$L^p \to L^q$ ESTIMATES FOR THE CIRCULAR MAXIMAL FUNCTION

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Abstract

In this thesis we establish sharp $L^p \to L^q$ bounds for the circular maximal function in the plane. This is accomplished by interpolating a $L^{5/2} \to L^5$ endpoint estimate with Bourgain's well–known $L^p \to L^p$ bounds. The endpoint estimate is proved by combining the geometric/combinatorial method of Kolasa–Wolff with a L^2 inequality for the maximal function on a small ball. The $L^p \to L^q$ estimates for the circular maximal function established in this thesis would be a consequence of C. Sogge's sharp local smoothing conjecture for the wave equation.

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Chapter 1

Introduction

1.1 Overview and general discussion

In [18] Stein introduced the maximal function

$$\mathcal{M}f(x) = \sup_{0<t<\infty} \int_{S^{d-1}} |f(x-ty)|\, d\sigma(y) \tag{1.1}$$

($d\sigma$ being surface measure on S^{d-1}) and showed that $M : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ for the optimal range of p's provided $d \geq 3$. The case $d = 2$ was settled by Bourgain [1]. It is easy to see that maximal functions of type (1.1) can never be bounded on any L^p with $p < \infty$ if we replace spheres by boundaries of cubes, say. Indeed, let

$$\widetilde{\mathcal{M}}f(x) = \sup_{1<t<2} \int_{\partial[-1,1]^d} |f(x-ty)|\, d\mu(y)$$

where $d\mu$ is surface measure on $\partial[-1,1]^d$. Choose $f : \mathbb{R}^d \to \mathbb{R}$ with the following properties: $f \geq 0$, $f(x', 0) = \infty$ for all $x' = (x_1, \ldots, x_{d-1})$, $\int |f|^p\, dx < \infty$ for any finite p. With this choice of f it is clear that $\widetilde{\mathcal{M}}(x', x_d) = \infty$ for all $x' \in \mathbb{R}^{d-1}$, $x_d \in (1,2)$. More generally, the same example shows that maximal averages over any hypersurface containing a piece of a plane can never be bounded on L^p for finite p. Typically, such hypersurfaces are ruled out by assuming nonvanishing Gaussian curvature. It turns out that this condition plays a crucial role in the analysis of $\mathcal{M}$, and also in other problems in harmonic analysis, cf. [20]. The reason for this is the following well–known result (which we state for spheres rather than general surfaces).

Proposition 1.1.1 *The Fourier transform of the surface measure $d\sigma$ of the unit sphere S^{d-1} has the following asymptotic expansion:*

$$\widehat{d\sigma}(\xi) = e^{2\pi i|\xi|}\omega_+(|\xi|) + e^{-2\pi i|\xi|}\omega_-(|\xi|) \tag{1.2}$$

with smooth functions $\omega_\pm$ on $\mathbb{R}^+$ satisfying

$$|\frac{d^k}{ds^k}\omega_\pm(s)| \leq C_k(1+s)^{-(d-1)/2-k} \quad k = 0, 1, 2, \ldots.$$

Proof: This is a standard application of the method of stationary phase to the integral

$$\widehat{d\sigma}(\xi) = \int_{S^{d-1}} e^{2\pi i x\cdot\xi}\, d\sigma(x). \tag{1.3}$$

Indeed, fix $v \in S^{d-1}$ and let $\xi = \lambda v$, $\lambda > 1$. By stationary phase, the main contributions to the integral in (1.3) will come from the critical points of the phase function $\phi : S^{d-1} \to \mathbb{R}$ defined by $\phi(x) = x \cdot \xi$. Clearly, those critical points are given by the points with normal parallel to v, i.e., $x = \pm v$. Furthermore, by the nonvanishing of the Gaussian curvature, these critical points are nondegenerate. Hence stationary phase implies that the main contributions to (1.3) come from $\lambda^{-1/2}$ neighborhoods of $\pm v$, more precisely

$$\widehat{d\sigma}(\lambda v) = c_1 e^{2\pi i \lambda} \lambda^{-(n-1)/2} + c_2 e^{-2\pi i \lambda} \lambda^{-(n-1)/2} + \text{lower order terms.}$$

Finally, the full statement (1.2) of the proposition follows by a more detailed analysis involving the parameter $v \in S^{d-1}$. We refer the reader to [6], Theorem 7.7.14, or [16], Theorem 1.2.1, for details. ■

Following Stein, we prove the L^2 boundedness of

$$\overline{\mathcal{M}} f(x) = \sup_{1<t<2} \int_{S^{d-1}} |f(x - ty)| \, d\sigma(y) \tag{1.4}$$

in dimensions $d \geq 3$ using (1.2), cf. [18], [20].

Proposition 1.1.2 *Suppose $d \geq 3$. Then*

$$\|\overline{\mathcal{M}} f\|_2 \leq C \|f\|_2 \tag{1.5}$$

for all $f \in C_c(\mathbb{R}^d)$ (= continuous functions with compact support).

Proof: Let

$$A_t f(x) = \int_{S^{d-1}} f(x - ty) \, d\sigma(y) = (f * d\sigma_t)(x)$$

where $d\sigma_t$ is the normalized surface measure on tS^{d-1}. We may assume that $f \geq 0$. Then

$$\begin{aligned} \sup_{1<t<2} A_t f(x) &\leq A_1 f(x) + \int_1^2 |\frac{d}{dt} A_t f(x)| \, dt \\ &= A_1 f(x) + \int_1^2 |f * (\frac{d}{dt} d\sigma_t)(x)| \, dt. \end{aligned}$$

Thus

$$\|\overline{\mathcal{M}} f\|_2 \leq \|A_1 f\|_2 + \int_1^2 \|f * \frac{d}{dt}(d\sigma_t)\|_2 \, dt. \tag{1.6}$$

By Young's inequality,

$$\|A_1 f\|_2 \leq \|f\|_2. \tag{1.7}$$

To estimate the integral we apply Plancherel's theorem.

$$\|f * \frac{d}{dt}(d\sigma_t)\|_2 = \|\hat{f} \frac{d}{dt} \widehat{d\sigma_t}\|_2 \leq \|f\|_2 \, \|\frac{d}{dt} \widehat{d\sigma_t}\|_\infty. \tag{1.8}$$

Since $d\sigma_t(x) = t^{-(d-1)} d\sigma(x/t)$ and thus $\widehat{d\sigma_t}(\xi) = \widehat{d\sigma}(t\xi)$, (1.2) implies that

$$\|\frac{d}{dt} \widehat{d\sigma_t}\|_\infty \leq C \tag{1.9}$$

provided $d \geq 3$. The proposition follows from (1.6)–(1.9). ■

Remarks: a) The operator $\overline{\mathcal{M}}$ is not bounded on L^2 if $d = 2$, cf. Stein's example (1.11) below.
b) Estimate (1.5) also holds for the global maximal function M. Indeed, given suitable inequalities for $\overline{\mathcal{M}}$ one can pass to corresponding bounds on M by scaling and Littlewood–Paley theory. This is done in detail in chapter 3, section 1 below. In this introduction we shall consider only $\overline{\mathcal{M}}$.
c) It is possible to extend Proposition 1.1.2 to general $f \in L^2$. Some care has to be taken in defining $\overline{\mathcal{M}}$, see, e.g., section 3.

It turns out that Proposition 1.1.2 is not optimal, in so far as $\overline{\mathcal{M}}$ is bounded on $L^p(\mathbb{R}^d)$ for some $p < 2$ provided $d \geq 3$. Indeed, we have the following result, which is due to Stein [18] in dimension $d \geq 3$ and Bourgain [1] in dimension 2.

Theorem 1.1.1 *Suppose $d \geq 2$, $\frac{d}{d-1} < p \leq \infty$. Then*

$$\|\overline{\mathcal{M}} f\|_p \leq C\|f\|_p \tag{1.10}$$

for all $f \in C_c(\mathbb{R}^d)$. (1.10) cannot hold for any $p \leq \frac{d}{d-1}$.

The sharpness is immediate from Stein's example

$$f(x) = |x|^{d-1} |\log|x||^{-1} \chi_{[|x| \leq 1/2]}(x). \tag{1.11}$$

Indeed, $f \in L^p(\mathbb{R}^d)$ for all $p \leq \frac{d}{d-1}$, but $\overline{\mathcal{M}} f(x) = \infty$ provided $1 < |x| < 2$.

For the original proofs we refer the reader to the aforementioned papers. Other proofs, based on the theory of Fourier integral operators, can be found in Sogge [15] and Stein [19]. See also section 2 below.

The main result in this thesis is the following generalization of Bourgain's theorem.

Theorem 1.1.2 *Suppose $(\frac{1}{p}, \frac{1}{q})$ lies in the interior of the triangle $\mathcal{T}$ with vertices (2/5,1/5), (1/2,1/2), (0,0). Then*

$$\|\overline{\mathcal{M}} f\|_{L^q(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)} \tag{1.12}$$

for any $f \in C_c(\mathbb{R}^2)$. Moreover, (1.12) cannot hold for any $(\frac{1}{p}, \frac{1}{q})$ in the exterior of $\mathcal{T}$.

In view of Bourgain's theorem, (1.12) also holds if $(\frac{1}{p}, \frac{1}{q})$ lies on the segment connecting $(0,0)$ with $(1/2, 1/2)$. The optimality statement is easy. Indeed, for small $\delta > 0$, let

$$f_0 = \chi_{[1-\delta<|x|<1+\delta]}.$$

Then $\overline{\mathcal{M}} f_0(x) = 1$ on $|x| < \delta$. Hence $\|f_0\|_p \simeq \delta^{\frac{1}{p}}$ and $\|\overline{\mathcal{M}} f_0\|_q \geq \delta^{\frac{2}{q}}$. Secondly, let

$$f_1 = \chi_{[|x_1|<\delta, |x_2|<\delta^{1/2}]}.$$

Then $\overline{\mathcal{M}} f_1(x) \simeq \delta^{1/2}$ if $1 < |x_1| < 2$ and $|x_2| < \delta^{1/2}$. Thus $\|f_1\|_p \simeq \delta^{\frac{3}{2p}}$ and $\|\overline{\mathcal{M}} f_1\|_q \simeq \delta^{\frac{1}{2}(1+\frac{1}{q})}$. Given these properties of f_0 and f_1 and the translation invariance of $\overline{\mathcal{M}}$ we conclude that (1.12) can hold only if $(\frac{1}{p}, \frac{1}{q})$ lies inside the region bounded by the lines $\frac{2}{q} = \frac{1}{p}$, $1 + \frac{1}{q} = \frac{3}{p}$, and $\frac{1}{p} = \frac{1}{q}$. However, it is easy to see that this region coincides with the closure of $\mathcal{T}$.

We shall obtain Theorem 1.1.2 by interpolating the known estimates for $\overline{\mathcal{M}}$ on the segment $(0,0)$, $(1/2, 1/2)$, cf. [1], [18], [10], with the following endpoint result (b is some positive constant)

$$\|\overline{\mathcal{M}}_\delta f\|_{5,\infty} \le C|\log\delta|^b \|f\|_{5/2,1}. \tag{1.13}$$

Here $\overline{\mathcal{M}}_\delta$ is the maximal average over thin δ annuli (see page 9) and the norms are those of the corresponding Lorentz spaces. In other words, $\overline{\mathcal{M}}_\delta$ is of restricted weak type (5/2,5). We do not know whether the $\log\delta$ term is an artefact of the proof or not.

The proof of (1.13) is a combination of the geometric/combinatorial methods from Kolasa–Wolff [8] and a localized L^2 inequality for $\overline{\mathcal{M}}_\delta$, which seems to be new. The details are in chapter 2 below.

1.2 Fourier integral operators

The proof of Proposition 1.1.2 suggests that it might be possible to obtain maximal function estimates from suitable $L^p \to L^q$ estimates for operators of the form

$$Ff(x,t) = \int_{\mathbb{R}^d} e^{2\pi i(x\cdot\xi + t|\xi|)} a(\xi)\hat{f}(\xi)\, d\xi \tag{1.14}$$

with symbol $a \in S^{-m}(\mathbb{R}^d)$. Note that the main ingredient in the proof of Proposition 1.1.2 is

$$\sup_{1\le t\le 2} \|Ff(\cdot,t)\|_2 \le C\|f\|_2$$

with $m = 0$, which is a direct consequence of Plancherel's theorem. It turns out that the optimal $L^p \to L^p$ estimates for fixed t (we set $t = 1$) are

$$\|Ff(\cdot,1)\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)} \quad \text{provided} \quad |\frac{1}{2} - \frac{1}{p}| \le \frac{m}{d-1}. \tag{1.15}$$

For the wave equation these inequalities were established by Peral [11]. More precisely, if u solves

$$\begin{aligned} \Box u &= 0 \\ u(\cdot,0) &= f, \quad u_t(\cdot,0) = 0 \end{aligned}$$

in $\mathbb{R}^d$ then Peral showed that

$$\|u(\cdot,1)\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{L^p_\gamma(\mathbb{R}^d)} \tag{1.16}$$

provided $|\frac{1}{2} - \frac{1}{p}| = \frac{\gamma}{d-1}$. Here L^p_γ is the usual Sobolev–Bessel space with norm

$$\|f\|_{L^p_\gamma} = \|(I - \triangle)^{\gamma/2} f\|_p.$$

The connection between (1.15) and (1.16) is given by the solution formula

$$u(x,t) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \cos(2\pi|\xi|t)\hat{f}(\xi)\, d\xi.$$

That one needs at least $(d-1)|\frac{1}{2} - \frac{1}{p}|$ many derivatives in (1.16) can be seen by taking initial data f which are localized to a thin shell of radius one, say, and a small ball, respectively.

In Seeger–Sogge–Stein [14] it was shown that (1.15) holds for Fourier integral operators with phase function $\Phi(x,\xi)$, homogeneous in ξ, satisfying the nondegeneracy condition

$$\det\left(\frac{\partial^2\Phi}{\partial x\partial\xi}\right) \neq 0.$$

See also Stein [19] and the numerous references to related work given there.

It is easy to see that Stein's theorem 1.1.1, i.e., the case $d \geq 3$ follows from (1.15). Indeed, by Proposition 1.1.1 spherical means can be written as

$$\begin{aligned} A_t f(x) &= \int_{\mathbb{R}^d} e^{2\pi i(x\cdot\xi + t|\xi|)}\omega_+(t|\xi|)\hat{f}(\xi)\,d\xi + \int_{\mathbb{R}^d} e^{2\pi i(x\cdot\xi - t|\xi|)}\omega_-(t|\xi|)\hat{f}(\xi)\,d\xi \\ &= F^+ f(x,t) + F^- f(x,t). \end{aligned}$$

It suffices to consider $F^+ f$. Let $\eta \in C_0^\infty(\mathbb{R})$ be equal to one on $(1,2)$. Applying Sobolev's theorem in the t variable yields

$$\sup_{1\leq t\leq 2} |F^+ f(x,t)| \leq C\|(I - \frac{\partial^2}{\partial t^2})^{\alpha/2}\{\eta(t)F^+ f(x,\cdot)\}\|_{L^p(\mathbb{R})} \tag{1.17}$$

with $\alpha > 1/p$. The right–hand side of (1.17) can be estimated by (1.15) since the operator inside the norm is of the form (1.14) with

$$m = -\alpha + \frac{d-1}{2}.$$

Indeed, $(d-1)/2$ is simply the rate of decay given by (1.2), whereas the $-\alpha$ term is due to taking α time derivatives, which essentially amounts to taking α derivatives in space. Hence (1.17) and (1.15) imply

$$\|\overline{\mathcal{M}}f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$

provided

$$\left|\frac{1}{2} - \frac{1}{p}\right| \leq \frac{1}{2} - \frac{\alpha}{d-1} < \frac{1}{2} - \frac{1}{p(d-1)},$$

i.e., if $d \geq 3$ and $p > \frac{d}{d-1}$, which agrees with Theorem 1.1.1 above. Note that this method fails in $d = 2$ since we need strict inequality in $\alpha > 1/p$. However, Mockenhaupt–Seeger–Sogge [10] observed that the above approach does yield Bourgain's circular maximal theorem provided one replaces (1.15) with the following local smoothing estimate

$$\int_1^2 \int_{\mathbb{R}^2} |Ff(x,t)|^p\,dx\,dt \leq C_\epsilon \|f\|^p_{L^p_{\alpha(p)+\epsilon}(\mathbb{R}^2)} \tag{1.18}$$

where $2 < p \leq \infty$, $\epsilon > 0$ and

$$\alpha(p) = \begin{cases} \frac{1}{2}(\frac{1}{2} - \frac{1}{p}) & 2 < p \leq 4 \\ \frac{1}{2} - \frac{3}{2p} & 4 < p \leq \infty. \end{cases} \tag{1.19}$$

In [10] inequality (1.18) was obtained by an adaptation of Córdoba's proof of the Carleson–Sjölin theorem [3]. Note that (1.19) represents a gain of a fixed number of derivatives over the fixed time estimate (1.15), e.g., in case $p = 4$ of a gain $< 1/8$. We therefore conclude from the previous discussion that Bourgain's circular maximal theorem is a consequence of (1.18). A conjecture of Sogge [15], known as the sharp local smoothing conjecture, maintains that (1.18) should hold with $2 < p \leq 4$ and $\alpha(p) = 0$. It is shown in [15] that this is best possible and that it would imply the Carleson–Sjölin theorem. Moreover, it is easy to see that the optimal $L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2)$ bounds for the circular maximal function would follow from Sogge's conjecture. Indeed, applying Sobolev imbedding in t as above, that conjecture would imply

$$\|\overline{\mathcal{M}}f\|_{L^4(\mathbb{R}^2)} \le C\|f\|_{L^4_{-1/4+\epsilon}(\mathbb{R}^2)}.$$

Interpolating this with the simple $L^1 \to L^\infty$ bound

$$\|\overline{\mathcal{M}}f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^1_1(\mathbb{R}^2)}$$

yields

$$\|\overline{\mathcal{M}}f\|_{L^q(\mathbb{R}^2)} \le C\|f\|_{L^p(\mathbb{R}^2)}$$

provided $(\frac{1}{p}, \frac{1}{q})$ lies on the half open segment connecting $(1/4, 1/4)$ and $(2/5, 1/5)$. This is precisely the statement of Theorem 1.1.2 above.

In conclusion we would like to mention that Schlag and Sogge [13] have recently established the following local smoothing estimate

$$\int_1^2 \int_{\mathbb{R}^2} |Ff(x,t)|^5 \, dxdt \le C_\epsilon \|f\|^5_{L^{5/2}_{3/10+\epsilon}}. \tag{1.20}$$

Firstly, (1.20) would again follow from the sharp local smoothing conjecture by interpolating with the appropriate $L^1_x \to L^\infty_{x,t}$ bound. Secondly, since $3/10 = 1/2 - 1/5$, Sobolev imbedding with $1/5 + \epsilon$ derivatives in t shows that Theorem 1.1.2 follows from (1.20). The proof of (1.20) is based in part on the Klainerman–Machedon estimate [7] for the wave equation. However, we shall not discuss the proof here, since the emphasis in this thesis lies on the combinatorial method of Kolasa–Wolff rather than on the Fourier integral operator methods. [13] also contains variable coefficient extensions of Theorem 1.1.2 as well as sharp $L^p \to L^q$ bounds for spherical maximal operators in higher dimensions.

1.3 Related questions from geometric measure theory

Suppose E and F are subsets of the plane with E closed and F measurable. Assume that for every $x \in F$ there is an $r_x \in (1, 2)$ so that

$$\mathcal{H}^1(C(x, r_x) \cap E) > 0 \quad \text{for all } x \in F$$

where $\mathcal{H}^1$ is linear Hausdorff measure (for properties of Hausdorff measure, see Falconer's book [5], chapter one. Dimension in this section will always mean Hausdorff dimension). In this section we shall always assume that E and F are as above. The following result is due to Marstrand [9] and Bourgain [1]. We will denote Lebesgue measure by $|\cdot|$.

Theorem 1.3.1 *If $|F| > 0$ then $|E| > 0$. In other words, the union of a family of circles has positive (planar) measure if their centers form a set of positive measure.*

Proof: This is a simple consequence of Bourgain's circular maximal theorem. Indeed, assume that $U \in \mathbb{R}^2$ is open. Then there is an increasing sequence of nonnegative functions $f_n \in C_c(\mathbb{R}^2)$ so that $\sup_n f_n = \chi_U$. Fix a $p \in (2, \infty)$. By monotone convergence and Theorem 1.1.1

$$\|\overline{\mathcal{M}}\chi_U\|_p \le C\|\chi_U\|_p = C|U|^{\frac{1}{p}}.$$

In case $|E| = 0$ there would be a decreasing sequence of open sets $U_n \supset E$ so that $\bigcap_n U_n = E$ and $|U_n| \to 0$. We could then conclude from the previous inequality that

$$\|\overline{\mathcal{M}}\chi_E\|_p = 0.$$

However, this contradicts $|F| > 0$. ■

One might ask whether the conclusion of Theorem 1.3.1 will still hold under the assumption that $\dim(F) > c_0$ for some $c_0 < 2$. The following result of Talagrand [21] shows that c_0 has to be at least one.

Proposition 1.3.1 *There exist E and F such that $\mathcal{H}^1(F) > 0$ but $|E| = 0$.*

On the other hand, Wolff [22] has shown recently that for F as in the proposition, E "barely fails to have positive measure". More precisely he showed

Theorem 1.3.2 *If $\mathcal{H}^1(F) > 0$ then $\dim(E) = 2$.*

His proof is based in part on an argument from combinatorial geometry that was developed in [4] to obtain bounds on the number of incidences between n spheres and m points in $\mathbb{R}^3$.

We do not know whether it is possible to deduce $c_0 = 1$ from [22]. Rather, we will show that $c_0 = 1$ would be a consequence of the sharp local smoothing conjecture from the previous section. This connection, which relies on the theory of capacities, was pointed out to me by Thomas Wolff. For the definition of capacity as well as the connection between Hausdorff measure and capacity we refer the reader to the appendix.

Proposition 1.3.2 *Given that the sharp local smoothing conjecture is true, suppose that $\dim(F) > 1$. Then $|E| > 0$.*

Proof: Firstly, we may assume that E is compact and that

$$\mathcal{H}^1(E \cap C(x, r_x)) \geq \gamma > 0$$

for all $x \in F$ with γ fixed (since $\mathcal{H}^1$ is a regular outer measure, see [5]). Since $\dim(F) > 1$ there is an $\epsilon > 0$ so that $\mathcal{H}^{1+\epsilon}(F) > 0$. Fix ϵ to be that number. Assume the proposition fails. Then there is a sequence f_j of nonnegative functions in $C_0^\infty(\mathbb{R}^2)$ so that $f_j = 1$ on a neighborhood of E and $\|f_j\|_4 \to 0$ as $j \to \infty$. Pick a cutoff function $\eta \in C_0^\infty(\mathbb{R})$ so that $\eta = 1$ on $(1, 2)$. Define

$$u_j(x, r) = (d\sigma_r * f_j)(x)\eta(r).$$

Then $u_j > \gamma/2$ on some neighborhood of $F' = \{(x, r_x) : x \in F\}$. If the sharp local smoothing conjecture is correct then

$$\|u_j\|_{L^4_{1/2-\epsilon/8}(\mathbb{R}^3)} \leq C_\epsilon \|f_j\|_{L^4(\mathbb{R}^2)}.$$

Therefore, by the definition of capacity (see appendix)

$$C_{4,\frac{1}{2}-\frac{\epsilon}{8}}(F') \leq C_\epsilon \|f_j\|_4.$$

Passing to the limit $j \to \infty$, this would imply that

$$C_{4,\frac{1}{2}-\frac{\epsilon}{8}}(F') = 0. \tag{1.21}$$

However, by the proposition in the appendix we conclude from (1.21) that

$$0 = \mathcal{H}^{3-4(\frac{1}{2}-\frac{\epsilon}{8})+\frac{\epsilon}{2}}(F') = \mathcal{H}^{1+\epsilon}(F') \geq \mathcal{H}^{1+\epsilon}(F)$$

which contradicts the choice of ϵ. ■

Chapter 2

The main argument

2.1 A reformulation of the main theorem

For the combinatorial argument below it is convenient to consider maximal averages over thin annuli rather than circles. More precisely, let $0 < \delta < \frac{1}{2}$, $1 < r < 2$, and define for $f \in \mathcal{S}$

$$\begin{aligned} C(x,r) &= \{x \in \mathbb{R}^2 \,:\, |x-y| = r\} \\ C_\delta(x,r) &= \{x \in \mathbb{R}^2 \,:\, r(1-\delta) < |x-y| < r(1+\delta)\} \\ \overline{\mathcal{M}}_\delta f(x) &= \sup_{1<r<2} \frac{1}{|C_\delta(x,r)|} \left| \int_{C_\delta(x,r)} f(y)\,dy \right| \end{aligned}$$

where $d\sigma_r$ is the normalized surface measure on $r\,S^1$. Unless stated to the contrary, we shall be dealing only with functions defined on $\mathbb{R}^2$.

We shall write $\lesssim$ to denote $\leq$ up to an absolute constant. Similarly with $\gtrsim$ and $\simeq$. For any measure ν on $\mathbb{R}^d$, we let $\nu_\lambda(x) = \lambda^{-d}\nu(\lambda^{-1}x)$. First we will consider some examples, two of which have already appeared in chapter 1, section 1.

Examples:

i. Let $f = \chi_{C_\delta(0,1)}$. Then $\overline{\mathcal{M}}_\delta f(x) \simeq 1$ on $|x| < \delta$. Hence $||f||_p \simeq \delta^{\frac{1}{p}}$ and $||\overline{\mathcal{M}}_\delta f||_q \gtrsim \delta^{\frac{2}{q}}$.

ii. Let $f = \chi_R$, where R is the rectangle centered at 0 with dimensions $\delta \times \delta^{\frac{1}{2}}$. Then $\overline{\mathcal{M}}_\delta f(x) \simeq \delta^{\frac{1}{2}}$ provided $|x_1| \simeq 1$ and $|x_2| < \delta^{\frac{1}{2}}$. Hence $||f||_p \simeq \delta^{\frac{3}{2p}}$ and $||\overline{\mathcal{M}}_\delta f||_q \simeq \delta^{\frac{1}{2}(1+\frac{1}{q})}$.

iii. Let $f = \chi_{B(0,\delta)}$. Then $\overline{\mathcal{M}}_\delta f(x) \simeq \delta$ for $|x| \simeq 1$ and thus $||f||_p \simeq \delta^{\frac{2}{p}}$, $||\overline{\mathcal{M}}_\delta f||_q \simeq \delta$.

iv. Let $f(x) = (|1-|x|| + \delta)^{-\frac{1}{2}} \chi_{B(0,2)}(x)$. Then

$$\overline{\mathcal{M}}_\delta f(x) \gtrsim \left| \log \frac{\delta}{|x|} \right| \, |x|^{-\frac{1}{2}} \;\text{ if }\; 2\delta \leq |x| < 1. \tag{2.1}$$

To see this write f as

$$f \simeq \delta^{-\frac{1}{2}} \sum_{1 \leq 2^j \leq \delta^{-1}} 2^{-j/2} \chi_{C_{2^j\delta}(0,1+(2^j-1)\delta)}. \tag{2.2}$$

Taking the average of f over the annulus $C_\delta(x, 1+|x|)$ and considering the contribution of each dyadic shell in (2.2) separately yields (2.1). Hence $||f||_2 \simeq |\log\delta|^{\frac{1}{2}}$ and $||\overline{\mathcal{M}}_\delta f||_2 \gtrsim |\log\delta|$.

In view of these examples one might make the following conjecture (see figure 1).

Conjecture: For any $f \in L^1 \cap L^\infty(\mathbb{R}^2)$

$$
\begin{aligned}
||\overline{\mathcal{M}} f||_q &\lesssim ||f||_p \qquad \text{in region I} && (2.3)\\
||\overline{\mathcal{M}}_\delta f||_q &\lesssim \delta^{\frac{2}{q}-\frac{1}{p}}\,||f||_p \qquad \text{in region II} && (2.4)\\
||\overline{\mathcal{M}}_\delta f||_q &\lesssim \delta^{\frac{1}{2}(1+\frac{1}{q}-\frac{3}{p})}\,||f||_p \qquad \text{in region III} && (2.5)\\
||\overline{\mathcal{M}}_\delta f||_q &\lesssim \delta^{1-\frac{2}{p}}\,||f||_p \qquad \text{in region IV.} && (2.6)
\end{aligned}
$$

Regions I,III, and IV do not contain the point $T = (\frac{1}{2}, \frac{1}{2})$, where we have the well–known, optimal inequality (see Bourgain [1] and [2] and example iv above)

$$||\overline{\mathcal{M}}_\delta f||_2 \lesssim |\log\delta|^{\frac{1}{2}}||f||_2. \tag{2.7}$$

Otherwise the boundaries are part of the regions. We will prove the following theorem (by C_ϵ we shall always mean a constant depending only on ϵ).

Theorem 2.1.1 *For any $f \in L^1 \cap L^\infty(\mathbb{R}^2)$ and any $\epsilon > 0$,*

$$
\begin{aligned}
||\overline{\mathcal{M}} f||_q &\lesssim ||f||_p \qquad \textit{in region } I \setminus (QP \cup PT) && (2.8)\\
||\overline{\mathcal{M}}_\delta f||_q &\leq C_\epsilon\,\delta^{\frac{2}{q}-\frac{1}{p}-\epsilon}\,||f||_p \qquad \textit{in region } II && (2.9)\\
||\overline{\mathcal{M}}_\delta f||_q &\leq C_\epsilon\,\delta^{\frac{1}{2}(1+\frac{1}{q}-\frac{3}{p})-\epsilon}\,||f||_p \qquad \textit{in region } III && (2.10)\\
||\overline{\mathcal{M}}_\delta f||_q &\lesssim \delta^{1-\frac{2}{p}}\,||f||_p \qquad \textit{in region } IV. && (2.11)
\end{aligned}
$$

Remarks: a) In certain cases the $\delta^{-\epsilon}$ term can be replaced by a suitable power of $|\log\delta|$, but we do not elaborate on this.
b) It can be shown by modifying the proof of Theorem 2.1.1 that the optimal estimates, i.e., (2.9) with $\epsilon = 0$ hold in the region $II \cap \{\frac{1}{q} < \frac{1}{6}\} \setminus (QP \cup PR)$. This (somewhat technical) argument is given in chapter 3, section 3.
c) The most interesting statement in Theorem 2.1.1 is probably the estimate at the point P (see figure 1), i.e., for all $f \in L^1 \cap L^\infty(\mathbb{R}^2)$ and any $\epsilon > 0$

$$||\overline{\mathcal{M}}_\delta f||_5 \leq C_\epsilon \delta^{-\epsilon}||f||_{5/2} \tag{2.12}$$

(note that the conjecture on page 10 says that this should hold with $\epsilon = 0$).

The proof of Theorem 2.1.1 is based on a combinatorial argument from Kolasa and Wolff [8] combined with a localized version of the L^2 estimate (2.7). For the δ–free bounds (2.8) we interpolate the (5/2,5) inequality with an estimate obtained from the local smoothing theorem in Mockenhaupt, Seeger, and Sogge [10].

This chapter is organized as follows. In Section 2 we introduce the notion of multiplicity μ of a family of annuli. It is shown that certain estimates for μ are equivalent to $L^p \to L^q$ bounds on $\overline{\mathcal{M}}_\delta$. Section 3 contains the localized L^2 inequality and a bound on the multiplicity is derived from it. In

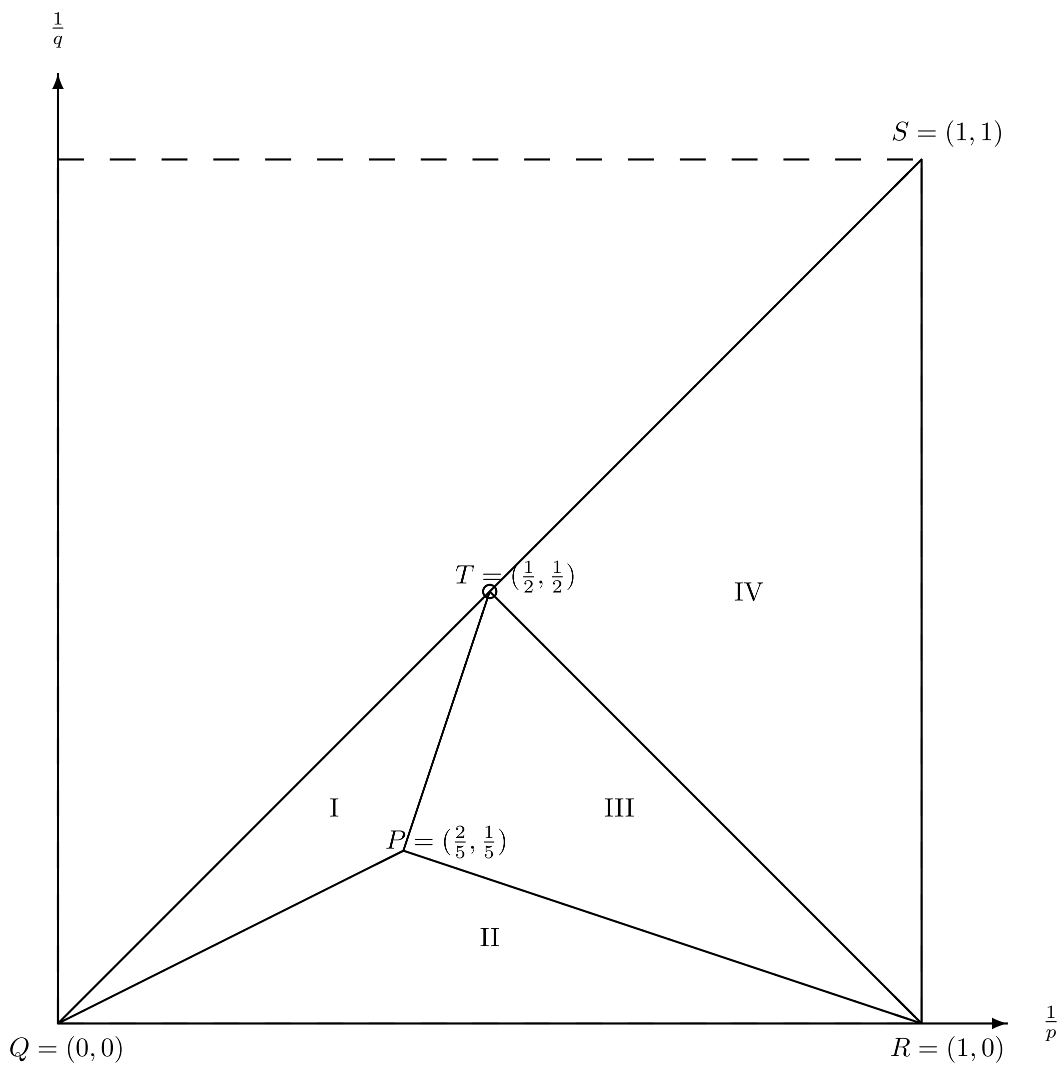

Figure 2.1: Regions of boundedness in Theorem 2.1.1

section 4 we establish the main result of this thesis, i.e., the restricted weak type (5/2,5) estimate. This is accomplished by combining the combinatorial argument from [8] (which is based on Marstrand's three circle lemma [9]) with the localized inequality from section 3. Theorem 2.1.1 then follows by various interpolation arguments. This is done in section 5.

2.2 The combinatorial method

Fix $E \subset [0,1]^2$ and $0 < \lambda \leq 1$. Let $\{x_j\}_{j=1}^M$ be a maximally δ–separated set in

$$F = \{x \in \mathbb{R}^2 \ : \ (\overline{\mathcal{M}}_\delta \chi_E)(x) > \lambda\}$$

and let $r_j \in (1,2)$ be chosen so that

$$|C_\delta(x_j, r_j) \cap E| > \lambda |C_\delta(x_j, r_j)|$$

for $j = 1, 2, \ldots, M$. Henceforth we shall write C_j^* instead of $C_\delta(x_j, r_j) \cap E$ and C_j instead of $C_\delta(x_j, r_j)$. We introduce the multiplicity function

$$\Phi = \sum_{j=1}^{M} \chi_{C_j^*}.$$

Following [8] we define μ to be the smallest integer for which there exist at least $M/2$ values of j such that

$$|\{C_j^* \ : \ \Phi \leq \mu\}| \geq \frac{\lambda}{2} |C_j|.$$

Clearly, we can then also find at least $M/2$ values of j for which

$$|\{C_j^* \ : \ \Phi \geq \mu\}| \geq \frac{\lambda}{2} |C_j|. \tag{2.13}$$

The combinatorial method attempts to bound μ from above, typically in terms of λ, M, and δ. Since

$$\mu |E| \geq \int_{\{E \ : \ \Phi \leq \mu\}} \Phi = \sum_j |\{C_j^* \ : \ \Phi \leq \mu\}| \gtrsim \lambda M \delta, \tag{2.14}$$

this will imply a lower bound on $|E|$. The following lemma characterizes the estimates of μ required for $L^p \to L^q$ boundedness of $\overline{\mathcal{M}}_\delta$.

Lemma 2.2.1 *Let $0 \leq \alpha$ and $\beta < 1$. Then $\mu \leq A\lambda^{-\alpha} M^{\beta}$ implies*

$$||\overline{\mathcal{M}}_\delta f||_{q,\infty} \lesssim A^{\frac{1}{p}} \delta^{-\gamma} ||f||_{p,1} \qquad \textit{for all } f \in L^1 \cap L^\infty$$

where $p = \alpha + 1$, $q = p(1-\beta)^{-1}$ and $\gamma = \frac{1}{p} - \frac{2}{q}$. We also have the following converse. Suppose that for some fixed $\rho > 0$, $1 \leq q \leq \infty$, $1 \leq p < \infty$, and all $f \in L^1 \cap L^\infty$

$$||\overline{\mathcal{M}}_\delta f||_q \lesssim \delta^{-\rho} ||f||_p. \tag{2.15}$$

Then

$$\mu \lesssim \lambda^{1-p} M^{1-\frac{p}{q}} \delta^{-p\rho+1-2\frac{p}{q}}. \tag{2.16}$$

Proof: For the first statement we need to show

$$|\{\overline{\mathcal{M}}_\delta \chi_E > \lambda\}|^{\frac{1}{q}} \lesssim \delta^{-\gamma} A^{\frac{1}{p}} \lambda^{-1} |E|^{\frac{1}{p}}. \tag{2.17}$$

Since $\{x_j\}_{j=1}^M$ was chosen to be a maximally δ–separated sequence in $\{\overline{\mathcal{M}}_\delta \chi_E > \lambda\}$, it follows that

$$|\{\overline{\mathcal{M}}_\delta \chi_E > \lambda\}| \lesssim M\delta^2.$$

In view of (2.14), i.e., $|E| \gtrsim \mu^{-1}\lambda M \delta$, and our assumption on μ we conclude that the right–hand side of (2.17) is

$$\gtrsim \delta^{-\gamma} A^{\frac{1}{p}} \lambda^{-1} (A^{-1}\lambda^{1+\alpha} M^{1-\beta}\delta)^{\frac{1}{p}} \simeq (M\delta^2)^{\frac{1}{q}}.$$

To prove the second statement, we distinguish two cases. First assume that

$$|E_1| = |\{E \,:\, \Phi \geq \mu\}| \leq \mu^{-1}\lambda M \delta.$$

Applying hypothesis (2.15) to the function $f = \chi_{E_1}$ and using (2.13) we obtain

$$\lambda(\delta^2 M)^{\frac{1}{q}} \lesssim \delta^{-\rho} |E_1|^{\frac{1}{p}} \lesssim \delta^{-\rho} (\mu^{-1}\lambda M \delta)^{\frac{1}{p}},$$

which implies the desired inequality (2.16).

In the other case, i.e., $|E_1| \geq \mu^{-1}\lambda M\delta$, we use duality. Note that the dual statement to (2.15) is

$$\| \sum_j a_j \chi_{C_\delta(y_j,\rho_j)} \|_{p'} \lesssim \delta^{-1-\rho} \tag{2.18}$$

for all δ-separated $\{y_j\}$ in $[0,1]^2$, all $\{a_j\}$ which satisfy $(\delta^2 \sum_j |a_j|^{q'})^{\frac{1}{q'}} \leq 1$, and all $\rho_j \in (1,2)$. Let $y_j = x_j$, $\rho_j = r_j$, and $a_j = (\delta^2 M)^{-\frac{1}{q'}}$ for $j = 1, 2, \ldots, M$. Then by (2.18)

$$\mu(\mu^{-1}\lambda M\delta)^{\frac{1}{p'}} \leq \mu |E_1|^{\frac{1}{p'}} \leq \|\Phi\|_{p'} \lesssim \delta^{-1-\rho} (\delta^2 M)^{\frac{1}{q'}},$$

which implies (2.16). $\blacksquare$

At this point it might be instructive to consider those bounds on μ that correspond to the points P, R, S, T in figure 1. By Lemma 2.2.1,

$$P: \quad p = 5/2,\ q = 5 \quad \mu \lesssim \lambda^{-\frac{3}{2}} M^{\frac{1}{2}} \tag{2.19}$$

$$R: \quad p = 1,\ q = \infty \quad \mu \lesssim M \tag{2.20}$$

$$S: \quad p = 1,\ q = 1 \quad \mu \lesssim \delta^{-2} \tag{2.21}$$

$$T: \quad p = 2,\ q = 2 \quad \mu \lesssim \lambda^{-1}\delta^{-1}. \tag{2.22}$$

Not surprisingly, inequalities (2.20) and (2.21) are trivial, whereas (2.22) follows (up to a $|\log\delta|^{\frac{1}{2}}$ factor) from (2.7). Our main goal will be to show (2.19) (the result below will involve a $|\log\delta|$ factor, though). In order to do this we shall need an improved version of the L^2 statement, i.e., inequality (2.22).

2.3 The L^2 theory

Before formulating the result, we consider an example.

Example: Let $10\,\delta < 10\rho < r < \frac{1}{2}$ and define

$$E = \{x \in \mathbb{R}^2 \ : \ 1-\rho < |x| < 1\}$$

and $\lambda = \sqrt{\frac{\rho}{r}}$. It is easy to see that $F = \{\overline{\mathcal{M}}_\delta \chi_E > \lambda\} \simeq B(0,r)$ and $M \simeq \frac{r^2}{\delta^2}$. To determine μ, note that Φ will be approximately constant on

$$E_1 = \{x \ : \ 1-\rho < |x| < 1-\rho/2\}.$$

Hence

$$\mu|E_1| \simeq \int_{E_1} \Phi \simeq \lambda M \delta.$$

Thus

$$\mu \simeq \delta^{-1}\rho^{-\frac{1}{2}} r^{\frac{3}{2}} = \lambda^{-1}\delta^{-1} r.$$

We shall prove below that this improved version of (2.22) holds in general (up to a $|\log\delta|$ factor) with r replaced by the typical distance of two intersecting annuli (for a precise version of this see page 18). To this end we need a refined version of the L^2 inequality (2.7). First we recall a result from [2].

Lemma 2.3.1 *Let $K \in L^1(\mathbb{R}^d)$ assuming $\widehat{K}$ differentiable. Define for $j \in \mathbb{Z}$*

$$\begin{aligned} \alpha_j &= \sup_{|\xi|\simeq 2^j} |\widehat{K}(\xi)| \\ \beta_j &= \sup_{|\xi|\simeq 2^j} |\langle \nabla \widehat{K}(\xi), \xi\rangle|. \end{aligned}$$

Then for any fixed j and $f \in \mathcal{S}$ such that supp($\hat{f}$) $\subset \{\mathbb{R}^d \ : \ 2^{j-1} < |\xi| < 2^{j+1}\}$

$$\|\sup_{t\simeq 1} |f * K_t|\|_{L^2(\mathbb{R}^d)} \lesssim \alpha_j^{\frac{1}{2}} (\alpha_j + \beta_j)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^d)}. \ \blacksquare$$

By well–known decay properties of $\widehat{d\sigma_r}$ (see Proposition 1.1.1 above) Lemma 2.3.1 implies that

$$\|\overline{\mathcal{M}} f\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)} \tag{2.23}$$

for any $f \in \mathcal{S}$ whose Fourier transform is supported in $\{\mathbb{R}^2 \ : \ 2^{j-1} < |\xi| < 2^{j+1}\}$ for some $j > 0$. The following proposition shows that this estimate can be improved if one restricts the maximal function to a small ball. We prove this fact by combining Bourgain's original argument with Lemma 2.3.2 below.

Proposition 2.3.1 *There exists an absolute constant C_0 so that for any $j = 1, 2, \ldots$, all $f \in \mathcal{S}$ with supp($\hat{f}$) $\subset \{\mathbb{R}^2 \ : \ 2^{j-1} < |\xi| < 2^{j+1}\}$, and all $0 < t \le 1$, $x_0 \in \mathbb{R}^2$*

$$\|\overline{\mathcal{M}} f\|_{L^2(B(x_0,t))} \le C_0 t^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^2)}. \tag{2.24}$$

Proof: We may assume that $x_0 = 0$. Choose cut–off functions $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\psi = 1$ on $B(0,1)$, $\eta \in C_0^\infty(1/2, 4)$ so that $\eta = 1$ on $(1,2)$, and $\phi \in \mathcal{S}$ such that supp($\hat{\phi}$) $\subset \{1/4 < |\xi| < 4\}$ and $\hat{\phi} = 1$ on $\{1/2 < |\xi| < 2\}$. Define

$$A_r^j f(x) = \psi(t^{-1}x)\eta(r) \int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\, \widehat{d\sigma}(r|\xi|)\hat{\phi}(2^{-j}\xi)\hat{f}(\xi)\, d\xi.$$

Let $\{r_\tau\}_\tau$ be a 2^{-j} net in $[1,2]$. Suppose $r_\tau \le r < r_{\tau+1}$. Then

$$|A_r^j f| \le 2^j \int_{r_\tau}^{r_{\tau+1}} |A_\rho^j f|\, d\rho + \int_{r_\tau}^{r_{\tau+1}} \left| \frac{d}{d\rho} A_\rho^j f \right| \, d\rho$$

and thus

$$\begin{aligned} \sup_{1<r<2} |A_r^j f|^2 &\le 2^j \int_1^2 |A_\rho^j f|^2 \, d\rho + 2^{-j} \int_1^2 \left| \frac{d}{d\rho} A_\rho^j f \right|^2 \, d\rho \\ &= 2^j A + 2^{-j} B. \end{aligned}$$

By Proposition 1.1.1 $\widehat{d\sigma}$ has the representation

$$\widehat{d\sigma}(\xi) = \Re\{e^{2\pi i|\xi|}\omega(|\xi|)\} \tag{2.25}$$

with $\omega \in C^\infty(0,\infty)$ and

$$\left| \frac{d^k\, \omega(s)}{ds^k} \right| \lesssim (1+|s|)^{-\frac{1}{2}-k} \tag{2.26}$$

for all $k = 0, 1, 2, \ldots$. Hence the integral of A can be written as

$$\int_{\mathbb{R}^2} A\, dx = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K(\xi,\tilde{\xi}) \hat{\phi}(2^{-j}\xi) \hat{\phi}(2^{-j}\tilde{\xi}) \hat{f}(\xi) \overline{\hat{f}(\tilde{\xi})}\, d\xi\, d\tilde{\xi} \tag{2.27}$$

where

$$K(\xi,\tilde{\xi}) = t^2 \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} e^{2\pi i(tx\cdot(\xi-\tilde{\xi}) + r(|\xi|-|\tilde{\xi}|))} \psi^2(x) \eta^2(r) \omega(r|\xi|) \overline{\omega(r|\tilde{\xi}|)}\, dx\, dr. \tag{2.28}$$

Integrating by parts and applying (2.25) shows that

$$|K(\xi,\tilde{\xi})| \lesssim 2^{-j} t^2 (1 + t|\xi - \tilde{\xi}|)^{-2} (1 + |\,|\xi| - |\tilde{\xi}|\,|)^{-2},$$

provided $|\xi| \simeq |\tilde{\xi}| \simeq 2^j$. Lemma 2.3.2 and Schur's lemma yield

$$\int_{\mathbb{R}^2} A\, dx \lesssim 2^{-j} t \|f\|_2^2.$$

Similarly,

$$\int_{\mathbb{R}^2} B\, dx \lesssim 2^{j} t \|f\|_2^2,$$

and the proposition follows. ■

The following lemma is true because the $|\,|\xi| - |\tilde{\xi}|\,|$ factor reduces the two–dimensional scaling in the integral below to one dimension.

Lemma 2.3.2 *Let* $0 < t < 1$. *Then*

$$\sup_{\tilde{\xi}} \int_{\mathbb{R}^2} \left(1 + t|\xi - \tilde{\xi}|\right)^{-2} \left(1 + |\,|\xi| - |\tilde{\xi}|\,|\right)^{-2} d\xi \lesssim t^{-1}.$$

Proof: Fix a $\tilde{\xi} \in \mathbb{R}^2$. Then, on the one hand,

$$
\begin{aligned}
& \int_{\{\xi\,:\,|\xi-\tilde{\xi}|\le|\tilde{\xi}|/2\}} \Big(1+t|\xi-\tilde{\xi}|\Big)^{-2} \Big(1+\big|\,|\xi|-|\tilde{\xi}|\,\big|\Big)^{-2}\, d\xi \\
\simeq\ & \sum_{2^j\le|\tilde{\xi}|/2} (1+t2^j)^{-2} \int_{\{\xi\,:\,|\xi-\tilde{\xi}|\simeq 2^j\}} (1+|\,|\xi|-|\tilde{\xi}|\,|)^{-2}\, d\xi \\
\simeq\ & \sum_{2^j\le|\tilde{\xi}|/2} (1+t2^j)^{-2} 2^j |\tilde{\xi}|^{-1} \int_{|\tilde{\xi}|-2^j}^{|\tilde{\xi}|+2^j} (1+|r-|\tilde{\xi}||)^{-2} r\, dr \\
\simeq\ & \sum_{2^j\le|\tilde{\xi}|} (1+t2^j)^{-2} 2^j \simeq \sum_{2^j t\le 1} 2^j + \sum_{t^{-1}\le 2^j\le|\tilde{\xi}|} t^{-2}2^{-j} \\
\simeq\ & t^{-1} + t^{-2}t \simeq t^{-1}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \int_{\{\xi\,:\,|\xi-\tilde{\xi}|\ge|\tilde{\xi}|/2\}} \Big(1+t|\xi-\tilde{\xi}|\Big)^{-2} \Big(1+\big|\,|\xi|-|\tilde{\xi}|\,\big|\Big)^{-2}\, d\xi \\
\simeq\ & \int_{\{\xi\,:\,|\xi-\tilde{\xi}|\ge|\tilde{\xi}|/2\,,\,|\,|\xi|-|\tilde{\xi}|\,|\le|\tilde{\xi}|/2\}} \Big(1+t|\tilde{\xi}|\Big)^{-2} \Big(1+\big|\,|\xi|-|\tilde{\xi}|\,\big|\Big)^{-2}\, d\xi \\
& \quad + \int_{\{\xi\,:\,|\,|\xi|-|\tilde{\xi}|\,|\ge|\tilde{\xi}|/2\}} \Big(1+t|\xi-\tilde{\xi}|\Big)^{-2} \Big(1+\big|\,|\xi|-|\tilde{\xi}|\,\big|\Big)^{-2}\, d\xi \\
=\ & A+B.
\end{aligned} \tag{2.29}
$$

The first term in (2.29) can be estimated as follows.

$$
\begin{aligned}
A \ &\simeq\ \Big(1+t|\tilde{\xi}|\Big)^{-2} \int_{|\tilde{\xi}|/2}^{3|\tilde{\xi}|/2} \Big(1+|r-|\tilde{\xi}||\Big)^{-2} r\, dr \\
&\simeq\ \Big(1+t|\tilde{\xi}|\Big)^{-2} |\tilde{\xi}| \lesssim t^{-1}.
\end{aligned}
$$

For the second term compute

$$
\begin{aligned}
B \ \simeq\ & \int_{\{\xi\,:\,\frac{3}{2}|\tilde{\xi}|\le|\xi|\}} (1+t|\xi|)^{-2} (1+|\xi|)^{-2}\, d\xi \,+ \\
& \qquad + \int_{\{\xi\,:\,\frac{1}{2}|\tilde{\xi}|\ge|\xi|\}} \Big(1+t|\tilde{\xi}|\Big)^{-2} \Big(1+|\tilde{\xi}|\Big)^{-2}\, d\xi \\
\simeq\ & \int_{\{\xi\,:\,t^{-1}\le|\xi|\}} (t|\xi|)^{-2} (1+|\xi|)^{-2}\, d\xi + \int_{\{\xi\,:\,\frac{3}{2}|\tilde{\xi}|\le|\xi|\le t^{-1}\}} (1+|\xi|)^{-2}\, d\xi \\
& \qquad + \Big(1+t|\tilde{\xi}|\Big)^{-2} \Big(1+|\tilde{\xi}|\Big)^{-2} |\tilde{\xi}|^2 \\
\lesssim\ & 1 + |\log t| + 1 \lesssim t^{-1}. \quad \blacksquare
\end{aligned}
$$

Remark: (2.28) above shows that Proposition 2.3.1 is essentially equivalent to the following estimate for the two–dimensional wave equation. Let u solve

$$\Box\, u = 0, \ \ u(0) = f, \ u_t(0) = 0.$$

Then there exists an absolute constant C_0 so that

$$\int_0^1 \int_{B(x_0,r)} |u(x,t)|^2\, dx\, dt \le C_0 r \|f\|^2_{L^2(\mathbb{R}^2)} \tag{2.30}$$

for all $0 < r \le 1$. It might be interesting to ask whether such an estimate can hold in L^p with $p \ne 2$. Interpolating (2.30) with Sogge's sharp local smoothing conjecture [15], i.e.

$$\int_0^1 \int_{\mathbb{R}^2} |u(x,t)|^4\, dx\, dt \le C_\epsilon\ \|f\|^4_{L^4_\epsilon} \tag{2.31}$$

with $\epsilon > 0$ yields

$$\left(\int_0^1 \int_{B(x_0,r)} |u(x,t)|^p\, dx\, dt\right)^{\frac{1}{p}} \le C_\epsilon\ r^{\frac{2}{p}-\frac{1}{2}} \|f\|_{L^p_\epsilon} \tag{2.32}$$

for $2 \le p \le 4$, $x_0 \in \mathbb{R}^2$, $0 < r \le 1$, $\epsilon > 0$ and all $f \in \mathcal{S}$. Solving the wave equation above with initial condition f equal to a smooth version of $\chi_{C_\delta(0,r)}$ shows that the exponent $\frac{2}{p} - \frac{1}{2}$ is optimal. Moreover, as in the case of local smoothing, (2.32) cannot hold for $p \notin [2,4]$ or with $\epsilon = 0$ if $p > 2$.

It is standard to pass from f as in the statement of Proposition 2.3.1 to general $f \in L^2$. This is done in the following corollary.

Corollary 2.3.1 *There exists an absolute constant C_0 so that for any $f \in L^2(\mathbb{R}^2)$, $x_0 \in \mathbb{R}^2$, and $0 < \delta, t < 1$,*

$$\|\overline{\mathcal{M}}_\delta f\|_{L^2(B(x_0,t))} \le C_0\, t^{\frac{1}{2}} |\log \delta|^{\frac{1}{2}} \|f\|_2. \tag{2.33}$$

The equivalent dual statement to (2.33) is:

$$\|\sum_j a_j \chi_{C_\delta(y_j,\rho_j)}\|_{L^2(\mathbb{R}^2)} \le C_0\, |\log \delta|^{\frac{1}{2}} \delta^{-1} t^{\frac{1}{2}} \tag{2.34}$$

for all δ–separated $\{y_j\}$ in $B(x_0,t)$, all $\{a_j\}$ for which $\delta^2 \sum_j |a_j|^2 \le 1$, and all $\rho_j \in (1,2)$.

Proof: Choose $\phi \in \mathcal{S}$ such that $\mathrm{supp}(\hat{\phi})$ compact, $\hat{\phi} \ge 0$, $\phi \ge 0$, $\hat{\phi} \ge 1$ on $B(0,1)$. Given $f \in L^2(\mathbb{R}^2)$, $f \ge 0$ let $f = \sum_{j=0}^\infty f_j$ be a Littlewood–Paley decomposition, i.e. $\mathrm{supp}(\hat{f}_0) \subset \{|\xi| < 2\}$ and $\mathrm{supp}(\hat{f}_j) \subset \{\mathbb{R}^2 \,:\, 2^{j-1} < |\xi| < 2^{j+1}\}$ for $j = 1, 2, \ldots$. Let $\chi_{\delta,r} = d\sigma_r * \phi_\delta$. Then clearly

$$\chi_{C_\delta(0,r)} \lesssim \chi_{\delta,r}.$$

If M denotes the usual Hardy–Littlewood maximal operator it is easy to see that

$$\begin{aligned}
\|\overline{\mathcal{M}}_\delta f\|_{L^2(B(x_0,t))} &\lesssim \|Mf_0\|_{L^2(B(x_0,t))} + \sum_{1<2^j \lesssim \delta^{-1}} \|\sup_{1<r<2} |\chi_{\delta,r} * f_j|\|_{L^2(B(x_0,t))} \\
&\lesssim t\|Mf_0\|_\infty + \sum_{1<2^j \lesssim \delta^{-1}} \|\overline{\mathcal{M}} f_j\|_{L^2(B(x_0,t))} \\
&\lesssim t\|f_0\|_\infty + t^{\frac{1}{2}} \sum_{1<2^j \lesssim \delta^{-1}} \|f_j\|_2 \\
&\lesssim t\|f_0\|_2 + t^{\frac{1}{2}} |\log \delta|^{\frac{1}{2}} \left(\sum_{1<2^j \lesssim \delta^{-1}} \|f_j\|_2^2\right)^{\frac{1}{2}} \qquad (2.35) \\
&\lesssim t^{\frac{1}{2}} |\log \delta|^{\frac{1}{2}} \|f\|_2.
\end{aligned}$$

In line (2.35) we have used a special case of Bernstein's inequality, namely

$$\|f_0\|_\infty \lesssim \|f_0\|_2. \quad \blacksquare$$

In order to obtain information on μ from (2.33) we will determine the typical distance of the centers of two intersecting annuli in any collection of annuli. More precisely, we can specify the distance of the centers and the angle of intersection of those annuli that contribute most to the multiplicity function Φ. Following [8], we will accomplish this by applying the pigeon hole principle to our family of annuli satisfying (2.13). Define $\bar{\lambda} = |\log\delta|^{-2}\lambda/2$, $\bar{\mu} = |\log\delta|^{-2}\mu$, $\overline{M} = |\log\delta|^{-2}M/2$. Furthermore, for all $i, j \in \{1, 2, \ldots, M\}$ we let (for the meaning of Δ see Lemma 2.4.2 below)

$$\begin{aligned} \Delta_{i,j} &= \max(\delta, \,||x_i - x_j| - |r_i - r_j||\,) \\ S^j_{t,\epsilon} &= \{i \;:\; C_i \cap C_j \neq \emptyset\;,\; t/2 \leq |x_i - x_j| \leq t\;,\; \epsilon \leq \Delta_{i,j} \leq 2\epsilon\} \\ \Phi^j_{t,\epsilon} &= \sum_{i \in S^j_{t,\epsilon}} \chi_{C_i^*} \end{aligned} \tag{2.36}$$

(recall that $C_i = C_\delta(x_i, r_i)$ and $C_i^* = E \cap C_i$). The pigeon hole principle asserts that there are numbers $t \in [\delta, 1]$, $\epsilon \in [\delta, 1]$ such that

$$|\{C_j^* \;:\; \Phi^j_{t,\epsilon} \geq \bar{\mu}\}| \geq \bar{\lambda}|C_j| \tag{2.37}$$

for at least $\overline{M}$ values of j, say $1 \leq j \leq \overline{M}$. Indeed, let j be one of the at least $M/2$ indices satisfying (2.13), i.e.,

$$|\{C_j^* \;:\; \Phi \geq \mu\}| \geq \frac{\lambda}{2}|C_j|.$$

Let $x \in C_j^*$ so that $\Phi(x) \geq \mu$. We conclude that for some choice of t and ϵ depending only on x and j we have

$$\Phi^j_{t,\epsilon}(x) \geq \bar{\mu}.$$

For if not, then (in the following sum t and ϵ are dyadic)

$$\Phi(x) = \sum_{t,\epsilon \in [\delta,1]} \Phi^j_{t,\epsilon}(x) < |\log\delta|^2\bar{\mu} = \mu,$$

contradicting the choice of x. Similarly, we see that (2.37) holds for any j as above and for some choice of t and ϵ depending only on j. Finally, applying the pigeon hole principle in j yields that there are t and ϵ so that (2.37) holds for at least $\overline{M}$ values of j. Otherwise, the number of j's satisfying (2.13) would have to be strictly less than $|\log\delta|^2\overline{M} = M/2$. Henceforth we will fix ϵ and t to be those numbers.

By essentially the same argument as in the second part of Lemma 2.2.1 we can now establish the refined version of (2.22).

Lemma 2.3.3 *The multiplicity μ satisfies the following apriori estimate with absolute constants C and b.*

$$\mu \leq C|\log\delta|^b\lambda^{-1}\delta^{-1}t. \tag{2.38}$$

Proof: Let $\{z_i\}$ be a t–net and consider the quantities

$$\begin{aligned} M_1(i) &= \mathrm{card}\{1 \le j \le \overline{M} \ : \ x_j \in B(z_i,t)\} \\ M_2(i) &= \mathrm{card}\{1 \le j \le M \ : \ x_j \in B(z_i,2t)\}. \end{aligned}$$

Then, clearly

$$\sum_i M_1(i) \simeq \overline{M} \quad \text{and} \quad \sum_i M_2(i) \simeq M. \tag{2.39}$$

Since $\overline{M} = |\log\delta|^{-2} M/2$ we conclude from (2.39) that there is a point of the net, say z_0, so that $M_1 = M_1(0)$ and $M_2 = M_2(0)$ satisfy $M_1 \gtrsim |\log\delta|^{-2} M_2$. Define

$$\Phi_1 = \sum_{j\,:\,|x_j - z_0| \le 2t} \chi_{C_j^*}.$$

As in Lemma 2.2.1 we distinguish two cases. If

$$|E_1| = |\{E \ : \ \Phi_1 \ge \bar\mu\}| \le \bar\mu^{-1}\bar\lambda M_2 \delta,$$

then by Corollary 2.3.1 (setting $x_0 = z_0$)

$$\|\overline{\mathcal{M}}_\delta \chi_{E_1}\|_{L^2(B(z_0,t))} \le C_0\, |\log\delta|^{\frac12} t^{\frac12} |E_1|^{\frac12}. \tag{2.40}$$

The expression on the left is $\gtrsim \bar\lambda(\delta^2 M_1)^{\frac12}$. Indeed, for any $1 \le j \le \overline{M}$ so that $x_j \in B(z_i,t)$ we have

$$\Phi_1 \ge \Phi^j_{t,\epsilon}.$$

Thus (2.37) implies that

$$|\{C_j^* \ : \ \Phi_1 \ge \bar\mu\}| \ge \bar\lambda |C_j|,$$

or equivalently

$$\begin{aligned} |C_j \cap E_1| &\ge \bar\lambda |C_j|, \quad \text{or} \\ \overline{\mathcal{M}}_\delta \chi_{E_1}(x_j) &\ge \bar\lambda. \end{aligned}$$

Since the $\{x_j\}$ are δ–separated, our claim follows. On the other hand, the right side of (2.40) is

$$\le C_0\, |\log\delta|^{\frac12} t^{\frac12} (\bar\mu^{-1}\bar\lambda M_2\delta)^{\frac12}$$

by our assumption on $|E_1|$. Recalling the definition of $M_1, M_2, \bar\lambda$ etc., we obtain (2.38).

If $|E_1| \ge \bar\mu^{-1}\bar\lambda M_2\delta$ we use duality, i.e., (2.34). Letting $x_0 = z_0$ in (2.34), replacing t by $2t$, setting $y_j = x_j$, $\rho_j = r_j$, and $a_j = (\delta^2 M_2)^{-\frac12}$ for $j = 1, \ldots, M_2$, or $a_j = 0$ otherwise, we obtain

$$\|\Phi_1\|_2 \le C_0\, |\log\delta|^{\frac12} \delta^{-1} t^{\frac12} (\delta^2 M_2)^{\frac12}.$$

The left–hand side is

$$\ge \bar\mu |\{E \ : \ \Phi_1 \ge \bar\mu\}|^{\frac12} = \bar\mu |E_1|^{\frac12} \ge \bar\mu(\bar\mu^{-1}\bar\lambda M_2\delta)^{\frac12}$$

and the lemma follows. ■

In section four of the following chapter it is shown how to obtain (2.38) (in a slightly different form) without using the Fourier transform. The main tool turns out to be a two circle lemma, cf. Lemma 3.4.4.

2.4 The three circle lemma

In the previous section implicit information about circles was used to prove an L^2 bound on the maximal function and thus a bound on the multiplicity μ. In this section we shall attempt to use explicit geometric properties of circles in order to bound μ. The procedure we apply here was discovered by L. Kolasa and T. Wolff [8], who in turn use Marstrand's three circle lemma, cf. [9] and Lemma 2.4.1 below. The underlying principle for that lemma is the following geometric observation. We call two tangent circles internally tangent if the smaller one is contained inside the larger one. By assuming that the centers of all circles under consideration are contained in a fixed set of diameter one, we shall henceforth rule out external tangencies.

Fact: Given any three circles which are not internally tangent at a single point, there are at most two circles which are internally tangent to the three given ones.

Kolasa and Wolff observed that this fact can be combined with a basic result from extremal graph theory to control the total number of possible tangencies in a large collection of circles of which no three are tangent at a single point. Indeed, we have the following

Proposition 2.4.1 *Suppose $\{C_j\}_1^N$ is a collection of distinct circles in the plane so that no three are tangent at a single point. Then*

$$\mathrm{card}\{(i,j)\ :\ C_i, C_j \text{ are tangent}\} \lesssim N^{5/3}.$$

In particular, at least half the circles will be tangent to no more than $\lesssim N^{2/3}$ other circles.

Proof: Let

$$Q = \{(C_i, C_{j_1}, C_{j_2}, C_{j_3})\ :\ C_i \text{ and } C_{j_k} \text{ are tangent for } k = 1,2,3\}.$$

On the one hand, the above fact implies that

$$\mathrm{card}(Q) \le 2\begin{pmatrix} N \\ 3 \end{pmatrix}.$$

On the other hand, with $n_i = \mathrm{card}\{C_j\ :\ C_i, C_j \text{ are tangent}\}$,

$$\mathrm{card}(Q) \ge \sum_{i=1}^N \begin{pmatrix} n_i \\ 3 \end{pmatrix} \ge \sum_{i=1}^N (n_i - 2)^3/6.$$

Thus

$$\begin{aligned}
\sum_{i=1}^N n_i &= \sum_{i=1}^N (n_i - 2) + 2N \\
&\le \Big(\sum_{i=1}^N (n_i-2)^3\Big)^{1/3} N^{2/3} + 2N \\
&\lesssim \begin{pmatrix} N \\ 3 \end{pmatrix}^{\frac{1}{3}} N^{2/3} + N \\
&\lesssim N^{5/3}.
\end{aligned}$$

Since $\mathrm{card}\{(i,j)\ :\ C_i, C_j \text{ are tangent}\} = \sum_{i=1}^N n_i$, we are done. ■

This proof is just a special case of a well–known argument that provides upper bounds for the maximal number of edges in a bipartite digraph with m edges and n sinks containing no $K_{s,t}$. For the meaning of this terminology, further details and applications, and references, see [4].

Although the argument below is motivated by the previous discussion, it does not directly use the fact or Proposition 2.4.1. Indeed, dealing only with tangencies is too restrictive for our purposes. Rather, we shall use the following quantitative version of the above fact, cf. [9], which allows for various degrees of tangency. For a more general and precise version, see [8].

The number ϵ in Lemma 2.4.1 controls the degree of internal tangency whereas λ separates the "points of tangency". $N_\delta(S)$ denotes the δ entropy of the set S, i.e., the cardinality of a maximally δ separated set in S.

Lemma 2.4.1 *Let* $(x_j, r_j)_{j=1}^3 \in \mathbb{R}^2 \times (1,2)$ *and fix* $0 < \lambda, \epsilon < 1$. *Consider the set*

$$\begin{aligned} S \;&=\; \{x \in \mathbb{R}^2 \setminus \bigcup_{j=1}^3 B(x_j,\epsilon) \;:\; \exists\, r \in (1,2) \quad \textit{with} \;\; ||x_i - x| - |r_i - r|| < \epsilon \\ &\qquad \textit{for } i = 1,2,3 \textit{ and } \quad |e_i(x,r) - e_j(x,r)| > \lambda \;\textit{ for } i \neq j\;, i,j = 1,2,3\}. \end{aligned}$$

Here

$$e_i(x,r) = \frac{x_i - x}{|x_i - x|}\operatorname{sgn}(r_i - r).$$

Then

$$N_\delta(S) \lesssim \left(\frac{\epsilon}{\delta}\right)^2 \lambda^{-3}$$

for any $0 < \delta \le \epsilon$.

Remark: It is easy to see that the bound on $N_\delta(S)$ can be attained.

Proof: Let

$$\begin{aligned} \Omega \;&=\; \{(x,r) \in \mathbb{R}^2 \times (1,2) \;:\; |x - x_j| > 3\epsilon, r \neq r_i, |e_i(x,r) - e_j(x,r)| > \lambda \\ &\qquad \text{for } i \neq j\;, i,j = 1,2,3\} \end{aligned}$$

and $F : \Omega \to \mathbb{R}^3$ be defined by

$$F(x,r) = (|x_i - x| - |r_i - r|)_{i=1}^3.$$

It is easy to see that the Jacobian JF of F satisfies

$$JF \simeq |e_1 - e_2||e_1 - e_3||e_2 - e_3| > \lambda^3.$$

Since $\operatorname{card}(F^{-1}(p)) \le C_0$ for some absolute constant C_0 and all $p \in \mathbb{R}^3$, we conclude that

$$|F^{-1}(B(0,2\epsilon))| \lesssim \epsilon^3 \lambda^{-3}.$$

According to the definition of S there exists a function $r : S \to (1,2)$ so that for every $x \in S$ we have $|F(x,r(x))| < \epsilon$. Then clearly

$$\begin{aligned} \{(x,r) : x \in S\,,\, |r - r(x)| < \epsilon\} \quad &\subset \quad F^{-1}(B(0,2\epsilon)) \cup \\ &\qquad \{(x,r) : x \in S \cap \bigcup_{j=1}^3 B(x_j,3\epsilon)\,,\, |r - r(x)| < \epsilon\} \end{aligned}$$

and thus $|S| \lesssim \epsilon^2 \lambda^{-3}$. ■

Figure 2.2: Marstrand's three circle lemma

The following lemma contains bounds on the diameter and the area of $C_\delta(x,r) \cap C_\delta(y,s)$. In various forms it appears in several papers on this subject, see, e.g., [1], [8], [9], [15]. Since the exact version we use here does not seem to be contained explicitly in any of these references, we provide a proof for the reader's convenience. Let

$$\Delta = \max(||x-y| - |r-s||, \delta).$$

Lemma 2.4.2 *Suppose $x, y \in \mathbb{R}^2$, $x \neq y$, $|x-y| \leq \frac{1}{2}$, and $r, s \in (1,2)$, $r \neq s$, $0 < \delta < 1$. Then there is an absolute constant A so that*

i. $C_\delta(x,r) \cap C_\delta(y,s)$ is contained in a δ neighborhood of an arc of length $\leq A\sqrt{\frac{\Delta}{|x-y|}}$ centered at the point $x - r\,\mathrm{sgn}(r-s)\frac{x-y}{|x-y|}$.

ii. the area of intersection satisfies

$$|C_\delta(x,r) \cap C_\delta(y,s)| \leq A\frac{\delta^2}{\sqrt{\Delta|x-y|}}.$$

Proof: Let $z \in C_\delta(x,r) \cap C_\delta(y,s)$. Then $|z-x| = r_1$ and $|z-y| = s_1$ where $|r-r_1| < \delta$ and $|s-s_1| < \delta$. By simple algebra

$$2(z-x)\cdot(y-x) = r_1^2 - s_1^2 + |y-x|^2. \tag{2.41}$$

Assume $r < s$. Then (2.41) implies

$$2r_1|x-y|(1-\cos\sphericalangle(z-x, x-y)) = (r_1 + |x-y|)^2 - s_1^2$$

and thus

$$\sphericalangle(z-x, x-y) \simeq \sqrt{\frac{|x-y| - (s_1 - r_1)}{|x-y|}} \lesssim \sqrt{\frac{\Delta}{|x-y|}}. \tag{2.42}$$

If $r > s$ one estimates $\sphericalangle(z-x, y-x)$ in a similar fashion.

If $\Delta \leq 10\delta$ the bound in ii follows from i. Otherwise consider $\alpha = \sphericalangle(z-x, x-y)$ as a function of r_1 and s_1. Taking partial derivatives in (2.41) yields

$$\begin{aligned}
\frac{\partial\alpha}{\partial r_1}\, r_1|x-y|\sin\alpha &= r_1 + |x-y|\cos\alpha \\
\frac{\partial\alpha}{\partial s_1}\, r_1|x-y|\sin\alpha &= -s_1.
\end{aligned}$$

Thus

$$|\frac{\partial\alpha}{\partial r_1}| + |\frac{\partial\alpha}{\partial s_1}| \lesssim (|\alpha||x-y|)^{-1} \simeq (\Delta|x-y|)^{-\frac{1}{2}}.$$

The last equality is true since $\Delta > 10\delta$ implies that (2.42) holds with $\simeq$ instead of $\lesssim$. Since r_1 and s_1 vary in a δ interval, α will be contained in an interval of length $\lesssim \frac{\delta}{\sqrt{\Delta|x-y|}}$ and ii follows. ∎

Proposition 2.4.2 below is the main result of this thesis.

Proposition 2.4.2 *$\overline{\mathcal{M}}_\delta$ is of restricted weak type $(5/2, 5)$, i.e., for any $f \in L^1 \cap L^\infty(\mathbb{R}^2)$*

$$\|\overline{\mathcal{M}}_\delta f\|_{5,\infty} \leq C\,|\log\delta|^b\,\|f\|_{5/2,1} \tag{2.43}$$

where b and C are absolute constants.

Proof: In this proof we let B_δ denote a constant of the form $C|\log\delta|^b$, where the values of C and b are allowed to vary depending on the context. By Lemma 2.2.1 we need to show

$$\mu \leq B_\delta\, \lambda^{-\frac{3}{2}} M^{\frac{1}{2}}. \tag{2.44}$$

C and b are determined implicitly in the calculation below. This will follow from the combinatorial argument in [8], which is based on the three circle lemma, and the refined L^2 bound from above. A is the absolute constant from Lemma 2.4.2.

$$\underline{\text{Case 1:}} \qquad \bar{\lambda} \leq 100A\left(\frac{\epsilon}{t}\right)^{\frac{1}{2}} \tag{2.45}$$

On the one hand, by (2.37) and Lemma 2.4.2,

$$\bar{\mu}\bar{\lambda}\delta \lesssim \int_{C_j} \sum_{i\in S^j_{t,\epsilon}} \chi_{C_i^*} \lesssim \operatorname{card}(S^j_{t,\epsilon})\frac{\delta^2}{\sqrt{\epsilon t}} \lesssim M\frac{\delta^2}{\sqrt{\epsilon t}}. \tag{2.46}$$

On the other hand, by Lemma 2.3.3

$$\bar{\mu} \leq B_\delta\, \bar{\lambda}^{-1}\delta^{-1} t.$$

Thus

$$\bar{\mu}\bar{\lambda}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{1}{2}} \leq B_\delta\, \min\left(M, \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}}\right).$$

Hence, if

$$M \leq \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}}, \tag{2.47}$$

then

$$\begin{aligned}
\bar{\mu} &\leq B_\delta\, \bar{\lambda}^{-1}\left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{\delta}{t}\right)^{\frac{1}{2}} M \\
&= B_\delta\, \bar{\lambda}^{-\frac{3}{2}} M^{\frac{1}{2}} \bar{\lambda}^{\frac{1}{2}} M^{\frac{1}{2}}\left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{\delta}{t}\right)^{\frac{1}{2}} \qquad (2.48)\\
&\leq B_\delta\, \bar{\lambda}^{-\frac{3}{2}} M^{\frac{1}{2}}\left(\frac{\epsilon}{t}\right)^{\frac{1}{4}}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{4}}\left(\frac{t}{\delta}\right)^{\frac{3}{4}}\left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{\delta}{t}\right)^{\frac{1}{2}} \\
&= B_\delta\, \bar{\lambda}^{-\frac{3}{2}} M^{\frac{1}{2}},
\end{aligned}$$

where we have used (2.45) and (2.47) in line (2.48) to replace $\bar{\lambda}^{\frac{1}{2}}$ and $M^{\frac{1}{2}}$, respectively. If, on the other hand,

$$M \geq \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}}, \tag{2.49}$$

then

$$\begin{aligned}
\bar{\mu} &\leq B_\delta\, \bar{\lambda}^{-1}\delta^{-1}t = B_\delta\, \bar{\lambda}^{-\frac{3}{2}}\bar{\lambda}^{\frac{1}{2}}\delta^{-1}t \\
&\leq B_\delta\, \bar{\lambda}^{-\frac{3}{2}}\left(\frac{\epsilon}{t}\right)^{\frac{1}{4}}\delta^{-1}t \leq B_\delta\, \bar{\lambda}^{-\frac{3}{2}}\left(\frac{\delta^2 M}{t^2}\right)^{\frac{1}{2}}\delta^{-1}t \qquad (2.50)\\
&= B_\delta\, \bar{\lambda}^{-\frac{3}{2}} M^{\frac{1}{2}}.
\end{aligned}$$

Here we have used (2.45) and then (2.49) in line (2.50).

$$\underline{\text{Case 2:}} \qquad \bar{\lambda} \geq 100A\left(\frac{\epsilon}{t}\right)^{\frac{1}{2}} \tag{2.51}$$

Following [8] we let

$$\begin{aligned} Q \quad = \quad & \{(j, i_1, i_2, i_3) \,:\, 1 \leq j \leq \overline{M},\ i_1, i_2, i_3 \in S^j_{t,\epsilon} \text{ and the distance between} \\ & \text{any two of the sets } \ C_j \cap C_{i_1}, C_j \cap C_{i_2}, C_j \cap C_{i_3} \text{ is at least } \bar{\lambda}/20\}. \end{aligned} \tag{2.52}$$

Suppose $(j, i_1, i_2, i_3) \in Q$. Then Lemma 2.4.2 implies that any two of the

$$e_i = x_j - r_j \operatorname{sgn}(r_j - r_i)\frac{x_j - x_i}{|x_j - x_i|}$$

for $i = i_1, i_2, i_3$ are separated by a distance $\bar{\lambda}/20$. Indeed, by that lemma, e_i is the center of $C_i \cap C_j$ and in view of (2.36), for any $i \in S^j_{t,\epsilon}$

$$\operatorname{diam}(C_i \cap C_j) \leq 2A\sqrt{\frac{\epsilon}{t}} \leq \bar{\lambda}/50 \tag{2.53}$$

by (2.51). Lemma 2.4.1 therefore implies that

$$\operatorname{card}(Q) \lesssim \left(\frac{\epsilon}{\delta}\right)^2 \bar{\lambda}^{-3} M^3. \tag{2.54}$$

On the other hand, we claim that

$$\operatorname{card}(Q) \gtrsim \overline{M}\left(\bar{\mu}\frac{\bar{\lambda}\delta}{\delta^2/\sqrt{\epsilon t}}\right)^3. \tag{2.55}$$

This would clearly follow from

$$\begin{aligned} & \min_{1 \leq j \leq \overline{M}} \operatorname{card}(\{(i_1, i_2, i_3) \in (S^j_{t,\epsilon})^3 \,:\, \text{The distance between any two of the sets} \\ & C_j \cap C_{i_1}, C_j \cap C_{i_2}, C_j \cap C_{i_3} \text{ is at least } \bar{\lambda}/20\}) \gtrsim \left(\bar{\mu}\frac{\bar{\lambda}\delta}{\delta^2/\sqrt{\epsilon t}}\right)^3. \end{aligned} \tag{2.56}$$

Denote the set on the left–hand side by $Q^{(j)}$ and fix any j as above. By (2.46) the number of possible choices of i_1 is

$$\operatorname{card}(S^j_{t,\epsilon}) \gtrsim \bar{\mu}\frac{\bar{\lambda}\delta}{\delta^2/\sqrt{\epsilon t}}.$$

Suppose that $(i_1, i_2, i_3) \in Q^{(j)}$. We claim that

$$\operatorname{card}(\{i \in S^j_{t,\epsilon} \,:\, (i_1, i_2, i) \in Q^{(j)}\}) \gtrsim \bar{\mu}\frac{\bar{\lambda}\delta}{\delta^2/\sqrt{\epsilon t}}. \tag{2.57}$$

To prove (2.57) let R_1 and R_2 be "rectangles" in C_j of length $\bar{\lambda}/5$ and width δ centered at e_{i_1} and e_{i_2}, respectively. Using (2.53) we conclude that

$$i \in S^j_{t,\epsilon}, \qquad C_i \cap R_\tau = \emptyset \quad \text{for} \quad \tau = 1, 2$$

implies that

$$\operatorname{dist}(C_j \cap C_{i_\tau}, C_j \cap C_i) > \bar{\lambda}/20 \quad \text{for} \quad \tau = 1, 2.$$

Since

$$|\{C_j^* \setminus (R_1 \cup R_2) \ : \ \Phi^j_{t,\epsilon} \geq \bar{\mu}\}| \geq \frac{\bar{\lambda}}{2}|C_j|,$$

(2.57) follows from (2.46) (simply replace (2.37) with the previous inequality). Estimating the number of admissible choices of i_2 given a fixed i_1 in a similar fashion proves (2.56) and thus (2.55). We infer from (2.54) and (2.55) that

$$\bar{\mu}^3 \leq B_\delta \, \bar{\lambda}^{-6} \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{3}{2}} M^{\frac{5}{4}}. \tag{2.58}$$

Combining (2.58) and (2.38) yields

$$\bar{\mu} \leq B_\delta \ \min\left(\bar{\lambda}^{-1}\delta^{-1}t, \bar{\lambda}^{-2}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{\frac{5}{12}}\right).$$

Hence, if

$$\bar{\lambda} \leq \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{3}{2}} M^{-\frac{1}{12}}, \tag{2.59}$$

we conclude that

$$\begin{aligned}
\bar{\mu} \quad &\leq \quad B_\delta \, \bar{\lambda}^{-1}\delta^{-1}t \leq B_\delta \, \bar{\lambda}^{-\frac{3}{2}}\bar{\lambda}^{\frac{3}{4}}\bar{\lambda}^{-\frac{1}{4}}\left(\frac{t}{\delta M^{\frac{1}{2}}}\right) M^{\frac{1}{2}} \\
&\leq \quad B_\delta \, \bar{\lambda}^{-\frac{3}{2}} \left(\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{3}{2}} M^{-\frac{1}{12}}\right)^{\frac{3}{4}} \\
&\qquad \cdot \left(\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{-\frac{1}{4}}\right)^{-\frac{1}{4}} \left(\frac{t}{\delta M^{\frac{1}{2}}}\right) M^{\frac{1}{2}} \\
&= \quad B_\delta \, \bar{\lambda}^{-\frac{3}{2}} M^{\frac{1}{2}}.
\end{aligned} \tag{2.60}$$

The expressions in (2.60) are obtained by estimating $\bar{\lambda}$ by (2.59) and (2.51), respectively.
If, on the other hand,

$$\bar{\lambda} \geq \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{3}{2}} M^{-\frac{1}{12}}, \tag{2.61}$$

then

$$\begin{aligned}
\bar{\mu} \quad &\leq \quad B_\delta \, \bar{\lambda}^{-2} \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{\frac{5}{12}} \\
&\leq \quad B_\delta \, \bar{\lambda}^{-2} \left(\bar{\lambda}\left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{-\frac{3}{2}} M^{\frac{1}{12}}\right)^{\frac{1}{4}}
\end{aligned} \tag{2.62}$$

$$
\begin{aligned}
& \cdot \left(\bar{\lambda}^{\frac{1}{3}} \left(\frac{\delta M^{\frac{1}{2}}}{t} \right)^{-\frac{1}{6}} M^{\frac{1}{12}} \right)^{\frac{3}{4}} \left(\frac{\delta M^{\frac{1}{2}}}{t} \right)^{\frac{1}{2}} M^{\frac{5}{12}} \\
= \quad & B_\delta \, \bar{\lambda}^{-\frac{3}{2}} M^{\frac{1}{2}}.
\end{aligned}
$$

To obtain (2.62), use (2.61) and the inequality

$$\left(\frac{\epsilon}{\delta} \right)^{\frac{1}{6}} \left(\frac{\delta M^{\frac{1}{2}}}{t} \right)^{\frac{1}{6}} \lesssim \bar{\lambda}^{\frac{1}{3}} M^{\frac{1}{12}},$$

which follows from (2.51). Consequently, we have established (2.44) and the proposition follows. ■

2.5 Proof of Theorem 2.1.1

The following lemma states that instead of averaging over δ annuli we can average over a mollified version of $d\sigma_r$ which is essentially concentrated on a δ annulus.

Lemma 2.5.1 *Fix a radial function $\phi \in \mathcal{S}(\mathbb{R}^2)$. Suppose that for fixed $1 \leq p \leq q \leq \infty$, $\alpha < 3$*

$$\|\overline{\mathcal{M}}_\delta f\|_q \lesssim \delta^{-\alpha} \|f\|_p$$

for all $0 < \delta < 1$, $f \in L^1 \cap L^\infty$. Then

$$\|\overline{\mathcal{M}}(f * \phi_\delta)\|_q \lesssim \delta^{-\alpha} \|f\|_p$$

for all $0 < \delta < 1$, $f \in \mathcal{S}$.

Proof: Write $\phi(|x|) = \phi(x)$. We construct a radial, non increasing majorant for ϕ as follows. Let $\rho(r) = r^2 |\phi\prime(r)|$ and define

$$\psi(|x|) = \int_{|x|}^{\infty} |\phi\prime(r)| \, dr$$

or equivalently

$$\psi(x) = \int_0^\infty (\chi_B)_r \, (x) \rho(r) \, dr$$

where B is the unit ball in $\mathbb{R}^2$. Note that

$$\int_{\mathbb{R}^2} \psi(x) \, dx = \int_0^\infty \rho(r) \, dr = \int_0^\infty \psi(r) r \, dr.$$

Let $f \in \mathcal{S}$. Then

$$
\begin{aligned}
\sup_{1<t2} |d\sigma_t * (\phi_\delta * f)| \quad & \leq \quad \left\{ \int_0^{\delta^{-1}} + \int_{\delta^{-1}}^\infty \right\} \sup_{1<t<2} |[d\sigma_t * (\chi_B)_{r\delta}] * |f|| \, \rho(r) \, dr \\
& = \quad A + B.
\end{aligned}
$$

On the one hand

$$
\begin{aligned}
\|A\|_q \quad & \lesssim \quad \int_0^{\delta^{-1}} \|\overline{\mathcal{M}}_{10r\delta} |f|\|_q \; \rho(r) \, dr \\
& \lesssim \quad \delta^{-\alpha} \int_0^{\delta^{-1}} r^{-\alpha} \rho(r) \, dr \; \|f\|_p \\
& \lesssim \quad \delta^{-\alpha} \|f\|_p
\end{aligned}
$$

since $\alpha < 3$. On the other hand, by Young's inequality with $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{s}$,

$$\begin{aligned}
\|B\|_q &\lesssim \int_{\delta^{-1}}^{\infty} \| (\chi_B)_{10r\delta} * |f|\|_q \, \rho(r)dr \\
&\lesssim \int_{\delta^{-1}}^{\infty} \| (\chi_B)_{10r\delta} \|_s \|f\|_p \, \rho(r)dr \\
&\lesssim \int_{\delta^{-1}}^{\infty} (\delta r)^{-\frac{2}{s'}} \rho(r)dr \, \|f\|_p \\
&\lesssim \|f\|_p
\end{aligned}$$

and the lemma follows. ■

Proof of Theorem 2.1.1: Statements (2.9), (2.10) of Theorem 2.1.1 follow via Marcinkiewicz's theorem from the estimates at the points Q,R,T,P (see figure 1). To prove (2.8), suppose we are given any $f \in \mathcal{S}$. Let

$$f = \sum_{j=0}^{\infty} f_j$$

be a Littlewood–Paley decomposition, i.e. $\mathrm{supp}(\hat{f}_0) \subset \{\mathbb{R}^2 \ : \ |\xi| < 2\}$ and $\mathrm{supp}(\hat{f}_j) \subset \{\mathbb{R}^2 \ : \ 2^{j-1} < |\xi| < 2^{j+1}\}$ for $j = 1, 2, \ldots$. On the one hand, (2.10), (2.9), and Lemma 2.5.1 imply

$$||\overline{\mathcal{M}} f_j||_q \leq C_\epsilon \, 2^{j\epsilon} ||f_j||_p \quad \text{if } (\tfrac{1}{p}, \tfrac{1}{q}) \in QP \cup PT \text{ (see figure 1)} \tag{2.63}$$

for any $\epsilon > 0$ and $j = 1, 2, \ldots$. On the other hand, by the local smoothing theorem in [10] (see also [2] and [15])

$$||\overline{\mathcal{M}} f_j||_p \lesssim 2^{-j\beta} \|f_j\|_p, \tag{2.64}$$

where $2 < p < \infty$, $\beta = \beta(p) > 0$, and $j = 1, 2, \ldots$. Interpolating (2.64) with (2.63) yields

$$||\overline{\mathcal{M}} f_j||_q \lesssim 2^{-j\gamma} ||f_j||_p, \ \text{ if } (\tfrac{1}{p}, \tfrac{1}{q}) \in \text{region I} \backslash QP \cup PT \tag{2.65}$$

for some $\gamma = \gamma(p, q) > 0$. Furthermore,

$$||\overline{\mathcal{M}} f_0||_q \lesssim ||f_0||_p \tag{2.66}$$

by the Hardy–Littlewood and Bernstein inequalities. Finally, (2.8) follows from (2.65) and (2.66) by the Littlewood–Paley inequality.

Up to a $|\log \delta|$ factor, (2.11) follows by interpolating the estimate at T, i.e. (2.7), with the ones at the endpoints R and S:

$$\|\overline{\mathcal{M}}_\delta f\|_1 + \|\overline{\mathcal{M}}_\delta f\|_\infty \lesssim \delta^{-1} \|f\|_1. \tag{2.67}$$

To obtain the sharp estimates, let $f = \sum_0^\infty f_j$ be as above. The analogue of (2.67) is

$$\|\overline{\mathcal{M}} f_j\|_1 + \|\overline{\mathcal{M}} f_j\|_\infty \lesssim 2^j \|f_j\|_1 \tag{2.68}$$

(see Lemma 3.2.1 below). Interpolating (2.68) with the L^2 bound (2.23) yields

$$||\overline{\mathcal{M}} f_j\|_q \lesssim 2^{j(\frac{2}{p}-1)} \|f_j\|_p \ \text{ if } (\tfrac{1}{p}, \tfrac{1}{q}) \in \text{region IV}. \tag{2.69}$$

(2.11) now follows from (2.69) by the same type of argument as in the proof of Corollary 2.3.1 provided $1 < p$. The estimates on the segment SR follow from the ones at the endpoints. We skip the details. ■

Chapter 3

Some remarks and a slight improvement

3.1 Estimates for the global maximal function

Theorem 2.1.1 holds for the global version of $\overline{\mathcal{M}}$. Let

$$\mathcal{M}f \quad = \quad \sup_{0<r<1} r^a |d\sigma_r * f|.$$

The number a can vary and will be specified below. One can pass from bounds on $\overline{\mathcal{M}}_\delta$ to bounds on $\mathcal{M}$ by Littlewood–Paley theory. In section two of this chapter we will then apply these global bounds to obtain a version of Theorem 2.1.1 for the wave equation. The following lemma is essentially a calculation from [1].

Lemma 3.1.1 *Let $1 \le p < \infty$, $1 \le p \le q \le \infty$ and $\beta \in \mathbb{R}$. Suppose that*

$$\|\overline{\mathcal{M}} f\|_q \lesssim 2^{j\beta} ||f||_p \tag{3.1}$$

for all $j = 1, 2, \dots$ and $f \in \mathcal{S}$ such that $\mathrm{supp}(\hat{f}) \subset \{\mathbb{R}^2 \ : \ 2^{j-1} < |\xi| < 2^{j+1}\}$ *. Then, with $a = 2(1/p - 1/q)$,*

$$\begin{aligned} \|\mathcal{M}f\|_q &\lesssim ||f||_p && \text{if } \beta < 0,\ q \ge 2, \text{ and } 1 < p \\ \|\mathcal{M}f\|_q &\le C_\epsilon \|f\|_{L^p_{\beta+\epsilon}} && \text{if } \beta \ge 0,\ \epsilon > 0 \end{aligned}$$

for all $f \in \mathcal{S}$. Here L^p_γ are the usual Sobolev spaces.

Proof: Let $f = \sum_0^\infty f_j$ be as in the proof of Theorem 2.1.1. Recall that $(g)_{2^k}(x) = 2^{-2k} g(2^{-k}x)$. Then (with M^a defined in Lemma 3.1.2)

$$\begin{aligned} \mathcal{M}f \quad &= \quad \sup_{k\ge 0} \sup_{r\sim 2^{-k}} r^a \, |d\sigma_r * f| \\ &\le \quad \sup_{k\ge 0} \sup_{r\sim 2^{-k}} r^a \left| d\sigma_r * (\sum_{j\le k} f_j) \right| + \sup_{k\ge 0} \sup_{r\sim 2^{-k}} 2^{-ka} \left| d\sigma_r * (\sum_{j>k} f_j) \right| \\ &\lesssim \quad M^a f + \sup_{k\ge 0} \sum_{j>k} 2^{-ka} \left(\overline{\mathcal{M}} (f_j)_{2^k} \right)_{2^{-k}} \end{aligned}$$

$$\lesssim \quad M^a f + \left(\sum_{k\geq 0} \left| \sum_{j>k} 2^{-ka} \left(\overline{\mathcal{M}}(f_j)_{2^k} \right)_{2^{-k}} \right|^q \right)^{\frac{1}{q}} . \tag{3.2}$$

The first term in (3.2) is bounded by Lemma 3.1.2. Assume first that $\beta < 0$. Let $s = \max(2, p)$. Using the inequalities of Young and Littlewood–Paley, we can then estimate the second term as follows.

$$\begin{aligned} \left\| \left(\sum_{k\geq 0} \left| \sum_{j>k} 2^{-ka} (\overline{\mathcal{M}}(f_j)_{2^k})_{2^{-k}} \right|^q \right)^{\frac{1}{q}} \right\|_q &= \left(\sum_{k\geq 0} \left(\sum_{j>k} 2^{-ka} ||(\overline{\mathcal{M}}(f_j)_{2^k})_{2^{-k}}||_q \right)^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{k\geq 0} \left(\sum_{j>k} 2^{(j-k)\beta} ||f_j||_p \right)^q \right)^{\frac{1}{q}} \qquad (3.3) \\ &\lesssim \left(\sum_j ||f_j||_p^s \right)^{\frac{1}{s}} \\ &\lesssim \|(\sum_j |f_j|^2)^{\frac{1}{2}}\|_p \lesssim ||f||_p. \end{aligned}$$

If $\beta \geq 0$ we compute, starting in line (3.3) above,

$$\begin{aligned} \left\| \left(\sum_{k\geq 0} \left| \sum_{j>k} 2^{-ka} (\overline{\mathcal{M}}(f_j)_{2^k})_{2^{-k}} \right|^q \right)^{\frac{1}{q}} \right\|_q &\lesssim \left(\sum_{k\geq 0} 2^{-k\beta q} \left(\sum_{j>k} 2^{-j\epsilon} 2^{j(\beta+\epsilon)} ||f_j||_p \right)^q \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{k\geq 0} 2^{-k\beta q} \left(\sum_{j>k} 2^{-j\epsilon} \|(1-\triangle)^{(\beta+\epsilon)/2} f\|_p \right)^q \right)^{\frac{1}{q}} \\ &\leq C_\epsilon \|f\|_{L^p_{\beta+\epsilon}} \end{aligned}$$

for any $\epsilon > 0$. ■

In the following lemma we recall a well–known fact about certain maximal averages.

Lemma 3.1.2 *Let $0 \leq a < 2$ and define for any $f \in L^1 \cap L^\infty(\mathbb{R}^2)$*

$$M^a f(x) = \sup_{0<r<\infty} r^{a-2} \int_{B_r(x)} |f(x-y)|\, dy.$$

Then

$$||M^a f||_q \lesssim ||f||_p \quad \textit{for } \frac{1}{q} = \frac{1}{p} - \frac{a}{2},\ 1 < p \leq q \leq \infty. \tag{3.4}$$

Proof: For $a = 0$ this is just the usual Hardy–Littlewood maximal function. In case $0 < a < 2$, $q < \infty$ we have the inequality

$$M^a f(x) \lesssim \int_{\mathbb{R}^2} \frac{|f(y)|}{|x-y|^{2-a}}\, dy.$$

Since the kernel is in weak $L^{\frac{2}{2-a}}$, (3.4) follows from Young's inequality. Finally, if $q = \infty$, (3.4) follows from Hölder's inequality. ■

We can now state the global version of Theorem 2.1.1.

Theorem 3.1.1 *For any $f \in \mathcal{S}(\mathbb{R}^2)$,*

$$\|\mathcal{M}f\|_q \lesssim ||f||_p \qquad \text{in region } I \setminus (QP \cup PT) \tag{3.5}$$

and for any $\epsilon > 0$

$$\|\mathcal{M}f\|_q \leq C_\epsilon \; ||f||_{L^p_{\gamma+\epsilon}}$$

where

$$\gamma = \frac{1}{p} - \frac{2}{q} \qquad \text{in region } II \tag{3.6}$$

$$\gamma = \frac{3}{2p} - \frac{1}{2} - \frac{1}{2q} \qquad \text{in region } III \tag{3.7}$$

$$\gamma = \frac{2}{p} - 1 \qquad \text{in region } IV. \tag{3.8}$$

We have set $a = 2(\frac{1}{p} - \frac{1}{q})$ throughout.

Proof: The above statements follow from Theorem 2.1.1 and Lemmas 2.5.1 and 3.1.1. ■

3.2 Some estimates for the wave equation

It is well–known that circular averages can be imbedded into an analytic family of operators. As in [18] we let for all $f \in \mathcal{S}$

$$\mathcal{A}_t^\alpha f = k_t^\alpha * f,$$

where

$$k^\alpha(x) = \frac{1}{\Gamma(\alpha)}(1 - |x|^2)_+^{\alpha-1}.$$

For $\Re(\alpha) \leq 0$ this is defined by analytic continuation. In particular,

$$d\sigma_r * f = \mathcal{A}_r^0 f$$

and $u(x,t) = t\mathcal{A}_t^{\frac{1}{2}} f$ solves

$$\Box\, u = 0, \quad u(0) = 0, \; u_t(0) = f.$$

By interpolation one can obtain analogues of Theorem 2.1.1 and Theorem 3.1.1 for the operators

$$\overline{\mathcal{M}}^\alpha f = \sup_{1<t\leq 2} |\mathcal{A}_t^\alpha f|$$

$$\mathcal{M}^\alpha f = \sup_{0<t\leq 1} t^a |\mathcal{A}_t^\alpha f|.$$

Estimates of this type go back to [18] (see also Stein [19], pp. 518-19 and the references given there). For simplicity we shall restrict ourselves to the wave equation, i.e., $\alpha = \frac{1}{2}$. Solving the wave equation above with initial conditions given by suitable modifications of examples i–iii on page 9 shows that the following theorem is optimal (up to ϵ). For the meaning of the points P_1 etc. see figure 2.

Theorem 3.2.1 *For any $f \in \mathcal{S}$ let u be a solution of the two–dimensional wave equation as above. Then*

$$\| \sup_{0<t\leq 1} t^{a-1} u(\cdot,t)\|_q \lesssim \|f\|_p \qquad \text{in region } A \setminus XP_1 \cup P_1Y \cup YZ \tag{3.9}$$

and

$$\| \sup_{0<t\leq 1} t^{a-1} u(\cdot,t)\|_q \leq C_\epsilon \, \|f\|_{L^p_{\gamma+\epsilon}}$$

with $\epsilon > 0$ and

$$\gamma = \frac{1}{p} - \frac{2}{q} - \frac{1}{2} \qquad \text{in region } B \tag{3.10}$$

$$\gamma = \frac{3}{2p} - 1 - \frac{1}{2q} \qquad \text{in region } C \tag{3.11}$$

$$\gamma = \frac{2}{p} - \frac{3}{2} \qquad \text{in region } D. \tag{3.12}$$

We have set $a = 2(\frac{1}{p} - \frac{1}{q})$ throughout.

The most interesting statement in Theorem 3.2.1 is probably the bound at $P_1 = (\frac{7}{10}, \frac{1}{10})$, i.e.

$$\| \sup_{1<t<2} u(\cdot,t)\|_{L^{10}(\mathbb{R}^2)} \leq C_\epsilon \, \|f\|_{L^{10/7}_\epsilon(\mathbb{R}^2)}.$$

It is easy to see that this is exactly what would follow from the local smoothing conjecture. The same remark applies to the operators $\overline{\mathcal{M}}^\alpha$. On the other hand, the position of all other points in fig. 2 can be explained fairly easily by invoking the decay properties of $\widehat{k^\alpha}$. This is done in the following two lemmas. They are proved in general dimensions.

Lemma 3.2.1 *For all $\alpha = \sigma + i\tau \in \mathbb{C}$ there exist constants C_α such that for any $j = 1, 2, \ldots$, all $f \in \mathcal{S}$ with $\operatorname{supp}(\hat{f}) \subset \{\mathbb{R}^d \,:\, 2^{j-1} < |\xi| < 2^{j+1}\}$, and $1 \leq p \leq \infty$*

$$\|\overline{\mathcal{M}}^\alpha f\|_{L^\infty(\mathbb{R}^d)} \leq C_\alpha 2^{j(\frac{1}{p}-\sigma)} \|f\|_{L^p(\mathbb{R}^d)} \tag{3.13}$$

$$\|\overline{\mathcal{M}}^\alpha f\|_{L^1(\mathbb{R}^d)} \leq C_\alpha 2^{j(1-\sigma)} \|f\|_{L^1(\mathbb{R}^d)} \tag{3.14}$$

$$\|\overline{\mathcal{M}}^\alpha f\|_{L^2(\mathbb{R}^d)} \leq C_\alpha 2^{-j(\sigma+(d-2)/2)} \|f\|_{L^2(\mathbb{R}^d)}. \tag{3.15}$$

Moreover, for every $|\sigma| \leq S$ there exists a constant C_S so that for all $-\infty < \tau < \infty$

$$|C_\alpha| \leq C_S \exp(C_S|\tau|). \tag{3.16}$$

Proof: Choose a radial function $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}(\hat{\phi}) \subset \{\mathbb{R}^d \;:\; \frac{1}{4} < |\xi| < 4\}$ and so that $f = f * \phi_{2^{-j}}$ for any f as in the statement of the lemma. By Lemma 3.2.2 below

$$\|\mathcal{A}^\alpha_t \phi_{2^{-j}}\|_{L^q(\mathbb{R}^d)} \lesssim 2^{j(1-\sigma-\frac{1}{q})} \tag{3.17}$$

$$\| \sup_{1<t<2} \mathcal{A}^\alpha_t \phi_{2^{-j}}\|_{L^1(\mathbb{R}^d)} \lesssim 2^{j(1-\sigma)} \tag{3.18}$$

for all $1 \leq q \leq \infty$. Since

$$\begin{aligned} |\mathcal{A}^\alpha_t f| &\leq |\mathcal{A}^\alpha_t \phi_{2^{-j}}| * |f| \\ \sup_{1<t<2} |\mathcal{A}^\alpha_t f| &\leq \left[\sup_{1<t<2} |\mathcal{A}^\alpha_t \phi_{2^{-j}}| \right] * |f|, \end{aligned}$$

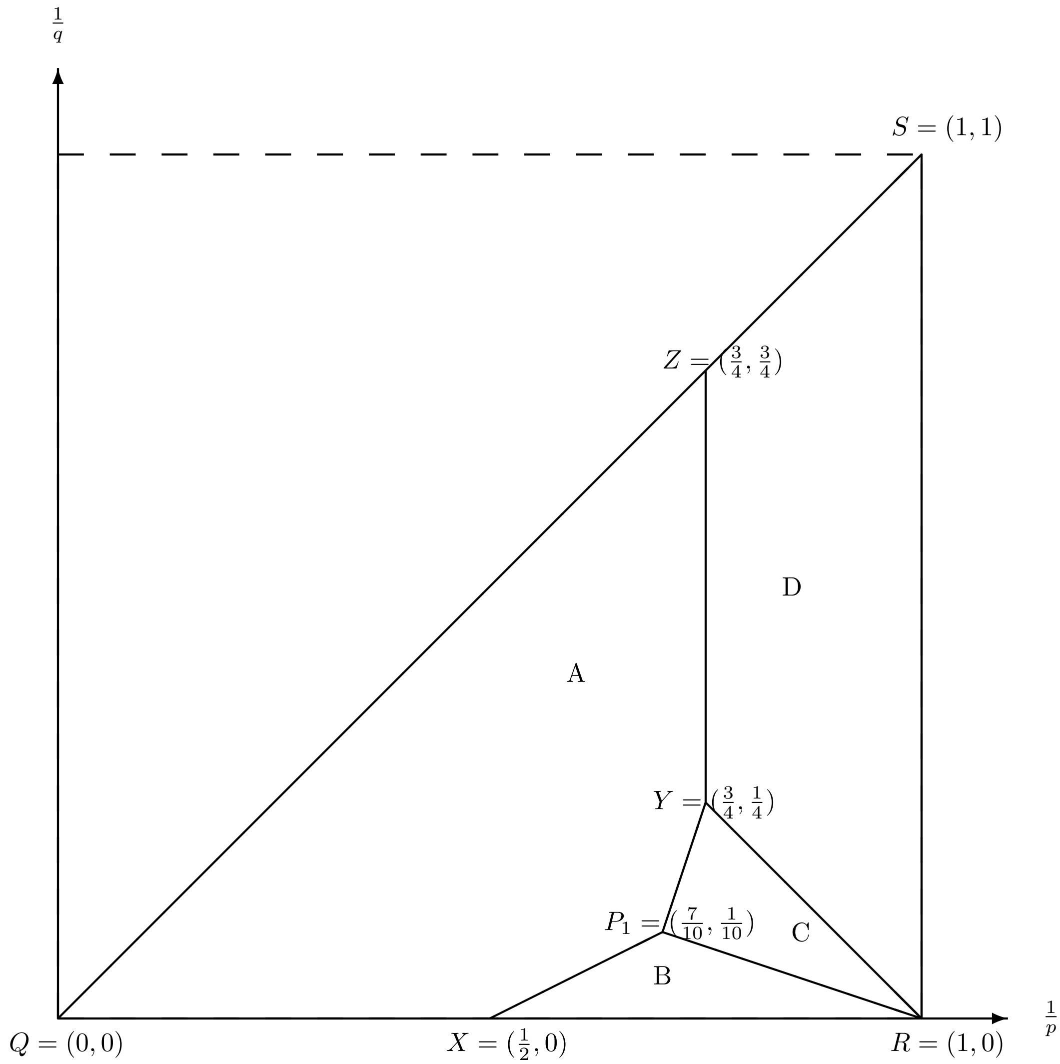

Figure 3.1: Regions of boundedness in Theorem 3.2.1

inequalities (3.13) and(3.14) follow from (3.17) and (3.18), respectively.

By Lemma 2.3.1

$$\|\overline{\mathcal{M}}^{\alpha} f\|_{L^2(\mathbb{R}^d)} \lesssim \alpha_j^{\frac{1}{2}}(\alpha_j+\beta_j)^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^d)}$$

where

$$\begin{aligned} \alpha_j &= \sup_{|\xi|\simeq 2^j} |\widehat{k^{\alpha}}(\xi)| \\ \beta_j &= \sup_{|\xi|\simeq 2^j} |\langle \nabla \widehat{k^{\alpha}}(\xi), \xi\rangle|. \end{aligned}$$

By (3.19) and (3.20) below

$$\begin{aligned} \alpha_j &\lesssim 2^{-j((d-1)/2+\sigma)} \\ \beta_j &\lesssim 2^{-j((d-3)/2+\sigma)}. \end{aligned}$$

Thus (3.15) follows. ■

Lemma 3.2.2 *Let $\phi \in \mathcal{S}$ be a radial function such that $\operatorname{supp}(\hat{\phi}) \subset \{\mathbb{R}^d : \frac{1}{4} < |\xi| < 4\}$. Then for any $\alpha = \sigma + i\tau \in \mathbb{C}$, $N = 0, 1, 2, \ldots$ there exist constants $C_{\alpha,N}$ so that*

$$|\mathcal{A}_t^{\alpha}\phi_{2^{-j}}(x)| \leq C_{\alpha,N} 2^{j(1-\sigma)}(1+2^j||x|-t|)^{-N}$$

for all $1 < t < 2$, $x \in \mathbb{R}^d$, and $j = 0, 1, \ldots$. Moreover, the constants $C_{\alpha,N}$ satisfy the growth condition (3.16).

Proof: This will follow from stationary phase. We shall use the asymptotic expansion of $\widehat{k^{\alpha}}$ derived in Lemma 2.2.3 in Sogge's book [16]:

$$\widehat{k^{\alpha}}(\xi) = e^{2\pi i|\xi|}\,\omega_{\alpha}^{+}(|\xi|) + e^{-2\pi i|\xi|}\,\omega_{\alpha}^{-}(|\xi|) \tag{3.19}$$

where ω_{α}^{+} and ω_{α}^{-} are $\in C^{\infty}(0,\infty)$. Their decay is given by

$$\left|\frac{d^k}{ds^k}\omega_{\alpha}^{\pm}(s)\right| \leq C_{k,\alpha}(1+s)^{-\frac{d-1}{2}-\Re\alpha-k}, \tag{3.20}$$

$k = 0, 1, \ldots$ where the $C_{k,\alpha}$ satisfy (3.16), as can be seen by Stirling's formula. Note that the representation (3.19) includes the surface measure $\widehat{d\sigma} = \widehat{k_0}$.

By definition of the $\mathcal{A}_t^{\alpha}$,

$$\begin{aligned} \mathcal{A}_t^{\alpha}\phi_{2^{-j}}(x) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\,\widehat{k^{\alpha}}(\xi t)\hat{\phi}(2^{-j}\xi)\,d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i(x\cdot\xi + t|\xi|)}\,\omega_{\alpha}^{+}(\xi t)\hat{\phi}(2^{-j}\xi)\,d\xi + \\ &\quad + \int_{\mathbb{R}^d} e^{2\pi i(x\cdot\xi - t|\xi|)}\,\omega_{\alpha}^{-}(\xi t)\hat{\phi}(2^{-j}\xi)\,d\xi \\ &= I_+ + I_-. \end{aligned}$$

It suffices to consider I_-. Introducing polar coordinates yields

$$\begin{aligned} I_- &= 2^{jd}\int_0^{\infty} e^{2\pi i 2^j(|x|-t)r}\,\omega_0^{+}(2^j|x|r)\omega_{\alpha}^{-}(2^j tr)\hat{\phi}(r)r^{d-1}\,dr + \\ &\quad + 2^{jd}\int_0^{\infty} e^{-2\pi i 2^j(|x|+t)r}\,\omega_0^{-}(2^j|x|r)\omega_{\alpha}^{-}(2^j tr)\hat{\phi}(r)r^{d-1}\,dr \\ &= A + B. \end{aligned}$$

Consider the first integral.

$$
\begin{aligned}
|A| &\leq C_N 2^{jd}(2^j||x|-t|)^{-N}\int_0^\infty \left|\left(\frac{d}{dr}\right)^N [\omega_0^+(2^j|x|r)\omega_\alpha^-(2^jtr)\hat{\phi}(r)r^{d-1}]\right| dr \\
&\leq C_{\alpha,N}2^{jd}(2^j||x|-t|)^{-N}\sum_{k+l\leq N}\int_{\frac{1}{4}}^4 (2^j|x|)^k(1+2^j|x|r)^{-\frac{d-1}{2}-k}(2^jt)^l \\
&\quad (1+2^jtr)^{-\frac{d-1}{2}-\sigma-l}\,dr \\
&\leq C_{\alpha,N}2^{jd}(2^j||x|-t|)^{-N}(1+2^j|x|)^{-\frac{d-1}{2}}\,2^{-j(\frac{d-1}{2}+\sigma)}.
\end{aligned}
$$

Hence

$$|A| \leq C_{\alpha,N}2^{j(1-\sigma)}(1+2^j||x|-t|)^{-N}$$

provided $1<t<2$. Estimating B in a similar fashion finishes the proof. ■

In the following proposition we move point P in figure 1 to position P_1 in figure 2 by Stein's interpolation theorem.

Proposition 3.2.1 *Let $f\in\mathcal{S}(\mathbb{R}^2)$ so that $\mathrm{supp}(\hat{f})\subset\{\mathbb{R}^2\ :\ 2^{j-1}<|\xi|<2^{j+1}\}$ for some $j=1,2,\ldots$. Then for any $\epsilon>0$*

$$\|\overline{\mathcal{M}}^{\frac{1}{2}}f\|_{L^{10}(\mathbb{R}^2)} \leq C_\epsilon\,2^{j\epsilon}\|f\|_{L^{10/7}(\mathbb{R}^2)}. \tag{3.21}$$

Proof: By equation (4) in [18],

$$\mathcal{A}_t^\alpha f(x) = \frac{2}{\Gamma(\alpha)}\int_0^1 \mathcal{A}_{st}^0 f(x)(1-s^2)^{\alpha-1}s\,ds \tag{3.22}$$

provided $\Re\alpha>0$. Let $\alpha=\epsilon+i\tau$. In view of (3.22) Theorem 3.1.1 implies that

$$\|\overline{\mathcal{M}}^\alpha f\|_5 \leq C_{\epsilon,\tau}\,2^{j\epsilon}\|f\|_{5/2}.$$

Interpolating this with

$$\|\overline{\mathcal{M}}^{1+i\tau}f\|_\infty \leq C_\tau 2^j\|f\|_1$$

(see inequality (3.13)) via Stein's theorem yields

$$\|\overline{\mathcal{M}}^{\frac{1}{2}}f\|_q \leq C_\epsilon\,2^{j\epsilon}\|f\|_p \tag{3.23}$$

where $p\to 10/7$ and $q\to 10$ as $\epsilon\to 0$. The proposition follows by interpolating (3.23) with

$$\|\overline{\mathcal{M}}^{\frac{1}{2}}f\|_\infty \lesssim 2^{\frac{j}{2}}\|f\|_1,$$

which is a special case of (3.13). ■

Remark: Just as in the case of circular means, the endpoint result (3.21) would follow from the sharp local smoothing conjecture [15]. Namely, by that conjecture

$$\|\overline{\mathcal{M}}^{\frac{1}{2}}f\|_4 \leq C_\epsilon\ 2^{-j(\frac{3}{4}-\epsilon)}\|f\|_4$$

for any f as in Proposition 3.2.1. Interpolating this with estimate (3.13), i.e.

$$\|\overline{\mathcal{M}}^{\frac{1}{2}} f\|_\infty \lesssim 2^{\frac{j}{2}} \|f\|_1$$

yields (3.21).

Proof of Theorem 3.2.1: Let $f \in \mathcal{S}$ such that $\mathrm{supp}(\hat{f}) \subset \{\mathbb{R}^2 \;:\; 2^{j-1} < |\xi| < 2^{j+1}\}$ for some $j = 1, 2, \ldots$. By (3.13), (3.14) and (3.15)

$$\|\overline{\mathcal{M}}^{\frac{1}{2}} f\|_{4/3} \lesssim \|f\|_{4/3}$$
$$\|\overline{\mathcal{M}}^{\frac{1}{2}} f\|_{4} \lesssim \|f\|_{4/3}$$

which correspond to the points Z and Y, respectively. Statements (3.10), (3.11), and (3.12) now follow from the estimates at the points X,R,S,Z,Y,P_1 (which we derived above) via interpolation and Lemma 3.1.1 (note that we can replace $\overline{\mathcal{M}}$ with $\overline{\mathcal{M}}^{\frac{1}{2}}$ in that lemma). Inequalities (3.13) and (3.15) of Lemma 3.2.1 imply that

$$\|\overline{\mathcal{M}}^{\frac{1}{2}} f\|_q \lesssim 2^{-j\beta} \|f\|_p \qquad \text{if } (\tfrac{1}{p}, \tfrac{1}{q}) \in QX \cup QZ \setminus \{X, Z\}$$

for some $\beta = \beta(p, q) > 0$. By what was shown in the first part of this proof

$$\|\overline{\mathcal{M}}^{\frac{1}{2}} f\|_q \le C_\epsilon \; 2^{j\epsilon} \|f\|_p \qquad \text{if } (\tfrac{1}{p}, \tfrac{1}{q}) \in XP_1 \cup P_1Y \cup YZ$$

for any $\epsilon > 0$. By interpolation,

$$\|\overline{\mathcal{M}}^{\frac{1}{2}} f\|_q \lesssim 2^{-j\gamma} \|f\|_p \qquad \text{if } (\tfrac{1}{p}, \tfrac{1}{q}) \in A \setminus XP_1 \cup P_1Y \cup YZ$$

for some $\gamma = \gamma(p, q) > 0$. Thus (3.9) follows from Lemma 3.1.1 provided $q \ge 2$. In [19], Stein proved (3.9) on the segment $QZ \setminus \{Z\}$. The theorem now follows by interpolating those estimates with the ones we just derived. ■

3.3 A slight improvement

In this section we prove remark b) from chapter 2, section 1. This will follow from an argument similar to the one in section 4. However, we will not use (2.38), which contains a logarithmic factor, but rather the following geometric observation. It might be worth noting that (3.25) is *not* sufficient for the (5/2,5) estimate. First we need to carry out the pigeon hole argument from page 18 with different weights. More precisely, let $a : [\delta, 1] \times [\delta, 1] \to [0, 1]$ be such that

$$\sum_{k \ge 0, l \ge 0} a(2^k \delta, 2^l \delta) = 1. \tag{3.24}$$

Define $\bar{\lambda} = \bar{\lambda}(t, \epsilon) = a(t, \epsilon)\bar{\lambda}/2$, $\bar{\mu} = \bar{\mu}(t, \epsilon) = a(t, \epsilon)\mu$, $\overline{M} = \overline{M}(t, \epsilon) = a(t, \epsilon)M/2$. The same discussion as on page 18 then shows that the inequalities (2.37) will hold for an appropriate choice of ϵ and t and with the parameters $\bar{\lambda}$, $\bar{\mu}$, $\overline{M}$ we have just defined. In this section we will always mean these parameters.

Lemma 3.3.1 *The multiplicity $\bar{\mu}$ satisfies the following apriori inequality:*

$$\bar{\mu} \lesssim \epsilon^{\frac{1}{2}} t^{\frac{3}{2}} \delta^{-2}. \tag{3.25}$$

The constant in (3.25) is absolute, in particular it is independent of the choice of a.

Proof: Let $x \in C_\delta(x_j, r_j) \cap C_\delta(x_i, r_i)$ where $\Delta_{ij} \simeq \epsilon$ and $|x_i - x_j| \simeq t$. Then $|r_i - r_j| \lesssim t$ and the angle $\sphericalangle(x_i, x, x_j) \simeq \sqrt{\epsilon t}$. Hence x_i has to lie in a rectangle of dimensions approximately $(\epsilon t)^{\frac{1}{2}} \times t$. By δ separatedness the maximal number of x_i's is bounded by (3.25). ■

Proof of remark b) on page 10:

Case 1:
$$\bar{\lambda} \le C_0 \left(\frac{\epsilon}{t}\right)^{\frac{1}{2}} \tag{3.26}$$

Here C_0 is the same constant that appeared in the case distinction in the proof of Proposition 2.4.2. For later purposes we rewrite (3.26) as

$$\lambda \lesssim a^{-1}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{-\frac{1}{4}}. \tag{3.27}$$

Since $\mathrm{card}(S^j_{t,\epsilon}) \lesssim \min(M, \frac{t^2}{\delta^2})$ by the definition (2.36) of the set $S^j_{t,\epsilon}$, we conclude from (2.46) that

$$\mu \lesssim a^{-2}\lambda^{-1}\left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}}\left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{\frac{3}{4}} \min\left(1, \frac{t^2}{M\delta^2}\right). \tag{3.28}$$

It is convenient to rewrite (3.25) as follows:

$$\mu \lesssim a^{-1}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta M^{\frac{1}{2}}}\right)^{\frac{3}{2}} M^{\frac{3}{4}}. \tag{3.29}$$

Let $0 < \beta < \frac{1}{2}$ and $0 < \eta < 1 - 2\beta$. Multiply (3.28) with the β^{th} power of (3.29) to wit

$$\begin{aligned}
\mu^{1+\beta} &\lesssim C\, a^{-2-\beta}\lambda^{-1}\left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}(1-\beta)}\left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}(1-3\beta)} M^{\frac{3}{4}(1+\beta)} \min\left(1, \frac{t^2}{M\delta^2}\right)\\
&\lesssim a^{-2-2\beta-\eta}\lambda^{-(1+\beta+\eta)}\left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}(1-2\beta-\eta)}\left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}(1-2\beta+\eta)} \times\\
&\quad \times M^{\frac{1}{4}(3+2\beta-\eta)} \min\left(1, \frac{t^2}{M\delta^2}\right)
\end{aligned} \tag{3.30}$$

where we have used (3.27) to obtain (3.30). Let

$$a(\epsilon, t) = C_1^{-1}\left(\frac{\delta}{\epsilon}\right)^{\tau}\left(\frac{\delta M^{\frac{1}{2}}}{t} + \frac{t}{\delta M^{\frac{1}{2}}}\right)^{-\tau} \tag{3.31}$$

for $\tau > 0$. Then we can choose C_1 sufficiently large (depending only on τ) so that a satisfies (3.24). For suitably small τ depending on β and η we conclude from (3.30) that

$$\mu \lesssim \lambda^{-(1+\frac{\eta}{1+\beta})} M^{\frac{3+2\beta-\eta}{4(1+\beta)}} = \lambda^{-\alpha} M^{\tilde{\beta}}, \tag{3.32}$$

where $1 < \alpha < 2$ and $\alpha \to 1$ as $\eta \to 0$, $\alpha \to 2$ as $\eta \to 1$ and $\beta \to 0$. Moreover,

$$\frac{1+\alpha}{1-\tilde{\beta}} = 6 + 2\frac{1-2\beta-\eta}{1+2\beta+\eta} > 6 \text{ and } \frac{1+\alpha}{1-\tilde{\beta}} \to 6$$

as $\eta \to 1 - 2\beta$. Thus, according to Lemma 2.2.1, (3.32) corresponds to weak type $p \to q$ estimates with $2 < p < 3$ and $q > 6$.

Case 2: $$\bar{\lambda} \geq C_0 \left(\frac{\epsilon}{t}\right)^{\frac{1}{2}} \tag{3.33}$$

We rewrite (3.33) as

$$\lambda \gtrsim a^{-1} \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}} M^{-\frac{1}{4}}. \tag{3.34}$$

With Q defined by (2.52), we infer from Lemma 2.4.1 that

$$\operatorname{card}(Q) \lesssim \left(\frac{\epsilon}{\delta}\right)^2 \bar{\lambda}^{-3} M \min\left(1, \frac{t^2}{M\delta^2}\right)^2. \tag{3.35}$$

The minimum occurs on the right–hand side because for a given choice of x_{i_1} we must have

$$|x_{i_2} - x_{i_1}| \leq |x_{i_1} - x_j| + |x_{i_2} - x_j| \leq 2t.$$

Here (j, i_1, i_2, i_3) is a typical element of Q (cf. (2.52)). This follows immediately from the definition (2.36) of $S^j_{t,\epsilon}$. Clearly, the same inequality also holds for x_{i_3}. Combining (3.35) with (2.55) we obtain therefore

$$\mu^3 \lesssim a^{-10} \lambda^{-6} \left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{3}{2}} M^{\frac{5}{4}} \min\left(1, \frac{t^2}{M\delta^2}\right)^2. \tag{3.36}$$

Suppose $\beta \geq 0$. Writing $\lambda^6 = \lambda^\beta \lambda^{6-\beta}$ in (3.36) and estimating λ^β by the β^{th} power of (3.34) yields

$$\mu^3 \lesssim a^{-10+\beta} \lambda^{-6+\beta} \left(\frac{\delta}{\epsilon}\right)^{\frac{1}{2}(\beta-1)} \left(\frac{\delta M^{\frac{1}{2}}}{t}\right)^{\frac{1}{2}(3-\beta)} M^{\frac{1}{4}(5+\beta)} \min\left(1, \frac{t^2}{M\delta^2}\right)^2. \tag{3.37}$$

If $1 < \beta < 3$ we can choose τ in (3.31) so that (3.37) implies

$$\mu \lesssim \lambda^{-\alpha} M^{1-(1+\alpha)/6} \quad \text{for } 1 < \alpha < 2. \tag{3.38}$$

Combining (3.32) and (3.38) and applying Lemma 2.2.1 shows that

$$||\overline{\mathcal{M}}_\delta f||_{q,\infty} \lesssim \delta^{\frac{2}{q}-\frac{1}{p}} ||f||_{p,1} \qquad \text{for } 2 < p < 3,\ 6 < q. \tag{3.39}$$

Inequality (2.9) of Theorem 2.1.1 with $\epsilon = 0$ now follows from (3.39) and the obvious estimates for $q = \infty$ via Marcinkiewicz's theorem. ■

3.4 The L^2 theory revisited

The purpose of this section is to rederive the crucial L^2 estimate, i.e., Lemma 2.3.3, by purely geometric/combinatorial arguments. This is accomplished by using a two circle lemma, see Lemma 3.4.4

below, and a suitable iteration scheme. First we introduce some notation, following [22]. By $\mathcal{C}$ we shall always mean a family of circles with δ–separated centers lying in some fixed compact set. Let

$$\begin{aligned}
C &= C(x,r) = \{y \in \mathbb{R}^2 \ : \ |x-y| = r\} \\
C^\rho &= \{y \in \mathbb{R}^2 \ : \ r-\rho < |x-y| < r+\rho\} \\
\Delta(C,\overline{C}) &= ||x-\bar{x}| - |r-\bar{r}|| \\
d(C,\overline{C}) &= |x-\bar{x}| + |r-\bar{r}| \\
\mu_\rho^{\mathcal{C}} &= \sum_{\overline{C}\in\mathcal{C}} \chi_{\overline{C}^\rho} \\
\mathcal{C}_{\epsilon t}^{C} &= \{\overline{C} \in \mathcal{C} \ : \ \epsilon/2 \le \Delta(C,\overline{C}) \le \epsilon, \ t \le |x-\bar{x}| \le 2t, \\
& \quad\ |r-\bar{r}| \le 4t\}.
\end{aligned} \tag{3.40}$$

The following proposition clearly implies that

$$\mu \le C_\epsilon \delta^{-\epsilon} \lambda^{-1} \delta^{-1} t$$

for any $\epsilon > 0$, which is essentially the same as (2.38). The only difference is that we have powers $\delta^{-\epsilon}$ instead of logarithms, which is inessential as far as the proof of Theorem 2.1.1 is concerned.

Proposition 3.4.1 *For all $\eta > 0$ there exist constants $C_\eta > 0$ and $\delta_0 > 0$ with the following properties: given any family $\mathcal{C}$ there exists $\mathcal{A} \subset \mathcal{C}$ with $|\mathcal{A}| > C_\eta^{-1}|\mathcal{C}|$ so that*

$$|\{x \in C^\delta \ : \ \mu_\delta^{\mathcal{A}_{\epsilon t}^{C}}(x) > \delta^{-\eta}\ \lambda^{-1}\delta^{-1}t\}| \le \lambda\delta \tag{3.41}$$

for all $C \in \mathcal{A}$, all $\delta \le \epsilon \le t$, $0 < \lambda$, provided $\delta < \delta_0$.

This will follow by iterating Lemma 3.4.5 below. The idea of considering weak type inequalities for the multiplicity in the full range of the parameters originates in [22]. First we establish some technical lemmas needed in the proof of Lemma 3.4.5.

Lemma 3.4.1 *Suppose $\Delta(C_1,C_2) = \beta$, $d(C_1,C_2) = \tau$, $t \ge 2\epsilon$, and that $\beta \ge 10\epsilon$. Then*

$$|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \lesssim \frac{t^2}{\delta^2}\frac{\epsilon}{\sqrt{\beta\tau}}. \tag{3.42}$$

<u>Proof:</u> Let $F(x,r) = (|x-x_1| - |r-r_1|, |x-x_2| - |r-r_2|)$ be defined on

$$\Omega = \{(x,r) \ : \ t \le |x-x_j| \le 2t, \ j = 1,2\}.$$

Suppose $(x,r) \in \Omega$ and let $e_i = \frac{x-x_i}{|x-x_i|}$ and $\sigma_j = \mathrm{sgn}(r-r_j)$. Then

$$DF(x,r) = \begin{pmatrix} e_1 & -\sigma_1 \\ e_2 & -\sigma_2 \end{pmatrix}$$

and thus

$$JF(x,r) \quad \simeq \quad |\sphericalangle(e_1\sigma_1, e_2\sigma_2)| \equiv \alpha.$$

Here JF^2 denotes the sum of the squares of all 2×2 subdeterminants of DF. Suppose $(x, r) \in \Omega$ and $|F(x, r)| < \epsilon$. Then there exist r_j' so that $|r_j - r_j'| < \epsilon$ and

$$|x - x_j| = |r - r_j'| \quad \text{for} \quad j = 1, 2.$$

Moreover, $|x - x_j| \geq t \geq 2\epsilon$ and $||r - r_j| - |x - x_j|| < \epsilon$ imply that $\operatorname{sgn}(r - r_j') = \sigma_j$. Consider first the case where $\sigma_1 = \sigma_2$. Then

$$\begin{aligned} |x_1 - x_2|^2 &= |x - x_1|^2 + |x - x_2|^2 - 2(x_1 - x) \cdot (x_2 - x) \\ &= |r - r_1'|^2 + |r - r_2'|^2 - 2|r - r_1'||r - r_2'| + \\ &\quad +2|x - x_1||x - x_2|(1 - \cos\alpha) \\ &= |r_1' - r_2'|^2 + 2|x - x_1||x - x_2|(1 - \cos\alpha), \end{aligned}$$

and thus

$$t^2\alpha^2 \gtrsim \beta\tau - 8\tau\epsilon \gtrsim \beta\tau.$$

If $\sigma_1 \neq \sigma_2$, then by a similar calculation $t^2\alpha^2 \gtrsim \beta\tau$. We conclude that

$$JF \gtrsim \frac{\sqrt{\beta\tau}}{t} \quad \text{on } \Omega \cap F^{-1}(D(0, \epsilon)).$$

By the coarea formula,

$$\begin{aligned} \int_{D(0,\epsilon)} \mathcal{H}^1(F^{-1}(y) \cap \Omega)\, dy &= \int_{\Omega \cap F^{-1}(D(0,\epsilon))} JF(x, r)\, dx dr \\ \epsilon^2 t &\gtrsim \frac{\sqrt{\beta\tau}}{t} |\Omega \cap F^{-1}(D(0, \epsilon))| \end{aligned}$$

which implies

$$|\operatorname{proj}_{\mathbb{R}^2}(\Omega \cap F^{-1}(D(0, \epsilon)))| \lesssim \epsilon \frac{t^2}{\sqrt{\beta\tau}}. \quad \blacksquare$$

For example, set $\beta = \epsilon$ and $t = \tau$ in (3.42). Then (3.42) says that the total number of circles in $\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}$ is no larger than the maximal number of circles in $\mathcal{C}_{\epsilon t}^{C_1}$ that pass through one of the points $C_1 \cap C_2$.

To estimate $|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}|$ in those cases where Lemma 3.4.1 does not apply we will use the following observation. Roughly speaking, it says that if $C_j = C(x_j, 3/4)$ are internally tangent to $C(0, 1)$ for $j = 1, 2$ with the points of tangency being far apart, then C_1 and C_2 cross each other.

Lemma 3.4.2 *Let $C_j = C(x_j, r_j)$ for $j = 0, 1, 2$. Suppose $\Delta(C_0, C_j) \leq \beta_j$ and $|r_0 - r_j| \leq 4\rho_j$, $\rho_j \leq |x_0 - x_j| \leq 2\rho_j$ for $j = 1, 2$. Assume*

$$\begin{aligned} \alpha &= \sphericalangle(\operatorname{sgn}(r_1 - r_0)(x_1 - x_0), \operatorname{sgn}(r_2 - r_0)(x_2 - x_0)) \\ &\geq A_0 \sqrt{\frac{(\beta_1 + \beta_2)(\rho_1 + \rho_2)}{\rho_1 \rho_2}} \end{aligned} \tag{3.43}$$

for some sufficiently large constant A_0. Then $\Delta(C_1, C_2) \geq \beta_1 + \beta_2$.

Proof: Let $\sigma_j = \mathrm{sgn}(r_j - r_0)$. Then

$$\begin{aligned} |x_1 - x_2|^2 &= |x_1 - x_0|^2 + |x_2 - x_0|^2 - 2(x_1 - x_0)\cdot(x_2 - x_0) \\ |r_1 - r_2|^2 &= |r_1 - r_0|^2 + |r_2 - r_0|^2 - 2(r_1 - r_0)(r_2 - r_0) \\ |x_1 - x_2|^2 - |r_1 - r_2|^2 &= |x_1 - x_0|^2 - |r_1 - r_0|^2 + |x_2 - x_0|^2 - |r_2 - r_0|^2 + \\ &\quad +2\sigma_1\sigma_2(|r_1 - r_0||r_2 - r_0| - |x_1 - x_0||x_2 - x_0|) + \\ &\quad +2\sigma_1\sigma_2|x_1 - x_0||x_2 - x_0|(1 - \cos\alpha). \end{aligned}$$

This implies that

$$\begin{aligned} \Delta(C_1, C_2)(|x_1 - x_2| + |r_1 - r_2|) &\gtrsim \rho_1\rho_2\alpha^2 - \beta_1\rho_1 - \beta_2\rho_2 - \rho_1\beta_2 - \rho_2\beta_1 \\ &= \rho_1\rho_2\alpha^2 - (\beta_1 + \beta_2)(\rho_1 + \rho_2) \\ &\gtrsim \rho_1\rho_2\alpha^2 \end{aligned}$$

where we have used (3.43) in the last step. Furthermore, $d(C_1, C_2) \lesssim \rho_1 + \rho_2$ and thus finally

$$\Delta(C_1, C_2) \gtrsim \frac{\rho_1\rho_2}{\rho_1 + \rho_2}\alpha^2 \gtrsim A_0^2(\beta_1 + \beta_2) \geq \beta_1 + \beta_2. \quad \blacksquare$$

Using Lemma 3.4.2 we can deal with the case $\beta \leq 10\epsilon$ that was left open in Lemma 3.4.1. As suggested by the case where C_1 and C_2 are tangent, we will show, roughly speaking, that any circle $C \in \mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}$ has to intersect the arc of minimal length on C_1 that contains $C_1 \cap C_2$.

Lemma 3.4.3 *Suppose $C_2 \in \mathcal{C}_{\beta\tau}^{C_1}$. Then*

$$|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \lesssim \frac{t^2}{\delta^2}\sqrt{\frac{\epsilon + \beta}{\tau}}. \tag{3.44}$$

Proof: We may assume that $\tau \leq 4t$ (otherwise $\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2} = \emptyset$). Let

$$\gamma_0 = \sqrt{\frac{\beta + \epsilon}{\tau}} \simeq \sqrt{\frac{(\beta + \epsilon)(t + \tau)}{t\tau}}.$$

Suppose $\overline{C} \in \mathcal{C}_{\epsilon t}^{C_1}$ satisfies

$$\min(\sphericalangle(\bar{x}, x_1, x_2), \sphericalangle(\bar{x}, x_1, -x_2)) > A_0\gamma_0,$$

A_0 being the constant in (3.43). In view of (3.40) we may then apply Lemma 3.4.2 with $C_0 = C_1$, $C_1 = C_2$, $C_2 = \overline{C}$, $\beta_1 = \beta$, $\beta_2 = \epsilon$, $\rho_1 = \tau$, and $\rho_2 = t$ to wit

$$\Delta(\overline{C}, C_2) \geq \epsilon + \beta > \epsilon.$$

In particular $\overline{C} \notin \mathcal{C}_{\epsilon t}^{C_2}$. We conclude that any $\overline{C} \in \mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}$ has to satisfy

$$\min(\sphericalangle(\bar{x}, x_1, x_2), \sphericalangle(\bar{x}, x_1, -x_2)) \leq A_0\gamma_0.$$

In particular, the centers of all circles in $\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}$ are contained in a $4t \times 2t\,A_0\gamma_0$ rectangle centered at x_1 and thus

$$|\mathcal{C}_{\epsilon t}^{C_1} \cap \mathcal{C}_{\epsilon t}^{C_2}| \lesssim \frac{t^2}{\delta^2}\gamma_0$$

as claimed. $\blacksquare$

The following result is the aforementioned two circle lemma.

Lemma 3.4.4 *Suppose $C_2 \in \mathcal{C}^{C_1}_{\beta\tau}$. Then*

$$|\mathcal{C}^{C_1}_{\epsilon t} \cap \mathcal{C}^{C_2}_{\epsilon t}| \lesssim \frac{t^2}{\delta^2} \min\Big(\sqrt{\frac{\epsilon}{\tau}}, \frac{\epsilon}{\sqrt{\beta\tau}}\Big). \tag{3.45}$$

Proof: As before we may assume that $\tau \leq 4t$. Moreover, we may also assume that $2\epsilon \leq t$. Indeed, since $|\mathcal{C}^{C_1}_{\epsilon t} \cap \mathcal{C}^{C_2}_{\epsilon t}| \lesssim \frac{t^2}{\delta^2}$ either

$$\sqrt{\frac{\epsilon}{\tau}} \leq \frac{1}{4} \quad \text{or} \quad \frac{\epsilon}{\sqrt{\beta\tau}} \leq \frac{1}{24}$$

without loss of generality. In the first case $16\epsilon \leq \tau \leq 4t$, whereas in the second case $24^2\epsilon^2 \leq \beta\tau \leq 6\tau^2 \leq 6 \cdot 24t^2$. Hence, if $\Delta(C_1, C_2) \geq 10\epsilon$ we may apply Lemma 3.4.1 to conclude

$$|\mathcal{C}^{C_1}_{\epsilon t} \cap \mathcal{C}^{C_2}_{\epsilon t}| \lesssim \frac{t^2}{\delta^2} \frac{\epsilon}{\sqrt{\beta\tau}}.$$

If on the other hand $\Delta(C_1, C_2) \leq 10\epsilon$, then (3.45) follows from (3.44). ∎

Lemma 3.4.5 *Let $\alpha \in (0, 1]$ be fixed and suppose that every $C \in \mathcal{C}$ satisfies*

$$|\{x \in C^\rho \; : \; \mu_\rho^{\mathcal{C}^C_{\epsilon t}}(x) \geq A\lambda^{-1}\delta^{-1}t\left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{2}} \frac{\rho}{\delta}\}| \leq \lambda\rho \tag{3.46}$$

for all $\delta \leq \rho \leq \epsilon \leq t$ and all $0 < \lambda$.

Then there exists $\mathcal{A} \subset \mathcal{C}$, $|\mathcal{A}| \geq \frac{1}{2}|\mathcal{C}|$ so that

$$|\{x \in C^\rho \; : \; \mu_\rho^{\mathcal{A}^C_{\epsilon t}}(x) \geq A_1|\log\delta|^{9/2}\lambda^{-1}\delta^{-1}t\left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{4}} \frac{\rho}{\delta}\}| \leq \lambda\rho \tag{3.47}$$

for all $C \in \mathcal{A}$, $\delta \leq \rho \leq \epsilon \leq t$, and $0 < \lambda$. Here A_1 is some absolute constant depending on A but not on $\mathcal{C}$ or the parameters (in fact, we can take $A_1 = c_0\sqrt{A}$ for some absolute constant c_0).

Proof: Suppose there are at least $\frac{1}{2}|\mathcal{C}|$ many circles $C \in \mathcal{C}$ with the property that

$$|\{x \in C^\rho \; : \; \mu_\rho^{\mathcal{C}^C_{\epsilon t}}(x) \geq A_1|\log\delta|^{9/2}\lambda^{-1}\delta^{-1}t\left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{4}} \frac{\rho}{\delta}\}| > \lambda\rho \tag{3.48}$$

for some choice of $\delta \leq \rho \leq \epsilon \leq t$ and $0 < \lambda$. Then there exist a family $\mathcal{B} \subset \mathcal{C}$ with $|\mathcal{B}| \gtrsim |\log\delta|^{-4}|\mathcal{C}|$ and numbers $\delta \leq \rho \leq \epsilon \leq t$ and $\lambda > 0$ so that (3.48) holds for this (fixed) choice of ρ, ϵ, t, λ, and all $C \in \mathcal{B}$.

Consider the set

$$\begin{aligned} S = \{(C, C_1, C_2) \; : \; & C \in \mathcal{B}, \; C_1, C_2 \in \mathcal{C}^C_{\epsilon t} \quad \text{for} \quad j = 1, 2, \; \mathrm{sgn}(r - r_1) = \\ & = \mathrm{sgn}(r - r_2), \; \beta/2 \leq \Delta(C_1, C_2) \leq \beta, \; \tau \leq |x_1 - x_2| \leq 2\tau\}. \end{aligned} \tag{3.49}$$

Here β and τ are chosen by pigeonholing so that the lower bound

$$\begin{aligned} \mathrm{card}(S) &\gtrsim |\log\delta|^{-2}|\mathcal{B}| \left(|\log\delta|^{-1}\lambda\frac{\sqrt{t\epsilon}}{\rho}A_1|\log\delta|^{9/2}\lambda^{-1}\frac{t}{\delta}\left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{4}}\frac{\rho}{\delta}\right)^2 \\ &\simeq |\log\delta|\,|\mathcal{C}|\; A_1^2\,\frac{t\epsilon}{\delta^2}\left(\frac{t}{\delta}\right)^2\left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{2}} \end{aligned} \tag{3.50}$$

holds.

To obtain an upper bound on card(S) we would like to estimate the number of choices for C given C_1 and C_2 by Lemma 3.4.4. However, (3.49) does not specify $|r_1-r_2|$ so it is not clear whether $C_2 \in \mathcal{C}^{C_1}_{\beta\tau}$ (see (3.40)). On the other hand, we only need to consider those (C_1, C_2) which also satisfy

$$|r_1 - r_2| \le |x_1 - x_2| + 2\epsilon.$$

Indeed, given any $(C, C_1, C_2) \in S$ we can estimate

$$\begin{aligned} |r_1 - r_2| &= |(r_1 - r) - (r_2 - r)| = ||r_1 - r| - |r_2 - r|| \\ &\le ||r_1 - r| - |x_1 - x|| + ||r_2 - r| - |x_2 - x|| + ||x_1 - x| - |x_2 - x|| \\ &\le 2\epsilon + |x_1 - x_2|, \end{aligned}$$

as claimed. Let us assume first that $\tau \ge \epsilon$. Then

$$|r_1 - r_2| \le |x_1 - x_2| + 2\tau \le 4\tau,$$

and thus $C_2 \in \mathcal{C}^{C_1}_{\beta\tau}$ for any (C_1, C_2) that can appear in an element of S. Also note that such (C_1, C_2) satisfy

$$C_1^\beta \cap C_2^\beta \neq \emptyset.$$

Hence

$$\begin{aligned} \mathrm{card}(S) &\le \sum_{C_1 \in \mathcal{C}} \sum_{C_2 \in \mathcal{C}^{C_1}_{\beta\tau} : C_1^\beta \cap C_2^\beta \neq \emptyset} |\mathcal{C}^{C_1}_{\epsilon t} \cap \mathcal{C}^{C_2}_{\epsilon t}| \qquad (3.51) \\ &\lesssim |\mathcal{C}|\ A|\log\delta| \frac{\sqrt{\tau\beta}}{\delta}\frac{\tau}{\delta}\left(\frac{\tau}{\beta}\right)^{\frac{\alpha}{2}} \frac{t^2}{\delta^2}\min\Big(\sqrt{\frac{\epsilon}{\tau}}, \frac{\epsilon}{\sqrt{\tau\beta}}\Big). \qquad (3.52) \end{aligned}$$

To pass from line (3.51) to line (3.52) we have applied Lemma 3.4.4 to bound the cardinality of the intersection whereas the number of terms in the second sum of line (3.51) can be estimated by invoking assumption (3.46). In fact, here we have used the special case $\rho = \beta$ of the following estimate

$$\mathrm{card}(\{C_2 \in \mathcal{C}^{C_1}_{\tau\beta} \ :\ C_1^\rho \cap C_2^\rho \neq \emptyset\}) \lesssim A\frac{\sqrt{\tau\beta}}{\delta}\frac{\tau}{\delta}\left(\frac{\tau}{\beta}\right)^{\frac{\alpha}{2}} |\log\delta|. \qquad (3.53)$$

To prove (3.53) write C_1^ρ as the union of $\frac{\rho}{\sqrt{\tau\beta}}$ rectangles $\{R_j\}_{j=1}^k$. Let

$$m = m(\lambda) = A\lambda^{-1}\tau\delta^{-1}\left(\frac{\tau}{\beta}\right)^{\frac{\alpha}{2}}\frac{\rho}{\delta}.$$

Then assumption (3.46) implies that (with $m = m(\lambda)$)

$$\begin{aligned} & m\ \mathrm{card}(\{j \ :\ \mathrm{card}(\{C_2 \in \mathcal{C}^{C_1}_{\tau\beta} : R_j \cap C_2^\rho \neq \emptyset\}) \in [m, 2m]\}) \qquad (3.54) \\ \lesssim\ & m\ \lambda\frac{\sqrt{\tau\beta}}{\rho} = A\frac{\tau}{\delta}\left(\frac{\tau}{\beta}\right)^{\frac{\alpha}{2}}\frac{\rho}{\delta}\frac{\sqrt{\tau\beta}}{\rho}. \end{aligned}$$

By summing (3.54) over the $|\log\delta|$ many dyadic values of $m \in [1, \delta^{-2}]$ we obtain (3.53).

We now show that the upper and lower bounds (3.50) and (3.52) are incompatible for large A_1. Consider first the case $\epsilon \le \beta$. Then (keeping in mind that $\tau \le 4t$) the right–hand side of (3.52) is

$$\begin{aligned} &\lesssim\ A\,|\mathcal{C}||\log\delta|\ \frac{\sqrt{\tau\beta}}{\delta}\ \frac{\tau}{\delta}\ \left(\frac{\tau}{\beta}\right)^{\frac{\alpha}{2}}\ \frac{t^2}{\delta^2}\ \frac{\epsilon}{\sqrt{\tau\beta}} \\ &\lesssim\ A\,|\mathcal{C}|\,|\log\delta|\ \frac{t\epsilon}{\delta^2}\ \frac{t^2}{\delta^2}\ \left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{2}}, \end{aligned}$$

which contradicts (3.50) for large A_1.

If $\beta \leq \epsilon$ then the right–hand side of (3.52) is

$$\begin{aligned} &\lesssim \quad A\,|\mathcal{C}|\,|\log\delta|\,\frac{\sqrt{\tau\beta}}{\delta}\,\frac{\tau}{\delta}\,\left(\frac{\tau}{\beta}\right)^{\frac{\alpha}{2}}\,\frac{t^2}{\delta^2}\,\sqrt{\frac{\epsilon}{\tau}} \\ &\lesssim \quad A\,|\mathcal{C}|\,|\log\delta|\,\frac{t\epsilon}{\delta^2}\left(\frac{t}{\delta}\right)^2\left(\frac{\tau}{\epsilon}\right)^{\frac{\alpha}{2}}\left(\frac{\epsilon}{\beta}\right)^{\frac{\alpha}{2}}\left(\frac{\beta}{\epsilon}\right)^{\frac{1}{2}} \\ &\lesssim \quad A\,|\mathcal{C}|\,|\log\delta|\,\frac{t\epsilon}{\delta^2}\,\left(\frac{t}{\delta}\right)^2\,\left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{2}} \end{aligned}$$

where we have used $\alpha \leq 1$ in order to pass to the last line. For large A_1 this will again result in a contradiction.

Recall that we have assumed $\tau \geq \epsilon$ throughout. We will now treat the case $\tau < \epsilon$. As before,

$$\mathrm{card}(S) \gtrsim |\log\delta|^5|\mathcal{B}|A_1^2\,\frac{t\epsilon}{\delta^2}\left(\frac{t}{\delta}\right)^2\left(\frac{t}{\epsilon}\right)^{\frac{\alpha}{2}}, \tag{3.55}$$

whereas trivially

$$\mathrm{card}(S) \lesssim |\mathcal{B}|\frac{t^2}{\delta^2}\frac{\tau^2}{\delta^2} \leq |\mathcal{B}|\frac{t^2}{\delta^2}\frac{\epsilon^2}{\delta^2}. \tag{3.56}$$

Inequalities (3.55) and (3.56) imply that $A_1^2 \lesssim 1$, which is a contradiction for large A_1. ■

Proof of Proposition 3.4.1: In view of Lemma 3.4.5 it will suffice to show the following.

Claim: There exists an absolute constant A so that for any family $\mathcal{C}$

$$|\{x \in C^\rho \;:\; \mu_\rho^{\mathcal{C}_{\epsilon t}^C}(x) \geq A\lambda^{-1}\delta^{-1}t\left(\frac{t}{\epsilon}\right)^{\frac{1}{2}}\frac{\rho}{\delta}\}| \leq \lambda\rho \tag{3.57}$$

for all $C \in \mathcal{C}$, all $\delta \leq \rho \leq \epsilon \leq t$ and $0 < \lambda$.

If (3.57) failed for one circle $C \in \mathcal{C}$ and one choice of the parameters then integrating $\mu_\rho^{\mathcal{C}_{\epsilon t}^C}$ over C^ρ (as in [8]) would yield

$$\lambda\rho\; A\lambda^{-1}\delta^{-1}t\left(\frac{t}{\epsilon}\right)^{\frac{1}{2}}\frac{\rho}{\delta} \lesssim |\mathcal{C}_{\epsilon t}^C|\frac{\rho^2}{\sqrt{\epsilon t}} \lesssim \frac{t^2}{\delta^2}\frac{\rho^2}{\sqrt{\epsilon t}},$$

which is a contradiction for large A. ■

Remark: It is possible to prove Lemma 3.4.5 without considering variable ρ, i.e., with $\rho = \delta$. One thus obtains a smaller power of $|\log\delta|$ in (3.47), but this is immaterial in the present context.

Suppose we fix η. Given any collection $\mathcal{C}$ of circles with δ–separated centers, Proposition 3.4.1 implies that there exists $\mathcal{A} \subset \mathcal{C}$, so that $|\mathcal{A}| \geq C_\eta^{-1}|\mathcal{C}|$ and that

$$\mathrm{card}(\{\tilde{\mathcal{C}} \in \mathcal{A}_{\epsilon t}^C \;:\; \tilde{\mathcal{C}}^\delta \cap C^\delta \neq \emptyset\}) \lesssim \delta^{-\eta}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}} \tag{3.58}$$

for all $C \in \mathcal{A}$ (this can be seen by the same type of argument that showed (3.53)). In particular, if $\mathcal{C}$ is the family of circles arising in the context of a restricted weak–type inequality for the maximal function, see page 12, we define the multiplicity μ using $\mathcal{A}$ instead of $\mathcal{C}$ – this will merely introduce a $\delta^{-\eta}$ in the norms. It turns out that (3.58) then allows for a more transparent proof of Proposition 2.4.2. Indeed, suppose we are in the transverse case (2.45), i.e.,

$$\bar{\lambda} \lesssim \sqrt{\frac{\epsilon}{t}}.$$

Then for any $C_j \in \mathcal{A}$, see (2.37),

$$\begin{aligned}\bar{\mu}\frac{\bar{\lambda}\delta}{\delta^2/\sqrt{\epsilon t}} &\lesssim \operatorname{card}(S^j_{t,\epsilon}) \\ &\lesssim \min(N, \delta^{-\eta}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{2}}\left(\frac{t}{\delta}\right)^{\frac{3}{2}}) \\ &\lesssim N^{\frac{1}{2}}\delta^{-\eta/2}\left(\frac{\epsilon}{\delta}\right)^{\frac{1}{4}}\left(\frac{t}{\delta}\right)^{\frac{3}{4}}.\end{aligned}$$

Thus, using our assumption on $\bar{\lambda}$, we conclude that

$$\bar{\mu} \lesssim \bar{\lambda}^{-1}N^{\frac{1}{2}}\delta^{-\eta/2}\left(\frac{t}{\epsilon}\right)^{\frac{1}{4}} \lesssim \delta^{-\eta/2}\bar{\lambda}^{-\frac{3}{2}}N^{\frac{1}{2}},$$

as desired. It is more complicated to deal with the tangential case in the same spirit. This seems to require a slightly different version of Marstrand's three circle lemma from the one we have used, i.e., Lemma 2.4.1. Since the argument does not improve on what we already have, we do not write it out.

Appendix A

Capacity and Hausdorff dimension

In this appendix we establish a special case of the well–known connection between capacity and Hausdorff dimension. For any set $E \subset \mathbb{R}^d$, $1 \le p < \infty$, and $\alpha > 0$ we let the (p, α) capacity of E (strictly speaking, the Bessel capacity) be

$$C_{p,\alpha}(E) = \inf\{\|u\|_{L^p_\alpha}^p \;:\; u \ge 1 \text{ on a neighborhood of E}\}.$$

Here $L^p_\alpha(\mathbb{R}^d)$ is the usual Sobolev–Bessel space with norm

$$\|f\|_{L^p_\alpha(\mathbb{R}^d)} = \|(I - \triangle)^{\alpha/2} f\|_p.$$

Suppose $p > 1$ and $\alpha p < d$. Then it is essentially a consequence of Sobolev's inequality that there exists a constant C_0 depending only on p, α, and d so that

$$C_0^{-1} r^{d-\alpha p} \le C_{p,\alpha}(B(x,r)) \le C_0 r^{d-\alpha p}$$

for all balls $B(x, r)$. This suggest that $C_{p,\alpha}$ and $\mathcal{H}^{d-\alpha p}$ are related. For a general statement in this direction see [23], Theorem 2.6.16 and the references to the original literature given there. Here we only deal with a special case which is sufficient for our purposes. The following proof is an adaptation of the argument on page 107 in [23] to the case of fractional Sobolev spaces.

Proposition: Let $2 \le p < \infty$, $0 < \alpha < 1$, and $\alpha p < d$. Then $C_{p,\alpha}(E) = 0$ implies that

$$\mathcal{H}^{d-\alpha p+\epsilon}(E) = 0 \quad \text{for all} \quad \epsilon > 0.$$

Proof: By the definition of capacity there exists a sequence $u_j \in L^p_\alpha$ so that $u_j \ge 1$ on a neighborhood of E for each j and $\|u_j\|_{L^p_\alpha} < 2^{-j}$. We will use the following result about the relation between Sobolev spaces defined in terms of Bessel potentials and Besov spaces. Let $\Lambda^{p,q}_\alpha(\mathbb{R}^d)$ be the space of functions for which the following norm is finite:

$$\|u\|_{\Lambda^{p,q}_\alpha(\mathbb{R}^d)} = \|u\|_p + \left(\int_{\mathbb{R}^d} \frac{\|u(\cdot + t) - u(\cdot)\|_p^q}{|t|^{d+\alpha q}}\, dt\right)^{\frac{1}{q}} \tag{A.1}$$

where $1 \le p, q < \infty$, $0 < \alpha < 1$. It is shown in Stein [18], page 155, that

$$L^p_\alpha(\mathbb{R}^d) \subset \Lambda^{p,p}_\alpha(\mathbb{R}^d) \;\; \text{provided } p \ge 2. \tag{A.2}$$

Now let

$$u = \sum_{j=1}^{\infty} u_j^+ \in \Lambda_\alpha^{p,p}(\mathbb{R}^d)$$

and define

$$\bar{u}(x,r) = r^{-d} \int_{B_r(x)} u(y)\, dy.$$

Since $u_j \geq 1$ on a neighborhood of E for each j, $\bar{u}$ satisfies

$$\liminf_{r\to\infty} \bar{u}(x,r) = \infty \text{ for all } x \in E. \tag{A.3}$$

Let μ be the measure

$$\mu(A) = \int_A \int_{\mathbb{R}^d} \frac{|u(y)-u(z)|^p}{|y-z|^{d+p\alpha}}\, dy\, dz.$$

Firstly, μ is finite by (A.1) and (A.2). Secondly we claim that

$$\limsup_{r\to 0} r^{\alpha p - d - \epsilon} \mu(B_r(x)) = \infty \text{ for all } x \in E. \tag{A.4}$$

Indeed, if for a fixed $x \in E$ there is a $M < \infty$ so that

$$r^{\alpha p - d} \mu(B_r(x)) \leq M r^{\epsilon},$$

then

$$\begin{aligned} r^{-d} \int_{B_r(x)} |u(y) - \bar{u}(x,r)|^p\, dy &\lesssim r^{-2d} \int_{B_r(x)} \int_{B_r(x)} |u(y)-u(z)|^p\, dy\, dz \\ &\lesssim r^{\alpha p - d} \int_{B_r(x)} \int_{\mathbb{R}^d} \frac{|u(y)-u(z)|^p}{|y-z|^{p\alpha+d}}\, dy\, dz \\ &= r^{\alpha p - d} \mu(B_r(x)) \leq M r^{\epsilon}. \end{aligned}$$

In particular,

$$|\bar{u}(x, 2^{-j-1}) - \bar{u}(x, 2^{-j})|^p \lesssim M 2^{-j\epsilon},$$

which would imply that $\lim_{j\to\infty} \bar{u}(x, 2^{-j})$ exists, contradicting (A.3).Thus (A.4) holds.

By a standard covering argument, see [23] Lemma 3.2.1, we conclude from (A.4) that

$$\mathcal{H}^{d-\alpha p+\epsilon}(E) = 0. \quad \blacksquare$$

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