Semi-linear wave equations

W. Schlag *

Abstract. This survey reviews some of the recent work on semilinear wave equations, in particular the wave map equation. We discuss wellposedness, as well as the construction of special solutions and their stability.

1. Introduction

In this article we shall survey some recent developments concerning the long-term dynamics of semi-linear wave equations. These results concern the well-known wave maps system, which is a geometric equation, as well as the semi-linear wave equation with a power nonlinearity. We begin with the basic variational formulation of these models.

1.1. Lagrangians. Consider the Lagrangian

$$\mathcal{L}(u, \partial_t u) := \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2}\big(- u_t^2 + |\nabla u|^2 \big)(t, x)\, dt dx \tag{1.1}$$

Substitute $u = u_0 + \varepsilon v$. Then

$$\mathcal{L}(u, \partial_t u) = \mathcal{L}(u_0, \partial_t u_0) + \varepsilon \int_{\mathbb{R}^{1+d}_{t,x}} (\Box u_0)(t, x) v(t, x)\, dt dx + O(\varepsilon^2)$$

where $\Box = \partial_{tt} - \Delta$. Thus u_0 is a critical point of $\mathcal{L}$ if and only if $\Box u_0 = 0$, the latter being the free wave equation on the flat Minkowski space $\mathbb{R}^{1+d}_{t,x}$. The wave equation is also a *Hamiltonian equation* with conserved energy

$$E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^d} \left(|u_t|^2 + |\nabla u|^2 \right) dx$$

Amongst other things, the Lagrangian formulation has the following significance:

- By Nöther's theorem underlying symmetries of the Lagrangian, more precisely 1-parameter groups of symmetries, yield continuity equations or *conservation laws*. The conservation of energy, momentum, angular momentum are a result of time-translation, space-translation, and rotation invariance of the Lagrangian, respectively.

- The Lagrangian formulation has a universal character, and is both flexible and versatile.

To illustrate the latter point, let (M, g) be a Riemannian manifold, and $u : \mathbb{R}_{t,x}^{1+d} \to M$ a smooth map. What does it mean for u to satisfy a wave equation?

While it is very non-obvious how to define such an object on the level of the equation, it is easy by modifying (1.1):

$$\mathcal{L}(u, \partial_t u) = \int_{\mathbb{R}_{t,x}^{1+d}} \frac{1}{2} \left(-|\partial_t u|_g^2 + \sum_{j=1}^{d} |\partial_j u|_g^2 \right) dt dx$$

The critical points $\mathcal{L}'(u, \partial_t u) = 0$ satisfy a manifold-valued wave equation. If $M \subset \mathbb{R}^N$ is imbedded, this equation is

$$\Box u \perp T_u M \quad \text{or} \quad \Box u = A(u)(\partial u, \partial u),$$

A being the second fundamental form. This is the *extrinsic formulation*. For example, if $M = \mathbb{S}^{n-1}$, then

$$\Box u = u(|\partial_t u|^2 - |\nabla u|^2) \tag{1.2}$$

This gives rise to a nonlinear wave equation in a canonical way, the nonlinearity exhibits a so-called null-form structure. Harmonic maps are time-independent solutions. The nonlinearity appears naturally, and is given by the geometry of the target.

There is also an *intrinsic formulation* of the wave map system, namely

$$D^\alpha \partial_\alpha u = \eta^{\alpha\beta} D_\beta \partial_\alpha u = 0,$$

where D_α is the covariant derivative on the pull-back bundle. This refers to the pull-back of the connection defined on M to the Minkowski space $\mathbb{R}_{t,x}^{1+d}$ using the wave map u itself. In coordinates we obtain

$$-u_{tt}^i + \Delta u^i + \Gamma_{jk}^i(u) \partial_\alpha u^j \partial^\alpha u^k = 0 \tag{1.3}$$

with $\eta = (-1, 1, 1, \ldots, 1)$ being the flat Minkowski metric. Note the following points:

- There is a formal similarity between (1.3) and the geodesic equation. This similarity yields the following conclusion concerning solutions: $u = \gamma \circ \varphi$ is a wave map provided $\Box \varphi = 0$ and γ is a geodesic in M.

- Energy conservation for wave maps:

$$E(u, \partial_t u) = \int_{\mathbb{R}^d} \big(|\partial_t u|_g^2 + \sum_{j=1}^d |\partial_j u|_g^2 \big) \, dx$$

  is constant in time.

Of central importance is the *Cauchy problem*, which we may now state in the following way for the extrinsic formulation:

$$\Box u = A(u)(\partial^\alpha u, \partial_\alpha u), \quad (u(0), \partial_t u(0)) = (u_0, u_1)$$

with smooth data, where u_0 is a fixed map into the manifold M, and u_1 a vectorfield in the pull-back tangent bundle. One typically also imposes a compact support assumption. For u_0 this means that outside of some compact set u_0 equals a fixed point $p \in M$, whereas for u_1 this requirement is just the usual vanishing condition.

The most basic question concerning the Cauchy problem is the following one: *Does there exist a smooth local or global-in-time solution?* In addition, we would like the solution to be robust in a suitable sense. This precise meaning of this is captured by well-posedness theory.

A very condensed answer to this question reads as follows:

- One has local existence for all data as above, and global well-posedness for small data. These results are not sensitive to the geometry of the target (such as positive vs. negative curvature).

- For large data, the question about global-in-time wellposedness is much more involved, and does depend crucially on the geometry of the target manifold and the dimension of the underlying Minkowski space.

Another Lagrangian relevant to this survey is the following one, which does not involve any curvature but rather a directly inserted nonlinearity:

$$\mathcal{L}(u, \partial_t u) := \int_{\mathbb{R}_{t,x}^{1+d}} \left\{ \frac{1}{2} \big(-u_t^2 + |\nabla u|^2 \big)(t, x) + \frac{k}{p+1} |u(t,x)|^{p+1} \right\} dt dx \qquad (1.4)$$

where $k \in \mathbb{R}$ is some constant. The critical points of this Lagrangian are given by the semi-linear wave equation

$$u_{tt} - \Delta u + k|u|^{p-1}u = 0$$

on $\mathbb{R}_{t,x}^{1+d}$. The sign of k is essential at least for large data. This is reflected in the conserved energy

$$E = \int_{\mathbb{R}^d} \left\{ \frac{1}{2} \big(u_t^2 + |\nabla u|^2 \big)(t, x) + \frac{k}{p+1} |u(t,x)|^{p+1} \right\} dx$$

If $k > 0$ then the energy is positive definite, whereas for $k < 0$ it is indefinite. In the latter case, which is referred to us the *focusing equation*, the dynamics is incomparably more complicated than for the defocusing equation ($k > 0$).

1.2. Symmetries and solvability. The wave equation is invariant under the Poincaré group. This group is the symmetry group of special relativity and is generated by the Lorentz transforms and rigid motions of $\mathbb{R}^3$.

However, conformal invariance is also essential for the understanding of these equations. Of particular importance to the well-posedness problem is the dilation symmetry. If $u(t, x)$ is a wave map, then so is $u(\lambda t, \lambda x)$ $\forall \lambda > 0$. Suppose the data belong to the Sobolev space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)$. The unique s for which this space remains invariant under the natural scaling is $s = \frac{d}{2}$. On the other hand, the energy remains invariant under the following scaling: $u(t, x) \mapsto \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x)$ same as $\dot{H}^1 \times L^2(\mathbb{R}^d)$. The interplay between the natural scaling of the wave-map equation, on the one hand, and the scaling of the energy, on the other hand, is essential for the solution theory.

- *Subcritical case $d = 1$.* The natural scaling is associated with less regularity than that of the conserved energy. We therefore expect global existence. The logic being that the local time of existence only depends on the energy of the data, which is preserved.

- *Critical case $d = 2$.* The conserved energy exactly keeps the balance with the natural scaling of the equation. The geometry of the target plays the decisive role. For example, for the sphere $\mathbb{S}^2$ large data may exhibit finite-time blowup (singularity formation), whereas for the hyperbolic plane $\mathbb{H}^2$ as a target we have global existence. These results are the culmination of many years' worth of developments, carried out by numerous researchers, see Klainerman, Selberg [41, 42], Tataru [74], Tao [70, 71], Krieger, Schlag, Tataru [47], Krieger, Schlag [46], Rodnianski, Sterbenz [61], Raphael, Rodnianski [59], Sterbenz, Tataru [67, 68], Tao [72].

- *Supercritical case $d \geq 3$.* For this, as well as energy supercritical equations in general, the dynamics is poorly understood. Self-similar blowup of the form $Q(r/t)$ for the sphere as target was observed in the 1980s by Shatah [62]. Negatively curved manifolds in high dimensions admit the same type of phenomenon, see [13]. Donninger [21] established the stability of the Shatah-type blowup relative to suitable norms.

From a mathematical perspective, the study of nonlinear Hamiltonian evolution equations focuses on the following problems, broadly stated:

- *Wellposedness:* Existence and uniqueness of solutions, the continuous dependence of these solutions on the data, and the persistence of regularity. At first, one needs to understand these properties locally in time.

- *Global behavior:* Does finite time break down occur? In the usual classical interpretation, this question means the following: does there exist a finite time T_* so that a smooth solution exists for all $0 < t < T_*$, but it is impossible to extend the solution smoothly beyond T_*. Typically, for semi-linear equations this property can be shown to reduce to the question whether or not some

norm, such as L^∞ or a suitable space-time norm, becomes unbounded in finite time.

If finite-time breakdown does not occur, then we have global existence: smooth solutions exist for all times for smooth data. In some instances, such as energy subcritical equations, this property can be deduced from two ingredients: (i) the local time of existence only depends on the size of the data as expressed by an "energy norm" (ii) this norm is dominated by a conserved quantity, typically the energy.

However, such a time-stepping scenario does not admit any conclusion about the nature of the long-term dynamics. Other methods are required in order to determine that.

- *Blow up dynamics:* If the solution breaks down in finite time, can one describe the mechanism by which it does so? For example, via energy concentration at the tip of a light cone? Often, symmetries (in a wider sense) play a crucial role in the process of singularity formation.

- *Scattering to a free wave:* If the solution exists for all times $t \geq 0$, does it approach a free wave? In more formal notation, suppose we are given a solution $u(t)$ to a nonlinear equation $\Box\, u = N(u)$, and we assume that u lies in a suitable space X. Does there then exist $v \in X$ with $\Box\, v = 0$ and such that $(\vec{u} - \vec{v})(t) \to 0$ as $t \to \infty$ in X? Here $\vec{u} = (u, \partial_t u)$. If scattering occurs, then we have local energy decay.

Of great importance are equations that admit special "soliton" solutions. This refers to standing waves in a wide sense, stationary solutions not depending on time being included. For wave maps, these would be given by harmonic maps.

- *Special solutions:* If a global solution does not approach a free wave, does it scatter to something else? A stationary nonzero solution, for example? Focusing equations often exhibit nonlinear bound states.

- *Stability theory:* If special solutions exist such as stationary or time-periodic ones, are they orbitally stable? Are they asymptotically stable?

- *Multi-bump solutions:* Is it possible to construct solutions which asymptotically split into moving "solitons" plus radiation? Lorentz invariance dictates the dynamics of the single solitons.

- *Resolution into multi-bumps:* Do all solutions decompose in this fashion (as in linear *asymptotic completeness*)? To rephrase the question: suppose solutions exist for all times $t \geq 0$. Is it then true that they either scatter to a free wave, or decompose into (moving) standing waves (solitons)? For the latter, the symmetries are essential: for the wave equation the movement would be determined by Lorentz symmetries, whereas for the Schrödinger equation Galilean symmetries determine the movement. Each soliton would consume a fixed quantum of energy, thus limiting the number of these (moving) standing waves.

For the remainder of this survey, we shall describe some of the answers to these questions that are known today for the wave map system and scalar semi-linear equations, respectively. But first, we present an indispensable tool in the study of wave equations, namely the pointwise decay of free waves.

1.3. Dispersion. In $\mathbb{R}^3$, the Cauchy problem $\Box u = 0$, $u(0) = 0, \partial_t u(0) = g \in C^\infty(\mathbb{R}^3)$ has the unique smooth solution

$$u(t, x) = t \fint_{tS^2} g(x + y)\, \sigma(dy)$$

If g is supported on the unit ball $B(0, 1)$, then the solution $u(t, x)$ is supported on $\big||t| - |x|\big| \leq 1$. This is a manifestation of Huygens' principle.

Since the energy is conserved and is spread out evenly over a volume of size t^2, we expect point-wise decay at the rate t^{-1}. The technical estimate in $\mathbb{R}^3$ reads as follows:

$$\|u(t, \cdot)\|_\infty \leq C t^{-1} \|Dg\|_1 \tag{1.5}$$

In dimension $= d$ the decay is $t^{-\frac{d-1}{2}}$. Generally speaking, (1.5) is not suitable for nonlinear problems, since $L^1(\mathbb{R}^d)$ is not invariant under the nonlinear flow. Rather, one uses the following energy based variant

$$\|u\|_{L_t^p L_x^q(\mathbb{R}^3)} \lesssim \|(u(0), \dot{u}(0))\|_{\dot{H}^1 \times L^2(\mathbb{R}^3)} + \|\Box u\|_{L_t^1 L_x^2(\mathbb{R}^3)}$$

where $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$, $\frac{1}{p} + \frac{3}{q} = \frac{1}{2}$. These are *Strichartz estimates* which play a fundamental role in the study of nonlinear problems. Examples of these estimates are given by $L_t^\infty L_x^6(\mathbb{R}^{1+3})$, $L_{t,x}^8(\mathbb{R}^{1+3})$. In principle, $L_t^2 L_x^\infty(\mathbb{R}^{1+3})$ also belongs to this class although this particular endpoint fails, see Keel, Tao [32].

2. Global well-posedness for wave maps

In this section, we give an overview of the known results on the Cauchy problem for the wave-map system. We begin with solutions obeying special rotational symmetry.

2.1. Equivariant solutions. Let $M \subset \mathbb{R}^{d+1}$ for $d \geq 2$ be a surface of revolution. This means that there exists a line in $\mathbb{R}^{d+1}$ so that rotations about this axis leave M invariant. Denote such rotations by R, and we identify the group of these symmetries with $\mathrm{SO}(d)$. We call a smooth map $u : \mathbb{R}^{1+d} \to M$ a k-equivariant map, where $k \geq 1$ is some integer provided

$$u \circ R = R^k \circ u$$

for any such $R \in \mathrm{SO}(d)$. The understanding here is that R on the left-hand side acts on the domain $\mathbb{R}^d$ by rotation about the origin.

Under this symmetry, $u = u(t,r)$ where $r = |x|$ which simplifies matters considerably. For $d = 2$ with ϕ being arc-length along a generator, the wave map equation now reads as follows:

$$\phi_{tt} - \phi_{rr} - \frac{1}{r}\phi_r + \frac{1}{r^2}f(\phi) = 0 \tag{2.1}$$

with the conserved energy

$$\mathcal{E}(\vec{\phi}) = \int_0^\infty \left(\phi_t^2(t,r) + \phi_r^2(t,r) + \frac{g(\phi(t,r))^2}{r^2} \right) r\, dr \tag{2.2}$$

The function g is defined by the metric on M which is of the form

$$ds^2 = d\phi^2 + g^2(\phi)ds_{S^2}^2.$$

The function f in (2.1) is $f = gg'$. For example, for $M = S^2$ we obtain $f(\phi) = \frac{1}{2}\sin(2\phi)$. Notice that unlike the full wave map system (2.1) does not contain any derivatives in the nonlinearity, further highlighting the semi-linear nature of these equations.

A special case are geodesically convex targets, i.e., those for which $f(\phi) > 0$ for $\phi > 0$. The one-sheeted hyperboloid is an example of such a surface.

Theorem 2.1. *For geodesically convex targets M, the equation (2.1) has smooth global solutions for all smooth data. In other words, the equivariant wave map system from $\mathbb{R}_{t,x}^{1+2} \to M$ is globally well-posed.*

This result goes back to the pioneering work of Christodoulou, Tahvildar-Zadeh, and Shatah from the 1990s, see [15, 14, 65, 66]. See the book by Shatah, Struwe [63] for an exposition of this work. As we shall see below, Theorem 2.1 does not hold for general targets such as the sphere for which singularities may form in finite time. So a geometric condition on the target such as that of being geodesically convex, is of intrinsic importance for a global regularity result. We remark that Theorem 2.1 deals with the energy critical case, which is $d = 2$. As noted above, this case stands out as being of special analytical as well as geometric interest; the latter being evidenced by the geometrical properties of the target M entering into the analysis. This is not to say that the wave map equations in dimensions $d \geq 3$ are not of interest. These supercritical equations are much less understood. From the equivariant formulation Shatah observed in the 1980s that there exists self-similar blowup solutions [62]. We shall now describe how the dynamics of equivariant wave maps exterior to a ball in $\mathbb{R}^3$ can be completely characterized in the energy class.

2.2. Exterior wave maps.
We now consider wave maps exterior to the unit ball $B(0,1)$. By this we mean a smooth equivariant map $u : \mathbb{R}^3 \setminus B(0,1) \to S^3$ satisfying the wave maps equation on $r > 1$ and a Dirichlet condition at $r = 1$; in other words, for all times we have $u(t,1) = p \in S^3$, a fixed point on the sphere.

In the equivariant formulation, we thus have an equation

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\sin(2\psi)}{r^2} = 0 \tag{2.3}$$

with $\psi(t,1) = 0$ for all times $t \geq 0$. The conserved energy is

$$\mathcal{E}(\psi, \psi_t) = \int_1^\infty \frac{1}{2} \big(\psi_t^2 + \psi_r^2 + 2 \frac{\sin^2(\psi)}{r^2} \big) r^2 \, dr \tag{2.4}$$

Any $\psi(t,r)$ of finite energy and continuous dependence on $t \in I := (t_0, t_1)$ must satisfy $\psi(t, \infty) = n\pi$ for all $t \in I$ where $n \geq 0$ is fixed. This integer n plays the role of the degree in this context.

The natural space to place the solution into for $n = 0$ is the *energy space* $\mathcal{H} := (\dot{H}_0^1 \times L^2)((1, \infty))$ with norm

$$\|(\psi, \dot{\psi})\|_{\mathcal{H}}^2 := \int_1^\infty (r^2 \psi_r^2(r) + \dot{\psi}^2(r)) \, dr \tag{2.5}$$

Here $\dot{H}_0^1((1,\infty))$ is the completion of the smooth functions on $(1, \infty)$ with compact support under the first norm on the right-hand side of (2.5).

The exterior equation (2.3) was proposed by Bizon, Chmaj, and Maliborski [10] as a model in which to study the problem of relaxation to the ground states given by the various equivariant harmonic maps. In the physics literature, this model was introduced in [5] as an easier alternative to the Skyrmion equation. Numerical simulations described in [10] indicate that in each equivariance class and topological class given by the boundary value $n\pi$ at $r = \infty$ *every solution* scatters to the unique harmonic map that lies in this class. The existence of these harmonic maps follows from a phase-plane analysis. The conjecture from [10] was verified by Lawrie and the author in [50] for the zero degree class, and then by Kenig, Lawrie and the author [33] in full generality.

Theorem 2.2. *For any smooth energy data in the class of degree n there exists a unique global and smooth evolution to* (2.3) *which scatters to the unique harmonic map in that degree class.*

The existence of a global smooth solutions is easy, since the removal of a ball around the origin renders the equation subcritical. The difficult part is to describe the asymptotic state of the solution. For degree zero solutions the scattering amounts to the property that viewed on any fixed compact set, the energy of the solution on the set tends to zero. In other words, the solutions tends to zero. For higher degrees, it means that the solution asymptotically tends to the unique harmonic map of that degree class.

The methods employed are those that fall under the name "concentration compactness", as developed in the works of Kenig, Merle [34, 35] and Duyckaerts, Kenig, Merle [22, 24, 23, 25, 26]. For the zero degree class, a variant of the Kenig, Merle argument in which the compact element is excluded via a virial identity based rigidity argument, suffices. But for the higher degrees this does not suffice, since the virial identity is not available. To circumvent this road block, one uses the exterior energy estimate method of [25].

2.3. Small data theory. We shall now briefly describe the by now classical *global-in-time* results on *non-equivariant* wave maps for small data. These were preceded by the local-in-time wellposedness obtained by Klainerman, Machedon [36, 37, 38, 39] and Klainerman, Selberg [41, 42] in the 1990s. These local results improved dramatically on the easy energy methods which required much regularity on the data since the nonlinearity was controlled by Sobolev embedding. More specifically, these authors were able to reduce the regularity requirement to $H^s \times H^{s-1}(\mathbb{R}^d)$ with $s > \frac{d}{2}$ for local wellposedness.

The latter condition goes all the way down to the critical scaling $\frac{d}{2}$, but does not achieve this endpoint. The argument relies on the contraction principle in a suitably chosen space. This turns out not to be energy and Strichartz spaces, but rather the $X^{s,b}$ spaces which take the geometry of the characteristic variety of the wave equation into account, which is the light-cone. This is crucial in order to capture the cancellations exhibited by the highly structured nonlinearity of the wavemap system. In fact, the right-hand side of (1.2) contains the term $|\partial_t u|^2 - |\nabla u|^2$ which is the Minkowski metric applied to the space-time gradient $\nabla_{t,x} u$. The relevance of this lies with the fact that the Minkowski metric vanishes on null-vectors, which are characterized by lying in a light-cone.

On the level of the wave equation, this means that the nonlinearity cancels self-interactions of plane waves. Without this property, it would not be possible to lower the regularity all the way down to $s > \frac{d}{2}$, which is called *subcritical regularity*.

After these important developments the focus shifted to the difficult question of wellposedness at the critical level $s = \frac{d}{2}$. The interest of this question hinges on the scaling invariance of the equation. Indeed, in contrast to any wellposedness theory at the subcritical level, wellposedness at the critical level is automatically global in time simply by rescaling the solution.

The key breakthroughs here were achieved by Tataru [74] and Tao [70, 71] about 15 years ago. In Tataru's work the regularity is expressed in terms of the Besov regularity $\dot{B}^1_{2,\frac{d}{2}} \times \dot{B}^1_{2,\frac{d}{2}-1}$ which is precisely at the scaling critical level. But it is stronger than the Sobolev regularity $\dot{H}^{\frac{d}{2}} \times \dot{H}^{\frac{d}{2}-1}$. In technical terms, Tataru solved the *division problem*, but not the *summation problem* (which refers to the summation over the dyadic frequency scales). Tao resolved the summation problem for the sphere as target by means of the important device of a *gauge transform*. Without going into too many details, this amounts to removing "dangerous" interactions in the nonlinearity by exploiting a freedom in the choice of coordinates (or the choice of a frame). In dimensions 5 and higher, Tao observed that Strichartz theory suffices to close the argument due to the stronger dispersion in those cases. In particular, the null form structure of the nonlinearity is not crucial. However, in low dimensions, especially in dimension $d = 2$ for which dispersion is very weak, much more technical heavy lifting is needed and the nullform becomes essential.

An important hallmark of Tao's work is the fact that the global wellposedness is *not achieved* by means of a contraction argument at the critical regularity. Rather, it is based on the device of *frequency envelope*. While one assume smallness of the data in the critical norm, the data are also assumed to have slightly more regularity than the critical level. The subcritical wellposedness theory then gives the local

existence of a solution. The key is now to show that this slightly higher degree of regularity is preserved by the flow; in this way the subcritical wellposedness theory allows one to solve globally in time. The Tataru, Tao theory was extended to more general target manifolds than the sphere, see Krieger [43, 44], Tataru [77, 78], Klainerman, Rodnianski [40]. Nahmod, Stepanov, Uhlenbeck [55] obtain a small data theorem in spatial dimensions $d \geq 4$ for targets given by compact groups or symmetric spaces.

Our current understanding is that small data wellposedness holds for all Riemannian manifolds as targets satisfying reasonable assumptions.

Shatah, Struwe [64] made the important observation that the *Coulomb gauge* can be used in dimensions $d \geq 4$ to obtain global regularity for small data. That particular gauge is natural form the theory of harmonic maps and exploits the formalism of a moving orthonormal frame. The Coulomb gauge refers to the choice of such frame which "twists" the least; in more technical terms, the Dirichlet energy is minimized by such a frame.

These developments set the stage for the next step, namely determining the different possible types of dynamics for large data wave maps. In the following two sections we will describe the two main phenomena that may appear in the energy critical setting, namely finite-time blowup on the one hand, vs. global regularity on the other hand. The energy supercritical wave map system is still very poorly understood.

2.4. Blowup for wave maps. In [47] Krieger, Tataru, and the author exhibited regularity breakdown for equivariant wave maps $u : \mathbb{R}^{2+1} \to S^2$ of co-rotation index 1 with certain H^{1+} regular initial data. More precisely, the data (u, u_t) are of class $H^{1+\delta} \times H^\delta$ for some $\delta > 0$. By a theorem of Struwe [69] such data result in unique local solutions of the same regularity until possible breakdown occurs via an *energy-concentration scenario*. More precisely, Struwe's result shows that if the solution is indeed C^∞–smooth before breakdown, such a scenario can only happen by the bubbling off of a harmonic map [69]: specifically, let $Q(r) : \mathbb{R}^2 \to S^2$ be an equivariant harmonic map, which can be constructed for every co-rotation index $k \in \mathbb{Z}$ (for example, for $k = 1$ one may use stereographic projection). We shall identify $Q(r)$ with the longitudinal angle, as above. Then according to [69], if an equivariant wave map u of co-rotation index $k = 1$, again identified with the longitudinal angle, with smooth initial data at some time $t_0 > 0$ breaks down at time $T = 0$, then energy focuses at the origin, and there is a decomposition

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r), \quad Q(r) \text{ a co-rotation } k = 1 \text{ index equivariant harmonic map}$$

where there is a sequence of times $t_i \to 0$, $t_i < 0$, $i = 1, 2, \ldots$, with $\lambda(t_i)|t_i| \to \infty$, such that the rescaled functions $u(t_i, \frac{r}{\lambda(t_i)})$ converge to $Q(r)$ locally in the strong energy topology.

We now describe the theorem of Krieger, Tataru and the author which *constructs* this type of non self-similar blowup for energy critical wave maps. We let $Q(r)$ represent the standard harmonic map of co-rotation $k = 1$, i.e., $Q(r) =$

$2 \arctan r$. Recall that in the equivariant formulation the energy is

$$\mathcal{E}(u) = \int_{\mathbb{R}^2} \left[\frac{1}{2}(u_t^2 + u_r^2) + \frac{\sin^2(u)}{2r^2} \right] r \, dr$$

The *local* energy relative to the origin is defined as

$$\mathcal{E}_{\mathrm{loc}}(u) = \int_{r<t} \left[\frac{1}{2}(u_t^2 + u_r^2) + \frac{\sin^2(u)}{2r^2} \right] r \, dr$$

It is well-known that for equivariant wave-maps singularities can only develop at the origin and this happens at time zero if and only if

$$\liminf_{t \to 0} \mathcal{E}_{\mathrm{loc}}(u)(t) > 0$$

One of the main features of the following theorem is that we need to "renormalize" the profile $Q(r\lambda(t))$ by means of a large perturbation (denoted u^e below). While this usage of the term "renormalize" may be at odds with the physics literature, it is quite common in applied mathematics and perturbation theory. What we mean here is that we can apply perturbative arguments only after a non-perturbative step that changes Q to $Q + u^e$, see Theorem 2.3. We find it convenient to solve backwards in time, with blow-up as $t \to 0+$. The equivariant formulation of the wave map equation from $\mathbb{R}^{1+2}_{t,x} \to S^2$ is

$$-u_{tt} + u_{rr} + \frac{u_r}{r} = \frac{\sin(2u)}{2r^2} \tag{2.6}$$

Theorem 2.3. *Let $\nu > \frac{1}{2}$ be arbitrary and $t_0 > 0$ be sufficiently small. Define $\lambda(t) = t^{-1-\nu}$ and fix a large integer N. Then there exists a function u^e satisfying*

$$u^e \in C^{\nu+1/2-}(\{t_0 > t > 0, \; |x| \le t\}), \qquad \mathcal{E}_{\mathrm{loc}}(u^e)(t) \lesssim (t\lambda(t))^{-2} \, |\log t|^2 \quad as \;\; t \to 0$$

and a blow-up solution u to (2.6) in $[0, t_0]$ which has the form

$$u(t,r) = Q(\lambda(t)r) + u^e(t,r) + \epsilon(t,r), \qquad 0 \le r \le t$$

where ϵ decays at $t = 0$; more precisely,

$$\epsilon \in t^N H_{\mathrm{loc}}^{1+\nu-}(\mathbb{R}^2), \qquad \epsilon_t \in t^{N-1} H_{\mathrm{loc}}^{\nu-}(\mathbb{R}^2), \qquad \mathcal{E}_{\mathrm{loc}}(\varepsilon)(t) \lesssim t^N \quad as \;\; t \to 0$$

with spatial norms that are uniformly controlled as $t \to 0$. Also, $u(t,0) = 0$ for all $0 < t < t_0$. The solution $u(t,r)$ extends as an $H^{1+\nu-}$ solution to all of $\mathbb{R}^2$ and the energy of u concentrates in the cuspidal region $0 \le r \lesssim \frac{1}{\lambda(t)}$ leading to blow-up at $r = t = 0$.

A somewhat surprising feature of our theorem is that the blow-up rate is prescribed as $\lambda(t) = t^{-1-\nu}$. This is in stark contrast to the usual modulation theoretic approach where the rate function is used to achieve orthogonality to all unstable modes of the linearized problem. Heuristically speaking, there are two types of

instabilities which typically arise in linearized problems: those due to symmetries of the nonlinear equation (typically leading to algebraic growth of the linear evolution) and those that produce exponential growth in the linear flow (due to some kind of discrete spectrum). For example, the latter arises in the recent work on "center-stable manifolds" for orbitally unstable equations (see the discussion of scalar semi linear Hamiltonian equations below), whereas for the former see [45]. Both types can lead to blow up.

In the case of Theorem 2.3 ones does not have any discrete spectrum in the linearized equation, but rather a *zero-energy resonance* which is due to the scaling symmetry. Intuitively speaking, it is unclear at this point which role the resonance plays in the formation of the blow-up, since the approach of [47] is based on a crucial non-perturbative component, namely the elliptic profile modifier produces a *large* perturbation of the basic profile Q. The perturbative component of our proof then deals with the removal of errors produced by the elliptic profile modifier (it is essential that these errors decay rapidly in time).

The restriction $\nu > \frac{1}{2}$ is a technical one and can be relaxed to $\nu > 0$ which is optimal by Struwe's aforementioned bubbling off theorem. For this see [12]. Due to the continuum of allowed blowup rates in Theorem 2.3 the solutions constructed are expected to be highly unstable; in fact, their stability should be associated to a finite codimension condition.

In contrast to these unstable solutions, Rodnianski, Sterbenz [60] and Raphael, Rodnianski [59] studied the problem of finding stable blowup regimes. The following theorem is the main result from [59]. The affine Sobolev space relative to the harmonic map Q is defined as.

$$\mathcal{H}_a^2 = \mathcal{H}^2 + Q. \tag{2.7}$$

For a pair of functions $(\epsilon(y), \sigma(y))$, we let

$$\|(\epsilon, \sigma)\|_{\mathcal{H}}^2 = \int \left[\sigma^2 + (\partial_y \epsilon)^2 + \frac{\epsilon^2}{y^2} \right] \tag{2.8}$$

define the energy space. The k-equivariant formulation of the wave map problem $\mathbb{R}_{t,r}^{1+2} \to S^2$ is

$$\begin{cases} \partial_t^2 u - \partial_r^2 u - \frac{\partial_r u}{r} + k^2 \frac{f(u)}{r^2} = 0, \\ u_{|t=0} = u_0, \quad (\partial_t u)_{|t=0} = v_0 \end{cases} \quad \text{with} \quad f = gg' \tag{2.9}$$

The main theorem in [59] also makes a reference to the equivariant (in a suitable sense) Yang-Mills equation, which we however we skip here. In effect, it amounts to the previous equation with $k = 2$.

Theorem 2.4 (Stable blow up dynamics of co-rotational Wave Maps). *Let $k \geq 1$. Let $\mathcal{H}_a^2$ denote the affine Sobolev space from above. There exists a set $\mathcal{O}$ of initial data which is open in $\mathcal{H}_a^2$ and a universal constant $c_k > 0$ such that the following holds true. For all $(u_0, v_0) \in \mathcal{O}$, the corresponding solution to (2.9) blows up in finite time $0 < T = T(u_0, v_0) < +\infty$ according to the following universal scenario:*

(i) Sharp description of the blow up speed: *There exists $\lambda(t) \in \mathcal{C}^1([0,T), \mathbb{R}_+^*)$ such that:*

$$u(t, \lambda(t)y) \to Q \quad in \ \ H_{r,loc}^1 \quad as \ \ t \to T \tag{2.10}$$

with the following asymptotics:

$$\lambda(t) = c_k(1 + o(1)) \frac{T - t}{|\log(T - t)|^{\frac{1}{2k-2}}} \quad as \ \ t \to T \ \ for \ \ k \geq 2, \tag{2.11}$$

$$\lambda(t) = (T - t)e^{-\sqrt{|\log(T-t)|} + O(1)} \quad as \ \ t \to T \ \ for \ \ k = 1. \tag{2.12}$$

Moreover,

$$b(t) := -\lambda_t(t) = \frac{\lambda(t)}{T - t}(1 + o(1)) \to 0 \quad as \ \ t \to T$$

(ii) Quantization of the focused energy: *Let $\mathcal{H}$ be the energy space (2.8), then there exist $(u^*, v^*) \in \mathcal{H}$ such that the following holds true. Pick a smooth cut off function χ with $\chi(y) = 1$ for $y \leq 1$ and let $\chi_{\frac{1}{b(t)}}(y) = \chi(b(t)y)$, then:*

$$\lim_{t \to T} \left\| u(t, r) - \left(\chi_{\frac{1}{b(t)}} Q \right)(\frac{r}{\lambda(t)}) - u^*, \partial_t \left[u(t, r) - \left(\chi_{\frac{1}{b(t)}} Q \right)(\frac{r}{\lambda(t)}) - v^* \right] \right\|_{\mathcal{H}} = 0. \tag{2.13}$$

Moreover, one has the following quantization of the focused energy:

$$E_0 = E(u, \partial_t u) = E(Q, 0) + E(u^*, v^*). \tag{2.14}$$

This theorem thus gives a complete description of a stable blow up regime for all homotopy numbers $k \geq 1$. Stable blow up solutions in $\mathcal{O}$ decompose into a singular part with a universal structure and a regular part which has a strong limit in the scale invariant space. Moreover, the amount of energy which is focused by the singular part is a universal amount independent of the Cauchy data.

2.5. Characterization of blowup for equivariant wave maps.
We now describe large data asymptotic behavior of equivariant wave maps taking values in $\mathbb{S}^2$. The setting is 1-equivariant (co-rotational), so u takes the special form $u(t, r, \phi) = (\psi(t, r), \phi)$ in polar coordinates, where ψ measures the angle from the north pole. This angle then satisfies the equivariant wave map equation

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin(2\psi)}{2r^2} = 0, \quad (\psi, \dot{\psi})(0) = (\psi_0, \psi_1),$$

Struwe's bubbling theorem [69] states: If a solution, $\psi(t, r)$, with smooth initial data $\vec{\psi}(0) = (\psi(0), \dot{\psi}(0))$, breaks down at $t = 1$, then the energy concentrates at the origin and there is a sequence of times $t_j \nearrow 1$ and scales $\lambda_j > 0$ with $\lambda_j \ll 1 - t_j$ so that the rescaled sequence of wave maps

$$\vec{\psi}_j(t, r) := \left(\psi(t_j + \lambda_j t, \lambda_j r), \lambda_j \dot{\psi}(t_j + \lambda_j t, \lambda_j r) \right)$$

converges *locally* to $\pm Q(r/\lambda_0)$ in the space-time norm $H^1_{\mathrm{loc}}((-1,1) \times \mathbb{R}^2; \mathbb{S}^2)$ for some $\lambda_0 > 0$. An important consequence is that any wave map that blows up must concentrate at least the energy of Q at the blow-up point.

Struwe's result gives a *local* characterization of blow-up behavior. To obtain a global picture, one needs to take into account the topological structure carried by an $\mathbb{S}^2$-valued wave map – in particular, each co-rotational wave map of finite energy has a fixed topological integer degree. Indeed, for the energy $\mathcal{E}(\psi, \dot\psi)$ to be finite for a solution $\vec\psi$ means that we must have $\psi(t,0) = m\pi$ and $\psi(t,\infty) = n\pi$. These integers are fixed by continuity and thus determine a homotopy class, or topological degree. Letting $m = 0$, n is the degree and $\mathcal{H}_n$ are all finite energy data of degree n, i.e.,

$$\mathcal{H}_n := \{(\psi_0, \psi_1) \mid \mathcal{E}(\psi_0, \psi_1) < \infty, \ \psi_0(0) = 0, \ \text{and} \ \psi_0(\infty) = n\pi\}.$$

2.5.1. Degree 0 initial data. An immediate consequence of Struwe's theorem is that a degree 0 solution $\vec\psi(t) \in \mathcal{H}_0$ is global-in-time if $\mathcal{E}(\vec\psi(0)) < 2\mathcal{E}(Q)$. Indeed, a wave map in $\mathcal{H}_0$ with energy below $2\mathcal{E}(Q)$ stays away from the south pole and hence cannot converge to a degree 1 rescaled harmonic map. $\mathcal{E}(Q)$ is the minimal energy in $\mathcal{H}_1$, so any map which sends $r = 0$ to the north pole uses at least the energy of Q to reach the south pole. One the other hand, since blow-up is a local phenomenon, one can modify the solutions constructed in the previous section outside the light cone to obtain a blow-up solution in $\mathcal{H}_0$ which has energy $\mathcal{E}(\vec\psi) = 2\mathcal{E}(Q) + \delta$ for any $\delta > 0$. Thus the energy $2\mathcal{E}(Q)$ forms a threshold for data in $\mathcal{H}_0$ under which every solution is global and above which blow-up may occur. In [17], Côte, Kenig, Lawrie and the author addressed the *global dymanics* of subthresold solutions in $\mathcal{H}_0$ by showing that in fact every solution with energy below $2\mathcal{E}(Q)$ must scatter to a free wave.

Theorem 2.5 (Global Existence and Scattering in $\mathcal{H}_0$ below $2\mathcal{E}(Q)$). *For any smooth data $\vec\psi(0) \in \mathcal{H}_0$ with $\mathcal{E}(\vec\psi(0)) < 2\mathcal{E}(Q)$, there exists a unique global evolution $\vec\psi \in C^0(\mathbb{R}; \mathcal{H}_0)$. Moreover, $\vec\psi(t)$ scatters to zero in the sense that the energy of $\vec\psi(t)$ on any arbitrary, but fixed compact region vanishes as $t \to \infty$. In other words, one has*

$$\vec\psi(t) = \vec\varphi_L(t) + o_{\mathcal{H}}(1) \quad as \quad t \to \infty \tag{2.15}$$

where $\vec\varphi_L \in \mathcal{H}$ solves the linearized equation, i.e.,

$$\varphi_{tt} - \varphi_{rr} - \frac{1}{r}\varphi_r + \frac{1}{r^2}\varphi = 0 \tag{2.16}$$

The proof follows the concentration compactness/rigidity method of Kenig and Merle, [34, 35]. The key ingredient in the proof is a rigidity statement: any equivariant wave maps with a pre-compact trajectory (modulo symmetries) must be a harmonic map. Before [17], Cote, Kenig and Merle [19] proved scattering in $\mathcal{H}_0$ for energies below $\mathcal{E}(Q) + \delta$ for small δ. We also note that Theorem 2.5 can be deduced from the more general work of Sterbenz and Tataru [67, 68] by restricting to the equivariant setting in $\mathcal{H}_0$.

2.5.2. Degree 1 initial data. Next consider initial data in $\mathcal{H}_1$. The harmonic map Q uniquely minimizes the energy in this degree class and thus there cannot be an energy threshold in $\mathcal{H}_1$ under which blow-up is excluded – indeed the blow up solutions from [47] described in the previous section can have energy $E(Q) + \delta$ for any $\delta > 0$. Note also that solutions in $\mathcal{H}_1$ cannot scatter to free waves – nontrivial degree is a topological obstruction to scattering.

The question of characterizing the possible dynamics in $\mathcal{H}_1$ is then one of determining the role that Q plays in asymptotic situations. In this regard there is an energy threshold under which this question of characterizing dynamics is most natural in $\mathcal{H}_1$, namely $3\mathcal{E}(Q)$. Indeed, consider a degree one wave map $\vec{\psi}(t)$ that blows up at $t = 1$. The result of Struwe [69] extracts the blow up profile $\pm Q_{\lambda_n} :=$ $\pm Q(\cdot/\lambda_n)$ along a sequence of times $t_n \to 1$. If $\vec{\psi}$ has $\mathcal{E} < 3\mathcal{E}(Q)$ the profile must be $+Q(\cdot/\lambda_n)$, and since $Q \in \mathcal{H}_1$ as well, we infer that $\psi(t_n) - Q_{\lambda_n} \in \mathcal{H}_0$. Since this object converges locally to zero, the energy of the difference is roughly the difference of the energies, at least for large n. Hence, if $\psi(t)$ has energy below $3\mathcal{E}(Q)$ the difference $\psi(t_n) - Q_{\lambda_n}$ is degree zero and has energy below $2\mathcal{E}(Q)$. More complicated dynamics are thus excluded by the degree zero scattering result, i.e., Theorem 2.5. The situation is similar in the case that the solution $\psi(t) \in \mathcal{H}_1$ is global in time.

Theorem 2.6. *[17, 18] Let $\vec{\psi}(0) := (\psi_0, \psi_1) \in \mathcal{H}_1$ be smooth, finite energy degree 1 data with energy $\mathcal{E}(\vec{\psi}) < 3\mathcal{E}(Q)$.*

1. ***Finite time blowup:*** *If the solution $\psi(t)$ blows up at, say, $t = 1$, then there exists a continuous function, $\lambda : [0, 1) \to (0, \infty)$ with $\lambda(t) = o(1 - t)$, a map $\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}_0$ with $\mathcal{E}(\vec{\varphi}) = \mathcal{E}(\vec{\psi}) - \mathcal{E}(Q, 0)$, and a decomposition*

$$\vec{\psi}(t) = \vec{\varphi} + (Q(\cdot/\lambda(t)), 0) + o_{\mathcal{H}_0}(1) \qquad as \ \ t \to 1$$

2. ***Global solutions:*** *If the solution $\vec{\psi}(t) \in \mathcal{H}_1$ exists globally in time then there exists a continuous function, $\lambda : [0, \infty) \to (0, \infty)$ with $\lambda(t) = o(t)$ as $t \to \infty$, a solution $\vec{\varphi}_L(t) \in \mathcal{H}_0$ to the linear wave equation (2.16), and*

$$\vec{\psi}(t) = \vec{\varphi}_L(t) + (Q(\cdot/\lambda(t)), 0) + o_{\mathcal{H}_0}(1) \ \ as \ \ t \to \infty$$

We remark that Duyckaerts, Kenig, and Merle in [22, 23, 25] established analogous classification results for $\Box u = u^5$ in $\dot{H}^1 \times L^2(\mathbb{R}^3)$ with $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$ instead of Q. The techniques developed there motivated the proof of Theorem 2.6 as certain elements of their ideology, in particular concentration compactness techniques, are essential. The proof also relies explicitly on several classical results in the field of equivariant wave maps. In particular, crucial roles are played by the vanishing of the kinetic energy proved by Shatah, Tahvildar-Zadeh [65], and Struwe's bubbling theorem, [69], in the finite time blow-up result.

A fundamental role in the degree 1 argument is played by a property of the linear wave equation. To be specific, consider $\Box u = 0$, $u(0) = f \in \dot{H}^1(\mathbb{R}^d)$, $u_t(0) = g \in L^2(\mathbb{R}^d)$ for radial functions f, g. Then Duyckaerts, Kenig, and Merle

showed the following: If the dimension d is odd, there exists $c > 0$ such that for all $t \geq 0$ or all $t \leq 0$ one has

$$E_{ext}(\vec{u}(t)) \geq cE(f, g) \qquad (2.17)$$

In *even dimensions* this property fails, see [20]. To be precise, in dimensions $d = 2, 6, 10, \ldots$ (2.17) holds for data $(0, g)$, but fails in general for data $(f, 0)$. On the other hand, for dimensions $d = 4, 8, 12, \ldots$ (2.17) holds for data $(f, 0)$ but fails in general for data $(0, g)$.

The proof of both the positive and negative results is based on the Fourier representation, which in our radial context becomes a Bessel transform. The dimension d is then reflected in the phase of the Bessel asymptotics. Due to the monotonicity of the energy over the regions $\{|x| \geq t\}$ the key calculation is that of the asymptotic exterior energy as $t \to \pm\infty$.

For the Theorem 2.6 we need the $d = 4$ result rather than $d = 2$ due to the repulsive $\frac{\psi}{r^2}$-potential coming from $\frac{\sin(2\psi)}{2r^2}$. Crucially, the result from [20] for $(f, 0)$ suffices for the argument because of the classical theory of Christodoulou, Tahvildar-Zadeh, and Shatah [14, 15, 65, 66] about equivariant wave maps; see also the book by Shatah, Struwe [63]. Amongst other things, these authors showed that at the blowup $t = 1$ one has vanishing kinetic energy:

$$\lim_{t \to 1} \frac{1}{1 - t} \int_t^1 \int_0^t |\dot{\psi}(t, r)|^2 \, r \, dr \, dt = 0$$

One has a similar averaged vanishing in the case of a global solution. This vanishing (modulo many other arguments) then allows us to work with the more restrictive form of (2.17) for data $(f, 0)$.

2.5.3. Characterization of dynamics at higher energies. Recently, a more general version of Theorem 2.6 has been established by Côte [16] which holds for data of arbitrary degree and energy.

Theorem 2.7. *[16] Let $\vec{\psi}(t)$ be a finite energy wave map with maximal forward time of existence $T^+(\vec{\psi})$. Then there exist a sequence of times $t_n \uparrow T^+(\vec{\psi})$, an integer $J \geq 0$, J sequences of scales $\lambda_{J,n} \ll \cdots \ll \lambda_{2,n} \ll \lambda_{1,n}$ and J harmonic maps $Q_1, \ldots, Q_J$ such that*

$$Q_J(0) = \psi(0), \quad Q_{j+1}(\infty) = Q_j(0) \quad \text{for } j = 1, \ldots, J - 1,$$

and that one of the following holds:

1. *If $T^+(\vec{\psi}) = +\infty$, denote $\ell = \lim_{r \to \infty} \psi(t, r) = Q_1(\infty)$. Then $\lambda_{1,n} \ll t_n$ and there exists a solution $\vec{\phi}_L(t)$ to the linearized wave equation (2.16) such that*

$$\vec{\psi}(t_n) = \sum_{j=1}^{J} (Q_j(\cdot/\lambda_{j,n}) - Q_j(\infty), 0) + (\ell, 0) + \vec{\phi}_L(t_n) + o_n(1),$$

2. If $T^+(\vec{\psi}) < +\infty$, then $\lambda_{1,n} \ll T^+(\vec{\psi}) - t_n$ and there exists a function $\vec{\phi} \in \mathcal{H}_0$ such that $\phi(0) = \lim_{t \uparrow T^+(\vec{\psi})} \psi(t, T^+(\vec{\psi}) - t)$ and $Q_1(\infty) = \phi(0)$ and

$$\vec{\psi}(t_n) = \sum_{j=1}^{J} (Q_j (\cdot/\lambda_{j,n}) - Q_j(\infty), 0) + \vec{\phi} + o_n(1) \ in \ \mathcal{H}_0,$$

Note that the energy of the of the wave map $\vec{\psi}$ together with the orthogonality of the scales $\lambda_{j,n}$ gives an upper bound on the number of large profiles J. An analogous result for the semi-linear equation $\Box u = u^5$ was proved in [23] and the proof of this result along with the proof of Theorem 2.6 provide a rough blueprint for the proof of Theorem 2.7. There are several key differences however, with perhaps the most significant being a generalization of Struwe's bubbling result proved in [16], which allows one to extract a bubble at each scale that carries a nontrivial amount of energy. The proof also relies on a generalization of the degree zero scattering theory, which states that any solution in $\mathcal{H}_0$ which stays bounded away from the south pole scatters to a free wave.

2.6. Large data global regularity. In contrast to the sphere as a target, for negatively curved manifolds as targets one has global existence of smooth solutions for smooth data of any size. This result is the culmination of many years' worth of effort, and was achieved in varying forms by three different groups. Sterbenz, Tataru [67, 68] proved the following very satisfactory result.

Theorem 2.8. *Consider the wave map equation for functions $u : \mathbb{R}_{t,x}^{1+2} \to M$ where M is a Riemannian manifold. Let E_0 be the infimum of all possible energies of nonconstant harmonic maps $\mathbb{R}^2 \to M$. Given smooth data $u_0 : \mathbb{R}^2 \to M$ and $u_1 : \mathbb{R}^2 \to TM$ so that the energy satisfies $E(u_0, u_1) < E_0$, there exists a unique global smooth evolution of the wave map u.*

In particular, if M is of negative curvature then it does not admit nontrivial harmonic maps by a result of Eels and Simpson, so we do indeed have the global regularity for all data. By different methods Tao [72] proved the global regularity theorem for hyperbolic spaces as targets, and Krieger and the author [46] obtained this result for the hyperbolic plane. The method of [46] is different from the aforementioned works since it relies on the concentration-compactness ideas of Bahouri, Gerard [4] and Kenig, Merle [34, 35]. To be more specific, one obtains both implicit space-time bounds which yield scattering for the derivative components in a suitable gauge, purely in terms of the energy, as well as a type of profile decomposition for sequences of wave maps with bounded energy. The key new aspect of the work [46] is that the Bahouri-Gérard approach to the profile decomposition, which depends crucially on the property that factors of widely separated frequency interact only weakly in the nonlinearity, needs to be replaced by a suitable form of a "covariant" Bahouri-Gérard approach. More precisely, the wave maps nonlinearity features certain low-high frequency interactions in the nonlinearity which cannot be shown to be negligible. In order to still be able to obtain a profile decomposition (precisely, one does so for the derivative components in the Coulomb gauge),

one needs to replace the free wave propagator by suitable magnetic potential wave operators of the form

$$\Box_A = \Box + 2iA_\alpha \partial^\alpha$$

with $\Box_A$ operating on (essentially) unit frequency functions, while A_α is of extremely low frequency. The precise theorem obtained in [46] then reads as follows.

Theorem 2.9. *There exists a function $K : (0,\infty) \to (0,\infty)$ with the following property: Let M be a hyperbolic Riemann surface. Suppose $(\mathbf{u}_0, \mathbf{u}_1) : \mathbb{R}^2 \to M \times TM$ are smooth and $\mathbf{u}_0 = const$, $\mathbf{u}_1 = 0$ outside of some compact set. Then the wave map evolution $\mathbf{u}$ of these data as a map $\mathbb{R}^{1+2} \to M$ exists globally as a smooth function and, moreover, for any $\frac{1}{p} + \frac{1}{2q} \leq \frac{1}{4}$ with $2 \leq q < \infty$, $\gamma = 1 - \frac{1}{p} - \frac{2}{q}$,*

$$\sum_{\alpha=0}^{2} \|(-\Delta)^{-\frac{\gamma}{2}} \partial_\alpha \mathbf{u}\|_{L_t^p L_x^q} \leq C_q \, K(E) \tag{2.18}$$

Moreover, in the case when $M \hookrightarrow \mathbb{R}^N$ is a compact Riemann surface, one has scattering:

$$\max_{\alpha=0,1,2} \|\partial_\alpha \mathbf{u}(t) - \partial_\alpha S(t)(f,g)\|_{L_x^2} \to 0 \quad as \ \ t \to \pm\infty$$

where $S(t)(f,g) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g$ and suitable $(f,g) \in (\dot{H}^1 \times L^2)(\mathbb{R}^2; \mathbb{R}^N)$. Alternatively, if M is non-compact, then lifting $\mathbf{u}$ to a map $\mathbb{R}^{1+2} \to \mathbb{H}^2$ with derivative components ϕ_α^j with respect to a suitable global frame, one has

$$\max_{\alpha=0,1,2} \|\phi_\alpha^j(t) - \partial_\alpha S(t)(f^j,g^j)\|_{L_x^2} \to 0 \quad as \ \ t \to \pm\infty$$

where $(f^j, g^j) \in (\dot{H}^1 \times L^2)(\mathbb{R}^2; \mathbb{R})$. Finally, denoting the derivative components of the Wave Map with respect to the Coulomb Gauge by ψ_α, $\alpha = 0,1,2$, then given a sequence of Wave Maps of bounded energy $\mathbf{u}_n : \mathbb{R}^{2+1} \to \mathbb{H}^2$, with corresponding components $\psi_{n,\alpha}$, there is an inductive procedure to construct concentration profiles, so that the $\psi_{n,\alpha}$ can be represented as sum of the suitably modulated concentration profiles, up to an error which can be made small in a suitable sense.

For a review of this work see Tao's Bulletin article [73].

2.7. Wave maps from curved Minkowski space. One can also consider the wave maps equation on a curved domain. Let $(\mathcal{N}, h)$ be a Riemannian manifold of dimension d. Denote by $(\tilde{\mathcal{N}}, \eta)$ the Lorentzian manifold $\tilde{\mathcal{N}} = \mathbb{R} \times \mathcal{N}$, with the metric η represented in local coordinates by $\eta = (\eta_{\alpha\beta}) = \operatorname{diag}(-1, h_{ij})$. Let $(\mathcal{M}, g)$ be a complete Riemannian manifold without boundary of dimension n and consider maps $u : \tilde{\mathcal{N}} \to \mathcal{M}$. Here wave maps are formal critical points of the Lagrangian

$$\mathcal{L}(u, du) = \frac{1}{2} \int_{\tilde{\mathcal{N}}} \langle du, du \rangle_{T^*\tilde{\mathcal{N}} \otimes u^*T\mathcal{M}} \, \operatorname{dvol}_\eta.$$

The differential, du, of the map u is a section of the vector bundle $(T^*\tilde{\mathcal{N}} \otimes u^*T\mathcal{M}, \eta \otimes u^*g)$, where $u^*T\mathcal{M}$ is the pullback of $T\mathcal{M}$ by u and u^*g is the pullback metric. In local coordinates (t, x) on $\tilde{\mathcal{N}}$ and $u = (u^1, \ldots, u^n)$ on $\mathcal{M}$ this becomes

$$\mathcal{L}(u, du) = \frac{1}{2} \int_{\tilde{\mathcal{N}}} \eta^{\alpha\beta}(t, x) g_{ij}(u(t, x)) \partial_\alpha u^i(t, x) \partial_\beta u^j(t, x) \sqrt{|h|} \, dt \, dx.$$

In this intrinsic formulation critical points satisfy

$$\Box_\eta u^k := -\partial_{tt} u^k + \Delta_h u^k = -\eta^{\alpha\beta} \Gamma^k_{ij}(u) \partial_\alpha u^i \partial_\beta u^j, \qquad (2.19)$$

where Δ_h is the Laplace-Beltami operator on $\mathcal{N}$. One can also consider the extrinsic formulation in which case the equation for u is given by

$$\Box_\eta u \perp T_u\mathcal{M} \quad \text{or} \quad \Box_\eta u = -\eta^{\alpha\beta} S(u)(\partial_\alpha u, \partial_\beta u). \qquad (2.20)$$

Here S is the second fundamental form of the embedding $\mathcal{M} \hookrightarrow \mathbb{R}^m$.

It is apparent from the left-hand sides of (2.19) and (2.20) that one must first understand the dynamics of the free wave equation

$$\Box_\eta u = 0 \qquad (2.21)$$

before considering the nonlinear wave maps equation. In general, (2.21) can present extremely challenging analytical aspects. For example, the presence of trapped geodesics can lead to the loss of a dispersive estimate such as (1.5). There are also difficulties from a purely technical point of view as important tools from harmonic analysis used to study dispersive equations on flat backgrounds, such as the Fourier transform, do not extend easily to the global geometric setting.

In light of these difficulties, we highlight two natural starting places where global-in-time dispersive estimates for the free equation (2.21) have been established: (1) $\mathcal{N}$ is a small, asymptotically flat perturbation of Euclidean space, $\mathbb{R}^d$; and (2) $\mathcal{N}$ is d-dimensional real hyperbolic space, $\mathbb{H}^d$.

2.7.1. Small asymptotically flat perturbations of Euclidean space. First consider the free wave equation (2.21) on a small, asymptotically flat perturbation of Euclidean space. The smallness assumption is used to avoid issues such as the trapping of bi-characteristic rays and also to handle the dynamics of low frequencies – in general one can expect significant departures from the Euclidean theory at the level of the low frequencies since these see the global geometry of $\mathcal{N}$. For $\mathcal{N}$ as in (1), *global* Strichartz estimates without loss are established by Metcalfe and Tataru in [52] for linear variable coefficient equations such as (2.21). Using a time dependent FBI transform and a delicate analysis of the evolution in phase space, the authors construct an outgoing parametrix that satisfies global-in-time dispersive estimates. These estimates are paired with a localized energy estimate to control the errors generated in the parametrix construction. See also [79] for a similar analysis of the Schrödinger evolution on curved space as well as [76] and the references therein for more background on phase space transforms and the microlocal framework in which these objects are considered.

Now consider the wave map equation (2.19) or (2.20) where $\mathcal{N}$ is a small, asymptotically flat perturbation of Euclidean space, $\mathbb{R}^d$ with $d = 4$ and $\mathcal{M}$ a smooth manifold with bounded geometry. Using the Shatah-Struwe approach from [64], the small data global theory in the critical space $H^2 \times H^1$ is established in [48]. The idea is to use the method of moving frames to derive a wave equation for the u^*TN-valued 1-form, du, with the Coulomb gauge as the choice of frame on u^*TN. In the global geometric setting, the resulting equation for du is an equation of 1-forms,

$$\Box(du) = d(\eta^{\alpha\beta} A_\alpha du_\beta) + \delta(-A \wedge du) \tag{2.22}$$

where d is the exterior derivative on $\tilde{\mathcal{N}}$, δ is its adjoint, $\Box = d\delta + \delta d$, and A is the connection form associated to the Coulomb gauge. The components of du satisfy a system of variable coefficient nonlinear wave equations for which the Metcalfe-Tataru Strichartz estimates can be used to obtain a-priori estimates in the case of small initial data. Here the Coloumb frame is crucial to estimate the right-hand side of (2.22) since the components of A satisfy a system of variable coefficient *elliptic* equations which are then used to estimate A in terms of du; see for example the classic work [80].

2.7.2. Hyperbolic space. If $\mathcal{N}$ is d-dimensional real hyperbolic space $\mathbb{H}^d$, the free wave equation on $\mathcal{N} = \mathbb{H}^d$ is an appealing object due to the geometric significance of $\mathbb{H}^d$, but also from a technical standpoint because of the existence of the Helgason-Fourier transform; see [28, 29]. In addition to this technical advantage, the negative curvature of $\mathbb{H}^d$ suggests that there should be better dispersion for solutions to (2.21) on $\mathbb{H}^d$ than for their Euclidean counterparts. Intuitively, the exponential volume growth of concentric spheres gives more room for a wave to spread out into. This is indeed the case, and Strichartz estimates for an improved range of admissible exponents have recently been established for the linear Schrödinger equation on $\mathbb{H}^d$ in [1, 31] and for the linear wave and Klein-Gordon equations in [2, 53, 54]; see also [3, 6, 7, 8, 9, 58, 75]. Moreover, in [30] a Bahouri-Gerard type profile decomposition was developed for the Schrödinger equation on $\mathbb{H}^d$.

Next, consider the wave maps equation on the domain $\mathcal{N} = \mathbb{R} \times \mathbb{H}^d$. Since $\mathbb{H}^d$ is rotationally symmetric one can consider *equivariant* wave maps as in the Euclidean case when the target manifold $\mathcal{M}$ is also rotationally symmetric. Restricting to the energy critical dimension, $d = 2$, the usual 1-equivariant formulation is

$$\psi_{tt} - \psi_{rr} - \coth r \, \psi_r + \frac{g(\psi)g'(\psi)}{\sinh^2 r} = 0, \tag{2.23}$$

where (ψ, θ) are geodesic polar coordinates on the target surface $\mathcal{M}$, and g determines the metric, $ds^2 = d\psi^2 + g^2(\psi)d\theta^2$.

Despite the relative simplicity of the equivariant model, this problem exhibits markedly different phenomena than its Euclidean counterpart. The cases of two model targets $\mathcal{M} = \mathbb{S}^2$ and $\mathcal{M} = \mathbb{H}^2$ are considered in [49].

When the target is $\mathbb{S}^2$, there exists a family of equivariant harmonic maps $Q_\lambda : \mathbb{H}^2 \to \mathbb{S}^2$, indexed by a parameter $\lambda \geq 0$ that measures how far the image of

each harmonic map wraps around the sphere, i.e., $Q_\lambda(r) \to 2\arctan(\lambda)$ as $r \to \infty$ and have energies

$$\mathcal{E}(Q_\lambda) \to 0 \;\; \text{as} \;\; \lambda \to 0, \;\; \text{and} \;\; \mathcal{E}(Q_\lambda) \to \mathcal{E}_{\text{euc}}(Q_{\text{euc}}) \;\; \text{as} \;\; \lambda \to 1$$

where Q_{euc} is the unique nontrivial co-rotational Euclidean harmonic map from $\mathbb{R}^2$ to $\mathbb{S}^2$, given by stereographic projection and $\mathcal{E}_{\text{euc}}(Q_{\text{euc}})$ is its energy. The Q_λ are asymptotically stable for values of λ smaller than a threshold that is large enough to allow for maps that wrap more than halfway around the sphere. However, as $\lambda \to \infty$, asymptotic stability via a perturbative argument based on Strichartz estimates is precluded by the existence of *gap eigenvalues* in the spectrum of the operator obtained by linearization about Q_λ. On the other hand, a Struwe-type bubbling argument as in [69] suggests that any solution $\vec{\psi}(t)$ to (2.23) that blows up in finite time must bubble off a Euclidean harmonic map Q_{euc}, and therefore must have enough energy to wrap completely around the sphere. Indeed finite time blow-up via energy concentration is a local phenomena and the global geometry of the domain plays little role. This gives evidence towards a conjecture that in fact every Q_λ is stable – as small perturbations of Q_λ will not have enough energy to bubble off a Q_{euc} – but for large λ, the stability manifests nonlinearly.

When the target is $\mathbb{H}^2$, there exists a continuous family of asymptotically stable equivariant harmonic maps $P_\lambda : \mathbb{H}^2 \to \mathbb{H}^2$ indexed by a parameter $\lambda \in (0,1)$ with $P_\lambda(r) \to 2\text{arctanh}(\lambda)$ as $r \to \infty$ and

$$\mathcal{E}(P_\lambda) \to 0 \;\; \text{as} \;\; \lambda \to 0, \;\; \text{and} \;\; \mathcal{E}(P_\lambda) \to \infty \;\; \text{as} \;\; \lambda \to 1.$$

This stands in sharp contrast to the corresponding problem on Euclidean space, where all finite energy solutions scatter to zero as time tends to infinity. The presence of the nontrivial stable, stationary solutions together with the lack of scaling symmetry and the defocusing nature of the nonlinearity make this an interesting setting to study the large data dynamics. In particular one may expect that any solution $\psi(t,r)$ with $\psi(t,r) \to 2\text{arctanh}(\lambda)$ as $r \to \infty$ scatters to P_λ as $t \to \infty$ – in other words, solution resolution.

3. Scalar semi-linear equations

In the section we present a very small selection of recent results on a much-studied family of problems, namely semilinear equations of the form

$$\Box u + f(u) = 0$$

where $f(u)$ is a suitable power nonlinearity. We will in fact consider only a very special model equation, which is however representative of the kind of phenomena we wish describe.

3.1. The defocusing cubic Klein-Gordon equation. In $\mathbb{R}^{1+3}_{t,x}$ consider the cubic defocusing Klein-Gordon equation

$$\Box u + u + u^3 = 0, \quad (u(0), \dot{u}(0)) = (f, g) \in \mathcal{H} := H^1 \times L^2(\mathbb{R}^3) \qquad (3.1)$$

with conserved energy

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + \frac{1}{4}|u|^4 \right) dx$$

With $S(t)$ denoting the linear propagator of $\Box + 1$ we have

$$\vec{u}(t) = (u, \dot{u})(t) = S(t)(f, g) - \int_0^t S(t-s)(0, u^3(s)) \, ds \qquad (3.2)$$

By contraction mapping for small times T one obtains local wellposedness for $\mathcal{H}$ data. The means that there is a unique solution $(u, \dot{u}) \in C([0,T]; H^1) \times C([0,T]; L^2)$ which satisfies (3.1) in the Duhamel sense. Note that T depends only on the $\mathcal{H}$-size of data. From energy conservation we obtain global existence by time-stepping. By Strichartz estimates, one can easily show that the solution in fact scatters to a free wave (solution of the Klein-Gordon equation without the nonlinearity) in the norm of $\mathcal{H}$ for all small data. For large data, the classical approach to proving scattering proceeds by means of Morawetz estimates, see Ginibre,Velo [27]. Alternatively, a general and elegant method is based on induction in energy, which was first introduced by Bourgain [11]. The modern blueprint of this method is due to Kenig and Merle [34, 35], a key component of which is the powerful concentration compactness decomposition of Bahouri, Gérard [4], see also Merle,Vega [51]. See [56, Capter 2] for the implementation of this method in the context of (3.1).

3.2. The focusing cubic Klein-Gordon equation. The dynamics of the focusing equation

$$\Box u + u - u^3 = 0, \quad (u(0), \dot{u}(0)) = (f, g) \in \mathcal{H} := H^1 \times L^2(\mathbb{R}^3) \qquad (3.3)$$

has been know for a long time to be very different from that of the defocusing equation. Note the conserved energy is indefinite,

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4 \right) dx$$

While small data again lead to global existence and scattering, large data may lead to blowup. Indeed, setting $u = u(t)$ (no spatial dependence) leads to an ODE which blows up in finite time. Truncation of the corresponding data by means of a smooth bump function which equals 1 on a large enough ball yields finite energy data that blow up in finite time. Equation (3.3) also admit time-independent solutions, which solve the elliptic PDE

$$-\Delta\varphi + \varphi - \varphi^3 = 0$$

which is the equation of a critical point of the stationary energy

$$J(\varphi) \int_{\mathbb{R}^3} \left(\frac{1}{2} |\nabla\varphi|^2 + \frac{1}{2} |\varphi|^2 - \frac{1}{4} |\varphi|^4 \right) dx$$

Amongst all nonzero solutions of this equation there exists a class which minimizes $J(\varphi)$. This class is of the form $\{\pm Q(\cdot + y) \mid y \in \mathbb{R}^3\}$ where $Q > 0$ is radial, exponentially decaying. It is also unique with this property. Many years ago, Payne and Sattinger [57] gave a characterization of all possible dynamics below the energy $E(Q,0)$. In the regime of energies above $E(Q,0)$ one has the following description of the dynamics for radial data, due to Nakanishi and the author [56].

Theorem 3.1. *Let* $E(u_0, u_1) < E(Q,0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{\mathrm{rad}}$. *Then for* $t \geq 0$ *the solutions to* (3.3) *exhibit the following trichotomy:*

1. *finite time blowup*

2. *global existence and scattering to* 0

3. *global existence and scattering to* Q: $u(t) = Q + v(t) + o_{H^1}(1)$ *as* $t \to \infty$, *and* $\dot{u}(t) = \dot{v}(t) + o_{L^2}(1)$ *as* $t \to \infty$, $\Box v + v = 0$, $(v, \dot{v}) \in \mathcal{H}$.

All nine combinations of this trichotomy occur as $t \to \pm\infty$.

Similar results can be established in the nonradial case (where Lorentz symmetries come into play), as well as several other nonlinear unstable Hamiltonian wave equations, see [56]. This theorem is not perturbative. In fact, a key step is to exclude almost homoclinic orbits emanating near $\pm Q$ and returning to these equilibria. This is based on an indirect argument using the hyperbolic dynamics near these points (which are a result of the unique negative eigenvalue of the linearized equation), together with the virial functional. The scattering statement is again obtained via concentration compactness arguments. The question of what happens for even larger energies is wide open. For the dissipative version of (3.3) one may obtain similar results, in fact a more complete picture of the dynamics emerges in that case, see the forthcoming work of Burq, Raugel, and the author.

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The University of Chicago, Department of Mathematics, 5734 South University Avenue, Chicago, IL 60637, U.S.A.

E-mail: schlag@math.uchicago.edu