + H_n^\pm \varepsilon_\pm(n) = F_\pm(n), \tag{7.21}$$

where

$$H_n^\pm = \partial_R^2 + \frac{1}{R}\partial_R - f_n(R) \pm g_n(R).$$

Call the wave operator on the left $-\Box_n$. Then we write the first term E_1 as

$$\begin{aligned} E_1 &= \Box_n\left(\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2<\lambda_3} \phi_{1,<\lambda_2}\phi_{2,\lambda_2}\phi_{3,\lambda_3}\right)\right) - \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\left(\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2<\lambda_3} \phi_{1,<\lambda_2}\phi_{2,\lambda_2}\phi_{3,\lambda_3}\right)\right) \\ &\quad + \left(\frac{1}{R}\partial_R - f_n(R) \pm g_n(R) + \frac{n^2}{R^2}\right)\left(\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2<\lambda_3} \phi_{1,<\lambda_2}\phi_{2,\lambda_2}\phi_{3,\lambda_3}\right)\right) \\ &=: \Box_n\psi + E_{11} + E_{12}. \end{aligned}$$

Here the first term on the right $\Box_n\psi$ will be incorporated into the left-hand part of the wave equation (7.21), while the remaining two terms are good source terms:

Lemma 7.32. *Assume that ϕ_j are angular momentum n_j functions, $|n_j| \geq 2$, $j = 1, 2, 3$, each of which admits a representation*

$$\phi_j(\tau, R) = \int_0^\infty \phi_{n_j}(R, \xi) \cdot \overline{x}_j(\tau, \xi) \cdot \rho_{n_j}(\xi)\, d\xi, \quad j = 1, 2, 3,$$

where the distorted Fourier transforms $\overline{x}_j$, $j = 1, 2$ each can be written as $\overline{x}_j = \overline{y}_j + \overline{z}_j$ with $\overline{y}_j$ admissibly singular (at angular momentum n_j) and $\overline{z}_j \in S_0^{\hbar_j}, \mathcal{D}_\tau \overline{z}_j \in S_1^{\hbar_j}$. Then if m, $|m| \geq 2$, is an admissible angular momentum for the output of each of E_{11}, E_{12}, then each of E_{11}, E_{12} satisfies

$$E_{1j}|_{R<\nu\tau} = g_j(\tau, R)|_{R<\nu\tau},$$

where g_j admits an angular momentum m distorted Fourier representation

$$g_j(\tau, R) = \int_0^\infty \phi_n(R, \xi) \cdot r(\tau, \xi)\rho_n(\xi)\, d\xi,$$

with $r = r_1 + r_2$ where r_1 is source admissibly singular, while $r_2 \in S_1^\hbar$.

Proof. Note that we avoid the key difficulty for the original null-form when the 'good' derivative hits a $S_0^\hbar$-term while the 'bad' derivative hits a singular term, because we have only one derivative to begin with here. The assertion for the term E_{11} then follows from the same considerations as in the proof of Prop. 7.25. For the term E_{12} we need in particular estimate the term(since $-f_n(R) \pm g_n(R) + \frac{n^2}{R^2} = O(\frac{n}{R^2})$)

$$\frac{n}{R^2}\chi_{R\gtrsim\tau} \cdot \left(\sum_{\lambda_2<\lambda_3} \phi_{1,<\lambda_2}\phi_{2,\lambda_2}\phi_{3,\lambda_3}\right),$$

where n refers to the angular momentum of the expression. Here we have to be careful not to lose in n, which we may assume in the most difficult case is the largest of all the angular momenta present, comparable to only one of the angular momenta n_j in the ϕ_j. In case all factors are in $\tilde{S}_0^\hbar$ this follows by the arguments used to prove Prop. 6.10. In case that one of the factors is admissibly singular, and the expression is a singular source term, we can absorb the outer weight n since the term is of level $l_1 \geq 2$. □

E_2: Here we have to be more precise about the output angular momentum n, which we fix as $n = \sum_{j=1}^{3} n_j$, the value it takes in the iterative scheme. Then we have

$$\sum_{k=1}^{3} \frac{n_k^2}{R^2} - \frac{n^2}{R^2} = \frac{1}{R^2} \cdot O\Big(\max\{|n_j|\} \cdot \max\{\{|n_j|\} \setminus \max\{|n_j|\}\}\Big)$$

and this shows that the contribution of E_2 can be handled analogously to the one of E_1.

E_3: This term is essentially

$$\frac{1}{R^2}\chi_{R\gtrsim\tau} \cdot \left(\sum_{\lambda_2<\lambda_3} \phi_{1,<\lambda_2}\phi_{2,\lambda_2}\phi_{3,\lambda_3} \right),$$

and hence a simpler variant of the preceding terms.

E_j, $j \in \{6,7\}$: These terms can be handled if we recall the equation satisfied by ϕ_1, ϕ_2, which will of course contain the same kinds of null-forms. However, the paraproduct structure inherent in these terms makes them better than the bare cubic null-form from before. To handle these terms, we (i) first have to move the operators $\Box'_{n_j}$ past the localizations $P_{<\lambda_2}, P_{\lambda_2}$, and then we (ii) have to invoke a paraproduct estimate to bound them.

(i): Commuting the wave operator at angular momentum n, $|n| \geq 2$, past localization operators. Here the following lemma mostly resolves the issue:

Lemma 7.33. *Let $P_\lambda, P_{<\lambda}$, $\lambda \geq 1$, be dyadic frequency localization operators acting on angular momentum n, $|n| \geq 2$ functions. Then if $f(\tau, R)$ is a function admitting the distorted Fourier representation*

$$f(\tau, R) = \int_0^\infty \phi_n(R, \xi) \cdot \overline{x}(\tau, \xi) \cdot \rho_n(\xi)\, d\xi,$$

with $\overline{x} = \overline{x}_1 + \overline{x}_2$ where $\overline{x}_1 \in S_0^\hbar$, $\mathcal{D}_\tau \overline{x}_1 \in S_1^\hbar$, and $\overline{x}_2$ admissibly singular of level $l_1 \geq 1$, then (recalling the notation from above)

$$\mathcal{F}\Big([\Box_n, P_\mu] f\Big) = \overline{y}_1 + \overline{y}_2$$

where $\overline{y}_1 \in S_1^\hbar$, and $\overline{y}_2$ is source admissibly singular. The same conclusion applies if P_λ is replaced by $P_{<\lambda}$.

Proof. Since $H_n^\pm$ commutes with P_λ, it suffices to consider the temporal part of $\Box_n$, which consists of the operators

$$\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big)^2, \quad \frac{\lambda_\tau}{\lambda}\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big).$$

Write

$$\begin{aligned}\left[\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big)^2, P_\mu\right] &= \Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big)\left[\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big), P_\mu\right] \\ &\quad + \left[\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big), P_\mu\right]\Big(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\Big)\end{aligned}$$

$$
\begin{aligned}
&= \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)\left[\frac{\lambda_\tau}{\lambda} R\partial_R, P_\mu\right] + \left[\frac{\lambda_\tau}{\lambda} R\partial_R, P_\mu\right]\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right) \\
&= \left[\frac{\lambda_\tau}{\lambda} R\partial_R, \left[\frac{\lambda_\tau}{\lambda} R\partial_R, P_\mu\right]\right] + 2\left[\frac{\lambda_\tau}{\lambda} R\partial_R, P_\mu\right]\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right) \\
&+ \left(\frac{\lambda_\tau}{\lambda}\right)' \left[R\partial_R, P_\mu\right]
\end{aligned}
$$

In order to proceed, we translate the commutator to the Fourier side:

$$
\mathcal{F} \circ \left[\frac{\lambda_\tau}{\lambda} R\partial_R, P_\mu\right] = \frac{\lambda_\tau}{\lambda}\left[-2\xi\partial_\xi + \mathcal{K}_\hbar^{(0)}, \chi_{\xi\simeq\lambda}\right] \circ \mathcal{F} = -2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi(\chi_{\xi\simeq\lambda})\mathcal{F} + \frac{\lambda_\tau}{\lambda}\left[\mathcal{K}_\hbar^{(0)}, \chi_{\xi\simeq\lambda}\right] \circ \mathcal{F}.
$$

Here the operator

$$
\left[\mathcal{K}_\hbar^{(0)}, \chi_{\xi\simeq\mu}\right]
$$

is seen to have similar properties to $\mathcal{K}_\hbar^{(0)}$. In fact, the kernel of this commutator is given by

$$
\frac{\left(\chi_{\eta\simeq\mu} - \chi_{\xi\simeq\mu}\right) F(\xi,\eta)\rho_n(\eta)}{\xi - \eta},
$$

which behaves at least as good as the kernel of $\mathcal{K}_\hbar^{(0)}$. If we denote by

$$
\mathcal{F} \circ \left[\frac{\lambda_\tau}{\lambda} R\partial_R, P_\mu\right] := \mathcal{K}_C,
$$

then

$$
\mathcal{F} \circ \left[\frac{\lambda_\tau}{\lambda} R\partial_R, \left[\frac{\lambda_\tau}{\lambda} R\partial_R, P_\lambda\right]\right] = \frac{\lambda_\tau}{\lambda}\left[-2\xi\partial_\xi + \mathcal{K}_\hbar^{(0)}, \mathcal{K}_C\right],
$$

which can be handled similarly as the commutator $\left[\mathcal{D}_\tau, \mathcal{K}_\hbar^{(0)}\right]$ (see Proposition 6.6). It easily follows that if $\overline{x}_1 \in S_0^\hbar$ (or $\overline{x}_1 \in S_1^\hbar$) , then

$$
\mathcal{F} \circ \left[\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)^2, P_\lambda\right] \circ \mathcal{F}^{-1}\overline{x}_1 \in S_1^\hbar.
$$

The conclusion for

$$
\mathcal{F} \circ \left[\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right)^2, P_\lambda\right] \circ \mathcal{F}^{-1}\overline{x}_2
$$

with $\overline{x}_2$ admissibly singular of level $l_1 \geq 1$ follows by combining the observation in subsection 7.5.2 and Proposition 7.15. □

(ii): Using the paraproduct structure. The preceding lemma is not strong enough to handle the case when $\overline{x}_2$ is of principal incoming singular type. This is where we have to take advantage of the paraproduct nature of the terms E_6, E_7:

Lemma 7.34. *Let $f(R)$ be an angular momentum n_1 function as in the preceding lemma, Let g, an angular momentum n_2 function with $|n_2| \geq 2$, have the same properties as f, and assume that $\{m, n_1, n_2\}$ with $|m| \geq 2$ is an admissible angular momentum triple. Then*

$$
\sum_{\lambda>0} [\Box_n, P_\lambda]\, f P_{\geq\lambda} g|_{R<\nu\tau} = h(\tau, R)|_{R<\nu\tau},
$$

where the distorted Fourier transform of $h(\tau, R)$ can be decomposed as the sum of a function $\overline{z}_1 \in S_1^{\hbar}$ and a function $\overline{z}_2$ which is source admissibly singular and of level $l_1 \geq 1$.

This lemma follows by using the preceding one, and combining it with Lemma 7.10 , Lemma 7.7.

Taking advantage of the preceding lemma, as well as the fact that the error terms arising upon replacing $\Box'_{n_j}$ by $\Box_{n_j}$ are again see to be admissible source terms of at least level $l_1 \geq 1$, we can now replace the terms E_6, E_7 by the following ones, respectively:

$$\chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3} P_{<\lambda_2}\left(\Box_{n_1}\phi_1\right)\phi_{2,\lambda_2}\phi_{3,\lambda_3}, \quad \chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3}\phi_{1,<\lambda_2}P_{\lambda_2}\left(\Box_{n_2}\phi_2\right)\phi_{3,\lambda_3}$$

Then we re-iterate application of the equations for ϕ_2, ϕ_1. The fact that these terms don't pose difficulties in spite of the fact that the same null-forms occur in $\Box_{n_1}\phi_1, \Box_{n_2}\phi_2$ are now consequences of the following

Lemma 7.35. *Let F be an angular momentum n_1 function, $|n_1| \geq 2$, whose distorted Fourier transform is the sum of a term in $\xi^{\frac{1}{4}}S_1^{\hbar_1}$ and an admissibly singular source term. Also, assume that ϕ_2 is an angular momentum $n_2, |n_2| \geq 2$ function, whose distorted Fourier transform is the sum of a function in $S_0^{\hbar_2}$ and an admissibly singular term. Finally, assume that $\{m, n_1, n_2\}$ is an admissible triple of angular momenta. Then*

$$\chi_{R\gtrsim\tau}\sum_{0<\lambda} P_{<\lambda}FP_{\lambda}\phi_2|_{R<\nu\tau} = g(\tau, R)|_{R<\nu\tau}$$

where g is an angular momentum m function whose distorted Fourier transform is the sum of a term in $S_1^{\hbar}$ and an admissibly singular source term of level $l_1 \geq 1$.

E_8. Here we finally arrive at a term (up to easier error terms) where the same procedure is re-iterated, but with additional factors which gain smallness. Again we may replace $\Box'_{n_3}$ by $\Box_{n_3}$ up to good source error terms, and we may commute the localizer P_{λ_3} and $\Box_{n_3}$, leading to the term

$$\chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}P_{\lambda_3}\left(\Box_{n_3}\phi_3\right)$$

Writing

$$\begin{aligned}\chi_{R\gtrsim\tau}\sum_{\lambda_2<\lambda_3}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}P_{\lambda_3}\left(\Box_{n_3}\phi_3\right) = &-\chi_{R\gtrsim\tau}\sum_{\lambda_2\geq\lambda_3}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}P_{\lambda_3}\left(\Box_{n_3}\phi_3\right)\\ &+\chi_{R\gtrsim\tau}\sum_{\lambda_2}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\left(\Box_{n_3}\phi_3\right),\end{aligned}$$

and taking advantage of Lemma 7.35, we can discard the first term on the right. Consider then the second term on the right, where we may assume that the term $\Box_{n_3}\phi_3$ is again given by a null-form of the type under consideration (the remaining terms in the source being admissible source terms). If we reduce this null-form to the paraproduct version which can again not be handled via Prop. 7.25, we arrive at the quintilinear expression

$$\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\right)\sum_{\lambda_5,\lambda_6}\mathcal{N}_0\left(P_{<\min\{\lambda_5,\lambda_6\}}\phi_4, \phi_{5,\lambda_5}, \phi_{6,\lambda_6}\right)$$

$$= \chi_{R\gtrsim\tau}\left(\sum_{\lambda_2}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\right)\sum_{\lambda_5<\lambda_6}\mathcal{N}_0\left(P_{<\lambda_5}\phi_4,\phi_{5,\lambda_5},\phi_{6,\lambda_6}\right)$$
$$+\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\right)\sum_{\lambda_5\geq\lambda_6}\mathcal{N}_0\left(P_{<\lambda_6}\phi_4,\phi_{5,\lambda_5},\phi_{6,\lambda_6}\right)$$

At this point, again taking advantage of Lemma 7.35 , it is clear how to proceed: taking the first term on the right (the second being handled analogously), if $\lambda_2 \geq \lambda_5$, this is a good source term (we can simply switch the positions between ϕ_2 and ϕ_4 in the quintilinear expression). Thus we reduce this term to

$$\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2<\lambda_5}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\right)\sum_{\lambda_5<\lambda_6}\mathcal{N}_0\left(P_{<\lambda_5}\phi_4,\phi_{5,\lambda_5},\phi_{6,\lambda_6}\right).$$

Using the same argument as the one deriving (7.20), the problem reduces to bounding

$$\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2<\lambda_5}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\right)\sum_{\lambda_5<\lambda_6}\phi_{4,<\lambda_5}\phi_{5,\lambda_5}P_{\lambda_6}\left(\Box_{n_6}\phi_6\right),$$

which, up to good error terms, can be replaced by

$$\chi_{R\gtrsim\tau}\left(\sum_{\lambda_2<\lambda_5}\phi_{1,<\lambda_2}\phi_{2,\lambda_2}\right)\left(\sum_{\lambda_5}\phi_{4,<\lambda_5}\phi_{5,\lambda_5}\right)\left(\Box_{n_6}\phi_6\right).$$

Now the process is repeated.

7.8. **Inclusion of the factors with exceptional angular momentum** $n \in \{0, \pm 1\}$**. Definition of admissibly singular terms.** Recall from subsection 6.5 the structure of functions with angular momentum $n = 0, \pm 1$, namely

$$f(\tau,R) = c_j(\tau)\cdot\phi_j(R) + \phi_j(R)\cdot\int_0^R\left[\phi_j(s)\right]^{-1}\cdot\mathcal{D}_j f(s)\,ds, \quad j = 0,\pm 1,$$

and furthermore, we have the Fourier representation

$$\mathcal{D}_j f(\tau,R) = \int_0^\infty \phi_j(R,\xi)\cdot\overline{x}(\tau,\xi)\cdot\tilde{\rho}_j(\xi)\,d\xi,$$

where the functions $\overline{x}(\tau,\xi)$ got measured for the bilinear estimates away from the light cone via the norms (6.26), (6.24), (6.22). Here we introduce the analogues of Definition 7.2 and Definition 7.8, which allow to characterize the singularity across the light cone for the exceptional angular momentum functions on the distorted Fourier side. Things are simplified somewhat due to the fact that in the low angular momentum setting we can completely neglect the n-dependence, and in particular the somewhat cumbersome terms

$$e^{\pm i\hbar^{-1}\rho\left(x_{\sigma'};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)}$$

disappear from the presentation. On the other hand, we will have to treat each exceptional case on its own, due to the different spectral measures. It is to be kept in mind that the functions $\overline{x}(\tau,\xi)$ in the following definition represent the derivative $\mathcal{D}_j f$. Observe that the improved temporal decay seen here compared to the angular modes $|n| \geq 2$ comes from the modulation theory developed later, and will play a crucial role in being able to control the evolution of the unstable modes, i. e. the functions $c_j(\tau)$, $j = 0, \pm 1$.

Definition 7.36. *For functions of angular momentum $n = 1$, we say that $\overline{x}(\tau, \xi)$ is a prototype singular function provided it admits the representation*

$$\overline{x}(\tau,\xi) = \sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\chi_{\xi\geq 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^i\cdot a_{k,i}(\tau) + \sum_{\pm}\sum_{l=1}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\chi_{\xi\geq 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}+\frac{l}{4}}}(\log\xi)^i\cdot F_{l,k,i}(\tau,\xi),$$

and furthermore we have the bounds with $k_1 \in \{0, 1\}$, $k_2 \in \{0, 1, \ldots, 5\}$

$$\left|\partial_\tau^{k_1}a_{k,i}(\tau)\right| \lesssim (\log\tau)^{N_1-i}\cdot\tau^{-3-\nu-k_1}$$
$$\left|\partial_\tau^{k_1}\partial_\xi^{k_2}F_{l,k,i}(\tau,\xi)\right| \lesssim (\log\tau)^{N_1-i}\cdot\tau^{-3-\nu-k_1}\cdot\xi^{-k_2},$$

as well as the 'closure bounds'

$$\left\|\xi^{5+\delta}\partial_\tau^{k_1}\partial_\xi^{5}F_{l,k,i}(\tau,\xi)\right\|_{\dot{C}^{\delta}_{\xi}} \lesssim (\log\tau)^{N_1-i}\cdot\tau^{-3-\nu-k_1}.$$

For functions of angular momentum $n = 0$, we say that $\overline{x}(\tau, \xi)$ is a prototype singular function provided it admits the representation

$$\overline{x}(\tau,\xi) = \sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\chi_{\xi\geq 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{3}{2}+k\frac{\nu}{2}}}(\log\xi)^i\cdot a_{k,i}(\tau) + \sum_{\pm}\sum_{l=1}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\chi_{\xi\geq 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{3}{2}+k\frac{\nu}{2}+\frac{l}{4}}}(\log\xi)^i\cdot F_{l,k,i}(\tau,\xi).$$

For functions of angular momentum $n = -1$, we say that $\overline{x}(\tau, \xi)$ is a prototype singular function provided it admits the representation

$$\overline{x}(\tau,\xi) = \sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\chi_{\xi\geq 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{2+k\frac{\nu}{2}}}(\log\xi)^i\cdot a_{k,i}(\tau) + \sum_{\pm}\sum_{l=1}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\chi_{\xi\geq 1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{2+k\frac{\nu}{2}+\frac{l}{4}}}(\log\xi)^i\cdot F_{l,k,i}(\tau,\xi)$$

The same bounds as in the case $n = 1$ apply.
Similarly, we have the more general concept of admissible functions: For functions of angular momentum $n = 1$, we say that $\overline{x}(\tau, \xi)$ is an admissible singular function provided it admits the representation

$$\overline{x} = \overline{x}_{in} + \overline{x}_{out} + \overline{x}_{proto},$$

where $\overline{x}_{proto}$ is a prototypical singular function at angular momentum $n = 1$, and we have the following representations for the incoming and outgoing parts:

$$\overline{x}_{in}(\tau,\xi) = \sum_{\pm}\sum_{l=1}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\chi^{(l)}_{\xi\geq 1}\langle\xi\rangle^{-\frac{l}{4}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^i\cdot\int_{\tau_0}^{\tau}F^{(\pm)}_{l,k,i}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,$$

where the $\pm$-signs in each expression on the right are synchronized, and we have the following bounds, where the δ_l are small positive numbers decreasing in l:

$$\left|\xi^{k_2}\partial_\tau^{\iota}\partial_\xi^{k_2}F^{(\pm)}_{l,k,i}(\tau,\sigma,\xi)\right| \lesssim (\log\tau)^{N_1-i}\,\tau^{-3-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\xi^{\frac{1}{2}}\right)\right],\quad 0\leq k_2\leq 5,\quad \iota\in\{0,1\}$$
$$\left\|\xi^{5+\delta_l}\partial_\tau^{\iota}\partial_\xi^{5}F^{(\pm)}_{l,k,i}(\tau,\sigma,\xi)\right\|_{\dot{C}^{\delta_l}_{\xi}(\xi\simeq\lambda)} \lesssim (\log\tau)^{N_1-i}\,\tau^{-3-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\lambda^{\frac{1}{2}}\right)\right],\quad \iota\in\{0,1\},$$

and further $\overline{x}_{out} = \overline{x}_{out,1} + \overline{x}_{out,2}$, *with*

$$\begin{aligned}\overline{x}_{out,1}(\tau,\xi) = \sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\chi_{\xi\geq 1}\,\langle\xi\rangle^{-\frac{l}{4}}\,\frac{(\log\xi)^i}{\xi^{1+\frac{k\nu}{2}}}\\ \cdot\int_{\tau_0}^{\tau} e^{\pm i\left[\left(\nu\tau-2\frac{\lambda(\tau)}{\lambda(\sigma)}\nu\sigma\right)\xi^{\frac12}\right]}\cdot F^{\pm}_{l,k,i}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,\end{aligned} \tag{7.22}$$

and the bounds

$$\left|\xi^r\partial_\tau^\iota\partial_\xi^r F^{(\pm)}_{l,k,i}(\tau,\sigma,\xi)\right| \lesssim (\log\tau)^{N_1-i}\,\tau^{-3-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\xi^{\frac12}\right)\right],\quad 0\leq r\leq 5,\quad \iota\in\{0,1\}$$

$$\left\|\xi^{5+\delta_l}\partial_\tau^\iota\partial_\xi^5 F^{(\pm)}_{l,k,i}(\tau,\sigma,\xi)\right\|_{\dot{C}^{\delta_l}_{\xi}(\xi\simeq\lambda)} \lesssim (\log\tau)^{N_1-i}\,\tau^{-3-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\lambda^{\frac12}\right)\right],\quad \iota\in\{0,1\}.$$

The second term $\overline{x}_{out,2}$, *which is the 'outgoing perpetuated singularity' admits the description*

$$\begin{aligned}\overline{x}_{out,2}(\tau,\xi) = \sum_{\pm}\sum_{l=0}^{7}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\chi_{\xi\geq 1}\,\langle\xi\rangle^{-\frac{l}{4}}\,\frac{(\log\xi)^i}{\xi^{1+k\frac{\nu}{2}}}\\ \cdot\int_0^\infty\int_{\tau_0}^{\tau} e^{\pm i\left[\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x+\tau\right)\xi^{\frac12}\right]}\cdot G^{\pm}_{l,k,i}\left(\tau,\sigma,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma dx,\end{aligned} \tag{7.23}$$

where $F^{\pm}_{0,k,i}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) = b^{\pm}_{k,i}(\tau,\sigma)$, $G^{\pm}_{0,k,i}\left(\tau,\sigma,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) = c^{\pm}_{k,i}(\tau,\sigma,x)$, *and we have the bounds*

$$\left\|\xi^{k_2}\partial_\xi^{k_2}G^{(\pm)}_{l,k,i}(\tau,\sigma,x,\xi)\right\|_{L^1_x} \lesssim (\log\tau)^{N_1-i}\,\tau^{-3-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\xi^{\frac12}\right)\right],\quad 0\leq k_2\leq 5,$$

$$\left\|\left\|\xi^{5+\delta_l}\partial_\xi^5 G^{(\pm)}_{l,k,i}(\tau,\sigma,x,\xi)\right\|_{\dot{C}^{\delta_l}_{\xi}(\xi\simeq\lambda)}\right\|_{L^1_x} \lesssim (\log\tau)^{N_1-i}\,\tau^{-3-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\lambda^{\frac12}\right)\right],$$

Moreover, we require that the same structure and bounds apply to $\xi^{-\frac12}\cdot\left(\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi\right)\overline{x}_{out,2}$ *up to terms of the kinds* $\overline{x}_{in}$, $\overline{x}_{out,1}$.
We similarly define admissibly singular functions at angular momentum $n = 0, n = -1$, *except that the expression* $\xi^{1+k\frac{\nu}{2}}$ *gets replaced by* $\xi^{\frac32+k\frac{\nu}{2}}, \xi^{2+k\frac{\nu}{2}}$, *respectively.*
We also define the norms $\|x\|_{adm}$ *for each of these types of angular momentum n,* $n\in\{0,\pm1\}$ *functions, in perfect analogy to Definition 7.9 but with the obvious modifications taking into account the different decay properties. Finally, we define the restricted type of principal singular function in analogy to the case of angular momentum* $|n|\geq 2$, *and the corresponding norms* $\|x\|_{adm(r)}$.

7.9. **Null-form estimates near the light cone for outputs at angular momentum** $|n|\geq 2$ **but with arbitrary factors.** The preceding considerations for the null-form $\mathcal{N}_0(\phi_1,\phi_2,\phi_3)$ can be rendered quantitative by taking advantage of the norm in Definition 7.9 and its natural analogue for the exceptional angular momenta $n\in\{0,\pm1\}$. Combining Prop. 7.25 as well as the considerations for the E_j following its proof, one infers the following preliminary null-form estimate, where certain details, such as the fact that exceptional angular momentum functions f are described in terms of a pair of functions, $(c_n, \mathcal{D}_n f)$, are simplified by assuming that $c_n = 0$. The fully general case will only be treated below in Proposition 8.12, after a final refinement of the spaces (so-called 'good functions' and associated norms). We note that the estimate in the following proposition would be too weak to get the desired temporal decay, which is part of the reason for the refinement later on concerning the non-singular part of the inputs:

Proposition 7.37. *Let $\phi_j,\ j = 1, 2, 3$ be angular momentum n_j functions. If $|n_j| \geq 2$, assume that*

$$\phi_j(\tau, R) = \int_0^\infty \phi_{n_j}(R, \xi)\overline{x}_j(\tau, \xi)\rho_{n_j}(\xi)\, d\xi,$$

where $\overline{x}_j = \overline{y}_j + \overline{z}_j$ and $\overline{y}_j$ admissibly singular (in the angular momentum n_j-sense), while $\overline{z}_j \in S_0^{\hbar_j}, \mathcal{D}_\tau z_j \in S_1^{\hbar_j}$. If $|n_j| < 2$, assume that ϕ_j has trivial $c_{n_j} = 0$, and that

$$\mathcal{D}_{n_j}\phi_j(\tau, R) = \int_0^\infty \phi_{n_j}(R, \xi)\overline{x}_j(\tau, \xi)\tilde{\rho}_{n_j}(\xi)\, d\xi,$$

where $\overline{x}_j = \overline{y}_j + \overline{z}_j$ and $\overline{y}_j$ admissibly singular (in the angular momentum n_j-sense of Definition 7.36), and $\overline{z}_j \in S_0^{(n_j)}, \mathcal{D}_\tau\overline{z}_j \in S_1^{(n_j)}$. Finally, assume that the angular momentum triples $\{m_1, n_1, n_2\}, \{m_2, m_1, n_3\}$ are admissible, where $|m_2| \geq 2$. Finally, denote[7] by $n_* = \max\{\{|n_j|\}_{j=1,2,3} \setminus \{\max|n_j|\}\}$. Then there exists an angular momentum m_2 function ψ such that (with $\hbar = \frac{1}{m_2+1}$) we have

$$\psi(\tau, R) = \int_0^\infty \phi_{m_2}(R, \xi)\tilde{\overline{x}}(\tau, \xi)\tilde{\rho}_{m_2}(\xi)\, d\xi,$$

with $\tilde{\overline{x}} = \tilde{\overline{y}} + \tilde{\overline{z}}$ and the bounds

$$\left\|\tilde{\overline{y}}\right\|_{adm} + \left\|\tilde{\overline{z}}\right\|_{S_0^\hbar} + \left\|\mathcal{D}_\tau\tilde{\overline{z}}\right\|_{S_1^\hbar} \lesssim \langle n_*\rangle^6 \prod_{j=1}^3 \left(\left\|\overline{y}_j\right\|_{adm} + \tau\left[\left\|\overline{z}_j\right\|_{S_0^{\hbar_j}} + \left\|\mathcal{D}_\tau z_j\right\|_{S_1^{\hbar_j}}\right]\right)$$

if all inputs have angular momentum $|n_j| \geq 2$, and with $\hbar_j$ replaced by n_j if the corresponding angular momentum is exceptional, and such that the following holds: there is a function $H(\tau, R)$ with

$$\chi_{R\gtrsim\tau}\left(\mathcal{N}_0(\phi_1, \phi_2, \phi_3) - \Box_{m_2}\psi\right)|_{R<\nu\tau} = H(\tau, R)|_{R<\nu\tau},$$

and such that we can write

$$H(\tau, R) = \int_0^\infty \overline{x}(\tau, \xi)\phi_{m_2}(R, \xi)\rho_{m_2}(\xi)\, d\xi$$

where $\overline{x} = \overline{y} + \overline{z}$ with $\overline{y}$ source admissibly singular at angular momentum m_2, $\overline{z} \in S_1^\hbar$, and finally the bound (recall Definition 7.24)

$$\|\overline{y}\|_{source\ adm} + \|\overline{z}\|_{S_1^\hbar} \lesssim \langle n_*\rangle^6 \prod_{j=1}^3 \left(\left\|\overline{y}_j\right\|_{adm} + \tau\left[\left\|\overline{z}_j\right\|_{S_0^{\hbar_j}} + \left\|\mathcal{D}_\tau\overline{z}_j\right\|_{S_1^{\hbar_j}}\right]\right)$$

if all inputs have angular momentum $|n_j| \geq 2$, and with $\hbar_j$ replaced by n_j if the corresponding angular momentum is exceptional. The norm $\|\cdot\|_{source\ adm}$ may be replaced by $\|\cdot\|_{source\ adm(r)}$, provided all inputs ϕ_j have restricted principal singular type.

[7]Thus if say $|n_1| \leq |n_2| \leq |n_3|$, then we set $n_* = |n_2|$.

7.10. **Estimates for the remaining source terms near the light cone, and for output at angular momentum $|n| \geq 2$.** To complete the first version of the source term estimates near the light cone, we now consider all terms arising in Prop. 6.23, Prop. 6.24, Prop. 6.27, Prop. 6.28, where we again assume that all inputs ϕ_j are the sum of an admissibly singular function and an (unstructured) smooth function. Specifically, we have

Proposition 7.38. *Write $\varphi_j = \sum_{n\in Z}\varphi_j(n)e^{in\theta}$, $j = 1,2$, as well as*

$$\varepsilon_\pm(n) = \varphi_1(n) \mp i\varphi_2(n), \quad \varepsilon_-(n) = \overline{\varepsilon_+(-n)}.$$

Assume that for $|n| \geq 2$ we have (with $\hbar = \frac{1}{n+1}$)

$$\varepsilon_+(n) = \int_0^\infty \phi_n(R,\xi)x_n(\tau,\xi)\rho_n(\xi)\,d\xi,$$

where $\overline{x}_n = \overline{y}_n + \overline{z}_n$ with $\overline{y}_n$ admissibly singular, $\overline{z}_n \in S_0^\hbar$, $\mathcal{D}_\tau\overline{z}_n \in S_1^\hbar$. For the exceptional angular momenta $n \in \{0,\pm1\}$, we write

$$\mathcal{D}_n\varepsilon_+(n) = \int_0^\infty \phi_n(R,\xi)\overline{x}_n(\tau,\xi)\tilde{\rho}_n(\xi)\,d\xi,$$

with $\overline{x}_n = \overline{y}_n + \overline{z}_n$ and $\overline{y}_n$ admissibly singular in the sense of Definition 7.36, and $\overline{z}_n \in S_0^{(n)}, \mathcal{D}_\tau\overline{z}_n \in S_1^{(n)}$. Finally, assume that

$$\varepsilon_+(n) = \phi_n(R)\cdot\int_0^R [\phi_n(s)]^{-1}\,\mathcal{D}_n\varepsilon_+(n)(\tau,s)\,ds,$$

i. e. the unstable part c_n is assumed to vanish. Finally, we assume that

$$\sum_{|n|\geq 2} n^{12}\left(\left\|\overline{y}_n\right\|_{adm} + \tau\left[\|\overline{z}_n\|_{S_0^\hbar} + \|\mathcal{D}_\tau\overline{z}_n\|_{S_1^\hbar}\right]\right) + \sum_{|n|<2}\left\|\overline{y}_n\right\|_{adm} + \tau\left[\|\overline{z}_n\|_{S_0^{(n)}} + \|\mathcal{D}_\tau\overline{z}_n\|_{S_1^{(n)}}\right] =: \Lambda \ll 1.$$

Then if F_j is any one of the source terms in Prop. 6.23, Prop. 6.24, Prop. 6.27, Prop. 6.28, writing

$$F_j = \sum_{n\in Z} F_j(n)e^{in\theta},$$

then for each n with $|n| \geq 2$, there is $\psi_n(\tau,R)$, $H_n(\tau,R)$, such that

$$\psi_n(\tau,R) = \int_0^\infty \phi_n(R,\xi)\tilde{\overline{x}}_n(\tau,\xi)\rho_n(\xi)\,d\xi, \quad \tilde{\overline{x}}_n = \tilde{\overline{y}}_n + \tilde{\overline{z}}_n,$$
$$H_n(\tau,R) = \int_0^\infty \phi_n(R,\xi)\tilde{\tilde{\overline{x}}}_n(\tau,\xi)\rho_n(\xi)\,d\xi, \quad \tilde{\tilde{\overline{x}}}_n = \tilde{\tilde{\overline{y}}}_n + \tilde{\tilde{\overline{z}}}_n,$$

with

$$\sum_{|n|\geq 2} n^{12}\left(\left\|\tilde{\overline{y}}_n\right\|_{adm} + \tau\left[\left\|\tilde{\overline{z}}_n\right\|_{S_0^\hbar} + \left\|\mathcal{D}_\tau\tilde{\overline{z}}_n\right\|_{S_1^\hbar}\right]\right) \leq \Lambda^2 + \tau_0^{-1}\Lambda,$$
$$\sum_{|n|\geq 2} n^{12}\left(\left\|\tilde{\tilde{\overline{y}}}_n\right\|_{source\ adm} + \tau\left\|\tilde{\tilde{\overline{z}}}_n\right\|_{S_1^\hbar}\right) \leq \Lambda^2 + \tau_0^{-1}\Lambda,$$

and such that

$$\left(F_j(n) - \Box_n\psi_n\right)|_{R<\nu\tau} = H_n(\tau,R)|_{R<\nu\tau}.$$

Remark 7.39. *We note that depending on the type of term F_j, the correction terms ψ_n may all vanish. In fact, they are only required for terms with the characteristic Q_0 null-form structure.*

8. Closing the estimates for angular momenta $|n| \geq 2$: parametrix bounds for source admissibly singular source terms, and solving the wave equation by Fourier methods

8.1. **Parametrix bounds for source admissibly singular source terms.** In this subsection, we finally show that the functional framework introduced in Definition 7.8, Definition 7.24, is compatible with the wave parametrix at angular momentum $n, |n| \geq 2$. Recall that this parametrix is explicitly given by

$$\int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot f\left(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) d\sigma, \quad U^{(n)}(\tau,\sigma,\xi) = \frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}\cdot\frac{\sin\left[\lambda(\tau)\xi^{\frac{1}{2}}\int_\sigma^\tau \lambda^{-1}(u)\,du\right]}{\xi^{\frac{1}{2}}}$$

Then we have the following

Proposition 8.1. *Let $\overline{y}(\tau,\xi)$ be source admissibly singular at angular momentum n, $|n| \geq 2$. Then*

$$\overline{x}(\tau,\xi) := \int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot \overline{y}\left(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) d\sigma$$

can be written as $\overline{x} = \overline{x}_1 + \overline{x}_2$ where $\overline{x}_1$ is an admissibly singular function and $\overline{x}_2 \in S_0^{\hbar}$, $\mathcal{D}_\tau \overline{x}_2 \in S_1^{\hbar}$. Moreover, if $\overline{y}$ is of restricted singular type, then so is $\overline{x}_1$.

Proof. We verify this for the various parts involved in an admissibly singular source term. In the following, when we talk about the 'contribution of a certain part' of $\overline{x}(\tau,\xi)$ as displayed in Definition 7.8, we mean the parametrix applied to $f(\tau,\xi) = \tau^{-1}\xi^{\frac{1}{2}}\cdot\overline{x}(\tau,\xi)$.

The contribution of the first term in (7.3). Observe that explicitly spelled out this is the function

$$\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\hbar^{-1}\frac{e^{\pm i\nu\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}}}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)^{\frac{1}{2}+k\frac{\nu}{2}}}\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)\right)^i$$
$$\cdot\left(\int_{\tau_0}^{\sigma} e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\lambda(\sigma)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\lambda(\sigma)}{\lambda(\sigma_1)},\hbar\right)}\cdot\sigma^{-1}a_{k,i}^{(\pm)}(\sigma,\sigma_1)\,d\sigma_1\right) d\sigma$$

Here it is important to observe that the inner phase simplifies, which suggests switching the orders of integration:

$$e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\lambda(\sigma)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\lambda(\sigma)}{\lambda(\sigma_1)},\hbar\right)} = e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}$$

The kernel $U^{(n)}(\tau,\sigma,\xi)$ of the parametrix has an oscillatory factor which can be written as

$$\frac{e^{-i\left(\nu\tau-\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\right)\xi^{\frac{1}{2}}} - e^{i\left(\nu\tau-\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\right)\xi^{\frac{1}{2}}}}{2i},$$

whence combined with the oscillatory phase $e^{\pm i\nu\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}}$ we either produce the phase $e^{\pm i\nu\tau\xi^{\frac{1}{2}}}$ corresponding to an incoming singular term, or else the phase $e^{\pm i\left(\nu\tau-2\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\right)\xi^{\frac{1}{2}}}$, corresponding to an outgoing singularity.

Incoming singularity: Explicitly, this is the expression

$$\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}$$
$$\cdot\left(\int_{\sigma_1}^{\tau}\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}\cdot\frac{\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)\right)^i}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)^{\frac{1}{2}+k\frac{\nu}{2}}}\cdot\sigma^{-1}a_{k,i}^{(\pm)}(\sigma,\sigma_1)\,d\sigma\right)d\sigma_1$$

We claim that up to better terms this is again of incoming principal type. To show this, we need to first expand $\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}$ in a Hankel type expansion towards $\xi=+\infty$, i. e. write

$$\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}=1+\left\langle\hbar\xi^{\frac{1}{2}}\right\rangle^{-1}\cdot g_n(\tau,\sigma,\xi)$$

where g_n has symbol type behavior and is bounded. The contribution if the term $\langle\hbar\xi^{\frac{1}{2}}\rangle^{-1}\cdot g_n(\tau,\sigma,\xi)$ is then easily seen to lead to a connecting incoming singular term. Further, we have

$$\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}\cdot\left(\int_{\sigma_1}^{\tau}\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\dots\right)d\sigma$$
$$=\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\hbar^{-1}\chi^{(l)}_{\xi\geq\hbar^{-2}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}\cdot\left(\int_{\sigma_1}^{\tau}\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\dots\right)d\sigma$$
$$+\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\hbar^{-1}\left(1-\chi^{(l)}_{\xi\geq\hbar^{-2}}\right)\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}\cdot\left(\int_{\sigma_1}^{\tau}\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\dots\right)d\sigma$$

and here the last term is easily seen to be of type $\overline{x}_2$ (as in the statement of the proposition), while the first term is decomposed as follows:

$$\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\hbar^{-1}\chi^{(l)}_{\xi\geq\hbar^{-2}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}\cdot\left(\int_{\sigma_1}^{\tau}\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\dots\right)d\sigma$$
$$=\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\hbar^{-1}\chi^{(l)}_{\xi\geq\hbar^{-2}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}\cdot\left(\int_{\sigma_1}^{\tau}\dots\right)d\sigma$$

$$-\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\hbar^{-1}\chi^{(l)}_{\xi\geq\hbar^{-2}}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{\frac12}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}\cdot\left(\int_{\sigma_1}^{\tau}\left(1-\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\right)\dots\right)d\sigma$$

and here the second term on the right is again of type $\overline{x}_2$ since the support of

$$\chi^{(l)}_{\xi\geq\hbar^{-2}}\cdot\left(1-\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\right),\quad \sigma\leq\tau,$$

is contained in $\xi\simeq\hbar^{-2}$.
We conclude that up to terms of type $\overline{x}_2$, the incoming singularity is given by the function

$$\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\hbar^{-1}\chi^{(l)}_{\xi\geq\hbar^{-2}}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{1+k\frac{\nu}{2}}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}$$
$$\cdot\left(\int_{\sigma_1}^{\tau}\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)\right)^i}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)^{\frac12+k\frac{\nu}{2}}}\cdot\sigma^{-1}a^{(\pm)}_{k,i}(\sigma,\sigma_1)\,d\sigma\right)d\sigma_1,$$

where we can expand out the logarithm as

$$\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)\right)^i=\sum_{k'+l=i}C_{k',l}\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)\right)^{k'}\cdot(\log\xi)^l$$

and it then suffices to set for fixed k,i and $p\leq i$

$$\tilde{a}^{(\pm)}_{k,p}(\tau,\sigma_1):=\int_{\sigma_1}^{\tau}\frac{C_{k,l}\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)\right)^{i-p}}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)^{k\frac{\nu}{2}}}\cdot\sigma^{-1}a^{(\pm)}_{k,i}(\sigma,\sigma_1)\,d\sigma,$$

and to verify that in light of Definition 7.24 we get the bounds needed according to Definition 7.8

$$\left|\tilde{a}^{(\pm)}_{k,p}(\tau,\sigma_1)\right|+\tau\left|\partial_\tau\tilde{a}^{(\pm)}_{k,p}(\tau,\sigma_1)\right|\lesssim(\log\tau)^{N_1-p}\,\tau^{-1-\nu}\cdot\sigma_1^{-3}.$$

Outgoing singularity: Explicitly this is the expression

$$\sum_{\pm}\sum_{k=1}^{N}\sum_{i=0}^{N_1}\hbar^{-1}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{\frac12}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}$$
$$\cdot\left(\int_{\sigma_1}^{\tau}e^{\mp2i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac12}}\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\rho_n^{\frac12}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac12}(\xi)}\cdot\frac{\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)\right)^i}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)^{\frac12+k\frac{\nu}{2}}}\cdot\sigma^{-1}a^{(\pm)}_{k,i}(\sigma,\sigma_1)\,d\sigma\right)d\sigma_1$$

We claim that this can be interpreted in the form $\overline{x}_{out,2}$, recalling Definition 7.8. In fact, write

$$e^{\pm i\nu\tau\xi^{\frac12}}\cdot e^{\mp2i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac12}}=e^{\mp i\nu\left(\tau+2\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}-2\tau\right)\xi^{\frac12}},$$

and introduce the variable

$$x := \frac{2\sigma\frac{\lambda(\tau)}{\lambda(\sigma)} - 2\tau}{\frac{\lambda(\tau)}{\lambda(\sigma_1)}},$$

which takes non-negative values for $\sigma \leq \tau$. Then set, with $\sigma = \sigma(x, \tau, \sigma_1)$

$$G_{k,0,p}(\tau, \sigma_1, x, \xi) := \chi^{(l)}_{\xi \geq \hbar^{-2}} \cdot \frac{\rho_n^{\frac{1}{2}}(\xi)}{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\sigma)}{\lambda^2(\tau)}\xi\right)} \frac{C_{i-p,l}\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)\right)^{i-p}}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)^{k\frac{\nu}{2}}} \cdot \sigma^{-1} a^{(\pm)}_{k,i}(\sigma, \sigma_1) \cdot \frac{\partial\sigma}{\partial x} \cdot \chi^{(sharp)}_{\sigma \in [\sigma_1, \tau]},$$

where $\chi^{(sharp)}_{\sigma \in [\sigma_1, \tau]}$ denotes the characteristic function corresponding to the interval $[\sigma_1, \tau]$ with respect to σ, which in turn is interpreted as a function of x, σ_1, τ via the above coordinate change. Here it may be objected that the function $G_{k,0,p}$ thus defined does not quite satisfy the requirements in Definition 7.8 that it should not depend on ξ (since $l = 0$), but this can be easily remedied by the arguments in the incoming case, by replacing

$$\frac{\rho_n^{\frac{1}{2}}(\xi)}{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\sigma)}{\lambda^2(\tau)}\xi\right)}$$

by 1 up to connecting singular terms, and abolishing the smooth cutoff $\chi^{(l)}_{\xi \geq \hbar^{-2}}$ up to terms of more regular type $S_0^{\hbar}$. It is now straightforward to check that the term can be written in the form (7.5)(but with σ_1 replacing the variable σ there), and that the required bounds for $G_{k,0,p}$ are satisfied.

The remaining terms in Definition 7.8 are of course handled similarly, let us consider

The contribution of the term (7.5). We again distinguish between an incoming and an outgoing term depending on the interaction of the oscillatory phase $U^{(n)}(\tau, \sigma, \xi)$ and the phase $e^{\pm 2i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}}$ from the source term.

Incoming singularity: In this case the combination of the phase from $U^{(n)}(\tau, \sigma, \xi)$ and the phase $e^{\pm 2i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}}$ simplifies to $e^{\pm i\nu\tau\xi^{\frac{1}{2}}}$, and so the phase

$$e^{\pm i\left[\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x+\tau\right)\xi^{\frac{1}{2}} + \hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}}; \alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}, \hbar\right)\right]}$$

after effecting the re-scaling $\xi \longrightarrow \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi$ gets transformed into

$$e^{\pm i\left[\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma_1)}x+\tau\right)\xi^{\frac{1}{2}} + \hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}}; \alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}, \hbar\right)\right]}$$

Considering the case $l = 0$ for simplicity, we obtain the following expression (after splitting up the power of the logarithm $\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)\right)^i$ as before)

$$\hbar^{-1}\chi^{(l)}_{\xi\geq\hbar^{-2}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^{i-p}\cdot\int_0^\infty\int_{\tau_0}^{\tau}e^{\pm i\left[\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma_1)}x+\tau\right)\xi^{\frac{1}{2}}+\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)\right]}$$
$$\cdot\left(\int_{\sigma_1}^{\tau}\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}\cdot\frac{\left(\log(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)})\right)^p}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)^{\frac{1}{2}+k\frac{\nu}{2}}}\cdot\sigma^{-1}G^{(\pm)}_{0,k,i}\left(\sigma,\sigma_1,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma_1)}\xi\right)d\sigma\right)d\sigma_1\,dx,$$

where inclusion of the outer cutoff $\chi^{(l)}_{\xi\geq\hbar^{-2}}$ is again legitimate up to an error in $S_0^{\hbar}$. But then setting

$$\tilde{G}^{(\pm)}_{0,k,i}\left(\tau,\sigma_1,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma_1)}\xi\right):=\int_{\sigma_1}^{\tau}\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}\cdot\frac{\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)\right)^p}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)^{\frac{1}{2}+k\frac{\nu}{2}}}\cdot\sigma^{-1}G^{(\pm)}_{0,k,i}\left(\sigma,\sigma_1,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma_1)}\xi\right)d\sigma,$$

the desired conclusion follows upon checking the routine bounds according to Definition 7.8. Furthermore, applying $\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi$ is seen to led to an expression which is up to a factor $\xi^{\frac{1}{2}}$ of the same form.

Outgoing singularity: Here we encounter the more complex expressions

$$\hbar^{-1}\chi^{(l)}_{\xi\geq\hbar^{-2}}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{1+k\frac{\nu}{2}}}(\log\xi)^{i-p}\cdot\int_0^\infty\int_{\tau_0}^{\tau}\int_{\sigma_1}^{\tau}e^{\mp i\nu\cdot 2\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}\mp i\nu\frac{\lambda(\tau)}{\lambda(\sigma_1)}x\xi^{\frac{1}{2}}\mp i\hbar^{-1}\rho\left(x_{\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)},\hbar\right)}$$
$$\cdot\chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\geq\hbar^{-2}}\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}\cdot\frac{\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)\right)^p}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)^{\frac{1}{2}+k\frac{\nu}{2}}}\cdot\sigma^{-1}G^{(\pm)}_{0,k,i}\left(\sigma,\sigma_1,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma_1)}\xi\right)d\sigma\,d\sigma_1\,dx,$$

where all the signs are synchronized. Before proceeding we note right away that applying $\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi$ reproduces a similar expression up to a factor $\xi^{\frac{1}{2}}$. In order to cast this in the mold required by Definition 7.8, introduce the new variable

$$\tilde{x} := x + \frac{2\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)} - 2\tau}{\frac{\lambda(\tau)}{\lambda(\sigma_1)}},$$

which only takes non-negative values if $x \geq 0$ and $\sigma \leq \tau$. Then if

$$\tilde{x} \geq \frac{2\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)} - 2\tau}{\frac{\lambda(\tau)}{\lambda(\sigma_1)}},$$

the restriction on σ remains $\tau \geq \sigma \geq \sigma_1$, while if

$$\tilde{x} < \frac{2\sigma_1\cdot\frac{\lambda(\tau)}{\lambda(\sigma_1)} - 2\tau}{\frac{\lambda(\tau)}{\lambda(\sigma_1)}},$$

the lower limit $\sigma_{min} = \sigma_{min}(\tau, \sigma_1, \tilde{x})$ for σ is given by the condition

$$\tilde{x} = \frac{2\sigma_{min} \cdot \frac{\lambda(\tau)}{\lambda(\sigma_{min})} - 2\tau}{\frac{\lambda(\tau)}{\lambda(\sigma_1)}}.$$

Denoting as before by χ^{sharp} the sharp (characteristic function) cutoff, we can then set

$$\begin{aligned}
&\tilde{G}^{(\pm)}_{0,k,i}\left(\tau, \sigma_1, \tilde{x}, \frac{\lambda^2(\tau)}{\lambda^2(\sigma_1)}\xi\right) := \\
&\int_{\sigma_1}^{\tau} \chi^{sharp}_{\tilde{x} \geq \frac{2\sigma_1 \cdot \frac{\lambda(\tau)}{\lambda(\sigma_1)} - 2\tau}{\frac{\lambda(\tau)}{\lambda(\sigma_1)}}} \chi^{(l)}_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi \geq \hbar^{-2}} \frac{\lambda(\tau)}{\lambda(\sigma)} \cdot \frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)} \cdot \frac{\left(\log\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)\right)^p}{\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\right)^{\frac{1}{2}+k\frac{\nu}{2}}} \\
&\qquad \cdot \sigma^{-1} G^{(\pm)}_{0,k,i}\left(\sigma, \sigma_1, \tilde{x} - \frac{2\sigma \cdot \frac{\lambda(\tau)}{\lambda(\sigma)} - 2\tau}{\frac{\lambda(\tau)}{\lambda(\sigma_1)}}, \frac{\lambda^2(\tau)}{\lambda^2(\sigma_1)}\xi\right) d\sigma \\
&+ \int_{\sigma_{min}}^{\tau} \chi^{sharp}_{\tilde{x} \geq \frac{2\sigma_1 \cdot \frac{\lambda(\tau)}{\lambda(\sigma_1)} - 2\tau}{\frac{\lambda(\tau)}{\lambda(\sigma_1)}}} \dots d\sigma,
\end{aligned}$$

which results in the desired representation

$$\hbar^{-1}\chi^{(l)}_{\xi \geq \hbar^{-2}} \frac{1}{\xi^{1+k\frac{\nu}{2}}} (\log \xi)^{i-p} \cdot \int_0^{\infty} \int_{\tau_0}^{\tau} e^{\mp i\left[\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}\tilde{x}+\tau\right)\xi^{\frac{1}{2}} + \hbar^{-1}\rho\left(x_{\sigma_1 \cdot \frac{\lambda(\tau)}{\lambda(\sigma_1)}}; \alpha \cdot \frac{\lambda(\tau)}{\lambda(\sigma_1)}, \hbar\right)\right]} \cdot \tilde{G}^{\pm}_{0,k,i}\left(\tau, \sigma_1, \tilde{x}, \frac{\lambda^2(\tau)}{\lambda^2(\sigma_1)}\xi\right) d\sigma_1 d\tilde{x}.$$

Verification of the required bounds for $\tilde{G}^{(\pm)}_{0,k,i}(\tau, \sigma_1, \tilde{x}, \xi)$ is straightforward and omitted. □

8.2. **Solution of the inhomogeneous wave equation on the distorted Fourier side.**

8.2.1. *The wave equation for angular momentum $|n| \geq 2$, formulation on the Fourier side.* Our point of departure of the fundamental equation (6.2), which we re-cast on the Fourier side. We emphasize at this point that for the technical reasons explained in subsection 7.7, it will be necessary to modify this equation by changing the variable $\varepsilon_{\pm}(n)$ to eliminate certain bad source terms. This, however, has no bearing on the subsequent Fourier methods, and so we shall formally work with the 'wrong equation' (6.2). Direct translation of this action to the Fourier side by applying $\mathcal{F}^{\hbar}$ and setting ($\hbar = \frac{1}{n+1}$)

$$\varepsilon_{\pm}(n) = \int_0^{\infty} \phi_n(R, \xi) \cdot \overline{x}^{\hbar}(\tau, \xi)\rho_n(\xi)\, d\xi$$

results in (see (6.6), (6.7) and (6.8))

$$-\left(\partial_\tau - 2\frac{\lambda'(\tau)}{\lambda(\tau)}\xi\partial_\xi + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_{\hbar}\right)^2 \overline{x}^{\hbar} - \frac{\lambda'(\tau)}{\lambda(\tau)}\left(\partial_\tau - 2\frac{\lambda'(\tau)}{\lambda(\tau)}\xi\partial_\xi + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_{\hbar}\right)\overline{x}^{\hbar} - \xi\overline{x}^{\hbar} = \mathcal{F}^{\hbar}\left(F_{\pm}(n)\right) \tag{8.1}$$

Introducing the important time derivative dilation type operator[8]

$$\mathcal{D}_\tau := \partial_\tau - 2\frac{\lambda'(\tau)}{\lambda(\tau)}\xi\partial_\xi - \frac{\lambda'(\tau)}{\lambda(\tau)}\frac{\rho_n'(\xi)\xi}{\rho_n(\xi)} - 2\frac{\lambda'(\tau)}{\lambda(\tau)}, \tag{8.2}$$

the preceding equation gets recast in the form

$$\begin{aligned}
&-\left(\mathcal{D}_\tau^2 + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{D}_\tau + \xi\right)\overline{x}^\hbar \\
&= \mathcal{F}^\hbar\left(F_\pm(n)\right) + 2\frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_\hbar^{(0)}\mathcal{D}_\tau\overline{x}^\hbar + \left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)'\mathcal{K}_\hbar^{(0)}\overline{x}^\hbar + \frac{\lambda'(\tau)}{\lambda(\tau)}\left[\mathcal{D}_\tau, \mathcal{K}_\hbar^{(0)}\right]\overline{x}^\hbar + \left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)^2\left(\left(\mathcal{K}_\hbar^{(0)}\right)^2 + \mathcal{K}_\hbar^{(0)}\right)\overline{x}^\hbar \\
&=: \sum_{j=1}^{5} f_j(\tau,\xi).
\end{aligned} \tag{8.3}$$

The first order of the day to solve this equation will be to show that applying the Duhamel parametrix to the last four terms on the right hand side will send admissible functions into admissible functions. In fact, due to the smoothing property of the operator $\mathcal{K}_\hbar^{(0)}$, the resulting functions won't have a principal incoming part anymore.

Proposition 8.2. *Assume that $\overline{x}^\hbar(\tau,\xi)$ is admissibly singular, and that $\mathcal{F}^\hbar\left(F_\pm(n)\right)$ is an admissibly singular source term. Then*

$$\int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot f_j\left(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma, \quad j = 1, 2, \ldots, 5,$$

is admissibly singular up to functions $\overline{z}_j(\tau,\xi)$ in $S_0^\hbar$ and with $\mathcal{D}_\tau\overline{z}_j \in S_1^\hbar$.

Proof. According to the preceding proposition it suffices to check that the terms f_j, $j = 2, \ldots, 5$ are source admissibly singular. It follows immediately from Definition 7.8 that $\mathcal{D}_\tau\overline{x}^\hbar$ is of the form $\xi^{\frac{1}{2}}\cdot\overline{y}(\tau,\xi)$ with $\overline{y}$ admissibly singular. Using a straightforward variant Proposition 7.15, and in particular the decay of the kernel $F(\xi,\eta)$ of the transference operator (at angular momentum n), we conclude that all the terms f_j, $j \geq 2$, are source admissibly singular or better (in $S_1^\hbar$). Then the desired conclusion follows upon application of Proposition 8.1. □

8.2.2. *A final refinement of the spaces, with a view toward dealing with the exceptional angular modes.* As seen in many instances before, admissibly singular terms lead to error terms in the good space $S_0^\hbar$, for example when applying the transference operator, or when translating things from the Fourier side to the physical side as in Lemma 7.10. Since admissibly singular terms have poor temporal decay properties (of the order $\tau^{-1-\nu}(\log\tau)^C$), this will mean the generation of poorly decaying terms with good regularity properties, but these will not be good enough to control the evolution of the instabilities of the exceptional modes. This problem forces us to refine the part of the distorted Fourier transform of $\varepsilon(n;\tau,R)$ which is in $S_0^\hbar$ by splitting it into a structured but poorly decaying part and an un-structured but well-decaying part, as follows:

[8]We suppress the dependence on n here to simplify the notation

Definition 8.3. *We say that the function* $\overline{x}(\tau,\xi)$ *is a good Fourier representation (or just a good function) at angular momentum* $|n|\geq 2$*, provided we can write*

$$\overline{x}=\overline{x}_{smooth}+\overline{x}_{singular},$$

where $\overline{x}_{singular}$ *is admissibly singular, and* $\overline{x}_{smooth}\in S_0^{\hbar}$*,* $\mathcal{D}_\tau\overline{x}_{smooth}\in S_1^{\hbar}$*, and furthermore, we can write*

$$\overline{x}_{smooth}(\tau,\xi)=\overline{x}_{1,smooth}+\overline{x}_{2,smooth},$$

where the first function on the right is structured but temporally slowly decaying. The structure is analogous to the one for the admissibly singular terms:

$$\begin{aligned}\overline{x}_{1,smooth}(\tau,\xi)&=\hbar^{-1}\langle\hbar^2\xi\rangle^{-4}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}+\delta}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)}\cdot F^{(\pm)}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\\&+\hbar^{-1}\langle\hbar^2\xi\rangle^{-4}\frac{1}{\xi^{\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}+\delta}}\\&\qquad\cdot\int_{\tau_0}^{\tau}e^{\pm i\left[\left(\nu\tau-2\frac{\lambda(\tau)}{\lambda(\sigma)}\nu\sigma\right)\xi^{\frac{1}{2}}-\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)\right]}\cdot\tilde{F}^{(\pm)}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\\&+\hbar^{-1}\langle\hbar^2\xi\rangle^{-4}\frac{1}{\xi^{\frac{1}{2}}\langle\xi\rangle^{\frac{1}{2}+\delta}}\\&\cdot\int_0^\infty\int_{\tau_0}^{\tau}e^{\pm i\left[\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x+\tau\right)\xi^{\frac{1}{2}}+\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)\right]}\cdot G^{\pm}\left(\tau,\sigma,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma dx,\end{aligned}$$

and we have the bounds

$$\begin{aligned}&\left|\xi^{k_2}\partial_\tau^\iota\partial_\xi^{k_2}F^{(\pm)}(\tau,\sigma,\xi)\right|\lesssim(\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right],\quad 0\leq k_2\leq 5,\quad \iota\in\{0,1\}\\&\left|\xi^{k_2}\partial_\tau^\iota\partial_\xi^{k_2}\tilde{F}^{(\pm)}(\tau,\sigma,\xi)\right|\lesssim(\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right],\quad 0\leq k_2\leq 5,\quad \iota\in\{0,1\}\\&\left\|\xi^{5+\delta_*}\partial_\tau^\iota\partial_\xi^5F^{(\pm)}(\tau,\sigma,\xi)\right\|_{\dot{C}_\xi^{\delta_*}(\xi\simeq\lambda)}\lesssim(\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\lambda^{\frac{1}{2}}\right)\right],\quad \iota\in\{0,1\}\\&\left\|\xi^{5+\delta_*}\partial_\tau^\iota\partial_\xi^5\tilde{F}^{(\pm)}(\tau,\sigma,\xi)\right\|_{\dot{C}_\xi^{\delta_*}(\xi\simeq\lambda)}\lesssim(\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\lambda^{\frac{1}{2}}\right)\right],\quad \iota\in\{0,1\}\\&\left\|\xi^{k_2}\partial_\xi^{k_2}G^{(\pm)}(\tau,\sigma,x,\xi)\right\|_{L_x^1}\lesssim(\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right],\quad 0\leq k_2\leq 5,\\&\left\|\left\|\xi^{5+\delta_l}\partial_\xi^5G^{(\pm)}(\tau,\sigma,x,\xi)\right\|_{\dot{C}_\xi^{\delta_*}(\xi\sim\lambda)}\right\|_{L_x^1}\lesssim(\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\lambda^{\frac{1}{2}}\right)\right],\end{aligned}$$

Moreover, applying $\partial_\tau-2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi$ *to the last term results in a term with the same structure and bounds as the preceding terms, up to a factor* $\xi^{\frac{1}{2}}$*. On the other hand, for the 'unstructured term'* $\overline{x}_{2,smooth}$ *we have the bounds*

$$\left\|\overline{x}_{2,smooth}(\tau,\cdot)\right\|_{S_0^{\hbar}}+\left\|\mathcal{D}_\tau\overline{x}_{2,smooth}(\tau,\cdot)\right\|_{S_1^{\hbar}}\lesssim\tau^{-3}.$$

Similarly, we call a function $\overline{x}(\tau,\xi)$ a good source term at angular momentum $|n|\geq 2$, provided it admits the representation

$$\overline{x} = \overline{x}_{smooth} + \overline{x}_{singular}$$

where $\overline{x}_{singular}$ is source admissibly singular, while the smooth part

$$\overline{x}_{smooth}(\tau,\xi) = \overline{x}_{1,smooth} + \overline{x}_{2,smooth} + \overline{x}_{3,smooth},$$

where

$$\overline{x}_{1,smooth}(\tau,\xi) = \hbar^{-1}\langle\hbar^2\xi\rangle^{-4}\frac{e^{\pm i\left(\nu\tau\xi^{\frac{1}{2}}+\hbar^{-1}\rho(x_\tau;\alpha,\hbar)\right)}}{\langle\xi\rangle^{\frac{1}{2}+\delta}}(\log\xi)^i\cdot\tilde{F}(\tau,\xi)$$

with the coefficient bounds

$$\left|\partial_\xi^{k_2}\tilde{F}(\tau,\xi)\right| \lesssim (\log\tau)^{N_1}\cdot\left[\tau^{-2-\nu-\delta}\cdot\xi^{-k_2} + \tau^{-2-\nu}\cdot\xi^{-k_2}\cdot\kappa\left(\hbar\frac{\lambda(\sigma)}{\lambda(\tau)}\xi^{\frac{1}{2}}\right)\right],\quad 0\leq k_2\leq 5,$$

as well as

$$\begin{aligned}\overline{x}_{2,smooth} &= \hbar^{-1}\langle\hbar^2\xi\rangle^{-4}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\langle\xi\rangle^{\frac{1}{2}+\delta}}\cdot\int_{\tau_0}^{\tau}e^{\pm i\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)}\cdot F^{(\pm)}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\\
&+\hbar^{-1}\langle\hbar^2\xi\rangle^{-4}\frac{1}{\langle\xi\rangle^{\frac{1}{2}+\delta}}\\
&\qquad\cdot\int_{\tau_0}^{\tau}e^{\pm i\left[\left(\nu\tau-2\frac{\lambda(\tau)}{\lambda(\sigma)}\nu\sigma\right)\xi^{\frac{1}{2}}-\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)\right]}\cdot\tilde{F}^{(\pm)}\left(\tau,\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\\
&+\hbar^{-1}\langle\hbar^2\xi\rangle^{-4}\frac{1}{\langle\xi\rangle^{\frac{1}{2}+\delta}}\\
&\cdot\int_0^\infty\int_{\tau_0}^{\tau}e^{\pm i\left[\nu\left(\frac{\lambda(\tau)}{\lambda(\sigma)}x+\tau\right)\xi^{\frac{1}{2}}+\hbar^{-1}\rho\left(x_{\sigma\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}};\alpha\cdot\frac{\lambda(\tau)}{\lambda(\sigma)},\hbar\right)\right]}\cdot G^{\pm}\left(\tau,\sigma,x,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma dx,\end{aligned}$$

and we have the bounds

$$\left|\xi^{k_2}\partial_\xi^{k_2}F^{(\pm)}(\tau,\sigma,\xi)\right| \lesssim (\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right]\cdot\left[\tau^{-1-\delta}+\kappa\left(\hbar\frac{\lambda(\sigma)}{\lambda(\tau)}\xi^{\frac{1}{2}}\right)\right],\quad 0\leq k_2\leq 5,$$

$$\left|\xi^{k_2}\partial_\xi^{k_2}\tilde{F}^{(\pm)}(\tau,\sigma,\xi)\right| \lesssim (\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right]\cdot\left[\tau^{-1-\delta}+\kappa\left(\hbar\frac{\lambda(\sigma)}{\lambda(\tau)}\xi^{\frac{1}{2}}\right)\right],\quad 0\leq k_2\leq 5,$$

$$\left\|\xi^{5+\delta_l}\partial_\tau^{\iota}\partial_\xi^5F^{(\pm)}(\tau,\sigma,\xi)\right\|_{\dot{C}_\xi^{\delta_*}(\xi\simeq\lambda)} \lesssim (\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\lambda^{\frac{1}{2}}\right)\right]\cdot\left[\tau^{-1-\delta}+\kappa\left(\hbar\frac{\lambda(\sigma)}{\lambda(\tau)}\xi^{\frac{1}{2}}\right)\right],$$

$$\left\|\xi^{5+\delta_l}\partial_\xi^5\tilde{F}^{(\pm)}(\tau,\sigma,\xi)\right\|_{\dot{C}_\xi^{\delta_*}(\xi\simeq\lambda)} \lesssim (\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\lambda^{\frac{1}{2}}\right)\right]\cdot\left[\tau^{-1-\delta}+\kappa\left(\hbar\frac{\lambda(\sigma)}{\lambda(\tau)}\xi^{\frac{1}{2}}\right)\right],$$

$$\left\|\xi^{k_2}\partial_\xi^{k_2}G^{(\pm)}(\tau,\sigma,x,\xi)\right\|_{L_x^1} \lesssim (\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right]\cdot\left[\tau^{-1-\delta}+\kappa\left(\hbar\frac{\lambda(\sigma)}{\lambda(\tau)}\xi^{\frac{1}{2}}\right)\right],\quad 0\leq k_2\leq 5,$$

$$\left\|\left\|\xi^{5+\delta_l}\partial_\xi^5 G^{(\pm)}(\tau,\sigma,x,\xi)\right\|_{\dot{C}_\xi^{\delta_*}(\xi\simeq\lambda)}\right\|_{L_x^1} \lesssim (\log\tau)^{N_1}\,\tau^{-1-\nu}\cdot\sigma^{-1}\cdot\left[\sigma^{-2}+\kappa\left(\hbar\lambda^{\frac{1}{2}}\right)\right]\cdot\left[\tau^{-1-\delta}+\kappa\left(\hbar\frac{\lambda(\sigma)}{\lambda(\tau)}\xi^{\frac{1}{2}}\right)\right].$$

Finally

$$\left\|\overline{x}_{3,smooth}(\tau,\cdot)\right\|_{S_1^\hbar} \lesssim \tau^{-4}.$$

We shall call $\overline{x}_{smooth}$ the regular or smooth part of the good function $\overline{x}(\tau,\xi)$, and similarly for good source function.
Finally, we shall say that $\overline{x}$ is a good function with restricted principal singular part, provided that $\overline{x}_{singular}$ has restricted singular part.

It is then natural to introduce a 'norm' on good functions, as well as source functions, as follows:

Definition 8.4. *Assume that $\overline{x}(\tau,\xi)$ is a good function at angular momentum $n, |n| \geq 2$, with the representation*

$$\overline{x} = \overline{x}_{smooth} + \overline{x}_{singular}$$

and the implied representation for $\overline{x}_{smooth}$. Then we set

$$\|\overline{x}\|_{good} := \|\overline{x}_{smooth}\|_{smooth} + \|\overline{x}_{singular}\|_{adm},$$

and we define

$$\begin{aligned}
&\|\overline{x}_{smooth}\|_{smooth} := \\
&\sum_{\substack{0\leq k_2\leq 5\\ \iota\in\{0,1\}}}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^{k_2}\partial_\xi^{k_2}\partial_\tau^\iota F^{(\pm)}(\tau,\sigma,\xi)}{\left(\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right)}\right\|_{L_\xi^\infty([0,\infty))}\right\|_{L_\tau^\infty([\tau_0,\infty))L_\sigma^\infty([\tau_0,\tau]}\\
&+\sum_{\substack{0\leq k_2\leq 5\\ \iota\in\{0,1\}}}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^{k_2}\partial_\xi^{k_2}\partial_\tau^\iota \tilde{F}^{(\pm)}(\tau,\sigma,\xi)}{\left(\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right)}\right\|_{L_\xi^\infty([0,\infty))}\right\|_{L_\tau^\infty([\tau_0,\infty))L_\sigma^\infty([\tau_0,\tau]}\\
&+\sum_{\iota}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^5\partial_\xi^5\partial_\tau^\iota F^{(\pm)}(\tau,\sigma,\xi)}{\left(\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right)}\right\|_{\dot{C}_\xi^{\delta_*}}\right\|_{L_\tau^\infty([\tau_0,\infty))L_\sigma^\infty([\tau_0,\tau]}\\
&+\sum_{\iota}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^5\partial_\xi^5\partial_\tau^\iota \tilde{F}^{(\pm)}(\tau,\sigma,\xi)}{\left(\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right)}\right\|_{\dot{C}_\xi^{\delta_*}}\right\|_{L_\tau^\infty([\tau_0,\infty))L_\sigma^\infty([\tau_0,\tau]}\\
&+\sum_{0\leq k_2\leq 5}\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^{k_2}\partial_\xi^{k_2}G^{(\pm)}(\tau,\sigma,x,\xi)}{\left(\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right)}\right\|_{L_\xi^\infty([0,\infty))}\right\|_{L_x^1([0,\infty)L_\tau^\infty([\tau_0,\infty))L_\sigma^\infty([\tau_0,\tau]}
\end{aligned}$$

$$+\left\|(\log\tau)^{-N_1}\,\tau^{1+\nu}\sigma\cdot\left\|\frac{\xi^5\partial_\xi^5\partial_\tau^\iota G^{(\pm)}(\tau,\sigma,x,\xi)}{\left(\sigma^{-2}+\kappa\left(\hbar\xi^{\frac{1}{2}}\right)\right)}\right\|_{\dot{C}_\xi^{\delta_*}}\right\|_{L_x^1L_\tau^\infty([\tau_0,\infty)L_\sigma^\infty([\tau_0,\tau]}$$
$$+\ldots,$$

where the term ... stands for the additional terms of a similar form which get included when applying $\partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi$ to the final term in the structure formula for $\overline{x}_{1,smooth}$.
Using obvious modifications we similarly define $\|x\|_{good\ source}$. Finally, we define the corresponding norms

$$\|\overline{x}\|_{good(r)},\quad \|\overline{x}\|_{good\ source(r)}$$

if the principal singular part is of restricted type, by replacing $\|\cdot\|_{adm}$ by $\|\cdot\|_{adm(r)}$, in accordance with Definition 7.9.

In perfect analogy to Proposition 8.1, we have the following

Proposition 8.5. *Let $\overline{y}(\tau,\xi)$ be a good source at angular momentum n, $|n|\geq 2$. Then*

$$\overline{x}(\tau,\xi) := \int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot\overline{y}\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma$$

is a good function. Moreover, we have the bound

$$\|\overline{x}\|_{good}\lesssim\|\overline{y}\|_{goodsource}$$

where the implied constant is uniform in n as well as $\tau_0\gg 1$.

Proof. In light of Proposition 8.1 and the earlier considerations on $S_1^{\hbar}$, it suffices to check this when the source term is of type $\overline{y}_{\text{smooth}}$, and here the argument follows exactly the same lines as the one for Prop. 8.2. □

We have the following analogue of Proposition 8.2:

Proposition 8.6. *Assume that $\overline{x}^{\hbar}(\tau,\xi)$ is the regular part of a good source function , and that $\mathcal{F}^{\hbar}(F_{\pm}(n))$ is an a regular good source term. Then referring to* (8.1)*,*

$$\int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot f_j\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,\quad j=1,2,\ldots,5,$$

is a regular good function, and we have the bound

$$\left\|\int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot f_j\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\right\|_{smooth}\lesssim\left\|\overline{x}^{\hbar}\right\|_{smooth},$$

with uniform implied constant. In conjunction with Prop. 8.2, we conclude that if $\overline{x}^{\hbar}(\tau,\xi)$ is good and $\mathcal{F}^{\hbar}(F_{\pm}(n))$ is a good source, then

$$\int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot f_j\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,\quad j=1,2,\ldots,5,$$

is a good function, and we have

$$\left\|\int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot f_j\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\right\|_{good} \lesssim \left\|\overline{x}^{\hbar}\right\|_{good}.$$

Proof. (Sketch) One uses Lemma 7.12 as well as Remark 7.13, which implies that applying the transference operator to a function of type $\overline{x}_{1,\text{smooth}}$ either reproduces a function of the same kind (in particular, preserving the Hölder property for its top order derivatives inherent in its definition), or it produces a less structured function but with better decay of type $\overline{x}_{2,\text{smooth}}$. □

8.2.3. *Solution of the Fourier wave equation* (8.1). Given the functional setup from the preceding, we can now detail the (re)-iterative scheme which leads to a solution of our Fourier wave equation, in analogy to [], [], []. As usual the main difficulty stems from the term

$$2\frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_{\hbar}^{(0)}\mathcal{D}_\tau\overline{x}^{\hbar},$$

since the temporal weight $\frac{\lambda'(\tau)}{\lambda(\tau)} \simeq \tau^{-1}$ only furnishes enough decay to recover any previous decay assumptions upon application of the wave parametrix, and in particular we cannot force a smallness gain by simply picking the initial time τ_0 large enough. This issue does not occur for the terms f_j, $j = 3,4,5$ in (8.1), as follows by the following simple sharpening of the preceding proposition:

Lemma 8.7. *Let $\overline{x}^{\hbar}(\tau,\xi)$ be a good function. Then*

$$\left\|\int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot f_j\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\right\|_{good} \lesssim \tau_0^{-1}\left\|\overline{x}^{\hbar}\right\|_{good}, \quad j=3,4,5.$$

In particular, given any $\gamma > 0$, there is $\tau_0 = \tau_0(\gamma)$ large enough such that

$$\left\|\int_{\tau_0}^{\tau} U^{(n)}(\tau,\sigma,\xi)\cdot f_j\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\right\|_{good} \leq \gamma\cdot\left\|\overline{x}^{\hbar}\right\|_{good}, \quad j=3,4,5.$$

The proof is of course a consequence of the one for Prop. 8.6, Prop. 8.2.

In order to cope with the 'bad term', we use the same method as in [3,15,16], namely manifold iteration, introduce the auxiliary composite operator

$$\Phi(f) := \int_{\tau_0}^{\tau}\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}\cdot\cos\left[\lambda(\tau)\xi^{\frac{1}{2}}\cdot\int_{\sigma}^{\tau}\lambda^{-1}(u)\,du\right]\cdot\beta_\nu(\sigma)\cdot\mathcal{K}_{\hbar}^{(0)}\left(\mathcal{D}_\tau f\right)\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma, \quad (8.4)$$

where we set $\beta_\nu(\sigma) = \frac{\lambda'(\sigma)}{\lambda(\sigma)} \simeq \sigma^{-1}$. Observe that this operator arises upon applying the operator $\mathcal{D}_\tau$ to the expression arising upon applying the parametrix to $\beta_\nu(\sigma)\cdot\mathcal{K}_{\hbar}^{(0)}\mathcal{D}_\tau f$. Then the following key proposition gives the desired smallness gain upon manifold application of Φ:

Proposition 8.8. *Let $\epsilon > 0$ sufficiently small. Then there is $\gamma_* > 0$ such that for any $k \geq 1$, there is $\tau_0 = \tau_0(\epsilon,k)$ large enough such that*

$$\left\|\beta_\nu(\tau)\mathcal{K}_{\hbar}^{(0)}\Phi^k f\right\|_{goodsource} \leq \epsilon^{\gamma k}\cdot e^{\epsilon^{-2}}\cdot\|f\|_{good}.$$

Proof. (sketch) The idea is to decompose the transference operator $\mathcal{K}_{\hbar}^{(0)}$ into a number of pieces, and most pivotally into a 'diagonal' and 'off diagonal' piece

$$\mathcal{K}_{\hbar}^{(0)} = \mathcal{K}_{\hbar,d}^{(0)} + \mathcal{K}_{\hbar,nd}^{(0)},$$

which arise as follows: fixing the $\epsilon > 0$, k, we pick ℓ sufficiently large (specifically $\ell \simeq k$) and define $\mathcal{K}_{\hbar,d}^{(0)}$ by including a smooth cutoff $\chi_{|1-\frac{\xi}{\eta}|\lesssim\frac{1}{\ell}}$ into the kernel of $\mathcal{K}_{\hbar}^{(0)}$, whence the off-diagonal part $\mathcal{K}_{\hbar,nd}^{(0)}$ is defined by including a cutoff $\chi_{|1-\frac{\xi}{\eta}|\gtrsim\frac{1}{\ell}}$. Call $\Phi^{(d)}$, $\Phi^{(nd)}$ the operators arising by these changes. Then the idea is to reduce things to the diagonal part, by observing that in any strings of Φs the presence of a single $\mathcal{K}_{\hbar,nd}^{(0)}$ results in a smallness gain by picking τ_0 large. This follows from the following

Lemma 8.9. *We have the bound*

$$\left\|\beta_\nu(\tau)\mathcal{K}_{\hbar}^{(0)}\Phi^{(nd)}\beta_\nu(\tau)\Phi f\right\|_{goodsource} \lesssim \tau_0^{-\gamma_1}\|f\|_{good}\,.$$

for some $\gamma_1 > 0$, where the implied constant is independent of the angular momentum n and the time τ_0, but will depend on ℓ.

Proof. (lemma, sketch) The idea is to perform integration by parts in the time variable for $\Phi^{(nd)}$. Specifically, write

$$\begin{aligned}&\left(\Phi^{(nd)}\beta_\nu(\tau)\Phi\right)(g)\\&= c\sum_{\pm,\pm}\int_{\tau_0}^{\tau}\tilde{\rho}_n(\tau,\sigma,\xi)\cdot e^{\pm i\left(\nu\tau-\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\right)\xi^{\frac{1}{2}}}\cdot\int_0^\infty\frac{F_n^{(nd)}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi,\eta\right)\cdot\rho_n(\eta)}{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi-\eta}\\&\qquad\cdot\int_{\tau_0}^{\sigma}\tilde{\rho}_n(\sigma,\sigma_1,\eta)\cdot e^{\pm i\left(\nu\sigma-\nu\sigma_1\frac{\lambda(\sigma)}{\lambda(\sigma_1)}\right)\eta^{\frac{1}{2}}}\cdot\tilde{g}\left(\sigma_1,\frac{\lambda^2(\sigma)}{\lambda^2(\sigma_1)}\eta\right)d\sigma_1\,d\eta\,d\sigma\end{aligned}$$

where c is a suitable constant, and we have set $\tilde{g} = \beta_\nu(\tau)\mathcal{K}_{\hbar}^{(0)}g$, $\tilde{\rho}_n(\tau,\sigma,\xi) = \beta_\nu(\sigma)\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\frac{\rho_n^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)}{\rho_n^{\frac{1}{2}}(\xi)}$. Also, $F^{(nd)}$ means inclusion of the cutoff $\chi_{\left|1-\frac{\xi}{\eta}\right|\gtrsim\frac{1}{\ell}}$, which means here inclusion of $\chi_{\left|1-\frac{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi}{\eta}\right|\gtrsim\frac{1}{\ell}}$ due to the change of scale. To begin with, we note that if we include a further smooth cutoff $\chi_{|\sigma-\sigma_1|\lesssim\tau_0^{-\frac{1}{2}}\sigma}$ in the second time integral, we can easily force smallness due to the restriction of the integration interval. This means we can include a smooth cutoff $\chi_{|\sigma-\sigma_1|\gtrsim\tau_0^{-\frac{1}{2}}\sigma}$. Furthermore, due to the asymptotic bounds for

$$F_n(\xi,\eta),$$

we may also include a cutoff $\chi_{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\gtrsim1}$, since else smallness can again be forced easily. The extra smallness gain also occurs if $\frac{\lambda(\tau)^2}{\lambda(\sigma)^2}\xi \gg \eta$ or $\frac{\lambda(\tau)^2}{\lambda(\sigma)^2}\xi \ll \eta$, in view of the bounds in Proposition 5.1. Therefore without loss of generality, we assume $\frac{\lambda(\tau)^2}{\lambda(\sigma)^2}\xi \simeq \eta$.

Switching to the new variable $\tilde{\eta}$ defined via

$$\eta = \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\tilde{\eta},$$

we replace the function $\tilde{g}\left(\sigma_1, \frac{\lambda^2(\sigma)}{\lambda^2(\sigma_1)}\eta\right)$ by $\tilde{g}\left(\sigma_1, \frac{\lambda^2(\tau)}{\lambda^2(\sigma_1)}\tilde{\eta}\right)$, and the integration kernel

$$\frac{F_n\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi, \eta\right)\cdot\rho_n(\eta)}{\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi - \eta}$$

is replaced by the more symmetrical

$$\frac{F_n\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\tilde{\eta}\right)\cdot\rho_n\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\tilde{\eta}\right)}{\xi - \tilde{\eta}}.$$

Finally, the oscillatory phase becomes

$$e^{\pm i\left(\nu\tau - \nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\right)\xi^{\frac{1}{2}}}\cdot e^{\pm i\left(\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)} - \nu\sigma_1\frac{\lambda(\tau)}{\lambda(\sigma_1)}\right)\tilde{\eta}^{\frac{1}{2}}} = e^{\pm i\nu\tau\xi^{\frac{1}{2}} \mp i\nu\sigma_1\frac{\lambda(\tau)}{\lambda(\sigma_1)}\tilde{\eta}^{\frac{1}{2}}}\cdot e^{i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\left(\mp\xi^{\frac{1}{2}} \pm \tilde{\eta}^{\frac{1}{2}}\right)}$$

For later reference, we observe that in case of anti-alignment of the pass, we get the σ-dependent phase $e^{\pm i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\cdot\left(\xi^{\frac{1}{2}} + \tilde{\eta}^{\frac{1}{2}}\right)}$, in which case we needn't even take advantage of the off-diagonal condition. In the worst case of destructive resonance, we get the σ-dependent phase $e^{\pm i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\left(\xi^{\frac{1}{2}} - \tilde{\eta}^{\frac{1}{2}}\right)}$, where our assumption implies that $\left|\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}} - \frac{\lambda(\tau)}{\lambda(\sigma)}\tilde{\eta}^{\frac{1}{2}}\right| \gtrsim \frac{1}{\ell}\cdot\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}$. Then we change the order of temporal integration in the above triple integral, so that the σ-integral comes first, and perform integration by parts with respect to σ. We have

$$\partial_\sigma\left(e^{\pm i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\left(\xi^{\frac{1}{2}} - \tilde{\eta}^{\frac{1}{2}}\right)}\right) = \mp i\frac{\lambda(\tau)}{\lambda(\sigma)}\left(\xi^{\frac{1}{2}} - \tilde{\eta}^{\frac{1}{2}}\right)\cdot e^{\pm i\nu\sigma\frac{\lambda(\tau)}{\lambda(\sigma)}\left(\xi^{\frac{1}{2}} - \tilde{\eta}^{\frac{1}{2}}\right)}$$

This produces an additional factor

$$\sim \ell\cdot\frac{1}{\sigma}\cdot\frac{1}{\frac{\lambda(\tau)}{\lambda(\sigma)}\xi^{\frac{1}{2}}} \lesssim \frac{1}{\sigma},$$

where we have taken into account the additional localization $\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi \gtrsim 1$, the derivative bounds of the transference kernel $F_n(\xi, \eta)$, as well as the fact that the additional time cutoff above prevents boundary terms. The smallness gain follows since $\sigma \geq \tau_0$, and the fact that one can place the resulting function in the good source space of functions by re-arranging the phases as before and invoking Prop. 8.6. □

Note that by the preceding proof, we may assume for the proof of Prop. 8.8 that the oscillatory phases (due to the factors $\cos\left[\lambda(\tau)\xi^{\frac{1}{2}}\int_\sigma^\tau \lambda^{-1}(u)\,du\right]$, written as a sum of exponentials), all have the same sign, since otherwise, two adjacent opposite signs allow us to utilize the gain from the preceding lemma to gain smallness. This implies (following the proof of Prop. 8.1) that starting with one of the three expressions in Definition 8.3, or alternatively one of the expressions in $\overline{x}_{in}, \overline{x}_{out}$ in Definition 7.8, we always reproduce the same kind of expression, with coefficient functions which are given by suitable iterated integrals. Moreover, the preceding lemma allows us to replace Φ by $\Phi^{(d)}$ throughout.

One then follows the proof of Prop. 11.2 in [16] to reduce the problem to integration over simplices which forces smallness via a combinatorial argument. □

The preceding finally entails the desired

Proposition 8.10. *There exists τ_0 large enough, independently of $\hbar$, such that if $\mathcal{F}^{\hbar}(F_{\pm}(n))$ is a good source function, then the problem* (8.1) *admits a solution on* $[\tau_0, \infty)$ *with vanishing data*

$$\left(\overline{x}^{\hbar}(\tau_0, \cdot),\ \mathcal{D}_\tau \overline{x}^{\hbar}(\tau_0, \cdot)\right) = (0, 0)$$

at time $\tau = \tau_0$, and such that

$$\left\|\overline{x}^{\hbar}\right\|_{good} \lesssim \left\|\mathcal{F}^{\hbar}(F_{\pm}(n))\right\|_{goodsource}.$$

The implied constant is uniform in $\hbar$.

Proof. Pick $\epsilon = \frac{1}{2}$ and choose k large enough and then τ_0 large enough such that $\epsilon^{\gamma k} \cdot e^{\epsilon^{-2}} < \frac{1}{2}$. Then set up the iterative scheme

$$-\left(\mathcal{D}_\tau^2 + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{D}_\tau + \xi\right)\overline{x}_{l+1}^{\hbar}$$
$$= \mathcal{F}^{\hbar}\left(F_{\pm}(n)\right) + 2\frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_{\hbar}^{(0)}\mathcal{D}_\tau \overline{x}_l^{\hbar} + \left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)'\mathcal{K}_{\hbar}^{(0)}\overline{x}_l^{\hbar} + \frac{\lambda'(\tau)}{\lambda(\tau)}\left[\mathcal{D}_\tau, \mathcal{K}_{\hbar}^{(0)}\right]\overline{x}_l^{\hbar} + \left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)^2\left(\left(\mathcal{K}_{\hbar}^{(0)}\right)^2 + \mathcal{K}_{\hbar}^{(0)}\right)\overline{x}_l^{\hbar}, \quad l \geq 0,$$

where $\overline{x}_0^{\hbar}$ solves the inhomogeneous problem

$$-\left(\mathcal{D}_\tau^2 + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{D}_\tau + \xi\right)\overline{x}_0^{\hbar} = \mathcal{F}^{\hbar}\left(F_{\pm}(n)\right)$$

with vanishing initial data throughout. Then writing explicitly the iterated Duhamel for $\overline{x}_k^{\hbar}$, one gains smallness for all arising expressions upon choosing τ_0 larger, if necessary, except for the expression $\Phi^k \mathcal{D}_\tau \overline{x}_0^{\hbar}$, for which smallness follows from the previous proposition. □

8.3. **The final estimates for all the source terms near the light cone in the case $|n| \geq 2$.** In order to finally wrap things up for the $|n| \geq 2$ modes, we of course also need to define the concept of a good (Fourier representation) function for the exceptional modes, and here we also need to take the coefficients $c_j(\tau)$ of the instabilities $\phi_j(R)$ in mind, recalling (6.21), (6.23), (6.25). Recalling from Proposition 6.14, Proposition 6.17 that we have already defined norms incorporating both the coefficients c_n and the Fourier coefficients of $\mathcal{D}_j f$, we generalize things naturally as follows:

Definition 8.11. *Let $n \in \{0, \pm 1\}$, and assume that the angular momentum n function $\phi(\tau, R)$ admits the representation*

$$\phi(\tau, R) = c_n(\tau)\phi_n(R) + \phi_n(R) \cdot \int_0^R \left[\phi_n(s)\right]^{-1} \mathcal{D}_n\phi(\tau, s)\, ds, \quad \mathcal{D}_n\phi(\tau, R) = \int_0^\infty \overline{x}(\tau, \xi)\phi_n(R, \xi)\tilde{\rho}_n(\xi)\, d\xi$$

Then we say that the function ϕ is a good function, or alternatively the pair $(c_n(\tau), \overline{x}(\tau, \xi))$ is good, provided

$$|c_n(\tau)| + \tau \cdot \left|c_n'(\tau)\right| \lesssim \tau^{-2+10\nu},$$

and the function $\overline{x}(\tau, \xi)$ admits the representation

$$\overline{x}(\tau, \xi) = \overline{x}_{smooth} + \overline{x}_{singular},$$

where we have

$$\|\overline{x}_{smooth}\|_{S_0^{(n)}} + \|\mathcal{D}_\tau \overline{x}_{smooth}\|_{S_1^{(n)}} \lesssim \tau^{-3+10\nu},$$

We then introduce the corresponding 'good norm' for the pair of functions $(c_n(\tau), x(\tau, \xi))$ *by*

$$\begin{aligned}\|(c_n(\tau), \overline{x}(\tau, \xi))\|_{good} := &\left\|\tau^{2-10\nu} \cdot \left[|c_n(\tau)| + \tau \cdot \left|c_n'(\tau)\right|\right]\right\|_{L^\infty_\tau([\tau_0,\infty)}\\ &+ \left\|\tau^{3-10\nu} \cdot \left[\|\overline{x}_{smooth}(\tau, \cdot)\|_{S_0^{(n)}} + \|\mathcal{D}_\tau \overline{x}_{smooth}(\tau, \cdot)\|_{S_1^{(n)}}\right]\right\|_{L^\infty_\tau([\tau_0,\infty)}\\ &+ \left\|\overline{x}_{singular}\right\|_{adm},\end{aligned}$$

where we refer to Definition 7.36 for the last norm on the right. Finally, for a function $\overline{y}(\tau, \xi)$*, we shall say that* $\overline{y}$ *is a good angular momentum* $n \in \{0, \pm 1\}$ *source function, provided*

$$\left(0, \tau^{1+\delta} \xi^{-\frac{1}{2}} \overline{y}\right)$$

is good. We also have the natural analogues of the concept of 'restricted principal singular part' with correspondingly modified norms.

Combining Definition 8.11, Definition 8.3, we finally have the tools that are sufficiently precise to derive the multilinear estimates to handle all the source terms arising for the $|n| \geq 2$ modes. Specifically, we strive to obtain analogues to Prop. 6.23, Prop. 6.27, Prop. 6.28, but here we shall have to refer to the more sophisticated functional framework developed in the preceding. Recall the formulae

$$\varphi_1 = \frac{1}{2}[\varepsilon_+ + \varepsilon_-], \quad \varphi_2 = \frac{1}{2i}[\varepsilon_- - \varepsilon_+].$$

as well as the decompositions

$$\varepsilon_+ = \sum_{n\in\mathbb{Z}} \varepsilon_+(n) e^{in\theta}, \quad \varepsilon_- = \sum_{n\in\mathbb{Z}} \varepsilon_-(n) e^{in\theta}.$$

and where we have $\varepsilon_-(-n) = \overline{\varepsilon_+(n)}$ since the solutions we consider are real valued. We shall assume that for $|n| \geq 2$ the function $\varepsilon_+(n)$ is a good angular momentum n function in the sense that

$$\varepsilon_+(n)(\tau, R) = \int_0^\infty \phi_n(R, \xi) \overline{x}_n(\tau, \xi) \rho_n(\xi)\, d\xi,$$

where $\overline{x}_n(\tau, \xi)$ is a good angular momentum n function in the sense of Definition 8.3. Similarly, if $n \in \{0, \pm 1\}$, we assume that

$$\varepsilon_+(n)(\tau, R) = c_n(\tau)\phi_n(R) + \phi_n(R) \cdot \int_0^R [\phi_n(s)]^{-1} \cdot \mathcal{D}_n \epsilon_+(n)(\tau, s)\, ds,$$

where

$$\mathcal{D}_n \varepsilon_+(n)(\tau, R) = \int_0^\infty \phi_n(R, \xi) \overline{x}_n(\tau, \xi) \tilde{\rho}_n(\xi)\, d\xi,$$

and the pair $(c_n(\tau),\ \overline{x}_n(\tau, \xi))$ is good in the sense of Definition 8.11. Finally, set

$$\Lambda := \sum_{n\in\mathbb{Z},\, |n|\geq 2} \langle n\rangle^{12} \|\overline{x}_n\|_{\text{good}} + \sum_{n\in\{0,\pm 1\}} \|(c_n(\tau),\ \overline{x}_n(\tau, \xi))\|_{\text{good}}. \tag{8.5}$$

We have the following

Proposition 8.12. *Assume that $\Lambda \ll 1$. Then for each $n \in \mathbb{Z}, |n| \geq 2$, $j = 1, \ldots,$ there exist angular momentum n functions*

$$\psi_j^{\pm}(n) = \int_0^\infty \overline{x}_j^{(n)}(\tau, \xi)\phi_n(R, \xi)\rho_n(\xi)\, d\xi,$$

with

$$\sum_{|n|\geq 2} n^{12} \left\|\overline{x}_j^{(n)}\right\|_{good} \lesssim \Lambda^{-3},$$

and such that if $F_j^{(\pm)}$ represents any one of the functions occurring in Prop. 6.23, Prop. 6.27, Prop. 6.28, and we write

$$F_j^{(\pm)} = \sum_{n\in\mathbb{Z}} F_j^{(\pm)}(n) e^{in\theta},$$

then we have for each $|n| \geq 2$

$$\left(F_j^{(\pm)}(n) - \Box_n \psi_j^{\pm}(n)\right)|_{R<\nu\tau} = G_j^{(\pm)}(n)|_{R<\nu\tau},$$

where $G_j^{(\pm)}(n)$ is a good angular momentum n source function and more quantitatively setting

$$G_j^{(\pm)}(n) = \int_0^\infty \phi_n(R, \xi)\overline{y}_n(\tau, \xi)\rho_n(\xi)\, d\xi,$$

we have

$$\sum_{|n|\geq 2} n^{12} \left\|\overline{y}_n(\tau, \xi)\right\|_{goodsource} \lesssim \left(\tau_0^{-1} + \Lambda\right) \cdot \Lambda \ll \Lambda.$$

If we restrict to functions $\overline{x}_n$ with restricted principal singular part, and correspondingly use $\|\overline{x}_n\|_{good(r)}$, then $\overline{y}_n$ also has restricted singular principal part, and we may replace $\left\|\overline{y}_n(\tau, \xi)\right\|_{goodsource}$ by $\left\|\overline{y}_n(\tau, \xi)\right\|_{goodsource(r)}$.

9. The exceptional angular momenta: $n \in \{0, \pm 1\}$ and modulation theory

9.1. The equations for the modes $n = 0, \pm 1$.

9.1.1. *The angular mode $n = 1$, equations on the physical side.* Recall the equation

$$-\left(\left(\partial_\tau + \frac{\lambda'}{\lambda} R\partial_R\right)^2 + \frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda} R\partial_R\right)\right)\varepsilon_+^1 + H_1^+\varepsilon_+^1 = F_+(1) \tag{9.1}$$

where we may as well specialize to the +-case by conjugation symmetry. We also recall the representation

$$\varepsilon_+^1 = \phi_+\left(\mathcal{D}_+\varepsilon_+^1\right) + c_+(\tau)\phi_1(R).$$

where we have $\mathcal{D}_+ = \partial_R + \frac{2R}{1+R^2}$, as well as

$$\phi_1(R) := \frac{1}{1+R^2}, \quad \phi_+(g) := \phi_1(R)\int_0^R (\phi_1(s))^{-1}\, g(s)\, ds.$$

From (9.1) we infer two equations, one for the 'better variable' $\mathcal{D}_+\varepsilon^1_+$ by applying $\mathcal{D}_+$ to it, and an ODE for $c_+(\tau)$ by analyzing the vanishing behavior of the original variable ε^1_+ at $R = 0$. To begin with, straightforward differentiation of (9.1) and computation of some commutators leads to the following equation for $\mathcal{D}_+\varepsilon^1_+$:

$$\begin{aligned}
&-\left(\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)^2 + 3\frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\right)\mathcal{D}_+\varepsilon^1_+ + \tilde{H}^+_1\mathcal{D}_+\varepsilon^1_+ - \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}_+\varepsilon^1_+ \\
=&-2\frac{\lambda'}{\lambda}\frac{4R}{(1+R^2)^2}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\varepsilon^1_+ + \left(\frac{\lambda'}{\lambda}\right)^2\left(\frac{4R}{(1+R^2)^2} - \frac{16R}{(1+R^2)^3}\right)\varepsilon^1_+ \\
&-\frac{4R}{(1+R^2)^2}\left(\frac{\lambda'}{\lambda}\right)'\varepsilon^1_+ + \mathcal{D}_+\left(F_+(1)\right) \\
=:&\mathcal{R}_+(\varepsilon^1_+, \mathcal{D}_+\varepsilon^1_+) + \mathcal{D}_+\left(F_+(1)\right).
\end{aligned} \tag{9.2}$$

Here we recall that the 'super-symmetrical' operator $\tilde{H}^+_1 = -\mathcal{D}_+\mathcal{D}^*_+$, i. e. the factors have been switched compared to H^+_1. The preceding equation gets complemented by one for $c_+(\tau)$, obtained by analyzing the first equation (9.1) at $R = 0$, and tracking the value of both sides at $R = 0$:

$$-\partial^2_\tau c_+ - \frac{\lambda_\tau}{\lambda}\partial_\tau c_+ + \lim_{R\to 0} H^+_1\varepsilon^1_+ = \lim_{R\to 0} F_+(1) \tag{9.3}$$

At this point it is crucial to observe that while the source term $F_+(1)$ does have components which depend linearly on $c(\tau)$ and are otherwise independent on the perturbation, namely those terms involving interactions of $\varepsilon_+(1)$ and ϵ, the latter quantity referring to the original blow up profile $Q(\lambda(t)r) + \epsilon(t,r) = U(t,r)$. These terms are explicitly given by

$$\begin{aligned}
&\frac{2\sin(2Q+\epsilon)\sin\epsilon}{R^2}\varepsilon^1_+, \quad \frac{4\sin\left(Q+\frac{\epsilon}{2}\right)\sin\left(\frac{\epsilon}{2}\right)}{R^2}i\varepsilon^1_+, \\
&\left(\frac{2\partial_R\epsilon}{1+R^2} + (\partial_R\epsilon)^2 - \frac{\lambda'}{\lambda}\cdot\frac{8R}{1+R^2}\left(\partial_\tau\epsilon + \frac{\lambda'}{\lambda}R\partial_R\epsilon\right) - \left(\partial_\tau\epsilon + \frac{\lambda'}{\lambda}R\partial_R\epsilon\right)^2\right)i\varepsilon^1_+
\end{aligned}$$

Since by their very construction the function ϵ vanishes to third order at $R = 0$, it is seen that each of these terms vanishes at the origin. This implies that the operator

$$\partial^2_\tau + \frac{\lambda_\tau}{\lambda}\partial_\tau$$

is responsible to leading order for the evolution of c_+. We observe that a fundamental system for this operator consists of the functions 1, $\tau^{-\nu^{-1}}$. The presence of the function 1 here distinguishes the mode $n = +1$ from the modes $n = 0, n = -1$ treated below, and for which the fundamental system consists of two rapidly decaying functions. Our way to deal with this will invoke modulation theory, by applying a carefully chosen rotation to the bulk part. For now, we relegate this issue for later, and deal with the challenges occurring in the multilinear estimates as well as the iterative step, after translating things to the Fourier side, and making the hypothesis that $c_+(\tau)$ does decay sufficiently toward $\tau = +\infty$, in accordance with the setup in subsection 7.8. Forcing this decay assumption will be the role of the final modulation step.

For later reference, we also observe a solution of the inhomogeneous problem associated to the above differential operator, namely

$$\partial_\tau^2 c + \frac{\lambda_\tau}{\lambda}\partial_\tau c = h$$

is solved by

$$c(\tau) = \nu\int_{\tau_0}^{\tau}\sigma h(\sigma)\,d\sigma - \nu\tau^{-\nu^{-1}}\int_{\tau_0}^{\tau}\sigma^{1+\nu^{-1}}h(\sigma)\,d\sigma. \tag{9.4}$$

9.1.2. *Translation of the equation* (9.2) *to the Fourier side.* The case $n = +1$ plays a somewhat special role here as well, since the spectral measure for the operator $\tilde{H}_1^+$ is particularly simple, namely a constant multiple of ξ, which simplifies both the dilation operator $\mathcal{D}_{\tau,1}$ as well as the formula for the wave parametrix. Even more importantly, the transference operator completely vanishes here, due to the extremely special structure of the Fourier basis. Specifically, we find that

$$\mathcal{D}_{\tau,1} := \partial_\tau - 2\frac{\lambda'}{\lambda}\xi\partial_\xi - 2\frac{\lambda'}{\lambda}. \tag{9.5}$$

Then we translate (9.2) into

$$-\left(\mathcal{D}_{\tau,1}^2 + \frac{\lambda'}{\lambda}\mathcal{D}_{\tau,1} + \xi\right)\overline{x}^1 = \mathcal{F}^{(1)}\left(\mathcal{R}_+\left(\varepsilon_+^1, \mathcal{D}_+\varepsilon_+^1\right)\right) + \mathcal{F}^{(1)}\left(\mathcal{D}_+\left(F_+(\varepsilon)(1)\right)\right). \tag{9.6}$$

For the solution of the homogeneous wave equation corresponding to the operator on the left hand side, we have the following

Lemma 9.1. *The homogeneous initial value problem*

$$\left(\mathcal{D}_{\tau,1} + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{D}_{\tau,1} + \xi\right)\overline{x}(\tau,\xi) = 0; \quad \overline{x}(\tau_0,\xi) = \overline{x}_0(\xi), \quad \mathcal{D}_{\tau,1}\overline{x}(\tau_0,\xi) = \overline{x}_1(\xi), \tag{9.7}$$

is solved by

$$\begin{aligned}\overline{x}(\tau,\xi) =& \frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\cos\left(\lambda(\tau)\xi^{\frac12}\int_{\tau_0}^{\tau}\lambda(u)^{-1}du\right)\overline{x}_0\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right)\\ &+ \xi^{-\frac12}\frac{\lambda(\tau)}{\lambda(\tau_0)}\sin\left(\lambda(\tau)\xi^{\frac12}\int_{\tau_0}^{\tau}\lambda(u)^{-1}du\right)\overline{x}_1\left(\frac{\lambda(\tau)^2}{\lambda(\tau_0)^2}\xi\right).\end{aligned} \tag{9.8}$$

This implies the fundamental $S^{(+1)}$-space propagation bounds

$$\left\|\overline{x}^1(\tau,\xi)\right\|_{S_0^{(1)}} + \left\|\mathcal{D}_{\tau,1}\overline{x}^1(\tau,\xi)\right\|_{S_1^{(1)}} \lesssim \left(\frac{\lambda(\tau_0)}{\lambda(\tau)}\right)^{1-\delta}\cdot\left[\|\overline{x}_0\|_{S_0^{(1)}} + \|\overline{x}_1\|_{S_1^{(1)}}\right].$$

In particular, choosing ν small enough, this quantity decays faster than any prescribed negative power of τ.

On the other hand, for the inhomogeneous problem, we have a direct analogue of Prop. 8.5:

Proposition 9.2. *Denoting by $U^{(1)}(\tau,\sigma,\xi)$ Duhamel propagator for the inhomogeneous problem associated to $\mathcal{D}_{\tau,1}^2 + \frac{\lambda'}{\lambda}\mathcal{D}_{\tau,1} + \xi$, we have*

$$\left\|\int_{\tau_0}^{\tau}U^{(1)}(\tau,\sigma,\xi)\cdot\overline{y}\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\right\|_{good} \lesssim \|\overline{y}\|_{goodsource},$$

where the norms are of course in the sense of angular momentum $n = 1$ functions. If $\overline{y}$ has principal singular part of restricted type, so does the left hand side, and the norms can be adjusted accordingly.

We next analyze more closely the equation for the coefficient $c_+(\tau)$ of the instability, and specifically the delicate source term

$$\lim_{R\to 0} H_1^+ \varepsilon_+^1 \tag{9.9}$$

there. Observe that this term is expected to decay at most like $\mathcal{D}_+\varepsilon_+^1$ which means solving the ordinary differential equation (9.3) will lead to a loss of two additional powers of τ-decay relative to the decay of $\mathcal{D}_+\varepsilon_+^1$.

To begin with, we render explicit the precise structure of the Fourier transform of the source term $\mathcal{D}_+\left(F_+(1)\right)$ as far as the top order singular terms are concerned, in order to prepare the modulation step. This proposition takes advantage of the fact that all bilinear null-form expressions in the non-linearity take real values, and also of the fact that the structure simplifies since high angular momentum terms which contribute to low angular momentum expressions have to come in pairs.

Proposition 9.3. *Denote by $\mathcal{F}^{(1)}$ the distorted Fourier transform at angular momentum $n = 1$ in the sense of subsection 6.5. Assume that we have* (6.54) *and define Λ as before, with $\Lambda \ll 1$. Finally, assume that the distorted Fourier transforms of the $\varepsilon_\pm(n)$ (both for the $|n| \geq 2$ and the exceptional modes) have restricted principal singular type (and in particular, we have to define Λ by using $\|\cdot\|_{good(r)}$). Then we can write*

$$\mathcal{F}^{(1)}\left(\mathcal{D}_+\left(F_+(1)\right)(\sigma,\cdot)\right)(\xi) = \sum_{\pm}\sum_{k=1,2}\sum_{i+j\leq N_1} C_{i,j}\frac{e^{\pm i\nu\sigma\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}+\frac{k\nu}{2}}}\left(\log\xi\right)^i\cdot\gamma^{(k,j,\pm)}\cdot\beta^{(k,i+j)}(\sigma) + \overline{y}(\sigma,\xi), \tag{9.10}$$

where $\overline{y}(\tau,\xi)$ is a good source term at angular momentum $n = 1$ and such that all terms with $k \in \{1,2\}, l = 0$ in the expansion of its singular part (according to Definition 7.36) vanish. The coefficients $\gamma^{(k,i,\pm)}$ are given by the formulae

$$\gamma^{(k,j,\pm)} = \int_0^\infty e^{\pm ix}\cdot(\log x)^j\cdot x^{-\frac{1}{2}+k\nu}\,dx$$

for certain real coefficients $C_{i,j}$

Finally, we have the bound

$$\left|\beta^{(k,i)}(\sigma)\right| \lesssim \sigma^{-\frac{7}{2}-3\nu}\left(\log\sigma\right)^{3N_1}\Lambda^2,$$

and the good source $\overline{y}(\tau,\xi)$ can be decomposed into two parts

$$\overline{y}(\tau,\xi) = \overline{y}_1(\tau,\xi) + \overline{y}_2(\tau,\xi)$$

such that

$$\left|\overline{y}_1(\sigma,\xi)\right| \lesssim \sigma^{-\frac{7}{2}-3\nu}\left(\log\sigma\right)^{3N_1}\Lambda^2, \quad \left\|\overline{y}_2(\sigma,\xi)\right\|_{goodsource} \lesssim \sigma^{-5}\Lambda^2.$$

Remark 9.4. *This proposition gives one precise obstruction to obtaining the desired decay for the $c_+(\tau)$-coefficient, as far as the contributions from the source are concerned. In fact, it is precisely the first term on the right in* (9.10) *which a priori leads to a poorly decaying contribution to $c_+(\tau)$, upon application of the wave parametrix stated in Lemma 9.1, as will become clear from the next lemma. However, this obstruction lives in a finite dimensional vector space, and suitable choice of modulation parameters below will allow us to eliminate it. We remark that the general setup we have used thus far can be rendered completely explicit for the blow up solutions of [18].*

To continue, we now observe the following simple identity which expresses the delicate source term (9.9) via the Fourier transform: setting

$$\mathcal{D}_+\varepsilon_+^1(\tau,R) = \int_0^\infty \phi_1(R,\xi)\cdot\overline{x}(\tau,\xi)\cdot\tilde{\rho}_1(\xi)\,d\xi,$$

we infer that (see (4.16) and (4.28))

$$-\lim_{R\to 0}\mathcal{D}_+^*\mathcal{D}_+\varepsilon_+^1 = -\frac{\pi}{2}\cdot\int_0^\infty \overline{x}(\tau,\xi)\cdot\tilde{\rho}_1(\xi)\,d\xi. \tag{9.11}$$

Let us apply the parametrix from Lemma 9.1 to the term (9.10), which is of course only the first approximation to the true Fourier transform of $\mathcal{D}_+\varepsilon_+^1$, into the preceding integral expression. We first make the following basic observation:

Lemma 9.5. *Denote by $U^{(1)}(\tau,\sigma,\xi)$ the Duhamel propagator according to Lemma 9.1. Then setting*

$$\overline{x}(\tau,\xi) = \int_{\tau_0}^{\tau} U^{(1)}(\tau,\sigma,\xi)\cdot\mathcal{F}^{(1)}\left(\mathcal{D}_+\left(F_+(1)\right)(\sigma,\cdot)\right)\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,$$

we have (for an arbitrary large but finite number M and a smooth cutoff $\chi_{\xi<M}$)

$$\begin{aligned}&\int_0^\infty \chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_1(\xi)\,d\xi\\ &=\sum_{\pm}\sum_{k=1,2}\sum_{0\le j_1\le N_1}\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{\frac{k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}\,d\xi\\ &\qquad\cdot\sum_{j_2+i\le N_1-j_1}C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\beta^{(k,j_1+j_2+i)}(\sigma)\,d\sigma\\ &+\tilde{c}_1(\tau),\end{aligned}$$

where the error satisfies

$$|\tilde{c}_1(\tau)|\lesssim\tau^{-4-}.$$

Note that the first integral can be written as (omitting certain constant coefficients)

$$\begin{aligned}\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{\frac{k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}\,d\xi \simeq&\,\partial_\tau\left(\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{(\pm i\nu)\cdot\xi^{\frac{1+k\nu}{2}}}[\log(\xi\lambda^2(\tau))]^{j_1}\,d\xi\right)\\ &-\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{(\pm i\nu)\,\xi^{\frac{1+k\nu}{2}}}\left(\log\left(\lambda(\tau)^2\xi\right)\right)^{j_1-1}\tau^{-1}\,d\xi\end{aligned}$$

where, upon change of variable $\xi\mapsto\tau^2\xi$ and integration by parts, we have the bound (uniformly in M)

$$\left|\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{(\pm i\nu)\cdot\xi^{\frac{1+k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}\,d\xi\right|\lesssim\tau^{-(1-k\nu)}\left(\log\tau\right)^{N_1}.$$

It follows that if we had for each $j_1 \in \{0, \dots, N_1\}$ the vanishing relations

$$\sum_{j_2+i\le N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)} \int_{\tau_0}^{\infty} (\log\sigma)^{j_2}\, [\lambda(\sigma)]^{\nu k}\, \beta^{(k,j_1+j_2+i)}(\sigma)\, d\sigma = 0, \tag{9.12}$$

we could conclude in light of Proposition 9.3 that

$$\int_0^\infty \chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_1(\xi)\, d\xi = \partial_\tau\gamma + \tilde{c}_+(\tau),$$

where $|\gamma(\tau)| \lesssim \tau^{-\frac{7}{2}-}$ uniformly in M, which would be enough to recover (more than) the required bound for c_+ (upon passing to the limit $M \to \infty$), up to a constant term[9] which will be dealt with by modulation theory as well. In fact, in light of (9.4), we get with $h(\tau) = \partial_\tau\gamma$ after integration by parts

$$\begin{aligned}\nu\int_{\tau_0}^{\tau}\sigma h(\sigma)\, d\sigma - \nu\tau^{-\nu^{-1}}\int_{\tau_0}^{\tau}\sigma^{1+\nu^{-1}}h(\sigma)\, d\sigma &= \nu\tau\gamma(\tau) - \left(\nu\tau_0\gamma(\tau_0) + \nu\int_{\tau_0}^{\infty}\gamma(\sigma)\, d\sigma\right)\\ &\quad + \nu\int_{\tau}^{\infty}\gamma(\sigma)\, d\sigma\\ &\quad + \tau^{-\nu^{-1}}\int_{\tau_0}^{\tau}(\nu+1)\,\sigma^{\nu^{-1}}\gamma(\sigma)\, d\sigma\\ &\quad + \nu\tau_0^{1+\nu^{-1}}\tau^{-\nu^{-1}}\gamma(\tau_0) - \tau\gamma(\tau).\end{aligned}$$

We shall achieve the desired canceling effects by exploiting suitable rotations on the target sphere.

9.1.3. *The angular mode $n = 0$, equations on the physical side.* .

Recall the equation

$$-\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2\varepsilon_+^0 - \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\varepsilon_+^0 + H_0^+\varepsilon_+^0 = F_+(0), \tag{9.13}$$

where we also focus on the +-case. Recall the representation

$$\varepsilon_+^0(\tau,R) = c_0(\tau)\phi_0(R) + \phi_0(R)\cdot\int_0^R[\phi_0(s)]^{-1}\mathcal{D}_0\varepsilon_+^0(\tau,s)\, ds,$$

[9] Recall that one function of the fundamental system is constant

where we set $\mathcal{D}_0 = \partial_R + \frac{1}{R} \cdot \frac{R^2-1}{R^2+1}$, and we recall $\phi_0(R) = \frac{R}{1+R^2}$. Commuting $\mathcal{D}_0$ past the equation results in

$$\begin{aligned}
&-\left(\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)^2 + 3\frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\right)\mathcal{D}_0\varepsilon_+^0 + \tilde{H}_0^+\mathcal{D}_0\varepsilon_+^0 \\
=&\mathcal{D}_0\left(F_+(0)\right) - \frac{4R}{(R^2+1)^2}\left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\varepsilon_+^0 \\
&- \frac{\lambda'}{\lambda}\frac{4R}{(R^2+1)^2}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\varepsilon_+^0 - \frac{\lambda'}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}R\partial_R\right)\left(\frac{4R}{(R^2+1)^2}\varepsilon_+^0\right) \\
&+ \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}\varepsilon_+^0 \\
=:&\mathcal{D}_0\left(F_+(0)\right) + \mathcal{R}^0(\varepsilon_+^0, \mathcal{D}\varepsilon_+^0) + \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}\varepsilon_+^0.
\end{aligned} \tag{9.14}$$

The operator $\tilde{H}_0^+$ is the super-symmetric cousin of the operator H_0^+, i. e. $\tilde{H}_0^+ = -\mathcal{D}_0\mathcal{D}_0^*$. To complete things, we analyze (9.13) around $R = 0$ to extract the evolution law governing the coefficient $c_0(\tau)$ of the instability, which becomes the following:

$$-\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)^2 c_+ - \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)c_+ + \lim_{R\to 0} R^{-1}H_0^+\varepsilon_+^0 = \lim_{R\to 0} R^{-1}F_+(0) \tag{9.15}$$

This time the key operator governing the evolution of the nonlinearity is given by

$$\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)^2 + \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)$$

with fundamental system given by $\tau^{-1-\nu^{-1}}$, $\tau^{-1-2\nu^{-1}}$. Moreover, we can solve the equation

$$\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)^2 c + \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)c = h$$

by means of

$$c(\tau) = \nu\left(\tau^{-1-\nu^{-1}}\int_{\tau_0}^{\tau}\sigma^{2+\nu^{-1}}h(\sigma)\,d\sigma - \tau^{-1-2\nu^{-1}}\int_{\tau_0}^{\tau}\sigma^{2+2\nu^{-1}}h(\sigma)\,d\sigma\right). \tag{9.16}$$

9.1.4. *Translation of the equation* (9.14) *to the Fourier side.* From an algebraic standpoint, the situation here is more complicated than the preceding case, since the transference operator is non-vanishing for $n = 0$, and the spectral measure is only implicit. Introduce the auxiliary operator

$$\mathcal{D}_\tau^{(0)} = \partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi - \frac{\lambda_\tau}{\lambda}\frac{(\tilde{\rho}_0(\xi))'\,\xi}{\tilde{\rho}_0(\xi)} - \frac{\lambda_\tau}{\lambda}.$$

By means of it, we translate the equation (9.14) to the Fourier variables $\overline{x}(\tau,\xi)$, which satisfies

$$\mathcal{D}_0\varepsilon_+^0(\tau, R) = \int_0^\infty \phi_0(R,\xi)\cdot\overline{x}(\tau,\xi)\cdot\tilde{\rho}_0(\xi)\,d\xi,$$

in perfect analogy to (8.1) as follows:

$$\begin{aligned}
&-\left(\left(\mathcal{D}_\tau^{(0)}\right)^2+\frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{D}_\tau^{(0)}+\xi\right)\overline{x}\\
&=\mathcal{F}^{(0)}\left(F_\pm(0)\right)+2\frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_0^{(0)}\mathcal{D}_\tau^{(0)}\overline{x}+\left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)'\mathcal{K}_0^{(0)}\overline{x}+\frac{\lambda'(\tau)}{\lambda(\tau)}\left[\mathcal{D}_\tau^{(0)},\mathcal{K}_0^{(0)}\right]\overline{x}+\left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)^2\left(\left(\mathcal{K}_0^{(0)}\right)^2+\mathcal{K}_0^{(0)}\right)\overline{x}\\
&+\mathcal{F}^{(0)}\left(\mathcal{R}^0\left(\varepsilon_+^0,\mathcal{D}\varepsilon_+^0\right)\right)
\end{aligned} \tag{9.17}$$

where $\mathcal{K}_0^{(0)}$ is the off-diagonal part of the transference operator associated with $\phi_0(R,\xi)$.
In analogy to Lemma 9.1, we have

Lemma 9.6. *The homogeneous initial value problem*

$$\left(\left(\mathcal{D}_\tau^{(0)}\right)^2+\frac{\lambda'}{\lambda}\mathcal{D}_\tau^{(0)}+\xi\right)\overline{x}(\tau,\xi)=0,\quad \overline{x}(\tau_0,\xi)=\overline{x}_0(\xi),\quad \mathcal{D}_\tau^{(0)}\overline{x}(\tau_0,\xi)=\overline{x}_1(\xi),$$

is solved by the function

$$\begin{aligned}
\overline{x}(\tau,\xi)&=\frac{\lambda(\tau)}{\lambda(\tau_0)}\cdot\frac{\left(\tilde{\rho}^{(0)}\right)^{\frac12}\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)}{\left(\tilde{\rho}^{(0)}\right)^{\frac12}(\xi)}\cdot\cos\left[\lambda(\tau)\xi^{\frac12}\int_{\tau_0}^{\tau}\lambda^{-1}(u)\,du\right]\cdot\overline{x}_0\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)\\
&+\frac{\left(\tilde{\rho}^{(0)}\right)^{\frac12}\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)}{\left(\tilde{\rho}^{(0)}\right)^{\frac12}(\xi)}\cdot\xi^{-\frac12}\sin\left[\lambda(\tau)\xi^{\frac12}\int_{\tau_0}^{\tau}\lambda^{-1}(u)\,du\right]\cdot\overline{x}_1\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)
\end{aligned}$$

This implies the fundamental $S^{(0)}$-space propagation bounds(recall the definition (6.24) *of this norm)*

$$\|\overline{x}(\tau,\xi)\|_{S_0^{(0)}}+\left\|\mathcal{D}_\tau^{(0)}\overline{x}(\tau,\xi)\right\|_{S_1^{(0)}}\lesssim\left(\frac{\lambda(\tau_0)}{\lambda(\tau)}\right)^{1-\delta}\cdot\left[\|\overline{x}_0\|_{S_0^{(0)}}+\|\overline{x}_1\|_{S_1^{(0)}}\right]$$

In particular, choosing ν small enough, this quantity decays faster than any prescribed negative power of τ.

Furthermore, recalling Definition 8.11 , we have the following analogue of Prop. 9.2:

Proposition 9.7. *Denoting by $U^{(0)}(\tau,\sigma,\xi)$ Duhamel propagator for the inhomogeneous problem associated to $\left(\mathcal{D}_\tau^{(0)}\right)^2+\frac{\lambda'}{\lambda}\mathcal{D}_\tau^{(0)}+\xi$, we have*

$$\left\|\int_{\tau_0}^{\tau}U^{(0)}(\tau,\sigma,\xi)\cdot\overline{y}\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\right\|_{good}\lesssim\|\overline{y}\|_{goodsource},$$

where the norms are of course in the sense of angular momentum $n=0$ functions. If $\overline{y}$ has principal singular part of restricted type, so does the left hand side, and the norms can be adjusted accordingly.

Continuing in the vein of the case $n=1$, we next consider the analogue of (9.9), which is the source term

$$\lim_{R\to0}R^{-1}H_0^+\varepsilon_+^0=\lim_{R\to0}R^{-1}\mathcal{D}_0^*\mathcal{D}_0\varepsilon_+^0=c\int_0^\infty\overline{x}(\tau,\xi)\cdot\tilde{\rho}_0(\xi)\,d\xi \tag{9.18}$$

for a suitable real number $c \neq 0$. To proceed, we use a proposition analogous to Prop. 9.3, which differs subtly in the low frequency regime:

Proposition 9.8. *Denote by $\mathcal{F}^{(0)}$ the distorted Fourier transform at angular momentum $n = 0$ in the sense of subsection 6.5. Assume that we have (6.54) and define Λ as before, with $\Lambda \ll 1$. Finally, assume that the distorted Fourier transforms of the $\varepsilon_{\pm}(n)$ (both for the $|n| \geq 2$ and the exceptional modes) have restricted principal singular type (and in particular, we have to define Λ by using $\|\cdot\|_{good(r)}$). Then we can write*

$$\mathcal{F}^{(0)}\left(\mathcal{D}_0\left(F_+(0)\right)(\sigma,\cdot)\right)(\xi) = \sum_{\pm}\sum_{k=1,2}\sum_{i+j\leq N_1} C_{i,j}\frac{e^{\pm i\nu\sigma\xi^{\frac{1}{2}}}}{\xi^{1+\frac{k\nu}{2}}}(\log\xi)^i\cdot\gamma^{(k,j,\pm)}\cdot\beta^{(k,i+j)}(\sigma) + \overline{y}(\sigma,\xi), \tag{9.19}$$

where $\overline{y}(\tau,\xi)$ is a good source term at angular momentum $n = 1$ and such that all terms with $k \in \{1,2\}, l = 0$ in the expansion of its singular part (according to Definition 7.36) vanish. The coefficients $\gamma^{(k,i,\pm)}$ are given by the formulae

$$\gamma^{(k,j,\pm)} = \int_0^\infty e^{\pm ix}\cdot(\log x)^j\cdot x^{-\frac{1}{2}+k\nu}\,dx$$

for certain real coefficients $C_{i,j}$

Finally, we have the bound

$$\left|\beta^{(k,i)}(\sigma)\right| \lesssim \sigma^{-4-3\nu}(\log\sigma)^{3N_1}\Lambda^2,$$

and the good source $\overline{y}(\tau,\xi)$ can be decomposed into two parts

$$\overline{y}(\tau,\xi) = \overline{y}_1(\tau,\xi) + \overline{y}_2(\tau,\xi)$$

such that

$$\left|\overline{y}_1(\sigma,\xi)\right| \lesssim \sigma^{-4-3\nu}(\log\sigma)^{3N_1}\Lambda^2, \quad \left\|\overline{y}_2(\sigma,\xi)\right\|_{goodsource} \lesssim \sigma^{-5}\Lambda^2.$$

From here we can again infer the main obstruction to obtaining a good bound for the source term (9.18) in the ODE (9.15):

Lemma 9.9. *Denote by $U^{(0)}(\tau,\sigma,\xi)$ the Duhamel propagator according to Lemma 9.6. Then setting*

$$\overline{x}(\tau,\xi) = \int_{\tau_0}^{\tau} U^{(0)}(\tau,\sigma,\xi)\cdot\mathcal{F}^{(0)}\left(\mathcal{D}_0\left(F_+(0)\right)(\sigma,\cdot)\right)\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,$$

we have (for an arbitrary large but finite number M and a smooth cutoff $\chi_{\xi<M}$)

$$\begin{aligned}&\int_0^\infty \chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_0(\xi)\,d\xi\\ &= \sum_{\pm}\sum_{k=1,2}\sum_{0\leq j_1\leq N_1}\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}+\frac{k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}(\tilde{\rho}_0)^{\frac{1}{2}}(\xi)\,d\xi\\ &\qquad\cdot\sum_{j_2+i\leq N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\beta^{(k,j_1+j_2+i)}(\sigma)\,d\sigma\\ &\quad+\tilde{c}_0(\tau),\end{aligned}$$

where the error satisfies

$$|\tilde{c}_0(\tau)| \lesssim \tau^{-4-}.$$

Observe that the smooth cutoff $\chi_{\xi<M}$ ensures that the first integral in fact has a uniform bound in M, and in fact we have

$$\left|\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}+\frac{k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}(\tilde{\rho}_0)^{\frac{1}{2}}(\xi)\,d\xi\right| \lesssim \tau^{-(1-k\nu)}(\log\tau)^{N_1},$$

as one sees by inserting the smooth cutoffs $\chi_{\tau\xi^{\frac{1}{2}}\lesssim 1}, \chi_{\tau\xi^{\frac{1}{2}}\gtrsim 1}$, and performing integration by parts in case of inserting the latter cutoff. As in the case $n = 1$, we could infer the good bound

$$\left|\int_0^\infty \chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_0(\xi)\,d\xi\right| \lesssim \tau^{-4-},$$

in case we could enforce the vanishing conditions

$$\sum_{j_2+i\leq N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^\infty(\log\sigma)^{j_2}\left[\lambda(\sigma)\right]^{\nu k}\beta^{(k,j_1+j_2+i)}(\sigma)\,d\sigma = 0, \tag{9.20}$$

in case $j_1 = 0, 1, \ldots, N_1$.

The required cancellations here shall be enforced by exploiting scaling invariance, as well as the one remaining rotation on the target.

9.1.5. *The angular mode $n = -1$, equations on the physical side.* Finally, we turn to the equation for the $n = -1$ angular momentum mode, which we recall is given by

$$-\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2\varepsilon_+^{-1} - \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\varepsilon_+^{-1} + H_{-1}^+\varepsilon_+^{-1} = F_+(-1), \tag{9.21}$$

where we also focus on the +-case. Recall the representation

$$\varepsilon_+^{-1}(\tau,R) = c_{-1}(\tau)\phi_{-1}(R) + \phi_{-1}(R)\cdot\int_0^R\left[\phi_{-1}(s)\right]^{-1}\mathcal{D}_-\varepsilon_+^{-1}(\tau,s)\,ds,$$

where we set $\mathcal{D}_- = \partial_R - \frac{2}{R} + \frac{2R}{1+R^2}$, and we recall $\phi_{-1}(R) = \frac{R^2}{1+R^2}$. Commuting $\mathcal{D}_-$ past the equation results in

$$\begin{aligned}
&-\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2\mathcal{D}_-\varepsilon_+^{-1} - 3\frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\mathcal{D}_-\varepsilon_+^{-1} + \tilde{H}_{-1}^+\varepsilon_+^{-1} - \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}_-\varepsilon_+^{-1}\\
&= -\frac{\lambda_\tau}{\lambda}\frac{8R}{(1+R^2)^2}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\varepsilon_+^{-1} + \left(\frac{\lambda'}{\lambda}\right)^2\cdot\left(\frac{4R}{(1+R^2)^2} - \frac{16R}{(1+R^2)^3}\right)\varepsilon_+^{-1}\\
&- \frac{4R}{(1+R^2)^2}\left(\frac{\lambda'}{\lambda}\right)'\varepsilon_+^{-1} + \mathcal{D}_-\left(F_+(-1)\right)\\
&=: \mathcal{R}_+^{-1}\left(\varepsilon_+^{-1}, \mathcal{D}_-\varepsilon_+^{-1}\right) + \mathcal{D}_-\left(F_+(-1)\right).
\end{aligned} \tag{9.22}$$

Here we recall that the 'super-symmetrical;' operator $\tilde{H}_{-1}^+\varepsilon_+^1 = -\mathcal{D}_-\mathcal{D}_-^*$, i. e. the factors have been switched compared to H_{-1}^+. The preceding equation gets complemented by the one for $c_-(\tau)$ which arises by analyzing the terms vanishing to lowest order at the origin in (9.21), namely those vanishing to second order:

$$-\left(\partial_\tau + 2\frac{\lambda'}{\lambda}\right)^2 c_- - \frac{\lambda'}{\lambda}\left(\partial_\tau + 2\frac{\lambda'}{\lambda}\right)c_- + \lim_{R\to 0} R^{-2}H_{-1}^{+}\varepsilon_{+}^{-1} = \lim_{R\to 0} R^{-2}F_{+}(-1) \tag{9.23}$$

The operator here occurring on the left

$$\left(\partial_\tau + 2\frac{\lambda'}{\lambda}\right)^2 + \frac{\lambda'}{\lambda}\left(\partial_\tau + 2\frac{\lambda'}{\lambda}\right)$$

admits the fundamental system $\{\tau^{-2-2\nu^{-1}}, \tau^{-2-3\nu^{-1}}\}$, and the corresponding inhomogeneous problem

$$\left(\partial_\tau + 2\frac{\lambda'}{\lambda}\right)^2 c_{-1} + \frac{\lambda'}{\lambda}\left(\partial_\tau + 2\frac{\lambda'}{\lambda}\right)c_{-1} = h$$

is solved by the explicit expression

$$c_{-1}(\tau) = \nu\left(\tau^{-2-2\nu^{-1}}\int_{\tau_0}^{\tau}\sigma^{3+2\nu^{-1}}h(\sigma)\,d\sigma - \tau^{-2-3\nu^{-1}}\int_{\tau_0}^{\tau}\sigma^{3+3\nu^{-1}}h(\sigma)\,d\sigma\right). \tag{9.24}$$

9.1.6. *Translation of the equation* (9.22) *to the Fourier side.* The situation here is formally quite analogous to the one in the case $n = 0$, except the asymptotics of the spectral measure are quite different. Introduce the auxiliary operator

$$\mathcal{D}_\tau^{(-1)} = \partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi - \frac{\lambda_\tau}{\lambda}\frac{(\tilde{\rho}_{-1}(\xi))'\,\xi}{\tilde{\rho}_{-1}(\xi)} - \frac{\lambda_\tau}{\lambda}.$$

By means of it, we translate the equation (9.14) to the Fourier variables $\overline{x}(\tau,\xi)$, which satisfies

$$\mathcal{D}_-\varepsilon_+^{-1}(\tau,R) = \int_0^\infty \phi_{-1}(R,\xi)\cdot\overline{x}(\tau,\xi)\cdot\tilde{\rho}_{-1}(\xi)\,d\xi,$$

in perfect analogy to (8.1) as follows:

$$\begin{aligned}
&-\left(\left(\mathcal{D}_\tau^{(-1)}\right)^2 + \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{D}_\tau^{(-1)} + \xi\right)\overline{x}\\
&= \mathcal{F}^{(-1)}\left(F_+(-1)\right) + 2\frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_0^{(-1)}\mathcal{D}_\tau^{(-1)}\overline{x} + \left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)'\mathcal{K}_0^{(-1)}\overline{x} + \frac{\lambda'(\tau)}{\lambda(\tau)}\left[\mathcal{D}_\tau^{(-1)},\mathcal{K}_0^{(-1)}\right]\overline{x} + \left(\frac{\lambda'(\tau)}{\lambda(\tau)}\right)^2\left(\left(\mathcal{K}_0^{(-1)}\right)^2 + \mathcal{K}_0^{(-1)}\right)\overline{x}\\
&+ \mathcal{F}^{(-1)}\left(\mathcal{R}^{-1}\left(\varepsilon_+^{-1},\mathcal{D}_-\varepsilon_+^{-1}\right)\right)
\end{aligned} \tag{9.25}$$

where $\mathcal{K}_0^{(-1)}$ is the off-diagonal part of the transference operator associated with $\phi_{-1}(R,\xi)$.
The evolution under the linear operator on the left is described by the following

Lemma 9.10. *The homogeneous initial value problem*

$$\left(\left(\mathcal{D}_\tau^{(-1)}\right)^2 + \frac{\lambda'}{\lambda}\mathcal{D}_\tau^{(-1)} + \xi\right)\overline{x}(\tau,\xi) = 0, \quad \overline{x}(\tau_0,\xi) = \overline{x}_0(\xi), \quad \mathcal{D}_\tau^{(-1)}\overline{x}(\tau_0,\xi) = \overline{x}_1(\xi)$$

is solved by the function

$$\overline{x}(\tau,\xi) = \frac{\lambda(\tau)}{\lambda(\tau_0)} \cdot \frac{(\tilde{\rho}_{-1})^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)}{(\tilde{\rho}_{-1})^{\frac{1}{2}}(\xi)} \cdot \cos\left[\lambda(\tau)\xi^{\frac{1}{2}}\int_{\tau_0}^{\tau}\lambda^{-1}(u)\,du\right] \cdot \overline{x}_0\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)$$
$$+ \frac{(\tilde{\rho}_{-1})^{\frac{1}{2}}\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)}{(\tilde{\rho}_{-1})^{\frac{1}{2}}(\xi)} \cdot \xi^{-\frac{1}{2}}\sin\left[\lambda(\tau)\xi^{\frac{1}{2}}\int_{\tau_0}^{\tau}\lambda^{-1}(u)\,du\right] \cdot \overline{x}_1\left(\frac{\lambda^2(\tau)}{\lambda^2(\tau_0)}\xi\right)$$

This implies the fundamental $S^{(0)}$-space propagation bounds (recall the definition (6.22) *of this norm)*

$$\|\overline{x}(\tau,\xi)\|_{S_0^{(-1)}} + \left\|\mathcal{D}_\tau^{(-1)}\overline{x}(\tau,\xi)\right\|_{S_1^{(-1)}} \lesssim \left(\frac{\lambda(\tau_0)}{\lambda(\tau)}\right)^{1-\delta} \cdot \left[\|\overline{x}_0\|_{S_0^{(-1)}} + \|\overline{x}_1\|_{S_1^{(-1)}}\right]$$

In particular, choosing ν small enough, this quantity decays faster than any prescribed negative power of τ.

Again in light of Definition 8.11 , we have the following analogue of Prop. 9.2:

Proposition 9.11. *Denoting by $U^{(-1)}(\tau,\sigma,\xi)$ Duhamel propagator for the inhomogeneous problem associated to $\left(\mathcal{D}_\tau^{(-1)}\right)^2 + \frac{\lambda'}{\lambda}\mathcal{D}_\tau^{(-1)} + \xi$, we have*

$$\left\|\int_{\tau_0}^{\tau} U^{(-1)}(\tau,\sigma,\xi)\cdot\overline{y}\left(\sigma,\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma\right\|_{good} \lesssim \|\overline{y}\|_{goodsource},$$

where the norms are of course in the sense of angular momentum $n = -1$ functions. If $\overline{y}$ has principal singular part of restricted type, so does the left hand side, and the norms can be adjusted accordingly.

Finally, we analyze the delicate term $\lim_{R\to 0} R^{-2}H^+_{-1}\varepsilon_+^{-1}$ in (9.23), for which we have the customary relation

$$\lim_{R\to 0} R^{-2}H^+_{-1}\varepsilon_+^{-1} = d\int_0^\infty \overline{x}(\tau,\xi)\cdot\tilde{\rho}_{-1}(\xi)\,d\xi$$

for suitable $d \neq 0$, provided $\overline{x}(\tau,\xi)$ represents the distorted Fourier transform of $\mathcal{D}_-\varepsilon_+^{-1}$. In order to control this, we again need to understand the most singular terms in the source, which result in poorly temporally decaying contributions to the preceding expression:

Proposition 9.12. *Denote by $\mathcal{F}^{(-1)}$ the distorted Fourier transform at angular momentum $n = -1$ in the sense of subsection 6.5. Assume that we have* (6.54) *and define Λ as before, with $\Lambda \ll 1$. Finally, assume that the distorted Fourier transforms of the $\varepsilon_\pm(n)$ (both for the $|n| \geq 2$ and the exceptional modes) have restricted principal singular type (and in particular, we have to define Λ by using $\|\cdot\|_{good(r)}$). Then we can write*

$$\mathcal{F}^{(-1)}\left(\mathcal{D}_0\left(F_+(-1)\right)(\sigma,\cdot)\right)(\xi) = \sum_\pm\sum_{k=1,2}\sum_{i+j\leq N_1} C_{i,j}\frac{e^{\pm i\nu\sigma\xi^{\frac{1}{2}}}}{\xi^{\frac{3}{2}+\frac{k\nu}{2}}}(\log\xi)^i\cdot\gamma^{(k,j,\pm)}\cdot\beta^{(k,i+j)}(\sigma) + \overline{y}(\sigma,\xi), \quad (9.26)$$

where $\overline{y}(\tau,\xi)$ is a good source term at angular momentum $n = -1$ and such that all terms with $k \in \{1,2\}, l = 0$ in the expansion of its singular part (according to Definition 7.36) vanish. The coefficients $\gamma^{(k,i,\pm)}$ are given *by the formulae*

$$\gamma^{(k,j,\pm)} = \int_0^\infty e^{\pm ix} \cdot (\log x)^j \cdot x^{-\frac{1}{2}+k\nu}\,dx$$

for certain real coefficients $C_{i,j}$

Finally, we have the bound

$$\left|\beta^{(k,i)}(\sigma)\right| \lesssim \sigma^{-4-3\nu}\,(\log\sigma)^{3N_1}\,\Lambda^2,$$

and the good source $\overline{y}(\tau,\xi)$ can be decomposed into two parts

$$\overline{y}(\tau,\xi) = \overline{y}_1(\tau,\xi) + \overline{y}_2(\tau,\xi)$$

such that

$$\left|\overline{y}_1(\sigma,\xi)\right| \lesssim \sigma^{-4-3\nu}\,(\log\sigma)^{3N_1}\,\Lambda^2, \quad \left\|\overline{y}_2(\sigma,\xi)\right\|_{goodsource} \lesssim \sigma^{-5}\Lambda^2.$$

As in the preceding cases, we then infer the following structure result for the truncated integral expressing $\lim_{R\to 0} R^{-2}H^{+}_{-1}\varepsilon_{+}^{-1}$:

Lemma 9.13. *Denote by $U^{(-1)}(\tau,\sigma,\xi)$ the Duhamel propagator according to Lemma 9.10. Then setting*

$$\overline{x}(\tau,\xi) = \int_{\tau_0}^{\tau} U^{(-1)}(\tau,\sigma,\xi)\cdot\mathcal{F}^{(-1)}\left(\mathcal{D}_-\left(F_+(-1)\right)(\sigma,\cdot)\right)\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,$$

we have (for an arbitrary large but finite number M and a smooth cutoff $\chi_{\xi<M}$)

$$\begin{aligned}
&\int_0^\infty \chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_{-1}(\xi)\,d\xi\\
&= \sum_{\pm}\sum_{k=1,2}\sum_{0\le j_1\le N_1}\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}+\frac{k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}(\tilde{\rho}_{-1})^{\frac{1}{2}}(\xi)\,d\xi\\
&\qquad\qquad\cdot\sum_{j_2+i\le N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\beta^{(k,j_1+j_2+i)}(\sigma)\,d\sigma\\
&+ \tilde{c}_{-1}(\tau),
\end{aligned}$$

where the error satisfies

$$|\tilde{c}_{-1}(\tau)| \lesssim \tau^{-4-}.$$

As before the preceding implies the bound

$$\left|\int_0^\infty \chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_{-1}(\xi)\,d\xi\right| \lesssim \tau^{-4-},$$

in case we could enforce the vanishing conditions

$$\sum_{j_2+i\le N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\infty}(\log\sigma)^{j_2}\,[\lambda(\sigma)]^{\nu k}\,\beta^{(k,j_1+j_2+i)}(\sigma)\,d\sigma = 0, \tag{9.27}$$

in case $j_1 = 0, 1, \ldots, N_1$.

9.2. **The modulation step for the exceptional modes; forcing the vanishing conditions.** In this subsection, we introduce the final tool which will allow us to close all the estimates introduced gradually in the preceding. Specifically, recalling the decomposition of our Wave Map

$$\begin{aligned}\Psi &= \Phi + \Pi_{\Phi^\perp}\varphi + a\,(\Pi_{\Phi^\perp}\varphi)\,\Phi\\ &= \Phi + \varphi_1 E_1 + \varphi_2 E_2 + a\,(\Pi_{\Phi^\perp}\varphi)\,\Phi,\end{aligned}$$

we modify this by applying a re-scaling $\mathcal{S}_{c(t)}$, and a rotation $\mathcal{R}_{h(t)}^{\alpha(t),\beta(t)}$ as well as a Lorentz transform $\mathcal{L}_{v(t)}$ to the full expression. For technical reasons, the Lorentz transform as well as the scaling will be chosen to be constant from some time $t_1 < t_0$, where t_0 is the initial time for the perturbation. In fact, there will be a lot of flexibility in choosing these modulation parameters, since their role will be to enforce certain moment conditions, which are in effect time integrals of these functions multiplied against certain weights[10]. In total, we shall then pass to the following ansatz

$$\Psi = \mathcal{L}_{v(t)}\mathcal{R}_{h(t)}^{\alpha(t),\beta(t)}\mathcal{S}_{c(t)}\left(\Phi + \varphi_1 E_1 + \varphi_2 E_2 + a\,(\Pi_{\Phi^\perp}\varphi)\,\Phi\right), \quad \Phi = \begin{pmatrix} \sin U\cos\theta \\ \sin U\sin\theta \\ \cos U\end{pmatrix}, \tag{9.28}$$

whence a complete description of the Wave Map evolution shall consist of the tuple of functions

$$\{\varphi_1,\ \varphi_2,\ \alpha(t),\beta(t),\ h(t),\ c(t),\ v_1(t),\ v_2(t)\},$$

where in turn $\phi_{1,2}$ are described in terms of $\varepsilon_\pm = \varphi_1 \mp i\varphi_2$, each in turn decomposed into angular momentum n pieces, which are described and measured as in the preceding. While the equation for $\varphi_{1,2}$ is exactly identical in the case of constant coefficients α etc, the time dependence of these will introduce additional source terms. In the sequel, we shall analyze the different modulations parameters and the effect they have to leading order on the equations. Since the leading order effect will be linear, we can treat the contribution of each modulation parameter separately, which will clarify the analysis. We emphasize that our version of modulation theory differs from the more standard kind, where usually the action of the symmetries on the bulk part Q are used to enforce various vanishing conditions on fixed time slices. In our version, the action of the symmetries on the singular part of the profile U (and thus, a lower order term) are used to counteract singular terms arising via interactions between the perturbation and U, and the vanishing conditions refer to suitable time integrals.

9.2.1. *The contribution of the rotations $\alpha(t),\beta(t)$.* Our convention shall be that the angle $\alpha(t)$ corresponds to the rotation

$$\begin{pmatrix} \cos\alpha(t) & 0 & \sin\alpha(t) \\ 0 & 1 & 0 \\ -\sin\alpha(t) & 0 & \cos\alpha(t)\end{pmatrix},$$

while the angle $\beta(t)$ corresponds to

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & \cos\beta(t) & \sin\beta(t) \\ 0 & -\sin\beta(t) & \cos\beta(t)\end{pmatrix}.$$

[10]However, their terminal values will be uniquely determined, of course

It turns out that modulating on α contributes both in terms of the effect on the bulk profile Q as well as on the singular terms in U, where we recall that the polar angle U of the unperturbed blow up solution is given by

$$U = Q(R) + \epsilon, \quad Q(R) = 2\arctan R.$$

In order to simplify the computations at first, we start by considering the effect of $\mathcal{R}_0^{\alpha(t),0}$. Denoting by $(\partial_t^2)'$ only those terms where at least one derivative falls on $\alpha(t)$, we compute

$$\begin{aligned}
\left(\partial_t^2\right)'\left(\mathcal{R}_0^{\alpha(t),0}\Phi\right) &= \mathcal{R}_0^{\alpha(t),0}\alpha''(t)\begin{pmatrix} \cos\alpha(t) & 0 & -\sin\alpha(t) \\ 0 & 1 & 0 \\ \sin\alpha(t) & 0 & \cos\alpha(t) \end{pmatrix}\begin{pmatrix} -\cos\theta\sin U\sin\alpha + \cos U\cos\alpha \\ 0 \\ -\cos\theta\sin U\cos\alpha - \cos U\sin\alpha \end{pmatrix} \\
&+ 2\alpha'(t)\lambda'(t)rQ'(R)\mathcal{R}_0^{\alpha(t),0}\begin{pmatrix} \cos\alpha(t) & 0 & -\sin\alpha(t) \\ 0 & 1 & 0 \\ \sin\alpha(t) & 0 & \cos\alpha(t) \end{pmatrix}\begin{pmatrix} -\cos\theta\cos U\sin\alpha - \sin U\cos\alpha \\ 0 \\ -\cos\theta\cos U\cos\alpha + \sin U\sin\alpha \end{pmatrix} \\
&+ 2\alpha'(t)\lambda\cdot[\epsilon_\tau + \epsilon_X]\cdot\mathcal{R}_0^{\alpha(t),0}\begin{pmatrix} \cos\alpha(t) & 0 & -\sin\alpha(t) \\ 0 & 1 & 0 \\ \sin\alpha(t) & 0 & \cos\alpha(t) \end{pmatrix}\begin{pmatrix} -\cos\theta\cos U\sin\alpha - \sin U\cos\alpha \\ 0 \\ -\cos\theta\cos U\cos\alpha + \sin U\sin\alpha \end{pmatrix}
\end{aligned}$$

Here we have introduce the new variable $X := \nu\tau - R = \lambda\cdot(t-r)$, whence the singularity of ϵ across the light cone gets expressed in terms of X. We simplify the terms as follows:

$$\begin{pmatrix} \cos\alpha(t) & 0 & -\sin\alpha(t) \\ 0 & 1 & 0 \\ \sin\alpha(t) & 0 & \cos\alpha(t) \end{pmatrix}\begin{pmatrix} -\cos\theta\sin U\sin\alpha + \cos U\cos\alpha \\ 0 \\ -\cos\theta\sin U\cos\alpha - \cos U\sin\alpha \end{pmatrix} = \begin{pmatrix} \cos U \\ 0 \\ -\cos\theta\sin U \end{pmatrix}$$

$$\begin{pmatrix} \cos\alpha(t) & 0 & -\sin\alpha(t) \\ 0 & 1 & 0 \\ \sin\alpha(t) & 0 & \cos\alpha(t) \end{pmatrix}\begin{pmatrix} -\cos\theta\cos U\sin\alpha - \sin U\cos\alpha \\ 0 \\ -\cos\theta\cos U\cos\alpha + \sin U\sin\alpha \end{pmatrix} = \begin{pmatrix} -\sin U \\ 0 \\ -\cos\theta\cos U \end{pmatrix}$$

Also recall the customary change of temporal variables

$$\tau = \int_t^\infty \lambda(s)\,ds \longrightarrow \frac{d\tau}{dt} = -\lambda,$$

whence

$$\begin{aligned}
&\alpha'(t) = -\lambda\alpha'(\tau),\ \alpha''(t) = \lambda^2\alpha''(\tau) + \lambda\lambda_\tau\alpha'(\tau), \\
&2\alpha'(t)\lambda'(t)rQ'(R) = 2\lambda^2\alpha'(\tau)\lambda_\tau rQ'(R) = 2\lambda^2\alpha'(\tau)\frac{\lambda_\tau}{\lambda}RQ'(R)
\end{aligned}$$

In order to translate things to the $(\varepsilon_{1,2})$-coordinates, we expand things in terms of the $(E_{1,2},\Phi)$-frame, where we recall the formulae

$$E_1 = \begin{pmatrix} \cos\theta\cos U \\ \sin\theta\cos U \\ -\sin U \end{pmatrix}, \quad E_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} \cos U \\ 0 \\ -\cos\theta\sin U \end{pmatrix} = \cos\theta\cdot E_1 + (-\sin\theta\cos U)\cdot E_2 + (*)\cdot\Phi,$$

$$\begin{pmatrix} -\sin U \\ 0 \\ -\cos\theta\cos U \end{pmatrix} = 0\cdot E_1 + \sin\theta\sin U\cdot E_2 + (*)\cdot\Phi.$$

This implies that the $n = 1$ component in the source term of the equation for $\varepsilon_+^1 = \varphi_1^{(1)} - i\varphi_2^{(1)}$ due to modulation in α is given by (after division by λ^2)

$$\lambda^{-2}\alpha''(t)\cdot\left(\frac{1}{2} - i\cdot\frac{i}{2}\cos U\right) + 2\alpha'(\tau)\frac{\lambda_\tau}{\lambda}RQ'(R)\cdot\left(0 + i\cdot\frac{i}{2}\sin U\right) \tag{9.29}$$

$$+\alpha'(\tau)\cdot[\epsilon_\tau + \epsilon_X]\cdot\sin U$$

$$= \frac{\alpha''(\tau) + \frac{\lambda_\tau}{\lambda}\alpha'(\tau)}{2}\cdot(1+\cos U) - \alpha'(\tau)\frac{\lambda_\tau}{\lambda}RQ'(R)\sin U \tag{9.30}$$

$$+\alpha'(\tau)\cdot[\epsilon_\tau + \epsilon_X]\cdot\sin U. \tag{9.31}$$

Recalling (9.3), we now analyze the principal contributions of the preceding expression to the term $\lim_{R\to 0} F_+(1)$, as well as the term $\lim_{R\to 0} H_1^+\varepsilon_+^1$, where the latter term requires evaluation of the wave parametrix (for the $n = +1$ mode) on $\mathcal{D}_+$ ((9.29)). We shall distinguish between the contribution of the first line (9.30) in the last expression in (9.29), i. e. the smooth part, and the last term (9.31) (involving $\epsilon_\tau + \epsilon_X$), which is the non-smooth part.

(i): Contribution of (9.30) *to* $\lim_{R\to 0} F_+(1)$. This follows directly from the fact that $U(\tau, 0) = 0$, and so this contribution equals

$$\alpha''(\tau) + \frac{\lambda_\tau}{\lambda}\alpha'(\tau).$$

This expression is that one source term for the ODE governing the evolution of $c_+(\tau)$, according to (9.3).

(ii): Contribution of (9.30) *to* $\lim_{R\to 0} H_1^+\varepsilon_+^1$. Here we apply $\mathcal{D}_+$ to the extra source term and then the wave parametrix (after translating things to the Fourier side), and then we extract the leading behavior. Since we only care about the leading order behavior, and ϵ decays in time like τ^{-1}, we replace U by $Q(R)$, the bulk profile. Note that

$$\cos Q(R) = \cos^2(\arctan R) - \sin^2(\arctan R) = \frac{1-R^2}{1+R^2},$$

and so $\frac{1+\cos U}{2} = \frac{1}{1+R^2}$, which is of course killed by $\mathcal{D}_+$. Next, we have

$$RQ'(R)\sin Q(R) = \frac{2R}{1+R^2}\cdot\frac{2R}{1+R^2} = \frac{4R^2}{(1+R^2)^2} = \frac{4}{1+R^2} - \frac{4}{(1+R^2)^2}.$$

We conclude that to leading order we have

$$\mathcal{D}_+\left(-\alpha'(\tau)\frac{\lambda_\tau}{\lambda}RQ'(R)\sin U\right) = +\alpha'(\tau)\frac{\lambda_\tau}{\lambda}\mathcal{D}_+\left(\frac{4}{(1+R^2)^2}\right),$$

which is a new source term in the wave equation for $\mathcal{D}_+\varepsilon_+^1$ due to modulating in α.
Apply now the wave parametrix according to Lemma 9.1, which results in the integral

$$\int_{\tau_0}^{\tau} \frac{\lambda(\tau)}{\lambda(\sigma)} \cdot \xi^{-\frac{1}{2}} \sin\left[\lambda(\tau)\xi^{\frac{1}{2}} \int_{\sigma}^{\tau} \lambda^{-1}(u)\, du\right] \cdot \mathcal{F}^{(1)}\left(\alpha'(\sigma)\frac{\lambda_\sigma}{\lambda}\mathcal{D}_+\left(\frac{4}{(1+R^2)^2}\right)\right)\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) d\sigma$$

To extract the main contribution from this term, we perform integration by parts with respect to σ, which produces a boundary term at $\sigma = \tau$, as well as a negligible boundary term at $\sigma = \tau_0$ and a better term with an extra σ-derivative falling on the Fourier transform. Relegating the treatment of all these errors to later, we then wind up with the principal term

$$-\frac{1}{\xi} \cdot \mathcal{F}^{(1)}\left(\alpha'(\tau)\frac{\lambda_\tau(\tau)}{\lambda(\tau)}\mathcal{D}_+\left(\frac{4}{(1+R^2)^2}\right)\right)(\xi),$$

which contributes the term

$$-\int_0^\infty \frac{1}{\xi} \cdot \mathcal{F}^{(1)}\left(\alpha'(\tau)\frac{\lambda_\tau(\tau)}{\lambda(\tau)}\mathcal{D}_+\left(\frac{4}{(1+R^2)^2}\right)\right)(\xi)\tilde{\rho}_1(\xi)\, d\xi$$

to the term $\lim_{R\to 0} H_1^+\varepsilon_+^1$. We claim that this can be rendered more explicit. To begin with, we can replace $\frac{4}{(1+R^2)^2}$ by $\frac{4}{(1+R^2)^2} - \frac{c}{1+R^2}$ for arbitrary c. Then we pick c in such a way that

$$\frac{4}{(1+R^2)^2} - \frac{c}{1+R^2} = \mathcal{D}_+^* g$$

where g is such that $\int_0^\infty \left(\mathcal{F}^{(1)}g\right)(\xi)\tilde{\rho}_1(\xi)\, d\xi$ converges. Specifically, recall that

$$\mathcal{D}_+^* = -\partial_R - \frac{1}{R} + \frac{2R}{1+R^2},$$

which annihilates $\psi(R) = \frac{1+R^2}{R}$, and so we can set

$$\begin{aligned} g(R) = -\frac{R^2+1}{R} \cdot \int_0^R \frac{s}{s^2+1} \cdot \left(\frac{4}{(1+s^2)^2} - \frac{c}{1+R^2}\right) ds &= \frac{R^2+1}{R} \cdot \left(\frac{1}{(1+R^2)^2} - \frac{c}{2(1+R^2)}\right)\Big|_0^R \\ &= \frac{R^2+1}{R} \cdot \frac{1-(1+R^2)}{(1+R^2)^2} \\ &= -\frac{R}{1+R^2}, \end{aligned}$$

provided we set $c = 2$. It follows that

$$\begin{aligned} &-\int_0^\infty \frac{1}{\xi} \cdot \mathcal{F}^{(1)}\left(\alpha'(\tau)\frac{\lambda_\tau(\tau)}{\lambda(\tau)}\mathcal{D}_+\left(\frac{4}{(1+R^2)^2}\right)\right)(\xi)\tilde{\rho}_1(\xi)\, d\xi \\ &= -\alpha'(\tau)\frac{\lambda_\tau(\tau)}{\lambda(\tau)} \cdot \int_0^\infty \frac{1}{\xi} \cdot \mathcal{F}^{(1)}\left(\mathcal{D}_+\mathcal{D}_+^* g\right)(\xi)\tilde{\rho}_1(\xi)\, d\xi \\ &= -\alpha'(\tau)\frac{\lambda_\tau(\tau)}{\lambda(\tau)} \cdot \int_0^\infty \mathcal{F}^{(1)}(g)(\xi)\tilde{\rho}_1(\xi)\, d\xi \\ &= c_0\alpha'(\tau)\frac{\lambda_\tau(\tau)}{\lambda(\tau)} \end{aligned}$$

for certain constant c_0. This is the leading order contribution to $\lim_{R\to 0} H_1^+ \varepsilon_+^1$ arising from modulating in $\alpha(t)$.

(iii): Combined leading order contribution of (9.30) *to* (9.3). From *(i)* and *(ii)*, this is seen to be the equation

$$c_+''(\tau) + \frac{\lambda_\tau}{\lambda} c_+'(\tau) = -\alpha''(\tau) + (c_0 - 1)\frac{\lambda'(\tau)}{\lambda(\tau)}\alpha'(\tau).$$

which in light of (9.4)

$$\begin{aligned} c_+(\tau) = &- \nu \int_{\tau_0}^{\tau} \sigma \left(\alpha''(\sigma) + (1-c_0)(1+\nu^{-1})\sigma^{-1}\alpha'(\sigma)\right) d\sigma \\ &+ \nu\tau^{-\nu^{-1}} \int_{\tau_0}^{\tau} \sigma^{1+\nu^{-1}} \left(\alpha''(\sigma) + (1-c_0)(1+\nu^{-1})\sigma^{-1}\alpha'(\sigma)\right) d\sigma. \end{aligned}$$

The parameter $\alpha(\tau)$ will be chosen such that $\alpha'(\tau)$ is compactly supported in (τ_0, ∞). Therefore if we denote by $\alpha_\infty := \alpha(\infty) - \alpha(\tau_0)$, we have

$$\begin{aligned} c_+(\tau) = ((1+2\nu) - c_0(1+\nu))\,\alpha_\infty &+ \nu\tau^{-\nu^{-1}} \int_{\tau_0}^{\infty} \sigma^{1+\nu^{-1}} \alpha''(\sigma)\, d\sigma + (c_0-1)(1+\nu)\tau^{-\nu^{-1}} \int_{\tau_0}^{\infty} \sigma^{\nu^{-1}} \alpha'(\sigma)\, d\sigma \\ &+ (1+\nu)\tau^{-\nu^{-1}} \int_{\tau}^{\infty} \sigma^{\nu^{-1}} \alpha'(\sigma)\, d\sigma + (1-c_0)(1+\nu)\tau^{-\nu^{-1}} \int_{\tau}^{\infty} \sigma^{\nu^{-1}} \alpha'(\sigma)\, d\sigma. \end{aligned}$$

Except the term involving α_∞, all the other terms on the RHS above decays rapidly in τ as $\tau \to \infty$. On the other hand, since the operator $\frac{d^2}{d\tau^2} + \frac{\lambda_\tau}{\lambda}\frac{d}{d\tau}$ admits a constant fundamental solution. Therefore we choose α_∞ appropriately (using the flexibility of the moment condition satisfied by α', which is weaker than the ones for other modulation parameters except β) to cancel the constant fundamental solution such that $c_+(\tau)$ decays rapidly as $\tau \to \infty$.

(iv): the contribution of (9.31) *to* $\lim_{R\to 0} F_+(1)$. This is negligible and in fact of order $\alpha'(\tau)\tau^{-N}$ due to the structure of ϵ.

(v): Contribution of (9.31) *to* $\lim_{R\to 0} H_1^+ \varepsilon_+^1$. Precisely, we shall want to use this term to cancel the troublesome terms on the right in (9.10). For this, we shall use

Lemma 9.14. *There is a function $H(\tau, R)$ coinciding with the function $\chi_{R\gtrsim\tau} \cdot$* (9.31) *near the light cone $R = \nu\tau$, and such that we have the formula*

$$\mathcal{F}^{(1)}\left((H)(\sigma,\cdot)\right)(\xi) = \sum_{\pm}\sum_{k=1,2}\sum_{i+j\leq N_1} C_{i,j}\frac{e^{\pm i\nu\sigma\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}+\frac{k\nu}{2}}}(\log\xi)^i \cdot \gamma^{(k,j,\pm)} \cdot \frac{\alpha'(\sigma)}{\sigma^{\frac{1}{2}+k\nu}}\left(\sum_{r\leq N_1-i-j} D_{i,j,r}(\sigma)\cdot \log^r\sigma\right) + \overline{y}(\sigma,\xi), \tag{9.32}$$

where $\overline{y}(\tau,\xi)$ is a good source term at angular momentum $n = 1$ and such that all terms with $k \in \{1,2\}, l = 0$ in the expansion of its singular part (according to Definition 7.36) vanish. Here $D_{i,j,r}(\sigma)$ are suitable complex valued functions, and the numbers $C_{i,j}, \gamma^{(k,j,\pm)}$ are defined exactly as in Prop. 9.3. In particular, *applying the propagator $U^{(1)}(\tau,\sigma,\xi)$, and setting*

$$\overline{x}(\tau,\xi) = \int_{\tau_0}^{\tau} U^{(1)}(\tau,\sigma,\xi)\,\mathcal{F}^{(1)}\left((H)(\sigma,\cdot)\right)\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,$$

we have

$$\begin{aligned}
&\int_0^\infty \chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_1(\xi)\,d\xi \\
&= \sum_{\pm}\sum_{k=1,2}\sum_{0\leq j_1\leq N_1}\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}\,d\xi \\
&\qquad\cdot\sum_{j_2+i\leq N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}\frac{\alpha'(\sigma)}{\sigma^{\frac{3}{2}+k\nu}}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\leq N_1-i-j_1-j_2} D_{i,j,r}(\sigma)\cdot\log^r\sigma\right)d\sigma \\
&\quad+\tilde{c}_+(\tau),
\end{aligned}$$

where the error satisfies

$$|\tilde{c}_+(\tau)| \lesssim \tau^{-4-}.$$

The preceding lemma is of course still not enough to force the vanishing relations (9.12), since $\alpha'(\sigma)$ is real valued, but the other coefficients are mostly complex valued. This has to do with the fact that we have neglected modulations in the angle $\beta(t)$ up to now, which we do next. Since we will eventually prescribe a number of moment conditions on α,β, we may as well assume that α is constant on the support of β in order to simplify the computation, and equals its limiting value there. Denoting by $\left(\partial_t^2\right)'$ only those terms where at least one derivative falls on $\beta(t)$,

$$\begin{aligned}
&\left(\partial_t^2\right)'\left(\mathcal{R}_0^{\alpha,\beta(t)}\Phi\right) \\
=&\mathcal{R}_0^{\alpha,\beta(t)}\beta''(t)\begin{pmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\beta(t) & -\sin\beta(t)\\ 0 & \sin\beta(t) & \cos\beta(t)\end{pmatrix} \\
&\qquad\cdot\begin{pmatrix}0\\ -\sin\beta\sin\theta\sin U-\cos\beta\sin\alpha\cos\theta\sin U+\cos U\cos\alpha\cos\beta\\ -\cos\beta\sin\theta\sin U+\sin\beta\sin\alpha\sin U\cos\theta-\cos U\cos\alpha\sin\beta\end{pmatrix} \\
&+\mathcal{R}_0^{\alpha,\beta(t)}2\beta'(t)\lambda'(t)rQ'(R)\begin{pmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\beta(t) & -\sin\beta(t)\\ 0 & \sin\beta(t) & \cos\beta(t)\end{pmatrix} \\
&\qquad\cdot\begin{pmatrix}0\\ -\sin\beta\sin\theta\cos U-\cos\beta\sin\alpha\cos\theta\cos U-\sin U\cos\alpha\cos\beta\\ -\cos\beta\sin\theta\cos U+\sin\beta\sin\alpha\cos U\cos\theta+\sin U\cos\alpha\sin\beta\end{pmatrix} \\
&+\mathcal{R}_0^{\alpha,\beta(t)}2\beta'(t)\lambda(t)\cdot[\epsilon_\tau+\epsilon_X]\cdot\begin{pmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\beta(t) & -\sin\beta(t)\\ 0 & \sin\beta(t) & \cos\beta(t)\end{pmatrix}
\end{aligned}$$

$$\cdot\begin{pmatrix} 0 \\ -\sin\beta\sin\theta\cos U-\cos\beta\sin\alpha\cos\theta\cos U-\sin U\cos\alpha\cos\beta \\ -\cos\beta\sin\theta\cos U+\sin\beta\sin\alpha\cos U\cos\theta+\sin U\cos\alpha\sin\beta \end{pmatrix}$$

We compute

$$\begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta(t) & -\sin\beta(t) \\ 0 & \sin\beta(t) & \cos\beta(t) \end{pmatrix}$$
$$\cdot\begin{pmatrix} 0 \\ -\sin\beta\sin\theta\sin U-\cos\beta\sin\alpha\cos\theta\sin U+\cos U\cos\alpha\cos\beta \\ -\cos\beta\sin\theta\sin U+\sin\beta\sin\alpha\sin U\cos\theta-\cos U\cos\alpha\sin\beta \end{pmatrix}$$
$$=\begin{pmatrix} \sin\alpha\sin\theta\sin U \\ -\sin\alpha\cos\theta\sin U+\cos\alpha\cos U \\ -\cos\alpha\sin\theta\sin U \end{pmatrix}$$

as well as

$$\begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta(t) & -\sin\beta(t) \\ 0 & \sin\beta(t) & \cos\beta(t) \end{pmatrix}$$
$$\cdot\begin{pmatrix} 0 \\ -\sin\beta\sin\theta\cos U-\cos\beta\sin\alpha\cos\theta\cos U-\sin U\cos\alpha\cos\beta \\ -\cos\beta\sin\theta\cos U+\sin\beta\sin\alpha\cos U\cos\theta+\sin U\cos\alpha\sin\beta \end{pmatrix}$$
$$=\begin{pmatrix} \sin\alpha\sin\theta\cos U \\ -\sin\alpha\cos\theta\cos U-\cos\alpha\sin U \\ -\cos\alpha\sin\theta\cos U \end{pmatrix}$$

We conclude that the contribution to the source term for ε_+^1 coming from modulating in β is given by the expression (after division by λ^2)

$$-i\cos\alpha\cdot\left[\frac{\beta''(\tau)+\frac{\lambda_\tau}{\lambda}\beta'(\tau)}{2}\cdot(1+\cos U)-\beta'(\tau)\frac{\lambda_\tau}{\lambda}RQ'(R)\sin U-\beta'(t)\lambda(t)\cdot[\epsilon_\tau+\epsilon_X]\sin U\right] \tag{9.33}$$

As a consequence, proceeding in exact analogy to the angle α, we infer

(vi) Combined leading order contribution of smooth source terms generated by modulating in α,β to (9.3). In analogy to *(iii)* before, this is given by the formula (setting $\beta_\infty := \beta(\infty)-\beta(\tau_0)$)

$$\begin{aligned} c_+(\tau) &= ((1+2\nu)-c_0(1+\nu))\,(\alpha_\infty-i\beta_\infty) \\ &\quad+\nu\tau^{-\nu^{-1}}\int_{\tau_0}^\infty\sigma^{1+\nu^{-1}}\,(\alpha''(\sigma)-i\beta''(\sigma))\,d\sigma+(c_0-1)(1+\nu)\tau^{-\nu^{-1}}\int_{\tau_0}^\infty\sigma^{\nu^{-1}}\,(\alpha'(\sigma)-i\beta'(\sigma))\,d\sigma \\ &\quad+(1+\nu)\tau^{-\nu^{-1}}\int_\tau^\infty\sigma^{\nu^{-1}}\,(\alpha'(\sigma)-i\beta'(\sigma))\,d\sigma+(1-c_0)(1+\nu)\tau^{-\nu^{-1}}\int_\tau^\infty\sigma^{\nu^{-1}}\,(\alpha'(\sigma)-i\beta'(\sigma))\,d\sigma. \end{aligned}$$

A similar analysis as in *(iii)* on the terminating value $\alpha_\infty - i\beta_\infty$ applies here. (*It seems that when we consider the combination of the α and β modulations, a coefficient like* $\cos\alpha_{infty}$ is missing in front of β.)

(vii): The combined effect of modulating on α, β on $\lim_{R\to 0} H_1^+ \varepsilon_+^1$. Here by proceeding analogously for β as for α, we conclude

Lemma 9.15. *The same conclusion as in Lemma 9.14 obtains but with α replaced by $\alpha - i\beta$.*

Up to this stage, we have only considered the effect that modulating in the angles α, β has on the $n = 1$ mode. However, there are also leading order effects on the $n = -1$ mode. Here only the effect of the singular term matters, due to the rapid decay of the fundamental system describing the $c_{-1}(\tau)$ evolution. Specifically, in light of (9.23), we need to determine the leading order effect on $\lim_{R\to 0} R^{-2} H_{-1}^+ \varepsilon_+^{-1}$ arising after applying the $n = -1$ parametrix to the singular source term generated by modulating in α, β:

(viii): The combined effect of modulating on α, β on the evolution on $c_{-1}(\tau)$ via $\lim_{R\to 0} R^{-2} H_{-1}^+ \varepsilon_+^{-1}$. We formulate this directly in analogy to the last part of Lemma 9.14:

Lemma 9.16. *There exists a function $H(\tau, R)$ coinciding with*

$$\mathcal{D}_-\left((i\beta'(\tau) + \alpha'(\tau)) \cdot \frac{\lambda(\tau)}{\lambda} \cdot [\epsilon_\tau + \epsilon_X] \cdot \sin U\right)$$

near the light cone $R < \nu\tau$, and such that setting

$$\overline{x}(\tau, \xi) := \int_{\tau_0}^{\tau} U^{(-1)}(\tau, \sigma, \xi) \cdot \mathcal{F}^{(-1)}(H(\sigma, \cdot))\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) d\sigma,$$

we have (where as usual $\chi_{\xi<M}$ is a smooth cutoff)

$$\begin{aligned}
&\int_0^\infty \chi_{\xi<M} \overline{x}(\tau, \xi) \tilde{\rho}_{-1}(\xi)\, d\xi \\
&= -\sum_{\pm} \sum_{k=1,2} \sum_{0\le j_1 \le N_1} \int_0^\infty \chi_{\xi<M} \frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}+k\nu}} \left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1} (\tilde{\rho}_{-1}(\xi))^{\frac{1}{2}}\, d\xi \\
&\qquad \cdot \sum_{j_2+i\le N_1-j_1} C_{i,j_1,j_2} \gamma^{(k,i,\pm)} \int_{\tau_0}^{\tau} \frac{\alpha'(\sigma) + i\beta'(\sigma)}{\sigma^{\frac{3}{2}+k\nu}} (\log\sigma)^{j_2} \left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k} \left(\sum_{r\le N_1-i-j_1-j_2} D_{i,j,r} \cdot \log^r\sigma\right) d\sigma \\
&+ \tilde{c}_-(\tau),
\end{aligned}$$

where the error satisfies

$$|\tilde{c}_-(\tau)| \lesssim \tau^{-4-}.$$

The coefficients $C_{i,j_1,j_2}, D_{i,j,r}, \gamma^{(k,i,\pm)}$ are the same as in Lemma 9.14.

Lemma 9.16 and Lemma 9.15 give the principal effect of modulating on the angles α, β on the principal ingoing singular part of the $n = \mp 1$ modes and from there to the delicate source terms

$$\lim_{R\to 0} R^{-2} H_{-1}^+ \varepsilon_+^{-1}, \quad \lim_{R\to 0} H_1^+ \varepsilon_+^1$$

in the ODEs for $c_{-1}(\tau), c_{+1}(\tau)$. In order to force the vanishing conditions (9.12), (9.27), we have to complement the effect of modulating on α, β by the effect of modulating on the Lorentz transform parameters v_1, v_2.

9.2.2. *The contribution of the Lorentz parameters $v_1(t), v_2(t)$.* Since the parameters will be very small (their size depending on the initial perturbation), we shall neglect terms quadratic in the v_j in the ensuing discussion. This means that for some estimates, it is permissible to replace the Lorentz transform by the simpler Galilean transform $\mathbf{x} \longrightarrow \mathbf{x} - \mathbf{t} \cdot \mathbf{v}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Of course as far as the effect of the Lorentz transform on the singular terms is concerned, the precise structure will be of crucial importance. Observe that modulating on the Lorentz parameters will affect both the $n = +1$ and the $n = -1$ modes. We first analyze the effect on the bulk term Q and thereby directly on the evolution of $c_+(\tau)$ via $\lim_{R\to 0} F_+(1)$ (compare with (9.3)), and which will turn out to be of small order $O(|\mathbf{v}|^2)$, and then we shall analyze the contribution to the source terms

$$\lim_{R\to 0} R^{-2} H^+_{-1} \varepsilon^{-1}_+, \quad \lim_{R\to 0} H^+_1 \varepsilon^1_+$$

via the effect on the singular part of U.

Effect of Lorentz modulating on the $n = +1$ mode via the bulk part. Here we may replace the bulk part $Q(R)$ by $2R$, and neglecting terms which vanish at the origin $R = 0$, the bulk part Φ gets replaced by

$$\begin{pmatrix} \lambda(t)x_1 \\ \lambda(t)x_2 \\ 1 \end{pmatrix}$$

which gets mapped into

$$\begin{pmatrix} \lambda(t - \mathbf{x}\cdot\mathbf{v})(x_1 - tv_1) \\ \lambda(t - \mathbf{x}\cdot\mathbf{v})(x_2 - tv_2) \\ 1 \end{pmatrix}.$$

Using $\left(\partial_t^2\right)'$ as usual the operator where at least one derivative falls on a component $v_j(t)$, we infer

$$\left(\partial_t^2\right)' \begin{pmatrix} \lambda(t - \mathbf{x}\cdot\mathbf{v})(x_1 - tv_1) \\ \lambda(t - \mathbf{x}\cdot\mathbf{v})(x_2 - tv_2) \\ 1 \end{pmatrix} = \begin{pmatrix} -2\lambda'(t)\cdot tv_1'(t) - \lambda(t)\cdot(tv_1)'' \\ -2\lambda'(t)\cdot tv_2'(t) - \lambda(t)\cdot(tv_2)'' \\ 0 \end{pmatrix} + \text{error},$$

where we again neglect terms quadratic in $\mathbf{v}$ and the terms which vanishes at $R = 0$. Projecting the preceding onto the frame $\{E_1, E_2\}$ and extracting the $\varphi_1^{(1)} - i\varphi_2^{(1)}$-component leads after division by λ^{-2} and conversion to the variable τ to

$$-\nu\left[\tau v_{1,\tau\tau} + 2v_{1,\tau} - i(\tau v_{2,\tau\tau} + 2v_{2,\tau})\right] - (1+\nu)\left[v_{1,\tau} - iv_{2,\tau}\right]$$

The contribution from the first bracket above to c_+ is trivial in the leading order. In fact, for any twice continuously differentiable function $f(\tau)$ with $f_\tau = 0$ for large enough τ and $f_\tau(\tau_0) = 0$, we have

$$\int_{\tau_0}^{\tau} \sigma\cdot[\sigma f_{\sigma\sigma} + 2f_\sigma]\, d\sigma = \int_{\tau_0}^{\tau} \sigma\cdot(\sigma f)_{\sigma\sigma}\, d\sigma$$

$$= \sigma(\sigma f)_\sigma|_{\tau_0}^{\tau} - \int_{\tau_0}^{\tau} (\sigma f)_\sigma \, d\sigma$$
$$= \sigma^2 f_\sigma|_{\tau_0}^{\tau} + \sigma f(\sigma)|_{\tau_0}^{\tau} - \sigma f(\sigma)|_{\tau_0}^{\tau} = 0$$

for large enough τ, and since we can impose these requirements on $v_{1,2}(\sigma)$. The contribution from the second bracket is

$$\begin{aligned}
&- (1+\nu) \int_{\tau_0}^{\tau} \sigma \left(v_{1,\sigma}(\sigma) - i v_{2,\sigma}(\sigma) \right) d\sigma \\
&= - (1+\nu) \left(\tau \left(v_1(\tau) - i v_2(\tau) \right) - \tau_0 \left(v_1(\tau_0) - i v_2(\tau_0) \right) \right) + (1+\nu) \int_{\tau_0}^{\tau} \left(v_1(\sigma) - i v_2(\sigma) \right) d\sigma.
\end{aligned}$$

In view of the moment condition satisfied by v_1, v_2, we don't force the expression on the RHS above to vanish, but instead we choose $\alpha_\infty - i\beta_\infty$ appropriately (since the moment conditions satisfied by α, β are weaker), to cancel both the non-trivial contribution of v_1, v_2 and the contribution from the constant fundamental solution of the operator $\frac{d^2}{d\tau^2} + \frac{\lambda'}{\lambda} \frac{d}{d\tau}$.

Effect of Lorentz modulating on the singular parts of the $n = \pm 1$ *modes.* Here we finally complete forcing the vanishing conditions (9.12), (9.27) to leading order by combining the effects of modulating on the angles α, β as well as on the Lorentz parameters. For this we have to analyze the leading order effect of Lorentz transforming the singular term ϵ inherent in the bulk term $U = Q + \epsilon$. Recall the definitions in subsection 7.1, and write $\epsilon = \epsilon(t, t - r) =: \epsilon(t, \tilde{X})$, where we have

$$\tilde{X} = t - r = \frac{t^2 - r^2}{t + r}$$

Then observe that setting $X = \frac{t^2 - r^2}{2t}$, we have

$$\epsilon(t, \tilde{X}) = \epsilon(t, X) + \text{error},$$

where the error term is one degree smoother than the principal term $\epsilon(t, X)$, and thus in terms of the top order singularity, it suffices to work with $\epsilon(t, X)$. We now determine the maximally singular error terms generated by Lorentz modulating. Recalling that

$$\Phi = \begin{pmatrix} \sin U \cos\theta \\ \sin U \sin\theta \\ \cos U \end{pmatrix} = \begin{pmatrix} \sin U \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \\ \sin U \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \\ \cos U \end{pmatrix}, \quad U = Q + \epsilon,$$

and (neglecting terms quadratic in v) replacing $x_1 \longrightarrow x_1 - t v_1(t)$, $x_2 \longrightarrow x_2 - t v_2(t)$, $t \longrightarrow t_{\mathbf{v}} := t - \mathbf{x} \cdot \mathbf{v}(\mathbf{t})$, while $\epsilon = \epsilon(t, X)$, and denoting by $(\partial_t^2)'$ the operator where at least one derivative falls on a factor $v_j(t)$, we compute

$$
\begin{aligned}
(\partial_t^2)'\,(\mathcal{L}_{v(t)}\Phi) = & \begin{pmatrix} \cos(Q+\epsilon)\epsilon_X\cdot\frac{\partial X}{\partial t}\cdot\frac{-tv_1'}{\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}} \\ \cos(Q+\epsilon)\epsilon_X\cdot\frac{\partial X}{\partial t}\cdot\frac{-tv_2'}{\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}} \\ 0 \end{pmatrix} \\
&+ \begin{pmatrix} \cos(Q+\epsilon)\epsilon_X\cdot\frac{\partial X}{\partial t}\cdot\frac{-(-tv_1')(x_1-tv_1(t))^2-(-tv_2')(x_1-tv_1(t))(x_2-tv_2(t))}{\left[\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}\right]^3} \\ \cos(Q+\epsilon)\epsilon_X\cdot\frac{\partial X}{\partial t}\cdot\frac{-(-tv_2')(x_2-tv_2(t))^2-(-tv_1')(x_1-tv_1(t))(x_2-tv_2(t))}{\left[\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}\right]^3} \\ 0 \end{pmatrix} \\
&+ \begin{pmatrix} \cos(Q+\epsilon)\mathbf{v}'(t)\cdot\nabla_{\mathbf{v}}(t_{\mathbf{v}})\epsilon_{tX}\frac{x_1-tv_1}{\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}} \\ \cos(Q+\epsilon)\mathbf{v}'(t)\cdot\nabla_{\mathbf{v}}(t_{\mathbf{v}})\epsilon_{tX}\frac{x_2-tv_2}{\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}} \\ -\sin(Q+\epsilon)\mathbf{v}'(t)\cdot\nabla_{\mathbf{v}}(t_{\mathbf{v}})\epsilon_{tX} \end{pmatrix} \\
&+ \begin{pmatrix} \cos(Q+\epsilon)\left[(\partial_t^2)'X\right]\epsilon_X\frac{x_1-tv_1}{\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}} \\ \cos(Q+\epsilon)\left[(\partial_t^2)'X\right]\epsilon_X\frac{x_2-tv_2}{\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}} \\ -\sin(Q+\epsilon)\left[(\partial_t^2)'X\right]\epsilon_X \end{pmatrix} \\
&+ \begin{pmatrix} \cos(Q+\epsilon)(\partial_t)'X\cdot\epsilon_{XX}\frac{x_1-tv_1}{\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}} \\ \cos(Q+\epsilon)(\partial_t)'X\cdot\epsilon_{XX}\frac{x_2-tv_2}{\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}} \\ -\sin(Q+\epsilon)(\partial_t)'X\epsilon_{XX} \end{pmatrix} \\
&+ \text{error},
\end{aligned}
\tag{9.34}
$$

where the last term denotes expressions that are either quadratic in v or one degree smoother than ϵ_X. Observe that we can simplify the sum of the first two vectors on the right to

$$
\begin{pmatrix} \cos(Q+\epsilon)\epsilon_X\cdot\frac{\partial X}{\partial t}\cdot\frac{(-tv_1')(x_2-tv_2(t))^2-(-tv_2')(x_1-tv_1(t))(x_2-tv_2(t))}{\left[\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}\right]^3} \\ \cos(Q+\epsilon)\epsilon_X\cdot\frac{\partial X}{\partial t}\cdot\frac{(-tv_2')(x_1-tv_1(t))^2-(-tv_1')(x_1-tv_1(t))(x_2-tv_2(t))}{[\sqrt{(x_1-tv_1(t))^2+(x_2-tv_2(t))^2}]^3} \\ 0 \end{pmatrix},
$$

which, up to more regular terms, is equal to

$$
\begin{pmatrix} \cos U\epsilon_X\cdot\left(\sin\theta\cos\theta v_2'-\sin^2\theta v_1'\right) \\ \cos U\epsilon_X\cdot\left(\sin\theta\cos\theta v_1'-\cos^2\theta v_2'\right) \\ 0 \end{pmatrix} = \cos U\epsilon_X\left(-\cos\theta v_2'(t)+\sin\theta v_1'(t)\right)E_2. \tag{9.35}
$$

Here we also used the fact that up to more regular terms and quadratic terms in $\mathbf{v}$, we have $\frac{\partial X}{\partial t}\simeq 1$. The last three terms in (9.34) can be written as

$$
\left(\mathbf{v}'(t)\cdot\nabla_{\mathbf{v}}(t_{\mathbf{v}})\epsilon_{tX}+\left[(\partial_t^2)'X\right]\epsilon_X+(\partial_t)'X\cdot\epsilon_{XX}\right)E_1.
$$

Again up to the more regular terms as well as the quadratic terms in $|\mathbf{v}|$, we have

$$\begin{aligned}(\partial_t)'X \simeq&(t-r)\cdot\left(\cos\theta v_1'(t)+\sin\theta v_2'(t)\right),\\ (\partial_t^2)'X \simeq&\left(\cos\theta v_1'(t)+\sin\theta v_2'(t)\right),\\ \mathbf{v}'(t)\cdot\nabla_{\mathbf{v}}(t_{\mathbf{v}}) \simeq&-t\left(\cos\theta v_1'(t)+\sin\theta v_2'(t)\right)\end{aligned}$$

Therefore the last three terms in (9.34) is equal to, up to the more regular terms and the quadratic terms in $|\mathbf{v}|$,

$$\left(-t\left(\cos\theta v_1'(t)+\sin\theta v_2'(t)\right)\epsilon_{tX}+\left(\cos\theta v_1'(t)+\sin\theta v_2'(t)\right)\epsilon_X+(t-r)\cdot\left(\cos\theta v_1'(t)+\sin\theta v_2'(t)\right)\epsilon_{XX}\right)E_1 \tag{9.36}$$

Moreover, a direct calculation shows

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\cos\theta v_1'(t)+\sin\theta v_2'(t)\right)e^{\pm i\theta}\,d\theta=\frac{1}{2}\left(v_1'(t)\pm iv_2'(t)\right),$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\left(-\cos\theta v_2'(t)+\sin\theta v_1'(t)\right)e^{\pm i\theta}\,d\theta=-\frac{1}{2}v_2'(t)\pm\frac{i}{2}v_1'(t).$$

Interpreting the terms in (9.34) as $\mathcal{L}_{v(t)}\tilde{\Phi}$, and writing

$$\tilde{\Phi}=\tilde{\varphi}_1E_1+\tilde{\varphi}_2E_2+(*)\Phi,$$

we easily infer the following formula for the contribution to $\tilde{\varphi}_1^{(1)}-i\tilde{\varphi}_2^{(1)}$:

$$\tilde{\varphi}_1^{(1)}-i\tilde{\varphi}_2^{(1)}=\frac{1}{2}\left((t-r)\epsilon_{XX}-t\epsilon_{tX}+\epsilon_X\right)\left(v_1'(t)-iv_2'(t)\right)-\frac{1}{2}(\cos U)\epsilon_X(v_1'(t)-iv_2'(t)). \tag{9.37}$$

Similarly, as for the contribution to the $n=-1$ mode, we find

$$\tilde{\varphi}_1^{(-1)}-i\tilde{\varphi}_2^{(-1)}=\frac{1}{2}\left((t-r)\epsilon_{XX}-t\epsilon_{tX}+\epsilon_X\right)\left(v_1'(t)+iv_2'(t)\right)-\frac{1}{2}(\cos U)\epsilon_X(v_1'(t)+iv_2'(t)). \tag{9.38}$$

Observe here that the expression $(t-r)\epsilon_{XX}-t\epsilon_{tX}+\epsilon_X$ is again a type of singularity as described in subsection 7.1, specifically of type Q'. Normalizing things by dividing by λ^{-2}, we can then formulate an analogue of Lemma 9.16 , Lemma 9.15, which takes into account the combined effect of modulating on the angles α,β as well as the Lorentz parameters v_1,v_2, in terms of the effect on the top order singular terms and their contribution to the source terms for the ODEs governing $c_+(\tau),c_-(\tau)$.

Lemma 9.17. *There exists a function $H(\tau,R)$ coinciding with*

$$\mathcal{D}_+\left((9.31)+(9.33)+\lambda^{-2}\cdot(9.37)\right)$$

on the light cone $R<\nu\tau$, and setting

$$\overline{x}(\tau,\xi):=\int_{\tau_0}^{\tau}U^{(1)}(\tau,\sigma,\xi)\cdot\mathcal{F}^{(1)}\left(H(\sigma,\cdot)\right)\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma,$$

we have (where as usual $\chi_{\xi<M}$ is a smooth cutoff)

$$\int_0^\infty\chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_1(\xi)\,d\xi$$

$$= -\sum_{\pm}\sum_{k=1,2}\sum_{0\le j_1\le N_1}\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{\frac{k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}\,d\xi$$

$$\cdot\Bigg[\sum_{j_2+i\le N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}\frac{\alpha'(\sigma)-i\beta'(\sigma)}{\sigma^{\frac32+k\nu}}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\le N_1-i-j_1-j_2} D_{i,j,r}(\sigma)\cdot\log^r\sigma\right)d\sigma$$

$$+\sum_{j_2+i\le N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}\frac{v_1'(\sigma)-iv_2'(\sigma)}{\sigma^{1+k\nu}}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\le N_1-i-j_1-j_2}\tilde{D}_{i,j,r}(\sigma)\cdot\log^r\sigma\right)d\sigma\Bigg]$$

$$+\,\tilde{c}_+(\tau),$$

where the error satisfies

$$|\tilde{c}_+(\tau)|\lesssim\tau^{-4-}.$$

The coefficients $C_{i,j_1,j_2}, D_{i,j,r}, \gamma^{(k,i,\pm)}$ are the same as in Lemma 9.14, while the $\tilde{D}_{i,j,r}$ are suitable complex valued functions of σ
Replacing $n = +1$ by $n = -1$ and proceeding analogously to the preceding, we infer mutatis mutandis the following formula (where we think of all parameters $\alpha, \beta, v_{1,2}$ as functions of τ, i. e. the re-scaled time):

$$\int_0^\infty \chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_{-1}(\xi)\,d\xi$$

$$= -\sum_{\pm}\sum_{k=1,2}\sum_{0\le j_1\le N_1}\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac12}}}{\xi^{\frac12+\frac{k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}(\tilde{\rho}_{-1}(\xi))^{\frac12}\,d\xi$$

$$\cdot\Bigg[\sum_{j_2+i\le N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}\frac{-\alpha'(\sigma)-i\beta'(\sigma)}{\sigma^{\frac32+k\nu}}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\le N_1-i-j_1-j_2} D_{i,j,r}(\sigma)\cdot\log^r\sigma\right)d\sigma$$

$$+\sum_{j_2+i\le N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}\frac{v_1'(\sigma)+iv_2'(\sigma)}{\sigma^{1+k\nu}}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\le N_1-i-j_1-j_2}\tilde{D}_{i,j,r}(\sigma)\cdot\log^r\sigma\right)d\sigma\Bigg]$$

$$+\,\tilde{c}_-(\tau),$$

where the error satisfies

$$|\tilde{c}_-(\tau)|\lesssim\tau^{-4-}.$$

By taking advantage of the freedom to specify the quadruple of real valued functions α, β, v_1, v_2, we can finally enforce the required vanishing conditions (9.27), (9.12):

Proposition 9.18. *There are finite dimensional vector spaces V_j, $j = 1, 2, 3, 4$ of C^∞ functions for each of α, β, v_1, v_2, such that $\alpha_\tau, \beta_\tau, v_{1,\tau}, v_{2,\tau}$ are compactly supported on some interval $[\tau_0, \tau_1]$, $\tau_0 < \tau_1$, and such that given complex valued functions $\beta^{(k,r)}(\tau)$, $k \in \{1, 2\}, r \in \{0, 1, \dots, N_1\}$, $\tilde{\beta}^{(k,r)}(\tau)$, $k \in \{1, 2\}, r \in \{0, 1, \dots, N_1\}$, there exist functions $(\alpha, \beta, v_1, v_2) \in \times_{j=1}^4 V_j$, and such that setting*

$$\zeta^{(i+j_1+j_2,k)}(\sigma) := \frac{\alpha'(\sigma)-i\beta'(\sigma)}{\sigma^{\frac32+k\nu}}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\le N_1-i-j_1-j_2} D_{i,j,r}(\sigma)\cdot\log^r\sigma\right)$$

$$+\frac{\nu_1'(\sigma) - i\nu_2'(\sigma)}{\sigma^{1+k\nu}}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\leq N_1-i-j_1-j_2}\tilde{D}_{i,j,r}(\sigma)\cdot\log^r\sigma\right)$$

$$\eta^{(i+j_1+j_2,k)}(\sigma) := \frac{-\alpha'(\sigma) - i\beta'(\sigma)}{\sigma^{\frac{3}{2}+k\nu}}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\leq N_1-i-j_1-j_2}D_{i,j,r}(\sigma)\cdot\log^r\sigma\right)$$
$$+\frac{\nu_1'(\sigma) + i\nu_2'(\sigma)}{\sigma^{1+k\nu}}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\leq N_1-i-j_1-j_2}\tilde{D}_{i,j,r}(\sigma)\cdot\log^r\sigma\right),$$

we haves the relations

$$\sum_{j_2+i\leq N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\infty}\left[\beta^{(k,j_1+j_2+i)}(\sigma) + \zeta^{(i+j_1+j_2,k)}(\sigma)\right]d\sigma = 0$$
$$\sum_{j_2+i\leq N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\infty}\left[\tilde{\beta}^{(k,j_1+j_2+i)}(\sigma) + \eta^{(i+j_1+j_2,k)}(\sigma)\right]d\sigma = 0$$

Moreover, one can also independently prescribe the values of $\alpha_\infty = \lim_{\tau\to\infty}\alpha(\tau)$*,* $\beta_\infty = \lim_{\beta\to\infty}\beta(\tau)$ *(compare with (vi) in the preceding sub-subsection).*

Verification of this proceeds by an elementary completely explicit computation, exploiting the fact that for $k = 1, 2$, there are in fact only two logarithmic powers to consider.

At this stage, we the only obstructions left to force sufficient decay for all parameters at time $\tau = \infty$ comes from (9.20), arising for the $n = 0$ mode. We shall force these by modulating in the remaining angle $h(t)$ as well as the scaling parameter, meaning replacing λ by $\tilde{\lambda}$.

9.2.3. *The contribution of the remaining angle* $h(t)$ *and scaling.* At this point we have not yet exploited the final remaining rotation which acts in terms of the angle $h(t)$ via

$$\begin{pmatrix}\cos h(t) & -\sin h(t) & 0\\ \sin h(t) & \cos h(t) & 0\\ 0 & 0 & 1\end{pmatrix}$$

Alternatively, the effect on the bulk part will be that the angle θ in the representation $\Phi = \begin{pmatrix}\cos\theta\sin U\\ \sin\theta\sin U\\ \cos U\end{pmatrix}$ gets replaced by $\theta + h(t)$. We compute $(\partial_t^2)'\mathcal{R}_{h(t)}\Phi$ to leading order, where $(\partial_t^2)'$ means at least one derivative hits $h(t)$, and we only keep track of terms linear in h, and which are maximally singular at the light cone, i. e. which contribute to the principal incoming part. This immediately leads to the term

$$h'(t)\mathcal{R}_{h(t)}\circ\mathcal{R}_{-h(t)}\begin{pmatrix}-\sin h(t) & -\cos h(t) & 0\\ \cos h(t) & -\sin h(t) & 0\\ 0 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix}\cos\theta\cos U\epsilon_X\\ \sin\theta\cos U\epsilon_X\\ -\sin U\epsilon_X\end{pmatrix}$$

$$= h'(t)\mathcal{R}_{h(t)} \begin{pmatrix} \sin\theta\cos U \\ -\cos\theta\cos U \\ 0 \end{pmatrix} \cdot \epsilon_X + \text{error}.$$

and in terms of the frame $\{E_1, E_2\}$, we have

$$h'(t) \begin{pmatrix} \sin\theta\cos U \\ -\cos\theta\cos U \\ 0 \end{pmatrix} \cdot \epsilon_X = -h'(t)\cos U \epsilon_X E_2,$$

whence modulating in h contributes to leading order a term $h'(t)\cos U\epsilon_X$ to $\varphi_2^{(0)}$, which constitutes the imaginary part of the quantity

$$\varepsilon_+^{(0)} = \varphi_1^{(0)} - i\varphi_2^{(0)}.$$

In order to handle the real part, we have to use our final modulation parameter, namely scaling. Precisely, let us set $\tilde{\lambda} = c(t)\cdot\lambda(t)$ where $\lambda(t) = t^{-1-\nu}$, and where $c(t_0) = 1$ and $c'(t)$ is compactly supported on some interval $[t_1, t_0]$ with $t_1 > 0$. Now computing

$$\left(\partial_t^2\right)' \Phi_{c(t)},$$

where the subscript denotes a space-time rescaling of the function, and $\left(\partial_t^2\right)'$ indicates that at least one derivative falls on $c(t)$, we compute (again to leading order both in terms of its dependence on $c(t)$ as well as the singularity at the light cone), we get

$$\left(\partial_t^2\right)' \Phi_{c(t)} = \begin{pmatrix} \cos\theta\cos U \\ \sin\theta\cos U \\ -\sin U \end{pmatrix}_{c(t)} \cdot (2c'\epsilon_X + 2tcc'\epsilon_{tX} + 2\epsilon_{XX}(t-r)cc') + \text{error},$$

where error also comprises top order singular terms depending linearly on c' but which have an extra τ^{-1} smallness factor. Then the main term on the right will clearly only contribute to $\varphi_1^{(0)}$, namely the term

$$2c'\epsilon_X + 2tcc'\epsilon_{tX} + 2\epsilon_{XX}(t-r)cc',$$

which up to smaller order terms can be equated with

$$c'\left(2\epsilon_X + 2t\epsilon_{tX} + 2(t-r)\epsilon_{XX}\right).$$

Combining the effects of modulating in h and in c, we find the following principal contribution to $\varphi_1^{(0)} - i\varphi_2^{(0)}$:

$$c'\left(2\epsilon_X + 2t\epsilon_{tX} + 2(t-r)\epsilon_{XX}\right) - ih'(t)\cos U\epsilon_X$$

We can then formulate the final lemma giving the remaining contribution to the principal singular part at angular momentum $n = 0$:

Lemma 9.19. *There exists a function $H(\tau, R)$ coinciding with*

$$\mathcal{D}_0\left(\lambda^{-2}\left[c'\left(2\epsilon_X + 2t\epsilon_{tX} + 2(t-r)\epsilon_{XX}\right) - ih'(t)\cos U\epsilon_X\right]\right)$$

near the light cone $R = \nu\tau$, and setting

$$\overline{x}(\tau,\xi) := \int_{\tau_0}^{\tau} U^{(0)}(\tau,\sigma,\xi)\cdot\mathcal{F}^{(0)}(H(\sigma,\cdot))\left(\frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right) d\sigma,$$

we have (where as usual $\chi_{\xi<M}$ is a smooth cutoff, and by abuse of notation, we interpret c, h as functions of the re-scaled variable τ)

$$
\begin{aligned}
&\int_0^\infty \chi_{\xi<M}\overline{x}(\tau,\xi)\tilde{\rho}_0(\xi)\,d\xi\\
&= -\sum_{\pm}\sum_{k=1,2}\sum_{0\leq j_1\leq N_1}\int_0^\infty \chi_{\xi<M}\frac{e^{\pm i\nu\tau\xi^{\frac{1}{2}}}}{\xi^{\frac{1}{2}+\frac{k\nu}{2}}}\left[\log\left(\xi\lambda^2(\tau)\right)\right]^{j_1}(\tilde{\rho}_0(\xi))^{\frac{1}{2}}\,d\xi\\
&\qquad\cdot\left[\sum_{j_2+i\leq N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}\frac{c'(\sigma)}{\sigma^{1+k\nu}}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\leq N_1-i-j_1-j_2}E_{i,j,r}(\sigma)\cdot\log^r\sigma\right)d\sigma\right.\\
&\qquad\left.+\sum_{j_2+i\leq N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\tau}\frac{ih'(\sigma)}{\sigma^{1+k\nu}}(\log\sigma)^{j_2}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\leq N_1-i-j_1-j_2}\tilde{E}_{i,j,r}(\sigma)\cdot\log^r\sigma\right)d\sigma\right]\\
&+c_0(\tau),
\end{aligned}
$$

where the error satisfies

$$|c_0(\tau)| \lesssim \tau^{-4-}.$$

Then we can formulate the analogue of Prop. 9.18 for the angular momentum $n = 0$ case:

Proposition 9.20. *There are finite dimensional vector spaces $W_j,\ j = 1,2$ of real valued C^∞ functions for each of c', h, such that c', h are compactly supported on some interval $[\tau_0, \tau_1]$, $\tau_0 < \tau_1$, and such that given a complex valued function $\beta^{(k,r)}(\tau)$, $k \in \{1,2\}, r \in \{0,1,\ldots,N_1\}$, there exist functions $(c', h) \in \times_{j=1}^2 W_j$, and such that setting*

$$
\begin{aligned}
\psi^{(i+j_1+j_2,k)}(\sigma) := {}&\frac{c'(\sigma)}{\sigma^{1+k\nu}}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\leq N_1-i-j_1-j_2}E_{i,j,r}(\sigma)\cdot\log^r\sigma\right)\\
&+\frac{ih'(\sigma)}{\sigma^{1+k\nu}}\left[\frac{\lambda(\sigma)}{\lambda(\tau)}\right]^{\nu k}\left(\sum_{r\leq N_1-i-j_1-j_2}\tilde{E}_{i,j,r}(\sigma)\cdot\log^r\sigma\right)
\end{aligned}
$$

we haves the relations

$$\sum_{j_2+i\leq N_1-j_1} C_{i,j_1,j_2}\gamma^{(k,i,\pm)}\int_{\tau_0}^{\infty}(\log\sigma)^{j_2}\left[\beta^{(k,j_1+j_2+i)}(\sigma)+\psi^{(i+j_1+j_2,k)}(\sigma)\right]d\sigma = 0.$$

Taking advantage of Prop. 9.20, Prop. 9.18, we can finally formulate the analogue of the key result Prop. 8.12 in the context of the exceptional modes $n = 0, \pm 1$.

9.3. **The final estimates for the source terms arising for the exceptional modes, both away from and near the light cone.** Recalling the tools and methods from section 6.7 as well as section 7, and the preceding modulation techniques, we can finally formulate the analogues of Prop. 6.23, Prop. 6.24, Prop. 6.28 as well as Prop. 8.12. Recall the ansatz (9.28). Then in analogy to (7.21) (which is in fact valid for all angular momenta), we have modified equations with our new representation of the solution, which take into account the time dependence of the modulation parameters. In case of angular momenta $|n| \geq 2$, where (7.21) is indeed the relevant equation, let us write the new equations in the final form

$$-\left(\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 + \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\right)\varepsilon_\pm(n) + H_n^\pm\varepsilon_\pm(n) = F_\pm(n) + G_\pm(n), \tag{9.39}$$

where the term $G_\pm(n)$ takes into account the additional terms generated by modulating and at angular momentum $|n| \geq 2$. For the exceptional modes, we write the corresponding equations in the form

$$\begin{aligned}&-\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 \mathcal{D}_n\varepsilon_+^n - 3\frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\mathcal{D}_n\varepsilon_+^n + \tilde{H}_n^+\mathcal{D}_n\varepsilon_+^n - \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}_n\varepsilon_+^n \\ &=: \mathcal{R}_+^n\left(\varepsilon_+^n, \mathcal{D}_n\varepsilon_+^n\right) + \mathcal{D}_n\left(F_+(n)\right) + \mathcal{D}_n\left(G_+(n)\right),\end{aligned} \tag{9.40}$$

to be complemented by the evolution equations for the unstable modes (9.3), (9.15), (9.23), where of course $F_+(n)$ gets replaced by $F_+(n) + G_+(n)$. Also introduce the notation

$$-\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 \mathcal{D}_n\varepsilon_+^n - 3\frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\mathcal{D}_n\varepsilon_+^n + \tilde{H}_1^+\mathcal{D}_n\varepsilon_+^n - \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}_n\varepsilon_+^n = \tilde{\Box}_n\left(\mathcal{D}_n\varepsilon_+^n\right)$$

Proposition 9.21. *Assume the representations* (6.54) *and the bounds* (8.5)*. Then, recalling Prop. 9.20, Prop. 9.18, there exist* $(\alpha, \beta, v_1, v_2) \in \times_{j=1}^4 V_j$ *as well as* $(c', h) \in \times_{j=1}^2 W_j$*, satisfying the bound*[11]

$$\tau_0^{-1}\left(|\alpha| + |\beta|\right) + \sum_{j=1}^2 \left|v_j\right| + \left|c'\right| + |h| \lesssim \tau_0^{-1}\Lambda,$$

and such that the following conclusion applies: for each $n \in \{0, \pm 1\}$*, there exist*

$$\psi^+(n) = \int_0^\infty \phi_n(R,\xi)\overline{z}_n(\tau,\xi)\tilde{\rho}_n(\xi)\,d\xi, \quad H_n(\tau, R),$$

with

$$\begin{aligned}&\left(\mathcal{R}_+^n\left(\varepsilon_+^n, \mathcal{D}_n\varepsilon_+^n\right) + \mathcal{D}_n\left(F_+(n)\right) + \mathcal{D}_n\left(G_+(n)\right) - \tilde{\Box}_n\psi^+(n)\right)|_{R<\nu\tau} = H_n(\tau,R)|_{R<\nu\tau}, \\ &H_n(\tau,R) = \int_0^\infty \overline{y}_n(\tau,\xi)\phi_n(R,\xi)\tilde{\rho}_n(\xi)\,d\xi,\end{aligned}$$

and the bounds (recalling Definition 8.11)

$$\|\overline{z}_n(\tau,\xi)\|_{good} \lesssim \Lambda^3, \quad \left\|\overline{y}_n\right\|_{goodsource} \lesssim \Lambda^2 + \tau_0^{-1}\Lambda.$$

Furthermore, setting

$$\overline{x}_n(\tau,\xi) = \int_{\tau_0}^\tau U^{(n)}(\tau,\sigma,\xi)\,\overline{y}_n\left(\sigma, \frac{\lambda^2(\tau)}{\lambda^2(\sigma)}\xi\right)d\sigma + \overline{z}_n$$

and defining $\tilde{c}_n$*,* $n = 0, \pm 1$ *as the the solution of* (9.23)*,* (9.15)*,* (9.3) *but with* $F_\pm(n)$ *replaced by* $F_\pm(n)+G_\pm(n)$ *and with initial data* $(c_n(\tau_0), c_n'(\tau_0))$ *at time* $\tau = \tau_0$ *and with* $c(\tau)$ *as in the representation of* $\varepsilon_+(n)$*,* $n = 0, \pm 1$*, we have (recalling Definition 8.11,* (8.5)*)*

$$\|(c_n, \overline{x}_n)\|_{good} \lesssim |c_n(\tau_0)| + \tau_0\left|c_n'(\tau_0)\right| + \Lambda^2 + \tau_0^{-1}\Lambda.$$

[11] Here any fixed norm on the finite dimensional vector spaces V_j, W_k can be used.

As for Prop. 8.12, the correction term $\psi^{+}(n)$ is required to control certain source terms near the light cone. The preceding proposition then entails in particular the analogues of Prop. 6.23, Prop. 6.24, Prop. 6.28, which can be separately formulated as

Proposition 9.22. *Letting $\chi_{R<\frac{\tau}{2}}$ a smooth cutoff localizing to the indicated region, and under the preceding assumptions, we can write (for $n = 0, \pm 1$)*

$$\chi_{R<\frac{\tau}{2}}\left(\mathcal{R}_{+}^{n}\left(\varepsilon_{+}^{n}, \mathcal{D}_{n}\varepsilon_{+}^{n}\right) + \mathcal{D}_{n}\left(F_{+}(n)\right) + \mathcal{D}_{n}\left(G_{+}(n)\right)\right) = \int_{0}^{\infty} \phi_{n}(R,\xi)\tilde{\bar{y}}_{n}(\tau,\xi)\tilde{\rho}_{n}(\xi)\, d\xi,$$

where (recall Definition 8.11)

$$\left\|\tilde{\bar{y}}_{n}\right\|_{goodsource} \lesssim \Lambda^{2} + \tau_{0}^{-1}\Lambda.$$

The combination of Proposition 9.21, Proposition 8.12 finally furnishes us with the tools to control an iterative process leading to the desired solution of the initial value problem.

10. The iterative process and construction of the solution

10.1. **Basic setup and zeroth iterate.** Let $\Phi = \begin{pmatrix} \cos\theta\sin U \\ \sin\theta\sin U \\ \cos U \end{pmatrix}$ the unperturbed co-rotational blow up solution, and at time $t = t_0$, corresponding to $\tau = \tau_0 = \int_{t_0}^{\infty} \lambda(s)\, ds$ consider a perturbation of its data $(\Phi, \Phi_t)\,|_{t=t_0}$, which we write in the form

$$(\Psi, \Psi_t) = \left(\Phi|_{t=t_0} + \sum_{j=1,2} \varphi_{j}^{ini} E_{j}|_{t=t_0} + (*)\Phi,\ \sum_{j=1,2} \tilde{\varphi}_{j}^{ini} E_{j}|_{t=t_0}\right). \tag{10.1}$$

Here recall that the frame $\{E_1, E_2\}$ is given by $\left\{\begin{pmatrix} \cos\theta\sin U \\ \sin\theta\sin U \\ \cos U \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}\right\}$. We shall make very conservative assumptions on the functions $\varphi_j, \tilde{\varphi}_j$, certainly far from optimal: first, we assume that all φ_j are supported on $r < \frac{t_0}{2}$, i. e. away from the light cone[12]. Further, we assume the bound

$$\sum_{j=1,2} \left\|\varphi_{j}^{ini}\right\|_{H^{100}} + \sum_{j=1,2} \left\|\tilde{\varphi}_{j}^{ini}\right\|_{H^{99}} \leq \tau_{0}^{-100}. \tag{10.2}$$

Expanding each component into angular modes, we write (using now the re-scaled variable $R = r\lambda(t_0)$)

$$\varphi_{j}^{ini} = \sum_{n\in\mathbb{Z}} \varphi_{j}^{ini}(n)(R)e^{in\theta}, \quad \tilde{\varphi}_{j} = \sum_{n\in\mathbb{Z}} \tilde{\varphi}_{j}^{ini}(n)(R)e^{in\theta}$$

and our assumption (10.2) easily implies, passing to the variables(which are each functions of R)

$$\varepsilon_{\pm}^{ini}(n) = \varphi_{1}^{ini}(n) \mp i\varphi_{2}^{ini}(n), \quad \tilde{\varepsilon}_{\pm}^{ini}(n) = \lambda^{-1}(t_0)\left[\tilde{\varphi}_{1}^{ini}(n) \mp i\tilde{\varphi}_{2}^{ini}(n)\right],$$

and further defining

$$\tilde{\tilde{\varepsilon}}_{\pm}^{ini}(n) = \tilde{\varepsilon}_{\pm}^{ini}(n) + \frac{\lambda_{\tau}}{\lambda}|_{\tau=\tau_0} \cdot (R\partial_R)\,\varepsilon_{\pm}^{ini}(n),$$

[12] This assumption is purely technical to simplify the details of the modulation step and can certainly be eliminated.

we consider the family of problems (recall (6.2))

$$\begin{aligned}&-\left(\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)^2+\frac{\lambda_\tau}{\lambda}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\right)\varepsilon_\pm(n)+H_n^\pm\varepsilon_\pm(n)=F_\pm(n)+G_\pm(n),\\ &\varepsilon_\pm(n)|_{\tau=\tau_0}=\varepsilon_\pm^{ini}(n),\quad \left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\varepsilon_\pm(n)|_{\tau=\tau_0}=\tilde{\tilde{\varepsilon}}_\pm^{ini}(n).\end{aligned}\tag{10.3}$$

The source terms $F_\pm(n)$ in turn are as in (6.3), while the term $G_\pm(n)$ reflects the additional source terms generated by modulating in the symmetries. These are the equations which we solve for the $|n|\geq 2$ angular momenta, the exceptional angular momenta $n\in\{0,\pm1\}$ requiring passage to the supersymmetric formulation and application of symmetry modulations. In order to apply the methods from subsection 8.2, everything has to be converted to the (distorted) Fourier side, for which we use the simple

Lemma 10.1. *Writing*

$$\varepsilon_\pm(n)(\tau,R)=\int_0^\infty \overline{x}_n(\tau,\xi)\phi_n(R,\xi)\rho_n(\xi)\,d\xi,$$

we have (recall (8.2)*)*

$$\begin{aligned}\overline{x}_n(\tau_0,\xi)=\left\langle\varepsilon_\pm^{ini}(n),\phi_n(R,\xi)\right\rangle_{L^2_{RdR}},\quad \mathcal{D}_\tau\overline{x}_n(\tau,\xi)|_{\tau=\tau_0}&=\left\langle\tilde{\tilde{\varepsilon}}_\pm^{ini}(n),\phi_n(R,\xi)\right\rangle_{L^2_{RdR}}-\frac{\lambda'(\tau)}{\lambda(\tau)}\left(\frac{\rho_n'(\xi)\xi}{\rho_n(\xi)}+2\right)\overline{x}_n(\tau_0,\xi)\\ &\quad-\frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_\hbar^{(0)}\overline{x}_n(\tau_0,\xi)\end{aligned}$$

Moreover, we have the bound (recall the assumption for Prop. 8.12)

$$\sum_{|n|\geq2}n^{12}\left[\|\overline{x}_n(\tau_0,\xi)\|_{S_0^{(n)}}+\|\mathcal{D}_\tau x_n(\tau_0,\xi)\|_{S_1^{(n)}}\right]\lesssim\tau_0^{-100}\ll1,$$

provided $\tau_0\gg1$. *In particular, letting* $\varepsilon_\pm^{(0)}(n)$ *be the solution of the auxiliary linear equation*

$$\begin{aligned}&-\left(\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)^2+\frac{\lambda_\tau}{\lambda}\left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\right)\varepsilon_\pm^{(0)}(n)+H_n^\pm\varepsilon_\pm^{(0)}(n)=0,\\ &\varepsilon_\pm(n)|_{\tau=\tau_0}=\varepsilon_\pm^{ini}(n),\quad \left(\partial_\tau+\frac{\lambda_\tau}{\lambda}R\partial_R\right)\varepsilon_\pm(n)|_{\tau=\tau_0}=\tilde{\tilde{\varepsilon}}_\pm^{ini}(n),\end{aligned}$$

and writing $\varepsilon_\pm^{(0)}(n)(\tau,R)=\int_0^\infty\overline{x}_n^{(0)}(\tau,\xi)\phi_n(R,\xi)\rho_n(\xi)\,d\xi$, *we have (always assuming* $|n|\geq2$*) the bound*

$$\sum_{|n|\geq2}n^{12}\left\|\overline{x}_n^{(0)}(\tau,\xi)\right\|_{good}\lesssim\tau_0^{-100},$$

where we keep in mind Definition 8.4 for the norm. In fact $\overline{x}_n^{(0)}(\tau,\xi)$ *only consists of a smooth part which decays (much faster than)* τ^{-3}.

Dealing with the exceptional modes is more complicated, but we still define the zeroth iterate in the simplest nontrivial possible way: recalling (9.40), as well as the representation of $\varepsilon_\pm(n)$, $n\in\{0,\pm1\}$, in terms of $(c_n, \mathcal{D}_n\varepsilon_\pm(n))$, we define the first iterate, as pair of functions $\left(c_n^{(0)}, \mathcal{D}_n\varepsilon_\pm^{(0)}(n)\right)$, via

$$\begin{aligned}
&-\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 \mathcal{D}_n\varepsilon_+^{(0)}(n) - 3\frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\mathcal{D}_n\varepsilon_+^{(0)}(n)\\
&+ \tilde{H}_n^+\mathcal{D}_n\varepsilon_+^{(0)}(n) - \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}_n\varepsilon_+^{(0)}(n) = 0,\\
&\mathcal{D}_n\varepsilon_\pm^{(0)}(n)|_{\tau=\tau_0} = \mathcal{D}_n\varepsilon_\pm^{ini}(n), \quad \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\mathcal{D}_n\varepsilon_\pm(n)|_{\tau=\tau_0} = \mathcal{D}_n\tilde{\tilde{\varepsilon}}_\pm^{ini}(n) + \frac{\lambda_\tau}{\lambda}\left[R\partial_R, \mathcal{D}_n\right]\varepsilon_\pm^{ini}(n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =: \widetilde{\varepsilon_\pm^{ini}}(n)\\
&c_n^{(0)} = 0.
\end{aligned} \tag{10.4}$$

Invoking the slightly different dilation type operator on the Fourier side

$$\mathcal{D}_\tau = \mathcal{D}_\tau^{(n)} = \partial_\tau - 2\frac{\lambda_\tau}{\lambda}\xi\partial_\xi - \frac{\lambda_\tau}{\lambda}\frac{(\tilde{\rho}_n(\xi))'\,\xi}{\tilde{\rho}_n(\xi)} - \frac{\lambda_\tau}{\lambda},$$

we can then formulate the analogue of the preceding lemma for the exceptional modes and the corresponding zeroth iterate. Note that at this stage modulation theory does not come in yet, since we force the vanishing of the instabilities:

Lemma 10.2. *Let $n \in \{0, \pm 1\}$. Writing*

$$\mathcal{D}_n\varepsilon_\pm^{(0)}(n)(\tau, R) = \int_0^\infty \overline{x}_n^{(0)}(\tau,\xi)\phi_n(R,\xi)\tilde{\rho}_n(\xi)\,d\xi,$$

we have

$$\begin{aligned}
\overline{x}_n^{(0)}(\tau_0,\xi) = \left\langle \mathcal{D}_n\varepsilon_\pm^{ini}(n), \phi_n(R,\xi)\right\rangle_{L^2_{R\,dR}}, \quad \mathcal{D}_\tau\overline{x}_n^{(0)}(\tau,\xi)|_{\tau=\tau_0} &= \left\langle \widetilde{\varepsilon_\pm^{ini}}(n), \phi_n(R,\xi)\right\rangle_{L^2_{R\,dR}}\\
&- \frac{\lambda'(\tau)}{\lambda(\tau)}\left(\frac{(\tilde{\rho}_n)'(\xi)\xi}{\tilde{\rho}_n(\xi)} + 1\right)\overline{x}_n(\tau_0,\xi)\\
&- \frac{\lambda'(\tau)}{\lambda(\tau)}\mathcal{K}_n^{(0)}\overline{x}_n(\tau_0,\xi).
\end{aligned}$$

Moreover, we have the bound (recall subsection 6.5 for the corresponding norms)

$$\left\|\overline{x}_n^{(0)}(\tau_0,\xi)\right\|_{S_0^{(n)}} + \left\|\mathcal{D}_\tau\overline{x}_n^{(0)}(\tau,\xi)\right\|_{S_1^{(n)}} \lesssim \tau_0^{-100} \ll 1,$$

provided $\tau_0 \gg 1$. Moreover, writing the solution of the first part of (10.4) *in the form*

$$\mathcal{D}_n\varepsilon_+^{(0)}(n) = \int_0^\infty \overline{x}_n^{(0)}(\tau,\xi)\phi_n(R,\xi)\tilde{\rho}_n(\xi)\,d\xi,$$

we have the bound (recalling Definition 8.11)

$$\left\|\left(0, \overline{x}_n^{(0)}(\tau,\xi)\right)\right\|_{good} \lesssim \tau_0^{-100}.$$

In fact $\overline{x}_n^{(0)}(\tau,\xi)$ only consists of a smooth part which decays (much faster than) τ^{-3}.

To complete the information for the zeroth step, we also add the (trivial) information on the modulation parameters, namely we set

$$\left(\alpha^{(0)}(t),\beta^{(0)}(t),\ h^{(0)}(t),\ (c^{(0)})'(t),\ v_1^{(0)}(t),\ v_2^{(0)}(t)\right) = (0,0,0,0,0,0)\,.$$

10.2. **The higher iterates.** For the iterative step, consider a tuple of functions

$$\left(\varphi_1^{(l-1)},\ \varphi_2^{(l-1)},\ \alpha^{(l-1)}(t),\beta^{(l-1)}(t),\ h(t),\ (c^{(l-1)})'(t),\ v_1^{(l-1)}(t),\ v_2^{(l-1)}(t)\right),\quad l\geq 1, \tag{10.5}$$

where $\left(\alpha^{(l-1)},\beta^{(l-1)},v_1^{(l-1)},\ v_2^{(l-1)}\right)\in\times_{j=1}^4 V_j$, $\left((c^{(l-1)})',h^{(l-1)}\right)\in\times_{j=1,2}W_j$, where we recall Prop. 9.18, Prop. 9.20 for the notation. Fixing a norm on each V_j, W_k once and for all, we assume the bound

$$\left|\alpha^{(l-1)}\right|+\left|\beta^{(l-1)}\right|+\left|v_1^{(l-1)}\right|+\left|v_2^{(l-1)}\right|+\left|h^{(l-1)}\right|+\left|\left(c^{(l-1)}\right)'\right|\leq\tau_0^{-50}. \tag{10.6}$$

Furthermore, setting

$$\varepsilon_\pm^{(l-1)}=\varphi_1^{(l-1)}\mp i\varphi_2^{(l-1)},$$

we use the decompositions (6.54), i. e. write (using the coordinates (τ,R,θ))

$$\varepsilon_\pm^{(l-1)}=\sum_{n\in\mathbb{Z}}\varepsilon_\pm^{(l-1)}(n)e^{in\theta},\quad \varepsilon_\pm^{(l-1)}(n)=\varepsilon_\pm^{(l-1)}(n)(\tau,R),$$

where for each $|n|\geq 2$ the function $\varepsilon_\pm^{(l-1)}(n)$ admits the representation

$$\varepsilon_\pm^{(l-1)}(n)(\tau,R)=\int_0^\infty \phi_n(R,\xi)\overline{x}_n^{(l-1)}(\tau,\xi)\rho_n(\xi)\,d\xi,$$

with $\overline{x}_n^{(l-1)}(\tau,\xi)$ good with restricted singular part in the sense of Definition 8.3, and with the bound

$$\sum_{|n|\geq 2} n^{12}\left\|\overline{x}_n^{(l-1)}(\tau,\xi)\right\|_{\text{good}(r)}\leq\tau_0^{-50},\quad |n|\geq 2. \tag{10.7}$$

We also assume that the functions $\varepsilon_\pm^{(l-1)}(n)$ satisfy the boundary conditions (10.3) at time $\tau=\tau_0$. As for the exceptional modes $n\in\{0,\pm1\}$, we write them in the form

$$\varepsilon_\pm^{(l-1)}(n)(\tau,R)=c_{n,\pm}^{(l-1)}(\tau)\phi_n(R)+\phi_n(R)\int_0^R[\phi_n(s)]^{-1}\,\mathcal{D}_n\varepsilon_\pm^{(l-1)}(n)(\tau,s)\,ds,\quad n\in\{0,\pm1\},$$

and we impose the initial conditions

$$c_{n,\pm}^{(l-1)}(\tau_0)=\lim_{R\to0}[\phi_n(R)]^{-1}\,\varepsilon_\pm^{ini}(n)(R),\quad \left(c_{n,\pm}^{(l-1)}\right)'(\tau_0)=\lim_{R\to0}[\phi_n(R)]^{-1}\,\tilde{\varepsilon}_\pm^{ini}(n)(R),$$

as well as the same initial conditions for $\mathcal{D}_n\varepsilon_\pm^{(l-1)}(n)$ as those stated for $\mathcal{D}_n\varepsilon_\pm^{(0)}(n)$ in (10.4). Further, writing

$$\mathcal{D}_n\varepsilon_\pm^{(l-1)}(n)(\tau,R)=\int_0^\infty\overline{x}_n^{(l-1)}(\tau,\xi)\phi_n(R,\xi)\tilde{\rho}_n(\xi)\,d\xi,$$

we assume that $\left(c_{n,\pm}^{(l-1)}(\tau),\overline{x}_n^{(l-1)}(\tau,\xi)\right)$ is good in the sense of Definition 8.11 and with $\overline{x}_n^{(l-1)}(\tau,\xi)$ of restricted principal singular type, and we assume the bound

$$\left\|\left(c_{n,\pm}^{(l-1)}(\tau),\overline{x}_n^{(l-1)}(\tau,\xi)\right)\right\|_{\text{good}(r)}\leq\tau_0^{-50}. \tag{10.8}$$

Finally, recalling (10.3), let $F_{\pm}^{(l-1)}(n)$ be defined in terms of $\phi_j^{(l-1)}$, $j = 1, 2$, using the expressions in (6.3), and similarly we let $G_{\pm}(n)^{(l-1)}$ be the errors generated by modulating with respect to the parameters and inputs at stage $l-1$. We can now formulate the final theorem giving the induction step and convergence to the desired solution:

Theorem 10.3. *Let $\tau_0 \gg 1$ sufficiently large. Given a tuple* (10.5) *satisfying the bounds* (10.6), (10.7),(10.8), *there exist $\left(\alpha^{(l)}, \beta^{(l)}, v_1^{(l)}, v_2^{(l)}\right) \in \times_{j=1}^4 V_j$, $\left(\left(c^{(l)}\right)', h^{(l)}\right) \in \times_{j=1,2} W_j$ satisfying the bound*

$$\left|\alpha^{(l)}\right| + \left|\beta^{(l)}\right| + \left|v_1^{(l)}\right| + \left|v_2^{(l)}\right| + \left|h^{(l)}\right| + \left|\left(c^{(l)}\right)'\right| \leq \tau_0^{-50}$$

such that for all $|n| \geq 2$ the equations

$$\begin{aligned}
&-\left(\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 + \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\right)\varepsilon_{\pm}^{(l)}(n) + H_n^{\pm}\varepsilon_{\pm}^{(l)}(n) = F_{\pm}^{(l-1)}(n) + G_{\pm}^{(l-1)}(n),\\
&\varepsilon_{\pm}^{(l)}(n)|_{\tau=\tau_0} = \varepsilon_{\pm}^{ini}(n), \quad \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\varepsilon_{\pm}^{(l)}(n)|_{\tau=\tau_0} = \tilde{\tilde{\varepsilon}}_{\pm}^{ini}(n).
\end{aligned} \tag{10.9}$$

admit solutions on the light cone $R < \nu\tau$ (meaning global solutions upon choice of suitable extensions of $\left(F_{\pm}^{(l-1)}(n) + G_{\pm}^{(l-1)}(n)\right)\Big|_{R<\nu\tau}$ beyond the light cone), which, using the representations

$$\varepsilon_{\pm}^{(l)}(n)(\tau, R) = \int_0^\infty \overline{x}_n^{(l)}(\tau, \xi)\phi_n(R, \xi)\rho_n(\xi)\, d\xi,$$

satisfy the bounds

$$\sum_{|n|\geq 2} n^{12} \left\|\overline{x}_n^{(l)}(\tau, \xi)\right\|_{good(r)} \leq \tau_0^{-50}, \quad |n| \geq 2. \tag{10.10}$$

Moreover, for the exceptional modes $n \in \{0, \pm 1\}$, the equations

$$\begin{aligned}
&-\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)^2 \mathcal{D}_n\varepsilon_+^{(l)}(n) - 3\frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\mathcal{D}_n\varepsilon_+^{(l)}(n) + \tilde{H}_n^{+}\mathcal{D}_n\varepsilon_+^{(l)}(n) - \left(2\left(\frac{\lambda'}{\lambda}\right)^2 + \left(\frac{\lambda'}{\lambda}\right)'\right)\mathcal{D}_n\varepsilon_+^{(l)}(n)\\
&=: \mathcal{R}_+^n\left(\varepsilon_+^{(l-1)}(n), \mathcal{D}_n\varepsilon_+^{(l-1)}(n)\right) + \mathcal{D}_n\left(F_+^{(l-1)}(n)\right) + \mathcal{D}_n\left(G_+^{(l-1)}(n)\right),\\
&\mathcal{D}_n\varepsilon_{\pm}^{(l)}(n)|_{\tau=\tau_0} = \mathcal{D}_n\varepsilon_{\pm}^{ini}(n), \quad \left(\partial_\tau + \frac{\lambda_\tau}{\lambda}R\partial_R\right)\mathcal{D}_n\varepsilon_{\pm}^{(l)}(n)|_{\tau=\tau_0} = \mathcal{D}_n\tilde{\tilde{\varepsilon}}_{\pm}^{ini}(n) + \frac{\lambda_\tau}{\lambda}\left[R\partial_R, \mathcal{D}_n\right]\varepsilon_{\pm}^{ini}(n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \widetilde{\varepsilon_{\pm}^{ini}}(n)
\end{aligned} \tag{10.11}$$

with suitable extensions of the right hand sides outside the light cone, in conjunction with the evolution for the unstable modes given by

$$\begin{aligned}
&-\left(\partial_\tau + 2\frac{\lambda'}{\lambda}\right)^2 c_-^{(l)} - \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + 2\frac{\lambda'}{\lambda}\right)c_-^{(l)} + \lim_{R\to 0} R^{-2}H_{-1}^{+}\varepsilon_+^{(l)}(-1) = \lim_{R\to 0} R^{-2}\left[F_+(-1) + G_+(-1)\right]\\
&c_-^{(l)}(\tau_0) = \lim_{R\to 0}\left[\phi_{-1}(R)\right]^{-1}\varepsilon_+^{ini}(-1)(R), \quad \left(c_-^{(l)}\right)'(\tau_0) = \lim_{R\to 0}\left[\phi_{-1}(R)\right]^{-1}\tilde{\varepsilon}_+^{ini}(-1)(R)\\
&-\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)^2 c_0^{(l)} - \frac{\lambda_\tau}{\lambda}\left(\partial_\tau + \frac{\lambda'}{\lambda}\right)c_0^{(l)} + \lim_{R\to 0} R^{-2}H_0^{+}\varepsilon_+^{(l)}(0) = \lim_{R\to 0} R^{-1}\left[F_+(0) + G_+(0)\right]
\end{aligned}$$

$$c_0^{(l)}(\tau_0) = \lim_{R\to 0} [\phi_0(R)]^{-1} \varepsilon_+^{ini}(0)(R), \quad \left(c_0^{(l)}\right)'(\tau_0) = \lim_{R\to 0} [\phi_0(R)]^{-1} \tilde{\varepsilon}_+^{ini}(0)(R)$$
$$-(\partial_\tau)^2 c_+^{(l)} - \frac{\lambda_\tau}{\lambda}(\partial_\tau) c_+^{(l)} + \lim_{R\to 0} R^{-2} H_1^+ \varepsilon_+^{(l)}(0) = \lim_{R\to 0} [F_+(1) + G_+(1)]$$
$$c_+^{(l)}(\tau_0) = \lim_{R\to 0} [\phi_1(R)]^{-1} \varepsilon_+^{ini}(1)(R), \quad \left(c_+^{(l)}\right)'(\tau_0) = \lim_{R\to 0} \phi_1(R)\tilde{\varepsilon}_+^{ini}(1)(R)$$

admit solutions on (τ_0, ∞), such that

$$\sum_{|n|<2} \left\| \left(c_n^{(l)}, \overline{x}_n^{(l)}(\tau, \xi)\right) \right\|_{good(r)} \leq \tau_0^{-50}, \quad |n| \geq 2. \tag{10.12}$$

Defining the zeroth iterate as in Lemma 10.1, Lemma 10.2, in particular $c_n^{(0)} = 0$, the preceding inductively constructed sequence of iterates converges, in the sense that the (obviously defined) difference norms are $\leq \delta^l$ for some $\delta < 1$.
Defining for $|n| \geq 2$

$$\varepsilon_\pm(n) = \lim_{l\to\infty} \varepsilon_+^{(l)}(n),$$

and for $|n| < 2$ setting $c_n = \lim_{l\to\infty} c_n^{(l)}$, $\mathcal{D}_n\varepsilon_+(n) = \lim_{l\to\infty} \mathcal{D}_n\epsilon_+^{(l)}(n)$, and

$$\varepsilon_+(n) = c_n\phi_n(R) + \phi_n(R)\int_0^R [\phi_n(s)]^{-1} \mathcal{D}_n\varepsilon_+(n)(\tau, s)\, ds,$$

and finally $\varepsilon_-(n) = \overline{\varepsilon_+(-n)}$, defining

$$\varphi_1(n) = \frac{1}{2}\sum_\pm \varepsilon_\pm(n), \quad \varphi_2(n) = \frac{1}{2i}[\varepsilon_-(n) - \varepsilon_+(n)],$$

and further

$$\varphi_j = \sum_{n\in Z} \varphi_j(n) e^{in\theta}, \quad j = 1, 2,$$

the function

$$\Psi = \mathcal{L}_{v(t)}\mathcal{R}_{h(t)}^{\alpha(t),\beta(t)}\mathcal{S}_{c(t)}(\Phi + \varphi_1 E_1 + \varphi_2 E_2 + a(\Pi_{\Phi^\perp}\varphi)\Phi), \quad \Phi = \begin{pmatrix} \sin U\cos\theta \\ \sin U\sin\theta \\ \cos U \end{pmatrix},$$

where the modulation parameters are the limits of $\alpha^{(l)}$ etc solves the Wave Maps equation with initial data (10.1).

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