ON MINIMA OF THE ABSOLUTE VALUE OF CERTAIN RANDOM EXPONENTIAL SUMS

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Abstract. Let $T_n(x) = \sum_{j=1}^n \pm e^{2\pi i j^2 x}$ where $\pm$ stands for a random choice of sign with equal probability. It is shown here that with high probability $\min_{x\in[0,1)} |T_n(x)| < n^{-\sigma}$ provided n is large and $\sigma < 1/12$. Similar results are proved for other powers than squares. The problem of determining the optimal σ is open. For the case $T_n(x) = \sum_{j=1}^n r_j e^{2\pi i j^d x}$, where $d = 2, 3, \dots$ is fixed and with standard normal r_j we show that the minima are typically on the order of $n^{-d+\frac{1}{2}}$ with high probability and for large n.

1. Introduction

Let $T_n(x) = \sum_{j=1}^n \pm e(j^2 x)$ where the signs are chosen independently with probability $\frac{1}{2}$ each. Throughout this paper $e(x) = e^{2\pi i x}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Theorem 1. *For any $\sigma < 1/12$, $\mathbb{P}(\min_{x\in\mathbb{T}} |T_n(x)| < n^{-\sigma}) \to 1$ as $n \to \infty$.*

The study of minima of random trigonometric polynomials originates in [9], where Littlewood posed the problem of showing that

$$\mathbb{P}\left(\min_{x\in\mathbb{T}}\left|\sum_{j=1}^n \pm e(jx)\right| < \epsilon\sqrt{n}\right) \to 1 \quad \text{as} \quad n \to \infty$$

for any $\epsilon > 0$. This was proved by Kashin [5]. In fact, a much stronger statement holds as shown by Konyagin [6]:

$$\mathbb{P}\left(\min_{x\in\mathbb{T}}\left|\sum_{j=1}^n \pm e(jx)\right| < n^{-\frac{1}{2}+\epsilon}\right) \to 1 \quad \text{as} \quad n \to \infty$$

for any $\epsilon > 0$. In [7] Konyagin and the author then showed that $n^{-\frac{1}{2}}$ is the correct order of magnitude, i.e.,

$$\limsup_{n\to\infty} \mathbb{P}\left(\min_{x\in\mathbb{T}}\left|\sum_{j=1}^n \pm e(jx)\right| < \epsilon n^{-\frac{1}{2}}\right) \le C\epsilon$$

for any $\epsilon > 0$. The purpose of this paper is to prove Theorem 1 and some related results. The basic approach will resemble that in [6], but at several crucial points completely different arguments are required. The most essential part of our argument involves bounding the discrepancy of the sequence

$$(1) \qquad \Big(\alpha_1(j^2 - k^2), \alpha_2(j^2 - k^2)\Big)_{j,k=1}^n \qquad \mod \mathbb{Z}^2$$

1991 *Mathematics Subject Classification.* Primary 42A05, 42A61 ; Secondary 30C15, 60F05.

The author is very grateful to S. V. Konyagin for several helpful comments on a preliminary version of this paper and for pointing out reference [12].

where α_1 and α_2 can be thought of as rational numbers with denominators of size about n (loosely speaking this is related to the pair correlations considered by Rudnik and Sarnak [11]. However, their objective is different from ours and they also consider only one coordinate). Obtaining a nontrivial bound requires a suitable independence condition on α_1 and α_2, see (18) below. Since we want this condition to hold for sufficiently many pairs (α_1, α_2), we obtain only weak bounds for the discrepancy of (1) and therefore only small σ in Theorem 1. However, even the optimal bounds on the discrepancy of (1) would only yield $\sigma < \frac{1}{2}$ in Theorem 1 using our methods. This is most likely not the correct order of magnitude for the minimum, as it is shown in Section 3 that in the Gaussian case the optimal $\sigma = \frac{3}{2}$, see Theorems 16 and 20 in Section 3 (Gaussian here means that the coeffcients are chosen to be standard normal rather than Rademacher variables). It is possible, however, that the order of magnitude of the minima depends on the distribution of the coefficients (notice that in the case of the polynomials of Littlewood mentioned above this is not the case, see [6] and [7]).

We now give a heuristic argument that should give some indication why the minima are on the order of $n^{-\frac{3}{2}}$. Let T_n be as above. By the Salem–Zygmund inequality [4]

$$\|T_n\|_\infty \le C\sqrt{n\log n}$$

with probability tending to 1 as $n \to \infty$ (we refer to this as asymptotically almost surely or a.a.s.). So up to a logarithm, which we want to ignore here, the values of T_n lie in a square of size about $\sqrt{n}$ a.a.s.. Split this square into about n^4 many squares of side length $n^{-\frac{3}{2}}$ and pick a minimal n^{-4}-net $\mathcal{N}$ on $\mathbb{T}$. The point is that one expects to hit most of the small squares with the values $\{T_n(\alpha)\}_{\alpha\in\mathcal{N}}$ and therefore also the square at the origin or one nearby. To see this notice firstly that one would expect that with high probability $|T_n'(x)|$ is of size approximately $n^2\sqrt{n}$ for typical x. Moreover, $\|T_n''\|_\infty \le Cn^4\sqrt{n\log n}$ a.a.s. by Salem–Zygmund. Suppressing logarithms, one therefore concludes from Taylor's theorem that

$$|T_n(\beta) - T_n(\alpha)| = Cn^2\sqrt{n}|\beta-\alpha|\Big[1 + O(n^2(\beta-\alpha))\Big].$$

This means that any two points in the net $\mathcal{N}$ of distance $< n^{-2}$ should fall into distinct squares. In particular, any n^2 adjacent points in our net should fall into different squares. Secondly, we claim that for typical points α, β at a distance greater than n^{-2} the random variables $T_n(\alpha)$ and $T_n(\beta)$ are weakly correlated. In fact,

$$\mathbb{E}[T_n(\alpha)\overline{T_n(\beta)}] = \sum_{j=1}^{n} e((\alpha-\beta)j^2).$$

It is a standard fact of analytic number theory that the Weyl sum on the right–hand side is much smaller in absolute value than n for $\alpha - \beta$ which do not come too close to fractions with small denominators, see [10]. Since $n = \sqrt{\mathbb{E}|T_n(\alpha)|^2}\sqrt{\mathbb{E}|T_n(\beta)|^2}$ our claim follows. Assuming for simplicity that the values of T_n at such points are actually independent, we are in the situation of picking n^2 of the small squares randomly and independently n^2 many times. It is a simple exercise to see that the expected number of the small squares we select this way is $\asymp n^4$, which is the total number of those squares.

Finally, we would like to point out that a similar heuristic argument also applies to other powers than squares. More precisely, one easily checks along the same lines as before that one might expect for any fixed $d = 2, 3, \ldots$ and with $\omega_d = (2d-1)/2$,

$$\min_{x\in\mathbb{T}} \Bigl| \sum_{j=1}^{n} \pm e(j^d x) \Bigr| < n^{-\omega_d+\epsilon}$$

for most choices of $\pm$ as $n \to \infty$. As far as rigorous results are concerned, it is not hard to see that the basic approach from Section 2 yields

$$\mathbb{P}\Bigl(\min_{x\in\mathbb{T}} \Bigl| \sum_{j=1}^{n} \pm e(j^d x) \Bigr| < n^{-\sigma_d} \Bigr) \to 1 \quad \text{as} \quad n \to \infty$$

for some $\sigma_d > 0$. See Section 2.4 for further discussion of higher powers. Contrary to the behavior suggested by the heuristic argument, the σ_d tend to zero with increasing d. Moreover, it is shown in Section 3 that ω_d is the correct power in the Gaussian case for all $d \ge 2$.

2. The Bernoulli case

In this section we prove Theorem 1 above. Following [6], we use the second moment method. More precisely, fix some $0 < \delta_1 < 1/20$ and pick a minimal n^{-2}–net $\mathcal{N} = \{\alpha_s\}_{s=1}^{M}$ in

$$\mathbb{T} \setminus \bigcup_{q=1}^{\lceil n^{1-\delta_1} \rceil} \bigcup_{p=1}^{q} \Bigl[\frac{p}{q} - \frac{1}{qn}, \frac{p}{q} + \frac{1}{qn} \Bigr].$$

Here $\lceil a \rceil$ is the smallest integer greater or equal to a. Clearly, $M \asymp n^2$ where the notation $a \ll b$ means $a \le Cb$ for some absolute constant C and $a \asymp b$ stands for $a \ll b$ and $b \ll a$. Fix $\sigma > 0$ and let

$$E_\alpha = \{ |T_n(\alpha)| < n^{-\sigma} \} \tag{2}$$

for all $\alpha \in \mathcal{N}$. By Cauchy-Schwarz (the “second moment method”),

$$\mathbb{P}\Bigl(\bigcup_{\alpha\in\mathcal{N}} E_\alpha \Bigr) \ge \frac{\Bigl(\sum_{\alpha\in\mathcal{N}} \mathbb{P}(E_\alpha) \Bigr)^2}{\sum_{\alpha,\beta\in\mathcal{N}} \mathbb{P}(E_\alpha \cap E_\beta)}. \tag{3}$$

The goal of this paper is to show that the right-hand side tends to one provided $\sigma < 1/12$. The number $1/12$ is most likely a technical artifact and it is possible that one can improve on it by being more economical in the following argument. A substantial improvement, however, will require new ideas. The harder part is of course bounding the variances (the denominator) and it is here that (1) becomes relevant. One might expect that $\mathbb{P}(E_\alpha) \asymp n^{-1-2\sigma}$ as in the Gaussian case (Gaussian here means that the coefficients are standard normal rather than $\pm$). However, it is easy to see that

$$\mathbb{P}\Bigl(T_{4n}\Bigl(\frac{1}{4}\Bigr) = 0 \Bigr) \asymp \frac{1}{n}.$$

This suggests that fractions with small denominators play a special role, which explains the choice of $\mathcal{N}$ above. Furthermore, one cannot use normal approximation to prove this bound since the error in the central limit theorem is $n^{-\frac{1}{2}}$ (the Berry–Esseen theorem, see [1]). For this reason such small probabilities are calculated in [6] and [7] directly by means of the characteristic functions (i.e., the Fourier transforms) of the random variables at hand. Finally, we would like to remark that the case $\sigma < 0$ in Theorem 1 is much easier than the case $\sigma > 0$. In fact, it is implicit in our argument below that $\sigma < 0$ is covered by some fairly standard central limit type theorems, see Theorem 8.4 in [1]. In particular, basically no number theory is required for that case.

2.1. **Estimating $\mathbb{P}(E_\alpha)$.** We turn to the numerator in (3). In that case the correct estimate only requires $\sigma < 1$.

Proposition 2. *Fix some $\sigma \in (0,1)$ and let E_α be given by (2). Then*

$$\mathbb{P}(E_\alpha) = c_0 n^{-1-2\sigma}(1+o(1)) \quad \textit{as } n \to \infty$$

uniformly in $\alpha \in \mathcal{N}$. Here c_0 is an absolute constant.

We shall prove this after establishing some lemmas. For any sequence $x_1, \ldots, x_n \in \mathbb{T}$ we let D_n be the usual discrepancy over intervals, i.e.,

$$D_n(\{x_j\}_{j=1}^n) = \sup_{I \subset \mathbb{T}} \Big| |I| - \frac{1}{n} \mathrm{card}\{j \in [1,n] : x_j \in I\} \Big|,$$

where the supremum is taken over intervals. The main tool for bounding D_n is the well–known Erdös–Turan inequality, see [8] and [10],

$$D_n(\{x_j\}_{j=1}^n) \ll \frac{1}{K} + \frac{1}{n} \sum_{k=1}^K \Big| \frac{1}{k} \sum_{\ell=1}^n e(kx_\ell) \Big|. \tag{4}$$

Throughout, $\|\cdot\|$ denotes the distance to the nearest integer. The following lemma is standard.

Lemma 3. $\max_{\alpha \in \mathcal{N}} D_n(\{\alpha j^2\}_{j=1}^n) \ll n^{-\frac{1}{2}+\delta_1}$

Proof. By the Erdös–Turan inequality with $K = n$

$$\begin{aligned} D_n(\{\alpha j^2\}_{j=1}^n) &\ll \frac{1}{n} + \sum_{\ell=1}^n \frac{1}{\ell} \left| \frac{1}{n} \sum_{j=1}^n e(\alpha \ell j^2) \right| \\ &\ll \frac{1}{n} + \frac{1}{n} \left(\sum_{\ell=1}^n \frac{1}{\ell} \Big| \sum_{j=1}^n e(\alpha \ell j^2) \Big|^2 \right)^{\frac{1}{2}} \sqrt{\log n} \\ &\ll \frac{1}{n} + \frac{\sqrt{\log n}}{n} \left(n \log n + \sum_{\ell=1}^n \frac{1}{\ell} \sum_{u=1}^{n-1} \min(n, \|2\alpha \ell u\|^{-1}) \right)^{\frac{1}{2}}. \qquad (5) \end{aligned}$$

To pass to the previous line one uses the standard Weyl differencing method [10], chapter 3:

$$\begin{aligned} \Big| \sum_{j=1}^n e(\alpha \ell j^2) \Big|^2 &= \Big| \sum_{j,k=1}^n e(\alpha \ell (j^2 - k^2)) \Big| \\ &\le n + \Big| \sum_{j=1}^n \sum_{u \in \mathbb{Z}, u \neq 0} \chi_{[j+u \in [1,n]]} e(\alpha \ell (j^2 - (j+u)^2)) \Big| \\ &\le n + 2 \sum_{u=1}^{n-1} \Big| \sum_{j=1}^n e(-2\alpha \ell j u) \Big| \le n + 2 \sum_{u=1}^{n-1} \min(n, \|2\alpha \ell u\|^{-1}), \qquad (6) \end{aligned}$$

and (5) follows. We now recall a standard estimate for reciprocal sums as in (5), see equation (9) in [10]:

$$\sum_{h=1}^{H} \min(N, \|h\alpha\|^{-1}) \ll \frac{HN}{q} + H\log q + N + q\log q \tag{7}$$

provided $|\alpha - \frac{p}{q}| \le q^{-2}$ with $(p,q)=1$. Let

$$s_\ell = \sum_{k=1}^{\ell}\sum_{u=1}^{n-1} \min(n, \|2\alpha k u\|^{-1}).$$

By Dirichlet's principle, $|\alpha - \frac{p}{q}| \le \frac{1}{nq}$ for some integers $1 \le p \le q \le n$. In view of the definition of $\mathcal{N}$ one has $q \ge n^{1-\delta_1}$. Therefore, for any $1 \le \ell \le n$,

$$\begin{aligned} s_\ell &\le \sum_{h=1}^{2\ell(n-1)} \sum_{k=1}^{\ell} \chi_{[k|h]} \min(n, \|\alpha h\|^{-1}) \le C_\epsilon n^\epsilon \sum_{h=1}^{2\ell(n-1)} \min(n, \|\alpha h\|^{-1}) \\ &\le C_\epsilon n^\epsilon \left(\frac{\ell n^2}{q} + (n\ell + q)\log q \right) \le C_\epsilon\, \ell n^{1+\delta_1+\epsilon}. \end{aligned}$$

To pass to the second inequality we used that the number of divisors of h grows at most like h^ϵ, see [3], and the third inequality follows from (7). Applying partial summation to (5) finally implies that

$$\begin{aligned} D_n(\{\alpha j^2\}_{j=1}^n) &\ll \frac{\log n}{\sqrt{n}} + \frac{\sqrt{\log n}}{n} \left(\log n \max_{1\le \ell \le n} \frac{s_\ell}{\ell} \right)^{\frac12} \\ &\ll \frac{\log n}{\sqrt{n}} + C_\epsilon \frac{\log n}{n} n^{(1+\delta_1+\epsilon)/2}. \end{aligned}$$

Hence the lemma for n large. ☐

Lemma 4 is the first of two essential technical statements about the number of elements in a particular sequence $\{\psi_j\}_{j=1}^n$ that come close to integers. The ψ_j arise naturally in the characteristic function of $T_n(\alpha)$, see the proof of Proposition 2 below.

Lemma 4. *Suppose $\alpha \in \mathcal{N}$, $1/4 \le v \le n^{1-2\delta_1}$, and $\theta \in [0, 2\pi)$ are arbitrary but fixed. For any $j \in \mathbb{Z}$ let $\psi_j = v\cos(2\pi\alpha j^2 + \theta)$. Then*

$$\operatorname{card}\{j \in [1,n] : \|\psi_j\| < n^{-\delta_1}\} \le n/2 \tag{8}$$

provided $n > n_0(\delta_1)$ (with n_0 depending only on δ_1).

Proof. Assume false. Then there exist integers m_j such that $|\psi_j - m_j| < n^{-\delta_1}$ for at least $n/2$ values of j. Thus

$$|\cos(2\pi\alpha j^2 + \theta) - \frac{m_j}{v}| < n^{-\delta_1} v^{-1} = \Delta$$

for those j. If $c_1 > 0$ is a small constant then the previous lemma implies that

$$\operatorname{card}\{j : |\sin(2\pi\alpha j^2 + \theta)| < c_1\} < n/4 \tag{9}$$

if n is large. Define the intervals I_ℓ and J_ℓ via

$$I_\ell \cup J_\ell = \{x \in \mathbb{T} : |\cos(2\pi x + \theta) - \ell/v| < \Delta\}.$$

Discarding the points falling into the set in (9) we conclude that

$$\sum_{\ell=-L}^{L}\sum_{j=1}^{n}\chi_{I_\ell\cup J_\ell}(\alpha j^2) > n/4$$

where $L=[v]+1$ and $|I_\ell|\asymp|J_\ell|\asymp\Delta$. We may assume that

$$\sum_{\ell=-L}^{L}\sum_{j=1}^{n}\chi_{I_\ell}(\alpha j^2) > n/8. \tag{10}$$

Pick a nonnegative bump function ϕ_Δ so that $\phi_\Delta\ge 1$ on $[-C\Delta, C\Delta]$ and

$$\mathrm{supp}(\widehat{\phi_\Delta})\subset[-C\Delta^{-1}, C\Delta^{-1}]$$

where C is a suitable constant. Applying Cauchy–Schwarz to (10) yields

$$\begin{aligned} n^2 &\ll \sum_{\ell=-L}^{L}\left(\sum_{j=1}^{n}\chi_{I_\ell}(\alpha j^2)\right)^2 v \le \sum_{j,k=1}^{n}\chi_{[-C\Delta,C\Delta]}(\alpha(j^2-k^2))\,v \\ &\ll vn + v\sum_{j\neq k=1}^{n}\phi_\Delta(\alpha(j^2-k^2)) \\ &= nv + vn(n-1)\widehat{\phi_\Delta}(0) + v\sum_{j\neq k=1}^{n}\sum_{\nu\neq 0}\widehat{\phi_\Delta}(\nu)e(\alpha\nu(j^2-k^2)). \end{aligned} \tag{11}$$

Here $\sum_{j\neq k=1}^{n}$ means that we sum over $j,k=1,\ldots,n$ with the restriction that $j\neq k$. Therefore, substituting $k=j+u$ as in the proof of the previous lemma (see (6)),

$$\begin{aligned} n^2/v &\ll \Delta n^2 + \Delta\sum_{\nu=1}^{C/\Delta}\sum_{j=1}^{n}\min(n, \|2\alpha\nu j\|^{-1}) \\ &\ll \Delta n^2 + C_\epsilon\, n^\epsilon\Delta\sum_{k=1}^{Cn/\Delta}\min(n, \|\alpha k\|^{-1}). \end{aligned} \tag{12}$$

To pass to the second line we use that the divisor function grows more slowly than any power, see [3]. Now let $|\alpha-\frac{p}{q}|\le\frac{1}{nq}$ for some $1\le p\le q\le n$. In view of the definition of $\mathcal{N}$ one has $q\ge n^{1-\delta_1}$. Applying (7) to the reciprocal sum in (12) yields

$$\begin{aligned} n^2/v &\ll n^2\Delta + C_\epsilon\, n^\epsilon\Delta\left(\frac{n^2}{\Delta q} + (\Delta^{-1}n+q)\log q\right) \\ &\ll n^2\Delta + C_\epsilon\, n^\epsilon\left(n^{1+\delta_1} + n\log n\right). \end{aligned}$$

This leads to the contradiction $n^{1-\delta_1-\epsilon}\ll C_\epsilon\, v$ for all $\epsilon>0$. □

Remark : What underlies the proof of the previous lemma is the fact that the discrepancy of the sequence $\{\alpha(j^2-k^2)\}_{j\neq k=1}^{n}$ is small. In fact, one easily checks along the lines of Lemma 3 that it is no bigger than $n^{-\frac{1}{2}+\delta_1+\epsilon}$. One can introduce this bound explicitly into the proof of Lemma 4. Indeed, line (11) implies that

$$v^{-1}\ll\Delta + D_{n(n-1)}\left(\{\alpha(j^2-k^2)\}_{j\neq k=1}^{n}\right).$$

This, however, would lead to $\sigma < \frac{1}{2}$ rather than $\sigma < 1$ as above. This is precisely the reason why Lemma 4 is only *implicitly* based on the discrepancy of the sequence of differences.

In the following lemma we compute the mean covariance matrices of $T_n(\alpha)$, which we shall need in the proof of Proposition 2.

Lemma 5. *Let I be the 2×2 identity matrix. Then* $\mathrm{cov}(\frac{1}{\sqrt{n}}T_n(\alpha)) = \frac{1}{2}I + O(n^{-\frac{1}{2}+\delta_1})$ *uniformly in $\alpha\in\mathcal{N}$ as $n\to\infty$.*

Proof. By independence of the summands in T_n,

$$\mathrm{cov}(T_n(\alpha)) = \sum_{j=1}^n \mathrm{cov}\big(\pm(\cos(2\pi\alpha j^2), \sin(2\pi\alpha j^2))\big).$$

Therefore,

$$\mathrm{cov}\Big(\frac{1}{\sqrt{n}}T_n(\alpha)\Big) = \frac{1}{2}I + \frac{1}{2n}\sum_{j=1}^n \begin{bmatrix} \cos(4\pi j^2\alpha) & \sin(4\pi j^2\alpha) \\ \sin(4\pi j^2\alpha) & -\cos(4\pi j^2\alpha) \end{bmatrix}.$$

By Koksma's inequality, see Theorem 5.1 in [8], and Lemma 3

$$\left| \frac{1}{n}\sum_{j=1}^n \cos(4\pi\alpha j^2) - \int_0^1 \cos(4\pi x)\,dx \right| \ll D_n(\{\alpha j^2\}_{j=1}^n) \ll n^{-\frac{1}{2}+\delta_1} \tag{13}$$

and similarly for the other terms. □

Proof of Proposition 2. Fix a $\sigma\in(0,1)$ and choose δ_1 and $\epsilon>0$ such that $\sigma+2\epsilon < 1-2\delta_1$. Let $0\le\phi_n\le\chi_{B(0,1)}\le\psi_n\le 1$ in $\mathbb{R}^2$ with $\phi_n(x)=1$ if $|x|<1-n^{-\epsilon}$ and $\psi_n(x)=0$ if $|x|>1+n^{-\epsilon}$. Here $B(0,1)$ is the unit ball in $\mathbb{R}^2$. Moreover, we require that

$$\sup_{\xi\in\mathbb{R}^2}(1+n^{-\epsilon}|\xi|)^m\Big(|\widehat{\phi_n}(\xi)| + |\widehat{\psi_n}(\xi)|\Big) \le C_m \tag{14}$$

for any $m>1$. It is standard to construct such functions. Let f_α denote the characteristic function of $\frac{1}{\sqrt{n}}T_n(\alpha)$. Then clearly

$$\begin{aligned} \mathbb{P}(E_\alpha) &\ge \int_{\mathbb{R}^2}\phi_n(n^{\frac{1}{2}+\sigma}x)\,d\mathbb{P}_{\frac{1}{\sqrt{n}}T_n(\alpha)}(x) \\ &= n^{-1-2\sigma}\int_{A\cup B\cup C}\widehat{\phi_n}(n^{-\frac{1}{2}-\sigma}\xi)f_\alpha(\xi)\,d\xi. \end{aligned} \tag{15}$$

Here $A=\{|\xi|<n^{1/6}\}$, $B=\{n^{1/6}<|\xi|<n^{\frac{1}{2}+\sigma+2\epsilon}\}$, and $C=\{|\xi|>n^{\frac{1}{2}+\sigma+2\epsilon}\}$. The main contribution will come from region A, whereas B and C are error terms. To compute the integral over A we use a general statement about convergence of the characteristic function of a sum of independent (but not necessarily identically distributed) random variables with finite third moments. More precisely, by Theorem 8.4 in Bhattacharya–Rao [1]

$$|f_\alpha(\xi) - \exp(-\langle V_\alpha\xi,\xi\rangle/2)| \le C|\xi|^3 n^{-\frac{1}{2}}\exp(-c|\xi|^2) \tag{16}$$

for all $|\xi|\le n^{1/6}$ (actually, the only important condition is that $|\xi|$ is much smaller than $\sqrt{n}$). Here $V_\alpha = \mathrm{cov}\Big(\frac{1}{\sqrt{n}}T_n(\alpha)\Big)$ and C,c are absolute constants. We would like to emphasize that $|f_\alpha(\xi)|\asymp 1$ for certain ξ with $|\xi|\asymp n^{\frac{1}{2}}$ if α comes close to rational numbers with small denominators. In particular, (16) fails for such ξ. It is therefore necessary to take the arithmetic nature of α into account for values of $|\xi| > \sqrt{n}$. In view of Lemma 5 and (16)

$$\int_A \widehat{\phi_n}(n^{-\frac{1}{2}-\sigma}\xi) f_\alpha(\xi)\, d\xi = c\widehat{\phi_n}(0)(1+o(1)) = c_0(1+o(1)). \tag{17}$$

To estimate the contribution of the region B we write $f_\alpha(\xi) = \prod_{j=1}^n \cos(\pi\psi_j(\xi))$ where

$$\psi_j(\xi) = \frac{2\xi_1}{\sqrt{n}}\cos(2\pi j^2\alpha) + \frac{2\xi_2}{\sqrt{n}}\sin(2\pi j^2\alpha) = v\cos(2\pi j^2\alpha + \theta).$$

Here we have set $v = 2|\xi|/\sqrt{n}$ and $\theta = \arg(\xi)$. If not too many ψ_j come close to $\mathbb{Z}$, then $|f_\alpha(\xi)|$ will be small. If $v < \frac{1}{4}$, then each $|\psi_j| < \pi/4$ and one simply expands cos around zero. Otherwise one uses Lemma 4. In B one has $2n^{-\frac{1}{3}} < v < 2n^{\sigma+2\epsilon}$. Consider first the case $n^{-\frac{1}{3}} < v < 1/4$. In view of Koksma's inequality and Lemma 3 (replace cos with $\cos^2$ in (13)) one has

$$|f_\alpha(\xi)| \le \exp(-c\sum_{j=1}^n v^2\cos^2(2\pi\alpha j^2 + \theta)) \le \exp(-cn^{\frac{1}{3}}).$$

If $1/4 < v < n^{\sigma+2\epsilon}$, then by Lemma 4 (recall $\sigma + 2\epsilon < 1 - 2\delta_1$)

$$|f_\alpha(\xi)| \le \exp(-c\sum_{j=1}^n \|\psi_j\|^2) \le \exp(-cn^{1-2\delta_1}).$$

These estimates show that the integral over B goes to zero. Finally, it is immediate from (14) that the same holds true for the integral over C. An upper bound is obtained in a similar fashion using ψ_n instead, and we are done. □

2.2. **Estimating $\mathbb{P}(E_\alpha \cap E_\beta)$.** Let $(\alpha_1, \alpha_2) \in \mathcal{N}^2$. We say that this pair is *bad* if

$$\alpha_1\ell_1 + \alpha_2\ell_2 \in \bigcup_{q=1}^{A}\bigcup_{p=1}^{q}\Big[\frac{p}{q} - \frac{1}{qn^{2-2\tau}}, \frac{p}{q} + \frac{1}{qn^{2-2\tau}}\Big] \mod 1 \tag{18}$$

for some integers ℓ_1, ℓ_2 with $1 \le |\ell_1|, |\ell_2| \le A = \lceil n^\tau \rceil$. The intervals on the right-hand side are basically the major arcs from the Hardy–Littlewood circle method [13] for the Waring problem with squares. The value of τ will be specified below. The set of bad pairs will be denoted by $\mathcal{B} = \mathcal{B}_\tau$. Clearly,

$$\mathrm{card}(\mathcal{B}) \ll \sum_{\alpha_1\in\mathcal{N}}\sum_{\pm\ell_1,\pm\ell_2=1}^{A}\sum_{q=1}^{A}\sum_{p=1}^{q}\frac{n^2}{q\ell_2 n^{2-2\tau}}\ell_2 \asymp MA^3n^{2\tau} \asymp n^{2+5\tau}. \tag{19}$$

This bound becomes trivial if $\tau \ge 2/5$, so we may assume that $0 < \tau < 2/5$. The independence condition between α_1, α_2 given by (18) will allow us to obtain nontrivial bounds on the discrepancy of the multidimensional sequence

$$(\alpha_1(j^2-k^2), \alpha_2(j^2-k^2))_{j\neq k=1}^n. \tag{20}$$

The notation means that $j, k = 1, \ldots, n$ but $j \neq k$. Discrepancy here refers to discrepancy over intervals, i.e., given a sequence $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{T}^d$ on the d–dimensional torus, we define its discrepancy to be

$$D_N = \sup_{I\subset\mathbb{T}^d}\Big||I| - \frac{1}{N}\mathrm{card}\{j \in [1,N] \,:\, \mathbf{x}_j \in I\}\Big|.$$

The supremum is taken over all rectangles $I = I_1 \times \ldots \times I_d \subset \mathbb{T}^d$. Our basic tool for bounding the discrepancy is the Erdös–Turan–Koksma inequality, see [8],

$$D_N \ll \frac{1}{m} + \sum_{0<\|\mathbf{h}\|\le m} \frac{1}{r(\mathbf{h})}\Big|\frac{1}{N}\sum_{n=1}^{N} e(\langle \mathbf{h}, \mathbf{x}_n\rangle)\Big| \tag{21}$$

for any integer m. Here $\mathbf{h} = (h_1, \ldots, h_d) \in \mathbb{Z}^d$ and $r(\mathbf{h}) = \prod_{j=1}^d \max(1, |h_j|)$.

Lemma 6. *For any $(\alpha_1, \alpha_2) \in \mathcal{N}^2 \setminus \mathcal{B}$ one has*

$$\begin{aligned} D_n(\{\alpha_1 j^2, \alpha_2 j^2\}_{j=1}^n) &\ll n^{-\tau/2}\log^2 n \\ D_{n(n-1)}\Big(\{\alpha_1(j^2-k^2), \alpha_2(j^2-k^2)\}_{j\ne k=1}^n\Big) &\ll n^{-\tau}\log^2 n. \end{aligned}$$

Proof. It is well–known that, see chapter 3 in [10],

$$\left|\sum_{j=1}^{n} e(\alpha j^2)\right| \ll \frac{n}{\sqrt{q}} + \sqrt{(n+q)\log q} \tag{22}$$

provided $|\alpha - \frac{p}{q}| \le q^{-2}$ with $(p,q) = 1$. This is proved by the exact same methods as Lemma 3. In view of the Erdös–Turan inequality one has

$$\begin{aligned} D_n(\{\alpha_1 j^2, \alpha_2 j^2\}_{j=1}^n) &\ll \frac{1}{A} + \sum_{0<\|\mathbf{h}\|\le A}\frac{1}{r(\mathbf{h})}\Big|\frac{1}{n}\sum_{j=1}^n e((\alpha_1 h_1 + \alpha_2 h_2)j^2)\Big| \\ &\ll \frac{1}{A} + \sum_{\ell=1}^{A}\frac{1}{\ell}\Big(\Big|\frac{1}{n}\sum_{j=1}^n e(\alpha_1 \ell j^2)\Big| + \Big|\frac{1}{n}\sum_{j=1}^n e(\alpha_2 \ell j^2)\Big|\Big) \\ &\quad + \sum_{\ell_1,\ell_2=1}^{A}\frac{1}{\ell_1\ell_2}\Big|\frac{1}{n}\sum_{j=1}^n e((\alpha_1\ell_1 + \alpha_2\ell_2)j^2)\Big| \qquad (23)\\ &\ll A^{-1} + n^{-\frac{1}{2}+\delta_1} + n^{-\tau/2}\log^2 n. \qquad (24) \end{aligned}$$

The estimate that leads to the second term in (24) was obtained in the proof of Lemma 3, and we refer the reader to (5). For the third term, note that for any ℓ_1 and ℓ_2 that appear in the sum in (23), there exist integers p, q with $(p,q) = 1$ and $1 \le q \le n^{2-2\tau}$ such that $|\alpha_1\ell_1 + \alpha_2\ell_2 - \frac{p}{q}| \le q^{-2}$. In view of (18) one has $q \ge n^\tau$. Applying (22) to (23) leads to (24). To obtain the second statement of the lemma, one applies the Erdös–Turan–Koksma inequality as follows:

$$\begin{aligned} &D_{n(n-1)}\Big(\{\alpha_1(j^2-k^2), \alpha_2(j^2-k^2)\}_{j\ne k=1}^n\Big) \\ \ll\ &\frac{1}{A} + \frac{1}{n^2}\sum_{\ell=1}^{A}\frac{1}{\ell}\Big|\sum_{j\ne k=1}^{n} e(\alpha_1\ell(k^2-j^2))\Big| + \frac{1}{n^2}\sum_{\ell=1}^{A}\frac{1}{\ell}\Big|\sum_{j\ne k=1}^{n} e(\alpha_2\ell(k^2-j^2))\Big| \\ &+\frac{1}{n^2}\sum_{\ell_1,\ell_2=1}^{A}\frac{1}{\ell_1\ell_2}\Big|\sum_{j\ne k=1}^{n} e((\alpha_1\ell_1+\alpha_2\ell_2)(k^2-j^2))\Big|. \end{aligned}$$

Estimating the first and second sums as in the proof of Lemma 3 one sees that they do not exceed $n^{-1+\delta_1}\log n$. For the third sum let $\Big|\alpha_1\ell_1 + \alpha_2\ell_2 - \frac{p}{q}\Big| \le \frac{1}{qn^{2-2\tau}}$ where

$1 \le p \le q \le n^{2-2\tau}$. By the definition of $\mathcal{B}$ one has $q \ge n^\tau$. Therefore, in view of (6) and (7),

$$\begin{aligned} &\sum_{\ell_1,\ell_2=1}^{A} \frac{1}{\ell_1\ell_2}\Big|\sum_{j\neq k=1}^{n} e((\alpha_1\ell_1+\alpha_2\ell_2)(k^2-j^2))\Big| \\ \ll\quad &\sum_{\ell_1,\ell_2=1}^{A} \frac{1}{\ell_1\ell_2}\sum_{j=1}^{n} \min\Big(n, \|2(\alpha_1\ell_1+\alpha_2\ell_2)j\|^{-1}\Big) \\ \ll\quad &\sum_{\ell_1,\ell_2=1}^{A} \frac{1}{\ell_1\ell_2}\left(\frac{n^2}{n^\tau} + n^{2-2\tau}\log n\right) \ll n^{2-\tau}\log^2 n. \end{aligned}$$

The lemma follows. □

The second moment method requires us to compute the probabilities $\mathbb{P}(E_\alpha \cap E_\beta)$ for typical $\alpha, \beta \in \mathcal{N}$, see (3). This will be done in Proposition 10 below. The proof of that proposition is similar to the proof of Proposition 2. In particular, one needs to bound the number of points

$$\psi_j = v_1\cos(2\pi j^2\alpha + \theta_1) + v_2\cos(2\pi j^2\beta + \theta_2)$$

with $j = 1, \ldots, n$ that can come very close to integers for arbitrary but fixed $1 \le v_1^2 + v_2^2 \le n^{2\sigma}$ and $\theta_1, \theta_2 \in [0, 2\pi)$. We start with a simple observation concerning the level curves of $v_1\cos(2\pi x) + v_2\cos(2\pi y)$.

Lemma 7. *Let $F(x,y) = v_1\cos(2\pi x) + v_2\cos(2\pi y)$ with $v_1^2 + v_2^2 = 1$ and let $0 < \epsilon < 1$. Define*

$$B_\epsilon = \Big\{(x,y) \in \mathbb{T}^2 : |x| < \epsilon \;\; or \; |x - \tfrac{1}{2}| < \epsilon \;\; or \; |y| < \epsilon \;\; or \; |y - \tfrac{1}{2}| < \epsilon\Big\}.$$

Then $|\nabla F| \gg \epsilon$ on $\mathbb{T}^2 \setminus B_\epsilon$ and the curvature κ on any component of any level–curve $\{F = a\} \setminus B_\epsilon$ satisfies

$$\kappa \le C\epsilon^{-3}|v_1v_2| \tag{25}$$

with some absolute constant C.

Proof. Clearly,

$$|\nabla F(x,y)| = 2\pi(v_1^2\sin^2(2\pi x) + v_2^2\sin^2(2\pi y))^{\frac{1}{2}} \gg \epsilon$$

for any $(x,y) \in \mathbb{T}^2 \setminus B_\epsilon$. By calculus the curvature of a level set of F at a point is given by the ratio

$$\kappa = \frac{|\langle D^2F(\nabla F)^\perp, (\nabla F)^\perp\rangle|}{|\nabla F|^3}$$

at that point. One easily verifies that with F as above this is the same as

$$|\nabla F|^3\kappa = |v_1v_2||v_2\cos(2\pi x)\sin^2(2\pi y) - v_1\cos(2\pi y)\sin^2(2\pi x)|,$$

and the lemma follows. □

The following lemma is the main technical statement of our proof.

Lemma 8. *Suppose that* $(\alpha_1,\alpha_2)\in\mathcal{N}^2\setminus\mathcal{B}_\tau$, $1\le v_1^2+v_2^2\le c_2 n^{2\sigma}$, *and* $\theta_1,\theta_2\in[0,2\pi)$ *are arbitrary but fixed* (c_2 *is a sufficiently small absolute constant). Let*

$$\psi_j = v_1\cos(2\pi\alpha_1 j^2+\theta_1)+v_2\cos(2\pi\alpha_2 j^2+\theta_2)$$

for all $j\in\mathbb{Z}$. *Then*

$$\mathrm{card}\{j\in[1,n]:\|\psi_j\|\le n^{-\sigma}\}\le n/2$$

provided $2\sigma<\tau$ *and* $n>n_0(\sigma,\tau)$.

Proof. Let $(v_1^2+v_2^2)^{\frac12}=v$. Fix some $2\sigma<\tau$ and suppose the lemma fails. Then there are integers m_j such that for at least $n/2$ values of j

$$\left|w_1\cos(2\pi\alpha_1 j^2+\theta_1)+w_2\cos(2\pi\alpha_2 j^2+\theta_2)-\frac{m_j}{v}\right|\le n^{-\sigma}v^{-1}=\Delta$$

where we have set $w_1=v_1/v$, $w_2=v_2/v$. For the sake of convenience we set $\theta_1=\theta_2=0$, since a translation does not matter. Thus

$$\mathrm{card}\left\{j\in[1,n]:|F(\alpha_1 j^2,\alpha_2 j^2)-m/v|\le\Delta\ \text{ for some }\ m\in\mathbb{Z}\right\}>\frac{n}{2}, \tag{26}$$

where $F(x,y)=w_1\cos(2\pi x)+w_2\cos(2\pi y)$. Let B_ϵ be the set from Lemma 7 for some fixed small ϵ (say $\epsilon=1/100$). By the definition of discrepancy and Lemma 6,

$$\mathrm{card}\{j\in[1,n]:(\alpha_1 j^2,\alpha_2 j^2)\in B_\epsilon\}<n/4 \tag{27}$$

for small ϵ and large n. By Lemma 7 one therefore has $|\nabla F(\alpha_1 j^2,\alpha_2 j^2)|>\epsilon$ for at least half the points satisfying (26). Furthermore, on those points the curvature of the level sets is no larger than $\kappa\asymp|w_1w_2|\asymp\min(|v_1/v_2|,|v_2/v_1|)\ll 1$. Thus (26) implies that

$$\sum_{j=1}^{n}\sum_{\ell=1}^{L}\chi_{C_\ell}(\alpha_1 j^2,\alpha_2 j^2)>n/4, \tag{28}$$

where the C_ℓ are $\asymp\Delta$–neighborhoods of the level sets $\{F=m/v\}\setminus B_\epsilon$ with $m\in\mathbb{Z}$ and $L\asymp v$. In order to introduce "pair correlations" as in the proof of Lemma 4 we now write each C_ℓ as a union of "Fefferman rectangles", cf. [2]. More precisely, one has

$$C_\ell\subset\bigcup_{\ell=1}^{L}\bigcup_{s=1}^{s_0}R_{\ell s},$$

where each $R_{\ell s}$ is a rectangle of dimensions $C\Delta\times C\sqrt{\frac{\Delta}{\kappa+\Delta}}$ that is tangent to the level curve. We can assume that each rectangle intersects at most its immediate neighbors and that $R_{1s},\dots,R_{Ls}$ are parallel for each s. Clearly, $s_0\asymp\sqrt{\frac{\kappa+\Delta}{\Delta}}$. These rectangles have the following important property: If $x,y\in R_{\ell s}$ for a fixed s but arbitrary ℓ, then $x-y\in R_s^*$, a rectangle of approximately the same dimensions as and parallel to each $R_{\ell s}$, but centered at the origin. Also,

$$\bigcup_{s=1}^{s_0}R_s^*\subset R_0,$$

where R_0 is an axis-parallel rectangle centered at the origin of dimensions roughly $\sqrt{\Delta(\kappa+\Delta)}\times\sqrt{\Delta/(\Delta+\kappa)}$. In particular, $|R_0|\asymp\Delta$. Applying Cauchy-Schwarz to (28) therefore yields

$$\begin{aligned}
n \ &\ll\ \sum_{j=1}^{n}\sum_{\ell=1}^{L}\chi_{C_\ell}(\alpha_1 j^2,\alpha_2 j^2) \le \sum_{j=1}^{n}\sum_{\ell=1}^{L}\sum_{s=1}^{s_0}\chi_{R_{\ell s}}(\alpha_1 j^2,\alpha_2 j^2)\\
&\le\ \sum_{s=1}^{s_0}\left(\sum_{\ell=1}^{L}\Big(\sum_{j=1}^{n}\chi_{R_{\ell s}}(\alpha_1 j^2,\alpha_2 j^2)\Big)^2\right)^{\frac12}\sqrt{L}\\
&\ll\ \sum_{s=1}^{s_0}\left(\sum_{j,k=1}^{n}\chi_{\bigcup_{\ell=1}^{L}R_{\ell s}}(\alpha_1 j^2,\alpha_2 j^2)\chi_{R_s^*}(\alpha_1(j^2-k^2),\alpha_2(j^2-k^2))\right)^{\frac12}\sqrt{v}\\
&\ll\ \left(\sum_{s=1}^{s_0}\sum_{j,k=1}^{n}\chi_{\bigcup_{\ell=1}^{L}R_{\ell s}}(\alpha_1 j^2,\alpha_2 j^2)\chi_{R_0}(\alpha_1(j^2-k^2),\alpha_2(j^2-k^2))\right)^{\frac12}\sqrt{vs_0}\\
&\ll\ \Big(\mathrm{card}\{j,k\in[1,n]:(\alpha_1(j^2-k^2),\alpha_2(j^2-k^2))\in R_0\}\Big)^{\frac12}\sqrt{vs_0}. \qquad (29)
\end{aligned}$$

By the definition of discrepancy and Lemma 6,

$$\mathrm{card}\{j,k\in[1,n]:(\alpha_1(j^2-k^2),\alpha_2(j^2-k^2))\in R_0\}\ll n^2\Delta+n^{2-\tau}\log^2 n. \qquad (30)$$

We conclude from (29) and (30) that

$$n^2\sqrt{\Delta}/v\ll n^2(vs_0)^{-1}\ll n^2\Delta+n^{2-\tau}\log^2 n. \qquad (31)$$

First notice that $\sqrt{\Delta}/v$ is much larger than Δ by definition of v,Δ provided c_2 is small. Thus (31) reduces to

$$\sqrt{\Delta}/v\ll n^{-\tau}\log^2 n \quad \text{or} \quad n^{\tau}\ll n^{\frac{\sigma}{2}}v^{\frac32}\log^2 n\ll n^{2\sigma}\log^2 n,$$

which is a contradiction for large n. □

Lemma 9. $\mathrm{cov}(\frac{1}{\sqrt n}T_n(\alpha),\frac{1}{\sqrt n}T_n(\beta))=\frac12 I+O(n^{-\tau})$ *uniformly in* $(\alpha,\beta)\in\mathcal{N}^2\setminus\mathcal{B}$. *Here* I *denotes the identity matrix in* $\mathbb{R}^4$.

Proof. We will only sketch the proof, see the proof of Lemma 5 for more details. We need to show that the 2×2 matrices in the upper right–hand and lower left–hand corners of the covariance matrix tend to zero. The entries of these matrices are controlled by the sums

$$\Big|\frac1n\sum_{\ell=1}^{n}e((\alpha\pm\beta)\ell^2)\Big|,$$

which go to zero as $n\to\infty$. In fact,

$$\Big|\frac1n\sum_{\ell=1}^{n}e((\alpha\pm\beta)\ell^2)\Big|<n^{-\tau/2}, \qquad (32)$$

see (18) and (22). □

Proposition 10. *One has uniformly in* $(\alpha,\beta)\in\mathcal{N}^2\setminus\mathcal{B}$

$$\mathbb{P}(E_\alpha\cap E_\beta)=c_0^2 n^{-2-4\sigma}(1+o(1)) \quad \textit{as } n\to\infty$$

provided $2\sigma<\tau$.

Proof. Let ϕ_n and ψ_n be as in the proof of Proposition 2, and define $\widetilde{\phi}_n(x,y) = \phi_n(x)\phi_n(y)$ and $\widetilde{\psi}_n(x,y) = \psi_n(x)\psi_n(y)$. We denote the characteristic function of $(\frac{1}{\sqrt{n}}T_n(\alpha), \frac{1}{\sqrt{n}}T_n(\beta))$ by $f_{\alpha\beta}$. Then

$$\begin{aligned} \mathbb{P}(E_\alpha \cap E_\beta) &\geq \int_{\mathbb{R}^4} \widetilde{\phi}_n(n^{\frac{1}{2}+\sigma}x)\, d\mathbb{P}_{(\frac{1}{\sqrt{n}}T_n(\alpha), \frac{1}{\sqrt{n}}T_n(\beta))}(x) \\ &= n^{-2-4\sigma} \int_{A\cup B\cup C} \widehat{\widetilde{\phi}_n}(n^{-\frac{1}{2}-\sigma}\xi) f_{\alpha\beta}(\xi)\, d\xi. \end{aligned} \tag{33}$$

Here $A = \{\xi \in \mathbb{R}^4 : |\xi| < n^{1/6}\}$, $B = \{\mathbb{R}^4 : n^{1/6} < |\xi| < n^{\frac{1}{2}+\sigma+2\epsilon}\}$, and $C = \{\mathbb{R}^4 : |\xi| > n^{\frac{1}{2}+\sigma+2\epsilon}\}$. By Theorem 8.4 in Bhattacharya–Rao and Lemma 9

$$\int_A \widehat{\widetilde{\phi}_n}(n^{-\frac{1}{2}-\sigma}\xi) f_{\alpha\beta}(\xi)\, d\xi = c_0^2(1+o(1)), \tag{34}$$

see (16) and (17). Here c_0 is the same constant as in (17). The contribution from region B is estimated via Lemma 8 as in the proof of Proposition 2. The only difference is that for small values of $v_1^2 + v_2^2$ one has

$$\begin{aligned} \sum_{j=1}^n \|\psi_j\|^2 &= \sum_{j=1}^n |\psi_j|^2 = \sum_{j=1}^n |v_1 \cos(2\pi\alpha j^2 + \theta_1) + v_2 \cos(2\pi\beta j^2 + \theta_2)|^2 \\ &= n(v_1^2 + v_2^2)/2 + o(n) \end{aligned}$$

since the product term goes to zero in view of (32). The contribution from C goes to zero by the decay of $\widehat{\widetilde{\phi}_n}$ and $\widehat{\widetilde{\psi}_n}$, see (14). Finally, using $\widetilde{\psi}_n$ one obtains an upper bound in a similar fashion. □

2.3. **The proof of Theorem 1.**

Proof. We bound $\mathbb{P}\Big(\bigcup_{\alpha\in\mathcal{N}} E_\alpha\Big)$ from below using the second moment method. More precisely, Cauchy-Schwarz implies that

$$\mathbb{P}\Big(\bigcup_{\alpha\in\mathcal{N}} E_\alpha\Big) \geq \frac{\big(\sum_{\alpha\in\mathcal{N}} \mathbb{P}(E_\alpha)\big)^2}{\sum_{\alpha,\beta\in\mathcal{N}} \mathbb{P}(E_\alpha \cap E_\beta)}.$$

The numerator is equal to $(Mc_0 n^{-1-2\sigma})^2(1+o(1))$ by Proposition 2. We bound the denominator from above as follows, using Propositions 2,10 and (19):

$$\begin{aligned} \sum_{\alpha,\beta\in\mathcal{N}} \mathbb{P}(E_\alpha \cap E_\beta) &= \sum_{\alpha,\beta\in\mathcal{N}\setminus\mathcal{B}} \mathbb{P}(E_\alpha \cap E_\beta) + \sum_{\alpha,\beta\in\mathcal{B}} \mathbb{P}(E_\alpha \cap E_\beta) \\ &\leq \sum_{\alpha,\beta\in\mathcal{N}\setminus\mathcal{B}} \mathbb{P}(E_\alpha \cap E_\beta) + \sum_{\alpha,\beta\in\mathcal{B}} \mathbb{P}(E_\alpha) \\ &\leq M^2 c_0^2 n^{-2-4\sigma}(1+o(1)) + O(n^{2+5\tau} n^{-1-2\sigma}) \\ &= M^2 c_0^2 n^{-2-4\sigma}\Big(1 + O(n^{-1+5\tau+2\sigma})\Big). \end{aligned} \tag{35}$$

Since $2\sigma < \tau$ the O–term goes to zero provided $\tau < 1/6$. Thus the theorem follows as long as $\sigma < 1/12$, as claimed. □

2.4. **The case of higher powers.** In this subsection we will sketch a proof of the following theorem.

Theorem 11. *Let $d = 2, 3, 4, \dots$ be fixed and define $T_n(x) = \sum_{j=1}^n \pm e(j^d x)$ where the $\pm$ are chosen independently with probability $\frac{1}{2}$ each. If $\sigma < \frac{d-1}{2+5\cdot 2^{d-1}}$, then*

$$\mathbb{P}(\min_{x\in\mathbb{T}} |T_n(x)| < n^{-\sigma}) \to 1 \quad \text{as } n \to \infty.$$

The case $d = 2$ is just Theorem 1, and we will simply indicate the changes that need to be made to adapt the proof from Sections 2.1–2.3 to this case. The basic strategy for the proof of Theorem 11 is the same as that for the proof of Theorem 1. In fact, the modifications are straightforward, and we will therefore not supply too many details. The restrictions on σ are far from the conjectured ones $\sigma < (2d-1)/2 = \omega_d$. One can improve on our result for large d by invoking Vinogradov's method rather than the cruder Weyl bounds (38) that we shall use here, cf. [10]. However, these improvements are still minor compared to the remaining gap with ω_d and we have therefore chosen to use the simpler but less precise estimates.
Let $\mathcal{N}$ be a minimal n^{-d}–net in $\mathbb{T}$ and define E_α as in (2) for all $\alpha \in \mathcal{N}$. Let $\omega = d/5$ and retain only those $\alpha \in \mathcal{N}$ that satisfy

$$\alpha\ell \notin \bigcup_{q=1}^{\lceil n^\omega\rceil} \bigcup_{p=1}^{q} \Big[\frac{p}{q} - \frac{1}{qn^{d-2\omega}}, \frac{p}{q} + \frac{1}{qn^{d-2\omega}}\Big] \mod 1 \tag{36}$$

for all $\ell = 1, 2, \dots, \lceil n^\omega\rceil$. The number of points that we remove is no larger than $O(n^{4\omega}) = o(n^d)$. So one can assume that all points $\alpha \in \mathcal{N}$ satisfy (36) and that $\#\mathcal{N} \asymp n^d$.
The "bad pairs" in $\mathcal{N}^2$ are defined as follows, cf. (18): $\mathcal{B} = \mathcal{B}_\tau \subset \mathcal{N}^2$ consists of all pairs $(\alpha_1, \alpha_2) \in \mathcal{N}^2$ such that

$$\alpha_1\ell_1 + \alpha_2\ell_2 \in \bigcup_{q=1}^{A} \bigcup_{p=1}^{q} \Big[\frac{p}{q} - \frac{1}{qn^{d-2\tau}}, \frac{p}{q} + \frac{1}{qn^{d-2\tau}}\Big] \mod 1 \tag{37}$$

for some integers ℓ_1, ℓ_2 with $1 \le |\ell_1|, |\ell_2| \le A = \lceil n^\tau\rceil$. It is clear that

$$\mathrm{card}(\mathcal{B}) \ll \sum_{\alpha_1\in\mathcal{N}} \sum_{\pm\ell_1,\pm\ell_2=1}^{A} \sum_{q=1}^{A}\sum_{p=1}^{q} \frac{n^d}{q\ell_2 n^{d-2\tau}}\ell_2 \asymp \#\mathcal{N}\cdot A^3 n^{2\tau} \asymp n^{d+5\tau}.$$

It suffices to consider that case $\tau < \omega = d/5$. We claim that Propositions 2 and 10 remain valid for higher d under the condition $2^{d-1}\sigma < \tau$. In fact, we have

Proposition 12. *Suppose $0 < 2^{d-1}\sigma < \tau$. Then*

$$\begin{aligned} \mathbb{P}(E_\alpha) &= c_0 n^{-1-2\sigma}(1+o(1)) \quad \text{uniformly in } \alpha\in\mathcal{N} \text{ and} \\ \mathbb{P}(E_\alpha\cap E_\beta) &= c_0^2 n^{-2-4\sigma}(1+o(1)) \quad \text{uniformly in } (\alpha,\beta)\in\mathcal{N}^2\setminus\mathcal{B}_\tau \end{aligned}$$

with an absolute constant c_0.

Inspection of the proof of Theorem 1 in the previous section now reveals that we only need to ensure that $d - 1 > 2\sigma + 5\tau > (2 + 5\cdot 2^{d-1})\sigma$, which leads to the bound stated in Theorem 11. Indeed, since $M \asymp n^d$ the O–term in (35) is of the form $n^{1-d+5\tau+2\sigma}$.

Proposition 12 is of course the analogue of Propositions 2 and 10 and will be proved similarly. More precisely, it will suffice to prove the analogue of the two main Lemmas 4 and 8 under the condition $2^{d-1}\sigma < \tau$. These in turn are based on the following discrepancy bounds, cf. the remark following Lemma 4 and Lemma 6.

Lemma 13. *For any $\alpha \in \mathcal{N}$ and $(\alpha_1, \alpha_2) \in \mathbb{T}^2 \setminus \mathcal{B}$, one has*

$$\begin{aligned} D_n\big(\{\alpha j^d\}_{j=1}^n\big) &\le C_\epsilon\, n^{-\omega\kappa+\epsilon} \\ D_{n(n-1)}\big(\{\alpha(j^d-k^d)\}_{j\neq k=1}^n\big) &\le C_\epsilon\, n^{-2\omega\kappa+\epsilon} \\ D_n\big(\{\alpha_1 j^d, \alpha_2 j^d\}_{j=1}^n\big) &\le C_\epsilon\, n^{-\tau\kappa+\epsilon} \\ D_{n(n-1)}\big(\{\alpha_1(j^d-k^d), \alpha_2(j^d-k^d)\}_{j\neq k=1}^n\big) &\le C_\epsilon\, n^{-2\tau\kappa+\epsilon} \end{aligned}$$

for any $\epsilon > 0$. Here $\kappa = 2^{1-d}$.

Proof. We shall use the inequality, see [10],

$$\Big|\sum_{\ell=1}^N e(\ell^d\alpha)\Big| \le C_{d,\epsilon}\, N^{1+\epsilon}\Big(\frac{1}{q}+\frac{1}{N}+\frac{q}{N^d}\Big)^\kappa \tag{38}$$

provided $|\alpha - \frac{p}{q}| \le 2q^{-2}$ with $(p,q)=1$. We will only consider the sequence $\{\alpha_1(j^d-k^d), \alpha_2(j^d-k^d)\}_{j\neq k=1}^n$ and leave the other cases to the reader. By the Erdös–Turan inequality (21) with $m = \lceil n^\omega \rceil$,

$$D_{n(n-1)}\Big(\{\alpha_1(j^d-k^d), \alpha_2(j^2-k^2)\}_{j\neq k=1}^n\Big)$$

$$\ll n^{-\omega} + \frac{1}{n^2}\sum_{\ell=1}^{\lceil n^\omega\rceil}\frac{1}{\ell}\Big|\sum_{j\neq k=1}^n e(\alpha_1\ell(k^d-j^d))\Big| + \frac{1}{n^2}\sum_{\ell=1}^{\lceil n^\omega\rceil}\frac{1}{\ell}\Big|\sum_{j\neq k=1}^n e(\alpha_2\ell(k^d-j^d))\Big| \tag{39}$$

$$+\frac{1}{n^2}\sum_{\ell_1,\ell_2=1}^{\lceil n^\omega\rceil}\frac{1}{\ell_1\ell_2}\Big|\sum_{j\neq k=1}^n e((\alpha_1\ell_1+\alpha_2\ell_2)(k^d-j^d))\Big|. \tag{40}$$

Consider the first sum in (39) and fix some positive integer $\ell \le \lceil n^\omega \rceil$. By Dirichlet's principle and the definition of $\mathcal{N}$, see (36), there exist integers p, q with $(p,q)=1$ such that $|\alpha_1\ell - \frac{p}{q}| \le q^{-2}$ and $n^\omega \le q \le n^{d-2\omega}$. Applying (38) one obtains

$$\begin{aligned} \Big|\sum_{j\neq k=1}^n e(\alpha_1\ell(k^d-j^d))\Big| &\le n + \Big|\sum_{j=1}^n e(\alpha_1\ell j^d)\Big|^2 \\ &\le n + C_\epsilon\, n^{2+\epsilon}\Big(n^{-\omega}+n^{-2\omega}\Big)^{2\kappa}. \end{aligned}$$

Therefore the contribution from (39) is no bigger than

$$n^{-\omega} + n^{-1}\log n + C_\epsilon\, n^{-2\omega\kappa+\epsilon}.$$

Now fix ℓ_1 and ℓ_2 as in (40) and apply Dirichlet's principle to $\alpha_1\ell_1+\alpha_2\ell_2$ to obtain integers p, q with $(p,q)=1$ such that $|\alpha_1\ell_1+\alpha_2\ell_2 - \frac{p}{q}| \le q^{-2}$ and $n^\tau \le q \le n^{d-2\tau}$, cf. (37). Thus, in view of (38),

$$\begin{aligned} \Big|\sum_{j\neq k=1}^n e((\alpha_1\ell_1+\alpha_2\ell_2)(k^d-j^d))\Big| &\le n + \Big|\sum_{j=1}^n e((\alpha_1\ell_1+\alpha_2\ell_2)j^d)\Big|^2 \\ &\le n + C_\epsilon\, n^{2-2\tau\kappa+\epsilon}. \end{aligned}$$

Summing over ℓ_1 and ℓ_2 in (40) and observing that $2\tau\kappa < 2\omega\kappa = 2d2^{1-d}/5 \le \frac{1}{2}$ leads to the stated bound on the discrepancy. □

We now turn to the generalization of Lemmas 4 and 8 to higher powers. As before, $\kappa = 2^{1-d}$.

Lemma 14. *For any $\alpha \in \mathcal{N}$ let $\psi_j = v\cos(2\pi\alpha j^d + \theta)$ for some $1/4 \le v \le n^\sigma$ and $\theta \in [0, 2\pi)$. Suppose $\sigma < 2\omega\kappa$. Then for any $\delta > 0$*

$$\mathrm{card}\{j \in [1,n] : \|\psi_j\| < n^{-\delta}\} \le n/2$$

provided $n > n_0(\delta, \sigma)$.

Proof. Assume the lemma fails and let $\Delta = (n^\delta v)^{-1}$. As in the proof of Lemma 4 one obtains inequality (11), i.e.,

$$n^2 \ll v \sum_{j,k=1}^{n} \chi_{[-C\Delta, C\Delta]}(\alpha(j^d - k^d)).$$

By the definition of discrepancy and Lemma 13

$$n^2/v \quad \ll \quad \Delta n^2 + n^2\, D_{n(n-1)}\big(\{\alpha(j^d - k^d)\}_{j\neq k=1}^n\big) \ll \Delta n^2 + C_\epsilon\, n^{2-2\omega\kappa+\epsilon}.$$

Hence $n^{-\sigma} \le C_\epsilon\, n^{-2\omega\kappa+\epsilon}$ for all $\epsilon > 0$ which contradicts our assumption on σ. $\square$

The analogue of Lemma 8 reads as follows.

Lemma 15. *Let $(\alpha_1, \alpha_2) \in \mathcal{N}^2 \setminus \mathcal{B}_\tau$ and set*

$$\psi_j = v_1 \cos(2\pi\alpha_1 j^2 + \theta_1) + v_2 \cos(2\pi\alpha_2 j^2 + \theta_2)$$

where $1 \le v_1^2 + v_2^2 \le c_2 n^{2\sigma}$ and $\theta_1, \theta_2 \in [0, 2\pi)$ (c_2 is a sufficiently small absolute constant). Then

$$\mathrm{card}\{j \in [1,n] : \|\psi_j\| \le n^{-\sigma}\} \le n/2$$

provided $\sigma < \tau\kappa$ and $n > n_0(\sigma, \tau)$.

Proof. Assume the lemma fails and let $\Delta = v^{-1} n^{-\sigma}$. As in the proof of Lemma 8 one obtains by means of simple geometric considerations that

$$n \ll \Big(\mathrm{card}\{j,k \in [1,n] : (\alpha_1(j^d - k^d), \alpha_2(j^d - k^d)) \in R_0\}\Big)^{\frac12} \sqrt{v\Delta^{-\frac12}} \tag{41}$$

where R_0 is a rectangle of area Δ. By the definition of discrepancy and Lemma 13,

$$\mathrm{card}\{j,k \in [1,n] : (\alpha_1(j^d - k^d), \alpha_2(j^d - k^d)) \in R_0\} \ll n^2\Delta + C_\epsilon\, n^{2-2\tau\kappa+\epsilon}. \tag{42}$$

We conclude from (41) and (42) that

$$n^2\sqrt{\Delta}/v \ll n^2\Delta + C_\epsilon\, n^{2-2\tau\kappa+\epsilon}. \tag{43}$$

First notice that $\sqrt{\Delta}/v$ is much larger than Δ by definition of v, Δ provided c_2 is small. Thus (43) reduces to $\sqrt{\Delta}/v \le C_\epsilon\, n^{-2\tau\kappa+\epsilon}$ for all $\epsilon > 0$ or to $\sigma \ge \tau\kappa$, which contradicts our assumption. $\square$

The proof of Proposition 12 above now proceeds along the same lines as the proofs of Propositions 2 and 10, with Lemmas 14 and 15 replacing Lemmas 4 and 8, respectively. We leave the details to the reader.

3. The Gaussian case

Fix any $d = 2, 3, \ldots$ and let $\omega_d = (2d-1)/2$. In this section we let $T_n(x) = \sum_{j=1}^n r_j e(j^d x)$ with standard normal and independent r_j. The dependence on d will not be indicated in T_n.

Theorem 16. *For any $\delta_0 > 0$*

$$\mathbb{P}\Big(\min_{x\in\mathbb{T}} |T_n(x)| < n^{-\omega_d+\delta_0}\Big) \to 1 \quad as \quad n\to\infty.$$

This will again require several lemmas. As already apparent from our heuristic derivation at the end of Section 1, freezing T_n at the points $\{\alpha_j\}_{j=1}^M$ and applying the second moment method with the events $E_j = \{|T_n(\alpha_j)| < n^{-\omega_d+\delta_0}\}$ would require $M \asymp n^{2d}$. The difficulty with this approach, however, is that the "independence length scale" is only n^{-d}. We therefore split the circle into intervals of length n^{-d}, but cannot expect to use the events E_j. Following Konyagin [6], we instead consider Taylor expansions of T_n of very high order around the points in $\mathcal{N}$. The idea is simply that the size of T_n on an interval of size n^{-d} can be controlled by a Taylor polynomial of sufficiently high degree around any point in that interval.

More precisely, we let $\mathcal{N} = \{\alpha_s\}_{s=1}^M$ be a minimal n^{-d}–net in

$$\mathbb{T} \setminus \bigcup_{q=1}^{\lceil n^{1-\delta_1}\rceil} \bigcup_{p=1}^{q} \Big[\frac{p}{q} - \frac{1}{qn}, \frac{p}{q} + \frac{1}{qn}\Big]$$

for some fixed $\delta_1 > 0$. The choice of the intervals on the right–hand side is somewhat arbitrary. The only important feature is that they are centered around fractions with small denominators and that their total length goes to zero. Fix some $\delta_0 \in (0,1)$ and define

$$h = \frac{1}{4} n^{-\omega_d+\delta_0}, \quad H = n^{\frac{1}{2}-\delta_1}, \quad \text{and } \Delta = n^{-d-\delta_2}$$

where $\delta_2 < \delta_0/4$. Furthermore, we choose r to be a large integer satisfying $r\delta_2 > d+1$ and δ_1 sufficiently small such that $r\delta_1 < \delta_0/10$, where r will be the order of the Taylor expansion. Let

$$\mathcal{C} = \Bigg\{ \prod_{j=0}^{r} [U_j h, (U_j+1)h) \times \prod_{\ell=0}^{r} [V_\ell h, (V_\ell+1)h) \subset [-H, H]^{2r+2} \subset \mathbb{C}^{r+1} : \\ U_j, V_j \in \mathbb{Z}, \inf_{|y|\le\Delta} \Big| \sum_{j=0}^{r} \frac{U_j h + i V_j h}{j!} (2\pi i n^d)^j y^j \Big| < \frac{h}{2} \Bigg\}.$$

We shall refer to the cubes in $\mathcal{C}$ as good cubes. The second moment method will be applied with the events

$$\begin{aligned} E_\alpha &= \{X_n(\alpha) \in \bigcup_{Q\in\mathcal{C}} Q\} \quad \text{where} \\ X_n(\alpha) &= (\Re T_n(\alpha), \Re[T_n'(\alpha)/(2\pi i n^d)], \ldots, \Re[T_n^{(r)}(\alpha)/(2\pi i n^d)^r], \\ &\quad \Im T_n(\alpha), \Im[T_n'(\alpha)/(2\pi i n^d)], \ldots, \Im[T_n^{(r)}(\alpha)/(2\pi i n^d)^r]) \end{aligned} \tag{44}$$

with $\alpha \in \mathcal{N}$.

Lemma 17. *If E_α occurs, then $\inf_{|x-\alpha|<\Delta} |T_n(x)| < n^{-\omega_d+\delta_0}$.*

Proof. Taylor expanding T_n around α to the order r shows that

$$\begin{aligned}\inf_{|x-\alpha|<\Delta}|T_n(x)| \quad &\leq \quad \inf_{|y|\leq\Delta}\Big|\sum_{\ell=0}^{r}\frac{U_\ell h+iV_\ell h}{\ell!}(2\pi i n^d)^\ell y^\ell\Big| + \sum_{\ell=0}^{r}\frac{2}{\ell!}h(2\pi n^d\Delta)^\ell \\ &\quad\quad + O\Big(\frac{n^{d(r+1)}n}{(r+1)!}\Delta^{r+1}\Big) \\ &< \quad \frac{h}{2}+2h\exp(2\pi n^d\Delta)+O\Big(\frac{n^{1-(r+1)\delta_2}}{(r+1)!}\Big) < n^{-\omega_d+\delta_0},\end{aligned}$$

as claimed. □

The following lemma establishes that the number of good cubes is sufficiently large. Naively speaking, the number of cubes in $\mathcal{C}$ should be $\asymp \left(\frac{H}{h}\right)^{2r+1}$ because one can think of these cubes belonging to a neighborhood of width $\asymp h$ of a hypersurface. This basically turns out to be true.

Lemma 18. *$\#\mathcal{C} \gg \left(\frac{H}{h}\right)^{2r+1} n^{-2\delta_2}$. In particular,*

$$\#\mathcal{C}\cdot n^d\Big(hn^{-\frac12}\Big)^{2(r+1)} \geq n^{\delta_0/2} \tag{45}$$

provided n is sufficiently large.

Proof. Let

$$\mathcal{G} = \Big\{(z_0,z_1,\ldots,z_r)\in\mathbb{C}^{r+1} : |z_0|<h/2, \max_{1\leq j\leq r}|z_j|<H/2, \Re z_1 > H/3\Big\}.$$

With $Z=(z_0,\ldots,z_r)$ we set

$$Q_Z(t) = \sum_{j=0}^{r} z_j\frac{i^j t^j}{j!}.$$

Taylor's formula implies that the mapping

$$\Phi_s(Z) = \Big(Q_Z(s), Q_Z'(s),\ldots,Q_Z^{(r)}(s)\Big)$$

defines a unimodular flow on $\mathbb{C}^{r+1}$. One has, for all $Z\in\mathcal{G}$,

$$\inf_{|t|<\Delta/2}\Big|Q_Z(2\pi t n^d)\Big| = \inf_{|t|<\pi n^{-\delta_2}}|Q_Z(t)| < \frac{h}{2}.$$

Therefore

$$\inf_{|t|<2\pi n^{-\delta_2}}\Big|Q_Z(t+s)\Big| = \inf_{|t|<2\pi n^{-\delta_2}}\Big|Q_{\Phi_s(Z)}(t)\Big| = \inf_{|t|<\Delta}\Big|Q_{\Phi_s(Z)}(2\pi n^d t)\Big| < \frac{h}{2}$$

for any $|s|<\pi n^{-\delta_2}$. On the other hand, one checks easily that $\Phi_t(\mathcal{G})\cap\mathcal{G}=\emptyset$ if

$$t\Re z_1 - \frac{H}{2}\frac{t^2}{1-t} - \frac{h}{2} \geq \frac{h}{2},$$

or if $\frac12 > t \geq 6h/H$. Also notice that $\Phi_t(\mathcal{G})\subset[-H,H]^{2r+2}$ as long as $|t|<\frac12$. It follows from these properties that the number of good cubes is ($|\mathcal{G}|$ is the measure of $\mathcal{G}$)

$$\#\mathcal{C} \gg \frac{n^{-\delta_2}}{h/H}\frac{|\mathcal{G}|}{h^{2(r+1)}} \asymp \left(\frac{H}{h}\right)^{2r+1} n^{-\delta_2}.$$

Hence

$$\#\mathcal{C} \cdot n^d \Big(hn^{-\frac{1}{2}}\Big)^{2(r+1)} \asymp H^{2r+1} n^{d-\delta_2-r-1} h = n^{\delta_0-\delta_2-\delta_1(2r+1)} \ge n^{\delta_0/2},$$

as claimed. □

In the next lemma we compute the covariance matrices of $\frac{1}{\sqrt{n}} X_n(\alpha)$ and the joint covariances of two such vectors.

Lemma 19. *Let the random vector $X_n(\alpha)$ be defined by (44). Let V be the $(r+1)\times(r+1)$ matrix with entries $V_{k\ell} = \frac{1}{1+d(k+\ell)}$ for $k, \ell = 0, 1, \dots, r$. Then $\det V \ne 0$ and for any $\alpha \in \mathcal{N}$*

$$\operatorname{cov}\Big(\frac{1}{\sqrt{n}} X_n(\alpha)\Big) \;=\; \frac{1}{2}\begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} + O\Big(n^{-\gamma}\Big) \quad \textit{as } n \to \infty \tag{46}$$

where $\gamma = \gamma(d, \delta_1) > 0$. Suppose $\alpha, \beta \in \mathcal{N}^2$ are such that

$$\alpha \pm \beta \notin \bigcup_{q=1}^{\lceil n^{\delta_1}\rceil} \bigcup_{p=1}^{q} \Big[\frac{p}{q} - \frac{1}{qn^{d-\delta_1}}, \frac{p}{q} + \frac{1}{qn^{d-\delta_1}}\Big] \quad \textit{mod 1.} \tag{47}$$

Then, as $n \to \infty$

$$\operatorname{cov}\Big(\frac{1}{\sqrt{n}} X_n(\alpha), \frac{1}{\sqrt{n}} X_n(\beta)\Big) \;=\; \frac{1}{2}\begin{bmatrix} V & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & V & 0 \\ 0 & 0 & 0 & V \end{bmatrix} + O\Big(n^{-\gamma}\Big) \tag{48}$$

where the constants in the O–notation only depends on δ_1 and d.

Proof. By independence of the r_j

$$\begin{aligned}\operatorname{cov}(X_n(\alpha)) \;&= \sum_{j=1}^{n} \operatorname{cov}\big[r_j(\cos(2\pi j^d \alpha), (j/n)^d \cos(2\pi j^d\alpha), \dots, (j/n)^{dr} \cos(2\pi j^d \alpha), \\ &\qquad \sin(2\pi j^d\alpha), (j/n)^d \sin(2\pi j^d \alpha), \dots, (j/n)^{dr} \sin(2\pi j^d\alpha))\big].\end{aligned}$$

The matrix V arises since

$$\frac{1}{n}\sum_{j=1}^{n} (j/n)^{d(\ell+k)} = \int_0^1 x^{d(\ell+k)}\,dx + O(1/n) = \frac{1}{1+d(\ell+k)} + O(1/n).$$

As the Gram matrix of the functions x^{2j} and $x^{2\ell}$ in $L^2[0,1]$ the matrix V is nondegenerate. Standard trigonometric identities therefore reduce (46) to showing that

$$\frac{1}{n}\left|\sum_{j=1}^{n} \Big(\frac{j}{n}\Big)^p e(2j^d\alpha)\right| = O(n^{-\gamma}) \tag{49}$$

for any nonnegative integer p. Let

$$s_j = \sum_{\ell=1}^{j} e(2\ell^d \alpha) \quad \text{for} \quad j = 0, 1, \dots, n.$$

To prove (49) we shall use the Weyl bound (38) above. By Dirichlet's principle and in view of the definition of $\mathcal{N}$, there are integers p, q with $n^{1-\delta_1} \le q \le n$ so that

$|\alpha - \frac{p}{q}| \le \frac{1}{qn}$. Clearly we may also assume that p, q are relatively prime. Hence $|2\alpha - \frac{2p}{q}| \le \frac{2}{qn} \le 2q^{-2}$ and by (38)

$$\frac{1}{j}|s_j| \quad \le \quad C_\epsilon n^\epsilon \Big[n^{(-1+\delta_1)\kappa} + j^{-\kappa} + (nj^{-d})^\kappa \Big] \tag{50}$$

for any $j = 1, 2, \ldots, n$. Summing by parts in (49) we obtain

$$\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^p e(2j^d\alpha) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^p (s_j - s_{j-1})$$

$$= \frac{1}{n}\sum_{j=1}^{n-1}\left[\left(\frac{j}{n}\right)^p - \left(\frac{j+1}{n}\right)^p\right] s_j + \frac{1}{n}s_n = O\left(\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^p \frac{1}{j}|s_j|\right) + \frac{1}{n}s_n. \tag{51}$$

Setting $j = n$ in (50) shows that the last term in (51) is $O\Big(n^{-\gamma}\Big)$. Splitting the sum in (51) into $j < n^{\frac{3}{2d}}$ and $j \ge n^{\frac{3}{2d}}$ and using (50) in the latter and $|s_j| \le j$ in the former finishes the proof of (49). We skip the details.

The proof of (48) is similar and will only be sketched. Here we need to show that for any nonnegative integer p

$$\frac{1}{n}\left|\sum_{j=1}^{n}\left(\frac{j}{n}\right)^p e(j^d(\alpha \pm \beta))\right| = O(n^{-\gamma}) \tag{52}$$

for all α, β as in (47). Consider $\alpha + \beta$ for simplicity. By Dirichlet's principle and (47) there exist relatively prime integers p, q so that $|\alpha + \beta - \frac{p}{q}| \le q^{-2}$ and $n^{\delta_1} \le q \le n^{d-\delta_1}$. Let $b = 1 - \frac{\delta_1}{2d}$. One checks from (38) that for any $j \ge n^b$

$$\frac{1}{j}|s_j| \le C_\epsilon j^\epsilon \Big[j^{-\kappa} + n^{-\delta_1\kappa/2} \Big]$$

where now $s_j = \sum_{\ell=1}^{j} e(\ell^d(\alpha + \beta))$ (recall that $\kappa = 2^{1-d}$). To finish the proof one sums by parts in (52) as above and then splits the sum in (51) into $j > n^b$ and $j \le n^b$. The details are left to the reader. □

Proof of Theorem 16. The random variable $X_n(\alpha)$ is a $2(r+1)$–dimensional Gaussian vector with covariance matrix given by (46), i.e., the distribution of $\frac{1}{\sqrt{n}}X_n(\alpha)$ has a density function of the form

$$\frac{\sqrt{\det \Sigma_\alpha(n)}}{(2\pi)^{r+1}} \exp\Big(-\frac{1}{2}\langle \Sigma_\alpha(n)\mathbf{x}, \mathbf{x}\rangle\Big) \qquad \text{with } \ \mathbf{x} \in \mathbb{R}^{2(r+1)}$$

where $\Sigma_\alpha(n)^{-1} = \mathrm{cov}\Big(\frac{1}{\sqrt{n}}X_n(\alpha)\Big)$. Since $\mathrm{dist}(Q, 0) \le H = n^{\frac{1}{2}-\delta_1}$, one obtains from Lemma 19

$$\begin{aligned}\mathbb{P}(X_n(\alpha) \in \mathcal{C}) \quad &= \quad \sum_{Q\in\mathcal{C}} \mathbb{P}(X_n(\alpha) \in Q) = \sum_{Q\in\mathcal{C}} c_0\Big(\frac{h}{\sqrt{n}}\Big)^{2(r+1)}(1 + o(1)) \\ &= \quad c_0 \#\mathcal{C}\Big(\frac{h}{\sqrt{n}}\Big)^{2(r+1)}(1 + o(1)) = p_n(1 + o(1)),\end{aligned} \tag{53}$$

uniformly in $\alpha \in \mathcal{N}$ as $n \to \infty$. Here p_n is defined by the last equality. Similarly by (48), for any $\alpha, \beta \in \mathcal{N}$ satisfying (47),

$$\mathbb{P}(X_n(\alpha) \in \mathcal{C} \ \text{ and } \ X_n(\beta) \in \mathcal{C}) = \left(c_0 \# \mathcal{C}\left(\frac{h}{\sqrt{n}}\right)^{2(r+1)} (1+o(1))\right)^2 = p_n^2(1+o(1)).$$

Thus

$$\sum_{\alpha,\beta \in \mathcal{N}} \mathbb{P}(X_n(\alpha) \in \mathcal{C} \ \text{ and } \ X_n(\beta) \in \mathcal{C}) \leq \Big[(\#\mathcal{N})^2 p_n^2 + \#\mathcal{N} \cdot n^{2\delta_1} p_n\Big](1+o(1)), \tag{54}$$

where the second summand are the pairs that violate (47). In fact, their number is no bigger than

$$\sum_{\alpha \in \mathcal{N}} \sum_{q=1}^{n^{\delta_1}} \sum_{p=1}^{q} \frac{2}{q} n^{\delta_1} \ll \#\mathcal{N} \cdot n^{2\delta_1},$$

as claimed. By Cauchy–Schwarz, (53) and (54),

$$\mathbb{P}\Big(\bigcup_{\alpha \in \mathcal{N}} E_\alpha\Big) \geq \frac{(\#\mathcal{N} p_n)^2}{(\#\mathcal{N})^2 p_n^2 + \#\mathcal{N} n^{2\delta_1} p_n} = \frac{1}{1 + (\#\mathcal{N} p_n)^{-1} n^{2\delta_1}}.$$

By Lemma 18,

$$\#\mathcal{N} p_n \asymp n^d \# \mathcal{C}\Big(h n^{-\frac{1}{2}}\Big)^{2(r+1)} \geq n^{\delta_0/2}.$$

Hence (recall that $\delta_1 < \delta_0/10$)

$$\mathbb{P}\Big(\bigcup_{\alpha \in \mathcal{N}} E_\alpha\Big) \geq \frac{1}{1 + O(n^{2\delta_1 - \delta_0/2})} = 1 - o(1),$$

which implies the theorem in view of Lemma 17. ☐

The following result shows that $n^{-\omega_d}$ is the correct order of magnitude in the Gaussian case.

Theorem 20. *For any $\epsilon > 0$, $\mathbb{P}\Big(\min_{x \in \mathbb{T}} |T_n(x)| < n^{-\omega_d - \epsilon}\Big) \to 0$ as $n \to \infty$.*

Proof. Fix some small $\epsilon > 0$ and let $\mathcal{N} = \{x_j\}_{j=1}^M$ be a minimal $n^{-2d-\epsilon}$–net in $\mathbb{T}$. Clearly, $M \asymp n^{2d+\epsilon}$, and

$$\begin{aligned} \mathbb{P}\Big(\inf_{|x-x_j|<n^{-2d-\epsilon}} |T_n(x)| < n^{-\omega_d-\epsilon}\Big) \quad &\leq \quad \mathbb{P}\big(|T_n(x_j)| < C n^{-\omega_d-\epsilon}\sqrt{\log n}\big) \\ &\quad + \mathbb{P}\big(\|T_n'\|_\infty > C n^{d+\frac{1}{2}}\sqrt{\log n}\big). \end{aligned} \tag{55}$$

Indeed, suppose $\|T_n'\|_\infty < C n^{d+\frac{1}{2}}\sqrt{\log n}$ and suppose that $|T_n(y_j)| < n^{-\omega_d-\epsilon}$ for some $|y_j - x_j| < n^{-2d-\epsilon}$. Then

$$|T_n(x_j)| \leq |T_n(y_j)| + |x_j - y_j| \|T_n'\|_\infty \ll n^{-\omega_d-\epsilon}\sqrt{\log n},$$

as claimed. By the Salem–Zygmund inequality [4], the second term in (55) goes to zero like n^{-6d} provided C is large enough. We claim that for all points $x_j \in \mathcal{N}$ that do not belong to a finite number of certain intervals

$$\begin{aligned} \mathbb{P}\Big(|T_n(x_j)| \leq C n^{-\omega_d-\epsilon}\sqrt{\log n}\Big) \quad &\leq \quad C\Big(n^{-\omega_d-\frac{1}{2}-\epsilon}\sqrt{\log n}\Big)^2 n^{\epsilon/2} \\ &\leq \quad C n^{-2d-3\epsilon/2}\log n = o(M^{-1}). \end{aligned} \tag{56}$$

More precisely, (56) will be shown to hold for all

$$x_j \in \mathcal{N} \setminus \bigcup_{q\le q_0} \bigcup_{p=1}^{q} \Big[\frac{p}{q} - \frac{1}{qn^{d+\epsilon/4}}, \frac{p}{q} + \frac{1}{qn^{d+\epsilon/4}}\Big], \tag{57}$$

where q_0 is some constant depending only on d (for $d = 2$ one can show that there are no exceptions other than $q = 1$ and $q = 2$ and possibly this is true for all d, but we do not need such an accurate statement). The random polynomial $\frac{1}{\sqrt{n}} T_n(x_j)$ is a two–dimensional Gaussian vector, i.e., its distribution has a two–dimensional Gaussian density

$$\frac{\sqrt{\det \Sigma_j}}{2\pi} \exp\big(-\frac{1}{2}\langle \Sigma_j \mathbf{x}, \mathbf{x}\rangle\big) \quad \text{with} \ \ \mathbf{x} \in \mathbb{R}^2 \tag{58}$$

where $\Sigma_j^{-1} = \mathrm{cov}\Big(\frac{1}{\sqrt{n}} T_n(x_j)\Big)$. By definition

$$\begin{aligned}
\mathrm{cov}\Big(\frac{1}{\sqrt{n}} T_n(x)\Big) &= \frac{1}{n} \sum_{j=1}^{n} \begin{bmatrix} \cos^2(2\pi j^d \alpha) & \frac{1}{2}\sin(4\pi j^d \alpha) \\ \frac{1}{2}\sin(4\pi j^d \alpha) & \sin^2(2\pi j^d \alpha) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \frac{1}{2n} \sum_{j=1}^{n} \begin{bmatrix} \cos(4\pi j^d \alpha) & \sin(4\pi j^d \alpha) \\ \sin(4\pi j^d \alpha) & -\cos(4\pi j^d \alpha) \end{bmatrix}.
\end{aligned}$$

In other words,

$$2\mathrm{cov}\Big(\frac{1}{\sqrt{n}} T_n(x)\Big) = \begin{bmatrix} 1 + r(x) & s(x) \\ s(x) & 1 - r(x) \end{bmatrix}, \tag{59}$$

where $r(x) + is(x) = \frac{1}{n}\sum_{\ell=1}^{n} e(2\ell^d x)$. If $x_j \in \mathcal{N}$ belongs to a minor arc, i.e.,

$$x_j \notin \bigcup_{q=1}^{\lceil n^\epsilon \rceil} \bigcup_{p=1}^{q} \Big[\frac{p}{q} - \frac{1}{qn^{d-\epsilon}}, \frac{p}{q} + \frac{1}{qn^{d-\epsilon}}\Big], \tag{60}$$

then in view of (38) and a standard application of Dirichlet's principle

$$|r(x_j) + is(x_j)| = O(n^{-\tau})$$

for some $\tau = \tau(\epsilon, d)$. Thus, if $x_j \in \mathcal{N}$ satisfies (60), then

$$\Sigma_j^{-1} = \mathrm{cov}\Big(\frac{1}{\sqrt{n}} T_n(x_j)\Big) = \frac{1}{2} I + O(n^{-\tau})$$

where I is the identity in $\mathbb{R}^2$. In view of (58) we have shown that (56) holds for such x_j without the $n^{\epsilon/2}$–factor. On the major arcs we proceed as usual [13], i.e., suppose that $|x - \frac{p}{q}| \le \frac{1}{qn^{d-\epsilon}}$ with $1 \le p \le q \le n^\epsilon$ and $(p, q) = 1$. Then $x = \frac{p}{q} + \beta$

with $|\beta| \le n^{\epsilon-d}$. Thus

$$\begin{aligned}
\sum_{\ell=1}^{n} e(2\ell^d x) &= \sum_{\ell=1}^{n} e\Big(2\ell^d \frac{p}{q}\Big) e(2\ell^d \beta) \\
&= \sum_{u=0}^{[n/q]} \sum_{v=1}^{q} e\Big(2v^d \frac{p}{q}\Big) e(2u^d q^d \beta)(1 + O(q^d|\beta| + |\beta| n^{d-1} q)) + O(q) \\
&= \sum_{u=0}^{[n/q]} \sum_{v=1}^{q} e\Big(2v^d \frac{p}{q}\Big) e(2u^d q^d \beta) + O(q + nq^d|\beta| + |\beta| n^d q) \\
&= S_{p,q} \sum_{u=0}^{[n/q]} e(2u^d q^d \beta) + O(n^{2\epsilon}) = \frac{n}{q} S_{p,q} \int_0^1 e(2n^d \beta x^d)\, dx + O(n^{2\epsilon}).
\end{aligned}$$

Here $S_{p,q} = \sum_{v=1}^{q} e\Big(2v^d \frac{p}{q}\Big)$ and we let $I(\lambda) = \int_0^1 e(2\lambda x^d)\, dx$. We therefore conclude from (59) that

$$\begin{aligned}
4 \det \operatorname{cov}\Big(\frac{1}{\sqrt{n}} T_n(x)\Big) &= 1 - \left|\frac{1}{q} \sum_{v=1}^{q} e\Big(2v^d \frac{p}{q}\Big)\right|^2 \Big|I(\beta n^d)\Big|^2 - O(n^{2\epsilon-1}) \\
&\ge \max\left[\frac{1}{2} - \left|\frac{1}{q} \sum_{v=1}^{q} e\Big(2v^d \frac{p}{q}\Big)\right|^2, 1 - |I(\beta n^d)|^2 - O(n^{2\epsilon-1})\right] \qquad (61)
\end{aligned}$$

for large n. By classical results on Gauss sums, see [13],

$$\max_{(p,q)=1} \left|\frac{1}{q} \sum_{v=1}^{q} e\Big(2v^d \frac{p}{q}\Big)\right| \to 0$$

as $q \to \infty$. In fact, it is known that

$$\max_{(p,q)=1} \left|\sum_{v=1}^{q} e\Big(2v^d \frac{p}{q}\Big)\right| \le C q^{1-\frac{1}{q}}$$

with some absolute constant C and all q, see [12] and the references cited there. Thus

$$\det \operatorname{cov}\Big(\frac{1}{\sqrt{n}} T_n(x)\Big) > \frac{1}{4}$$

for all but finitely many values of q, say if $q > q_0 = q_0(d)$. Since

$$\operatorname{trace} \operatorname{cov}\Big(\frac{1}{\sqrt{n}} T_n(x)\Big) = 1, \qquad (62)$$

see (59), both eigenvalues of Σ_j^{-1} are $\asymp 1$ and (56) continues to hold (even without the $n^{\epsilon/2}$–factor) for all points in $\mathcal{N}$ that belong to major arcs provided $q > q_0$. Now assume that $1 \le q \le q_0$. Then we can use the second term in (61). Indeed, one easily checks that $|I(\lambda)| \le 1 - c\min(\lambda^2, 1)$ for all λ and some small constant $c > 0$. Therefore,

$$\det \operatorname{cov}\Big(\frac{1}{\sqrt{n}} T_n(x)\Big) \gg n^{-\epsilon/2}$$

provided $|\beta| > n^{-d-\epsilon/4}$, which means that $|x - \frac{p}{q}| > n^{-d-\epsilon/4}$. In view of (62) the covariance matrix has one eigenvalue of size $\asymp 1$ and another that is at least $n^{-\epsilon/2}$.

We conclude that (56) holds for all x_j satisfying (57). To deal with the remaining intervals as defined by the right–hand side of (57), we let $I \subset \mathbb{T}$ be any interval of length $\asymp n^{-d-\epsilon/4}$ and fix some $y_0 \in I$. Then

$$\mathbb{P}\Big(\min_{x\in I}|T_n(x)| < n^{-\omega_d-\epsilon}\Big) \leq \mathbb{P}\big(|T_n(y_0)| < Cn^{\frac{1}{2}-\epsilon/4}\sqrt{\log n}\big) + \mathbb{P}\big(\|T_n'\|_\infty > Cn^{d+\frac{1}{2}}\sqrt{\log n}\big). \tag{63}$$

The second term is again $O(n^{-6d})$ by the Salem–Zygmund inequality. We claim that the first term goes to zero as $n \to \infty$. In view of (62), $\mathrm{cov}\Big(\frac{1}{\sqrt{n}}T_n(y_0)\Big)$ has a unit eigenvector, say e_0, with corresponding eigenvalue at least $\frac{1}{2}$. Thus

$$\mathbb{P}\big(|T_n(y_0)| < Cn^{\frac{1}{2}-\epsilon/4}\sqrt{\log n}\big) \leq \mathbb{P}\Big(\big|\langle n^{-1/2}T_n(y_0), e_0\rangle\big| < Cn^{-\epsilon/4}\sqrt{\log n}\Big) \ll n^{-\epsilon/8}.$$

Summing over the probabilities in (56) as well as (63) over the $\asymp q_0^2$ many intervals given by (57) yields

$$\mathbb{P}\Big(\min_{x\in\mathbb{T}}|T_n(x)| < n^{-\omega_d-\epsilon}\Big) \ll n^{-\epsilon/2}\log n + n^{-\epsilon/8},$$

and the theorem is proved. □

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