CHAPTER 1

Fourier Series: Convergence and Summability

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus (circle). We consider various function spaces on it, namely $C(\mathbb{T}), C^\alpha(\mathbb{T})$, and $L^p(\mathbb{T})$. The space of complex Borel measures on $\mathbb{T}$ will be denoted by $\mathcal{M}(\mathbb{T})$. Any $\mu \in \mathcal{M}(\mathbb{T})$ has associated with it a Fourier series

$$\mu \sim \sum_{n=-\infty}^{\infty} \hat{\mu}(n) e(nx)$$

where we let $e(x) = e^{2\pi i x}$ and

$$\hat{\mu}(n) = \int_0^1 e(-nx)\, d\mu(x) \ .$$

We first consider the classical question of convergence of Fourier series. The partial sums of $f \in L^1(\mathbb{T})$ are defined as

$$\begin{aligned} S_N f(x) &= \sum_{n=-N}^{N} \hat{f}(n) e(nx) \\ &= \sum_{n=-N}^{N} \int_0^1 e(-ny) f(y)\, dy\; e(nx) \\ &= \int_0^1 \sum_{n=-N}^{N} e(n(x-y)) f(y)\, dy \\ &= \int_0^1 D_N(x-y) f(y)\, dy \end{aligned}$$

where $D_N(x) = \sum_{n=-N}^{N} e(nx)$ is the *Dirichlet kernel*.

Exercise 1. *Check that*

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)} \ ,$$

and draw the graph of D_N*.*

One can also write

$$S_N f(x) = (D_N * f)(x)$$

where $f * g(x) := \int_{\mathbb{T}} f(x-y) g(y)\, dy$ *is the* convolution *of* f *and* g*. You should think of* $f * g$ *as an average of translates of* f*.*

Exercise 2. *Prove the following properties of the convolution:*

a) $\|f * g\|_p \leq \|f\|_p \|g\|_1$ *for all* $1 \leq p \leq \infty$, $f \in L^p$, $g \in L^1$. *This is called Young's inequality. You should pay careful attention to the fact that the integral defining* $f * g$ *is not necessarily absolutely convergent for every* x.

b) *More generally, if* $f \in C(\mathbb{T})$, $\mu \in \mathcal{M}(\mathbb{T})$ *then* $f * \mu$ *is well defined. Show that, for* $1 \leq p \leq \infty$,

$$\|f * \mu\|_p \leq \|f\|_p \|\mu\|$$

which allows you to extend $f * \mu$ *to* $f \in L^1$.

c) *If* $f \in L^p(\mathbb{T})$ *and* $g \in L^{p'}(\mathbb{T})$ *where* $1 < p < \infty$, *and* $\frac{1}{p} + \frac{1}{p'} = 1$ *then* $f * g$, *originally defined only almost every where, extends to a continuous function on* $\mathbb{T}$ *and*

$$\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'} \ .$$

Is this still true of $p = 1$ *or* $p = \infty$*?*

d) *For* $f, g \in L^1(\mathbb{T})$ *show that for all* $n \in \mathbb{Z}$

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n) \ .$$

It is typically difficult to understand convergence of $S_N f$. This can be seen as an instance of the fact that the Dirichlet kernel is *not* an approximate identity, see below. One (standard) positive result is the following theorem.

THEOREM 1. *If* $f \in C^\alpha(\mathbb{T})$, $0 < \alpha \leq 1$, *then* $\|S_N f - f\|_\infty \longrightarrow 0$ *as* $N \longrightarrow \infty$.

PROOF. One has, with $\delta > 0$ to be determined,

$$\begin{aligned} S_N f(x) - f(x) &= \int_0^1 (f(x-y) - f(x)) D_N(y) \, dy \\ &= \int_{|y| \leq \delta} (f(x-y) - f(x)) D_N(y) \, dy \\ &\quad + \int_{|y| > \delta} (f(x-y) - f(x)) D_N(y) \, dy \ . \end{aligned} \tag{1}$$

There is the obvious bound

$$|D_N(y)| \leq C \min\left(N, \frac{1}{|y|}\right) \ .$$

Here and in what follows, C will denote a numerical constant that can change from line to line. The first integral in (1) can be estimated as follows

$$\int_{|y| \leq \delta} |f(x) - f(x-y)| \frac{1}{|y|} \, dy \leq [f]_\alpha \int_{|y| \leq \delta} |y|^{\alpha - 1} \, dy \leq C [f]_\alpha \delta^\alpha \tag{2}$$

where we have set

$$[f]_\alpha = \sup_{x,y} \frac{|f(x) - f(x-y)|}{|y|^\alpha} \ .$$

To bound the second term in (1) one needs to exploit the oscillation of $D_N(y)$. In fact,

$$
\begin{aligned}
B &:= \int_{|y|>\delta} (f(x-y)-f(x))D_N(y)\,dy = \\
&= \int_{|y|>\delta} \frac{f(x-y)-f(x)}{\sin(\pi y)} \sin((2N+1)\pi y)\,dy \\
&= -\int_{|y|>\delta} h_x(y)\sin((2N+1)\pi(y+\frac{1}{2N+1}))\,dy
\end{aligned}
$$

where $h_x(y) := \frac{f(x-y)-f(x)}{\sin(\pi y)}$.

Therefore,

$$
\begin{aligned}
2B &= \int_{|y|>\delta} h_x(y)\sin((2N+1)\pi y)\,dy \\
&\quad - \int_{|y-\frac{1}{2N+1}|>\delta} h_x\left(y-\frac{1}{2N+1}\right)\sin((2N+1)\pi y)\,dy \\
&= \int_{|y|>\delta} (h_x(y)-h_x(y-\frac{1}{2N+1})), \sin((2N+1)\pi y)\,dy \\
&\quad + \int_{[-\delta,-\delta+\frac{1}{2N+1}]} h_x(y-\frac{1}{2N+1})\sin((2N+1)\pi y)\,dy \\
&\quad - \int_{[\delta,\delta+\frac{1}{2N+1}]} h_x(y-\frac{1}{2N+1})\sin((2N+1)\pi y)\,dy\,.
\end{aligned}
$$

These integrals are estimated by putting absolute values inside. To do so we use the bounds

$$
\begin{aligned}
|h_x(y)| &< C\frac{\|f\|_\infty}{\delta}\,, \\
|h_x(y)-h_x(y+\tau)| &< C\left(\frac{|\tau|^\alpha[f]_\alpha}{\delta}+\frac{\|f\|_\infty}{\delta^2}|\tau|\right)
\end{aligned}
$$

if $|y|>\delta>2\tau$.

In view of the preceding one checks

$$
|B| \le C\left(\frac{N^{-\alpha}[f]_\alpha}{\delta}+\frac{N^{-1}\|f\|_\infty}{\delta^2}\right)\,, \tag{3}
$$

provided $\delta > \frac{1}{N}$. Choosing $\delta = N^{-\alpha/2}$ one concludes from (1), (2), and (3) that

$$
|(S_N f)(x)-f(x)| \le C\left(N^{-\alpha^2/2}+N^{-\alpha/2}+N^{-1+\alpha}\right)(\|f\|_\infty+[f]_\alpha)\,,
$$

which proves the theorem. □

Remark. *We shall see later that the theorem fails for continuous functions i.e., $\alpha = 0$.*

Better convergence properties are achieved by means of Cesaro means, i.e.,

$$\sigma_N f := \frac{1}{N} \sum_{n=0}^{N-1} S_n f \ .$$

Setting $K_N = \frac{1}{N} \sum_{n=0}^{N-1} D_n$, which is called the *Fejer kernel*, one therefore has $\sigma_N f = K_N * f$.

EXERCISE 3. *Check that* $K_N(x) = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)}\right)^2$.

It is important to realize that $\widehat{K}_N$ *looks like a triangle, i.e., for all* $n \in \mathbb{Z}$

$$\widehat{K}_N(n) = \left(1 - \frac{|n|}{N}\right)^+ \ .$$

The importance of K_N *with respect to convergence properties lies with the fact that the Fejer kernels form an approximate identity (abbreviated a.i.).*

DEFINITION 1. $\{\Phi_N\}_{N=1}^\infty \subset L^\infty(\mathbb{T})$ *are an approximate identity provided*

A1) $\int_0^1 \Phi_N(x)\, dx = 1$ *for all* N
A2) $\sup_N \int_0^1 |\Phi_N(x)|\, dx < \infty$
A3) *for all* $\delta > 0$ *one has* $\int_{|x|>\delta} |\Phi_N(x)| dx \longrightarrow 0$ *as* $N \longrightarrow \infty$.

LEMMA 1. *The Fejer kernels* $\{K_N\}_{N=1}^\infty$ *form an a.i.*

PROOF. We clearly have $\int_0^1 K_N(x)\, dx = 1$ (why?) and $K_N(x) \geq 0$ so that A1) and A2) hold. A3) follows from the bound $|K_N(x)| \leq \frac{C}{N} \min\left(N^2 \ , \frac{1}{x^2}\right)$. $\square$

LEMMA 2. *For any a.i.* $\{\Phi_N\}_{N=1}^\infty$ *one has*

a) *If* $f \in C(\mathbb{T})$, *then* $\|\Phi_N * f - f\|_\infty \longrightarrow 0$ *as* $N \longrightarrow \infty$
b) *If* $f \in L^p(\mathbb{T})$ *where* $1 \leq p < \infty$, *then* $\|\Phi_N * f - f\|_p \longrightarrow 0$ *as* $N \longrightarrow \infty$.

PROOF.

a) Since $\mathbb{T}$ is compact, f is uniformly continuous. Given $\epsilon > 0$, let $\delta > 0$ be such that

$$\sup_x \sup_{|y|<\delta} |f(x-y) - f(x)| < \epsilon \ .$$

Then, by A1)–A3),

$$\begin{aligned} &|(\Phi_N * f)(x) - f(x)| = \left| \int_{\mathbb{T}} (f(x-y) - f(x))\Phi_N(y)\, dy \right| \\ \leq &\sup_{x\in\mathbb{T}} \sup_{|y|<\delta} |f(x-y) - f(x)| \int_{\mathbb{T}} \|\Phi_N(t)| dt + \int_{|y|\geq\delta} |\Phi_N(y)| 2\|f\|_\infty\, dy \\ < &C\epsilon \end{aligned}$$

provided N is large.

b) Let $g \in C(\mathbb{T})$ with $\|f-g\|_p < \epsilon$. Then

$$\begin{aligned}\|\Phi_N * f - f\|_p &\le \|\Phi_N * (f-g)\|_p + \|f-g\|_p + \|\Phi_N * g - g\|_p \\ &\le \left(\sup_N \|\Phi_N\|_1 + 1\right) \|f-g\|_p + \|\Phi_N * g - g\|_\infty\end{aligned}$$

where we have used Young's inequality (Exercise 2 part a)) to obtain the first term on the right-hand side. Using A2), the assumption on g, as well as part a) finishes the proof.

□

COROLLARY 1.

a) *Trigonmetric polynomials are dense in* $C(\mathbb{T})$, $L^p(\mathbb{T})$, $1 \le p < \infty$.
b) *For any* $f \in L^2(\mathbb{T})$

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

c) $\{e(nx)\}_{n\in\mathbb{Z}}$ *form a complete orthonormal basis in* $L^2(\mathbb{T})$.
d) $\int_{\mathbb{T} f(x)\bar{g}(x)\, dx = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}}$ *for all* $f, g \in L^2(\mathbb{T})$

PROOF.

a) By Lemma 1, $\{K_N\}_{N=1}^\infty$ form an a.i. and Lemma 2 applies. Since $\sigma_N f = K_N * f$ is a trigonometric polynomial, we are done.

b), *c*), *d*) are well-known to be equivalent by basic Hilbert space theory. The point to make here is of course that

$$\int_0^1 e_n(x)\overline{e_m(x)}\, dx = \delta_0(n-m)$$

(where $\delta_0(j) = 1$ if $j = 0$ and $\delta_0(j) = 0$ otherwise). Generally speaking one thus has Bessel's inequality

$$\sum |\hat{f}(n)|^2 \le \|f\|_2^2$$

and equality is equivalent to span $\{e_n\}$ being dense in $L^2(\mathbb{T})$. That, however, is guaranteed by part *a*). □

REMARK. *Parts b), c), d) go under the name Plancherel, Riesz-Fischer, and Parseval.*

COROLLARY 2. *(uniqueness theorem): If* $f \in L^1(\mathbb{T})$ *and* $\hat{f}(n) = 0$ *for all* $n \in \mathbb{Z}$, *then* $f = 0$.

PROOF. $\sigma_N f = 0$ for all N by assumption and $\|\sigma_N f - f\|_1 \longrightarrow 0$. □

COROLLARY 3. *(Riemann-Lebesgue): If* $f \in L^1(\mathbb{T})$, *then* $\hat{f}(n) \longrightarrow 0$ *as* $n \longrightarrow \infty$

PROOF. Given $\epsilon > 0$, let N be so large that $\|\sigma_N f - f\|_1 < \epsilon$.

Then $|\hat{f}(n)| = |\widehat{\sigma_N f}(n) - \hat{f}(n)| \le \|\sigma_N f - f\|_1 < \epsilon$ for $|n| > N$. □

Try to do problem 41 from the appendix.

We now return to the issue of convergence of the partial sums $S_N f$ in $L^p(\mathbb{T})$ or $C(\mathbb{T})$ (observe that it makes no sense to ask about uniform convergence of $S_N f$ for general $f \in L^\infty(\mathbb{T})$ because uniform limits of continuous functions are continuous).

LEMMA 3. *The following statements are equivalent: For any* $1 \leq p \leq \infty$

a) For every $f \in L^p(\mathbb{T})$ *(or* $f \in C(\mathbb{T})$ *if* $p = \infty$*)*

$$\|S_N f - f\|_p \longrightarrow 0$$

as $N \longrightarrow \infty$.

b) $\sup_N \|S_N\|_{p \to p} < \infty$

PROOF. The implication $b) \Longrightarrow a)$ follows from the fact that trigonometric polynomials are dense. The implication $a) \Longrightarrow b)$ can be deduced immediately from the uniform boundedness principle of functional analysis. Alternatively, there is the following elementary argument (the method of the "gliding hump"): Suppose $\sup_N \|S_N\|_{p \longrightarrow p} = \infty$. For every positive integer ℓ one can therefore find a (large) integer N_ℓ such that

$$\|S_{N_\ell} f_\ell\|_p > 2^\ell$$

where f_ℓ is a trigonometric polynomial with $\|f_\ell\|_p = 1$. Now let

$$f(x) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} e(M_\ell x) f_\ell(x)$$

with some integers $\{M_\ell\}$ to be specified. Notice that

$$\|f\|_p \leq \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \|f_\ell\|_p < \infty \,.$$

Now choose $\{M_\ell\}$ tending to infinity so rapidly that the Fourier support of

$$e(M_j x) f_j(x)$$

lies to the right of the Fourier support of

$$\sum_{\ell=1}^{j-1} \frac{1}{\ell^2} e(M_\ell x) f_\ell(x)$$

for every $j \geq 2$ (here Fourier support means those integers for which the corresponding Fourier coefficients are non-zero). Then

$$\|(S_{N_\ell + M_\ell} - S_{M_\ell - N_\ell - 1}) f\|_p = \frac{1}{\ell^2} \|S_{N_\ell} f_\ell\|_p > \frac{2^\ell}{\ell^2}$$

which $\longrightarrow \infty$ as $\ell \longrightarrow \infty$. On the other hand, since $N_\ell + M_\ell \longrightarrow \infty$ and $M_\ell - N_\ell - 1 \longrightarrow \infty$ (why?), the left-hand side $\longrightarrow 0$ as $\ell \longrightarrow \infty$. This contradiction finishes the proof. $\square$

COROLLARY 4. *Fourier series do not converge on* $C(\mathbb{T})$ *and* $L^1(\mathbb{T})$*, i.e., there exists* $f \in C(\mathbb{T})$ *so that* $\|S_N f - f\|_\infty \nrightarrow 0$ *and* $g \in L^1(\mathbb{T})$ *so that* $\|S_N g - g\|_1 \nrightarrow 0$ *.*

Proof. By Lemma 3 it suffices to show that

$$\begin{aligned} \sup_N \|S_N\|_{\infty\longrightarrow\infty} &= \infty \text{ and} \\ \sup_N \|S_N\|_{1\longrightarrow 1} &= \infty\ . \end{aligned} \tag{4}$$

Both properties follow from the fact that

$$\|D_N\|_1 \ge C\log N \longrightarrow \infty,$$

which you should check. To deduce (4) from this, notice that

$$\begin{aligned} \|S_N\|_{\infty\longrightarrow\infty} &= \sup_{\|f\|_\infty=1} \|D_N * f\|_\infty \\ &\ge \sup_{\|f\|_\infty=1} |(D_N * f)(0)| \\ &= \|D_N\|_1\ . \end{aligned}$$

Furthermore, with $\{K_M\}_{M=1}^\infty$ being the Fejer kernels,

$$\|S_N\|_{1\longrightarrow 1} \ge \|D_N * K_M\|_1 \longrightarrow \|D_N\|_1$$

as $M \longrightarrow \infty$. □

Exercise 4. *Show that* $\|S_N\|_{\infty\longrightarrow\infty} = \|S_N\|_{1\longrightarrow 1} = \|D_N\|_1$.

Much finer statements about failure of point-wise convergence of Fourier series are known, see Katznelson. We shall see below that for $1 < p < \infty$

$$\sup_N \|S_N\|_{p\longrightarrow p} < \infty$$

so that

$$S_N f \longrightarrow f \text{ in } L^p(\mathbb{T})\ .$$

The case $p = 2$ is clear, see Corollary 1, but $p \ne 2$ is a deep result. We will develop the theory of the conjugate function to obtain it.

Before doing so, we digress a little to present two basic results due to Bernstein. Both introduce important ideas.

Lemma 4. *Let f be a trigonometric polynomial with $\hat{f}(k) = 0$ if $|k| > n$. Then*

$$\|f'\|_p \le Cn\|f\|_p$$

for any $1 \le p \le \infty$. The constant C is absolute.

Proof. Let $V_n(x) = (1 + e(nx) + e(-nx))K_n(x)$ be de la Vallée Poussin's kernel. In problem 14 in the appendix you are asked to check that

$$\widehat{V}_n(j) = 1 \text{ if } |j| \le n$$

and

$$\|V_n'\|_1 \le Cn\ .$$

Then $f = V_n * f$ and thus $f' = V_n' * f$ so that by Young's inequality

$$\|f'\|_p \le \|V_n'\|_1 \|f\|_p \le Cn\|f\|_p$$

as claimed. $\square$

REMARK. *It is known that one can take $C = 1$ here.*

The next lemma, also due to Bernstein, addresses the question when

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty \ . \tag{5}$$

Applying Cauchy-Schwarz yields

$$\sum_{n\neq 0} |\hat{f}(n)| \leq \left(\sum_{n\neq 0} |\hat{f}(n)|^2 |n|^{1+\epsilon} \right)^{\frac{1}{2}} \left(\sum_{n\neq 0} |n|^{-1-\epsilon} \right)^{\frac{1}{2}}$$

so that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 |n|^{1+\epsilon} < \infty \tag{6}$$

is a sufficient condition for (5) to hold. It turns out that it is a better idea to apply Cauchy-Schwarz only on dyadic blocks. This yields

THEOREM 2. *For any $f \in C^\alpha(\mathbb{T})$ with $\alpha > \frac{1}{2}$ one has*

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$$

PROOF. Let $[f]_\alpha \leq 1$. We claim that for every $j \geq 0$,

$$\left(\sum_{2^j \leq |n| < 2^{j+1}} |\hat{f}(n)|^2 \right)^{\frac{1}{2}} \leq C 2^{-j\alpha} \ . \tag{7}$$

If (7) is true, then

$$\begin{aligned} \sum_{n\neq 0} |\hat{f}(n)| &\leq C \sum_{j=0}^{\infty} \left(\sum_{2^j \leq |n| < 2^{j+1}} |\hat{f}(n)|^2 \right)^{\frac{1}{2}} 2^{j/2} \\ &\leq C \sum_{j=0}^{\infty} 2^{-j(\alpha - \frac{1}{2})} < \infty \ . \end{aligned}$$

To prove (7) we choose a kernel φ_j so that

$$\widehat{\varphi}_j(n) = 1 \text{ if } 2^j \leq |n| \leq 2^{j+1}$$

and

$$\widehat{\varphi}_j(n) = 0 \text{ if } |n| \ll 2^j \text{ or } |n| \gg 2^{j+1}$$

(here $\ll$ and $\gg$ mean "much smaller" and "much bigger", respectively). The point is of course that then

$$\sum_{2^j \leq |n| < 2^{j+1}} |\hat{f}(n)|^2 \leq \|\varphi_j * f\|_2^2 \tag{8}$$

so that it remains to bound the right hand side. There are various ways to construct φ_j. We use de la Vallée Poussin's kernel for this purpose. Set

$$\varphi_j(x) = V_{2^{j-1}}(x) \cdot (e((3 \cdot 2^{j-1} - 1)x) + e(-(3 \cdot 2^{j-1} - 1)x))$$

We leave it to the reader to check that

$$\widehat{\varphi}_j(n) = 1 \text{ for } 2^j \leq |n| \leq 2^{j+1}$$

and that $\widehat{\varphi}_j(0) =, 0$ which is the same as

$$\int_0^1 \varphi_j(x)\, dx = 0 \; .$$

Moreover, since φ_j is constructed from Fejer kernels one easily checks that

$$|\varphi_j(x)| \leq C \frac{1}{2^j} \min\left(2^{2j} \; , \; \frac{1}{|x|^2} \right) \; .$$

Therefore,

$$\begin{aligned}
|(\varphi_j * f)(x)| &= |\int_{\mathbb{T}} \varphi_j(y)(f(x-y) - f(x))\, dy| \\
&\leq \int_{\mathbb{T}} |\varphi_j(y)||f(x-y) - f(x)|\, dy \\
&\leq C \int_0^1 |\varphi_j(y)||y|^\alpha\, dy \\
&\leq C 2^{-j} \int_{|y|>2^{-j}} |y|^{\alpha-2}\, dy + C 2^j \int_{|y|\leq 2^{-j}} |y|^\alpha\, dy \\
&\leq C 2^{-\alpha j} \; .
\end{aligned}$$

The theorem now follows from this bound by means of (7) and (8). $\square$

Remark. *For a proof that avoids using the kernels $\{\varphi_j\}$, see Katznelson. See the same reference for the fact that the theorem fails for $\alpha = \frac{1}{2}$.*

Exercise 5. *Show that under the conditions of Theorem 2 the previous proof implies that (6) holds for some $\varepsilon > 0$. Also show that for any $\alpha < \frac{1}{2}$ there exists $f \in C^\alpha(\mathbb{T})$ so that $\sum_{n\in\mathbb{Z}} |n||\hat{f}(n)|^2 = \infty$.*

CHAPTER 2

Harmonic Functions on the Disk and Poisson Kernel

There is a close connection between Fourier series and analytic (harmonic) functions on the disc $\mathbb{D} := \{z \in \mathbb{C} | |z| \leq 1\}$. In fact, a Fourier series can be viewed as the "boundary values" of a Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n \ .$$

Alternatively, suppose we are given a function f on $\mathbb{T}$ and wish to find the harmonic extension u of f into $\mathbb{D}$, i.e.,

$$\triangle u = 0 \text{ and } u = f \text{ on } \partial\mathbb{D} = \mathbb{T} \ .$$

Since $\triangle z^n = 0$ and $\triangle \bar{z}^n = 0$ for every integer $n \geq 0$, we are lead to defining

$$u(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n + sum_{n=-\infty}^{-1} \hat{f}(n) \bar{z}^{|n|} \tag{9}$$

which at least formally satisfies $u(e(\theta)) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e(n\theta) = f(\theta)$. Inserting $z = re(\theta)$ and $\hat{f}(n) = \int_0^1 e(-n\varphi) f(\varphi)\, d\varphi$ into (9) yields

$$u(re(\theta)) = \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} r^{|n|} e(n(\theta - \varphi)) f(\varphi)\, d\varphi$$

EXERCISE 6. *Check that, for $0 \leq r < 1$,*

$$P_r(\theta) := \sum_{n \in \mathbb{Z}} r^{|n|} e(n\theta) = \frac{1 - r^2}{1 - 2r\cos(2\pi\theta) + r^2} \ .$$

This is the Poisson kernel. *Based on our (formal) calculation above, we therefore expect to obtain the harmonic extension of a "nice enough" function f on $\mathbb{T}$ by means of*

$$u(re(\theta)) = \int_0^1 P_r(\theta - \varphi) f(\varphi)\, dy = (P_r * f)(\theta) \ .$$

for $0 \leq r < 1$.

Note that $P_r(\theta)$, for $0 \leq r < 1$, is a harmonic function of the variables $x + iy = re(\theta)$. Moreover, for any finite measure $\mu \in \mathcal{M}(\mathbb{T})$ the expression $(P_r * \mu)(\theta)$ is not only well-defined, but defines a harmonic function on $\mathbb{D}$.

EXERCISE 7. *Check that $\{P_r\}_{0<r<1}$ is an approximate identity. The role of $N \in \mathbb{Z}^+$ in Definition 1 is played here by $0 < r < 1$ and $N \longrightarrow \infty$ is replaced with $r \longrightarrow 1$.*

An important role is played by the kernel $Q_r(\theta)$ which is the *harmonic conjugate* of $P_r(\theta)$. Recall that this means that $P_r(\theta) + iQ_r(\theta)$ is analytic in $z = re(\theta)$ and $Q_0 = 0$. In this case it is easy to find $Q_r(\theta)$ since

$$P_r(\theta) = \Re\left(\frac{1+z}{1-z}\right)$$

and therefore

$$Q_r(\theta) = \Im\left(\frac{1+z}{1-z}\right) = \frac{2r\sin(2\pi\theta)}{1 - 2r\cos(2\pi\theta) + r^2} \ .$$

Exercise 8.

a) *Show that* $\{Q_r\}_{0<r<1}$ *is* not *an approximate identity.*
b) *Check that* $Q_1(\theta) = \cot(\pi\theta)$. *Draw the graph of* $Q_1(\theta)$. *What is the asymptotic behavior of* $Q_1(\theta)$ *for* θ *close to zero?*

We will study conjugate harmonic functions later. First, we clarify in what sense the harmonic extension $P_r * f$ of f attains f as its boundary values.

Definition 2. *For any* $1 \le p \le \infty$ *define*

$$h^p(\mathbb{D}) := \left\{ u : \mathbb{D} \longrightarrow \mathbb{C} \text{ harmonic} \mid \sup_{0<r<1} \int_0^1 |u(re(\theta))|^p d\theta < \infty \right\} .$$

These are the "little" Hardy spaces with norm

$$|||u|||_p := \sup_{0<r<1} \|u(re(\cdot))\|_{L^p(\mathbb{T})} \ .$$

It is important to observe that $P_r(\theta) \in h^1(\mathbb{D})$. Observe that this function has "boundary values" δ_0 (the Dirac mass at $\theta = 0$) since $P_r = P_r * \delta_0$.

Theorem 3. *There is a one-to-one correspondence between* $h^1(\mathbb{D})$ *and* $\mathcal{M}(\mathbb{T})$, *given by* $\mu \in \mathcal{M}(\mathbb{T}) \longmapsto F_r(\theta) := (P_r * \mu)(\theta)$. *Furthermore,*

$$\|\mu\| = \sup_{0<r<1} \|F_r\|_1 = \lim_{r \longrightarrow 1} \|F_r\|_1 \ , \tag{10}$$

and

a) μ *is absolutely continuous with respect to Lebesgue measure* ($\mu \ll d\theta$) *if and only if* $\{F_r\}$ *converges in* $L^1(\mathbb{T})$. *If so, then* $d\mu = f\, d\theta$ *where* $f = L^1$*-limit of* F_r.
b) *The following are equivalent for* $1 < p \le \infty$: $d\mu = f d\theta$ *with* $f \in L^p(\mathbb{T})$

$$\begin{aligned} &\Longleftrightarrow \{F_r\}_{0<r<1} \text{ is } L^p\text{- bounded} \\ &\Longleftrightarrow \{F_r\} \text{ converges in } L^p \text{ if } 1 < p < \infty \text{ and in } \sigma^* \text{ sense in } L^\infty \text{ if} \\ &\quad\ p = \infty \text{ as } r \longrightarrow 1 \end{aligned}$$

c) f *is continuous* $\Leftrightarrow$ F *extends to a continuous function on* $\overline{\mathbb{D}}$ $\Leftrightarrow$ F_r *converges uniformly as* $r \longrightarrow 1-$.

This theorem identifies $h^1(\mathbb{D})$ with $\mathcal{M}(\mathbb{T})$, and $h^p(\mathbb{D})$ with $L^p(\mathbb{T})$ for $1 < p \le \infty$. Moreover, $h^\infty(\mathbb{D})$ contains the subclass of harmonic function that can be extended continuously onto $\overline{\mathbb{D}}$; this subclass is the same as $C(\mathbb{T})$. Before proving the theorem we present two simple lemmas. In what follows we use the notation $F_r(\theta) := F\big(re(\theta)\big)$.

Lemma 5.

a) *If $F \in C(\overline{\mathbb{D}})$ and $\triangle F = 0$ in $\mathbb{D}$, then $F_r = P_r * F_1$ for any $0 \le r < 1$.*
b) *If $\triangle F = 0$ in $\mathbb{D}$, then $F_{rs} = P_r * F_s$ for any $0 \le r, s < 1$.*
c) *As a function of $r \in (0,1)$ the norms $\|F_r\|_p$ are non-decreasing for any $1 \le p \le \infty$.*

Proof.

a) Let $u(re(\theta)) := (P_r * F_1)(\theta)$ for any $0 \le r < 1$, θ. Then $\triangle u = 0$ in $\mathbb{D}$. By Lemma 2 a) and Exercise 7, $\|u_r - F_1\|_\infty \longrightarrow 0$ as $r \longrightarrow 1$. Hence, u extends to a continuous function on $\overline{\mathbb{D}}$ with the same boundary values as F. By the maximum principle, $u = F$ as claimed.
b) Rescaling the disc $s\mathbb{D}$ to $\mathbb{D}$ reduces *b)* to *a)*.
c) By b) and Young's inequality

$$\|F_{rs}\|_p \le \|P_r\|_1 \|F_s\|_p = \|F_s\|_p$$

as claimed.

□

Lemma 6. *Let $F \in h^1(\mathbb{D})$. Then there exists a unique measure $\mu \in \mathcal{M}(\mathbb{T})$ such that $F_r = P_r * \mu$.*

Proof. Since the unit ball of $\mathcal{M}(\mathbb{T})$ is σ^*-compact there exists a subsequence $r_j \longrightarrow 1$ with $F_{r_j} \longrightarrow \mu$ in σ^*-sense to some $\mu \in \mathcal{M}(\mathbb{T})$. Then, for any $0 < r < 1$,

$$P_r * \mu = \lim_{j\to\infty} (F_{r_j} * P_r) = \lim_{j\longrightarrow\infty} F_{rr_j} = F_r$$

by Lemma 5, *b)*. Let $f \in C(\mathbb{T})$. Then $\langle F_r, f\rangle = \langle P_r * \mu, f\rangle = \langle \mu, P_r * f\rangle \longrightarrow \langle \mu, f\rangle$ as $r \to 1$ (where we again use Lemma 2 *a)*. This shows that

$$\mu = \sigma^* - \lim_{r\to 1} F_r \,, \tag{11}$$

which implies uniqueness of μ. □

Proof. If $\mu \in \mathcal{M}(\mathbb{T})$, then $P_r * \mu \in h^1(\mathbb{D})$. Conversely, given $F \in h^1(\mathbb{D})$ then by Lemma 6 there is a unique μ so that $F_r = P_r * \mu$. This gives the one-to-one correspondence. Moreover, (11) and Lemma 5 *c)* show that

$$\|\mu\| \le \limsup_{r\to 1} \|F_r\|_1 = \sup_{0<r<1} \|F_r\|_1 = \lim_{r\to 1} \|F_r\|_1 \,.$$

Since clearly also

$$\sup_{0<r<1} \|F_r\|_1 \le \sup_{0<r<1} \|P_r\|_1 \|\mu\| = \|\mu\| \,,$$

(10) follows. If $f \in L^1(\mathbb{T})$ and $d\mu = fd\theta$, then Lemma 2 *b*) shows that $F_r \to f$ in $L^1(\mathbb{T})$. Conversely, if $F_r \to f$ in the sense of $L^1(\mathbb{T})$, then because of (11) necessarily $d\mu = fd\theta$ which proves *a*), *b*), and *c*) are equally easy and we skip the details—simply invoke Lemma 2 *b*) for $1 < p < \infty$ and Lemma22 a) if $p = \infty$. $\square$

Next, we turn to the issue of almost everywhere convergence of $P_r * f$ to f as $r \to 1$. This is an instance of the general fact that radially bounded approximate identities are dominated by the Hardy-Littlewood maximal function Mf, see below. Recall that (with $|I| = \text{mes}\,(I)$ =Lebesgue measure of I)

$$Mf(x) = \sup_{x \in I \subset \mathbb{T}} \frac{1}{|I|} \int_I |f(y)|\, dy$$

where $I \subset \mathbb{T}$ is an (open) interval. The basic fact here is

PROPOSITION 1.

a) *M is bounded from L^1 to weak L^1, i.e.,*

$$\text{mes}[x \in \mathbb{T} | Mf(x) > \lambda] \leq \frac{3}{\lambda} \|f\|_1$$

for all $\lambda > 0$.

b) *For any $1 < p \leq \infty$, M is bounded on L^p.*

PROOF. Fix some $\lambda > 0$ and any compact

$$K \subset \{x | Mf(x) > \lambda\}\ . \tag{12}$$

There exists a finite cover $\{I_j\}_{j=1}^N$ of $\mathbb{T}$ by open arcs I_j such that

$$\int_{I_j} |f(y)|\, dy > \lambda |I_j| \tag{13}$$

for each j. We now apply Wiener's covering lemma to pass to a more convenient subcover: Select an arc of maximal length from $\{I_j\}$; call it J_1. Observe that any I_j such that $I_j \cap J_1 \neq \emptyset$ satisfies $I_j \subset 3 \cdot J_1$ where $3 \cdot J_1$ is the arc with the same center as J_1 and three times the length (if $3 \cdot J_1$ has length larger than 1, then set $3 \cdot J_1 = \mathbb{T}$). Now remove all arcs from $\{I_j\}_{j=1}^N$ that intersect J_1. Let J_2 be one of the remaining ones with maximal length. Continuing in this fashion we obtain arcs $\{J_\ell\}_{\ell=1}^L$ which are pair-wise disjoint and so that

$$\bigcup_{j=1}^N I_j \subset \bigcup_{\ell=1}^L 3 \cdot J_\ell\ .$$

In view of (12) and (13) therefore,

$$\begin{aligned} \text{mes}(K) &\leq \text{mes}\left(\bigcup_{\ell=1}^L 3 \cdot J_\ell\right) \leq 3 \sum_{\ell=1}^L \text{mes}(J_\ell) \\ &\leq \frac{3}{\lambda} \sum_{\ell=1}^L \int_{J_\ell} |f(y)|\, dy \leq \frac{3}{\lambda} \|f\|_1\ , \end{aligned}$$

as claimed.

To prove part *b*), one interpolates the bound from *a*) with the trivial L^∞ bound

$$\|Mf\|_\infty \le \|f\|_\infty \ .$$

by means of Marcinkiewicz's interpolation theorem, see Stein-Weiss, Stein 1. $\square$

DEFINITION 3. *Let* $\{\Phi_n\}_{n=1}^\infty$ *be an approximate identity as in Definition 1. We say that it is* radially bounded *if there exist functions* $\{\Psi_n\}_{n=1}^\infty$ *on* $\mathbb{T}$ *so that the following additional property holds:*

A4) $|\Phi_n| \le \Psi_n$, Ψ_n *is even and decreasing, i.e.,* $\Psi_n(x) \le \Psi_n(y)$ *for* $0 \le y \le x \le \frac{1}{2}$, *for all* $n \ge 1$. *Finally, we require that* $\sup_n \|\Psi_n\|_1 < \infty$.

Examples: Fejer-, Poisson-kernels, and the box kernels (with $\epsilon \to 0$ instead of $n \to \infty$)

$$\left\{ \frac{1}{2\epsilon} \chi_{[-\epsilon,\epsilon]} \right\}_{0<\epsilon<\frac{1}{2}} \ .$$

LEMMA 7. *If* $\{\Phi_n\}_{n=1}^\infty$ *satisfies* A4), *then for any* $f \in L^1(\mathbb{T})$ *one has*

$$\sup_n |(\Phi_n * f)(x)| \le \sup_n \|\Psi_n\|_1 Mf(x)$$

for all $x \in \mathbb{T}$.

PROOF. It clearly suffices to show the following statement: let $K : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}^+ \cup \{0\}$ be even and decreasing. Then for any $f \in L^1(\mathbb{T})$

$$|(K * f)(x)| \le \|K\|_1 Mf(x) \ . \tag{14}$$

Indeed, assume that (14) holds. Then

$$\sup_n |(\Phi_n * f)(x)| \le \sup_n (\Psi_n * |f|)(x) \le \sup_n \|\Psi_n\|_1 Mf(x)$$

and the lemma follows. The idea behind (14) is to show that K can be written as an average of box kernels, i.e., for some positive measure μ

$$K(x) = \int_0^{\frac{1}{2}} \chi_{[-y,y]}(x) \, d\mu(y) \ . \tag{15}$$

We leave it to the reader to check that

$$d\mu = -dK + K\left(\frac{1}{2}\right) \delta_{\frac{1}{2}}$$

is a suitable choice. Notice that (15) implies that

$$\int_0^1 K(x) \, dx = \int_0^{\frac{1}{2}} 2y \, d\mu(y) \ .$$

Moreover, by (15),

$$\begin{aligned}
|(K * f)(x)| &= |\int_0^{\frac{1}{2}} \left(\frac{1}{2y}\chi_{[-y,y]} * f\right)(x)\, 2y\, d\mu(y)| \\
&\le \int_0^{\frac{1}{2}} Mf(x) 2y\, d\mu(y) \\
&= Mf(x)\|K\|_1
\end{aligned}$$

which is (14). □

This lemma establishes the uniform control that is needed for almost everywhere convergence.

THEOREM 4. *If $\{\Phi_n\}_{n=1}^{\infty}$ satisfies A1)–A4), then for any $f \in L^1(\mathbb{T})$ one has $\Phi_n * f \longrightarrow f$ almost everywhere as $n \longrightarrow \infty$.*

PROOF. Pick $\epsilon > 0$ and let $g \in C(\mathbb{T})$ with $\|f - g\|_1 < \epsilon$. By Lemma 2 a), with $h = f - g$ one has

$$\begin{aligned}
&\mathrm{mes}\left[x \in \mathbb{T} | \limsup_{n\longrightarrow\infty} |\left(\Phi_n * f\right)(x) - f(x)| > \sqrt{\epsilon}\right] \\
\le &\mathrm{mes}\left[x \in \mathbb{T} | \limsup_{n\to\infty} |\left(\Phi_n * h\right)(x)| > \sqrt{\epsilon}/2\right] + \mathrm{mes}\left[x \in \mathbb{T} | h(x)| > \sqrt{\epsilon}/2\right] \\
\le &\mathrm{mes}\left[x \in \mathbb{T} | \sup_n |\left(\Phi_n * h\right)(x)| > \sqrt{\epsilon}/2\right] + \mathrm{mes}\left[x \in \mathbb{T} || h(x)| > \sqrt{\epsilon}/2\right] \\
\le &\mathrm{mes}\left[x \in \mathbb{T} | CMh(x) > \sqrt{\epsilon}/2\right] + \mathrm{mes}\left[x \in \mathbb{T} || h(x)| > \sqrt{\epsilon}/2\right] \\
\le &C\sqrt{\epsilon}\ .
\end{aligned}$$

To pass to the final inequality we used Proposition 1 as well as Markov's inequality (recall $\|h\|_1 < \epsilon$). □

As a corollary we not only obtain the classical Lebesgue differentiation theorem, but also almost everywhere convergence of the Cesaro means $\sigma_N f$, as well as of the Poisson integrals $P_r * f$ to f for any $f \in L^1(\mathbb{T})$. It is a famous theorem of Kolmogoroff that this *fails* for the partial sums $S_N f$.

EXERCISE 9. *It is natural to ask whether there is an analogue of Theorem 4 for measures $\mu \in \mathcal{M}(\mathbb{T})$. Prove the following:*

a) *If $\mu \in \mathcal{M}(\mathbb{T})$ is singular with respect to Lebesgue measure ($\mu \perp d\theta$), then for a.e. $x \in \mathbb{T}$ (with respect to Lebesgue measure)*

$$\frac{\mu([x - \epsilon, x + \epsilon])}{2\epsilon} \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0$$

you should compare this with problems 6 and 9 from the appendix.

b) *Let* $\{\Phi_n\}_{n=1}^{\infty}$ *satisfy A1)–A4), and assume that the* $\{\Psi_n\}_{n=1}^{\infty}$ *from Definition 2 also satisfy*

$$\sup_{\delta<|\theta|<\frac{1}{2}} |\Psi_n(\theta)| \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

for all $\delta > 0$. *Under these assumptions show that for any* $\mu \in \mathcal{M}(\mathbb{T})$

$$\Phi_n * \mu \longrightarrow f \text{ } a.e. \text{ as } n \longrightarrow \infty$$

where $d\mu = f d\theta + d\nu_s$ *is the Lebesgue decomposition, i.e.,* $f \in L^1(\mathbb{T})$ *and* $\nu_s \perp d\theta$.

CHAPTER 3

L^1 bounded analytic functions and the F. & M. Riesz theorem

We now turn to functions $F = u + iv \in h^1(\mathbb{D})$ which are *analytic* in $\mathbb{D}$ (note that analytic functions are complex valued harmonic functions). These functions form the class $\mathbb{H}^1(\mathbb{D})$, the "big" Hardy space. We have shown that $F_r = P_r * \mu$ for some $\mu \in \mathcal{M}(\mathbb{T})$. It is important to note that by analyticity $\hat{\mu}(n) = 0$ if $n < 0$. A famous result by F. & M. Riesz asserts that such measures are necessarily absolutely continuous. We shall prove this theorem by means of subharmonic functions.

Definition 4. *Let $\Omega \subset \mathbb{R}^2$ be a region (i.e., open and connected) and let $f : \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$. We say that f is* subharmonic *if*

a) f is continuous
b) for all $z \in \Omega$ there exists $r_z > 0$ so that

$$f(z) = \int_0^1 f\big(z + re(\theta)\big)\, d\theta$$

for all $0 < r < r_z$ (we refer to this as the "submean value property").

Lemma 8.

a) If f and g are subharmonic, then $f \vee g = \max(f, g)$ is subharmonic.
b) If $f \in C^2(\Omega)$ then f is subharmonic $\Longleftrightarrow \triangle f \geq 0$ in Ω
c) F analytic $\Longrightarrow \log|F|$ and $|F|^\alpha$ with $\alpha > 0$ are subharmonic
d) If f is subharmonic and φ is increasing and convex, then $\varphi \circ f$ is subharmonic (we set $\varphi(-\infty) := \lim_{x\to-\infty} \varphi(x)$.

Proof. 1) is immediate. For 2) use Jensen's formula

$$\int_0^1 f(z + re(\theta))\, d\theta - f(z) = \iint_{D(z,r)} \log \frac{r}{|w - z|} \triangle f(w)\, dm(w) \tag{16}$$

where dm stands for two-dimensional Lebesgue measure and $D(z, r) = \{w \in C \,||w - z| < r\}$. As an exercise, you should verify this formula (from Green's formula) for all $f \in C^2(\Omega)$. If $\triangle f \geq 0$, then the submean value property holds. If $\triangle f(z_0) < 0$, then let $r_0 > 0$ be sufficiently small so that $\triangle f < 0$ on $D(z_0, r_0)$ Since $\log \frac{r_0}{|w-z_0|} > 0$ on this disk, Jensen's formula implies that the submean value property fails. Next, we verify 4) by means of Jensen's inequality:

$$\varphi(f(z)) \leq \varphi\left(\int_0^1 f(z + re(\theta))\, d\theta\right) \leq \int_0^1 \varphi(f(z + re(\theta))\, d\theta\ .$$

The first inequality sign uses that φ is increasing, whereas the second uses convexity of φ (this second inequality is called Jensen's inequality). If F is analytic, then $\log|F|$ is continuous with values in $\mathbb{R}\cup\{-\infty\}$. If $F(z_0)\neq 0$, then $\log|F(z)|$ is harmonic on some disk $D(z_0,r_0)$. Thus, one has the stronger mean value property on this disk. If $F(z_0)=0$, then $\log|F(z_0)|=-\infty$, and there is nothing to prove. To see that $|F|^\alpha$ is subharmonic, apply 4) to $\log|F(z)|$ with $\varphi(x)=\exp(\alpha x)$. □

REMARK. *It is helpful to keep in mind that in one dimension "harmonic = linear" and "subharmonic = convex".*

EXERCISE 10. *Let u be subharmonic on a domain Ω. Show that there exist a unique measure μ on Ω such that $\mu(K)<\infty$ for every $K\subset\subset\Omega$ (i.e., K is a compact subset of Ω) and so that*

$$u(z)=\iint \log|z-\zeta|\,d\mu(\zeta)+h(z)$$

where h is harmonic on Ω. (This is "Riesz's representation of subharmonic functions").

EXERCISE 11. *With u and μ as in the previous exercise, show that*

$$\int_0^1 u(z+re(\theta))\,d\theta-u(z)=\int_0^r \frac{\mu(D(z,t))}{t}\,dt$$

for all $D(z,r)\subset\Omega$ (this is "Jensen's formula").

LEMMA 9. *Let Ω be a bounded region. Suppose f is subharmonic on Ω, $f\in C(\bar\Omega)$ and let u be harmonic on Ω, $u\in C(\bar\Omega)$. If $f\le u$ on $\partial\Omega$, then $f\le u$ on Ω.*

PROOF. We may take $u=0$, so $f\le 0$ on $\partial\Omega$. Let $M=\max_{\bar\Omega} f$ and assume that $M>0$. Set

$$S=\{z\in\bar\Omega | f(z)=M\}\ .$$

Then $S\subset\Omega$ and S is closed in Ω. If $z\in S$, then by the submean value property there exists $r_z>0$ so that $D(z,r_z)\subset\Omega$. Hence S is also open. Since Ω is assumed to be connected, one obtains $S=\Omega$. This is a contradiction. □

The following lemma shows that the submean value property holds on any disk in Ω.

LEMMA 10. *Let f be subharmonic in Ω, $z_0\in\Omega$, $\overline{D(z_0,r)}\subset\Omega$. Then*

$$f(z_0)\le\int_0^1 f(z_0+re(\theta))\,d\theta\ .$$

PROOF. Let $g_n=\max(f,-n)$, where $n\ge 1$. Without loss of generality $z_0=0$. Define $u_n(z)$ to be the harmonic extension of g_n restricted to $\partial D(z_0,r)$ where $r>0$ is as in the statement of the lemma. By the previous lemma,

$$f(0)\le g_n(0)\le u_n(0)=\int_0^1 u_n(re(\theta))\,d\theta$$

the last equality being the mean value property of harmonic functions. Since

$$\max_{|z|\le r} u_n(z)\le\max_{|z|\le r} f(z)$$

the monotone convergence theorem for decreasing sequences yields

$$f(0) \leq \int_0^1 f(re(\theta))\, d\theta \ ,$$

as claimed. $\square$

Corollary 5. *If g is subharmonic on $\mathbb{D}$, then for all θ*

$$g(rse(\theta)) \leq (P_r * g_s)(\theta)$$

for any $0 < r, s < 1$.

Proof. If $g > -\infty$ everywhere on $\mathbb{D}$, then this follows from Lemma 9. If not, then set $g_n = g \vee n$. Thus

$$g(rse(\theta)) \leq g_n(rse(\theta)) \leq (P_r * (g_n)_s)(\theta) \ ,$$

and consequently

$$g(rse(\theta)) \leq \limsup_{n \longrightarrow \infty} (P_r * (g_n)_s)(\theta) \leq (P_r * g_s)(\theta)$$

where the final inequality follows from Fatou's lemma (which can be applied in the "reverse form" here since the (g_n)'s have a uniform upper bound). $\square$

Remark. *If $g_s \notin L^1(\mathbb{T})$, then $g \equiv -\infty$ on $D(0,s)$ and so $g \equiv -\infty$ on $D(0,1)$.*

Definition 5. *Let F be any function on $\mathbb{D}$ then $F^* : \mathbb{T} \longrightarrow \mathbb{R}$ is defined as*

$$F^*(\theta) = \sup_{0<r<1} F(re(\theta)) \ .$$

We showed in the previous lecture that any $u \in h^1(\mathbb{D})$ satisfies $u^* \leq CM\mu$ where μ is the boundary measure of u, i.e., $u_r = P_r * \mu$.

Proposition 2. *Suppose g is subharmonic on $\mathbb{D}, g \geq 0$ and g is L^1-bounded, i.e.,*

$$|||g|||_1 := \sup_{0<r<1} \int_0^1 , g(re(\theta))\, d\theta < \infty \ .$$

Then

a) mes$[\theta \in \mathbb{T} | g^*(\theta) > \lambda] \leq \frac{3}{\lambda} |||g|||_1$ for $\forall \lambda > 0$.
b) If g is L^p bounded, with $1 < p \leq \infty$, then

$$\|g^*\|_{L^p(\mathbb{T})} \leq C_p |||g|||_p \ .$$

Proof.

a) Let $g_{r_n} \rightharpoonup \mu \in \mathcal{M}(\mathbb{T})$ in the σ^*-sense. Then $\|\mu\| \leq |||g|||_1$ and

$$g_s \longleftarrow g_{r_n s} \leq g_{r_n} * P_s \longrightarrow P_s * \mu \ .$$

Thus, by Lemma 7,

$$g^* \leq \sup_{0<s<1} P_s * \mu \leq M\mu \ ,$$

and the desired bound now follows from Proposition 1.

b) If $|||g|||_p < \infty$, then $\frac{d\mu}{d\theta} \in L^p(\mathbb{T})$ with $\|\frac{d\mu}{d\theta}\|_p \leq |||g|||_p$ and thus

$$g^* \leq CM\left(\frac{d\mu}{d\theta}\right) \in L^p(\mathbb{T})$$

by Proposition 1, as claimed.

□

We now present three versions of a well-known theorem due to F. & M. Riesz. You should pay careful attention to the fact that the following result fails without analyticity.

Theorem 5 (First Version of F. & M. Riesz Theorem). *Suppose $F \in h^1(\mathbb{D})$ is analytic. Then $F^* \in L^1(\mathbb{T})$.*

Proof. $|F|^{\frac{1}{2}}$ is subharmonic and L^2-bounded. By Proposition 2 therefore $|F|^{\frac{1}{2}*} \in L^2(\mathbb{T})$. But $|F|^{\frac{1}{2}*} = |F^*|^{\frac{1}{2}}$ and thus $F^* \in L^1(\mathbb{T})$. □

Let $F \in h^1(\mathbb{D})$. By Theorem 3, $F_r = P_r * \mu$ where $\mu \in \mathcal{M}(\mathbb{T})$ has a Lebesgue decomposition $d\mu = fd\theta + \nu_s$, ν_s singular and $f \in L^1(\mathbb{T})$. By Exercise 9 *b*) one has $P_r * \mu \to f$ a.e. as $r \longrightarrow 1$. Thus, $\lim_{r\to 1} F(re(\theta)) = f(\theta)$ exists for a.e. $\theta \in \mathbb{T}$. This justifies the statement of the following theorem.

Theorem 5 (Second Version). *Assume $F \in h^1(\mathbb{D})$ and F analytic. Let $f(\theta) = \lim_{r\to 1} F\big(re(\theta)\big)$. Then $F_r = P_r * f$ for all $0 < r < 1$.*

Proof. We have $F_r \to f$ a.e. and $|F_r| \leq F^* \in L^1$ by the previous theorem. Therefore, $F_r \to f$ in $L^1(\mathbb{T})$ and Theorem 3 a) finishes the proof. □

Theorem 5 (Third Version). *Suppose $\mu \in \mathcal{M}(\mathbb{T})$, $\hat{\mu}(n) = 0$ if $n < 0$. Then $\mu \ll d\theta$.*

Proof. Since $\hat{\mu}(n) = 0$ for $n \in \mathbb{Z}^-$ one has that $F_r = P_r * \mu$ is analytic on $\mathbb{D}$. By the second version above and the remark preceding it, one concludes that $d\mu = f\ d\theta$ with $f = \lim_{r\to 1-} F\big(re(\theta)\big) \in L^1(\mathbb{T})$, as claimed. □

Remark. *The logic of this argument shows that if $\mu \perp d\theta$, then the harmonic extension u_μ of μ satisfies $u^*_\mu \notin L^1(\mathbb{T})$. It is possible to give a more quantitative version of this fact. Indeed, suppose that μ is a positive measure. Then for some absolute constant C,*

$$C^{-1}M\mu < u^*_\mu < CM\mu \tag{17}$$

where the upper bound is Lemma 7 (applied to the Poisson kernel) and the lower bound follows from the assumption $\mu \geq 0$ and the fact that the Poisson kernel dominates the box kernel. Problem 9 from the appendix therefore implies the quantitative non-L^1 statement

$$\text{mes}[\theta \in \mathbb{T} | u^*_\mu(\theta) \geq \lambda] \geq \frac{C}{\lambda}\|\mu\|$$

for $\mu \perp d\theta, \mu \geq 0$.

The F. & M. Riesz theorem raises the following deep question: Given $f \in L^1(\mathbb{T})$, how can one decide if

$$P_r * f + iQ_r * f \in h^1(\mathbb{D}) \ ?$$

We know that necessarily $u_f^* = (P_r * f)^* \in L^1(\mathbb{T})$. A famous theorem by Burkholder-Gundy-Silverstein, see Koosis, states that this is also sufficient (they proved this for the non-tangential *maximal function). It is important that you understand the difference from* (17)*, i.e., this is* not *the same as $\mathcal{M}f \in L^1(\mathbb{T})$ due to possible cancellation in f (it is known that $\mathcal{M}f \in L^1(\mathbb{T}) \iff |f|\log(2+|f|) \in L^1$, see Stein 1, page 23).*

Theorem 6 (Second F. & M. Riesz Theorem). *Let F be analytic on $\mathbb{D}$ and L^1-bounded, i.e., $F \in h^1(\mathbb{D})$. Assume $F \not\equiv 0$ and set $f = \lim_{r\to 1-} F_r$. Then $\log|f| \in L^1(\mathbb{T})$. In particular, f does not vanish on a set of positive measure.*

Proof. The idea is that if $F(0) \neq 0$, then

$$\int_{\mathbb{T}} \log|f| \geq \log|F(0)| > -\infty \ .$$

Since $\log_+|f| \leq |f| \in L^1(\mathbb{T})$ by Theorem 5, we should be done. Some care needs to be taken, though, as F attains the boundary values f only in the almost everywhere sense. This issue can easily be handled by means of Fatou's lemma: First, $F^* \in L^1(\mathbb{T})$, so $\log_+|F_r| \leq F^*$ implies that $\log_+|f| \in L^1(\mathbb{T})$ by Lebesgue dominated convergence. Second, by subharmonicity,

$$\int \log|F_r(\theta)|\, d\theta \geq \log|F(0)|$$

so that

$$\int \log|f(\theta)|\, d\theta = \int \lim_{r\to 1} \log|F_r(\theta)|\, d\theta \geq \limsup_{r\to 1} \int \log|F_r(\theta)|\, d\theta \geq \log|F(0)| \ .$$

If $F(0) \neq 0$, then we are done. If $F(0) = 0$, then choose another point $z_0 \in \mathbb{D}$ for which $F(z_0) \neq 0$. Now one either repeats the previous argument with the Poisson kernel instead of the submean value property, or one composes F with an automorphism of the unit disk that moves 0 to z_0. Then the previous argument applies. $\square$

Theorem 6 should of course be thought of as a version of the uniqueness theorem for analytic functions.

CHAPTER 4

The Conjugate Harmonic Function

DEFINITION 6. *Let u be real-valued and harmonic in $\mathbb{D}$. Then we define $\tilde{u}$ to be that unique real-valued and harmonic function in $\mathbb{D}$ for which $u + i\tilde{u}$ is analytic and $\tilde{u}(0) = 0$. If u is complex-valued and harmonic, then we set $\tilde{u} := (\Re u)\tilde{} + i(\Im u)\tilde{}$.*

The following lemma presents some properties of the harmonic conjugate $\tilde{u}$.

LEMMA 11.

a) *If u is constant, then $\tilde{u} = 0$.*
b) *If u is analytic in $\mathbb{D}$ and $u(0) = 0$, then $\tilde{u} = -iu$. If u is co-analytic (meaning that $\bar{u}$ is analytic), then $\tilde{u} = iu$.*
c) *Any harmonic function u can be written uniquely as $u = c + f + \bar{g}$ with $c =$constant, f, g analytic, and $f(0) = g(0) = 0$.*

PROOF. 1) and 2) follow immediately from the definition, whereas 3) is given by

$$u = u(0) + \frac{1}{2}(u - u(0) + i\tilde{u}) + \frac{1}{2}(u - u(0) - i\tilde{u}) \ .$$

Uniqueness of c, f, g is also clear. □

LEMMA 12. *Suppose u is harmonic on $\mathbb{D}$. Then for all $n \in \mathbb{Z}$, $n \neq 0$,*

$$\widehat{(\tilde{u}_r)}(n) = -i\mathrm{sign}(n)\widehat{u_r}(n) \ . \tag{18}$$

PROOF. By Lemma 11 part 3) it suffices to consider $u =$ constant, analytic, co-analytic. We present the case $u =$ analytic, $u(0) = 0$. Then $\tilde{u} = -iu$ so that $\widehat{(\tilde{u}_r)}(n) = -i\widehat{u_r}(n)$ for all $u \in \mathbb{Z}$. But $\widehat{u_r}(n) = 0$ for $u \leq 0$ and thus (18) holds in this case. □

COROLLARY 6. *Let $u \in h^2(\mathbb{D})$ be real. Then $\tilde{u} \in h^2(\mathbb{D})$. In fact,*

$$\|\tilde{u}_r\|_2^2 = \|u_r\|_2^2 - |u(0)|^2 \ .$$

PROOF. By Cauchy's theorem,

$$\int_0^1 (u_r + i\tilde{u}_r)^2(\theta)d\theta = (u + i\tilde{u})^2(0) = u^2(0) \ .$$

Since the right-hand side is real, the left-hand side is also necessarily real and thus

$$u^2(0) = \int_0^1 u_r^2(\theta)\ d\theta - \int_0^1 \tilde{u}_r^2(\theta)\ d\theta \ ,$$

as claimed. □

COROLLARY 7. *If $u \in h^2(\mathbb{D})$, then $\lim_{r\to 1}\ \tilde{u}(re(\theta))$ exists for a.e. $\theta \in \mathbb{T}$.*

PROOF. Combine Corollary 7 with Theorems 3 and 4. $\square$

Now we consider the case of $h^1(\mathbb{D})$. The following theorem is a famous result due to Besicovitch and Kolmogoroff around 1920. The proof presented here is a well-known argument involving harmonic measure.

THEOREM 7. *Let $u \in h^1(\mathbb{D})$. Then*

$$\mathrm{mes}[\theta \in \mathbb{T} | |\tilde{u}^*(\theta)| > \lambda] \leq \frac{C}{\lambda} |||u|||_1$$

with some absolute constant C (we get $C = \frac{48\sqrt{2}}{\pi}$).

PROOF. By Theorem 3, $u_r = P_r * \mu$. Splitting μ into real and imaginary parts, and then each piece into its positive and negative parts, we reduce ourselves to the case $u \geq 0$. Let $E_\lambda = \{\theta | \tilde{u}^*(\theta) > \lambda\}$ and set $F = -\tilde{u} + iu$. Then F is analytic and $F(0) = iu(0)$. Define a function

$$\omega_\lambda(x,y) = \frac{1}{\pi} \int_{(-\infty,-\lambda)\cup(\lambda,\infty)} \frac{y}{(x-t)^2 + y^2}\, dt\ ,$$

which is harmonic for $y > 0$ and non negative. The following two properties of ω_λ will be important:

a) $\omega_\lambda(x,y) \geq \frac{1}{2}$ if $|x| > \lambda$
b) $\omega_\lambda(0,y) \leq \frac{2y}{\pi\lambda}$.

For the first property compute

$$\begin{aligned}\omega_\lambda(x,y) &= 1 - \frac{1}{\pi} \int_{-\lambda}^{\lambda} \frac{y}{(x-t)^2 + y^2}\, dt = 1 - \frac{1}{\pi} \int_{-\lambda/y}^{\lambda/y} \frac{dt}{1 + (\frac{x}{y} - t)^2} \\ &= 1 - \frac{1}{\pi} \left(\arctan \frac{\lambda + x}{y} - \arctan \frac{x - \lambda}{y} \right) \geq \frac{1}{2}\end{aligned}$$

provided (x,y) lies outside the semi-circle with radius λ and center 0. For the second property compute

$$\omega_\lambda(0,y) = \frac{1}{\pi} \int_{(-\infty,-\lambda)\cup(\lambda,\infty)} \frac{y}{t^2 + y^2}\, dt \leq \frac{2}{\pi} \int_{\lambda/y}^{\infty} \frac{dt}{1+t^2} \leq \frac{2y}{\pi\lambda}\ ,$$

as claimed.

Observe that $\omega_\lambda \circ F$ is harmonic and that $\theta \in E_\lambda$ implies $(\omega_\lambda \circ F)\big(re(\theta)\big) \geq \frac{1}{2}$ for some $0 < r < 1$. Thus

$$|E_\lambda| \leq \mathrm{mes}\left[\theta | (\omega_\lambda \circ F)^*(\theta) \geq \frac{1}{2}\right] \leq \frac{3}{1/2} |||\omega_\lambda \circ F|||_1\ , \tag{19}$$

by Proposition 2. Since $\omega_\lambda \circ F \geq 0$, the mean value property implies that

$$|||\omega_\lambda \circ F|||_1 = (\omega_\lambda \circ F)(0) = \omega_\lambda(iu(0)) \leq \frac{2}{\pi} \frac{u(0)}{\lambda} = \frac{2}{\pi} \frac{|||u|||_1}{\lambda}\ .$$

Combining this with (19) yields

$$|E_\lambda| \leq \frac{12}{\pi} \frac{|||u|||_1}{\lambda} ,$$

as claimed. $\square$

The following result introduces the *Hilbert transform* and establishes a weak-L^1 bound for it. Formally speaking, the Hilbert transform $H\mu$ of $\mu \in M(\mathbb{T})$ is defined by

$$\mu \mapsto u_\mu \mapsto \widetilde{u_\mu} \mapsto \lim_{r\to 1} (\widetilde{u_\mu})_r =: H\mu ,$$

i.e., the Hilbert transform of a function on $\mathbb{T}$ is the boundary values of the conjugate function of its harmonic extension. By Corollary 7 this is well defined if $d\mu = f d\theta$, $f \in L^2(\mathbb{T})$. We now consider the case $f \in L^1(\mathbb{T})$.

Corollary 8. *Given $u \in h^1(\mathbb{D})$ the limit $\lim_{r\to 1} \tilde{u}(re(\theta))$ exists for a.e. θ. With $u = P_r * \mu$, $\mu \in M(\mathbb{T})$, this limit is denoted by $H\mu$. There is the weak-L^1 bound*

$$\mathrm{mes}[\theta | |H\mu(\theta)| > \lambda] \leq \frac{C}{\lambda} \|\mu\| .$$

Proof. If $d\mu = f d\theta$ with $f \in L^2(\mathbb{T})$ then $\lim\limits_{r\to 1} \widetilde{u_f}(re(\theta))$ exists for a.e. θ by Corollary 7. If $f \in L^1(\mathbb{T})$ and $\epsilon > 0$, then let $g \in L^2(\mathbb{T})$ such that $\|f - g\|_1 < \epsilon$. Denote, for any $\delta > 0$,

$$E_\delta = \{\theta | \limsup_{r,s\to 1} |\widetilde{u_f}(re(\theta)) - \widetilde{u_f}(se(\theta))| > \delta\}$$

and

$$F_\delta = \{\theta | \limsup_{r,s\to 1} |\widetilde{u_h}(re(\theta)) - \widetilde{u_h}(se(\theta))| > \delta\}$$

where $h = f - g$. In view of the preceding theorem and the L^2-case,

$$\begin{aligned} |E_\delta| = |F_\delta| &\leq \mathrm{mes}\, [\theta | (\widetilde{u_h})^*(\theta) > \delta/2] \\ &\leq \frac{C}{\delta} |||u_h|||_1 \leq \frac{C}{\delta} \|f - g\|_1 \to 0 \end{aligned}$$

as $\epsilon \to 0$. This finishes the case where $\mu \ll d\theta$. To treat singular measures, we first consider measures $\mu = \nu$ for which $|\mathrm{supp}(\nu)| = 0$. Here $\mathrm{supp}(\nu) := \mathbb{T} \setminus \cup\{I \subset \mathbb{T} | \nu(I) = 0\}$, I being an arc. Observe that for any $\theta \notin \mathrm{supp}(\nu)$ the limit $\lim_{r\to 1} \widetilde{u_r}(\theta)$ exists since the analytic function $u + i\tilde{u}$ can be continued across that interval J on $\mathbb{T}$ for which $\mu(J) = 0$ and which contains θ. Hence $\lim_{r\to 1} \widetilde{u_r}$ exist a.e. by the assumption $|\mathrm{supp}(\nu)| = 0$. If $\mu \in \mathcal{M}(\mathbb{T})$ is an arbitrary singular measure, then use inner regularity to say that for every $\epsilon > 0$ there is $\nu \in \mathcal{M}(\mathbb{T})$ with $\|\mu - \nu\| < \epsilon$ and $|\mathrm{supp}(\nu)| = 0$. Indeed, set $\nu(A) := \mu(A \cap K)$ for all Borel sets A where K is compact and $|\mu|(\mathbb{T} \setminus K) < \epsilon$. The theorem now follows by passing from the statement for ν to that for μ by means of the same argument that was used in the absolutely continuous case above. $\square$

Theorem 8 (Marcel Riesz). *If $1 < p < \infty$, then $\|Hf\|_p \leq C_p \|f\|_p$. Consequently, if $u \in h^p(\mathbb{D})$ with $1 < p < \infty$, then $\tilde{u} \in h^p(\mathbb{D})$ and $|||\tilde{u}|||_p \leq C_p |||u|||_p$.*

PROOF. By Corollary 6, $\|Hu\|_2 \le \|u\|_2$ (with equality if and only if $\int_0^1 u(\theta)\, d\theta = 0$). Interpolating this with the weak-L^1 bound from Corollary 8 by means of Marcinkiewicz finishes the case $1 < p \le 2$. If $2 < p < \infty$, then we use duality. More precisely, if $f, g \in L^2(\mathbb{T})$, then

$$\begin{aligned}\langle f, Hg\rangle &= \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\widehat{Hg}(n)} = \sum_{n\in\mathbb{Z}} i\mathrm{sign}(n)\hat{f}(n)\overline{\hat{g}(n)} \\ &= \sum_{n\in\mathbb{Z}} -\widehat{Hf}(n)\overline{\hat{g}(n)} = -\langle Hf, g\rangle\ .\end{aligned}$$

This shows that $H^* = -H$. Hence, if $f \in L^p(\mathbb{T}) \subset L^2(\mathbb{T})$ and $g \in L^2(\mathbb{T}) \subset L^{p'}(\mathbb{T})$, then

$$|\langle Hf, g\rangle| = |\langle f, Hg\rangle| \le \|f\|_p \|Hg\|_{p'} \le C_{p'}\|f\|_p\|g\|_{p'}\ .$$

and thus $\|Hf\|_p \le C_{p'}\|f\|_p$ as claimed. $\square$

REMARK. *Consider the analytic mapping $F = u + iv$ that takes $\mathbb{D}$ onto the strip $\{z \mid |\Re z| < 1\}$. Then $u \in h^\infty(\mathbb{D})$ but clearly $v \notin h^\infty(\mathbb{D})$ so that Theorem 8 has to fail on $L^\infty(\mathbb{T})$. By duality, it also fails on $L^1(\mathbb{T})$. The correct substitute for L^1 in this context is the space of real parts of functions in $\mathbb{H}^1(\mathbb{D})$. This is a deep result that goes much further than the F. & M. Riesz theorem. The statement is that*

$$\|Hf\|_1 \le C\|u_f^*\|_1 \tag{20}$$

where u_f^* is the non-tangential maximal function of the harmonic extension u_f of f (by the Burkholder-Gundy-Silverstein theorem the right-hand side in (20) is finite if and only if f is the real part of an analytic L^1-bounded function), see Koosis.

Next, we turn to the problem of expressing Hf in terms of a kernel. By Exercise 8, it is clear that one would expect that

$$(H\mu)(\theta) = \int_{\mathbb{T}} \cot(\pi(\theta-\varphi))\, d\mu(\varphi) \tag{21}$$

for any $\mu \in \mathcal{M}(\mathbb{T})$. This, however, requires justification as the integral on the right-hand side is not necessarily convergent.

PROPOSITION 3. *If $\mu \in \mathcal{M}(\mathbb{T})$, then*

$$\lim_{\epsilon\to 0}\int_{|\theta-\varphi|>\epsilon} \cot(\pi(\theta-\varphi))\, d\mu(\varphi) = (H\mu)(\theta) \tag{22}$$

for a.e. $\theta \in \mathbb{T}$. In other words, (21) holds in the principal value sense.

PROOF. As an exercise you should check that the limit in (22) exists for all $d\mu = f\, d\theta + d\nu$ where $f \in C^1(\mathbb{T})$ and $\mathrm{mes}(\mathrm{supp}(\nu)) = 0$ and that these measures are dense in $\mathcal{M}(\mathbb{T})$. We will now obtain the theorem by representing a general measure as a limit of such measures. As always, this requires a bound on an appropriate maximal function. In this case the natural bound is of the form

$$\mathrm{mes}\left[\theta \in \mathbb{T} \;\Big|\; \sup_{0<\epsilon<\frac12}\Big|\int_{|\theta-\varphi|>\epsilon} \cot(\pi(\theta-\varphi))\, d\mu(\varphi)\Big| > \lambda\right] \le \frac{C}{\lambda}\|\mu\| \tag{23}$$

for all $\lambda > 0$. We leave it to the reader to check that (23) implies the theorem. In order to prove (23) we invoke our strongest result on the conjugate function, namely Theorem 7. More precisely, we claim that

$$\sup_{0<r<1} \left| (Q_r * \mu)(\theta) - \int_{|\theta-\varphi|>1-r} \cot(\pi(\theta-\varphi))\, d\mu(\varphi) \right| \le CM\mu(\theta) \,, \tag{24}$$

where $M\mu$ is the Hardy-Littlewood maximal function. Since

$$\sup_{0<r<1} |(Q_r * \mu)(\theta)| = \widetilde{u_\mu}^*(\theta) \,,$$

(23) follows from (24) by means of Theorem 7 and Proposition 1. To verify (24) write the difference inside the absolute value signs as $(K_r * \mu)(\theta)$, where

$$K_r(\theta) = \begin{cases} Q_r(\theta) - \cot(\pi\theta) & \text{if } \ 1-r < |\theta| < \frac{1}{2} \\ Q_r(\theta) & \text{if } \ |\theta| \le 1-r \,. \end{cases}$$

By means of calculus one checks that (exercise!)

$$|K_r(\theta)| \le \begin{cases} C\frac{(1-r)^2}{|\theta|^3} & \text{if } \ |\theta| > 1-r \\ C(1-r)^{-1} & \text{if } \ |\theta| \le 1-r \,. \end{cases}$$

This proves that $\{K_r\}_{0<r<1}$ form a radially bounded approximate identity and (24) therefore follows from Lemma 7. ☐

Exercise 12. *Show by means of* (22) *that* H *is not bounded on* $L^\infty(\mathbb{T})$. *(Hint: consider* $H\chi_{[0,\frac{1}{2})}$*).*

The following proposition shows that Hf is exponentially integrable for bounded f. In Exercise 12, you should find that $H\chi_{[0,\frac{1}{2}]}$ has logarithmic behavior at 0 and $\frac{1}{2}$. If you find the precise asymptotics at those points it should show that the condition on α below is sharp.

Proposition 4. *Let* f *be a real-valued function on* $\mathbb{T}$ *with* $|f| \le 1$. *Then for any* $0 \le \alpha < \frac{\pi}{2}$

$$\int_0^1 e^{\alpha|Hf(\theta)|}\, d\theta \le \frac{2}{\cos\alpha} \,.$$

Proof. Let $u = u_f$ be the harmonic extension of f to $\mathbb{D}$ and set $F = \tilde{u} - iu$. Then $|u| \le 1$ by the maximum principle and hence $\cos(\alpha u) \ge \cos\alpha$. Therefore,

$$\Re(e^{\alpha F}) = \Re\left(e^{\alpha\tilde{u}} \cdot e^{-i\alpha u}\right) = \cos(\alpha u)e^{\alpha\tilde{u}} \ge \cos(\alpha)e^{\alpha\tilde{u}} \tag{25}$$

By the mean value property,

$$\int_0^1 \Re e^{\alpha F_r(\theta)}\, d\theta = \Re\, e^{\alpha F(0)} = \Re\, e^{-i\alpha(u(0))} = \cos(\alpha u(0)) \le 1 \,.$$

Combining this with (25) yields

$$\int_0^1 e^{\alpha\tilde{u}_r(\theta)} d\theta \ \le \frac{1}{\cos\alpha}$$

and by Fatou's lemma therefore

$$\int_0^1 e^{\alpha(Hf)(\theta)}\, d\theta \leq \frac{1}{\cos\alpha}\ .$$

Since this inequality also holds for $-f$, the proposition follows. $\square$

Remark. *In the next sections we will develop the real variable theory of singular integrals which contains the results on the Hilbert transform obtained above. The basic theorem, due to Calderon-Zygmund states that singular integrals are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ thus generalizing Theorem 8.*

The analogue of Proposition 4 for singular integrals is given by the fact that they are bounded from L^∞ to BMO and that BMO functions are exponentially integrable (John-Nirenberg inequality). The dual question of what happens on L^1 leads into the real variable theory of Hardy spaces. The analogue of (20) is then that singular integrals are bounded on $H^1(\mathbb{R}^n)$. Finally, we would like to point out that the dual space of H^1 is BMO, a well-known theorem of Charles Fefferman. See Stein 2 for these deep results, which will not be covered in these lectures.

We conclude the theory of the conjugate function by returning to the issue of $L^p(\mathbb{T})$ convergence of Fourier series. Recall from Lecture 1 that this fails for $p = 1$ and $p = \infty$ but we will now deduce from Theorem 8 that it holds for $1 < p < \infty$.

Theorem 9. *Let S_N denote the partial sums of Fourier series. Then for any $1 < p < \infty$ the partial sums are uniformly bounded on $L^p(\mathbb{T})$, i.e.,*

$$\sup_N \|S_N\|_{p\to p} < \infty\ .$$

By Lemma 3 this implies convergence of $S_N f \longrightarrow f$ in $L^p(\mathbb{T})$, $1 < p < \infty$.

Proof. The point is simply that S_N can be written in terms of the Hilbert transform. Indeed, recall that

$$\widehat{Hf}(n) = -i \text{ sign } (n)\hat{f}(n)$$

so that

$$Tf := \frac{1}{2}(1 + iH)f = \sum_n \chi_{(0,\infty)}(n)\hat{f}(n)e(n\cdot)\ .$$

In other words, on the Fourier side T is multiplication by $\chi_{(0,\infty)}$ whereas S_N is multiplication by $\chi_{[-N,N]}$. It remains to write $\chi_{[-N,N]}$ as the difference of two shifted $\chi_{(0,\infty)}$, i.e.,

$$\chi_{[-N,N]} = \chi_{(-N-1,\infty)} - \chi_{(N,\infty)}$$

or in terms of H and T,

$$(S_N f)(\theta) = e(-(N+1)\theta)[T(e((N+1)\cdot)f)](\theta) - e(N\theta)[T(e(-N\cdot)f)](\theta)\ .$$

Hence, for $1 < p < \infty$

$$\|S_N\|_{p\to p} \leq 2\|T\|_{p\to p} \leq 1 + \|H\|_{p\to p}$$

uniformly in N, as claimed. $\square$

Exercise 13.

a) *Show that, for any* $\lambda > 0$

$$\sup_N \operatorname{mes}[\theta \| (S_N f)(\theta) \| > \lambda] \le \frac{C}{\lambda} \|f\|_1$$

with some absolute constant C.

b) *What would such an inequality mean with the* $\sup_N$ *inside, i.e.,*

$$\operatorname{mes}[\theta | \sup_N |(S_N f)(\theta)| > \lambda] \le \frac{C}{\lambda} \|f\|_1$$

for all $\lambda > 0$*? Can this be true?*

c) *Using a) show that for every* $f \in L^1(\mathbb{T})$ *there exists a subsequence* $\{N_j\} \longrightarrow \infty$ *depending on* f *such that*

$$S_{N_j} f \longrightarrow f \text{ a.e.}$$

Remark. *It is an* open problem *to decide whether or not one can choose* $\{N_j\}$ *in part c) above in such a way that the growth of* N_j *is uniformly controlled, say* $N_j \le 2^j$ *for all* j. *It is an old and easy result for Walsh series that* $S_{2^j} f \to f$ *a.e. for every* $f \in L^1(\mathbb{T})$ *where* S_{2^j} *is the* 2^j *the partial sum of the Walsh series.*

Remark (Final). *The complex variable methods developed in Sections 2,3, and 4 equally well apply to the upper half plane instead of the disk. For example, the Poisson kernel is*

$$P_t(x) = \frac{1}{\pi} \frac{t}{x^2 + t^2} \text{ with } x \in \mathbb{R}, \ t > 0$$

and its conjugate is

$$Q_t(x) = \frac{1}{\pi} \frac{x}{x^2 + t^2}.$$

Observe that $P_t(x) + iQ_t(x) = \frac{1}{\pi} \frac{1}{x+it} = \frac{1}{\pi z}$ *with* $z = x + it$*, which should remind you of Cauchy's formula. The Hilbert transform is now*

$$(Hf)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x - y} \, dy$$

in the principal value sense, the precise statement being just as in Proposition 3. You can try to transfer various basic results from above to the half plane, or consult Koosis or Garnett.

CHAPTER 5

Calderon-Zygmund Theory of Singular Integrals

In this section we will present the basic result on singular integrals due to Calderon and Zygmund.

Definition 7. *Let* $K : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{C}$ *satisfy, for some constant* B,

i) $|K(x)| \leq B|x|^{-n}$
ii) $\int_{r<|x|<s} K(x)\, dx = 0$ *for all* $0 < r < s < \infty$
iii) $\int_{|x|>2|y|} |K(x) - K(x-y)|\, dx \leq B$ *for all* $y \neq 0$.

Then we define the *singular integral* operator with kernel K to be

$$Tf(x) := \lim_{\epsilon \longrightarrow 0} \int_{|x-y|>\epsilon} K(x-y) f(y)\, dy \tag{26}$$

for all $f \in C_0^1(\mathbb{R}^n)$.

Exercise 14.

a) *Check that the limit (i.e., principal value) exists in* (26) *for all* $f \in C_0^1(\mathbb{R}^n)$ *(these are* C^1 *functions with compact support).*
b) *Check that the Hilbert transform is a singular integral operator.*

There is a simple condition that guarantees iii) (which is the so-called "Hörmander condition") given in the following lemma.

Lemma 13. *Suppose* $|\nabla K(x)| \leq B|x|^{-n-1}$ *for all* $x \neq 0$ *and some constant* B. *Then*

$$\int_{|x|>2|y|} |K(x) - K(x-y)|\, dx \leq CB \tag{27}$$

with $C = C(n)$.

Proof. Fix $x, y \in \mathbb{R}^n$ with $|x| > 2|y|$. Connect x and $x-y$ by the line segment $x - ty$, $0 \leq t \leq 1$. This line segment lies entirely inside the ball $B(x, |x|/2)$. Hence

$$\begin{aligned} |K(x) - K(x-y)| &= \Big| -\int_0^1 \nabla K(x-ty) y\, dt\Big| \\ &\leq \int_0^1 |\nabla K(x-ty)||y|\, dt \leq B2^{n+1}|x|^{-n-1}|y|\,. \end{aligned}$$

Inserting this bound into the left-hand side of (27) yields the desired bound. □

Exercise 15. *Check that, for any fixed $0 < \alpha \le 1$, and all $x \neq 0$,*

$$\sup_{|y|<\frac{|x|}{2}} \frac{|K(x) - K(x-y)|}{|y|^\alpha} \le B|x|^{-n-\alpha}$$

also implies (27)*.*

The cancellation condition ii) implies L^2-boundedness of T, as shown in the following proposition.

Proposition 5. *Let K be as in Definition 7. Then $\|T\|_{2\to 2} \le CB$ with $C = C(n)$.*

Proof. Fix $0 < r < s < \infty$ and consider

$$(T_{r,s}f)(x) = \int_{\mathbb{R}^n} K(y)\chi_{[r<|y|<s]}(y) f(x-y)\, dy \; .$$

let

$$m_{r,s}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}\chi_{[r<|x|<s]} K(x)\, dx$$

be the Fourier transform of the restricted kernel. By Plancherel's theorem it suffices to prove that

$$\sup_{0<r<s} \|m_{r,s}\|_\infty \le CB \; . \tag{28}$$

Indeed, if (28) holds, then

$$\|T_{r,s}\|_{2\longrightarrow 2} = \|m_{r,s}\|_\infty \le CB$$

uniformly in r, s. Moreover, for any $f \in C_0^1(\mathbb{R}^n)$ one has

$$Tf(x) = \lim_{\substack{r\to 0\\ s\to\infty}} (T_{r,s}f)(x)$$

pointwise in $x \in \mathbb{R}^n$. Fatou's lemma therefore implies that $\|Tf\|_2 \le CB\|f\|_2$ for any $f \in C_0^1(\mathbb{R}^n)$. To verify (28) we split the integration in the Fourier transform into the regions $|x| < |\xi|^{-1}$ and $|x| \ge |\xi|^{-1}$. In the former case

$$\begin{aligned}
\left|\int_{r<|x|<|\xi|^{-1}} e^{-2\pi i x\cdot\xi} K(x)\, dx\right| &= \left|\int_{r<|x|<|\xi|^{-1}} (e^{-2\pi i x\cdot\xi} - 1) K(x)\, dx\right| \\
&\le \int_{|x|<|\xi|^{-1}} 2\pi|x||\xi||K(x)|\, dx \le \\
&\le 2\pi|\xi| \int_{|x|<|\xi|^{-1}} B|x|^{-n+1}\, dx \le CB|\xi||\xi|^{-1} \le CB \; ,
\end{aligned}$$

as desired. Notice that we used the cancellation condition ii) in the first equality sign. To deal with the case $|x| > |\xi|^{-1}$ one uses the cancellation in $e^{-2\pi i x\cdot\xi}$ which in turn requires smoothness of K, i.e., condition *iii*) (but compare Lemma 13). Firstly, observe that

$$\int_{s>|x|>|\xi|^{-1}} K(x) e^{-2\pi i x\cdot\xi}\, dx = -\int_{s>|x|>|\xi|^{-1}} K(x) e^{-2\pi i \left(x+\frac{\xi}{2|\xi|^2}\right)\cdot\xi}\, dx \tag{29}$$

$$= -\int_{s>|x-\frac{\xi}{2|\xi|^2}|>|\xi|^{-1}} K(x - \frac{\xi}{2|\xi|^2})\, e^{-2\pi i x\cdot\xi}\, dx \tag{30}$$

Denoting the expression on the left-hand side of (29) by F one thus has

$$2F = \int_{s>|x|>|\xi|^{-1}} \Big(K(x) - K(x - \frac{\xi}{2|\xi|^2})\Big) e^{-2\pi i x\cdot\xi}\, dx + 0(1) \tag{31}$$

The 0(1) term here stands for a term bounded by CB. Its origin is of course the difference between the regions of integration in (29) and (30). We leave it to the reader to check that condition *i*) implies that this error term is really no larger than CB. Estimating the integral in (31) by means of *iii*) now yields

$$|2F| \le \int_{|x|>|\xi|^{-1}} \left| K(x) - K\Big(x - \frac{\xi}{2|\xi|^2}\Big)\right| \, dx + CB \le CB\ ,$$

as claimed. We have shown (28) and the Proposition follows. $\square$

Our next goal is to show that singular integrals are bounded on weak-L^1. This requires the following basic decomposition lemma due to Calderon-Zygmund for L^1 functions. The proof uses a stopping time argument.

Lemma 14. *Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$. Then one can write $f = g + b$ where $|g| \le \lambda$ and $b = \sum_Q \chi_Q f$ where the sum runs over a collection $\mathcal{B} = \{Q\}$ of disjoint cubes such that for each Q one has*

$$\lambda < \frac{1}{|Q|}\int_Q |f| \le 2^n\lambda\ . \tag{32}$$

Furthermore,

$$\text{mes}\left(\bigcup_{Q\in\mathcal{B}} Q\right) < \frac{1}{\lambda}\|f\|_1\ . \tag{33}$$

Proof. For each $\ell \in \mathbb{Z}$ we define a collection $\mathcal{D}_\ell$ of dyadic cubes by means of

$$\mathcal{D}_\ell = \Big\{\Pi_{i=1}^n [2^\ell m_i, 2^\ell(m_i+1))\Big| m_1 \ldots, m_n \in \mathbb{Z}\Big\}\ .$$

Notice that if $Q \in \mathcal{D}_\ell$ and $Q' \in \mathcal{D}'_\ell$, then either $Q \cap Q' = \emptyset$ or $Q \subset Q'$ or $Q' \subset Q$. Now pick ℓ_0 so large that

$$\frac{1}{|Q|}\int_Q |f|\, dx \le \lambda$$

for every $Q \in \mathcal{D}_{\ell_0}$. For each such cube consider its 2^n "children" of size 2^{ℓ_0-1}. Any such cube Q' will then have the property that either

$$\frac{1}{|Q'|}\int_{Q'} |f|\, dx \le \lambda \text{ or } \frac{1}{|Q'|}\int_{Q'} |f|\, dx > \lambda\ . \tag{34}$$

In the latter case we stop, and include Q' in the family $\mathcal{B}$ of "bad cubes". Observe that in this case

$$\frac{1}{|Q'|}\int_{Q'} |f| \le \frac{2^n}{|Q|}\int_Q |f| \le 2^n\lambda$$

where Q denotes the parent of Q'. Thus (32) holds. If, however, the first inequality in (34) holds, then subdivide Q' again into its children of half the size. Continuing in this fashion produces a collection of disjoint (dyadic) cubes $\mathcal{B}$ satisfying (32). Consequently, (33) also holds, since

$$\mathrm{mes}\left(\bigcup_{\mathcal{B}} Q\right) \leq \sum_{\mathcal{B}} \mathrm{mes}(Q) < \sum_{\mathcal{B}} \frac{1}{\lambda}\int_Q |f| \leq \frac{1}{\lambda}\int_{\mathbb{R}^n} |f| \, .$$

Now let $x_0 \in \mathbb{R}^n \backslash \bigcup_{\mathcal{B}} \overline{Q}$. Then x_0 is contained in a decreasing sequence $\{Q_j\}$ of dyadic cubes each of which satisfies

$$\frac{1}{|Q_j|}\int_{Q_j} |f| \, dx \leq \lambda \, .$$

By Lebesgue's theorem $|f(x_0)| \leq \lambda$ for a.e. such x_0. Since moreover $\mathbb{R}^n \backslash \cup_{\mathcal{B}} Q$ and $\mathbb{R}^n \backslash \cup_{\mathcal{B}} \overline{Q}$ differ only by a set of measure zero, we can set

$$g = f - \sum_{Q \in \mathcal{B}} \chi_{\overline{Q}} f$$

so that $|g| \leq \lambda$ a.e. as desired. $\square$

We can now state the crucial weak-L^1 bound for singular integrals.

Proposition 6. *Let K be as in Definition 7. Then for every $f \in C_0^1(\mathbb{R}^n)$ there is the weak-L^1 bound*

$$\mathrm{mes}[x \in \mathbb{R}^n \,|\, |Tf(x)| > \lambda] \leq \frac{CB}{\lambda}\|f\|_1$$

where $C = C(n)$.

Proof. Dividing by B if necessary, we may assume that $B = 1$. Now fix $f \in C_0^1(\mathbb{R}^n)$ and let $\lambda > 0$ be arbitrary. By Lemma 14 one can write $f = g + b$ with this value of λ. We now set

$$f_1 = g + \sum_{Q \in \mathcal{B}} \chi_Q \frac{1}{|Q|}\int_Q f \, dx$$

$$f_2 = b - \sum_{Q \in \mathcal{B}} \chi_Q \frac{1}{|Q|}\int_Q f \, dx = \sum_{Q \in \mathcal{B}} f_Q$$

where we have set

$$f_Q := \chi_Q\Big(f - \frac{1}{|Q|}\int_Q f \, dx\Big) \, .$$

Notice that $f = f_1 + f_2$, $\|f_1\|_\infty \leq C\lambda\|f\|_1$, $\|f_2\|_1 \leq 2\|f\|_1$, $\|f_1\|_1 \leq 2\|f\|_1$, and

$$\int_Q f_Q \, dx = 0$$

for all $Q \in \mathcal{B}$. We now proceed as follows

$$\begin{aligned} \mathrm{mes}[x \in \mathbb{R}^n \,|\, |(Tf)(x)| > \lambda] &\leq \mathrm{mes}\Big[x \big| (Tf_1)(x)| > \frac{\lambda}{2}\Big] + \mathrm{mes}\Big[x \big| |(Tf_2)(x)| > \frac{\lambda}{2}\Big] \\ &\leq \frac{C}{\lambda^2}\|Tf_1\|_2^2 + \mathrm{mes}\Big[x \big| |(Tf_2)(x)| > \frac{\lambda}{2}\Big] \, . \end{aligned} \tag{35}$$

The first term in (35) is controlled by Proposition 5:

$$\frac{C}{\lambda^2}\|Tf_1\|_2^2 \le \frac{C}{\lambda^2}\|f_1\|_2^2 \le \frac{C}{\lambda^2}\|f_1\|_\infty\|f_1\|_1 \le \frac{C}{\lambda}\|f\|_1 \;.$$

To estimate the second term in (35) we define, for any $Q \in \mathcal{B}$, the cube Q^* to be the dilate of Q by a fixed factor depending only on the dimension (i.e., Q^* has the same center as Q but side length equal to b times that of Q with $b = b(n)$). Thus

$$\begin{aligned}
\operatorname{mes}\Big[x\big||(Tf_2)(x)| > \frac{\lambda}{2}\Big] &\le \operatorname{mes}(\cup_{\mathcal{B}} Q^*) + \operatorname{mes}\Big[x \in \mathbb{R}^n \setminus \cup_{\mathcal{B}} Q^* \big| |(Tf_2)(x)| > \frac{\lambda}{2}\Big] \\
&\le C\sum_Q \operatorname{mes}(Q) + \frac{2}{\lambda}\int_{\mathbb{R}^n\setminus\cup Q^*} |(Tf_2)(x)|\, dx \\
&\le \frac{C}{\lambda}\|f\|_1 + \frac{2}{\lambda}\sum_{Q\in\mathcal{B}}\int_{\mathbb{R}^n\setminus Q^*} |(Tf_Q)(x)|\, dx \;.
\end{aligned}$$

The crucial point of this entire proof is the fact that f_Q has mean zero which allows one to exploit the smoothness of the kernel K. More precisely, for any $x \in \mathbb{R}^n\setminus Q^*$,

$$\begin{aligned}
(Tf_Q)(x) &= \int_Q K(x-y) f_Q(y)\, dy \\
&= \int_Q [K(x-y) - K(x-y_Q)] f_Q(y)\, dy
\end{aligned} \tag{36}$$

where y_Q denotes the center of Q. Thus

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus Q^*} |(Tf_Q)(x)|\, dx &\le \int_{\mathbb{R}^n\setminus Q^*}\int_Q |K(x-y) - K(x-y_Q)||f_Q(y)|\, dy\, dx \\
&\le \int_Q |f_Q(y)|\, dy \le 2\int_Q |f(y)|\, dy \;.
\end{aligned}$$

To pass to the second inequality sign we used condition *iii*) in Definition 7. Hence the second term on the right bound side of (36) is no larger than

$$\frac{C}{\lambda}\sum_Q \int_Q |f(y)| dy \;\le \frac{C}{\lambda}\|f\|_1 \;,$$

and we are done. □

Remark. *The assumption $f \in C_0^1(\mathbb{R}^n)$ in the previous proposition was for convenience only. It ensured that one could define Tf by means of* (26)*. However, observe that Proposition 5 allows one to extend T to a bounded operator on L^2. This in turn implies that the weak-L^1 bound in Proposition 6 holds for all $f \in L^1 \cap L^2(\mathbb{R}^n)$. Indeed, inspection of the proof reveals that apart from the L^2 boundedness of T, see* (35)*, the definition of T in terms of K was only used in* (36) *where x and y are assumed to be sufficiently separated so that the integrals are absolutely convergent.*

Theorem 10 (Calderon-Zygmund). *Let T be a singular integral operator as in Definition 7. Then for every $1 < p < \infty$ one can extend T to a bounded operator on $L^p(\mathbb{R}^n)$ with the bound $\|T\|_{p\to p} \le CB$ with $C = C(p,n)$.*

Proof. By Proposition 5 and 6 (and the previous remark) one obtains this statement for the range $1 < p \leq 2$ from the Marcinkiewicz interpolation theorem. The range $2 \leq p < \infty$ now follows by duality. Indeed, we leave it to the reader to check that for $f, g \in C_0^1(\mathbb{R}^n)$ one has

$$\langle Tf, g\rangle = \langle f, T^* g\rangle \text{ where } T^* g(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} K^*(x-y) g(y)\, dy$$

and $K^*(x) := \overline{K(-x)}$. Since K^* clearly verifies conditions $i)$–$iii)$ in Definition 77, we are done. □

Remark. *It is important to realize that the cancellation condition ii) was only used to prove L^2 boundedness, but did not appear in the proof of Proposition 6. Therefore, T is bounded on $L^p(\mathbb{R}^n)$ provided it is bounded for $p = 2$ and conditions i) and iii) hold.*

We now present some of the most basic examples of singular integrals. Consider the equation $\triangle u = f$ in $\mathbb{R}^n$ where $f \in C^0(\mathbb{R}^n)$, $n \geq 2$. Then it is well-known that the unique bounded solution is given by, if $n \geq 3$, (and similarly with a logarithmic potential if $n = 2$)

$$u(x) = C_n \int_{\mathbb{R}^n} |x-y|^{2-n} f(y)\, dy \tag{37}$$

with some dimensional constant C_n. We would like to express the second derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ in terms of f.

Exercise 16.

a) With u as in (37), show that for $f \in C_0^1(\mathbb{R}^n)$

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = C_n \cdot (n-2)(n-1) \int_{\mathbb{R}^n} K_{ij}(x-y) f(y)\, dy$$

in the principal value sense, where

$$K_{ij}(x) = \begin{cases} \frac{x_i x_j}{|x|^{n+2}} & \text{if } \ i \neq j \\ \frac{x_i^2 - \frac{1}{n}|x|^2}{|x|^{n+2}} & \text{if } \ i = j\ . \end{cases}$$

b) Verify that K_{ij} as above are singular integral kernels. Also show that $K_i(x) = \frac{x_i}{|x|^{n+1}}$ is a singular integral kernel.

The operators R_i and R_{ij} defined in terms of the kernels K_i and K_{ij} respectively, from above are called the Riesz transforms or double Riesz transforms respectively. By Exercise 16

$$R_{ij}(\triangle\varphi) = \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \tag{38}$$

for any $\varphi \in C_0^2(\mathbb{R}^n)$.

Corollary 9. *Let $u \in C_0^2(\mathbb{R}^n)$. Then*

$$\sup_{1\leq i,j\leq n} \|\frac{\partial^2 u}{\partial x_i \partial x_i}\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \|\triangle u\|_{L^p(\mathbb{R}^n)} \tag{39}$$

for any $1 < p < n$. Here $C_{p,n}$ only depends on p and n.

Proof. This is an immediate consequence of (38) and Theorem 10. $\square$

This is the most basic formulation of many results in elliptic equations that state that solutions of elliptic PDE are two derivatives better than the inhomogeneity, see Gilbarg-Trudinger. In fact, most of those statements, at least on L^p, follow from Corollary 9. The reader should appreciate the depth of Corollary 9 in view of its failure on L^1 and L^∞. We now indicate why (39) fails on L^1. The idea is simply to take u to be the fundamental solution of $\triangle$ in $\mathbb{R}^n$, i.e., $u(x) = c|x|^{2-n}$ if $n \geq 3$ and $u(x) = \log|x|$ if $n = 2$. Then $\triangle u = \delta_0$ which, at least heuristically, belongs to $L^1(\mathbb{R}^n)$. However, one checks that $\frac{\partial^2 u}{\partial x_i \partial x_j} \notin L^1(B(0,1))$ for any i, j. Indeed, see the kernels K_{ij} form Exercise 16). This can be made precise in the usual way, i.e., by means of an approximate identity.

Exercise 17.

a) *Let $\varphi \geq 0$ be in $C_0^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi \, dx = 1$. Set $\varphi_\epsilon(x) := \epsilon^{-n}\varphi(\frac{x}{\epsilon})$ for any $0 < \epsilon < 1$. Clearly, $\{\varphi_\epsilon\}_{0<\epsilon<1}$ form an approximate identity provided the latter are defined analogously to Definition 1 on $\mathbb{R}^n$. Moreover, let $\chi \in C_0^\infty(\mathbb{R}^n)$ be arbitrary with $\chi(0) \neq 0$. Verify that, with $\Gamma_n(x) = |x|^{2-n}$ for $n \geq 3$ and $\Gamma_2(x) = \log|x|$, $u_\epsilon(x) := (\varphi_\epsilon * \Gamma_n)(x)\chi(x)$ has the following properties:*
$$\sup_{\epsilon>0} \|\triangle u_\epsilon\|_{L^1} < \infty$$
and
$$\limsup_{\epsilon \longrightarrow 0} \|\frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j}\|_{L^1} = \infty$$
for any $1 \leq i, j \leq n$. Thus Corollary 9 fails on $L^1(\mathbb{R}^n)$.

b) *Now show that Corollary 9 also fails on L^∞.*

We conclude this section by addressing the question whether a singular integral operator can be defined by means of formula (26) even if $f \in L^p(\mathbb{R}^n)$ rather than $f \in C_0^1(\mathbb{R}^n)$. This question is the analogue of Proposition 3 and should be understood as follows: For $1 < p < \infty$ we defined T “abstractly” as an operator on $L^p(\mathbb{R}^n)$ by extension from $C_0^1(\mathbb{R}^n)$ via the a priori bounds $\|Tf\|_p \leq C_{p,n}\|f\|_p$ for all $f \in C_0^1(\mathbb{R}^n)$ (the latter space is dense in $L^p(\mathbb{R}^n)$—cf. Lemma 2). We now ask if the principal value (26) converges almost everywhere to this extension T for any $f \in L^p(\mathbb{R}^n)$. Clearly, resolving this question requires controlling the maximal operator
$$T_* f(x) = \sup_{\epsilon>0} |\int_{|y|>\epsilon} K(y) f(x-y) \, dy| \ .$$
In what follows we shall need the Hardy-Littlewood maximal function
$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| \, dy \ ,$$
where the supremum runs over all balls B containing x. M satisfies basically the same bounds as in Proposition 1 (check!).

We prove the desired bounds on T_* only for a subclass of kernels, namely the homogeneous one. This class includes the Riesz transforms from above. More precisely, set

$$K(x) := \frac{\Omega(\frac{x}{|x|})}{|x|^n} \quad \text{for} \quad x \neq 0 \tag{40}$$

where $\Omega : S^{n-1} \longrightarrow \mathbb{C}$, $\Omega \in C^1(S^{n-1})$ and $\int_{S^{n-1}} \Omega(x) d\sigma(x) = 0$ where σ is the surface measure on S^{n-1}. Observe that $K(tx) = t^{-n} K(x)$.

EXERCISE 18. *Check that any such K satisfies the conditions in Definition 7.*

PROPOSITION 7. *Suppose K is of the form* (40). *Then T_* satisfies*

$$(T_* f)(x) \leq C[M(Tf)(x) + Mf(x)]$$

with some absolute constant C. In particular, T_* is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Furthermore, T_* is also weak-L^1 bounded.

PROOF. Let $\tilde{K}(x) := K(x)\chi_{[|x|\geq 1]}$ and more generally $\tilde{K}_\epsilon(x) = \epsilon^{-n}\tilde{K}(\frac{x}{\epsilon}) = K(x)\chi_{[|x|\geq \epsilon]}$. Pick a smooth bump function $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi \, dx = 1$. Define $\Phi := \varphi * K - \tilde{K}$. Observe that $\varphi * K$ is well-defined in the principal value sense. For any function F on $\mathbb{R}^n$ let $F_\epsilon(x) := \epsilon^{-n} F\left(\frac{x}{\epsilon}\right)$ be its L^1-normalized rescaling. Then $K_\epsilon = K$, $\tilde{K}_\epsilon = (\tilde{K})_\epsilon$, and thus $\Phi_\epsilon = (\varphi * K)_\epsilon - \tilde{K}_\epsilon = \varphi_\epsilon * K_\epsilon - \tilde{K}_\epsilon = \varphi_\epsilon * K - \tilde{K}_\epsilon$. Hence, for any $f \in C_0^1(\mathbb{R}^n)$, $K_\epsilon * f = \varphi_\epsilon * (K * f) - \Phi_\epsilon * f$. We will now invoke the analogue of Lemma 7 for radially bounded approximate identities. This of course requires that $\{\Phi_\epsilon\}_{\epsilon>0}$ from such a radially bounded a.i., which can be easily deduced (the case of φ_ϵ is obvious). Indeed, we leave it to the reader to verify that

$$|\Phi(x)| \leq C \min(1, |x|^{-n-1}) ,$$

which implies the desired property. Therefore,

$$T_* f \leq C(M(Tf) + Mf) ,$$

as claimed. The boundedness of T_* on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ now follows from that of T and M. The proof of the weak-L^1 boundedness of T_* is a variation of the same property of T, of Proposition 6. We shall not present the details, see Stein 1. ☐

COROLLARY 10. *Let K be a homogeneous singular integral kernel as in* (40). *Then for any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the limit in* (26) *exists almost everywhere.*

PROOF. Let

$$\Lambda(f)(x) = |\limsup_{\epsilon \to 0} (T_\epsilon f)(x) - \liminf_{\epsilon \to 0} (T_\epsilon f)(x)| .$$

Observe that $\Lambda(f) \leq 2T_* f$. Fix $f \in L^p$ and let $f \in C_0^1(\mathbb{R}^n)$ so that $\|f - f_1\|_p < \delta$ for a given small $\delta > 0$. Then $\Lambda(f) = \Lambda(f - f_1)$ and therefore $\|\Lambda f\|_p \leq C\|\Lambda(f - f_1)\|_p < C\delta$ if $1 < p < \infty$. Hence, $\Lambda f = 0$. We leave the similar case $p = 1$ to the reader. ☐

CHAPTER 6

Almost Orthogonality; Schauder Estimates

In this section we present further results on singular integrals. Needless to say, the subject has undergone vast development beyond the material of the previous section, (see Christ's book and Stein 2). We are not able to cover much of it, but will discuss two issues here namely an alternative approach to L^2 theory and the fact that singular integrals are bounded on Hölder spaces. This latter fact predates the Calderon-Zygmund theorem from the previous section, at least for the special case of the double Riesz transform.

Let us start with some very simple comments about L^p bounds, which explain why one is mainly concerned with $L^p \to L^p$ bounds.

Exercise 19.

a) *Show that a homogeneous kernel as that in* (40) *can only be bounded from* $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ *if* $p = q$.
b) *Show that for* any *translation invariant non-zero operator* $T\colon L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n)$ *one necessarily has* $q \geq p$.

The proof of the L^2 bound in the previous section was based on the Fourier transform. This is very restrictive as it requires translation invariance. Although we do not develop non translationally invariant singular integrals here (see Christ or Stein 2), we now present a very useful tool that avoids the Fourier transform. The idea is to break up the singular integral operator T into a sum $\sum_j T_j$ (usually by a partition of K over dyadic shells). A trivial estimate would be $\|T\|_{2\to 2} \leq \sum_j \|T_j\|_{2\to 2}$, but this is useless. Much rather, we want something like

$$\|T\|_{2\to 2} \lesssim \sup_j \|T_j\|_{2\to 2} \ .$$

In analogy with block matrices, with the T_j being blocks with pair-wise disjoint rows and columns, we would want the *ranges* and *co-ranges* to be perpendicular, i.e., for $j \neq k$

$$T_j^* T_k = 0 \text{ and } T_j T_k^* = 0 \ . \tag{41}$$

Exercise 20. *Show that under condition* (41) *one has*

$$\|\sum_{j=1}^{N} T_j\|_{2\to 2} \leq \sup_{1\leq j\leq N} \|T_j\|_{2\to 2} \ .$$

For applications conditions (41) *are usually too strong. The point of the following lemma (the Cotlar-Stein lemma) is to show that it is enough if they hold "almost" (hence "almost orthogonality").*

LEMMA 15. *Let $\{T_j\}_{j=1}^N$ be operators on L^2 such that, for some function $\gamma : \mathbb{Z} \to \mathbb{R}$, $\|T_j^* T_k\| \le \gamma^2(j-k)$, $\|T_j T_k^*\| \le \gamma^2(j-k)$ for any j, k. Let*

$$\sum_{\ell=-\infty}^{\infty} \gamma(\ell) =: A < \infty \ .$$

Then $\| \sum_{j=1}^N T_j\| \le A$.

PROOF. For any positive integer n,

$$(T^*T)^n = \sum_{\substack{j_1,\ldots,j_n=1 \\ k_1,\ldots,k_n=1}}^N T_{j_1}^* T_{k_1} T_{j_2}^* T_{k_2} \ldots T_{j_n}^* T_{k_n}$$

Therefore, with $\sup_{1\le j\le N} \|T_j\| =: B$

$$\begin{aligned}
\|(T^*T)^n\| &\le \sum_{\substack{j_1,\ldots,j_n=1 \\ k_1,\ldots k_n=1}}^N \|T_{j_1}\|^{\frac{1}{2}} \|T_{j_1}^* T_{k_1}\|^{\frac{1}{2}} \|T_{k_1} T_{j_2}^*\|^{\frac{1}{2}} \cdots \\
&\quad \cdots \|T_{k_{n-1}} T_{j_n}^*\|^{\frac{1}{2}} \|T_{j_n}^* T_{k_n}\|^{\frac{1}{2}} \|T_{k_n}\|^{\frac{1}{2}} \\
&\le \sum_{\substack{j_1,\ldots,j_n=1 \\ k_1,\ldots,k_n=1}}^N \sqrt{B}\gamma(j_1-k_1)\gamma(k_1-j_2)\gamma(j_2-k_2)\ldots\gamma(k_{k_1}-j_n)\gamma(j_n-k_n)\sqrt{B} \\
&\le NBA^{2n-1} \ .
\end{aligned}$$

Since T^*T is self-adjoint, the spectral theorem implies that $\|(T^*T)^n\| = \|T^*T\|^n = \|T\|^{2n}$. Hence,

$$\|T\| \le (N\ BA^{-1})^{\frac{1}{2n}} \cdot A \ .$$

Letting $n \to \infty$ yields the desired bound. □

We will now give an alternative proof of $L^2(\mathbb{R}^n)$ boundedness of singular integrals for kernels which satisfy

$$|\nabla K(x)| \le B|x|^{-n-1} \ ,$$

of Lemma 13. This requires a certain standard partition of unity which we now present.

LEMMA 16. *There exists $\psi \in C_0^\infty(\mathbb{R}^n)$ with the property that supp $\psi \subset \mathbb{R}^n\backslash\{0\}$ and that*

$$\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1 \tag{42}$$

for any $x \ne 0$. Moreover, ψ can be chosen to be a radial function, nonegative function and no more than two terms in (42) are nonzero for any given $\xi \ne 0$.

PROOF. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ so that $\chi(x) = 1$ for all $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$, say.

Set $\psi(x) := \chi(x) - \chi(2x)$. Clearly, for any positive N,

$$\sum_{j=-N}^{N} \psi(2^{-j}x) = \chi(2^{-N}x) - \chi(2^{N+1}x) \ .$$

If $x \neq 0$ is given, then we take N so large that $\chi(2^{-N}x) = 1$ and $\chi(2^{N+1}x) = 0$. This implies (42), as desired (note that the sum in (42) is finite for any $x \neq 0$). $\square$

The point of the following corollary is the method of proof rather than the statement (which is weaker than the one in the previous section).

Corollary 11. *Let K be as in Definition 7 with the additional assumption that $|\nabla K(x)| \leq B|x|^{-n-1}$. Then*

$$\|T\|_{2\to 2} \leq CB$$

with $C = C(n)$.

Proof. Let ψ be a radial function as in Lemma 16 and set $K_j(x) = K(x)\psi(2^{-j}x)$. It is easy to see that these kernels have the following properties:

$$\int K_j(x)\, dx = 0, \qquad \|\nabla K_j\|_\infty \leq C 2^{-j} 2^{-jn} \ \text{ for all } \ j \in \mathbb{Z} \ ,$$

$$\sup_{j\in\mathbb{Z}} \int |K_j(x)|\, dx < \infty, \ \text{ and } \ \sup_{j\in\mathbb{Z}} 2^{-j} \int |x||K_j(x)|\, dx < \infty \ .$$

Define

$$(T_j f)(x) = \int_{\mathbb{R}^n} K_j(x-y) f(y)\, dy \ .$$

Observe that this integral is absolutely convergent for any $f \in L^1_{\ell oc}(\mathbb{R}^n)$. We shall now check the conditions in Lemma 15. Let $\widetilde{K}_j(x) := \overline{K}_j(-x)$. Then it is easy to see that

$$(T_j^* T_k f)(x) := \int_{\mathbb{R}^n} (\widetilde{K}_j * K_k)(y) f(x-y)\, dy$$

and

$$(T_j T_k^* f)(x) = \int_{\mathbb{R}^n} (K_j * \widetilde{K}_k)(y) f(x-y)\, dy \ .$$

Hence, by Young's inequality,

$$\|T_j^* T_k\|_{2\to 2} \leq \|\widetilde{K}_j * K_k\|_1$$

and

$$\|T_j T_k^*\|_{2\to 2} \leq \|K_j * \widetilde{K}_k\|_1 \ .$$

It suffices to consider the case $j \geq k$. Then, using $\int K_k(y)\, dy = 0$, since obtains

$$\begin{aligned}\left|(\widetilde{K}_j * K_k)(x)\right| &= \left|\int_{\mathbb{R}^n} \overline{K}_j(y-x) K_k(y) dy\right| = \left|\int_{\mathbb{R}^n} \left[\overline{K}_j(y-x) - \overline{K}_j(-x)\right] K_k(y)\, dy\right| \\ &\leq \int_{\mathbb{R}^n} \|\nabla K_j\|_\infty |y||K_k(y)|\, dy \leq CB^2 2^{-j} 2^{-jn} 2^k \ .\end{aligned}$$

Since

$$\text{supp } (\widetilde{K}_j * K_k) \subset \text{supp}(\widetilde{K}_j) + \text{supp}(K_k) \subset \ B(0, C\cdot 2^j) \ ,$$

we have

$$\|\widetilde{K}_j * K_k\|_1 \le CB^2 2^{k-j} = CB^2 2^{-|j-k|} .$$

Therefore, Lemma 15 applies with

$$\gamma^2(\ell) = CB^2 2^{-|\ell|}$$

and the corollary follows. $\square$

EXERCISE 21.

a) In the previous proof it suffices to consider the case $j > k = 0$. Provide the details of this reduction.
b) Observe that Corollary 11 covers the Hilbert transform. In that case, you should draw the graph of $K(x) = \frac{1}{x}$ and also $K_j(x) = \frac{1}{x}\psi(2^{-j}x)$ and explain the previous argument by means of pictures.

Next, we turn to the question of boundedness of singular integrals on Hölder spaces. At this point you are strongly encouraged to try problem 16 from the appendix. You should not *use the Fourier transform for this problem.*

We will now show in the full generality of Definition 7 that singular integrals are bounded on $C^\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$. This requires a little bit of knowledge of tempered distributions. Recall that $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of functions $f \in C^\infty$ for which for any α, β

$$\sup_{x\in\mathbb{R}^n} |x^\alpha||\partial^\beta f(x)| < \infty$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ are multi-indices and

$$x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \ , \ \partial^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial_{x_n}^{\beta_n}} \ , \ |\beta| = \sum_{j=1}^n |\beta_j| \ .$$

EXERCISE 22. *Show that the Fourier transform takes $\mathcal{S}$ onto $\mathcal{S}$. The dual space $\mathcal{S}'$ of $\mathcal{S}$ is called the space of* tempered distributions, *see Rudin 2. If $u \in \mathcal{S}'$ and $f \in \mathcal{S}$, then $\hat{u}$ is defined via $\langle \hat{u}, f\rangle := \langle u, \hat{f}\rangle$. In what follows we work with the projections $\triangle_j$ defined via*

$$\triangle_j u := (\psi_j \hat{u})^\vee = \hat{\psi}_j * u \tag{43}$$

for any $u \in \mathcal{S}'$. Here $\psi_j(\xi) := \psi(2^{-j}\xi)$ for any $j \in \mathbb{Z}$ where ψ is as in Lemma 16. You should pay attention to the fact that (43) *is meaningful since $u \in \mathcal{S}' \Longrightarrow \hat{u} \in \mathcal{S}' \Longrightarrow \psi_j\hat{u} \in \mathcal{S}' \Longrightarrow (\psi_j\hat{u})^\vee \in \mathcal{S}'$. The final equality in* (43) *is a simple fact about distributions that we leave to the reader to check.*

We require a characterization of $C^\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1$ in terms of the $\triangle_j$. The reader should realize that the proof of the following lemma is reminiscent of the proof of Bernstein's Theorem 2.

Lemma 17. *Let $|f| \le 1$. Then $f \in C^\alpha(\mathbb{R}^n)$ for $0 < \alpha < 1$ if and only if*

$$\sup_{j\in\mathbb{Z}} 2^{j\alpha} \|\triangle_j f\|_\infty \le A \, . \tag{44}$$

Moreover, the smallest A for which (44) holds is comparable to $[f]_\alpha$.

Proof. Set $\check{\psi}_j(x) = 2^{jn}\ \check{\psi}(2^j x) =: \varphi_j(x)$.

Hence

$$\|\varphi_j\|_1 = \|\check{\psi}\|_1 \ \text{ for all } \ j \in \mathbb{Z}$$

and

$$\int_{\mathbb{R}^n} \varphi_j(x) x^\gamma \, dx = 0$$

for all multi-indices γ. First assume $f \in C^\alpha$. Then

$$\begin{aligned} |\triangle_j f(x)| &\le \int_{\mathbb{R}^n} |f(x-y) - f(x)||\varphi_j(y)| \, dy \\ &\le \int_{\mathbb{R}^n} [f]_\alpha |y|^\alpha |\varphi_j(y)| \, dy \\ &= 2^{-j\alpha}[f]_\alpha \int_{\mathbb{R}^n} |y|^\alpha |\check{\psi}(y)| \, dy \, . \end{aligned}$$

Hence $A \le C[f]_\alpha$, as claimed.

Conversely, for any positive integer ℓ define

$$g_\ell(x) = \sum_{-\ell \le j \le \ell} (\triangle_j f)(x) \, .$$

We need to show that, for all $y \in \mathbb{R}^n$,

$$\sup_\ell |g_\ell(x-y) - g_\ell(x)| \le CA|y|^\alpha$$

with some constant $C = C(n)$.

Now fix $y \neq 0$ and estimate

$$\Big| \sum_{|y|^{-1} < 2^j \le 2^\ell} (\triangle_j f)(x) \Big| \le \sum_{2^j > |y|^{-1}} A 2^{-j\alpha} \le CA|y|^\alpha \, . \tag{45}$$

Secondly, observe that

$$\begin{aligned} |\triangle_j f(x-y) - \triangle_j f(x)| &\le \|\nabla \triangle_j f\|_\infty |y| \le C 2^j \|\triangle_j f\|_\infty |y| \\ &\le C 2^{j(1-\alpha)} A |y| \end{aligned} \tag{46}$$

where we invoked Bernstein's inequality (Lemma 18 below, with $|\gamma| = 1$) to pass to the second inequality sign. Combining (45) and (46) yields

$$\begin{aligned} |g_\ell(x-y) - g_\ell(x)| &\le \sum_{2^{-\ell} \le 2^j \le |y|^{-1}} |\triangle_j f(x-y) - \triangle_j f(x)| + \sum_{|y|^{-1} < 2^j \le 2^\ell} 2\|\triangle_j f\|_\infty \\ &\le \sum_{2^j \le |y|^{-1}} CA 2^{j(1-\alpha)} |y| + \sum_{|y|^{-1} < 2^j} 2A 2^{-j\alpha} \le CA|y|^\alpha \, , \end{aligned} \tag{47}$$

uniformly in $\ell \geq 1$.

So $\{g_\ell - g_\ell(0)\}_{\ell=1}^\infty$ are uniformly bounded on $C^\alpha(K)$ for any compact K. By the Arzela-Ascoli theorem one concludes that

$$g_\ell - g_\ell(0) \longrightarrow g$$

uniformly on any compact set and therefore $[g]_\alpha \leq CA$ by (47) (strictly speaking, we should pass to a subsequence $\{\ell_i\}$, but we suppress this detail). It remains to show that f has the same property. This follows from the

Claim: $f = g+$ constant.

To verify this, note that $g_\ell - g_\ell(0) \to g$ is $\mathcal{S}'$. Thus, also $\hat{g}_\ell - \delta_0 g_\ell(0) \to \hat{g}$ in $\mathcal{S}'$ which is the same as

$$\sum_{-\ell \leq j \leq \ell} \psi(2^{-j}\xi)\hat{f}(\xi) - \delta_0 g_\ell(0) \to \hat{g} \text{ in } \mathcal{S}' \ .$$

So if $h \in \mathcal{S}$ with $\operatorname{supp}(h) \subset \mathbb{R}^n \backslash \{0\}$, then $\langle \hat{f} - \hat{g}, h\rangle = 0$, i.e., $\operatorname{supp}(\hat{f} - \hat{g}) = \{0\}$.

By an elementary theorem about tempered distributions, therefore

$$(f-g)(x) = \sum_{|\gamma| \leq M} C_\gamma x^\gamma \ , \tag{48}$$

See Theorem 6.25 in Rudin 2. On the other hand, since $g(0) = 0$,

$$\begin{aligned} |(f-g)(x)| &\leq \|f\|_\infty + |g(x) - g(0)| \\ &\leq 1 + CA|x|^\alpha \ . \end{aligned}$$

Since $\alpha < 1$, comparing this bound with (48) shows that the polynomial in (48) has to have degree zero. Thus, $f - g =$ constant, as claimed. □

Lemma 18 (Bernstein's inequality). *Suppose $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$ satisfies* supp $(\hat{f}) \subset B(0,R)$. *Then*

$$\|D^\gamma f\|_p \leq C_\gamma R^{|\gamma|} \|f\|_p$$

for any multiindex γ where $C_\gamma = C(n,\gamma)$.

Proof. Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ with supp $\widehat{\chi} \subset B(0,2)$ and $\widehat{\chi}(\xi) = 1$ on $|\xi| \leq 1$. Then $\widehat{f}(\xi) = \widehat{\chi}(\xi/R)\widehat{f}(\xi)$ so that $f = R^n \chi(R\cdot) * f$. Hence, by Young,

$$\begin{aligned} \|D^\gamma f\|_p &\leq R^{|\gamma|} \|R^n (D^\gamma \chi)(R\cdot)\|_{L^1} \|f\|_p \\ &= C_\gamma R^{|\gamma|} \ \|f\|_p \ , \end{aligned}$$

as claimed. □

We now present a proposition that might seem a little unmotivated for now. Nevertheless, it is not only very natural, but will also allow us to conclude the desired Hölder bound for singular integrals. We postpone the proof of the proposition.

PROPOSITION 8. *Let K be a singular integral kernel as in Definition 7. For any $\eta \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \eta(x)\, dx = 0$ one has*

$$\|T\eta\|_1 \leq C(\eta)B$$

with a constant depending on η.

THEOREM 11. *Let K be as in Definition 7 and $0 < \alpha < 1$. Then for any $f \in L^2 \cap C^\alpha(\mathbb{R}^n)$ one has $Tf \in C^\alpha(\mathbb{R}^n)$ and*

$$[Tf]_\alpha \leq C_\alpha B[f]_\alpha$$

with $C_\alpha = C(\alpha, n)$.

PROOF. We use Lemma 17. In order to do so, notice first that

$$\|Tf\|_\infty \leq CB([f]_\alpha + \|f\|_2) ,$$

which we leave to the reader to check. Therefore, it suffices to show that

$$\sup_j 2^{j\alpha} \|\triangle_j Tf\|_\infty \leq CB[f]_\alpha \ . \tag{49}$$

Let $\widetilde{\triangle}_j$ be defined as

$$\widetilde{\triangle}_j u = \Big(\tilde{\psi}(2^{-j}\cdot)\hat{u}(\xi)\Big)^\vee \ ,$$

where $\tilde{\psi} \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ and $\tilde{\psi}\psi = \psi$, cf. (43). Thus, $\widetilde{\triangle}_j \triangle_j = \triangle_j$. Hence,

$$\begin{aligned} \|\triangle_j Tf\|_\infty &= \|\widetilde{\triangle}_j \triangle_j Tf\|_\infty = \|\widetilde{\triangle}_j T \triangle_j f\|_\infty \\ &\leq \|\widetilde{\triangle}_j T\|_{\infty\to\infty} \|\triangle_j f\|_\infty \\ &\leq \|\widetilde{\triangle}_j T\|_{\infty\longrightarrow\infty} C[f]_\alpha 2^{-j\alpha} \ . \end{aligned}$$

It remains to show that $\sup_j \|\widetilde{\triangle}_j T\|_{\infty\longrightarrow\infty} \leq CB$, see (49). Clearly, the kernel of $\widetilde{\triangle}_j T$ is $2^{jn}\check{\tilde{\psi}}(2^j\cdot) * K$ so that

$$\begin{aligned} \|\widetilde{\triangle}_j T\|_{\infty\longrightarrow\infty} &\leq \|2^{jn}\check{\tilde{\psi}}(2^j\cdot) * K\|_1 \\ &= \|\check{\tilde{\psi}} * 2^{-jn} K\Big(2^{-j}\cdot\Big)\|_1 \\ &\leq CB \end{aligned} \tag{50}$$

by Proposition 8. Indeed, set $\eta = \check{\tilde{\psi}}$ in that proposition so that

$$\int_{\mathbb{R}^n} \eta(x)\, dx = \tilde{\psi}(0) = 0 \ ,$$

as required. Furthermore, we apply Proposition 8 with the kernel $2^{-jn}K(2^{-j}\cdot)$. As this kernel satisfies the conditions in Definition 7 *uniformly* in j, one obtains (50) and the theorem is proved. $\square$

It remains to show Proposition 8, which will be accomplished by means of two lemmas.

LEMMA 19. *Let $f \in L^\infty(\mathbb{R}^n)$ with $\in f(x)\, dx = 0$,* supp $(f) \subset B(0,R)$ *and $\|f\|_\infty, \leq R^{-n}$. Then*

$$\|Tf\|_1 \leq CB \ .$$

PROOF. By Cauchy-Schwarz,

$$\int_{|x|\le 2R} |(Tf)(x)| \ dx \le \|Tf\|_2 C R^{\frac{n}{2}}$$
$$\le CB\|f\|_2 R^{\frac{n}{2}} \le CBR^{-n}R^{\frac{n}{2}}R^{\frac{n}{2}} = CB \ .$$

Furthermore,

$$\int_{|x|>2R} |(Tf)(x)| \ dx \le \int_{\mathbb{R}^n}\int_{|x|>2|y|} |K(x-y)-K(x)| \ dx|f(y)| \ dy \le B\|f\|_1 \le CB \ ,$$

as desired. □

REMARK. *A function f as in Lemma 19 is called an H^1-*atom*. The next lemma is an instance of an* atomic decomposition*.*

LEMMA 20. *Let $\eta \in \mathcal{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta(x) \ dx = 0$. Then one can write*

$$\eta = \sum_{\ell=1}^{\infty} c_\ell a_\ell$$

with $\int_{\mathbb{R}^n} a_\ell(x) \ dx = 0$, $\|a_\ell\|_\infty \le \ell^{-n}$, $\mathrm{supp}(a_\ell) \subset B(0;\ell)$ *for all $\ell \ge 1$ and*

$$\sum_{\ell=1}^{\infty} |c_\ell| \le C(\eta)$$

with a constant $C(\eta)$ depending on η.

PROOF. In this proof, we let

$$\langle g\rangle_S := \frac{1}{|S|}\int_S g(x) \ dx$$

for any $g \in L^1(\mathbb{R}^n)$ and $S \subset \mathbb{R}^n$ with $0 < |S| < \infty$. Moreover, $B_\ell := B(0,\ell)$ for $\ell \ge 1$, and $\chi_\ell = \chi_{B_\ell}$ (indicator of B_ℓ). Define

$$f_1 := (\eta - \langle\eta\rangle_{B_1})\chi_1 \ , \eta_1 := \eta - f_1$$

and set inductively

$$\begin{cases} f_{\ell+1} & := & (\eta_\ell - \langle\eta_\ell\rangle_{B_{\ell+1}})\,\chi_{\ell+1} \quad \text{and} \\ \eta_{\ell+1} & := & \eta_\ell - f_{\ell+1} \end{cases} \tag{51}$$

for $\ell \ge 1$ (one can take this also with $\ell = 0$ and $\eta_0 := \eta$).

Observe that

$$\eta = \eta_1 + f_1 = f_1 + f_2 + \eta_2 = \ldots = \sum_{\ell=1}^{M} f_\ell + \eta_{M+1} \ . \tag{52}$$

We need to show that we can pass to the limit $M \to \infty$ and that

$$a_\ell := f_\ell \cdot \frac{\ell^{-n}}{\|f_\ell\|_\infty} \quad \text{for} \quad \ell \ge 1 \tag{53}$$

have the desired properties. By construction, for all $\ell \geq 1$,

$$\int_{\mathbb{R}^n} a_\ell(x)\, dx = 0\ ,\ \operatorname{supp}(a_\ell) \subset B_\ell\ ,$$

and $\|a_\ell\|_\infty \leq \ell^{-n}$. It remains to show that

$$c_\ell := \ell^n \|f_\ell\|_\infty \tag{54}$$

satisfies $\sum_{\ell=1}^\infty c_\ell < \infty$ and that $\|\eta_{M+1}\|_\infty \to 0$, see (52). Clearly,

$$\eta_1 = \begin{cases} \langle \eta \rangle_{B_1} & \text{on} \quad B_1 \\ \eta & \text{on} \quad \mathbb{R}^n \backslash B_1\ . \end{cases}$$

Hence,

$$\eta_2 = \begin{cases} \langle \eta_1 \rangle_{B_2} & \text{on} \quad B_2 \\ \eta_1 = \eta & \text{on} \quad \mathbb{R}^n \backslash B_2\ . \end{cases}$$

By induction, for $\ell \geq 1$, one checks that

$$\eta_{\ell+1} = \begin{cases} \langle \eta_\ell \rangle_{B_{\ell+1}} & \text{on} \quad B_{\ell+1} \\ \eta & \text{on} \quad \mathbb{R}^n \backslash B_{\ell+1}. \end{cases} \tag{55}$$

Moreover, induction shows that

$$\int_{\mathbb{R}^n} \eta_\ell(x)\, dx = 0 \tag{56}$$

for all $\ell \geq 0$. Indeed, this is assumed for $\eta_0 = \eta$. Since $\int f_\ell(x)dx = 0$ for $\ell \geq 1$ by construction, one now proceeds inductively via the formula

$$\eta_{\ell+1} = \eta_\ell - f_{\ell+1}\ .$$

Property (56) implies that

$$\begin{aligned} \left|\langle \eta_\ell \rangle_{B_{\ell+1}}\right| &\leq \frac{1}{|B_{\ell+1}|} \int_{\mathbb{R}^n \backslash B_{\ell+1}} |\eta_\ell(x)|\ dx \\ &\leq \frac{1}{|B_{\ell+1}} \int_{\mathbb{R}^n \backslash B_{\ell+1}} |\eta(x)|\ dx \leq C\ell^{-20n} \end{aligned} \tag{57}$$

since $\eta_\ell = \eta$ on $\mathbb{R}^n \backslash B_\ell$, see (55), and since η has rapid decay. (57) implies that

$$\|\eta_{\ell+1}\|_\infty \leq C\ell^{-20n}\ ,$$

see (55), and also

$$\|f_{\ell+1}\|_\infty \leq C\ell^{-20n}\ ,$$

see (51). We now conclude from (52)–(54) that

$$\eta = \sum_{\ell=1}^\infty f_\ell = \sum_{\ell=1}^\infty c_\ell a_\ell$$

with, see (54) and (53),

$$\sum_{\ell=1}^\infty |c_\ell| = \sum_{\ell=1}^\infty \ell^n \|f_\ell\|_\infty \leq \sum_{\ell=1}^\infty C\ell^{-19n} < \infty\ ,$$

and the lemma follows. $\square$

Proof of Proposition 8. Let η be as in the statement of the proposition. By Lemma 20,

$$\eta = \sum_{\ell=1}^{\infty} c_\ell a_\ell$$

with c_ℓ and a_ℓ as stated there. By Lemma 19,

$$\|T(c_\ell a_\ell)\|_1 \leq |c_\ell| \|Ta_\ell\|_1 \leq CB|c_\ell|$$

for $\ell \geq 1$. Hence,

$$\sum_{\ell=1}^{\infty} \|T(c_\ell a_\ell)\|_1 \leq C_\eta B$$

and this easily implies that $\|T\eta\|_1 \leq C_\eta B$, is claimed. □

A typical application of Theorem 11 is to the so-called "Schauder estimate". More precisely, one has the following corollary.

Corollary 12. *Let $f \in C_0^{2,\alpha}(\mathbb{R}^n)$. Then*

$$\sup_{1 \leq i,j \leq n} \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_\alpha \leq C(\alpha, n)[\triangle f]_\alpha \tag{58}$$

for any $0 < \alpha < 1$.

Proof. As in the L^p case, see Corollary 9, this follows from the fact that

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = R_{ij}(\triangle f) \quad \text{for} \quad 1 \leq i, j \leq n$$

where R_{ij} are the double Riesz transforms. Now apply Theorem 11 to R_{ij}. □

This in turn implies estimates for elliptic equations on a region $\Omega \subset \mathbb{R}^n$, i.e.,

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) = f(x) \quad \text{in} \quad \Omega$$

where $a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ and $a_{ij} \in C^\alpha(\Omega)$. By "freezing x", one concludes from (58) that

$$\sup_{1 \leq i,j \leq \eta} \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{C^\alpha(K)} \leq C(\alpha, n, K)([f]_\alpha + \|f\|_\infty)$$

for any compact $K \subset \Omega$. See Gilbarg-Trudinger for this estimate and much more.

Remark. (58) *fails for $\alpha = 0$ and $\alpha = 1$. Try to show this—it follows immediately from the failure of the L^p bounds for $p = \infty$.*

CHAPTER 7

L^p-multipliers, Mikhlin theorem and Littlewood-Paley theory

One of the central concerns of harmonic analysis is the study of multiplier operators, i.e., operators T which are of the form

$$(Tf)(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} m(\xi)\, \hat{f}(\xi)\, d\xi \tag{59}$$

where $m : \mathbb{R}^n \longrightarrow \mathbb{C}$ is bounded. By Plancherel's theorem $\|T\|_{2\longrightarrow 2} \le \|m\|_\infty$. In fact, one has equality here, see Davis and Chang. The same reference contains other basic theorems on operators of the form (59), for example: There is a distribution $K \in \mathcal{S}'$ so that $Tf = K * f$ for any $f \in \mathcal{S}$ (of course $K = \check{m}$). Secondly, T is bounded on L^1 if and only if the associated kernel is in L^1, in which case

$$\|T\|_{1\longrightarrow 1} = \|K\|_1 \ .$$

There are many cases, though, where $K \notin L^1(\mathbb{R}^n)$ but still $\|T\|_{p\longrightarrow p}$ for some (or all) $1 < p < \infty$. The Hilbert transform is one such example. We shall now discuss one of the basic results in the field, which describes a large class of multipliers m that give rise to L^p bounded operators for $1 < p < \infty$.

Theorem 12 (Mikhlin). *Let $m : \mathbb{R}^n\backslash\{0\} \longrightarrow \mathbb{C}$ satisfy, for any multi-index γ of length $|\gamma| \le n+2$*

$$|D^\gamma m(\xi)| \le B|\xi|^{-|\gamma|}$$

for all $\xi \ne 0$. Then for any $1 < p < \infty$ there is a constant $C = C(n,p)$ so that

$$\|(m\hat{f})^\vee\|_p \le CB\|f\|_p$$

for all $f \in \mathcal{S}$.

Proof. Let ψ give rise to a dyadic partition of unity as in Lemma 1616. Define for any $j \in \mathbb{Z}$

$$m_j(\xi), = \psi(2^{-j}\xi)m(\xi)$$

and set $K_j = \check{m}_j$. Now fix some large positive integer N and set

$$K(x) = \sum_{j=-N}^{N} K_j(x) \ .$$

We claim that under our smoothness assumption on m one has

$$|K(x)| \le CB|x|^{-n} \quad \text{and} \quad |\nabla K(x)| \le CB|x|^{-n-1} \ , \tag{60}$$

where $C = C(n)$. One then applies the Calderon-Zygmund theorem and lets $N \longrightarrow \infty$.

We will verify the first inequality in (60). The second one is similar, and will be only sketched. By assumption, $\|D^\gamma m_j\|_\infty \le CB2^{-j|\gamma|}$ and thus

$$\|D^\gamma m_j\|_1 \le CB2^{-j|\gamma|}2^{jn}$$

for any multi-index $|\gamma| \le n+2$. Similarly,

$$\|D^\gamma(\xi_i m_j)\|_1 \le CB2^{-j(|\gamma|-1)}2^{jn}$$

for the same γ. Hence,

$$\|x^\gamma\, \check{m}_j(x)\|_\infty \le CB2^{j(n-|\gamma|)}$$

and

$$\|x^\gamma D\check{m}_j(x)\|_\infty \le CB2^{j(n+1-|\gamma|)}\ .$$

Since $|x|^k \le C(k,n)\sum_{|\gamma|=k}|x^\gamma|$ one concludes that

$$|\check{m}_j(x)| \le CB2^{j(n-k)}|x|^{-k} \tag{61}$$

and

$$|D\check{m}_j(x)| \le CB2^{j(n+1-k)}|x|^{-k} \tag{62}$$

for any $0 \le k \le n+2$ and all $j \in \mathbb{Z}$, $x \in \mathbb{R}^n\backslash\{0\}$. We shall use this with $k=0$ and $k=n+2$. Indeed,

$$\begin{aligned} |K(x)| \le \sum_j |\check{m}_j(x)| &\le \sum_{2^j \le |x|^{-1}} |\check{m}_j(x)| + \sum_{2^j > |x|^{-1}} |\check{m}_j(x)| \\ &\le CB \sum_{2^j \le |x|^{-1}} 2^{jn} + CB \sum_{2^j > |x|^{-1}} 2^{jn}(2^j|x|)^{-(n+2)} \\ &\le CB|x|^{-n} + CB|x|^2|x|^{-n-2} = CB|x|^{-n}\ , \end{aligned}$$

as claimed. To obtain the second inequality in (60) one uses (62) instead of (61). Otherwise the argument is unchanged. Thus we have verified that K satisfies the conditions *i*) and *iii*) of Definition 7, see Lemma 13. Furthermore, $\|m\|_\infty \le B$ so that $\|(m\hat{f})^\vee\|_2 \le B\|f\|_2$. By Theorem 10, and the remark following it, one concludes that

$$\|(m\hat{f})^\vee\|_p \le C(p,n)\|f\|_p$$

for all $f \in \mathcal{S}$ and $1 < p < \infty$, as claimed. $\square$

REMARK. *The conditions in Theorem 12 can be relaxed. Indeed, it suffices to assume the derivative bounds on* m *for all* $|\gamma| \le \left[\frac{n}{2}\right]+1$*, see Stein 1. The point is to verify the Hörmander condition directly rather than going through Lemma 13.*

COROLLARY 13. *Under the same assumptions as in Theorem 12 one has*

$$[(m\hat{f})^\vee]_\alpha \le C_\alpha B\,[f]_\alpha$$

for any $0 < \alpha < 1$ *and* $f \in C^\alpha(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

PROOF. This is really a corollary of the proof of Theorem 12. Indeed, we verified there that $K = \check{m}$ is a kernel that satisfies *i*) and *iii*) from Definition 7. The Hölder theory from the previous section does not require *ii*), but only L^2-boundedness of $f \mapsto (m\hat{f})^\vee$. As $\|m\|_\infty \le B$ this holds and the corollary therefore follows from (60) and Theorem 11. $\square$

Exercise 23. *Provide all details in the previous proof.*

Theorem 12 and Corollary 13 allow one to give proofs of Corollaries 9 and 12 without going through Exercise 16. Indeed, one has

$$\widehat{\frac{\partial^2 u}{\partial x_i \partial x_j}}(\xi) = \frac{\xi_i \xi_j}{|\xi|^2}\widehat{\triangle u}(\xi) \ .$$

Since $m(\xi) = \frac{\xi_i \xi_j}{|\xi|^2}$ satisfies the conditions of Theorem 1212 (this is obvious, as m is homogeneous of degree 0 and smooth away from $\xi = 0$), we are done.

Observe that this example shows that Theorem 12 and Corollary 13 fail if $p = 1$ or $p = \infty$, see Exercise 17.

We now present an important application of Mikhlin's theorem, namely to Littlewood-Paley theory. With ψ as in Lemma 16 set $\psi_j(\xi) = \psi(2^{-j\xi})$ so that

$$1 = \sum_{j \in \mathbb{Z}} \psi_j(\xi) \text{ if } \ \xi \neq 0 \ .$$

As above, we define $\triangle_j f = (\psi_j \hat{f})^\vee$. Then by Plancherel,

$$C^{-1}\|f\|_2^2 \leq \sum_{j \in \mathbb{Z}} \|\triangle_j f\|_2^2 \leq \|f\|_2^2 \tag{63}$$

for any $f \in L^2(\mathbb{R}^n)$. Observe that the middle expression is equal to $\|Sf\|_2^2$ with

$$Sf = \left(\sum_j |\triangle_j f|^2 \right)^{\frac{1}{2}}$$

This is called the Littlewood-Paley *square-function*. It is a famous result of theirs that (63) generalizes to

$$C^{-1}\|f\|_p \leq \|Sf\|_p \leq C\|f\|_p$$

for any $f \in L^p(\mathbb{R}^n)$ provided $1 < p < \infty$ and with $C = C(p, n)$. We shall now derive this result by means of a standard randomization technique. In what follows we let $\{r_j\}$ be a sequence of independent random variables with $\mathbb{P}[r_j = 1] = \mathbb{P}[r_j = -1] = \frac{1}{2}$, for all j (in other words, the r_j are a coin tossing sequence).

Lemma 21. *For any positive integer N and $\{a_j\}_{j=1}^N \in \mathbb{C}$ one has*

$$\mathbb{P}\left[\Big| \sum_{j=1}^N r_j a_j \Big| > \lambda \left(\sum_{j=1}^N |a_j|^2 \right)^{\frac{1}{2}} \right] \leq 4e^{-\lambda^2/2} \tag{64}$$

for all $\lambda > 0$.

Proof. Assume first that $a_j \in \mathbb{R}$. Then

$$\mathbb{E}e^{tS_N} = \Pi_{j=i}^N \mathbb{E}\Big(e^{t\, r_j a_j} \Big) = \Pi_{j=1}^N \cosh(t a_j)$$

where we have set $S_N = \sum_{j=1}^N r_j a_j$. Now invoke the calculus fact

$$\cosh x \le e^{x^2/2} \text{ for all } x \in \mathbb{R}$$

to conclude that

$$\mathbb{E}e^{tS_N} \le \Pi_{j=1}^N e^{t^2 a_j^2/2} = \exp\left(t^2 \sum_{j=1}^N a_j^2/2\right) .$$

Hence, with $\sigma^2 = \sum_{j=1}^N a_j^2$,

$$\begin{aligned}\mathbb{P}\Big[S_N > \lambda\sigma\Big] &\le e^{t^2\sigma^2/2} e^{-\lambda t\sigma} \\ &\le e^{-\lambda^2/2}\end{aligned}$$

where the final inequality follows by minimizing in t, i.e., $= \frac{\lambda}{\sigma}$. Similarly,

$$\mathbb{P}[S_N < -\lambda\sigma] \le e^{-\lambda^2/2}$$

so that

$$\mathbb{P}[|S_N| > \lambda\sigma] \le 2e^{-\lambda^2/2} .$$

The case of $a_j \in \mathbb{C}$ now follows by means of a decomposition into real and imaginary parts. □

The appearance of the Gaussian on the right-hand side of (64) should be very natural to anybody who is familiar with the central limit theorem. It should also explain why the following lemma holds (Khinchin's inequality).

Lemma 22. *For any $1 \le p < \infty$ there exist constants $C = C(p)$ so that*

$$C^{-1}\left(\sum_{j=1}^N |a_j|^2\right)^{\frac{p}{2}} \le \mathbb{E}\Big|\sum_{j=1}^N r_j a_j\Big|^p \le C\left(\sum_{j=1}^N |a_j|^2\right)^{\frac{p}{2}} \tag{65}$$

for any choice of positive integer N and $\{a_j\}_{j=1}^N \in \mathbb{C}$.

Proof. We start with the upper bound in (65). It suffices to consider the case $\sum_{j=1}^N |a_j|^2 = 1$. Setting $\sum_{j=1}^N a_j r_j = S_N$ one has

$$\mathbb{E}|S_N|^p = \int_0^\infty \mathbb{P}[|S_N| > \lambda] p\lambda^{p-1}\, d\lambda \le \int_0^\infty 4e^{-\lambda^2/2} p\lambda^{p-1}\, d\lambda =: C(p) < \infty .$$

For the lower bound it suffices to assume that $1 \le p \le 2$, in fact, $p = 1$. By Hölder's inequality,

$$\begin{aligned}\mathbb{E}|S_N|^2 &= \mathbb{E}|S_n|^{2/3}|S_N|^{4/3} \\ &\le (\mathbb{E}|S_N|)^{2/3}(\mathbb{E}|S_n|^4)^{1/3} \\ &\le C(\mathbb{E}|S_N|)^{2/3}(\mathbb{E}, |S_N|^2)^{4/3}\end{aligned}$$

where the final inequality follows from the case $2 \le p < \infty$ just considered. This implies that

$$\mathbb{E}|S_N|^2 \le C(\mathbb{E}|S_N|)^2 ,$$

and we are done. □

REMARK. *Khinchin's inequality is usually formulated for the* Rademacher *functions, which are a concrete realization of the sequence* $\{r_j\}$ *on the interval* $[0,1]$. *The explicit form of the Rademacher functions allows for a different proof of Khinchin's inequality, but those proofs are somewhat less transparent.*

We are now ready to prove the Littlewood-Paley theorem.

THEOREM 13 (Littlewood-Paley). *For any* $1 < p < \infty$ *there are constants* $C = C(p,n)$ *such that*

$$C^{-1}\|f\|_p \le \|Sf\|_p \le C\|f\|_p$$

for any $f \in \mathcal{S}$.

PROOF. Let $\{r_j\}$ be as above. The proof rests on the fact that

$$m(\xi) := \sum_{j=-N}^{N} r_j \psi_j(\xi)$$

satisfies the conditions of the Mikhlin multiplier theorem uniformly in N and uniformly in the realization of the random variables $\{r_j\}$. Indeed, for any γ,

$$\begin{aligned}
|D^\gamma m(\xi)| &\le \sum_{j=-N}^{N} |D^\gamma \psi_j(\xi)| \\
&\le C \sum_{j=-N}^{N} |\xi|^{-\gamma} |(D^\gamma \psi)(2^{-j}\xi)| \\
&\le C|\xi|^{-\gamma} .
\end{aligned}$$

To pass to the final inequality one uses that only an absolutely bounded number of terms is non-zero in the sum preceding it for any $\xi \neq 0$. Hence, in view of Lemma 22,

$$\begin{aligned}
\int_{\mathbb{R}^n} \Big|(Sf)(x)\Big|^p \, dx &\le C \limsup_{N\longrightarrow\infty} \mathbb{E} \int_{\mathbb{R}^n} \Big| \sum_{j=-N}^{N} r_j (\triangle_j f)(x)\Big|^p \, dx \\
&\le C\|f\|_p^p ,
\end{aligned}$$

as desired.

To prove the lower bound we use duality: choose a function $\tilde{\psi}$ so that $\tilde{\psi} = 1$ on $\mathrm{supp}\,(\psi)$ and $\tilde{\psi}$ is compactly supported with supp $\tilde{\psi} \subset \mathbb{R}^d \backslash \{0\}$. Defining $\tilde{\triangle}_j$ like $\triangle_j$ with $\tilde{\psi}$ instead of ψ yields $\{\tilde{\triangle}_j\}$ satisfying $\tilde{\triangle}_j \triangle_j = \triangle_j$. Therefore, for any $f, g \in \mathcal{S}$, and any $1 < p < \infty$,

$$\begin{aligned}
|\langle f, g\rangle| &= |\sum_j \langle \triangle_j f, \tilde{\triangle}_j g\rangle| \\
&\le \int_{\mathbb{R}^n} \Big(\sum_j |\triangle_j f|^2\Big)^{\frac12} \Big(\sum_k |\tilde{\triangle}_k g|^2\Big)^{\frac12} \, dx \\
&\le \|Sf\|_p \|\tilde{S}g\|_{p'} \le C\|Sf\|_p \|g\|_{p'} .
\end{aligned}$$

Thus, $\|f\|_p \leq C\|Sf\|_p$, as claimed (observe that the argument for the upper bound equally well applies to $\tilde{S}$ instead of S). $\square$

Remark. *Theorem 12 is formulated as an a priori inequality for $f \in \mathcal{S}$. In that case $m\hat{f} \in \mathcal{S}'$ so that $(m\hat{f})^\vee \in \mathcal{S}'$. The question arises whether or not $(m\hat{f})^\vee$ is meaningful for $f \in L^p(\mathbb{R}^n)$. If $1 < p \leq 2$, then $\hat{f} \in L^2 + L^\infty(\mathbb{R}^n)$ (in fact $\hat{f} \in L^{p'}$ by Hausdorff-Young), so that $m\hat{f} \in \mathcal{S}'$ and therefore $(m\hat{f})^\vee \in \mathcal{S}'$ is well-defined. If $p > 2$, however, it is known that there are $f \in L^p(\mathbb{R}^n)$ such that $\hat{f} \in \mathcal{S}'$ has positive order (see Theorem 7.6.6 in Hörmander). For such f it is in general not possible to define $(m\hat{f})^\vee$ in $\mathcal{S}'$. Similarly, it is desirable to formulate Theorem 13 on $L^p(\mathbb{R}^n)$ rather than on $\mathcal{S}$. This is done in the following corollary. Observe that Sf is defined pointwise if $f \in \mathcal{S}'(\mathbb{R}^n)$.*

Corollary 14.

i) *Let $1 < p < \infty$. Then for any $f \in L^p(\mathbb{R}^n)$ one has $Sf \in L^p$ and*

$$C_{p,n}^{-1}\|f\|_p \leq \|Sf\|_p \leq C_{p,n}\|f\|_p \,.$$

ii) *Suppose that $f \in \mathcal{S}'$ and that $Sf \in L^p(\mathbb{R}^n)$ with some $1 < p < \infty$. Then $f = g + P$ where P is a polynomial and $g \in L^p(\mathbb{R}^n)$. Moreover, $Sf = Sg$ and*

$$C_{p,n}^{-I}\|Sf\|_p \leq \|g\|_p \leq C_{p,n}\|Sf\|_p \,.$$

Proof. To prove part *i*), let $f_k \in \mathcal{S}$ so that $\|f_k - f\|_p \to 0$ as $k \to \infty$. We claim that

$$\lim_{k\to\infty} \|Sf_k - Sf\|_p = 0 \,. \tag{66}$$

If (66) holds, the passing to the limit $k \to \infty$ is

$$C_{p,n}^{-1}\|f_k\|_p \leq \|S(f_k)\|_p \leq C_{p,n}\|f_k\|_p$$

implies part *i*).

To prove (66) one applies Fatou repeatedly. Fix $x \in \mathbb{R}^n$. Then

$$\begin{aligned} |Sf_k(x) - Sf(x)| &= |\|\{\triangle_j f_k(x)\}_{k=-\infty}^\infty\|_{\ell^2} - \|\{\triangle_j f(x)\}_{j=-\infty}^\infty\|_{\ell^2}| \\ &\leq \|\{\triangle_j f_k(x) - \triangle_j f(x)\}_{j=-\infty}^\infty\|_{\ell^2} \\ &= S(f_k - f)(x) \leq \liminf_{m\to\infty} S(f_k - f_m)(x) \,. \end{aligned}$$

Therefore,

$$\|Sf_k - Sf\|_p \leq \liminf_{m\to\infty} \|S(f_k - f_m)\|_p \leq C_{p,n} \liminf_{m\to\infty} \|f_k - f_m\|_p \,.$$

The claim (66) now follows by letting $k \to \infty$. For the second part we rely on the following fact about distributions in $\mathcal{S}'$ (see the proof of Lemma 17 as well as Theorem 6.25 in Rudin 2): Let $u, v \in \mathcal{S}'$ with $\langle u, \varphi\rangle = \langle v, \varphi\rangle$ for all $\varphi \in \mathcal{S}$ with supp $\hat{\varphi} \subset \mathbb{R}^n\backslash\{0\}$. Then

$$u - v = P$$

for some polynomial P.

We now argue as in the proof of the lower bound in Theorem 13: Let $f \in \mathcal{S}'$ and $h \in \mathcal{S}$ with sup $\hat{h} \subset \mathbb{R}^d \backslash \{0\}$. Then

$$|\langle f, \bar{h}\rangle| \leq C_p \|Sf\|_p \|\tilde{S}h\|_{p'} \leq C_p \|Sf\|_p \|h\|_{p'} ,$$

where $\tilde{S}$ is the modified square function from the proof of Theorem 13. By the Hahn-Banach theorem there exists $g \in L^p$ so that $\langle f, \bar{h}\rangle = \langle g, h\rangle$ for all h as above satisfying $\|g\|_p \leq C_p \|Sf\|_p$. By the above mentioned fact about distributions, $f = g + P$. Moreover, $Sf = Sg$ and

$$\|Sf\|_p = \|Sg\|_p \leq C\|g\|_p$$

by part *i*). □

EXERCISE 24.

a) *Show that Theorem 13 fails for* $p = 1$. *The intuition is of course to take* $f = \delta_0$. *For that case you should check that*

$$(Sf)(x) \sim |x|^{-n} ,$$

so that $Sf \notin L^1(\mathbb{R}^n)$. *Now transfer this to* $L^1(\mathbb{R}^n)$ *by means of approximate identities.*

b) *Show that Theorem 13 fails for* $p = \infty$

It is natural to ask at this point whether one can develop a Littlewood-Paley theory for square functions defined in terms of sharp cut-offs rather than smooth ones as above. More precisely, suppose we set

$$(S_{\mathrm{new}} f)^2 = \sum_{j\in\mathbb{Z}} |(\chi_{[2^{j-1}\leq|\xi|<2^j]} \hat{f}(\xi))^\vee|^2 .$$

Is it true that

$$C_p^{-1}\|f\|_p \leq \|S_{\mathrm{new}} f\|_p \leq C_p\|f\|_p \tag{67}$$

for $1 < p < \infty$*?*

It is a relatively easy consequence of the L^p*-boundedness of the Hilbert transform that the answer is "yes" for the case* $n = 1$ *(one dimension). On the other hand, it is a very non-trivial and famous result of Charles Fefferman that the answer is "no" for dimensions* $n \geq 2$. *This latter result is based on the existence of Kakeya sets and will not be discussed in this lecture, see Davis-Chang and the notes by Tom Wolff.*

For the courageous reader we have presented the proof of (67) for $n = 1$ as an exercise. Alternatively, look into Davis-Chang, Christ.

EXERCISE 25. *In this exercise* $n = 1$.

a) *Deduce* (67) *from Theorem 13 by means of the following "vector-valued" inequality: Let* $\{I_j\}_{j\in\mathbb{Z}}$ *be an arbitrary collection of intervals. Then for any* $1 < p < \infty$

$$\Big\|\Big\{\Big(\chi_{I_j}\hat{f}_j\Big)^\vee\Big\}_j\Big\|_{L^p(\ell^2)} := \Big\|\Big(\sum_j \Big|\Big(\chi_{I_j}\hat{f}_j\Big)^\vee\Big|^2\Big)^{\frac{1}{2}}\Big\|_p \leq C_p\|\{f_j\}\|_{L^p(\ell^2)} \tag{68}$$

where $\{f_j\}_{j\in\mathbb{Z}}$ *are an arbitrary collection of functions in* $\mathcal{S}(\mathbb{R}^1)$, *say.*

b) Now deduce (68) *from Theorem 8 by means of Khinchin's inequality (***Hint:** *express the operator $f \mapsto (\chi_I \hat{f})^\vee$ by means of the Hilbert transform via the same procedure which was used in the proof of Theorem 9).*

c) By similar means prove the following Littlewood-Paley theorem for functions in $L^p(\mathbb{T})$: For any $f \in L^1(\mathbb{T})$ let

$$Sf = \Big(\sum_{j=0}^{\infty} |\triangle_j f|^2 \Big)^{\frac{1}{2}}$$

where

$$(\triangle_j f)(\theta) = \sum_{2^{j-1} \le |n| < 2^j} \hat{f}(n) e(n\theta)$$

for $j \ge 1$ and $\triangle_0 f(\theta) = \hat{f}(0)$. Show that, for any $1 < p < \infty$

$$C_p^{-1} \|f\|_{L^p(\mathbb{T})} \le \|Sf\|_{L^p(\mathbb{T})} \le C_p \|f\|_{L^p(\mathbb{T})} \tag{69}$$

for all $f \in L^p(\mathbb{T})$.

d) As a consequence of (67) *with $n = 1$ show the following multiplier theorem: Let $m : \mathbb{R} \backslash \{0\} \to \mathbb{C}$ have the property that, for each $j \in \mathbb{Z}$*

$$m(\xi) = m_j = \text{ constant}$$

for all $2^{j-1} \le |\xi| < 2^j$. Then, for any $1 < p < \infty$

$$\|(m\hat{f})^\vee\|_{L^p(\mathbb{R})} \le C_p \sup_{j \in \mathbb{Z}} |m_j| \|f\|_p$$

for all $f \in \mathcal{S}(\mathbb{R})$. Prove a similar theorem for $L^p(\mathbb{T})$ using (69)*.*

We now prove the “fractional Leibnitz rule” as an application of the Littlewood-Paley theorem.

Theorem 14. *Let $\langle \nabla \rangle$ denote the Fourier multiplier operator with symbol $(1+|\xi|^2)^{\frac{1}{2}}$, and similarly for $\langle \nabla \rangle^s$ for every real number s. Then the following “fractional Leibnitz rule” holds for $s > 0$: For any Schwartz functions $f, g \in \mathcal{S}(\mathbb{R}^d)$*

$$\|\langle \nabla \rangle^s (fg)\|_p \le C(s, d, p, p_1, p_2, p_3, p_4) \big[\|\langle \nabla \rangle^s f\|_{p_1} \|g\|_{p_2} + \|\langle \nabla \rangle^s f\|_{p_3} \|g\|_{p_4} \big] \tag{70}$$

where $1 < p < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, $p_1, p_2, p_3, p_4 \in (1, \infty)$.

In the notation of (fractional) Sobolev spaces $W^{s,p}(\mathbb{R}^d)$, the inequality (70) is of course the same as

$$\|fg\|_{W^{s,p}} \le C(s, d, p, p_1, p_2, p_3, p_4) \big[\|f\|_{W^{s,p_1}} \|g\|_{p_2} + \|f\|_{W^{s,p_3}} \|g\|_{p_4} \big].$$

The idea of the proof of Theorem 14 is as follows: Let $\{\Delta_j\}_{j \in \mathbb{Z}}$ be the Littlewood-Paley multipliers as above. Since the multiplier of $\langle \nabla \rangle$ is smooth around $\xi = 0$, we will not need to worry about small frequencies and therefore replace Δ_0 with the entire sum $\sum_{k \le 0} \Delta_k$. Now represent the functions f, g in the form:

$$f = \sum_{j \ge 0} \Delta_j f \ , \quad g = \sum_{k \ge 0} \Delta_k g \, .$$

Then

$$\begin{aligned}
&\|\langle\nabla\rangle^s(fg)\|_p = \big\|\sum_{j,k\ge 0}\langle\nabla\rangle^s(\Delta_j f\Delta_k g)\big\|_p \\
&= \big\|\sum_{\substack{j,k\ge 0\\ k\le j-10}}\langle\nabla\rangle^s(\Delta_j f\Delta_k g)\big\|_p + \big\|\sum_{\substack{j,k\ge 0\\ j\le k-10}}\langle\nabla\rangle^s(\Delta_j f\Delta_k g)\big\|_p + \big\|\sum_{\substack{j,k\ge 0\\ |j-k|<10}}\langle\nabla\rangle^s(\Delta_j f\Delta_k g)\big\|_p \\
&=: P_1 + P_2 + P_3.
\end{aligned}$$

It should be possible to bound P_1 by means of the Littlewood-Paley theorem as follows, at least heuristically:

$$P_1 \lesssim \Big\|\Big(\sum_{j\ge 0} 2^{2js}|\Delta_j f|^2\ |\sum_{k\le j-10}\Delta_k g|^2\Big)^{\frac12}\Big\|_p \tag{71}$$

$$\lesssim \Big\|\Big(\sum_{j\le 0} 2^{2js}|\Delta_j f|^2\Big)^{\frac12} Mg\Big\|_p \lesssim \Big\|\Big(\sum_{j>0} 2^{2js}|\Delta_j f|^2\Big)^{\frac12}\Big\|_{p_1}\|Mg\|_{p_2}$$

$$\lesssim \|f\|_{W^{s,p_1}}\|g\|p_2 \tag{72}$$

Here M is the Hardy-Littlewood maximal operators, which is bounded on L^{p_2} because of $p_2 > 1$. The expression in (71) is an example of a so-called "paraproduct". To pass from (71) to the line below is a simple fact using convolutions, whereas the inequality sign in (71) should be a rather direct consequence of the Littlewood-Paley theorem since the Fourier support of

$$\sum_{k\le j-10}\langle\nabla\rangle^s(\Delta_j f\Delta_k g)$$

for fixed $j\ge 0$ is a shell of size 2^j. That this is indeed so (even without the restriction $s>0$), will be shown in Lemma 23. The term P_2 is the same, whereas P_3 is different in the sense that for fixed j the summand

$$\sum_{|k-j|\le 10}\langle\nabla\rangle^s(\Delta_j f\Delta_k g)$$

no longer has Fourier support in a shell, but rather in a ball of size 2^j. Nevertheless, because of $s>0$ one would expect to still be able to estimate

$$\begin{aligned}
P_3 &\lesssim \Big\|\sum_{|j-k|<10} 2^{js}|\Delta_j f|\,|\Delta_k g|\,\Big\|_p \\
&\lesssim \Big\|\Big(\sum_j 2^{2js}|\Delta_j f|^2\Big)^{\frac12}\Big(\sum_k |\Delta_k g|^2\Big)^{\frac12}\Big\|_p \\
&\lesssim \Big\|\Big(\sum_j 2^{2js}|\Delta_j f|^2\Big)^{\frac12}\Big\|_{p_1}\Big\|\Big(\sum_k |\Delta_k g|^2\Big)^{\frac12}\Big\|_{p_2} \lesssim \|f\|_{W^{s,p_1}}\|g\|_{p_2}
\end{aligned}$$

To pass to the second line here, we use the Cauchy-Schwartz inequality.

Exercise 26. *Use the following Lemma 23 in order to make these estimates for P_1, P_2, P_3 rigorous. Hint: For P_3, do not use Cauchy-Schwartz, but rather resort to M*

again as follows:

$$\Big\| \sum_{\substack{j,k\geq 0\\ |j-k|\leq 10}} \langle\nabla\rangle^s \Delta_j f \Delta_k g \Big\|_p \lesssim \Big\| \Big(\sum_{\substack{j,k\geq 0\\ |j-k|\leq 10}} 2^{2sj} |\Delta_j f|^2 \, |\Delta_k g|^2 \Big)^{\frac{1}{2}} \Big\|_p$$
$$\lesssim \Big\| \Big(\sum_{j\geq 0} 2^{2sj} |\Delta_j f|^2 \Big)^{\frac{1}{2}} Mg \Big\|_p \lesssim \|f\|_{W^{s,p_1}} \|g\|_{p_2}.$$

The following lemma is the main ingredient in the proof of Theorem 14.

LEMMA 23. *Fix some constant $L > 1$. Then the following properties hold:*

i) Let $\{f_k\}_{k\geq 0}$ be Schwartz functions so that $\mathrm{supp}(\hat{f}_k) \subset \{L^{-1}\, 2^k \leq |\xi| \leq L\, 2^k\}$ *if $k \geq 1$ and* $\mathrm{supp}(\hat{f}_0) \subset \{|\xi| \leq L\}$*. Then*

$$\Big\| \langle\nabla\rangle^s \sum_{k\geq 0} f_k \Big\|_p \leq C(s,p,d,L) \Big\| \Big(\sum_{k\geq 0} 2^{2sk} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p$$

if $1 < p < \infty$ and $s \in \mathbb{R}$. Conversely, if $f_k = \Delta_k f$ for some $f \in \mathcal{S}(\mathbb{R}^d)$, then the converse holds, i.e.,

$$\Big\| \Big(\sum_{k\geq 0} 2^{2sk} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p \leq C(s,p,d) \big\| \langle\nabla\rangle^s f \big\|_p$$

if $1 < p < \infty$ and $s \in \mathbb{R}$.

ii) If $\{f_k\}_{k\geq 0}$ are Schwartz functions such that $\mathrm{supp}(\hat{f}_k) \subset \{|\xi| \leq L\, 2^k\}$ *for all $k \geq 0$, then*

$$\|\langle\nabla\rangle^s \sum_{k\geq 0} f_k\|_p \leq C(s,p,d,L) \Big\| \Big(\sum_{k\geq 0} 2^{2sk} |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_p$$

if $s > 0$ and $1 < p < \infty$.

PROOF. Fix a positive integer ν so that $2^\nu \gg L^2$. This insures that

$$\mathrm{dist}\big(\mathrm{supp}(\widehat{f_k}), \mathrm{supp}(\widehat{f_{k+\nu}})\big) \gg L\, 2^k$$

for all $k \geq 0$. Also, let ψ_1 be an even Schwartz function with the property that

$$\mathrm{supp}(\psi_1) \subset \{\xi \in \mathbb{R}^d \mid L^{-1}/2 < |\xi| < 2L\}, \qquad \psi_1(\xi) = 1 \text{ if } L^{-1} < |\xi| < L$$

and define ψ_k via $\psi_k(\xi) = \psi_1(2^{-k+1}\xi)$ for all $k \geq 1$, whereas ψ_0 is some Schwartz functions so that $\mathrm{supp}(\psi_0)$ is compact and $\psi_0(\xi) = 1$ if $|\xi| \leq L^{-1}$. Then, by construction, $f_k = \widehat{\psi_k} * f_k$ for all $k \geq 0$. We split the sum into congruence classes modulo ν. Thus,

$$\|\langle\nabla\rangle^s \sum_{k\geq 0} f_k\|_p \leq \sum_{j=0}^{\nu-1} \Big\| \langle\nabla\rangle^s \sum_{k\equiv j} f_k \Big\|_p = \sum_{j=0}^{\nu-1} \Big\| \sum_{k\equiv j} \langle\nabla\rangle^s 2^{-ks} \widehat{\psi_k} * 2^{ks} f_k \Big\|_p$$
$$= \sum_{j=0}^{\nu-1} \|(m_{s,j} \hat{F}_j)^\vee\|_p,$$

where

$$F_j := \sum_{k\equiv j} 2^{ks} f_k$$

$$m_{s,j}(\xi) := \sum_{k\equiv j} (1+|\xi|^2)^{\frac{s}{2}} 2^{-ks} \widehat{\psi_k}(\xi).$$

Since ψ_k is obtained by scaling, it is easy to check that $m_{s,j}$ is a Mikhlin multiplier and we therefore have by the Mikhlin and Littlewood-Paley theorems

$$\begin{aligned}\|\langle\nabla\rangle^s \sum_{k\geq 0} f_k\|_p &\lesssim \sum_{j=0}^{\nu-1} \|(m_{sj}\hat{F}_j)^\vee\|_p \lesssim \sum_{j=0}^{\nu-1} \|F_j\|_p \lesssim \sum_{j=0}^{\nu-1} \Big\|(\sum_{k\equiv j} 2^{2ks}|f_k|^2)^{\frac{1}{2}}\Big\|_p \\ &\lesssim \Big\|(\sum_{k\geq 0} 2^{2ks}|f_k|^2)^{\frac{1}{2}}\Big\|_p,\end{aligned}$$

as claimed. To prove the converse, let $f_k = \Delta_k f = \widehat{\psi_k} * f$, where now ψ_k are the *standard* Littlewood-Paley functions, cf. Lemma 16. In particular, the supports of ψ_k are disjoint if k runs through all even or odd integers, respectively. Define $\tilde{\psi}_k$ to be "fattened up" versions of ψ_k, i.e., $\tilde{\psi}_k\psi_k = \psi_k$ and $\mathrm{supp}(\tilde{\psi}_k)$ and $\mathrm{supp}(\tilde{\psi}_\ell)$ are disjoint if $k-\ell \geq 3$. Observe that this implies that

$$\psi_\ell \sum_{k\equiv j} \tilde{\psi}_k = \psi_\ell \tilde{\psi}_\ell = \psi_\ell \tag{73}$$

for every $j = 0,1,2$ provided $\ell \equiv j$. Here congruences are to be undestood modulo 3. Finally, we can assume that $\tilde{\psi}_k(\xi) = \tilde{\psi}_1(2^{-k+1}\xi)$ for all $k \geq 1$. Hence,

$$\begin{aligned}&\|\Big(\sum_{k\geq 0} 2^{2sk}|\Delta_k f|^2\Big)^{\frac{1}{2}}\|_p \\ &\lesssim \sum_{j=0}^{2} \Big\|\Big(\sum_{\substack{k\geq 0\\ k\equiv j \mod 3}} 2^{2sk}|\Delta_k f|^2\Big)^{\frac{1}{2}}\Big\|_p = \sum_{j=0}^{2} \Big\|\Big(\sum_{\substack{k\geq 0\\ k\equiv j \mod 3}} 2^{2sk}|\Delta_k(\widehat{\tilde{\psi}_k} \star f)|^2\Big)^{\frac{1}{2}}\Big\|_p \\ &= \sum_{j=0}^{2} \|\Big(\sum_{\substack{k\geq 0\\ k\equiv j \mod 3}} |\Delta_k(F_j)|^2\Big)^{\frac{1}{2}}\|_p \lesssim \sum_{j=0}^{2} \|F_j\|_p,\end{aligned} \tag{74}$$

by the Littlewood-Paley theorem and with the definition

$$F_j := \sum_{\substack{\ell\geq 0\\ \ell\equiv j \mod 3}} 2^{s\ell}\, \widehat{\tilde{\psi}}_\ell * f.$$

Equality in (74) holds because of (73), since the latter implies that $\Delta_k(F_j) = 2^{sk}\widehat{\tilde{\psi}_k} * f$. It remains to show that

$$\|F_j\|_p \lesssim \|f\|_{W^{s,p}} \tag{75}$$

for each $0 \leq j \leq 2$. To this end define

$$m_{s,j}(\xi) := \sum_{\substack{k\geq 0\\ k\equiv j \mod 3}} 2^{sk}(1+|\xi|^2)^{-\frac{s}{2}} \widehat{\tilde{\psi}}(2^{-k}\xi)$$

so that

$$F_j = \left[m_{s,j}(1+|\xi|^2)^{\frac{s}{2}}\hat{f}(\xi)\right]^{\vee}.$$

In order to prove (75) it suffices to show that $m_{s,j}$ is a Mikhlin multiplier. In fact, it is easy to check that $\|m_s\|_\infty \lesssim 1$ and $|\partial^\alpha m_s(\xi)| \le C_\alpha(1+|\xi|)^{-|\alpha|}$. Indeed,

$$\begin{aligned}|\nabla m_s(\xi)| &\lesssim \sum(1+|\xi|^2)^{-\frac{s}{2}}\left\{(1+|\xi|)^{-1}2^{sk}|\hat{\tilde{\psi}}(2^{-k}\xi)| + 2^{sk}2^{-k}|(\nabla\hat{\tilde{\psi}})(2^{-k}\xi)\right\} \\ &\lesssim (1+|\xi|)^{-1},\end{aligned}$$

and similarly for higher derivatives. This finishes the proof of part (i).

Remark for (ii), what about $s=0$?

$$\rho = 2, s = 0\ , \qquad \|\sum_k f_k\|_2^2 \overset{?}{\lesssim} \sum_k \|f_k\|_2$$

would be true only if there was some orthogonality, not in general. (e.g., take $\hat{f}_k = \chi_{[0,1]}$, $N^2 \overset{?}{\lesssim} N$, no.)

Proof of (ii). Use fact that weight $(1+|\xi|^2)^{\frac{s}{2}}$ is bigger at the endpoints of the interval $[-C_0 2^k, C_0 2^k]$ in 1-D.

From (i),

$$\begin{aligned}\|\sum_{k\ge 0} f_k\|_{W^{s,\rho}} &\lesssim \|\Big(\sum_{l\ge 0} 2^{2sl}\Big|\Delta_l \sum_{k\ge l} f_k\Big|^2\Big)^{\frac{1}{2}}\|_\rho \\ &= \|\sum_{l\ge 0}\Big|\sum_{k\ge l} 2^{s(l-k)}\psi_l \star \underbrace{2^{sk} f_k}_{w_k}\Big|^2\Big)^{\frac{1}{2}}\|_\rho \\ &\overset{?}{\lesssim} \|\Big(\sum_{k\ge 0}\underbrace{2^{2sk}|f_k|^2}_{|w_k|^2}\Big)^{\frac{1}{2}}\|_\rho\end{aligned}$$

is the statement

$$\begin{aligned}\|T\big(\{w_k\}_{k\ge 0}\big)\|_{L^\rho(l^2)} &\overset{?}{\lesssim} \|\{w_k\}_{k\ge 0}\|_{L^\rho(l^2)} \\ T\big(\{w_k\}_{k\ge 0}\big) &= \{\sum_{k\ge l\ge 0} 2^{s(l-k)}\psi_l \star w_k\}_{l\ge 0}\end{aligned}$$

$\rightarrow$ vector-valued singular integral.

By Hilbert space-valued Calderon-Zygmund theory,

α) $T : L^2(l^2) \to L^2(\ell^2)$ bounded
β) kernel matrix element $K_{k\ell} = 2^{s(\ell-k)}\psi_\ell$,
$\|K(x)\|_{\ell^2\to\ell^2} \le B.|x|^{-d}$
γ) $\|\nabla K(x)\|_{\ell^2\to\ell^2} \le B.|x|^{-d-1}$

why γ) is needed, $1/|x| \star f$ does not make sense.

C2-theorem: α), β), γ) $\Rightarrow \|T\|_{L^\rho(\ell^2)\to L^\rho(\ell^2)} \le C_{d,\rho} B,$

$$1 < \rho < \infty$$

Show α)

$$\begin{aligned}
\|T\{w_k\}_{k\ge 0}\|^2_{L^2(l^2)} &= \int \Big(\sum_{l\ge 0} \Big| \sum_{k\ge l\ge 0} K_{kl} \star w_k \Big|^2 \Big)\, dx \\
&= \sum_{l\ge 0} \int \Big| \sum_{k\ge l} 2^{s(l-k)} \widehat{\psi(2^{-l}\xi)} \hat{w}_k(\xi) \Big|^2 \, d\xi \\
&\lesssim \int \underbrace{\sum_{l\ge 0} \big|\hat{\psi}(2^{-l}\xi)\big|^2}_{\le C} \underbrace{\sum_{k\ge l} 2^{2s(l-k)}}_{\le C(s)} \sum_{k\ge l} |\hat{w}_k(\xi)|^2 \, d\xi \\
&\le C(s). \|\{w_k\}_{k\ge 0}\|^2_{L^2(l^2)} \ .
\end{aligned}$$

Show β): Schur's test

$$(1) \quad \sup_l \sum_{k\ge l} |K_{kl}(x)| \overset{?}{\le} B.|x|^{-d}$$

$$(2) \quad \sup_k \sum_{k\ge l\ge 0} |K_{kl}(x)| \overset{?}{\le} B.|x|^{-d}$$

$$\begin{aligned}
(1) \quad & \sum_{k\ge l} 2^{s(l-k)} |\psi_l(x)| \\
& \le C(s)|\psi_l(x)| = C(s) 2^{ld} |\psi(2^l x)| \\
& \le C_N(s) 2^{ld} (1 + 2^k|x|)^{-N} \qquad N \text{ big.} \\
& \lesssim \begin{cases} |x|^{-d} & \text{for } 2^l|x| < 1 \\ 2^{l(d-N)}|x|^{-N} < |x|^{-d} & \text{for } 2^l|x| > 1\ . \end{cases}
\end{aligned}$$

$$\begin{aligned}
(2) \quad & \sum_{k\ge l\ge 0} 2^{s(l-k)} |\psi_l(x)| \le C_N \sum_{k\ge l\ge 0} 2^{s(l-k)} (1 + 2^l|x|)^{-N} \\
& \lesssim \underbrace{\sum_{\substack{k\ge l\ge 0\\ 2^l|x|<1}} \underbrace{2^{s(l-k)}}_{\le 1} 2^{ld}}_{\lesssim |x|^{-d}} + \underbrace{\sum_{\substack{k\ge l\ge 0\\ 2^l|x|>1}} \underbrace{2^{s(l-k)}}_{\le 1} 2^{ld} (2^l|x|)^{-N}}_{\lesssim |x|^{-d-1}\left(|x|^{-1}\right)^{-1} = |x|^{-d}} \\
& \qquad \text{take } N = d+1,
\end{aligned}$$

Show γ) idem, 2^{ld} becomes $2^{l(d+1)}$, etc.

Note: Vector-valued $C2$ theory trivially follows from scalar $C2$ theory.

$\square$

CHAPTER 8

Restriction of the Fourier transform, Stein-Tomas theorem, Strichartz estimates

In the late 1960's, Elias M. Stein posed the question whether it is possible to restrict the Fourier transform $\hat{f}$ of a function $f \in (\mathbb{R}^n)$ with $1 \le p \le 2$ to the sphere S^{n-1} as a function in $L^q(S^{n-1})$ for some $1 \le q \le \infty$. In other words, is there a bound

$$\|\hat{f} \restriction S^{n-1}\|_{L^q(S^{n-1})} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{76}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$, with a constant $C = C(n,p,q)$? As an example, take $p = 1$, $q = \infty$ and $C = 1$. On the other hand, $p = 2$ is impossible, as $\hat{f}$ is no better than a general L^2-function by Plancherel. Stein asked whether it is possible to find $1 < p < 2$ so that for some finite q one has the estimate (76). The following theorem settles the important case $q = 2$.

THEOREM 15 (Tomas-Stein). *For every dimension $n \ge 2$ there is a constant $C(n)$ such that for all $f \in L^p(\mathbb{R}^n)$*

$$\|\hat{f} \restriction S^{n-1}\|_{L^2(S^{n-1})} \le C(n)\|f\|_{L^p(\mathbb{R}^n)} \tag{77}$$

*with $p \le p_n =$
$frac2n+2n+3$. Moreover, this bound fails for $p > p_n$.*

The left-hand side in (77) is $\left(\int_{S^{n-1}} |\hat{f}(w)|^2 d\sigma(w)\right)^{\frac{1}{2}}$, where σ is the surface measure on S^{n-1}.

We shall prove this theorem in this section, but first some remarks are in order. Firstly, there is nothing special about the sphere. In fact, if S_0 is a compact subset of a hypersurface S with nonvanishing Gaussian curvature, then

$$\|\hat{f} \restriction S_0\|_{L^2(S_0)} \le C(n, S_0)\|f\|_{L^{\frac{2n+2}{n+3}}(\mathbb{R}^n)} \tag{78}$$

for any $f \in \mathcal{S}(\mathbb{R}^n)$. For example, take the truncated paraboloid

$$S_0 := \{(\xi', |\xi'|^2) | \xi' \in \mathbb{R}^{n-1}, |\xi'| \le 1\}$$

which is important for the Schrödinger equation. On the other hand, (78) fails for

$$S_0 := \{(\xi', |\xi'|) | \xi' \in \mathbb{R}^{n-1}\ , 1 \le |\xi'| \le 2\}$$

since this piece of the cone has exactly one vanishing principal curvature, namely the one along a generator of the one. This latter example is of course relevant for the wave equation, and we will need to find a substitute of (78) for the wave equation. A much simpler remark concerns the range $1 \leq p \leq p_n$ in Theorem 15: for $p = 1$, one has

$$\|\hat{f} \restriction S^{n-1}\|_{L^2(S^{n-1})} \leq \|\hat{f} \restriction S^{n-1}\|_\infty |S^{n-1}|^{\frac{1}{2}} \leq \|f\|_{L^1(\mathbb{R}^n)} |S^{n-1}|^{\frac{1}{2}} \ . \tag{79}$$

Hence, it suffices to prove Theorem 15 for $p = p_n$ since the cases $1 \leq p < p_n$ follows by interpolation with (79). The Stein-Tomas theorem is more accessible than the general restriction conjecture, see below, because of the appearance of $L^2(S^{n-1})$ on the left-hand side. This allows one to use duality in the proof. To do so, we need to identify the adjoint of the restriction operator

$$R : f \mapsto \hat{f} \restriction S^{n-1} \ .$$

LEMMA 24. *For any finite measure μ in $\mathbb{R}^n$, and any $f, g \in \mathcal{S}(\mathbb{R}^n)$ one has the identity*

$$\int_{\mathbb{R}^n} \hat{f}(\xi)\bar{\hat{g}}(\xi) d\mu(\xi) = \int_{\mathbb{R}^n} f(x)(\bar{g} * \hat{\mu})(x) dx \ .$$

PROOF. We use the following elementary identity for tempered distributions: If μ is a finite measure, and $\phi \in \mathcal{S}$, then

$$\widehat{\phi\mu} = \hat{\phi} * \hat{\mu} \ .$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} \hat{f}(\xi)\bar{\hat{g}}(\xi)\, d\mu(\xi) &= \int_{\mathbb{R}^n} f(x)\widehat{\bar{\hat{g}}\mu}(x)\, dx = \\ &= \int_{\mathbb{R}^n} f(x)(\hat{\bar{\hat{g}}} * \hat{\mu})(x)\, dx = \int_{\mathbb{R}^n} f(x)(\bar{g} * \hat{\mu})(x)\, dx \end{aligned}$$

since $\hat{\bar{\hat{g}}} = \bar{\hat{\hat{g}}} = \bar{g}$. $\square$

LEMMA 25. *Let μ be a finite measure on $\mathbb{R}^n$, and $g \geq 2$. Then the following are equivalent:*

a) $\|\widehat{f\mu}\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mu)}$ *for all* $f \in \mathcal{S}\ (\mathbb{R}^n)$
b) $\|\hat{g}\|_{L^2(\mu)} \leq C\|g\|_{L^{q'}(\mathbb{R}^n)}$ *for all* $g \in \mathcal{S}\ (\mathbb{R}^n)$
c) $\|\hat{\mu} * f\|_{L^q(\mathbb{R}^n)} \leq C^2\|f\|_{L^{q'}(\mathbb{R}^n)}$ *for all* $f \in \mathcal{S}\ (\mathbb{R}^n)$.

PROOF. By the previous lemma, for any $g \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\hat{g}\|_{L^2(\mu)} = \sup_{\|f\|_{L^2(\mu)}=1, f\in\mathcal{S}} \left| \int_{\mathbb{R}^n} \hat{g}(\xi) f(\xi)\, d\mu(\xi) \right| = \sup_{f\in\mathcal{S}, \|f\|_{L^2(\mu)}=1} \left| \int_{\mathbb{R}^n} g(x)\widehat{f\mu}(x)\, dx \right| \tag{80}$$

Hence, if a) holds, then the right-hand side of (80) is no larger than $\|g\|_{L^{q'}(\mathbb{R}^n)}$ and b) follows. Conversely, if b) holds, then the entire expression in (80) is no larger than $C\|g\|_{L^{q'}(\mathbb{R}^n)}$, which implies a). Thus a) and b) are equivalent with the same choice of C. Clearly, applying first b) and then a) with $f = \hat{g}$ yields

$$\|\widehat{\hat{g}\mu}\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^{q'}(\mathbb{R}^n)}$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$. Since $\widehat{g\mu} = g(-\cdot) * \hat{\mu}$, c) follows.

$$\int g(x)\,(\hat{\mu}, *f)\,(x)\;dx = \int g(x)\widehat{(\mu \check{f})}(x)\;dx, = \int \hat{g}(\xi)\check{f}(\xi)\;d\mu(\xi)\;.$$

for any $f, g \in \mathcal{S}(\mathbb{R}^n)$. Hence, if c) holds then

$$\left|\int_{\mathbb{R}^n} \hat{g}(\xi)\check{f}(\xi)\;d\mu(\xi)\right| \le C^2 \|g\|_{L^{q'}(\mathbb{R}^n)} \|f\|_{L^{q'}(\mathbb{R}^n)}\;.$$

Now set $f(x) = g(-x)$. Then

$$\int_{\mathbb{R}^n} |\hat{g}(\xi)|^2\;d\mu(\xi) \le C^2 \|g\|^2_{L^{q'}(\mathbb{R}^n)}\;,$$

which is b). □

Setting $\mu = \sigma = \sigma_{S^{n-1}}$, the surface measure of the unit sphere in $\mathbb{R}^n$, one now obtains the following:

COROLLARY 15. *The following assertions are equivalent*

a) The Stein-Tomas theorem in the "restriction form":

$$\|\hat{f} \restriction S^{n-1}\|_{L^2(\sigma)} \le C\|f\|_{L^{q'}(\mathbb{R}^n)}$$

for $q' = \frac{2n+2}{n+3}$ *and all* $f \in \mathcal{S}(\mathbb{R}^n)$

b) the "extension form" of the Stein-Tomas theorem

$$\|\widehat{g\sigma_{S^{n-1}}}\|_{L^q(\mathbb{R}^n)} \le C\|g\|_{L^2(\sigma)}$$

for $q = \frac{2n+2}{n-1}$ *and all* $g \in \mathcal{S}(\mathbb{R}^n)$.

c) The composition of a) and b): for all $f \in \mathcal{S}(\mathbb{R}^n)$

$$\|f * \widehat{\sigma_{S^{n-1}}}\|_{L^q(\mathbb{R}^n)} \le C^2\|f\|_{L^{q'}(\mathbb{R}^n)}$$

with $q = \frac{2n+2}{n-1}$.

PROOF. Set $\mu = \sigma_{S^{n-1}} = \sigma$ in Lemma 21. □

EXERCISE 27. *In general, a) and b) above remain true, whereas c) requires* $L^2(\sigma)$. *More precisely, show the following: The restriction estimate*

$$\|\hat{f} \restriction S^{n-1}\|_{L^p(\sigma)} \le C\|f\|_{L^{q'}(\mathbb{R}^n)} \forall g \in \mathcal{S}$$

is equivalent to the extension estimate

$$\|\widehat{g\sigma_{S^{n-1}}}\|_{L^q(\mathbb{R}^n)} \le C\|g\|_{L^{p'}(\sigma)} \forall g \in \mathcal{S}\;.$$

We now show via part b) of Corollary 15 that the Stein-Tomas theorem is optimal. This is the well-known Knapp example.

LEMMA 26. *The exponent* $p_n = \frac{2n+2}{n+3}$ *in Theorem 15 is optimal.*

Proof. This is equivalent to saying that the exponent $q = \frac{2n+2}{n-1}$ in part $b)$ of the previous corollary is optimal. Fix a small $\delta > 0$ and let $g \in \mathcal{S}$ such that $g = 1$ on $B(e_n; \sqrt{\delta})$, $g \geq 0$, and $\mathrm{supp}(g) \subset B(e_n; 2\sqrt{\delta})$ where $e_n = (0, \ldots, 0, 1)$

Then

$$
\begin{aligned}
|\widehat{g\sigma}(\xi)| &= \left| \int e^{-2\pi i [x' \cdot \xi' + \xi_n (\sqrt{1-|x'|^2} - 1)]} \frac{g(x', \sqrt{1-|x'|^2})}{\sqrt{1-|x'|^2}}\, dx' \right| \\
&\geq \left| \int \cos(2\pi (x' \cdot \xi' + \xi_n (\sqrt{1-|x'|^2} - 1))) \frac{g(x', \sqrt{1-|x'|^2})}{\sqrt{1-|x'|^2}}\, dx' \right| \qquad (81) \\
&\geq \cos\frac{\pi}{4} \int g \, d\sigma \geq C^{-1} \left(\sqrt{\delta}\right)^{n-1}
\end{aligned}
$$

provided $|\xi'| \leq \frac{(\sqrt{\delta})^{-1}}{100}$, $|\xi_n| \leq \frac{\delta^{-1}}{100}$. Indeed, under these assumptions, and for $\delta > 0$ small,

$$
\begin{aligned}
|x' \cdot \xi' + \xi_n \left(\sqrt{1-|x'|^2} - 1 \right) | &\leq \sqrt{\delta} \cdot \frac{(\sqrt{\delta})^{-1}}{100} + \frac{\delta^{-1}}{100} \left(\sqrt{\delta}\right)^2 \\
&\leq \frac{1}{50} \,,
\end{aligned}
$$

so that the argument of the cosine in (81) is smaller than $\frac{2\pi}{50} \leq \frac{\pi}{4}$ in absolute value, as claimed.

Hence,

$$
\begin{aligned}
\|\widehat{g\sigma_{S^{n-1}}}\|_{L^q(\mathbb{R}^n)} &\geq C^{-1} \delta^{\frac{(n-1)}{2}} \cdot \left(\delta^{-\frac{(n-1)}{2}} \cdot \delta^{-1} \right)^{\frac{1}{q}} \\
&= C^{-1} \delta^{\frac{n-1}{2}} \delta^{-\frac{(n+1)}{2q}}
\end{aligned}
$$

whereas $\|g\|_{L^2(\sigma)} \leq C\delta^{\frac{n-1}{4}}$. It is therefore necessary that

$$
\frac{n-1}{4} \leq \frac{n-1}{2} - \frac{n+1}{2q}
$$

or $q \geq \frac{2n+2}{n-1}$, as claimed. $\square$

For the proof of Theorem 15 we need the following decay estimate for the Fourier transform of the surface measure $\sigma_{S^{n-1}}$:

$$
\left| \widehat{\sigma_{S^{n-1}}}(\xi) \right| \leq C(1 + |\xi|)^{-(\frac{n-1}{2})} \qquad (82)
$$

This is a well-known fact that is most easily proved by means of the method of stationary phase, see Tom Wolff's notes, for example, or Hömander's book. It is easy to see that (82) imposes a restriction on the possible exponents for an extension theorem of the form

$$
\|\widehat{f\sigma_{S^{n-1}}}\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(S^{n-1})} \,. \qquad (83)
$$

Indeed, setting $f = 1$ implies that one needs

$$
q > \frac{2n}{n-1}
$$

by (82). On the other hand, one has

EXERCISE 28. *Check by means of Knapps's example from Lemma 22 that* (83) *can only hold for*

$$q \geq \frac{n+1}{n-1} p' \ . \tag{84}$$

It is a famous conjecture of Elias Stein that (83) *holds under these conditions, i.e., provided*

$$\infty \geq q > \frac{2n}{n-1} \quad \text{and} \quad q \geq \frac{n+1}{n-1} p' \ .$$

Observe that the Stein-Tomas theorem with $q = \frac{2n+2}{n-1}$ *and* $p = 2$ *is a partial result in this direction. In two dimensions the conjecture was proved in the early 1970's by Fefferman and Stein. But in higher dimensions it appears to be a very difficult problem with only little progress for* $n \geq 3$. *It is known, see Wolff's notes, that the full restriction conjecture implies the Kakeya conjecture on the Hausdorff dimensions of Kakeya sets in dimension* $n \geq 3$. *This latter conjecture appears to be also very difficult.*

PROOF OF THE STEIN-TOMAS THEOREM FOR $p < \frac{2n+2}{n+3}$. Let $\sum_{j \in \mathbb{Z}} \psi(2^{-j}x) =$ for $x \neq 0$ be the usual partition of unity. By (Lemma 21 and) Corollary 15 it is necessary and sufficient to prove

$$\|f * \widehat{\sigma_{S^{n-1}}}\|_{L^{p'}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Firstly, let $\varphi(x) = 1 - \sum_{j \geq 0} \psi(2^{-j}x)$; clearly, $\varphi \in C_0^\infty(\mathbb{R}^n)$, and

$$1 = \varphi(x), + \sum_{j \geq 0} \psi(2^{-j}x) \ \text{ for all } \ x \in \mathbb{R}^n \ .$$

Now observe that $\varphi \widehat{\sigma_{S^{n-1}}} \in C_0^\infty$ so that

$$\|f * \varphi \widehat{\sigma_{S^{n-1}}}\|_{L^{p'}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \tag{85}$$

with $C = \|\varphi \widehat{\sigma_{S^{n-1}}}\|_{L^r}$ where $1 + \frac{1}{p'} = \frac{1}{r} + \frac{1}{p}$, i.e., $\frac{2}{p'} = \frac{1}{r}$. It therefore remains to control $K_j := \psi(2^{-j}x) \widehat{\sigma_{S^{n-1}}}(x)$ in the sense

$$\|f * K_j\|_{L^{p'}(\mathbb{R}^n)} \leq C 2^{-j\epsilon} \|f\|_{L^p(\mathbb{R}^n)} \forall f \in \mathcal{S}(\mathbb{R}^n) \tag{86}$$

for all $j \geq 0$ and some small $\epsilon > 0$. It is clear that the desired bound follows by summing (85) and (86) over $j \geq 0$. To prove (86) we interpolate a $2 \longrightarrow 2$ and $1 \longrightarrow \infty$ bound as follows:

$$\begin{aligned} \|f * K_j\|_{L^2} &= \|\hat{f}\|_{L^2} \|\hat{K}_j\|_{L^\infty} = \\ &= \|f\|_{L^2} \|2^{nj} \hat{\psi}(2^j \cdot) * \sigma_{S^{n-1}}\|_{L^\infty} \\ &\leq C \|f\|_{L^2} 2^{nj} \cdot 2^{-j(n-1)} = C 2^j \|f\|_{L^2} \ . \end{aligned} \tag{87}$$

To pass to the estimate (87) one basically uses that $\sup_x \sigma_{S^{n-1}}(B(x,r)) \leq C r^{n-1}$.

On the other hand,

$$\begin{aligned} \|f * K_j\|_\infty &\leq \|K_j\|_\infty \|f\|_1 \\ &\leq C 2^{-j \frac{(n-1)}{2}} \|f\|_1 \ , \end{aligned} \tag{88}$$

since the size of K_j is controlled by (82). Interpolating (87) with (88) yields

$$\|f * K_j\|_{p'} \leq C 2^{-j\frac{n-1}{2}\theta} 2^{j(1-\theta)} \|f\|_p$$

where $\frac{1}{p'} = \frac{\theta}{\infty} + \frac{1-\theta}{2} = \frac{1-\theta}{2}$. We thus obtain (86) provided

$$\begin{aligned} 0 <& \frac{n-1}{2}\theta - (1-\theta) = \frac{n+1}{2}\theta - 1 = \\ &= (n+1)\left(\frac{1}{2} - \frac{1}{p'}\right) - 1 = \frac{n-1}{2} - \frac{n+1}{p'} \ . \end{aligned}$$

This is the same as $p' > \frac{2n+2}{n-1}$ or $p < \frac{2n+2}{n+3}$, as claimed. $\square$

CHAPTER 9

The endpoint for the Stein-Tomas theorem and Strichartz estimates

The proof of the Stein-Tomas theorem in Chapter 88 fails because (86) leads to a divergent series with $\epsilon = 0$, which is the case of $p = \frac{2n+2}{n+3}$. One therefore has to avoid interpolating the operator bounds on each dyadic piece separately. The idea is basically to sum *first* and *then* to interpolate, rather than interpolating first and *then* summing. Of course, one needs to explain what it means to sum first: Recall that the proof of the Riesz-Thorin interpolation theorem is based on the three lines theorem from complex analysis. The key idea in our context is to sum the dyadic pieces $T_j : f \mapsto f * K_j$ together with *complex weights* $w_j(z)$ in such a way that

$$T_z := \sum_{j\geq 0} w_j(z) T_j$$

converges on the strip $0 \leq \Re z \leq 1$ to an analytic, operator valued function with the property that

$$T_z : L^1 \longrightarrow L^\infty \text{ for } \Re z = 1$$

and

$$T_z : L^2 \longrightarrow L^2 \text{ for } \Re z = 0$$

It then follows that $T_\theta = L^p(\mathbb{R}^n) \longrightarrow L^{p'}(\mathbb{R}^n)$ for $\frac{1}{p'} = \frac{1-\theta}{2}$.

Although this describes the idea behind complex interpolation, it is rarely implemented in this fashion. Rather, one tries to embed the operator under consideration into an analytic family that is analytically tractable.

Proof of the endpoint for Tomas-Stein. We shall consider a surface of non-zero curvature which can be written as a graph: $\xi_n = h(\xi')$, $\xi' \in \mathbb{R}^{n-1}$. Define

$$M_z(\xi) = \frac{1}{\Gamma(z)}\big(\xi_n - h(\xi')\big)_+^{z-1}\chi_1(\xi')\chi_2\big(\xi_n - h(\xi')\big) \tag{89}$$

where $\chi_1 \in C_0^\infty(\mathbb{R}^{n-1})$, $\chi_2 \in C_0^\infty(\mathbb{R})$ are smooth cut-off-functions, Γ is the Gamma-function, and $\Re z > 0$. We will show that

$$T_z f := (M_z \hat{f})^\vee \tag{90}$$

can be defined by means of analytic continuation to $\Re z \leq 0$. Moreover,

$$\|T_z\|_{2\longrightarrow 2} \leq B(z) \quad \text{for } \Re z = 1 \ . \tag{91}$$

$$\|T_z\|_{1\longrightarrow\infty} \le A(z) \quad \text{for} \quad \Re z = -\frac{n-1}{2} \tag{92}$$

where $A(z)$, $B(z)$ grow now faster than $e^{C|z|^2}$ as $|\Im z| \longrightarrow \infty$.

Since one easily checks that

$$M_0(\xi) = \chi_1(\xi')\delta_0(\xi_n - h(\xi'))\, d\xi' \;, \tag{93}$$

see (96), it follows from Stein's complex interpolation theorem that

$$f \mapsto \widehat{M_0} * f$$

dent is bounded from $L^p \longrightarrow L^{p'}$ where

$$\frac{1}{p'} = \frac{\theta}{\infty} + \frac{1-\theta}{2} \;,\; 0 = -\theta\frac{n-1}{2} + 1 - \theta$$

which implies that

$$\frac{1}{p'} = \frac{n-1}{2n+2} \;,$$

as desired. It remains to check (91)–(93). To do so, recall firstly that $\frac{1}{\Gamma(z)}$ is an entire function with zeros at $z = 0,\,-1,\,-2\,,\ldots$. It has the product representation

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{\nu=1}^{\infty}\left(1+\frac{z}{\nu}\right)e^{-z/\nu} \;,$$

t where $z = x + iy$. Thus,

$$\begin{aligned}
\left|\frac{1}{\Gamma(z)}\right|^2 &\le |z|^2, e^{2\gamma x}\prod_{\nu=1}^{\infty}\left[\left(1+\frac{x}{\nu}\right)^2+\frac{y^2}{\nu^2}\right]e^{-\frac{2}{\nu}x} \\
&\le |z|^2e^{2\gamma x}\prod_{\nu=1}^{\infty}\left[e^{\frac{2x}{\nu}+\frac{|z|^2}{\nu^2}}e^{\frac{-2x}{\nu}}\right] \\
&= |z|^2e^{2\gamma x}e^{|z|^2\frac{\pi^2}{6}} \;.
\end{aligned} \tag{94}$$

In particular, if $\Re z = 1$, then

$$\begin{aligned}
|M_z(\xi)| &\le (1+y^2)e^{2\gamma}e^{(1+y^2)\frac{\pi^2}{6}}\chi_1(\xi')\chi_2\big(\xi_n - h(\xi')\big) \\
&\le Ce^{cy^2} \qquad\qquad \text{for all } \xi \;.
\end{aligned}$$

Therefore, (91) holds with the stated bound on $B(z)$. Now let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Thus, for $\Re z > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^n} M_z(\xi)\varphi(\xi)\, d\xi &= \frac{1}{\Gamma(z)}\int_{\mathbb{R}^{n-1}}\int_0^\infty \chi_2(t)\varphi(\xi', t+h(\xi'))t^{z-1}\, dt\chi_1(\xi')\, d\xi' \\
&= -\frac{1}{z\Gamma(z)}\int_{\mathbb{R}^{n-1}}\int_0^\infty \frac{d}{dt}\left[\chi_2(t)\varphi\big(\xi', t+h(\xi')\big)\right]t^z\, dt\chi_1(\xi')\, d\xi'
\end{aligned} \tag{95}$$

Observe that the right-hand side is well-defined for $\Re z > -1$. Furthermore, at $z = 0$, using $z\Gamma(z)\big|_{z=0} = 1$

$$\int_{\mathbb{R}^n} M_0(\xi)\varphi(\xi)\, d\xi = \int_{\mathbb{R}^{n-1}} \chi_2(0)\varphi(\xi', h(\xi'))\chi_1(\xi')\, d\xi'$$

which shows that the analytic continuation of M_z to $z=0$ is equal to (setting $\chi_2(0)=1$)

$$M_0(\xi) = \chi_1(\xi')\, d\xi' \delta_0(\xi_n - h(\xi'))\ . \tag{96}$$

Clearly, M_0 is proportional to surface measure on a piece of the surface

$$S = \left\{(\xi', h(\xi'))|\xi' \in \mathbb{R}^{n-1}\right\}\ .$$

This is exactly what we want, since we need to bound $\widehat{\sigma_S} * f$.

Now recall that (95) defined the analytic continuation to $\Re z > -1$. Integrating by parts again extends this to $\Re z > -2$ and so forth. Indeed, the right-hand side of

$$\begin{aligned}\int_{\mathbb{R}^n} M_z(\xi)\varphi(\xi)\, d\xi = \frac{(-1)^k}{z(z+1)\cdot\ldots\cdot(z+k-1)\Gamma(z)} \int_{\mathbb{R}^{n-1}} \chi_1(\xi') \int_0^\infty t^{z+k-1} \\ \frac{d^k}{dt^k}\left[\chi_2(t)\varphi(\xi', \xi_n + k(\xi'))\right] dt\, d\xi'\end{aligned}$$

is well-defined for all $\Re z > -k$.

Next we prove (92) by means of an estimate on $\|\widehat{M_z}\|_\infty$. This requires the following preliminary calculation:

Let N be a positive integer such that $N > \Re z + 1 > 0$. Then we claim that

$$\left|\int_0^\infty e^{-2\pi i t\tau} t^z \chi_2(t)\, dt\right| \le \frac{C_N(1+|z|)^N}{1+\Re z}(1+|\tau|)^{-\Re z - 1}\ . \tag{97}$$

To prove (97) we will distinguish large and small $t\tau$. Let $\psi \in C_0^\infty(\mathbb{R})$ be such that $\psi(t) = 1$ for $|t| \le 1$ and $\psi(t) = 0$ for $|t| \ge 2$. Then, since $0 \le \chi_2 \le 1$,

$$\begin{aligned}&\left|\int_0^\infty e^{-2\pi i t\tau} t^z \psi(t\tau)\chi_2(t)\, dt\right| \le \\ &\le \int_0^\infty t^{\Re z}\psi(t|\tau|)\, dt \le |\tau|^{-\Re z - 1}\int_0^2 t^{\Re z}\, dt \\ &\le \frac{C}{\Re z + 1}, |\tau|^{-\Re z - 1}\ .\end{aligned} \tag{98}$$

If $|\tau| \le\ 1$, then (98) is no larger than

$$\int_0^\infty t^{\Re z}\chi_2(t)\, dt \le \frac{C}{\Re z + 1}\ .$$

Hence

$$(98)\ \le \frac{C}{1+\Re z}(1+|\tau|)^{-\Re z - 1} \tag{99}$$

in all cases. To treat the case $t\tau$ large, which implies that τ is large, we exploit cancellation in the phase. More precisely,

$$
\begin{aligned}
&\left|\int_0^\infty e^{-2\pi i t\tau} t^z (1-\psi(t\tau))\chi_2(t)\,dt\right| \leq \\
&\leq \left(\frac{1}{2\pi|\tau|}\right)^N \int_0^\infty \left|\frac{d^N}{d\tau^N}\left[t^z(1-\psi(t\tau))\chi_2(t)\right]\right| dt \\
&\leq C_N \left(\frac{1}{2\pi|\tau|}\right)^N \int_0^\infty dt \left[|z(z-1)\cdot\ldots\cdot(z-N+1)| t^{\Re z - N}(1-\psi(t\tau))\chi_2(t)\right. \\
&\quad \left. + t^{\Re z}|\psi^{(N)}(t\tau)|\tau^N \chi_2(t) + t^{\Re z}(1-\psi(t\tau))|\chi_2^{(N)}(t)|\right] \\
&\leq C_N \left(\frac{1}{2\pi|\tau|}\right)^N |z(z-1)\cdot\ldots\cdot(z-N+1)| \int_{\frac{1}{\tau}}^\infty t^{\Re z - N}\,dt + \frac{C}{(2\pi|\tau|)^N}|\tau|^N|\tau|^{-\Re z - 1} \\
&+ C|\tau|^{-N}\int_0^1 t^{\Re z}|\chi_2^{(N)}(t)|\,dt
\end{aligned} \tag{100}
$$

$$
\leq C_N\left[|z(z-1)\cdot\ldots\cdot(z-N+1)|+1\right]|\tau|^{-\Re z-1} + C_N|\tau|^{-N}\,. \tag{101}
$$

Observe that the indefinite integral in (100) converges because of $\Re z - N < -1$. Moreover, the second term in (101) is $\leq |\tau|^{\Re z - 1}$ by the same condition (recall that we are taking τ to be large). Hence (97) follows from(99) and (101). We now compute $\widehat{M_z}(x)$. Let k be a positive integer with $\Re z > -k$. Then

$$
\begin{aligned}
\widehat{M_z}(x) &= \int e^{-2\pi i x\cdot\xi}\frac{1}{\Gamma(z)}(\xi_n - h(\xi'))_+^{z-1}\chi_1(\xi')\chi_2(\xi_n - h(\xi'))\,d\xi'\,d\xi_n \\
&= \frac{1}{\Gamma(z)}\int_0^\infty e^{-2\pi i x_n t}t^{z-1}\chi_2(t)\,dt \int_{\mathbb{R}^{n-1}} e^{-2\pi i[x'\cdot\xi' + x_n h(\xi')]}\chi_1(\xi')\,d\xi' \\
&= \frac{(-1)^k}{\Gamma(z) z\cdot(z-1)\cdot\ldots\cdot(z-k+1)}\int_0^\infty \left(e^{-2\pi i x_n t}\chi_2(t)\right)^{(k)} t^{z+k-1}\,dt\cdot \\
&\cdot \int_{\mathbb{R}^{n-1}} e^{-2\pi i[x'\cdot\xi' + x_n h(\xi')]}\chi_1(\xi')\,d\xi'
\end{aligned} \tag{102}
$$

where the final expression is well-defined for $\Re z > -k$. We are interested in $\Re z = -\frac{n-1}{2}$; so pick $k \in \mathbb{Z}^+$ such that $1-k \leq \Re z < -k+2$, ie, $\frac{n+1}{2} \leq k < \frac{n+1}{2}+1$. Now apply (97) with $z+k-1$ instead of z and with $N=2$. Then the first integral (102) is bounded by

$$
\begin{aligned}
&\left|\int_0^\infty \left(e^{-2\pi i x_n t}\chi_2(t)\right)^{(k)} t^{z+k-1}\,dt\right| \leq \\
&\leq \frac{C(1+|z|)^2}{\Re z + k}(1+|x_n|)^{-\Re z - k}\cdot C_k(1+|x_n|)^k \\
&\leq C_k(1+|z|)^2(1+|x_n|)^{-\Re z}\,.
\end{aligned} \tag{103}
$$

On the other hand, the second integral in (102) is controlled by the stationary phase estimate, cf. (82),

$$\left| \int_{\mathbb{R}^{n-1}} e^{-2\pi i[x'\cdot\xi' + x_n h(\xi')]} \chi_1(\xi')\, d\xi' \right| \leq C(1+|x|)^{-\left(\frac{n-1}{2}\right)} \tag{104}$$

Observe that the growth in $|x_n|$ for $\Re z = -\frac{n-1}{2}$ is exactly balanced by the decay in (104). One concludes that for $\Re z = -\frac{(n-1)}{2}$ and with k as in (103)

$$\|\widehat{M_z}\|_\infty \leq C_n \left| \frac{1}{\Gamma(z) z \cdot (z-1) \cdot \ldots \cdot (z-k+1)} \right| (1+|z|)^2 ,$$

see (102)–(104). Thus (92) follows from the growth estimate (94). $\square$

Now suppose we have the Schrödinger equation

$$\left[\begin{array}{rcl} \frac{1}{i}\partial_t u + \frac{1}{2\pi} \triangle_{\mathbb{R}^n} u & = & 0 \\ u|_{t=0} & = & f . \end{array} \right.$$

Then

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi + t|\xi|^2)} \hat{f}(\xi)\, d\xi \\ &= (\hat{f}\mu)^\vee(t,x) \end{aligned} \tag{105}$$

where μ is the measure in $\mathbb{R}^{n+1}$ defined

$$\int_{\mathbb{R}^{n+1}} F(\xi,\tau)\, d\mu(\xi,\tau) = \int_{\mathbb{R}^n} F(\xi,|\xi|^2)\, d\xi$$

for all $F \in C^0(\mathbb{R}^{n+1})$.

Now let $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$, $\varphi(\xi,\tau) = 1$ if $|\xi|+|\tau| \leq 1$. Then the endpoint of Stein-Tomas applies and one has

$$\|(\hat{f}\varphi\mu)^\vee\|_{L^q(\mathbb{R}^{n+1})} \leq C\|\hat{f}\|_{L^2(\varphi\mu)}$$

where $q = \frac{2n+4}{n} = 2 + \frac{4}{n}$. In other words, if $\mathrm{supp}\hat{f} \subset B(0,1)$, then

$$\|(\hat{f}\mu)^\vee\|_{L^{2+\frac{4}{n}}(\mathbb{R}^{n+1})} \leq C\|\hat{f}\|_{L^2(\mathbb{R}^n)} = C\|f\|_{L^2(\mathbb{R}^n)} . \tag{106}$$

This is the well-known Strichartz estimate under the additional condition $\mathrm{supp}\hat{f} \subset B(0,1)$. To remove it, one can rescale. Let

$$f_\lambda(x) = f(x/\lambda) \text{ and}$$
$$u_\lambda(x,t) = u(x/\lambda, t/\lambda^2) .$$

Then

$$\left[\begin{array}{rcl} \frac{1}{i}\partial_t u_\lambda + \frac{1}{2\pi} \triangle u_\lambda & = & 0 \\ u_\lambda|_{t=0} & = & f_\lambda \end{array} \right.$$

If $\mathrm{supp}\hat{f}$ is compact, then $\mathrm{supp}\hat{f}_\lambda \subset B(0,1)$ if λ is large. Hence, in view of (105) and (106),

$$\|u_\lambda\|_{L^q(\mathbb{R}^{n+1})} \leq C\|f_\lambda\|_{L^2(\mathbb{R}^n)} . \tag{107}$$

However,

$$\|f_\lambda\|_{L^2(\mathbb{R}^n)} = \lambda^{\frac{n}{2}} \|f\|_{L^2(\mathbb{R}^n)} \ ,$$

and

$$\|u_\lambda\|_{L^q(\mathbb{R}^{n+1})} = \lambda^{\frac{n+2}{q}} \|u\|_{L^q} = \lambda^{\frac{n}{2}} \|u\|_{L^q} \ .$$

Hence (107) is the same as

$$\|u\|_{L^gq(\mathbb{R}^{n+1})} \leq C\|f\|_{L^2(\mathbb{R}^n)} \tag{108}$$

for all $f \in \mathcal{S}$ will supp$\hat{f}$ compact. These functions are dense in $L^2(\mathbb{R}^n)$, and (108), which is the Strichartz bound for the Schrödinger equation in $n+1$ dimensions, follows for all $f \in L^2(\mathbb{R}^n)$.

For the inhomogeneous equation

$$\left[\begin{array}{rcl} \frac{1}{i}\partial_t u + \frac{1}{2\pi} \triangle_{\mathbb{R}^n} u &=& F \\ u|_{t=0} &=& 0 \end{array} \right. \tag{109}$$

use Duhamel's principle and dispersive inequality for the free evolution, i.e.,

$$\|e^{-\frac{i}{2\pi}\triangle t} f\|_{L^{p'}(\mathbb{R}^n)} \leq C|t|^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{p'})} \|f\|_{L^p(\mathbb{R}^n)}$$

for $1 \leq p \leq 2$. Thus, if u solves (109), then

$$\begin{aligned} \|u(t,\cdot)\|_{L^{p'}} &\leq \int_0^t C|t-s|^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{p})} \|F(s,\cdot)\|_p \, ds \\ &\leq C \int_{-\infty}^{\infty} |t-s|^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{p'})} \|F(s,\cdot)\|_p \, ds \end{aligned}$$

Hence

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} \leq C\|F\|_{L^p(\mathbb{R}^{n+1})}$$

provided

$$1 + \frac{1}{p'} = \frac{1}{p} + \frac{n}{2}\left(\frac{1}{p} - \frac{1}{p'}\right) \tag{110}$$

and

$$0 < \frac{n}{q}\left(\frac{1}{p} - \frac{1}{p'}\right) < 1 \ . \tag{111}$$

Now (110) is the same as

$$\frac{2}{p'} = \frac{n}{2}\left(1 - \frac{2}{p'}\right) = \frac{n}{2} - \frac{n}{p'}$$

or

$$\frac{1}{p'} = \frac{n}{2(n+2)} \ , \ p' = 2 + \frac{4}{n}$$

and (111) also holds.

Hence, solutions of the general equation

$$\left\{ \begin{array}{rcl} \frac{1}{i}\partial_t u + \frac{1}{2\pi} \triangle u &=& F \\ u|_{t=0} &=& f \end{array} \right.$$

satisfy

$$\|u\|_{L^{2+\frac{4}{n}}(\mathbb{R}^{n+1})} \leq C\big[\|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\frac{2n+4}{n+4}}(\mathbb{R}^{n+1})}\big]$$

for every $n \geq 1$.

The case of the wave equation:

Let $\Gamma = \{x \in \mathbb{R}^n \mid 1 \leq |x'| = x_n \leq 2\}$ be a section of a light cone in $\mathbb{R}^n$. Since the cone has one vanishing principal curvature, the Fourier transform of the surface measure on Γ decays as follows:

$$|\widehat{\varphi\sigma_\Gamma}(\xi)| \leq C(1+|\xi|)^{-\frac{(n-2)}{2}} \ . \tag{112}$$

It is not hard to see that (112) is optimal for all directions ξ belonging to the dual cone Γ^* (which is equal to Γ if the opening angle is $90°$). It is quite clear that the complex interpolation method from above therefore implies that there is the following restriction estimate for Γ:

$$\|\hat{f} \restriction \Gamma\|_{L^2(\sigma_\Gamma)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

where $p = \frac{2n}{n+2}$ and $n \geq 3$.

Theorem 16. *Let* Γ *be the cone in* $\mathbb{R}^n, n \geq 3$, *equipped with the measure* $d\mu(\xi) = \frac{d\xi'}{|\xi'|}$.

Then

$$\left(\int_\Gamma |\hat{f}(\xi)|^2 \, d\mu(\xi)\right)^{\frac{1}{2}} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{113}$$

with $p = \frac{2n}{n+2}$.

Proof. Let Γ be the cone restricted to $1 \leq |\xi_n| \leq 2$. Then

$$\begin{aligned}\left(\int_{\lambda\leq|\xi|\leq 2\lambda} |\hat{f}(\xi)|^2 \, d\mu(\xi)\right)^{\frac{1}{2}} &\leq \left(\int_\Gamma |\hat{f}(\lambda\xi)|^2 d\mu(\lambda\xi)\right)^{\frac{1}{2}} \\ &\leq C\lambda^{\frac{n-2}{2}} \|\frac{1}{\lambda^n} f\left(\frac{1}{\lambda}\cdot\right)\|_{L^p(\mathbb{R}^n)} \\ &= C\lambda^{\frac{n-2}{2}} \cdot \lambda^{-n} \cdot \lambda^{\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)} = C\|f\|_{L^p(\mathbb{R}^n)}\end{aligned} \tag{114}$$

since

$$\frac{n-2}{2} - n + \frac{n}{p} = -\frac{n+2}{n} + \frac{n+2}{n} = 0 \ .$$

To sum up (114) for $\lambda = 2^j$ we use the Littlewood-Paley theorem: Since $p < 2$,

$$\begin{aligned}\|\hat{f}\|_{L^2(\mu)} &\leq C\left(\sum_j \| \triangle_j f\|_{L^p}^2\right)^{\frac{1}{2}} \\ &\leq C\left\|\left(\sum_j |\triangle_j f|^2\right)^{\frac{1}{2}}\right\|_{L^p} \\ &\leq C\|f\|_{L^p(\mathbb{R}^n)} \ ,\end{aligned}$$

which is (113). ☐

Now suppose we have the wave equation

$$\begin{cases} (\partial_t^2 - \triangle_{\mathbb{R}^n})\, u = 0 \\ u\,|_{t=0} = 0\ , \quad \partial_t u|_{t=0} = f \end{cases}$$

then

$$\begin{aligned} u(t,x) &= \int e^{2\pi i(t|\xi|+x\xi)} \hat{f}(\xi) \frac{d\xi}{4\pi i|\xi|} + \int e^{2\pi i(-t|\xi|+x\cdot\xi)} \hat{f}(\xi) \frac{d\xi}{4\pi i|\xi|} \\ &= (F\mu)^{\vee}(x,t) \text{ where } F(\xi, \pm|\xi|) = \hat{f}(\xi),\ d\mu(\xi, \pm|\xi|) = \frac{d\xi}{|\xi|}\ . \end{aligned}$$

By the dual to Theorem 16,

$$\|(F\,\mu)^{\vee}\|_{L^{p'}(\mathbb{R}^{n+1})} \le C\|F\|_{F^2(\mu)} \tag{115}$$

where $p' = \frac{2n+2}{n-1}$. Clearly,

$$\|F\|_{L^2(\mu)} = \left(\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \frac{d\xi}{|\xi|} \right)^{\frac{1}{2}} = \|f\|_{\dot{H}^{-\frac{1}{2}}}$$

so that (115) implies that

$$\|u\|_{L^{\frac{2n+2}{n-1}}(\mathbb{R}^{n+1})} \le C\|f\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^n)}\ ,$$

which is the standard Strichartz estimate for the wave equation in $n+1$ dimensions, $n \ge 2$.

Remark. *We have followed the original derivation of the Strichartz estimates. The are other ways of proving these inequalities which are perhaps somewhat simpler. For example, let*

$$T : L^2(\mathbb{R}^n) \to L^p(\mathbb{R}^{n+1}), \qquad Tf := e^{-\frac{i}{2\pi}\triangle} f$$

be the evolution operator of the free Schrödinger equation. Then it is easy to check that the dispersive inequality for the free Schrödinger equation, viz.

$$\|e^{-\frac{i}{2\pi}\triangle t} f\|_{L^{p'}(\mathbb{R}^n)} \le C|t|^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{p'})} \|f\|_{L^p(\mathbb{R}^n)}$$

together with the fractional integration theorem implies that

$$TT^* : L^{p'}(\mathbb{R}^{n+1}) \to L^p(\mathbb{R}^{n+1})$$

provided $p' = 2 + \frac{4}{n}$ *and* $n \ge 1$.

CHAPTER 10

Some pointwise questions for Fourier series: the case of $L^1(\mathbb{T})$

In this lecture we return to Fourier series on the circle. In contrast to the our previous discussion in the first four lecture, which dealt with $L^p(\mathbb{T})$ convergence, we now turn to pointwise questions. This area is regarded as hard. Indeed, a famous theorem of Carleson [**2**] shows that Fourier series of L^2 functions converge almost everywhere. This was extended by Hunt to $L^p(\mathbb{T})$, $1 < p < \infty$. Long before these results, Kolmogoroff proved that Fourier series of functions on $L^1(\mathbb{T})$ do not converge almost everywhere. It is this result which we discuss in Chapter 10. The proof strategy used in this chapter is the standard one of Kolmogoroff, although the specific approach which we follow is due to Stein [**12**]. The following lemma is a consequence of the Borel-Cantelli lemma.

Lemma 27. *Let $\{E_n\}_{n=1}^\infty$ be a sequence of measurable subsets of $\mathbb{T}$ such that*

$$\sum_{n=1}^\infty |E_n| = \infty \ .$$

Then there exists a sequence $\{x_n\}_{n=1}^\infty \in \mathbb{T}$ so that

$$\sum_{n=1}^\infty \chi_{E_n}(x + x_n) = \infty \tag{116}$$

for almost every $x \in \mathbb{T}$.

Proof. View $\Omega := \prod_{n=1}^\infty \mathbb{T}$ as a probability space equipped with the infinite product measure. Given $x \in \mathbb{T}$, let $A_x \subset \Omega$ be the event characterized by (116). We claim that

$$\mathbb{P}(A_x) = 1 \quad \forall x \in \mathbb{T} \ . \tag{117}$$

By Fubini, it then follows that for almost every $\{x_n\}_{n=1}^\infty \in \Omega$, the event (116) holds for almost every $x \in \mathbb{T}$. Hence fix an arbitrary $x \in \mathbb{T}$. Then

$$\begin{aligned} A_x &= \Big\{ \{x_n\}_{n=1}^\infty \mid x \text{ belongs to infinitely many } E_n(\cdot + x_n) \Big\} \\ &= \Big\{ \{x_n\}_{n=1}^\infty \mid x \in \bigcap_{N=1}^\infty \bigcup_{m=N}^\infty E_m(\cdot + x_m) \Big\} \\ &= \bigcap_{N=1}^\infty A_N \ , \end{aligned}$$

where

$$\begin{aligned}A_N^c &:= \Big\{\{x_n\}_{n=1}^\infty \mid x \in \bigcap_{m=N}^\infty E_m^c(\cdot + x_m)\Big\}\\ &= \Big\{\{x_n\}_{n=1}^\infty \mid x_m \in E_m^c(\cdot + x) \quad \forall m \ge N\Big\}\,.\end{aligned}$$

By definition of the product measure on Ω, it therefore follows that

$$\mathbb{P}(A_N^c) = \prod_{m=N}^\infty (1 - |E_m|) = 0$$

by assumption (116). Hence (117) holds, and the lemma follows. □

It will come as no surprise to the reader that the theory of almost everywhere convergence of Fourier series is intimately tied up with the theory of the maximal function (known as Carleson maximal operator)

$$\mathcal{C}f(x) := \sup_n |S_n f(x)|\,, \tag{118}$$

where S_n are the partial sum operators of Fourier series. Previously, we encountered the Hardy-Littlewood maximal function, which controlled the almost everywhere convergence of aproximate identities, see Theorem 4. Recall that the underlying bound on the maximal function was the weak-L^1 bound. Stein [**12**] discovered that this property is also *necessary* for almost everywhere convergence. More precisely, we have the following fact. It will be formulated in greater generality than Fourier series.

Lemma 28. *Let T_n be a sequence of operators, bounded on $L^1(\mathbb{T})$, and translation invariant. Define*

$$\mathcal{M}f(x) := \sup_{n\ge 1} |T_n f(x)|\,,$$

and assume that $\|\mathcal{M}f\|_\infty < \infty$ for every trigonometric polynomial f on $\mathbb{T}$. Now suppose that for any $f \in L^1(\mathbb{T})$,

$$|\{x \in \mathbb{T} \mid \mathcal{M}f(x) < \infty\}| > 0\,.$$

Then there exists a constant A so that

$$|\{x \in \mathbb{T} \mid \mathcal{M}f(x) > \lambda\}| \le \frac{A}{\lambda}\|f\|_1$$

for any $f \in L^1(\mathbb{T})$ and $\lambda > 0$.

Proof. We will prove this by contradiction. Hence, assume that there exists a sequence $\{f_j\}_{j=1}^\infty \subset L^1(\mathbb{T})$ with $\|f_j\|_1 = 1$ for all $j \ge 1$, as well as $\lambda_j > 0$ so that

$$E_j := \{x \in \mathbb{T} \mid \mathcal{M}f_j(x) > \lambda_j\}$$

satisfies

$$|E_j| > \frac{2^j}{\lambda_j}$$

for each $j \ge 1$. Be definition of $\mathcal{M}$ we then also have

$$\lim_{m\to\infty} |\{x \in \mathbb{T} \mid \sup_{1\le k\le m} |T_k f_j(x)| > \lambda_j\}| > \frac{2^j}{\lambda_j}\,.$$

for each $j \geq 1$. Hence there are $M_j < \infty$ with the property that

$$|\{x \in \mathbb{T} \mid \sup_{1 \leq k \leq M_j} |T_k f_j(x)| > \lambda_j\}| > \frac{2^j}{\lambda_j} .$$

for each $j \geq 1$. Let σ_N denote the N^{th} Cesaro sum, i.e., $\sigma_N f = K_N * f$, where K_N is the Fejer kernel. Since each T_j is bounded on L^1, we conclude that

$$\lim_{N \to \infty} |\{x \in \mathbb{T} \mid \sup_{1 \leq k \leq M_j} |T_k \sigma_N f_j(x)| > \lambda_j\}| > \frac{2^j}{\lambda_j} .$$

Hence, we assume from now on that each f_j is a trigonometric polynomial. Let m_j be a positive integer with the property that

$$m_j \leq \frac{\lambda_j}{2^j} < m_j + 1 .$$

Then

$$\sum_{j=1}^{\infty} m_j |E_j| = \infty$$

by construction. Counting each of the sets E_j with multiplicity m_j, the previous lemma implies that there exists a sequence of points $x_{j,\ell}$, $j \geq 1$, $1 \leq \ell \leq m_j$, so that

$$\sum_{j=1}^{\infty} \sum_{\ell=1}^{m_j} \chi_{E_j}(x - x_{j,\ell}) = \infty \tag{119}$$

for almost every $x \in \mathbb{T}$. Let

$$\delta_j := \frac{1}{j^2 m_j}$$

and define

$$f(x) := \sum_{j=1}^{\infty} \sum_{\ell=1}^{m_j} \pm \delta_j f_j(x - x_{j,\ell}) ,$$

where the signs $\pm$ will be chosen randomly. First note that irrespective of the choice of these signs,

$$\|f\|_1 \leq \sum_{j=1}^{\infty} m_j \delta_j < \infty.$$

The point is now to choose the signs so that

$$\mathcal{M} f(x) = \infty$$

for almost every $x \in \mathbb{T}$. For this purpose, select $x \in \mathbb{T}$ such that $x \in E_j + x_{j,\ell}$ for infinitely many j and $\ell = \ell(j)$ (just pick one such $\ell(j)$ if there are more than one). Since the T_n are translation invariant, so is $\mathcal{M}$. Hence

$$\mathcal{M} f_j(x - x_{j,\ell}) > \lambda_j$$

for infinitely many j. We conclude that for those j there is a positive integer $n(j,x)$ so that

$$|T_{n(j,x)} f_j(x - x_{j,\ell})| > \lambda_j.$$

At the cost of removing another set of measure zero we may assume that

$$T_n f(x) = \sum_{j=1}^{\infty} \sum_{\ell=1}^{m_j} \pm \delta_j T_n f(x - x_{j,\ell})$$

for all positive integers n. In particular, we have that

$$T_{n(j,x)} f(x) = \sum_{j=1}^{\infty} \sum_{\ell=1}^{m_j} \pm \delta_j T_{n(j,x)} f(x - x_{j,\ell})$$

which implies that

$$\mathbb{P}[|T_{n(j,x)} f(x)| > \delta_j \lambda_j] \geq \frac{1}{2},$$

where the probability measure is with respect to the choice of signs $\pm$. Since $\delta_j \lambda_j \to \infty$, we obtain that

$$\mathbb{P}[|T_{n(j,x)} f(x)| = \infty] \geq \frac{1}{2}.$$

We claim that the event on the left-hand side is a tail event. Indeed, this holds since

$$\mathcal{M} f(x) \leq \sum_{j=1}^{\infty} \mathcal{M} f_j(x)$$

and each summand on the right-hand side here is finite (f_j is a trigonometric polynomial and we are assuming that $\mathcal{M}$ is uniformly bounded on trigonometric polynomials). By Kolmogoroff's zero-one law we therefore have

$$\mathbb{P}[|T_{n(j,x)} f(x)| = \infty] = 1.$$

It follows from Fubini's theorem that almost surely (in the choice of $\pm$)

$$\mathcal{M} f(x) = \infty.$$

This would contradict our main hypothesis, and we are done. $\square$

The previous lemma, which is due to Stein, reduces Kolmogoroff's theorem on the failure of almost everywhere convergence of Fourier series of L^1 functions to disproving a weak-L^1 bound for the Carleson maximal operator. More precisely, we arrive at the following corollary.

Corollary 16. *Suppose $\{S_n f\}_{n=1}^{\infty}$ converges almost everywhere for every $f \in L^1(\mathbb{T})$. Then there exists a constant A such that*

$$|\{x \in \mathbb{T} \mid \mathcal{C}\mu(x) > \lambda\}| \leq \frac{A}{\lambda} \|\mu\| \tag{120}$$

for any complex Borel measure μ on $\mathbb{T}$ and $\lambda > 0$, where $\mathcal{C}$ is as in (118).

Proof. By the previous lemma, our assumption implies that there exists a constant A such that

$$|\{x \in \mathbb{T} \mid \mathcal{C} f(x) > \lambda\}| \leq \frac{A}{\lambda} \|f\|_1$$

for all $\lambda > 0$. This is the same as

$$|\{x \in \mathbb{T} \mid |S_n f(x)| > \lambda\}| \leq \frac{A}{\lambda} \|f\|_1$$

for all $n \geq 1$ and $\lambda > 0$. If μ is a complex measure, then we set $f = V_N * \mu$, where V_N is de la Vallée-Poussin's kernel. It follows that

$$\sup_{N\geq 1} |\{x \in \mathbb{T} \mid |S_n[V_N * \mu](x)| > \lambda\}| \leq \frac{A}{\lambda}\|\mu\|$$

for all $n \geq 1$ and $\lambda > 0$. Passing to the limit $N \to \infty$, we obtain our desired conclusion. □

The idea behind Kolmogoroff's theorem is to find a measure μ which would violate (120). This measure will be chosen to create resonances, i.e., so that the peaks of the Dirichlet kernel all appear with the same sign. More precisely, for every positive integer N we will choose

$$\mu_N := \frac{1}{N}\sum_{j=1}^{N} \delta_{x_{j,N}} \tag{121}$$

where the $x_{j,N}$ are close to $\frac{j}{N}$. Then

$$(S_n\mu_N)(x) = \frac{1}{N}\sum_{j=1}^{N} \frac{\sin((2n+1)\pi(x - x_{j,N}))}{\sin(\pi(x - x_{j,N}))}. \tag{122}$$

If x is fixed, we will then argue that there exists n so that the summands on the right-hand have the same sign (for this, we will need to make a careful choice of the $x_{j,N}$). Thus, the size of the entire sum will be about

$$\sum_{j=1}^{N} \frac{1}{j} \asymp \log N$$

because of the denominators in (122), which clearly contradicts (120).

The choice of the points $x_{j,N}$ is based on the following lemma due to Kronecker.

Lemma 29. *Assume that $(\theta_1, \ldots, \theta_d) \in \mathbb{T}^d$ is* incommensurate*, i.e., that for any $(n_1, \ldots, n_d) \in \mathbb{Z}^d \setminus \{0\}$ one has that*

$$n_1\theta_1 + \ldots + n_d\theta_d \notin \mathbb{Z}.$$

Then the orbit

$$\big\{(n\theta_1, \ldots, n\theta_d) \mod \mathbb{Z}^d \mid n \in \mathbb{Z}\big\} \subset \mathbb{T}^d \tag{123}$$

is dense in $\mathbb{T}^d$.

Proof. It will suffice to show that for any smooth function f on $\mathbb{T}^d$

$$\frac{1}{N}\sum_{n=1}^{N} f(n\theta_1, \ldots, n\theta_d) \to \int_{\mathbb{T}^d} f(x)\, dx. \tag{124}$$

Indeed, if the orbit (123) is not dense, then we could find $f \geq 0$ so that the set $\{\mathbb{T}^d | f > 0\}$ does not intersect it. Cleary, this would contradict (124). To prove (124), expand f into a Fourier series. Then

$$\frac{1}{N}\sum_{n=1}^{N} f(n\theta_1,\dots,n\theta_d) = \frac{1}{N}\sum_{n=1}^{N}\sum_{\nu\in\mathbb{Z}^d}\hat f(\nu)e^{2\pi i\theta\cdot\nu}$$
$$= \hat f(0) + \sum_{\nu\in\mathbb{Z}^d\setminus\{0\}}\hat f(\nu)\frac{1}{N}\frac{1-e^{2\pi i(N+1)\theta\cdot\nu}}{1-e^{2\pi i\theta\cdot\nu}},$$

where the ratio on the right-hand side is well-defined by our assumption. Clearly, $\hat f(\nu)$ is rapidly decaying in $|\nu|$ since $f\in C^\infty(\mathbb{T}^d)$. Thus, since

$$\Big|\frac{1}{N}\frac{1-e^{2\pi i(N+1)\theta\cdot\nu}}{1-e^{2\pi i\theta\cdot\nu}}\Big|\le 1$$

for all $N\ge 1$, $\nu\neq 0$, it follows that

$$\lim_{N\to\infty}\sum_{\nu\in\mathbb{Z}^d\setminus\{0\}}\hat f(\nu)\frac{1}{N}\frac{1-e^{2\pi i(N+1)\theta\cdot\nu}}{1-e^{2\pi i\theta\cdot\nu}} = 0,$$

and we are done. $\square$

We can now carry out our construction of the μ_N.

Lemma 30. *There exists a sequence μ_N of probability measures on $\mathbb{T}$ with the property that*

$$\limsup_{n\to\infty}\,(\log N)^{-1}|S_n\mu_N(x)|>0$$

for almost every $x\in\mathbb{T}$.

Proof. For every $N\ge 1$ and $1\le j\le N$ choose $x_{j,N}\in\mathbb{T}$ which satisfy

$$|x_{j,N}-\frac{j}{N}|\le N^{-2}$$

and so that $\{x_{j,N}\}_{j=1}^N\in\mathbb{T}^N$ is an incommensurate vector. This can be done since the commensurate vectors have measure zero. Clearly, the set of $x\in\mathbb{T}$ such that $\{2(x-x_{j,N})\}_{j=1}^N\in\mathbb{T}^N$ is a commensurate vector is at most countable. It follows that for almost every $x\in\mathbb{T}$,

$$\big\{\{2n(x-x_{j,N})\}_{j=1}^N \mod \mathbb{Z}^N \mid n\in\mathbb{Z}\big\}$$

is dense in $\mathbb{T}^N$. Hence, for almost every $x\in\mathbb{T}$,

$$\big\{\{(2n+1)(x-x_{j,N})\}_{j=1}^N \mod \mathbb{Z}^N \mid n\in\mathbb{Z}\big\}$$

is also dense in $\mathbb{T}^N$. It follows that for almost every $x\in\mathbb{T}$ there are infinitely many choices of $n\ge 1$ so that

$$\sin((2n+1)\pi(x-x_{j,N}))\ge\frac{1}{2}$$

for all $1\le j\le N$. In particular, for those n the sum in (122) satisfies

$$|S_n\mu_N(x)|\ge\frac{1}{2N}\sum_{j=1}^N\frac{1}{|\sin(\pi(x-x_{j,N}))|}\ge C\,\frac{1}{N}\sum_{j=1}^N\frac{1}{j/N}\ge C\,\log N,$$

as desired. $\square$

Finally, combining Lemma 30 with Corollary 16 yields Kolmogoroff's theorem.

THEOREM 17. *There exists $f \in L^1(\mathbb{T})$ so that $S_n f$ does not converge almost everywhere.*

It is known that this statement also holds with everywhere divergence. There has been considerable interest in the question of how much the condition $f \in L^1(\mathbb{T})$ can be relaxed for the conclusion of everywhere divergence of the Fourier series. It is known that

$$\int_{\mathbb{T}} |f(x)| \log(1 + |f(x)|) \log\log(100 + |f(x)|)\, dx < \infty$$

is sufficient for a.e. convergence, see Sjölin [**11**] (and Antonov [**1**] for a sharper condition), whereas Konyagin [**10**] showed the following: Let $\phi : [0, \infty) \to [0, \infty)$ be non-decreasing and satisfy $\phi(u) = o(u\sqrt{\log u}/\sqrt{\log\log u})$ as $u \to \infty$. Then there exists $f \in L^1(\mathbb{T})$ so that

$$\int \phi(|f(x)|)\, dx < \infty$$

and $\limsup_{m\to\infty} S_m f(x) = \infty$ for all $x \in \mathbb{T}$.

EXERCISE 29. *Consider the following variants of Lemma 28:*

1) Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of positive measures on $\mathbb{R}^d$ supported in a common compact set. Define

$$\mathcal{M}f(x) := \sup_{n\ge 1} |(f * \mu_n)(x)|.$$

Let $1 \le p < \infty$ and assume that for each $f \in L^p(\mathbb{R}^d)$

$$\mathcal{M}f(x) < \infty \quad \text{on a set of positive measure.}$$

Then show that $f \mapsto \mathcal{M}f$ is of weak-type (p, p).

2) Now suppose μ_n are complex *measures of the form $d\mu_n(x) = K_n(x)dx$, but again with common compact support. Show that the conclusion of part 1) holds, but only for $1 \le p \le 2$.*

APPENDIX A

Homework Problems 1–41, Introduction to Harmonic Analysis

Some of these homework problems are quite challenging, and you should probably only attempt them after you have worked through all eight lectures. Only very few of the following problems are referred to in the text. Moreover, some of them are only loosely connected with the contents of these lectures, like those on uniform distribution. Generally speaking, it is not necessary to work through the problems to understand the lectures, but they should be thought of as—hopefully interesting—extensions and applications of the ideas in the lectures.

(1) Let $\phi : [0, \infty) \to [0, \infty)$ be a continuously differentiable and monotone function with $\phi(0) = 0$. Prove

$$\int_X \phi\big(f(x)\big)\, d\mu(x) = \int_0^\infty \phi'(t)\mu(\{x \in X : f(x) > t\})\, dt$$

where $f \geq 0$, $f \in L^1(X, \mu)$ and (X, μ) is some σ-finite measure space.

(2) We say a sequence $\{f_n\}_{n=1}^\infty \in L^1(\mu)$ is *uniformly integrable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) < \delta \Longrightarrow \sup_n \left| \int_E f_n\, du \right| < \epsilon\ .$$

Suppose μ is a finite measure. Let $\phi : [0, \infty) \to [0, \infty)$ be a continuous increasing function with $\lim_{t\to\infty} \frac{\phi(t)}{t} = +\infty$. Prove that

$$\sup_n \int \phi(|f_n|)\, d\mu < \infty$$

implies that $\{f_n\}$ is uniformly integrable.

(3) Suppose $\{f_n\}_{n=1}^\infty$ is a sequence in $L^1([0, 1],\ dx)$. Show that there is a subsequence $\{f_{n_j}\}_{j=1}^\infty$ and a measure μ with $f_{n_j} \xrightarrow{\sigma^*} \mu$ provided $\sup_n \|f_n\|_1 < \infty$. Here σ^* is the weak-star convergence of measures. Show that in general $\mu \notin L^1([0, 1],\ dx)$. However, if we assume that, in addition, $\{f_n\}_1^\infty$ is uniformly integrable, then $d\mu = f dx$ for some $f \in L^1([0, 1])$. Can we conclude anything about strong convergence (ie, in the L^1-norm) of $\{f_n\}$? Consider the analogous question on $L^p([0, 1]), p > 1$.

(4) Let m denote Lebesgue measure on $\mathbb{R}^d$. Fix some $f \in L^p(m)$, $1 \leq p < \infty$. Define

$$\Phi_f : \mathbb{R}^d \longrightarrow L^p(m) \text{ by } \Phi_f(y)(x) = f(x + y)\ .$$

Show that Φ_f is continuous.

(5) Let μ be a finite Borel measure on $\mathbb{R}^n$. Prove that

$$\varphi(x) = \mu\big(B(x,r)\big)$$

is a lower semi-continuous function in x. Here $B(x,r)$ is the *open* ball of radius r and center x ($r > 0$ is fixed).

(6) Let μ be a finite Borel measure on $\mathbb{R}^d$. Recall

$$M\mu(x) = \sup_{r>0} \frac{\mu\big(B(x,r)\big)}{m\big(B(x,r)\big)}$$

a) Show that $\mu \perp m$ implies $\mu(\{x : M\mu(x) < \infty\}) = 0$

b) Show that if $\mu \perp m$, then $\limsup\limits_{r\to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = \infty \ \mu - a.e.$

(7) For any $f \in L^1(\mathbb{R}^d)$ and $1 \le k \le d$ let

$$M_k f(x) = \sup_{r>0} r^{-k} \int_{B(x,r)} |f(y)| \, dy \, .$$

Show that

$$\mathrm{mes}(\{x \in L : M_k f(x) > \lambda\}) \le \frac{C}{\lambda} \|f\|_{L^1}$$

where L is an arbitrary affine k-dimensional subspace and "mes" stands for Lebesgue measure (i.e., k-dimensional measure) on this space. C is an absolute constant.

(8) Prove the Besicovitch covering lemma on the circle: Suppose $\{I_j\}$ are finitely many arcs with $|I_j| < 1$. Then there is a sub-collection $\{I_{j_k}\}$ such that

a) $\cup_k I_{j_k} = \cup_j I_j$

b) No point belongs to more than C I_{j_k}'s where C is a *numerical* constant. Give an explicit value for C, as good as you can.

(9) *a*) Prove that if $\mu \in \mathcal{M}(\mathbb{T}) \setminus \{0\}$ satisfies $d\mu \perp d\theta$, then $M\mu \notin L^1$ (M is the Hardy-Littlewood maximal function). In fact, show that

$$\mathrm{mes}\{\theta \in \mathbb{T} \ : \ M\mu(\theta) > \lambda\} \ge \frac{c}{\lambda}\|\mu\|$$

provided $\lambda > \|\mu\|$ with an absolute constant $c > 0$.

b) Prove that there is a numerical constant C such that if $\mu \in \mathcal{M}(\mathbb{T})$ is a *positive* measure and F the associated harmonic function, then $M\mu \le CF^*$. Conclude that if μ is singular, then $F^* \notin L^1$.

(10) *a*) Let f be the standard Cantor-Lebesque function on the middle-third Cantor set on $[0,1]$. Show that f is Hölder continuous with exponent $\alpha = \frac{\log 2}{\log 3}$.

b) Let C be the usual middle-third Cantor set on $[0,1]$. Show that $C + C \supset [0,1]$. Can you find a larger interval than $[0,1]$ with this property?

(11) Let μ be a measure on X with $\mu(X) = 1$. Let f, g be two nonnegative measurable functions with $\int g d\mu = 1$. Prove

$$\int fg \, d\mu \le \int g \log g d\mu + \log \int e^f \, d\mu \, .$$

(12) Suppose $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, f is absolutely continuous with $f' \in L^p(\mathbb{R})$. Prove

$$\lim_{h\to 0} \int \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right|^p \, dx = 0$$

(13) Suppose that $f \in L^1(\mathbb{T})$ and that $\{S_n f\}_{n=1}^{\infty}$ (the sequence of partial sums of the Fourier series) converges in $L^p(\mathbb{T})$ to g for some $p \in [1, \infty]$ and some $g \in L^p$. Prove that $f = g$. If $p = \infty$ conclude that f is continuous.

(14) Let K_n denote the Fejer kernel with Fourier support $[-(n-1), n-1]$. Show that de la Vallée Poussin's kernel

$$V_n(\theta) = \left(1 + e^{2\pi i n\theta} + e^{-2\pi i n\theta}\right) K_n(\theta)$$

satisfies

a) $\widehat{V}_n(j) = 1$ where $|j| \le n$

b) $\|V_n'\|_1 \le Cn$ with C independent of n.

(15) Prove the following result (Bohr's inequality) which is a sort of converse to Bernstein's inequality: Suppose that $f \in C^1(\mathbb{T})$ and that $\hat{f}(j) = 0$ for all j with $|j| < n$.

Then

$$\left\| \frac{df}{d\theta} \right\|_p \ge Cn\|f\|_p$$

for all $p \in [1, \infty]$, where C is independent of $n \in \mathbb{Z}^+$, f and p.

(16) Show that the Hilbert transform preserves the Hölder class $C^\alpha(\mathbb{T})$, $0 < \alpha < 1$.

(17) If ω is an irrational number, show that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} f(\cdot + n\omega) - \int_{\mathbb{T}} f(\theta)\, d\theta \right\|_{L^2} \to 0$$

for any $f \in L^2(\mathbb{T})$. In particular, if $f \in L^2$ is such that $f(x+\omega) = f(x)$ for a.e. x, then $f = \text{const}$.

(18) Let α be an irrational number. Can there be a non-constant function $f \in L^2(\mathbb{T}^2)$ so that

$$f(x_1 + \alpha, x_1 + x_2) = f(x_1, x_2)$$

for a.e. $(x_1, x_2) \in \mathbb{T}^2$?

(19) For a real-valued function φ on $\mathbb{T}$ let A_φ denote the multiplication operator $(A_\varphi f)(x) = \varphi(x) f(x)$. Let $P_N : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the projection onto $\operatorname{span}\{1, e^{2\pi i\theta}, \ldots, e^{2\pi i(N-1)\theta}\}$. Let $\varphi(\theta) = \cos(2\pi\theta)$. Denote the eigenvalues of $P_N A_\varphi P_N$ by $\{\lambda_{j,N}\}_{j=1}^{N}$. Show that

$$\frac{1}{N} \sum_{j=1}^{N} \lambda_{j,N}^k = a_k + O\left(\frac{1}{N}\right) \quad k = 0, 1, 2, \ldots \tag{A.1}$$

for some constants a_k, which you should compute. Also show that

$$\frac{1}{N} \# \{j : \lambda_{j,N} \le E\} \to \rho(E) \text{ as } N \to \infty \tag{A.2}$$

uniformly in $E \in \mathbb{R}$. Find the function ρ.

(20) Now let $\varphi \in C^\infty(\mathbb{T})$ be *arbitrary* and define A_φ and $\lambda_{j,N}$ as above. Show that (A.3) below and (A.2) hold and find $\{a_k\}_{k=0}^\infty$ and ρ in terms of φ. If φ is non-degenerate of order s (i.e. $\sum_{\ell=0}^s |\varphi^{(\ell)}(x)| \neq 0$ on $\mathbb{T}$), show that $\rho \in C^{1/s}(\mathbb{R})$. Here (A.3) means

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^N \lambda_{j,N}^k = a_k, \ \ k = 0, 1, 2, \dots \tag{A.3}$$

(21) Let $\varphi \in C^\infty(\mathbb{T})$ and denote A_φ as in No. 19. If H is the Hilbert transform on $\mathbb{T}$, show that

$$[A_\varphi, H] = A_\varphi \circ H - H \circ A_\varphi$$

is a *smoothing* operator, i.e., if $\mu \in \mathcal{M}(\mathbb{T})$ is an arbitrary measure, then

$$[A_\varphi, H]\mu \in C^\infty(\mathbb{T})\ .$$

(22) In No. 20 show that not only (A.3) holds, but also (A.1). I.e., show that for $\varphi \in C^\infty(\mathbb{T})$

$$\frac{1}{N}\sum_{j=1}^N \lambda_{j,N}^k - a_k = 0\left(\frac{1}{N}\right) \ \text{ as } N \to \infty$$

with the same a_k as in 20.

(23) Let $f \in L^1(\mathbb{T})$. Given $\lambda > 1$, show that there exists $E \subset \mathbb{T}$ (depending on λ and f) so that $\mathrm{mes}(E) < \lambda^{-1}$ and for all $N \in \mathbb{Z}^+$

$$\frac{1}{N}\int_{\mathbb{T}\setminus E} \sum_{n=1}^N |S_n f(x)|^2 \ dx \le C\lambda \|f\|_1^2\ .$$

C is a constant independent of f, N, λ.

(24) Let $\{r_j\}_{j=1}^\infty$ be a sequence of independent, identically distributed random variables with $\mathbb{P}(r_1 = 1) = \mathbb{P}(r_1 = -1) = \frac{1}{2}$ (coin tossing sequence). Show that for $N = 1, 2, \dots$

$$\mathbb{P}\left(\left|\sum_{j=1}^N r_j a_j\right| > \lambda \left(\sum_{j=1}^N a_j^2\right)^{\frac{1}{2}}\right) \le 2e^{-\lambda^2/2}$$

for any $\{a_j\}_{j=1}^\infty \in \mathbb{R}$ and $\lambda > 0$.

(25) Suppose $\{X_n\}_{n=1}^\infty$ is a martingale difference sequence adopted to some filtration $\{\mathcal{F}_n\}_{n=1}^\infty$. Show that

$$\mathbb{P}\left(\left|\sum_{n=1}^N X_n\right| > \lambda \left(\sum_{n=1}^N \|X_n\|_\infty^2\right)^{\frac{1}{2}}\right) \le Ce^{-c\lambda^2}$$

for any $N = 1, 2, \dots$, $\lambda > 0$. $C, c > 0$ are absolute constants.

(26) Let $f \in C^1(\mathbb{T})$ be such that $\|f\|_\infty \le 1$ and $\|f'\|_\infty \le K$ (with some $K \ge 1$). Identifying $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ we let $x \mapsto 2x \bmod 1$ be the doubling map on $\mathbb{T}$. Using the previous exercise show that for any $N = 1, 2, \dots$

$$\mathrm{mes}\left\{x \in \mathbb{T} : \left|\frac{1}{N}\sum_{n=1}^N f(2^n x) - \int_{\mathbb{T}} f\right| > \lambda\right\} \le C\exp\left(-c\frac{\lambda^2 N}{\log^2(K/\lambda)}\right)$$

for some absolute constants c, C. Can you obtain $\log(K/\lambda)$ instead of $\log^2(K/\lambda)$? Would this be optimal?

(27) Let $\{r_j\}_1^\infty$ be as in No. 24. Show that for any $\{a_j\} \in \mathbb{C}$, and $N \in \mathbb{Z}^+$

$$\mathbb{P}\left(\sup_{0\le\theta\le1}\left|\sum_{j=1}^N r_j\, a_j e^{2\pi i j\theta}\right| > C_0\left(\sum_{j=1}^N |a_j|^2\right)^{\frac12}\sqrt{\log N}\right) \le C_0 N^{-2}$$

provided C_0 is a sufficiently large absolute constant.

(28) Let $T_N(x) = \sum_{n=0}^N [a_n\cos(2\pi nx) + b_n\sin(2\pi nx)]$ be an arbitrary trigonometric polynomial with real coefficients $a_0,\dots,a_N,b_0,\dots,b_N$. Show that there is a polynomial $P(z) = \sum_{\ell=0}^{2N} u_\ell z^\ell$ so that $T_N(x) = e^{-2\pi i N x}\, P(e^{2\pi i x})$ and such that $P(z) = z^{2N}\,\overline{P(\bar z^{-1})}$. How are the zeros of P distributed in the complex plane?

(29) Suppose $T_N(x) = \sum_{n=0}^N [a_n\cos(2\pi nx) + b_n\sin(2\pi nx)]$ is such that $T_N \ge 0$ everywhere and $a_n, b_n \in \mathbb{R}$ for all $n = 0, 1, \dots, N$. Show that there are $c_0, \dots, c_N \in \mathbb{C}$ such that

$$T_N(x) = \left|\sum_{n=0}^N c_n e^{2\pi i n x}\right|^2 \quad \text{for all } x\ .$$

(30) Suppose that $T(x) = a_0 + \sum_{h=1}^H a_h\cos(2\pi hx)$ satisfies $T(x) \ge 0$ for all x and $T(0) = 1$. Show that for any complex numbers $y_1, y_2, \dots, y_N$,

$$\left|\sum_{n=1}^N y_n\right|^2 \le (N+H)\left(a_0\sum_{n=1}^N |y_n|^2 + \sum_{h=1}^H |a_h|\left|\sum_{n=1}^{N-h} y_{n+h}\bar y_n\right|\right)\ .$$

(31) Let $\{x_n\}_{n=1}^\infty$ be an infinite sequence of real numbers. Show that the following three conditions are equivalent:

a) For any $f \in C(\mathbb{T})$,

$$\frac1N\sum_{n=1}^N f(x_n) \to \int_{\mathbb{T}} f\,dx$$

b) $\frac1N\sum_{n=1}^N e(kx_n) \to 0$ for all $k \in \mathbb{Z}^+$.

c) $\lim_{N\to\infty}\sup_{I\subset\mathbb{T}} |\frac1N \#\{1\le j\le N : x_j \in I \mod 1\} - |I|| = 0$.

If these conditions hold we say that $\{x_n\}_{n=1}^\infty$ is uniformly distributed modulo 1.

(32) Using #30 with a suitable choice of T, prove the following: If $\{x_n\}_{n=1}^\infty$ is a sequence for which $\{x_{n+k} - x_n\}_{n=1}^\infty$ is u.d. modulo 1 for any $k \in \mathbb{Z}^+$, then $\{x_n\}_1^\infty$ is also u.d. mod 1. In particular, show that $\{n^d\omega\}_{n=1}^\infty$ is u.d. mod 1 for any irrational ω and $d \in \mathbb{Z}^+$.

(33) *a*) Let $p \ge 2$ be a positive integer. Show that for a.e. $x \in \mathbb{T}$ $\{p^n x\}_{n=1}^\infty$ is u.d. modulo 1.

b) Can you characterize those x which have this property?

(34) Suppose $\sum_{n=1}^\infty a_n^2 n < \infty$ and $\sum_1^\infty a_n$ is Cesaro summable. Show that $\sum_{n=1}^\infty a_n$ converges. Use this to prove that for any $f \in C(\mathbb{T})$ for which $\sum_n |\hat f(n)|^2\cdot|n| < \infty$ one has $S_n f \to f$ uniformly.

(35) Show that there exists an absolute constant C so that

$$C^{-1}\sum_{n\neq 0}|\hat{f}(n)|^2|n| \le \int_0^1\int_0^1 \frac{|f(x)-f(y)|^2}{\sin^2(\pi(x-y))}\,dx\,dy \le C\sum_{n\neq 0}|\hat{f}(n)|^2|n|$$

for any $f\in H^{1/2}(\mathbb{T})$.

(36) Use #34 and #35 to prove the following theorem of Pal-Bohr: For any real function $f\in C(\mathbb{T})$ there exists a homeomorphism $\varphi:\mathbb{T}\to\mathbb{T}$ such that

$$S_n(f\circ\varphi)\longrightarrow f\circ\varphi$$

uniformly. Hint: Wlog $f>0$. Consider the domain defined in terms of polar coordinates by means of $r(\theta)=f(\theta/2\pi)$. Then apply the Riemann mapping theorem to the unit disc.

(37) Show that

$$\|f*g\|_{L^2(\mathbb{R})}^2 \le \|f*f\|_{L^2(\mathbb{R})}\|g*g\|_{L^2(\mathbb{R})} \text{ for all } f,g\in L^2(\mathbb{R}) .$$

Can there be such an inequality with $L^1(\mathbb{R})$ instead of $L^2(\mathbb{R})$?

(38) Prove Poincaré's inequality:

$$\int_{D(0,R)}|f(x)-f_{D(0,R)}|^2\,dx \le CR^2\int_{D(0,R)}|\nabla f(x)|^2\,dx$$

for all $f\in\mathcal{S}$. Here C depends only on the dimension. $f_{D(0,R)}$ denotes the mean of f over $D(0,R)$.

(39) Prove the following weak form of the Logvinenko-Sereda theorem by means of Poincaré's inequality and Bernstein's inequality:

Suppose $|F\cap D|\le\gamma|D|$ for all disks of radius R. Show that for small $\gamma>0$ there exists $\delta=\delta(\gamma)$ so that $\delta(\gamma)\to 0$ as $\gamma\to 0$, and such that

$$\|f\|_{L^2(F)}\le\delta(\gamma)\|f\|_2 \text{ if } \operatorname{supp}(\hat{f})\subset D(0,R^{-1}).$$

(40) Given N disjoint arcs $\{I_\alpha\}_{\alpha=1}^N\subset\mathbb{T}$, set $f=\sum_{\alpha=1}^N\chi_{I_\alpha}$. Show that

$$\sum_{|\nu|>k}|\hat{f}(\nu)|^2\lesssim\frac{N}{k}.$$

(41) Given any function $\psi:\mathbb{Z}^+\to\mathbb{R}^+$ so that $\psi(n)\to 0$ as $n\to\infty$, show that you can find a measurable set $E\subset\mathbb{T}$ for which

$$\limsup_{n\to\infty}\frac{|\widehat{\chi_E}(n)|}{\psi(n)}=\infty.$$

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