On the integrated density of states for Schrödinger operators on $\mathbb{Z}^2$ with quasi periodic potential

Wilhelm Schlag

Abstract
In this paper we consider discrete Schrödinger operators on the lattice $\mathbb{Z}^2$ with quasi-periodic potential. We establish new regularity results for the integrated density of states, as well as a quantitative version of a "Thouless formula", as previously considered by Craig-Simon, for real energies and with rates of convergence. The main ingredient is a large deviation theorem for the Green's function that was recently established by Bourgain, Goldstein, and the author. For the integrated density of states an argument of Bourgain is used. Finally, we establish certain fine properties of separately subharmonic functions of two variables that might be of independent interest.

1 Introduction
The purpose of this note is mainly to establish certain regularity properties of the integrated density of states for the two-dimensional discrete quasi periodic model

$$H = -\triangle_{\mathbb{Z}^2} + \lambda V(\theta_1 + n_1 \omega_1, \theta_2 + n_2 \omega_2)$$

for large $\lambda$. In [6] it was shown that for analytic $V$ that are non constant on any vertical and horizontal line, and large $\lambda$, Anderson localization holds for a large (in measure) set of $(\omega_1, \omega_2) \in \mathbb{T}^2$ and $(\theta_1, \theta_2) \in \mathbb{T}^2$. Our first result shows that under the same hypotheses the integrated density of states (IDS) has a modulus of continuity $\exp(-|\log \epsilon|^c)$ for some small $c > 0$. It is reasonable to believe that the IDS should have better regularity properties, but our current methods do not allow us to conclude that. Studying the regularity of the IDS has a long history that we will not review in detail. The IDS is known to be continuous under very mild assumptions, see Delyon and Souillard [9] or Figotin and Pastur [10], Theorem 3.4. It is also well-known that this is equivalent to saying that any given number has zero probability of being an eigenvalue, see Craig and Simon [8]. In the case of the two-dimensional quasi periodic model considered above Craig and Simon [8] showed that the IDS is log-Hölder continuous (in fact, their argument is valid in any dimension).
Recall that the IDS is the limiting distribution of the eigenvalues. More precisely, if \( \Lambda \subset \mathbb{Z}^2 \) is a square centered at the origin, then let \( E_j^{(\Lambda)}(\theta) \) denote the eigenvalues of \( H(\theta) \) restricted to the square \( \Lambda \) with Dirichlet boundary conditions. Set

\[
N^{(\Lambda)}_\theta(E) := \frac{1}{|\Lambda|} \# \{ j : E_j^{(\Lambda)}(\theta) \leq E \}.
\]

It is a simple and well-known consequence of the ergodic theorem (in this case with two commuting shifts \( T_{\omega_1} \) and \( T_{\omega_2} \)) that the limit

\[
\lim_{\text{diam}(\Lambda) \to \infty} N^{(\Lambda)}_\theta = k
\]

exists for almost every \( \theta \in \mathbb{T}^2 \). The probability measure given by \( k \) is called the IDS. For further details see [1] or [10]. In fact, combining [1] with [9] implies that this limit procedure converges for every \( \theta \in \mathbb{T}^2 \). Clearly, one also has

\[
\lim_{\text{diam}(\Lambda) \to \infty} \int_{\mathbb{T}^2} N^{(\Lambda)}_\theta \, d\theta = k.
\]

For one dimensional models (such as almost Mathieu) the IDS is naturally connected with the Lyapunov exponent by means of the Thouless formula

\[
\gamma(E) = \int \log |E - E'| \, dk(E'),
\]

see Avron and Simon [1]. On \( \mathbb{Z}^2 \) there is no Lyapunov exponent. Nevertheless, Craig and Simon [8] made successful use of the integral on the right-hand side for \( E \) with \( \text{Im}(E) \neq 0 \). Amongst other things they showed that one has

\[
\int \log |E - E'| \, dk(E') \geq 0
\]

which was crucial for their proof of log-Hölder continuity. Below we study this mean for real \( E \) and show that it is (as one would expect) the limit of logarithms of determinants. Rates of convergence as well as a large deviation theorem for the determinants are also obtained. We want to emphasize though, that all aforementioned results are obtained only for large disorders (because we use large deviation theorems for the Green's function from [6], see the following section). Part of our original motivation was to understand if large deviation estimates could be obtained for \( \log |\det(H_\Lambda(\theta) - E)| \). This is natural in view of recent nonperturbative arguments by Jitomirskaya [12], Bourgain and Goldstein [5], and Bourgain and Jitomirskaya [7]. Although a nonperturbative proof for the \( \mathbb{Z}^2 \) model seems rather remote at this point,
it might be helpful to know what should be true based on the results obtained here for large disorders. Throughout this paper $C$ will stand for a numerical constant that can change from line to line. Usually we will indicate which parameters various constants depend on. Also, $a \lesssim b$ stands for $a \leq Cb$ and similarly with $a \gtrsim b$. Finally, $a \asymp b$ means both $a \lesssim b$ and $a \gtrsim b$.

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2 The IDS for large disorder

Let

$$H(\varrho) = -\Delta + \lambda V(\varrho),$$

where $V(\varrho)(n_1, n_2) = V(\varrho_1 + n_1 \omega_1, \varrho_2 + n_2 \omega_2)$ and $\lambda$ is a large parameter. As in [6] we assume that the real-analytic function $\nu : \mathbb{T}^2 \to \mathbb{R}$ satisfies:

$$\theta_1 \mapsto \nu(\theta_1, \theta_2) \quad \text{and} \quad \theta_2 \mapsto \nu(\theta_1, \theta_2)$$

are nonconstant functions for any choice of the other variable. Most of the work in [6] was devoted to proving large deviation estimates for the Green’s functions

$$G_\lambda(\varrho, E) := (H_\lambda(\varrho) - E)^{-1},$$

where $H_\lambda(\varrho) := R_\Lambda H(\varrho) R_\Lambda$, $R_\Lambda$ being the restriction operator to $\Lambda$. Here $\Lambda$ should be thought of as a square in $\mathbb{Z}^2$, but for technical reasons it was necessary to consider a larger class of $L$-shaped sets in [6], which were referred to as “elementary regions”.

We shall not dwell on this point here, see Section 2 in [6] for details. Returning to the large deviation estimates (LDE), we call $G_\lambda(\varrho, E)$ good, provided for some fixed $0 < b < 1$ and $\gamma > 0$

$$||G_\lambda(\varrho, E)|| \leq \lambda^{-1} e^{\ell b}$$

$$|G_\lambda(\varrho, E)(x, y)| \leq e^{-\gamma |x - y|} \quad \text{for all} \quad x, y \in \Lambda, \quad |x - y| > \ell/4$$

where $\ell = \text{diam}(\Lambda)$. Otherwise, $G_\lambda(\varrho, E)$ is called bad. The main technical statement in [6] is the following proposition, see Section 4 there. We set

$$\mathcal{E}_\lambda = [-4 - 2\lambda ||\nu||_\infty, 4 + 2\lambda ||\nu||_\infty]$$

so that the spectrum of $H$ is always strictly contained inside $\mathcal{E}_\lambda$.  

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Proposition 2.1. Let $v$ be a real-analytic function satisfying (2.2). Given $\varepsilon > 0$ there exist $\Omega_\varepsilon \subset \mathbb{T}^2$ so that $\operatorname{mes}(\mathbb{T}^2 \setminus \Omega_\varepsilon) < \varepsilon$, and (large) numbers $\lambda_0 = \lambda_0(v, \varepsilon)$, and $N_0 = N_0(v, \varepsilon)$ with the following property: For any $\mathbf{q} \in \Omega_\varepsilon$, all $\lambda \geq \lambda_0$ and all $N \geq N_0$ there is the estimate

$$
\sup_{E \in E_\lambda} \operatorname{mes}(\{ \mathbf{q} \in \mathbb{T}^2 : G_\lambda(\mathbf{q}, E) \text{ is bad}\}) \leq \exp(-N^\rho) \quad \text{for } i = 1, 2
$$

for any square $\Lambda \subset \mathbb{Z}^2$, $\operatorname{diam}(\Lambda) = N$, with $\gamma = \frac{1}{4} \log \lambda$ and some constants $0 < \delta, \rho < 1$.

This LDE was a crucial ingredient in the proof of localization, the other being a technique of energy elimination via properties of semi-algebraic sets. The latter will not concern us here, but we will rely heavily on Proposition 2.1. Clearly, we may assume that there is some (large) constant $C_\varepsilon$ so that for all $\mathbf{q} \in \Omega_\varepsilon$

$$
\|\omega_1n_1 + \omega_2n_2\| \geq C_\varepsilon^{-1} |n_1 + n_2|^{-3} \quad \text{for all } (n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}.
$$

Here $\| \cdot \|$ denotes the distance to the nearest integer. This Diophantine condition will be used below without further mention. Fix some small $\varepsilon > 0$, and let $\Omega_\varepsilon$, $\lambda_0$, and $N_0$ be as above. For any $N \geq N_0$ and $\Lambda \subset \mathbb{Z}^2$ a square of side length $N$, the operator $H_\Lambda := H_\Lambda(\mathbf{q})$ has eigenvectors $\xi_\ell^\lambda = \xi_\ell^\lambda(\mathbf{q})$ with corresponding eigenvalues $E_\ell^\lambda = E_\ell^\lambda(\mathbf{q})$ (here $\mathbf{q}$ is fixed but arbitrary). We will show that for any interval $J \subset \mathbb{R}$ and large $N$

$$
\#[\ell : E_\ell^\lambda \in J] \leq C \exp(-|J|^{\rho}|[\Lambda]|
$$

with constants $c, C$ that only depend on $\varepsilon, v$. This of course shows that the IDS has the modulus of continuity stated in the introduction. The proof of (2.5) is based on the following idea (for the details see the following proposition): Let $\delta = |J|$ and choose $N_0 \asymp |\log \delta|^{1/3}$ (this notation means comparable up to constants). Let $E$ be the center of $J$. Partitioning $\Lambda$ into squares $\{Q_j\}$ of size $N_0$, it follows from Proposition 2.1, and the fact that the set in (2.4) is contained in a semi-algebraic set of degree $N_0^{C_1}$ for some constant $C_1$ and comparable measure, that

$$
\#[\ell : G_{Q_j}(\mathbf{q}, E) \text{ is bad}] \leq N_0^{C_1} \exp(-N_0^{\rho}) N^2.
$$

Now denote $\Lambda_* = \bigcup_{\text{bad}} Q_j$ where the union runs over those squares in $\Lambda$ that are bad. Using the resolvent identity it can be shown that

$$
\|G_{\Lambda \setminus \Lambda_*}(\mathbf{q}, E)\| \leq \exp(N_0^\rho).
$$

By choice of $N_0$ one has

$$
\delta e^{N_0^\rho} \ll 1.
$$
It is then easy to see from (2.7) that any eigenfunction $\xi = \xi^A_\ell$ with eigenvalue belonging to $J$ has most of its $L^2$-mass on the set $A_\ell$. Since the $\xi$'s under consideration are pairwise orthogonal, in view of (2.6) there can basically be no more than

$$|A_\ell| \lesssim N_0^{C_1} \exp(-N_0^\rho) N^2 \times \exp(-|\log \delta|^6)|A|$$

many of these eigenfunctions, as claimed. This argument is taken from [3], where Bourgain studied the regularity of the IDS for the almost Mathieu (and more general) equations by means of this method. The main point of his paper was to show that the norm of the Green's function can be controlled by quadratic polynomials. This allows him to prove Hölder-$\left(\frac{1}{2}\right)$ regularity for the IDS. Using Proposition 2.1 instead of the explicit control via (quadratic) polynomials gives a correspondingly weaker result. Presently it is not clear how to improve on it. In [4], the same argument is also used. We now turn to a more detailed account of the regularity result for the IDS in the $\mathbb{Z}^2$ model.

**Proposition 2.2.** Given $\varepsilon$, let $\Omega_\varepsilon$, $\lambda_0$, and $N_0$ be as in Proposition 2.1. There exist constants $c = c(\varepsilon, v), C = C(\varepsilon, v)$ so that for any $\omega \in \Omega_\varepsilon, \lambda \geq \lambda_0$, and any interval $J \subset \mathbb{R}$ the bound (2.5) holds for sufficiently large $N$, i.e.,

$$\sup_{\ell \in \mathbb{Z}^2} \#\{\ell \in J \subset \mathbb{R} : E^{(\varepsilon)}_\ell (\ell) \in J\} \leq C \exp(-|\log \delta|^6) N$$

where $\delta = |J|$. In particular, for any such $\omega$ and $\lambda$, the IDS has modulus of continuity $\exp(-|\log \delta|^6)$.

**Proof.** Assume $\varepsilon > 0, \omega \in \Omega_\varepsilon$ fixed. Fix some small interval of energies $J$ of length $\delta$ and center $E$. Further, let $\Lambda \subset \mathbb{Z}^2$ be a large square centered at 0. Let $N_0 \gg |\log \delta|^6$ be an integer, where $b$ is as in (2.4) and the multiplicative constant in this notation is taken sufficiently small. Increasing $N$ if necessary, we can assume that $\Lambda$ can be partitioned into squares $\{Q_j\}$ of side length $N_0$. Let $A_0$ denote the square centered at 0 with side length $N_0$ and set

$$B_0 := \{\ell \in \mathbb{T}^2 : G^{(\varepsilon)}_\Lambda (\ell, E) \text{ is bad}\},$$

see (2.3). Clearly, $G_{Q_j}(0, E)$ is good iff $(m_j \omega_1, n_j \omega_2) \notin B_0 \mod \mathbb{Z}^2$ where $(m_j, n_j) \in \mathbb{Z}^2$ is the center of $Q_j$. We have set the phase $\ell = 0$ merely for convenience. Any other phase works just as well. Since Proposition 2.1 provides the measure estimate

$$\text{mes}(B_0) \leq \exp(-N_0^\rho),$$

and $B_0$ is semi-algebraic of degree at most $N_0^{C_1}$, one has

$$\# \{(n_1, n_2) \in \Lambda : (n_1 \omega_1, n_2 \omega_2) \in B_0 \mod \mathbb{Z}^2 \} \lesssim N_0^{C_1} \exp(-N_0^\rho)|A|.$$
To verify this claim, observe firstly that we may replace the potential function with a trigonometric polynomial of degree \( N_0^2 \), thus providing the semi-algebraic property of \( B_0 \). More precisely, by analyticity of \( v \) there is a trigonometric polynomial \( P_N \) of degree at most \( N^2 \), say, such that \( \| v - P_N \|_\infty < e^{-N} \) when \( N \) is large. This introduces at most an exponentially small error that is negligible, whereas the set \( B_0^c \) defined in terms of \( P_N \) rather than \( v \) is semi-algebraic of degree at most \( N^{10} \), see [6], Remark 4.3 for details. For convenience, we do not distinguish between \( B_0 \) and \( B_0^c \). In light of this fact, we may cover \( B_0 \) by at most \( N_0^{C_1} \) many disks of size \( \asymp \exp(-\frac{1}{2}N_0^p) \). Since the vector \( (\omega_1,\omega_2) \) satisfies a Diophantine condition, one concludes (2.8) by means of standard discrepancy considerations, say. Therefore, the number of bad squares \( Q_j \) in \( \Lambda \) cannot exceed the right-hand side of (2.8). In particular, \( \Lambda_\ast = \bigcup_{\text{bad}} Q_j \) satisfies

\[
|\Lambda_\ast| \lesssim N_0^{C_1} \exp(-N_0^p)|\Lambda|.
\]

On the other hand,

\[
\|G_{\Lambda \setminus \Lambda_\ast}(0,E)\| \lesssim e^{N_0^0}.
\]

This follows by means of a straightforward application of the resolvent identity. The details can be found in Lemma 2.2 and Lemma 4.4 of [6] \(^1\). Denote the eigenfunctions of \( H_\Lambda = H_\Lambda(0) \) with eigenvalues falling into the interval \( J \) by \( \{\xi_j\}_{j=1}^M \). Let \( \xi \) be one of them with eigenvalue \( E' \). By definition,

\[ R_{\Lambda \setminus \Lambda_\ast}(H_\Lambda - E)R_{\Lambda \setminus \Lambda_\ast}\xi + R_{\Lambda \setminus \Lambda_\ast}(H_\Lambda - E)R_{\Lambda_\ast}\xi = (E' - E)R_{\Lambda \setminus \Lambda_\ast}\xi. \]

Applying \( G_{\Lambda \setminus \Lambda_\ast}(0,E) \) to this line yields

\[
R_{\Lambda \setminus \Lambda_\ast}\xi + G_{\Lambda \setminus \Lambda_\ast}(0,E)(H_\Lambda - E)R_{\Lambda_\ast}\xi = (E' - E)G_{\Lambda \setminus \Lambda_\ast}(0,E)R_{\Lambda \setminus \Lambda_\ast}\xi.
\]

Let \( P \) denote the projection onto the range of \( G_{\Lambda \setminus \Lambda_\ast}(0,E)(H_\Lambda - E)R_{\Lambda_\ast} \). Clearly, the dimension of this range does not exceed \( |\Lambda_\ast| \). Thus \( \text{rank}(P) \leq |\Lambda_\ast| \). In view of (2.11) and (2.10),

\[
\|R_{\Lambda \setminus \Lambda_\ast}\xi - PR_{\Lambda \setminus \Lambda_\ast}\xi\| \lesssim \delta e^{N_0^0}.
\]

\(^1\)Strictly speaking, one needs to define a good square so that every point in it is surrounded by a good elementary, i.e., \( L \)-shaped, region of a certain size. But this only brings in another factor of \( N_0^p \). For more details concerning elementary regions as well as the details of the resolvent identity argument we refer the reader to [6].
By taking \( N_0 \) to be a small multiple of \( |\log \delta|^{1/2} \), the right-hand side of (2.12) can be made less than \( \frac{1}{10} \), say. Invoking (2.12) for each of the \( \xi_j \) shows that

\[
M = \sum_{j=1}^M |\xi_j|^2 \leq \frac{M}{2} + \sum_{j=1}^M \|PR_{A_\lambda} \xi_j\|^2 + \sum_{j=1}^M \|R_{A_\lambda} \xi_j\|^2 \\
\leq \frac{M}{2} + 2 \sum_{j=1}^M \|P \xi_j\|^2 + 3 \sum_{j=1}^M \|R_{A_\lambda} \xi_j\|^2 \\
\leq \frac{M}{2} + 2 \text{trace}(P) + 3 \text{trace}(R_{A_\lambda}) \leq \frac{M}{2} + 2 \text{rank}(P) + 3|\Lambda| \\
\leq \frac{M}{2} + C N_0^C \exp(-N_0^\rho)|\Lambda|.
\]

This yields the desired bound (2.5). \( \square \)

Inspection of this proof shows that a LDE (2.4) with \( b = \rho = 1 \) implies Hölder continuity of the IDS (in any dimension). This should explain why it was possible to establish Hölder continuity of the IDS in [11] from the “sharp LDEs” established there. Note, however, that the previous argument is more satisfactory as it bounds the number of eigenvalues inside a small interval. In contrast, [11] uses the Thouless formula that only applies to the limit. As far as LDEs with \( b = \rho = 1 \) are concerned, they have been established only for the case of a one-dimensional equation with one frequency. In all other cases where LDEs are known, the value of \( \rho \) is rather small. For the remainder of this section we discuss LDEs for logarithms of determinants and the “Thouless formula”. More precisely, we shall consider squares \( \Lambda \subset \mathbb{Z}^2 \) centered at the origin and we define

\[
f_{\Lambda, E}(\varrho) := \frac{1}{|\Lambda|} \log |\det(H_{\Lambda}(\varrho) - E)|.
\]

Let

\[
\gamma_{\Lambda}(E) := \int_{\mathbb{T}^2} f_{\Lambda, E}(\varrho) d\varrho \quad \text{and} \quad \gamma(E) := \int \log |E - E'| \, d\lambda(E').
\]

For the case of \( \text{Im}(E) \neq 0 \) the quantity \( \gamma(E) \) was introduced by Craig and Simon [8]. Their objective was to prove the log-Hölder continuity of the IDS (in all dimensions). They accomplished this by showing that

\[
\gamma(E) \geq 0 \quad \text{for all } E \text{ with } \text{Im}(E) \neq 0.
\]

Of course, it follows from the fact that the IDS exists that for a.e. \( \varrho \)

\[
\lim_{\text{diam}(\Lambda) \to \infty} f_{\Lambda, E}(\varrho) \to \gamma(E) \quad \text{for all } E \text{ with } \text{Im}(E) \neq 0.
\]
However, it is much harder to show that this limit is always nonnegative, and Craig and Simon achieved this by means of a reduction to strips. In the latter case one has an interpretation of \( \gamma(E) \) as an average of all nonegative Lyapunov exponents. Observe that their result implies by means of Fatou’s lemma that

\[
\gamma(E) \geq 0 \quad \text{for all real } E.
\]

As already apparent in the proof of the Thouless formula in [1] it is more subtle to understand whether or not the limit of \( f_{\lambda,E}(\theta) \) exists for real \( E \) and equals this integral. By general principles one can easily conclude that for a.e. \( \theta \) it exists in an \( L^2 \) sense in \( E \), cf. Proposition 2.7 below. We show here that for large disorders and most \( \omega \) one has for all \( E \) that \( \gamma_\lambda(E) \rightarrow \gamma(E) \). Moreover, we obtain the rate of convergence

\[
|\gamma_\lambda(E) - \gamma(E)| \lesssim |\lambda|^{-\delta}
\]

with some constant \( \delta > 0 \). Finally, we establish a LDE for \( f_{\lambda,E} \) for large disorders, see Proposition 2.5 below. The argument proving this proposition is again very general and applies to all cases (in any dimension) where a LDE for the Green’s function is known. By means of this LDE one concludes that for all \( E \)

\[
f_{\lambda,E}(\theta) \rightarrow \gamma(E) \quad \text{for a.e. } \theta.
\]

Presently it is not clear whether this can be true for a.e. \( \theta \) and all \( E \).

The following lemma is Weyl’s well-known eigenvalue comparison theorem for Hermitian matrices. The proof is an immediate consequence of the min-max characterization of eigenvalues, see Theorem 8.4 in [2].

**Lemma 2.3.** Let \( A, B \) be Hermitian \( d \times d \) matrices. Suppose \( \text{rank}(A-B) \leq k \). If \( a_1 \leq a_2 \leq \ldots \leq a_d \), and \( b_1 \leq b_2 \leq \ldots \leq b_d \) denote the eigenvalues of \( A \) and \( B \), respectively, then

\[
a_{\ell+k} \geq b_\ell \quad \text{for any } d \geq \ell + k \geq 1 \]

\[
b_\ell \geq a_{\ell-k} \quad \text{for any } d \geq \ell \geq \ell - k \geq 1.
\]

This lemma is used to compare determinants, as stated in the following result.

**Corollary 2.4.** Suppose \( A, B \) are Hermitian with \( \text{rank}(A-B) \leq k \). If

\[
\text{dist}(\text{Spec}(A), 0) \geq \rho > 0,
\]

then for all \( t \in \mathbb{R} \)

\[
|\det(B + it)| \leq \rho^{-4k} |B + it|^4 |\det(A + it)|.
\]

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Proof. Consider first the case \( t = 0 \). Let the eigenvalues of \( A \) be given by

\[
  a_1 \leq a_2 \leq a_{t-1} \leq a_t \leq 0 < a_{t+1} \leq a_{t+2} \leq \ldots \leq a_d.
\]

The eigenvalues of \( B \) are denoted by \( b_j \). Then, by Weyl's lemma,

\[
  \left| \det A \right| = a_d a_{d-1} \cdots a_{t+1} a_{t} \cdots a_1 \geq b_{t-k} b_{t-k-1} \cdots b_{t+2k} \cdots a_{t+1} \geq \rho^t b_{t-k} b_{t-k-1} \cdots b_{t+2k} \cdots b_{t-k+1} \cdot |a_d a_{d-1} \cdots a_{t-2k+1} b_{t-k} b_{t-k-1} \cdots b_{t-k+2} b_{t-k+1}|.
\]

Therefore

\[
  | \det B | \leq \rho^{-tk} | \det A | |B|^{tk},
\]

as claimed. For the case \( t \neq 0 \) simply note that for real numbers \( a \geq b \geq 0 \) one has

\[
  |a + it| \geq |b + it|.
\]

Hence the same arguments apply to the general case as well. \( \square \)

The following proposition shows that \( f_{\Lambda, E} \) does not deviate much from its mean \( \gamma_{\Lambda}(E) \). The proof uses the LDEs for the Green's function. In fact, passing from the Green's functions to the determinants is rather straightforward.

**Proposition 2.5.** Let \( \varepsilon > 0 \) and let \( \Omega_\varepsilon, \lambda_0, N_0 \) be as in Proposition 2.1. Fix any \( \omega \in \Omega_\varepsilon \), and let \( \lambda \geq \lambda_0 \) and \( N \geq N_0 + (\log \lambda)^C \). Then for any square \( \Lambda \) of size \( N \) and any \( E \in [-C \log \lambda, C \log \lambda] \),

\[
  \mes \{ \theta \in \mathbb{T}^2 : |f_{\Lambda, E}(\theta) - \gamma_{\Lambda}(E)| > N^{-\delta} \} < \exp(-N^\delta).
\]

Here \( \delta > 0 \) is some small constant.

**Proof.** Let \( \Lambda_1 \subset \mathbb{Z}^2 \) be a large square of size \( N \) and set \( \Lambda_2 := \Lambda_1 + (1, 1) \). Let \( \Lambda \) be the smallest square containing \( \Lambda_1 \cup \Lambda_2 \) (it has size \( N + 1 \)), see Figure 1. Define \( H_\Lambda^{(j)}(\theta) \) to be the operator that is obtained from \( H_\Lambda(\theta) \) by cutting the bonds along the boundary of \( \Lambda_j \) that lies inside \( \Lambda \). More precisely, \( H_\Lambda^{(j)}(\theta) \) is the direct sum of \( H_{\Lambda_j}(\theta) \) and the operator on \( \ell^2(\Lambda \setminus \Lambda_j) \) that acts solely by multiplication with the potential at any site in \( \Lambda \setminus \Lambda_j \). Observe that for all \( \theta \)

\[
  \text{rank}(H_\Lambda^{(j)}(\theta) - H_\Lambda(\theta)) \leq 10N
\]

for \( j = 1, 2 \). By Proposition 2.1 we know that
\[
\|G_\lambda(\theta, E)\| + \max_{j=1,2} \|G_{\lambda_j}(\theta, E)\| < e^{N^b}
\]
up to a \(\theta\)-set of measure less than \(e^{-N^p}\). Moreover, in view of (2.2) one easily concludes that up to a \(\theta\)-set of measure \(\lesssim e^{-cN^b}\),

\[
\|(H^{(j)}_\lambda(\theta) - E)^{-1}\| \lesssim e^{N^b}.
\]

This holds because the eigenvalues of the operator on \(l^2(\Lambda \setminus \Lambda_j)\) are simply the values of the potential along the boundary. For such \(\theta\) Corollary 2.4 implies that

\[
\begin{align*}
|\det(H_\lambda(\theta) - E)| &< (C\lambda)^{40N^b} e^{40N^b+1} |\det(H_{\lambda_j}(\theta) - E)| \\
|\det(H_\lambda(\theta) - E)| &> (C\lambda)^{-40N^b} e^{-40N^b+1} |\det(H_{\lambda_j}(\theta) - E)|,
\end{align*}
\]

where we have absorbed the contribution from the boundary strip into the error terms. Therefore,

\[
\left| \frac{1}{|\Lambda|} \log |\det(H_\lambda(\theta) - E)| - \frac{1}{|\Lambda|} \log |\det(H_{\lambda_j}(\theta) - E)| \right| \lesssim N^{b-1} + \frac{\log \lambda}{N}
\]
for $j = 1, 2$ and such $\theta$. This clearly implies that
\[ \left| \frac{1}{|A_1|} \log \left| \det(H_{A_1}(\theta) - E) \right| - \frac{1}{|A_2|} \log \left| \det(H_{A_2}(\theta) - E) \right| \right| \lesssim N^{b-1} + \frac{\log \lambda}{N} \]
up to a $\theta$-set of measure less than $e^{-N^\delta}$. Hence, for any (large) square $\Lambda$ of size $N \gg (\log \lambda)^{\frac{1}{\delta}}$

\[ (2.15) \quad \left| f_{\Lambda,E}(\theta) - f_{\Lambda,E}(\theta + \omega) \right| \lesssim N^{-\delta} \]
up to a $\theta$-set of measure less than $e^{-N^\delta}$ (with $\delta = 1 - b$). This is an almost invariance property like the one used in [5] and [11] for the monodromy matrices. It is clear that such an invariance property cannot hold uniformly in $\theta$ in case of the determinant. We shall now apply Theorem 3.7 to a suitably normalized version of the function $f_{\Lambda,E}$. Firstly, observe that
\[ \sup_{|z_1|,|z_2| \leq 2} f_{\Lambda,E}(z_1, z_2) \leq C \log \lambda \]

since $|E| \leq C \log \lambda$. It therefore remains to check that $f_{\Lambda,E}(\theta_1, \theta_2)$ is not too negative for some $\theta \in \mathbb{T}^2$. This can be seen as follows: Let $N_1 = \lceil (\log N)^\delta \rceil$ and for every point $x \in \Lambda$ consider a square $\Lambda_{N_1}(x) \subset \Lambda$ of size about $N_1$. By Proposition 2.1 the Green's function $G_{\Lambda_{N_1}(x)}(\theta, E)$ is good for every $x \in \Lambda$ up to a $\theta$-set of measure less than $N^2 e^{-N_1^2} \lesssim N^2 e^{-(\log N)^2} < \frac{1}{2}$, say, for large $N$. This implies by means of the resolvent identity that
\[ \inf_{\theta \in \mathbb{T}^2} \|G_{\Lambda}(\theta, E)\| \leq e^{N_1} \]
see Lemma 2.2 in [6]. Hence,
\[ \sup_{\theta \in \mathbb{T}^2} f_{\Lambda,E}(\theta) \gtrsim -N_1. \]

In view of the preceding,
\[ \frac{f_{\Lambda,E} + C(\log N)^{2/\rho}}{C(\log N)^{2/\rho} + C \log \lambda} \]
is a separately subharmonic function, see Definition 3.5. One now applies Theorem 3.7 below with $r = N^{-\frac{\varepsilon}{2}}$, $\gamma = \frac{1}{3}$, and $\rho = \frac{1}{2}$. Thus, there is a $N^{-\varepsilon} \times N^{-\varepsilon}$-rectangle $R \subset \mathbb{T}^2$ with the property that
\[ (2.16) \quad \left| f_{\Lambda,E}(\theta) - f_{\Lambda,E}(\theta') \right| \lesssim N^{-\varepsilon/2}(\log \lambda + (\log N)^{2/\rho}) \text{ for any } \theta, \theta' \in R \setminus B \]
where \( \text{mes}(\mathcal{B}) < e^{-N^2/8} \). Here we take \( \epsilon = \frac{4}{100} \). By the Diophantine property of \( \omega \) any point of \( \mathbb{T}^2 \) can be moved into \( R \) by no more than \( N^{3 \varepsilon} \omega \)-steps. In view of (2.15) and (2.16) this implies that

\[
|f_{\Lambda, E}(\theta) - f_{\Lambda', E}(\theta')| < N^{-\frac{3}{100}} \quad \text{for any} \quad \theta, \theta' \in \mathbb{T}^2 \setminus \mathcal{B}
\]

where \( \text{mes}(\mathcal{B}) < \exp(-N^{100}) \) and \( N > (\log \lambda)^C \). In view of Lemma 3.6

\[
\int_{\mathbb{T}^2} f_{\Lambda, E}^2(\theta) \, d\theta < C(\log \lambda + (\log N)^2/\rho)\lambda.
\]

The desired bound (2.14) now follows from (2.17) and Cauchy-Schwarz.

Next we turn to considerations involving the convergence of the \( \gamma_\Lambda \). For technical reasons, we also allow complex energies. In what follows, \( E \) and \( \eta \) are always real.

**Lemma 2.6.** Under the assumptions of Proposition 2.5 there are constants \( \delta > 0 \) and \( C(\lambda, |E| + |\eta|) \) such that

\[
|\gamma_\Lambda(E + i\eta) - \gamma_{2\Lambda}(E + i\eta)| \leq C(\lambda, |E| + |\eta|) (\text{diam } \Lambda)^{-\delta}
\]

for all squares \( \Lambda \subset \mathbb{Z}^2 \). Here \( 2\Lambda \) denotes the double of \( \Lambda \). In particular, the limit

\[
\lim_{\ell \to \infty} \gamma_{2^\ell \Lambda}(E + i\eta) =: \gamma_\infty^\Lambda(E + i\eta)
\]

exists for every \( \Lambda \) and \( E + i\eta \) and for all \( \ell \geq 0 \)

\[
|\gamma_{2^\ell \Lambda}(E + i\eta) - \gamma_\infty^\Lambda(E + i\eta)| \lesssim (2^\ell \text{diam } \Lambda)^{-\delta}
\]

uniformly in \( E + i\eta \) in bounded sets.

**Proof.** Fix a large square \( 2\Lambda \) of size \( 2N \), say. Partition it into four congruent squares \( \{\Lambda_j\}_{j=1}^4 \) of size \( N \). Let \( H_\Lambda(\theta) \) denote the operator which is the direct sum of the \( H_{\Lambda_j}(\theta) \). Then \( \text{rank}[H_\Lambda(\theta) - \widetilde{H}_\Lambda(\theta)] \leq 10N \). By Proposition 2.1 one has

\[
||G_\Lambda(\theta, E)|| + \max_{j=1,2,3,4} ||G_{\Lambda_j}(\theta, E)|| < e^{N^2}
\]

up to a \( \varepsilon \)-set of measure not exceeding \( e^{-N^p} \). Corollary 2.4 therefore implies that

\[
\bigg| \log |\det(H_{2\Lambda}(\theta) - (E + i\eta))| - \sum_{j=1}^4 \log |\det(H_{\Lambda_j}(\theta) - (E + i\eta))| \bigg| \lesssim

N^{b+1} + N \log(\lambda + |E| + |\eta|)
\]

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for all $\theta \in \mathcal{G}_\Lambda(E)$, where $\text{mes}[\mathbb{T}^2 \setminus \mathcal{G}_\Lambda(E)] < e^{-N^p}$. Integrating this last line over $\mathcal{G}_\Lambda(E)$ and applying Cauchy-Schwarz to the integral over $\mathbb{T}^2 \setminus \mathcal{G}_\Lambda(E)$ (as in the previous proof) yields

$$|\gamma_{2\Lambda}(E + \hat{\eta}) - \gamma_\Lambda(E + \hat{\eta})| \leq C(\lambda, |E| + |\eta|)N^{-\delta}$$

with $\delta = 1 - b$, as claimed. $\Box$

In the following proposition we identify the limit $\gamma^\infty_\Lambda$.

**Proposition 2.7.** Assume that $\omega \in \Omega_\varepsilon$ and that $\lambda$ is large, cf. Proposition 2.5. Then the limit $\gamma^\infty_\Lambda$ from the previous lemma does not depend on $\Lambda$. In fact,

$$\gamma^\infty_\Lambda(E) = \int \log |E - E'| \, dk(E')$$

for all $E$. Moreover, for every $E$

$$f_{\lambda,E}(\theta) \to \int \log |E - E'| \, dk(E')$$

for a.e. $\theta$.

**Proof.** This is basically the same as Section 4 in [1]. Denote

$$\int_{\mathbb{T}^2} N^{(\Lambda)}_\theta(\cdot) \, d\theta =: N^{(\Lambda)}(\cdot),$$

see Section 1. Then

$$N^{(\Lambda)}(\{E\}) \to k(\{E\}) \quad \text{as} \quad \text{diam}(\Lambda) \to \infty$$

for every $E$ that is not an atom of $k$ (in particular, a.e.). Hence they also converge in $L^2$. By definition and standard properties of the Hilbert transform,

$$\gamma_\Lambda(E) = \int \log |E - E'| \, dN^{(\Lambda)}(E') = \int \frac{N^{(\Lambda)}(E')}{E - E'} \, dE',$$

where the second equality holds for a.e. $E$. By $L^2$ boundedness of the Hilbert transform one has

$$\int \frac{N^{(\Lambda)}(E')}{E - E'} \, dE' \to \int \frac{k(E')}{E - E'} \, dE' = \int \log |E - E'| \, dk(E') \quad \text{as} \quad \text{diam}(\Lambda) \to \infty$$

in the $L^2$ sense w.r.t. $E$. By the previous lemma therefore

$$(2.18) \quad \gamma^\infty_\Lambda(E) = \int \frac{k(E')}{E - E'} \, dE' = \int \log |E - E'| \, dk(E')$$

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for a.e. $E$ (note that the previous equality holds for all $E+i\eta$ with $\eta \neq 0$ by virtue of the existence of the IDS as a weak limit). The right-hand side is clearly subharmonic in (complex) $E$. It is important to recall at this point that subharmonicity requires both the sub mean value property and upper semi-continuity (the latter being Fatou in the case of the logarithmic integral). As a uniform limit of continuous subharmonic functions $\gamma^\infty_h(E)$ is also subharmonic. Indeed, Lemma 2.6 guarantees that uniform convergence takes place in bounded sets of the complex $E+i\eta$ plane. Since two subharmonic functions that are equal a.e. are equal everywhere (which is an immediate consequence of the aforementioned two properties of subharmonic functions), it follows that (2.18) holds for all $E$. The final statement of the proposition is obtained by means by combining the previous one with the LDE Proposition 2.5. □

3 Polar sets and Cartan's theorem

In this section we present some material that it basically already contained in [11], see Section 8 there. However, the two-dimensional Cartan theorem proved there is not strong enough for our purposes because the functions are assumed to be bounded. It is simple to remove that assumption, though. As the resulting theorem, see Theorem 3.7 below, has both a stronger conclusion and weaker assumptions, we have decided to include it here with all details.

The following lemma is Cartan’s theorem, see [14] Section 11.2. It differs from the statement there only by allowing for the parameter $\varepsilon$.

**Lemma 3.1.** Let

$$u(z) = \int_{\mathbb{C}} \log |z - \zeta| d\mu(\zeta)$$

for some positive finite measure $\mu$. For any $0 < \varepsilon, H < 1$ there exist disks $\{D(z_j, r_j)\}_{j=1}^\infty$ with the property that

$$\sum_j r_j^\varepsilon \leq (5H)^\varepsilon$$

(3.2)

$$u(z) > -||\mu|| \left[\varepsilon^{-1} + \log \frac{1}{H}\right] \quad \text{for all} \quad z \in \mathbb{C} \setminus \bigcup_{j=1}^\infty D(z_j, r_j).$$

(3.3)

**Proof.** Fix $\varepsilon > 0$. For any $p > 0$ we say that $z$ is $p$-good if

$$\mu(D(z, r)) \leq pr^\varepsilon \quad \forall r > 0.$$
By a well-known covering theorem, see Stein [15] page 9, there are pairwise disjoint disks \( \{D(z_j, r_j/5)\}_{j=1}^{\infty} \) (possible empty) with the property that

\[
B_{z_p} := \{ z \in \mathbb{C} \mid z \text{ is } p\text{-bad} \} \subset \bigcup_{j=1}^{\infty} D(z_j, r_j)
\]

and

\[
\sum r_j^p \leq 5 \frac{1}{p} \|\mu\|.
\]

Setting \( p = H^{-\varepsilon} \|\mu\| \), this latter inequality is exactly (3.2). Furthermore, if \( z \notin B_{z_p} \), then

\[
u(z) \geq \int_{|z| \leq 1} \log |z - \zeta| \, d\mu(\zeta) = -\int_0^1 \frac{\mu(D(z, r))}{r} \, dr
\]

\[
\geq -\int_0^H p r^{-\varepsilon} \, dr - \int_H^1 \|\mu\| \, dr
\]

\[
= -\|\mu\| (e^{-1} + \log \frac{1}{H}),
\]

as claimed. \( \square \)

Observe that this has the following well-known

**Corollary 3.2.** Let \( u \) be as in (3.1). Then

\[ \dim[u = -\infty] = 0 \]

where \( \dim \) refers to Hausdorff dimension.

**Definition 3.3.** Let \( 0 < H < 1 \). For any subset \( B \subset \mathbb{C} \) we say that \( B \in \text{Car}_1(H) \) if \( B \subset \bigcup_j D(z_j, r_j) \) with

\[ \sum_j r_j \leq H. \] (3.4)

If \( d \) is a positive integer greater than one and \( B \subset \mathbb{C}^d \) we define inductively that \( B \in \text{Car}_d(H) \) if there exists some \( B_0 \in \text{Car}_{d-1}(H) \) so that

\[
B = \{(z_1, z_2, \ldots, z_d) : (z_2, \ldots, z_d) \in B_0 \text{ or } z_1 \in B(z_2, \ldots, z_d) \text{ for some } B(z_2, \ldots, z_d) \in \text{Car}_1(H)\}.
\]

We refer to the sets in \( \text{Car}_d(H) \) for any \( d \) and \( H \) collectively as Cartan sets.
The following lemma collects some well-known facts, see [14] and [13]. The proof of this lemma is in [11] (we assume there that \( u \) is bounded but this assumption is irrelevant).

**Lemma 3.4.** Suppose \( u : D(0, 2) \mapsto \mathbb{R} \cup \{-\infty\} \) is a subharmonic function satisfying

\[
\sup_{z \in D(0, 2)} u(z) \leq 1 \quad \text{and} \quad \sup_{-1 < r < 1} u(r) = 0.
\]

Let \( \mu \) be the Riesz measure of \( u \). For any \( z_0 \in D(0, \frac{1}{2}) \), \( 0 < r < \frac{1}{2} \), and \( H \in (0, 1) \) there exists \( B \in \text{Car}_1(H) \) so that

\[
|u(z) - u(z')| < C \left[ \mu(D(z_0, r)) \log \frac{1}{H} + |z - z'| \left( 1 + \int_{D(0, 1) \setminus D(z_0, r)} \frac{d\mu(\zeta)}{|z_0 - \zeta|} \right) \right]
\]

for all \( z, z' \in D(z_0, r/2) \setminus B \). In particular, if for some \( A \geq 1 \)

\[
M_1 \mu(z_0) = \sup_{0 < t < A} \frac{\mu(D(z_0, t))}{t} \leq A,
\]

then

\[
|u(z) - u(z')| < C A \left[ r \log \frac{1}{H} + |z - z'| \log \frac{1}{r} \right]
\]

for all \( z, z' \in D(z_0, r/2) \setminus B \).

Our main concern in this section is to obtain a suitable analogue of the previous lemma that applies to functions of two variables which are subharmonic in each variable. The precise meaning of this is given in the following definition.

**Definition 3.5.** Let \( u \) be a continuous function on \( \Pi(0, 2) := D(0, 2) \times D(0, 2) \subset \mathbb{C}^2 \) with values in \( \mathbb{R} \cup \{-\infty\} \) so that

\[
\sup_{\Pi(0, 2)} u \leq 1 \quad \text{and} \quad \sup_{-1 < x_1, x_2 < 1} u(x_1, x_2) = 0.
\]

Suppose further that

\[
\begin{aligned}
\{ z_1 \mapsto u(z_1, z_2) \} & \quad \text{is subharmonic for each} \quad z_2 \in D(0, 2) \\
\{ z_2 \mapsto u(z_1, z_2) \} & \quad \text{is subharmonic for each} \quad z_1 \in D(0, 2).
\end{aligned}
\]

Then \( u \) will be called separately subharmonic.

We first dispense with a small technical lemma.

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Lemma 3.6. For any separately subharmonic function $u$ one has

$$\int_{-1}^{1} \int_{-1}^{1} u^2(x_1, x_2) \, dx_1 \, dx_2 < C$$

where $C$ is some absolute constant.

Proof. Applying conformal transformations in each variable separately, one may assume that $u(0, 0) = 0$. Then $z_2 \mapsto u(0, z_2)$ is a subharmonic function with bounded Riesz measure and harmonic part. In particular,

$$-C < \int_{-1}^{1} u(0, x_2) \, dx_2 < C.$$

Then the subharmonic function

$$v(z_1) := \int_{-1}^{1} u(z_1, x_2) \, dx_2$$

satisfies

$$v(0) > -C \quad \text{and} \quad \sup_{|z_1| \leq 2} v(z_1) \leq 2.$$

Hence, by Cartan’s estimate

$$\text{mes}[x_1 \in [-1, 1] : v(x_1) \leq -Ct] \leq e^{-ct}$$

for every $t \geq 1$. If $v(x_1) \geq -Ct$, then also $\sup_{-1 < x_2 < 1} u(x_1, x_2) \geq -Ct$. Hence, via the Riesz representation in the second variable

$$\int_{-1}^{1} u^2(x_1, x_2) \, dx_2 < Ct^2$$

in that case. One concludes from (3.9) that therefore

$$\text{mes}\left[x_1 \in [-1, 1] : \int_{-1}^{1} u^2(x_1, x_2) \, dx_2 > Ct\right] \leq e^{-ct}$$

for $t \geq 1$. Clearly, this implies the lemma. \qed

Theorem 3.7. Let $u$ be a separately subharmonic function as in Definition 3.5. Fix some $\gamma \in (0, \frac{1}{2})$. Given $r \in (0, 1)$ and $r < \rho < 1$ there exists $\Omega_0 \subset [-1, 1]^2$ with

$$\text{mes}([-1, 1]^2 \setminus \Omega_0) \leq \rho$$

and a set $B \in \text{Car}_2\left(e^{-r^{-\gamma}}\right)$, as defined in Definition 3.3, such that for every choice of $(x_1^{(0)}, x_2^{(0)}) \in \Omega_0$ the polydisk $\Pi = D(x_1^{(0)}, r^{1-\gamma}) \times D(x_2^{(0)}, r)$ satisfies

$$\begin{aligned}
|u(z_1, z_2) - u(z_1', z_2')| &< \frac{C_r}{\rho^2} r^{1-2\gamma} \log \frac{1}{r} \quad \text{for all} \quad (z_1, z_2), (z_1', z_2') \in \Pi \setminus B.
\end{aligned}$$

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Proof. Applying conformal transformations in each variable separately, one may assume that $u(0,0) = 0$. Then the subharmonic function $z_2 \mapsto u(0,z_2)$ has finite Riesz measure and bounded harmonic part. The Riesz representation therefore implies that

$$
(3.11) \quad \int_{-1}^{1} u(0,x_2) \, dx_2 > -C
$$

with some absolute constant $C$. For any $z_1 \in D(0,2)$ define

$$
(3.12) \quad v(z_1) = \int_{-1}^{1} u(z_1,x_2) \, dx_2.
$$

The subharmonic function $v : D(0,2) \to \mathbb{R}$ satisfies $v(0) > -C$, see (3.11), and $v \leq 1$. Hence its Riesz measure $\mu_v$ is bounded, and

$$
(3.13) \quad \int_{-1}^{1} v(x_1) \, dx_1 > -C
$$

for some absolute constant. Let $M_1 \mu_v$ be the maximal function given by (3.6). Clearly, $M_1$ satisfies the usual weak-type $L^1$ inequality

$$
(3.14) \quad \text{mes} \left[ x_1 \in [-1,1] : M_1 \mu_v(x_1) > \lambda \right] \leq \frac{C}{\lambda} \mu_v(D(0,\frac{3}{2})).
$$

In view of (3.13) and (3.11), up to a set of $x_1$-measure at most $\rho$ one has both $M_1 \mu_v(x_1) \approx \rho^{-1}$ and $v(x_1) \approx -\rho^{-1}$. Pick one such $x_1^{(0)}$. For any $z_2 \in D(0,2)$ let

$$
(3.15) \quad g_t(z_2) = \int_{0}^{1} u(x_1^{(0)} + te^{2\pi i \theta},z_2) \, d\theta - u(x_1^{(0)},z_2).
$$

By Jensen's formula, see Theorem 2 in Section 7.2 of [14],

$$
(3.16) \quad g_t(z_2) = \int_{\{z_2 - x_1^{(0)}\} \cap \{|z_2 - x_1^{(0)}| \leq t\}} \log \frac{t}{|z_2 - x_1^{(0)}|} \, \mu(dz_1, z_2) = \int_{0}^{t} \frac{n(s,z_2)}{s} \, ds
$$

where $n(s,z_2) = \mu(D(x_1^{(0)},s),z_2)$ with the Riesz measure $\mu(\cdot,z_2)$ of $u(\cdot,z_2)$. Clearly,

$$
\mu_v(D(x_1^{(0)},s)) = \int_{-1}^{1} n(s,x_2) \, dx_2.
$$

Therefore, in view of (3.15), (3.16), and our choice of $x_1^{(0)}$,

$$
(3.17) \quad \int_{-1}^{1} g_t(x_2) \, dx_2 = \int_{0}^{t} \frac{\mu_v(D(x_1^{(0)},s))}{s} \, ds \leq C \frac{t}{\rho}.
$$
Now fix some \( r \in (0, 1/2) \) and define
\[
G = \sum_{0 \leq j < C \log \frac{1}{r}} 2^{-j} g_{2j^r}.
\]
The subharmonicity of \( z_1 \mapsto u(z_1, z_2) \) implies that \( g_t \geq 0 \) so that \( G \) is the sum of nonnegative terms. By (3.17)
\[
\int_{-1}^{1} G(x_2) dx_2 \leq C \frac{\log \frac{1}{r}}{r}.
\]
and thus
\[
\text{mes}\{x_2 \in [-1, 1]: G(x_2) > \frac{C}{\rho^2} r \log \frac{1}{r}\} < \rho
\]
provided \( C \) is a sufficiently large absolute constant. For technical reasons we introduce the auxiliary subharmonic function
\[
h(z_2) = \int_0^1 u(x_1^{(0)} + r^2 e^{2\pi i \theta}, z_2) d\theta \quad \text{for any} \; z_2 \in D(0, 2).
\]
We denote the Riesz measure of \( h \) by \( \mu_h \). The function \( g_t \) introduced in (3.15) is the difference of two subharmonic functions on \( D(0, 2) \). Let \( \mu_t \) and \( \mu_0 \) be their respective Riesz measures. By our choice of \( x_1^{(0)}, v(x_1^{(0)}) > -\frac{C}{\rho} \) and thus
\[
\sup_{-1 < x_2 < 1} u(x_1^{(0)}, x_2) > -\frac{C}{\rho}.
\]
Since also \( \sup_{z_2 \in D(0, 2)} u(x_1^{(0)}, z_2) \leq 1 \), the Riesz measure \( \mu_0 \) satisfies the bound
\[
\mu_0(D(0, 3/2)) \leq \frac{C}{\rho}.
\]
Since the integral in (3.15) is point wise bigger than \( u(x_1^{(0)}, z_2) \) for any choice of \( z_2 \), one concludes that (3.20) also holds for \( \mu_t \) for any \( t \) (and thus in particular for \( \mu_h \)). By the weak-\( L^1 \) bound on \( M_1 \) and (3.20) all points \( x_2 \in [-1, 1] \) up to a set of measure at most \( \rho \) satisfy
\[
M_1 \left( \sum_{0 \leq j < C \log \frac{1}{r}} \mu_{2j^r} + \mu_0 + \mu_h \right)(x_2) \leq \frac{C}{\rho^2} \log \frac{1}{r}.
\]
In view of Lemma 3.4, there exists $B_0 = B_0(x_1^{(0)}) \in \text{Car}_1(\exp(-r^{-\gamma}))$ so that for any such $x_2$

\begin{equation}
\sup_{0 \leq j < C \log \frac{1}{r}} |g_{2j}r(z_2) - g_{2j}r(z'_2)| < \frac{C}{\rho^2} r^{1-\gamma} \log \frac{1}{r} \quad \text{for all } z_2, z'_2 \in D(x_2, r) \setminus B_0.
\end{equation}

Combining (3.18) and (3.22) shows that up to an $x_2$-set of measure at most $\rho$,

\begin{equation}
g_{2j}r(z_2) \leq \frac{C}{\rho^2} [2^j r + r^{1-\gamma}] \log \frac{1}{r} \quad \text{for all } z_2 \in D(x_2^{(0)}, r) \setminus B_0 \quad \text{and} \quad 0 \leq j < C \log \frac{1}{r}.
\end{equation}

Now fix such a point $x_2^{(0)} \in [-1, 1]$. Using (3.16) one concludes from (3.23) that

\[ \mu(D(x_1^{(0)}, 2^j r), z_2) \leq \frac{C}{\rho^2} [2^j r + r^{1-\gamma}] \log \frac{1}{r} \]

for all $z_2$ and $j$ as before. Inserting this bound into (3.5) with $H = \exp(-r^{-\gamma})$ and $r^{1-\gamma}$ instead of $r$ one obtains for any such $z_2$ a Cartan set $B(z_2) \in \text{Car}_1(H)$ so that

\begin{equation}
|u(z_1, z_2) - u(z'_1, z_2)| \leq \frac{C}{\rho^2} [r^{1-\gamma} \log \frac{1}{r} \log \frac{1}{r} + |z_1 - z'_1| \log \frac{1}{r}]
\end{equation}

\[ \leq \frac{C}{\rho^2} r^{1-2\gamma} \log \frac{1}{r} \quad \text{for any } z_1, z'_1 \in D(x_1^{(0)}, r^{1-\gamma}) \setminus B(z_2). \]

To control the deviation in $z_2$ we invoke the auxiliary subharmonic function $h$ from above. Because of (3.21) Lemma 3.4 implies that

\begin{equation}
|h(z_2) - h(z'_2)| \leq \frac{C}{\rho^2} r^{1-\gamma} \log \frac{1}{r} \quad \text{for all } z_2, z'_2 \in D(x_2^{(0)}, r) \setminus B_1
\end{equation}

where $B_1 = B_1(x_1^{(0)}) \in \text{Car}_1(H) \supseteq \exp(-r^{-\gamma})$. By the definition of a Cartan set and (3.24),

\begin{equation}
|h(z_2) - u(z_1, z_2)| \leq \frac{C}{\rho^2} [r^{1-2\gamma} \log \frac{1}{r} + r^{2\gamma} H] \quad \text{for all } z_2 \in D(x_2^{(0)}, r) \setminus B_0,
\end{equation}

\[z_1 \in D(x_1^{(0)}, r) \setminus B(z_2).\]

Let $\Pi = D(x_1^{(0)}, r^{1-\gamma}) \times D(x_2^{(0)}, r)$ and

\[ \mathcal{B} = \{(z_1, z_2) : z_2 \in B_0 \cup B_1 \text{ or } z_2 \in D(x_2^{(0)}, r) \setminus B_0 \cup B_1 \text{ and } z_1 \in B(z_2)\}. \]
In view of Definition 3.3, $B \in \text{Car}_2(H)$ with $H = \exp(-r^{-\gamma})$. Combining (3.26) with (3.25) implies that

$$|u(z_1, z_2) - u(z'_1, z'_2)| \leq \frac{C}{\rho^2} r^{1-2\gamma} \log \frac{1}{r}$$

for all $(z_1, z_2), (z'_1, z'_2) \in \Pi \setminus B$,

as claimed. Let $\Omega_0$ be the set of all $(x^{(0)}_1, x^{(0)}_2)$ as above. By construction, $\text{mes}([-1, 1]^2 \setminus \Omega_0) \lesssim \rho$. Notice that the set $B$ depends on $\Pi$. To finish the proof cover $\Omega_0$ by about $r^{-2}$ many such polydiscs and take the union of the resulting sets $B$. \hfill \Box

References


DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, FINE HALL, PRINCETON N.J. 08544, U.S.A.
email: schlag@math.princeton.edu