Time decay for solutions of Schrödinger equations with rough and time-dependent potentials.

Igor Rodnianski and Wilhelm Schlag

October 10, 2003

Abstract

In this paper we establish dispersive estimates for solutions to the linear Schrödinger equation in three dimension

$$\frac{1}{i}\partial_t\psi - \triangle\psi + V\psi = 0, \qquad \psi(s) = f \tag{0.1}$$

where $V(t,x)$ is a time-dependent potential that satisfies the conditions

$$\sup_t \|V(t,\cdot)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{-\infty}^{\infty}\frac{|V(\hat{\tau},x)|}{|x-y|}\,d\tau\,dy < c_0.$$

Here c_0 is some small constant and $V(\hat{\tau},x)$ denotes the Fourier transform with respect to the first variable. We show that under these conditions (0.1) admits solutions $\psi(\cdot)\in L^\infty_t(L^2_x(\mathbb{R}^3))\cap L^2_t(L^6_x(\mathbb{R}^3))$ for any $f\in L^2(\mathbb{R}^3)$ satisfying the dispersive inequality

$$\|\psi(t)\|_\infty \le C|t-s|^{-\frac{3}{2}}\,\|f\|_1 \ \text{ for all times } t,s. \tag{0.3}$$

For the case of time independent potentials $V(x)$, (0.3) remains true if

$$\int_{\mathbb{R}^6}\frac{|V(x)|\,|V(y)|}{|x-y|^2}\,dxdy < (4\pi)^2 \quad \text{and} \quad \|V\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\,dy < 4\pi.$$

We also establish the dispersive estimate with an ε-loss for large energies provided $\|V\|_{\mathcal{K}} + \|V\|_2 < \infty$.

Finally, we prove Strichartz estimates for the Schrödinger equations with potentials that decay like $|x|^{-2-\varepsilon}$ in dimensions $n\ge 3$, thus solving an open problem posed by Journé, Soffer, and Sogge.

1 Introduction

It follows from the explicit expression for the kernel of $e^{-it\triangle}$ that the free Schrödinger evolution in $\mathbb{R}^n$, $n\ge 1$, satisfies the dispersive inequality

$$\|e^{-it\triangle}f\|_{L^\infty_x} \le Ct^{-\frac{n}{2}}\|f\|_{L^1_x}. \tag{1.1}$$

Closely related are the classical Strichartz estimate [Str]

$$\|e^{-it\triangle}f\|_{L^{2+\frac{4}{n}}(\mathbb{R}^{n+1})} \le C\|f\|_{L^2(\mathbb{R}^n)}$$

or more generally

$$\|e^{-it\triangle}f\|_{L^p_tL^q_x(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)} \tag{1.2}$$

for any $\frac{n}{q}+\frac{2}{p}=\frac{n}{2}$, $2\le p\le\infty$. The case $p=\infty$, $q=2$ is the energy estimate (in fact $\|e^{-itH}f\|_2 = \|f\|_2$), whereas the range $2<p<\infty$ can be obtained from the case $p=2$ and (1.1) by means of a well-known argument (see for example [KT]). The endpoint $p=2, q=\frac{2n}{n-2}$ result, which in fact fails in dimension $n=2$, is more difficult and was recently settled for $n\ge 3$ by Keel and Tao [KT].

The question whether these bounds also hold for more general Schrödinger equations has been considered by various authors. From a physical perspective it is of course natural to consider the case of e^{itH} with $H=-\triangle+V$. For the purposes of the present discussion we assume that the potential V is real and has enough regularity to ensure that H is a self-adjoint operator on $L^2(\mathbb{R}^n)$, see Simon's review [Si2] for explicit conditions on V. One obstacle to having decay in time for e^{itH} are eigenvalues of the operator $H=-\triangle+V$ and a result as in (1.1) and (1.2) therefore requires that f be orthogonal to any eigenfunction of H. In fact, Journé, Soffer, and Sogge [JSS] have shown that, with P_c being the projection onto the continuous subspace of $L^2(\mathbb{R}^n)$ with respect to H,

$$\|e^{it(-\triangle+V)}P_cf\|_\infty \le C\,t^{-\frac{n}{2}}\|f\|_{L^1(\mathbb{R}^n)} \tag{1.3}$$

for all dimensions $n\ge 3$ provided that zero is neither an eigenvalue nor a resonance of H. In addition, they need to assume that, roughly speaking, $|V(x)|\lesssim(1+|x|)^{-n-4}$ and $\hat V\in L^1(\mathbb{R}^n)$. Recall that a resonance is a distributional solution of $H\psi=0$ so that $\psi\notin L^2$ but $(1+|x|^2)^{-\frac{\sigma}{2}}\psi(x)\in L^2$ for any $\sigma>\frac12$, see [JK]. It is well-known that under the assumptions on V used in [JSS] the spectrum $\sigma(H)$ satisfies

$$\sigma(H)=[0,\infty)\cup\{\lambda_j \mid j=1,\ldots,N\}$$

where $[0,\infty)=\sigma_{a.c.}(H)$ and $\lambda_N<\lambda_{N-1}<\ldots<\lambda_1\le 0$ are a discrete and finite set of eigenvalues of finite multiplicity. Indeed, since V is bounded and decays at infinity Weyl's criterion (Theorem XIII.14 in [RS]) implies that $\sigma_{ess}(H)=\sigma_{ess}(-\triangle)=[0,\infty)$, whereas the finiteness follows from the Cwikel-Lieb-Rosebljum bound. Furthermore, since V is bounded and decays faster than $|x|^{-1}$ at infinity it follows from Kato's theorem (Theorem XIII.58 in [RS]) that there are no positive eigenvalues of H. Finally, since any V as in [JSS] is an Agmon potential, $\sigma_{sing}(H)=\emptyset$ by the Agmon-Kato-Kuroda theorem (Theorem XIII.33 in [RS]).

The work by Journé, Soffer, and Sogge was preceded by related results of Rauch [R], Jensen, Kato [JK], and Jensen [J1],[J2]. The fact that one cannot have $t^{-\frac32}$ decay in the presence of a resonance at zero energy was observed by these authors. Moreover, the small energy asymptotic expansions of the resolvent developed in [JK], [J1], [J2] are used in [JSS]. However, the actual time decay estimates obtained by Rauch, Jensen, and Kato are formulated in terms of weighted L^2-spaces rather than in the much stronger $L^1\to L^\infty$ sense of Journé, Soffer, and Sogge. The appearance of weighted L^2 spaces is natural in view of the so called limiting absorption principle. This refers to boundedness of the resolvents $(-\triangle-\lambda\pm i0)^{-1}$ for $\lambda>0$ on certain weighted L^2 spaces as proved by Agmon [Ag] and Kuroda [Ku2], [Ku1]. It is also with respect to these weighted norms that the asymptotic expansions of the resolvents $(H-z)^{-1}$ as $z\to 0$ with $\Im(z)\ge 0$, $\Re(z)>0$ in [JK], [J1],[J2] hold. Jensen and Kato need to assume that $|V(x)|\lesssim(1+|x|)^{-\beta}$ for certain $\beta>1$ (most of their results require $\beta>3$). For a more detailed discussion of the limiting absorption principle see our Strichartz estimates in Section 4.

Another approach to decay estimates for $e^{it(-\triangle+V)}$ was taken by Yajima [Y2], [Y3], and Artbazar and Yajima [AY], who relied on scattering theory. Recall that if the so called wave-operator

$$W = s - \lim_{t\to\infty} e^{-it(-\triangle+V)} e^{-it\triangle}$$

exists, where the limit is understood in the strong L^2 sense, then it is an isometry that intertwines the evolutions, i.e.,

$$W e^{-it\triangle} = e^{it(-\triangle+V)} W \quad \text{for all times } t.$$

In [Y2] Yajima proved that the wave operators W are bounded from $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ with $n \geq 3$ for $1 \leq p \leq \infty$ provided V has a certain explicit amount of decay, and provided zero is neither an eigenvalue nor a resonance. Since $WW^* = P_{a.c.}$, he concludes from the free dispersive estimate (1.1) that

$$\|e^{it(-\triangle+V)} P_{a.c.}\|_{L^1\to L^\infty} = \|W e^{-it\triangle} W^*\|_{L^1\to L^\infty} \leq C\, t^{-\frac{n}{2}}$$

under the usual assumption on the zero energy but imposing weaker conditions on V than [JSS]. Moreover, [Y3] contains the first results on dispersive and Strichartz estimates for $e^{it(-\triangle+V)}$ in two dimensions.

The relatively strong decay and regularity assumptions that appear in all aforementioned works are by far sufficient to ensure scattering, i.e, the existence of wave operators on L^2, even though Yajima was the first to exploit this link explicitly in the context of dispersive estimates. The connection with scattering is of course natural, as the decay of V (and possibly that of derivatives of V) at infinity allows one to reduce matters to the free equation by methods that are to a large extent perturbative.

On the other hand, the existence of scattering (in the traditional L^2 sense) is known for potentials that are small in some global sense, but without any explicit rate of decay. Indeed, it is a classical result of Kato [Ka] that under the sole assumption that the real potential V satisfies

$$\int_{\mathbb{R}^6} \frac{|V(x)|\,|V(y)|}{|x-y|^2}\, dxdy < (4\pi)^2 \tag{1.4}$$

the operator $H = -\triangle + V$ on $\mathbb{R}^3$ is self-adjoint and unitarily equivalent to $-\triangle$ via the wave operators. The left–hand side of (1.4) is usually referred to as the Rollnik norm, see [Si1]. Observe that (1.4) roughly corresponds to the potential decaying at infinity as $|x|^{-2-\varepsilon}$.

The appearance of the Rollnik norm in the context of small potentials is natural from several perspectives, one of which is scaling. The Rollnik norm is invariant under the scaling $R^2V(Rx)$ forced by the Schrödinger operator H onto the potential V. It is well–known that the Rollnik norm defines a class of potentials that is slightly wider than $L^{\frac{3}{2}}(\mathbb{R}^3)$, which is also scaling invariant. Another natural occurrence of a scaling invariant condition arises in connection with bounds on the number of negative eigenstates. Indeed, in dimension n it is precisely the scaling invariant $L^{\frac{n}{2}}$ norm of the negative part of the potential that governs the number of negative eigenvalues of $-\Delta + V$ via the Cwikel-Lieb-Rosebljum bound.

We show in this work that dispersive estimates lead naturally to what we call the "global Kato norm" of the potential. Recall that the Kato norm of V is defined to be

$$\sup_{x\in\mathbb{R}^3} \int_{|x-y|\leq 1} \frac{|V(y)|}{|x-y|}\, dy,$$

whereas the scaling invariant analogue is given by (1.5) below. The Kato norm, or more precisely the closely related Kato class, arise in the study of self-adjoint extensions of H, as well as in the study of the properties of the heat semigroup e^{-tH}, see [AS], [Si2], and Section 3 below.

One of the goals of our paper is to bridge the gap between the "classical" perturbation results of spectral theory that involve Rollnik and Kato classes of potentials (or other scaling invariant classes) and the results concernning the dispersive properties of the time-dependent Schrödinger equation.

In our first result, see Theorem 2.6 below, we show that the dispersive estimates are stable under perturbations by small potentials that belong to the intersection of the Rollnik and the global Kato classes.

Theorem 1.1. *Suppose V is real and satisfies* (1.4). *Suppose in addition that*

$$\sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\,dy<4\pi. \tag{1.5}$$

Then one has the estimate

$$\|e^{it(-\triangle+V)}\|_{L^1\to L^\infty}\lesssim t^{-\frac{3}{2}}$$

for all $t>0$.

The proof relies on a Born series expansion for the resolvent with a subsequent estimate of an arising oscillatory integral. The convergence of the resulting geometric series is guaranteed by (1.5). See Section 2 for details.

The main focus of this paper is on the dispersive properties of solutions of the Schrödinger equation (0.1) with time dependent potentials, see Sections 5–7. It appears that not much is known on the long time behavior of solutions to Schrödinger equations with time dependent potentials. See, however, Bourgain [Bo2], [Bo2] on the issue of slow growth of higher Sobolev norms in the space-periodic setting. In this paper we establish dispersive and Strichartz estimates for a class of scaling invariant small potentials on $\mathbb{R}^3$.

Theorem 1.2. *Let $V(t,x)$ be a real-valued measurable function on $\mathbb{R}^4$ such that*

$$\sup_t\|V(t,\cdot)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}+\sup_{y\in\mathbb{R}^3}\int_{\mathbb{R}^3}\int\frac{|V(\hat{\tau},x)|}{|x-y|}\,d\tau\,dx<c_0 \tag{1.6}$$

for some small constant $c_0>0$. Here $V(\hat{\tau},x)$ denotes the Fourier transform in the first variable, and if $V(\hat{\tau},x)$ happens to be a measure then the L^1–norm in τ gets replaced with the norm in the sense of measures. Then for every initial time s and every $\psi_s\in L^2(\mathbb{R}^3)$ the equation

$$\begin{aligned}&\frac{1}{i}\partial_t\psi-\Delta\psi+V(t,x)\psi=0,\\&\psi|_{t=s}(x)=\psi_s(x)\end{aligned} \tag{1.7}$$

admits a (weak) solution $\psi(t,\cdot)=U(t,s)\psi_s$ (via the Duhamel formula). The propagator $U(\cdot,s)$ satisfies $U(\cdot,s):L^2(\mathbb{R}^3)\to L^\infty_t(L^2_x(\mathbb{R}^3))\cap L^2_t(L^6_x(\mathbb{R}^3))$, $t\mapsto\psi(t,\cdot)$ is weakly continuous as a map into $L^2(\mathbb{R}^3)$, and $\|U(t,s)\psi_s\|_2\le\|\psi_s\|_2$. Finally, $U(t,s)$ satisfies the dispersive inequality

$$\|U(t,s)\psi_s\|_{L^\infty}\le C|t-s|^{-\frac{3}{2}}\,\|\psi_s\|_{L^1}\quad\textit{for all times } t,s \textit{ and any } \psi_s\in L^1. \tag{1.8}$$

Examples of potentials to which the theorem applies are $V(t,x) = \cos(t)V_0(x)$ where $\|V_0\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < c_0$, and for which (1.5) holds. More generally, one can take potentials that are quasi-periodic in time, such as $V(t,x) = \phi(t)V_0(x)$ with

$$\phi(t) = \sum_{\nu \in Z^d} c_\nu e^{2\pi i t \omega \cdot \nu}$$

and $\sum_{\nu\in\mathbb{Z}^d} |c_\nu| < \infty$, $\omega \in [0,1)$ arbitrary.
Note that Theorem 1.2 also applies to time independent potentials $V_0(x)$ via $V(t,x) := V_0(x)$. Clearly, in that case the conditions become

$$\|V_0\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\, dy < c_0.$$

Since by fractional integration

$$\int_{\mathbb{R}^6} \frac{|V(x)|\,|V(y)|}{|x-y|^2}\, dxdy \le C\|V\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)},$$

it follows that these conditions are strictly stronger than those in Theorem 1.1.
Whereas our main emphasis is of course on the decay estimate (1.8), it appears that even the easier question of solvability of equation (1.7) for rough potentials that do not decay in time had not been addressed before, at least under the conditions of Theorem 1.2. Yajima [Y1] considered the problem of existence of solutions to the Schrödinger equation with time-dependent potentials. In his paper he proves the existence of the strongly continuous semigroup $U(t,s)$ on $L^2(\mathbb{R}^n)$ provided that the potential satisfies $V \in L^q_t L^p_x$ for $0 \le \frac{1}{q} < 1 - \frac{n}{2p}$. Notice that in our case $q = \infty$, $p = \frac{n}{2}$, which corresponds to the endpoint of this condition not covered in [Y1]. We use the endpoint Strichartz estimate [KT] for the *free* problem for that purpose, which automatically yields the endpoint Strichartz estimate in the context of Theorem 1.2.
For time-dependent potentials the analogue of Kato's scattering result [Ka] was proved by Howland [H1]. More precisely, under the condition that for a sufficiently large time $t_0 > 0$, $V(t,x) \le V_0(x)$ for some *time independent* potential $V_0(x)$ obeying the small Rollnik condition (1.4), there exist a unitary wave operator W intertwining $U(t,s)$ and $e^{it(-\Delta)}$. In case $V(t,x)$ does decay in time (in the sense of a small amount of integrability), wave operators were constructed by Howland [H2] and Davies [D]. In contrast to Theorem 1.2 they do not require smallness (the latter being replaced by time decay of the potential) and they also obtain strong continuity of the evolution.
One of the difficulties in this case is the absence of the connection between the semigroup generated by the Schrödinger equation and the spectral properties of the operator $-\Delta + V$. Recall that for time independent potentials V,

$$e^{itH} f = \int e^{it\lambda} dE(\lambda) f$$

where $dE(\lambda)$ is the spectral measure of the operator $-\Delta + V$. This is no longer available for time-dependent potentials.
The proof of (1.8) is similar to that of Theorem 1.1 but much more involved. Since we can no longer rely on the spectral theorem, resolvents, and Born series to construct the evolution of (1.7), we use the Duhamel formula instead (we note in passing that the Fourier transform in the spectral parameter establishes an equivalence between the representation of the evolution in terms of a Born series and an

infinite expansion of the solution by means of Duhamel's principle). One of the novelties in our paper is the formula representing the time evolution of the Schrödinger equation with a time-dependent potential as an infinite series of oscillatory integrals involving the resolvents of the *free* problem. Most of the work in the proof of Theorem 1.2 is devoted to estimating these oscillatory integrals, whose phases typically have a critical point with degeneracies of the third order. See Sections 5–7 for details.

Two sections of this paper are devoted to time independent potentials without any restrictions on their sizes. In Section 3 we prove the following result. As before, $H = -\triangle + V$ and $P_{a.c.}$ refers to the projection onto the absolutely continuous subspace of L^2 relative to H.

Proposition 1.3. *Let*

$$\|\!|V|\!\| := \|V\|_2 + \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\, dy < \infty.$$

Then for every $\varepsilon > 0$ there exists some positive $\lambda_0 = \lambda_0(\|\!|V|\!\|, \varepsilon)$ so that

$$\|e^{itH}\chi(H/\lambda_0)P_{a.c.}\|_{L^1\to L^\infty} \le Ct^{-\frac{3}{2}+\varepsilon} \tag{1.9}$$

for all $t > 0$.

The proof is again perturbative. For the case of large energies, *and for those only*, the required smallness is provided by the following estimate on the resolvents, which can be viewed as some instance of the limited absorption principle:

$$\|(-\triangle - \lambda + i0)^{-1} f\|_{L^4(\mathbb{R}^3)} \le C\lambda^{-\frac{1}{4}}\|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}. \tag{1.10}$$

The proof of (1.10) is an immediate consequence of the Stein-Tomas theorem [St]. The appearance of the Stein-Tomas theorem in this context is most natural, as the resolvent $(-\triangle - \lambda + i0)^{-1}$ of the free problem is closely related to the restriction of the Fourier transform to the sphere $|x| = \sqrt{\lambda}$ for $\lambda > 0$. In contrast to (1.10), which heavily relies on the nonvanishing Gaussian curvature of the sphere, the classical limiting absorption principle of Agmon and Kuroda [Ag], [Ku1], and [Ku2] only uses the most elementary restriction property of the Fourier transform to arbitrary surfaces which leads to a loss of $\frac{1}{2} + \varepsilon$ derivatives in L^2 (on the physical side this translates into the weights $|x|^{\frac{1}{2}+\varepsilon}$ in L^2 that appear in [Ag], [JK] etc.). For further details of the proof of Proposition 1.3 we refer the reader to Section 3.

It is common knowledge that the case of large energies should be the most accessible one. From the perspective of scattering the intuition is that particles with high energies will escape the scatterer and thus lead to extended states (absolutely continuous spectrum) whereas particles with smaller energies can be trapped and create bound states (pure point spectrum). It is of course a most interesting problem to extend Proposition 1.3 to small energies under similar conditions. Recall that [JSS] and particularly [Y2] have accomplished exactly that, but under conditions on V that are by far stronger than those in Proposition 1.3.

We also address the question of Strichartz estimates for $e^{it(-\triangle+V)}$ in dimensions greater or equal than three. Traditionally the mixed norm Strichartz estimates (1.2) are shown to be a consequence of the dispersive estimates. In fact, in [JSS], Journé, Soffer, and Sogge establish the $L^1 \to L^\infty$ dispersive bound and therefore also Strichartz estimates under strong decay and regularity assumptions on V, see (1.3). However, they conjecture that Strichartz estimates hold for potentials that decay only faster than $(1+|x|)^{-2}$. In this paper we prove this conjecture assuming only this rate of decay. In particular, we do not require any regularity. More precisely, the following theorem holds.

Theorem 1.4. *Suppose that for some $\varepsilon > 0$ one has $|V(x)| \lesssim (1+|x|)^{-2-\varepsilon}$ for all $x \in \mathbb{R}^n$ with $n \geq 3$. Then*

$$\|e^{itH} P_c f\|_{L^q_t L^r_x(\mathbb{R}^n)} \lesssim \|f\|_{L^2_x(\mathbb{R}^n)} \qquad \forall (q, r, n), \quad \frac{2}{q} = n(\frac{1}{2} - \frac{1}{r})$$

provided the zero energy is neither an eigenvalue nor a resonance of the operator $H = -\Delta + V$. Here P_c denotes the spectral projection onto the continuous states.

The decay condition $|V(x)| \lesssim (1+|x|)^{-2-\varepsilon}$ is very natural from the perspective of Kato's smoothing theory [Ka]. In contrast to [JSS] we prove the Strichartz estimates directly, i.e., without relying on dispersive estimates. In fact, we do not know if the $L^1 \to L^\infty$ estimates hold under the conditions of Theorem 1.4. It is known that (local in time) Strichartz estimates can hold even if the $L^1 \to L^\infty$ dispersive property fails, see Bourgain [Bo3] for the case of the torus, Staffilani, Tataru [ST] for variable coefficients, and Burq, Gerard, Tzvetkov [BGT] for the case of equations on Riemannian manifolds.

This paper is organized as follows: Section 2 to 4 deal with time independent potentials. Section 2 establishes dispersive estimates for small Rollnik potentials in $\mathbb{R}^3$. Section 3 considers the high energy case for low regularity potentials, and in Section 4 we establish mixed norm Strichartz estimates for potentials that decay like $(1+|x|)^{-2-\varepsilon}$. The remaining sections 5-7 are devoted to small time-dependent potentials. In Section 5 we show that solutions exist for potentials that do not necessarily decay in time by means of the Keel-Tao [KT] endpoint. We then proceed to represent the solution by means of an infinite Duhamel expansion and we derive a formula for each term in the Duhamel series. The most technical part are Sections 6 that provide the necessary bounds on the oscillatory integrals that arise in this context. We combine all the pieces in the final Section 7.

Acknowledgment: The authors thank Alexander Pushnitski for valuable discussions on the Agmon-Kato-Kuroda theory, Thomas Spencer for his interest in the problem of time dependent potentials, as well as Elias Stein for a discussion on Section 6. The first author was supported by an NSF grant. The second author was supported by an NSF grant and a Sloan fellowship. Part of this work was done while he was a member at the Institute for Advanced Study, Princeton. The authors are indebted to an anonymous referee whose insightful comments lead to significant simplifications of Section 6.

2 Small time independent potentials in $\mathbb{R}^3$

The purpose of this section is to prove the $L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ dispersive inequality for e^{itH} where $H = -\triangle + V$ in $\mathbb{R}^3$. The following definition states the properties of the real potential V that we will need.

Definition 2.1. *We require that both*

$$\|V\|_R^2 := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|V(x)|\,|V(y)|}{|x-y|^2}\, dx\, dy \quad < \quad (4\pi)^2 \quad \textit{and} \tag{2.1}$$

$$\|V\|_{\mathcal{K}} := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\, dy \quad < \quad 4\pi. \tag{2.2}$$

The norm $\|\cdot\|_R$ on the left-hand side of (2.1) is usually referred to as the *Rollnik norm.* Kato [Ka] showed that under the condition (2.1) the operator H admits a self-adjoint extension which is unitarily equivalent to $H_0 = -\triangle$. In particular, the spectrum of H is purely absolutely continuous. Many properties of the Rollnik norm, which can be seen to be majorized by the norm of $L^{\frac{3}{2}}(\mathbb{R}^3)$ via fractional integration, can be found in Simon's monograph [Si1]. The norm $\|\cdot\|_{\mathcal{K}}$ in (2.2) is closely related to the well-known *Kato norm*, see Aizenman and Simon [AS], [Si2] and we refer to it as the *global Kato norm.*

The main result in this section is Theorem 2.6. The proof splits into several lemmas, the first of which presents some well-known properties of the resolvents $R_V(z) = (-\triangle + V - z)^{-1}$ under the condition (2.1). We begin by recalling that a potential with finite (but not necessarily small) Rollnik norm is *Kato smoothing*, i.e.,

$$\sup_{\varepsilon>0} \| \,|V|^{\frac12} R_0(\lambda \pm i\varepsilon) f\|_{L^2_\lambda L^2_x} \le C\|f\|_{L^2}, \qquad \sup_{\varepsilon>0} \| \, R_0(\lambda \pm i\varepsilon)\,|V|^{\frac12} f\|_{L^2_\lambda L^2_x} \le C\|f\|_{L^2} \tag{2.3}$$

for any $f \in L^2(\mathbb{R}^3)$ and with $R_0(z) = (-\triangle - z)^{-1}$. This implies, in particular, that $\mathcal{D}(|V|^{\frac12}) \supset H^2$. The Rollnik norm arises in this context as a majorant for the Hilbert-Schmidt norm $\|\cdot\|_{HS}$ of the operators

$$K(\lambda \pm i\varepsilon) := |V|^{\frac12} R_0(\lambda \pm i\varepsilon)|V|^{\frac12}. \tag{2.4}$$

Indeed, it is well-known that the resolvent $R_0(z)$ for $\Im z \ge 0$ has the kernel

$$R_0(z)(x,y) = \frac{\exp(i\sqrt{z}|x-y|)}{4\pi|x-y|} \tag{2.5}$$

where $\Im(\sqrt{z}) \ge 0$. Thus

$$\|K(z)\|_{L^2\to L^2} \le \|K(z)\|_{HS} \le (4\pi)^{-1}\|V\|_R, \tag{2.6}$$

for every $z \in \mathbb{C}$ with $\Im z \ge 0$. This allows one to check immediately that $S_z := |V|^{\frac12} R_0(z) : L^2 \to L^2$ for every $z \in \mathbb{C}\setminus\mathbb{R}$. Indeed, by the resolvent identity,

$$S_z S_z^* = \frac{1}{-2i\Im z}\big[|V|^{\frac12} R_0(z)|V|^{\frac12} - |V|^{\frac12} R_0(\bar z)|V|^{\frac12}\big].$$

In view of (2.6) therefore

$$\|S_z\|^2 = \|S_z S_z^*\| \lesssim \frac{1}{|\Im z|}\|K(z)\| \lesssim \frac{1}{|\Im z|}\|V\|_R, \tag{2.7}$$

as desired. One of the main observations of Kato [Ka] was the relation between this pointwise condition in $z = \lambda \pm i\varepsilon$ and the L^2_λ boundedness that appears in (2.3). We present a short proof of this fact for the sake of completeness. Although it is standard, the following argument is somewhat different from the usual one which can be found in basic references like Kato [Ka] and Reed, Simon [RS]. Denote $T_\varepsilon := |V|^{\frac12} R_0(\lambda + i\varepsilon)$ for $\varepsilon > 0$. Truncating the large values of V and then passing to the limit we may assume that V is bounded. Then $T_\varepsilon : L^2 \to L^2_\lambda L^2_x$ for every $\varepsilon > 0$ and one checks that

$$T^* F = \int R_0(\lambda - i\varepsilon)|V|^{\frac12} F(\lambda)\, d\lambda$$

for every $F \in L^2_\lambda(L^2_x)$. Thus

$$\begin{aligned} T_\varepsilon T_\varepsilon^* F &= \int |V|^{\frac12} R_0(\lambda+i\varepsilon) R_0(\mu-i\varepsilon)|V|^{\frac12} F(\mu)\, d\mu \\ &= -\int |V|^{\frac12} \frac{R_0(\lambda+i\varepsilon)-R_0(\mu-i\varepsilon)}{\lambda-\mu+2i\varepsilon}|V|^{\frac12} F(\mu)\, d\mu \qquad (2.8)\\ &= -\int \frac{K(\lambda+i\varepsilon)F(\mu)}{\lambda-\mu+2i\varepsilon}\, d\mu + \int \frac{K(\mu-i\varepsilon)F(\mu)}{\lambda-\mu+2i\varepsilon}\, d\mu, \qquad (2.9) \end{aligned}$$

where we used the resolvent identity to pass to (2.8). By the L^2 boundedness of the (vector valued) Hilbert transform,

$$\sup_{\varepsilon>0} \Big\| \int \frac{F(\mu)}{\lambda+i\varepsilon-\mu}\, d\mu \Big\|_{L^2_\lambda L^2_x} \lesssim \|F\|_{L^2_\lambda(L^2_x)}.$$

Using this bound and (2.6) in (2.9) yields

$$\sup_{\varepsilon>0} \|T_\varepsilon T_\varepsilon^* F\|_{L^2_x} \lesssim \|F\|_{L^2_\lambda L^2_x} \|V\|_R$$

which implies (2.3) with a constant of the form $C\|V\|_R^{\frac12}$.

Lemma 2.2. *Let $\|V\|_R < 4\pi$ as in Definition 2.1. Then for all $f, g \in L^2(\mathbb{R}^3)$*

$$\langle R_V(\lambda \pm i\varepsilon) f,\, g\rangle - \langle R_0(\lambda \pm i\varepsilon) f,\, g\rangle = \sum_{\ell=1}^{\infty} (-1)^\ell \langle R_0(\lambda \pm i\varepsilon)(V R_0(\lambda \pm i\varepsilon))^\ell f, g\rangle \qquad (2.10)$$

where the right-hand side of (2.10) *is an absolutely convergent series in the norm of $L^1(d\lambda)$ uniformly in $\varepsilon > 0$. Furthermore, if $\|V - V_m\|_R \to 0$ as $m \to \infty$, then*

$$\sup_{\varepsilon>0} \int \Big| \langle R_{V_m}(\lambda \pm i\varepsilon) f,\, g\rangle - \langle R_V(\lambda \pm i\varepsilon) f,\, g\rangle \Big|\, d\lambda \to 0 \qquad (2.11)$$

as $m \to \infty$.

Proof. We start from the resolvent identity

$$R_V(z) - R_0(z) = -R_0(z) V R_V(z) = -R_V(z) V R_0(z) \qquad (2.12)$$

which holds in the sense of bounded operators on L^2 for any $\Im z \neq 0$, see (2.7). It is a standard fact, see [Ka], that the Kato smoothing property (2.3) remains valid with R_V instead of R_0 provided that $\|V\|_R < 4\pi$. Indeed, multiplying (2.12) by $|V|^{\frac12}$ leads to

$$(1 + Q(z)) A R_V(z) = A R_0(z) \qquad (2.13)$$

where $Q(z) := A R_0(z) B$, $A = |V|^{\frac12}$, and $B = |V|^{\frac12}\mathrm{sign} V$. In view of (2.6) one has

$$\sup_{\Im z \neq 0} \|Q(z)\|_{L^2 \to L^2} =: \rho < 1 \quad \text{so that} \quad \sup_{\Im z \neq 0} \|(1 + Q(z))^{-1}\|_{L^2 \to L^2} \leq (1-\rho)^{-1}. \qquad (2.14)$$

In conjunction with (2.13) and (2.3) this implies that

$$\sup_{\varepsilon>0} \| A R_V(\lambda \pm i\varepsilon) f\|_{L^2_\lambda L^2_x} \le C\|f\|_{L^2}, \qquad \sup_{\varepsilon>0} \| R_V(\lambda \pm i\varepsilon)\, B f\|_{L^2_\lambda L^2_x} \le C\|f\|_{L^2} \tag{2.15}$$

for any $f \in L^2$. Fix $f, g \in L^2$. Iterating (2.12) leads to
(2.16)

$$\langle R_V(\lambda\pm i\varepsilon)f,\, g\rangle = \sum_{\ell=0}^{N} (-1)^\ell \langle R_0(\lambda\pm i\varepsilon)(VR_0(\lambda\pm i\varepsilon))^\ell f, g\rangle + (-1)^{N+1}\langle R_V(\lambda\pm i\varepsilon)(VR_0(\lambda\pm i\varepsilon))^{N+1} f, g\rangle$$

for any positive integer N. By (2.15) the error term is

$$\langle R_V(\lambda \pm i\varepsilon) B (A R_0(\lambda \pm i\varepsilon) B)^N A R_0(\lambda \pm i\varepsilon) f, g\rangle = \langle R_V(\lambda \pm i\varepsilon) B Q(\lambda \pm i\varepsilon)^N A R_0(\lambda \pm i\varepsilon) f, g\rangle$$

and thus has $L^1(d\lambda)$ norm bounded by $C\,\rho^N$, see (2.15) and (2.14). Similarly, each of the terms in the sum for $1 \le \ell \le N$ has $L^1(d\lambda)$ norm at most $C\,\rho^{\ell-1}$. Thus (2.10) holds for any V which satisfies (2.1). If m is sufficiently large, then the series expansion (2.10) holds for both V and V_m. Subtracting these series termwise and invoking the previous bounds yields that the left-hand side of (2.11) is bounded by

$$\sum_{\ell=1}^{\infty} C\ell\rho^{\ell-1}\, \|V - V_m\|_R \le C(1-\rho)^{-2}\, \|V - V_m\|_R,$$

and the lemma follows. □

The following technical corollary deals with the case $\varepsilon = 0$ in Lemma 2.2. We state it in the form in which it is used later on. In particular, we did not strive for the greatest generality. Below $C^0_b(\mathbb{R})$ refers to the bounded continuous functions on $\mathbb{R}$ with the supremum norm.

Corollary 2.3. *Let $V \in C^\infty_0(\mathbb{R}^3)$ satisfy $\|V\|_R < 4\pi$. Then for all $f, g \in C^\infty_0(\mathbb{R}^3)$ the limit*

$$\langle R_V(\lambda + i0)f, g\rangle = \lim_{\varepsilon\to 0} \langle R_V(\lambda + i\varepsilon)f, g\rangle$$

exists for every $\lambda \in \mathbb{R}$ and is a continuous function in λ. Moreover, for each λ one can pass to the limit $\varepsilon \to 0$ in all other terms in (2.10) *and*

$$\langle R_V(\lambda + i0)f,\, g\rangle - \langle R_0(\lambda + i0)f,\, g\rangle = \sum_{\ell=1}^{\infty} (-1)^\ell \langle R_0(\lambda + i0)(VR_0(\lambda + i0))^\ell f, g\rangle \tag{2.17}$$

holds for every λ and the series converges absolutely in the norm of $C^0_b(\mathbb{R}) \cap L^1(d\lambda)$.

Proof. Fix $f, g \in C^\infty_0(\mathbb{R}^3)$. By our assumptions on V and the explicit representation (2.5), $VR_0(z)f \in C^\infty_0$, and thus also $R_0(z)(VR_0(z))^\ell f$ for every $z \in \mathbb{C}$ with $\Im z \ge 0$. Moreover, $z \mapsto \langle R_0(z)(VR_0(z))^\ell f, g\rangle$ is a continuous function in $\Im z \ge 0$ for every $\ell \ge 0$. As in the previous proof one obtains the Kato smoothing bound

$$\sup_{\varepsilon\ge 0} \int \Big|\langle R_0(\lambda + i\varepsilon)(VR_0(\lambda + i\varepsilon))^\ell f, g\rangle\Big|\, d\lambda \le C(\|V\|_R/4\pi)^{\ell-1} \|f\|_{L^2_x} \|g\|_{L^2_x} \tag{2.18}$$

for each $\ell \geq 1$ (note that the case $\varepsilon = 0$ is included here). Moreover, see (2.4) and (2.6),

$$\begin{aligned}
\sup_{\Im z \geq 0} \Big| \langle R_0(z)(VR_0(z))^\ell f, g\rangle \Big| &\leq \sup_{\Im z\geq 0} \| \, |V|^{\frac12} R_0(z) g\|_2 \, \|K(z)\|^{\ell-1} \, \| \, |V|^{\frac12} R_0(z) f\|_2 \\
&\leq C(f,g,V) \, (\|V\|_R/4\pi)^{\ell-1}.
\end{aligned}$$

This implies that

$$S_{f,g}(\lambda) := \sum_{\ell=0}^{\infty} \langle R_0(\lambda+i0)(VR_0(\lambda+i0))^\ell f, g\rangle, \tag{2.19}$$

converges uniformly and thus defines a continuous function. Furthermore, one concludes that the series in (2.10) converges uniformly in the closed upper half-plane (i.e., for all $\lambda \in \mathbb{R}$ and $\varepsilon \geq 0$) and therefore defines the limit $\langle R_V(\lambda+i0)f, g\rangle$ pointwise in $\lambda \in \mathbb{R}$. Also note that, by (2.18), the series for $S_{f,g}(\lambda) - \langle R_0(\lambda+i0)f, g\rangle$ converges absolutely in $L^1(d\lambda)$, and similarly for every $z \in \mathbb{C}$ with $\Im z \geq 0$. In view of (2.10), (2.18), and with an arbitrary $N \geq 1$,

$$\begin{aligned}
& \int \liminf_{\varepsilon\to 0} |\langle R_V(\lambda+i\varepsilon)f,\, g\rangle - S_{f,g}(\lambda)| \, d\lambda \\
\leq \; & \int \liminf_{\varepsilon\to 0} |\langle S_{f,g}(\lambda+i\varepsilon)f,\, g\rangle - S_{f,g}(\lambda)| \, d\lambda \\
\leq \; & \int \sum_{\ell=1}^{N} \limsup_{\varepsilon\to 0} \Big| \langle R_0(\lambda+i\varepsilon)(VR_0(\lambda+i\varepsilon))^\ell f, g\rangle - \langle R_0(\lambda+i0)(VR_0(\lambda+i0))^\ell f, g\rangle \Big| \, d\lambda \\
& \qquad + C \sum_{\ell=N+1}^{\infty} (\|V\|_R/4\pi)^{\ell-1} \|f\|_2 \|g\|_2 \\
\leq \; & C(1-\|V\|_R/4\pi)^{-1} \, (\|V\|_R/4\pi)^N \|f\|_2\|g\|_2,
\end{aligned}$$

and we are done. □

Next we turn to a simple lemma that is basically an instance of stationary phase.

Lemma 2.4. *Let ψ be a smooth, even bump function with $\psi(\lambda) = 1$ for $-1 \leq \lambda \leq 1$ and $\mathrm{supp}(\psi) \subset [-2,2]$. Then for all $t \geq 1$ and any real a,*

$$\sup_{L\geq 1} \Big| \int_0^\infty e^{it\lambda} \sin(a\sqrt{\lambda}) \, \psi(\frac{\sqrt{\lambda}}{L}) \, d\lambda \Big| \leq C \, t^{-\frac32} \, |a| \tag{2.20}$$

where C only depends on ψ.

Proof. Denote the integral in (2.20) by $I_L(a,t)$. Clearly, $I_L(a,t)$ is a smooth function of a, t for any $L > 0$ and $I_L(0,t) = 0$. The change of variables $\lambda \to \lambda^2$ leads to the expression

$$I_L(a,t) = 2\int_0^\infty \lambda \, e^{it\lambda^2} \sin(a\lambda) \, \psi(\lambda/L) \, d\lambda$$

Integrating by parts we obtain

$$I_L(a,t) = \frac{i}{t} \int_0^\infty e^{it\lambda^2} \Big(a \, \cos(a\lambda) \, \psi(\lambda/L) + \frac{1}{L} \sin(a\lambda) \, \psi'(\lambda/L) \Big) \, d\lambda.$$

Since ψ is assumed to be even, ψ' is odd. Hence,

$$
\begin{aligned}
I_L(a,t) &= \frac{i}{2t}\int_{-\infty}^{\infty} e^{it\lambda^2}\Big(a\,\cos(a\lambda)\,\psi(\lambda/L) + \frac{1}{L}\sin(a\lambda)\,\psi'(\lambda/L)\Big)\,d\lambda \\
&= \frac{a}{4t}\,i\int_{-\infty}^{\infty} e^{it\lambda^2}\big(\,e^{ia\lambda}+e^{-ia\lambda}\big)\,\psi(\lambda/L)\,d\lambda + \int_0^a \frac{i}{4t}\int_{-\infty}^{\infty} e^{it\lambda^2}\big(\,e^{ib\lambda}+e^{-ib\lambda}\big)\,\frac{\lambda}{L}\,\psi'(\lambda/L)\,d\lambda\,db.
\end{aligned}
$$

Thus it suffices to show that

$$
J_L(a,t) = \int_{-\infty}^{\infty} e^{i(t\lambda^2+a\lambda)}\,\phi(\lambda/L)\,d\lambda
$$

obeys the estimate $|J_L(a\,t)| \le Ct^{-\frac{1}{2}}$ for any smooth bump function ϕ satisfying the same properties as ψ. The change of variables $\lambda \to \lambda/L$ further reduces the problem to the estimate $|J(a,t)| \le Ct^{-\frac{1}{2}}$ with

$$
J(a,t) = \int_{-\infty}^{\infty} e^{i(t\lambda^2+a\lambda)}\,\phi(\lambda)\,d\lambda
$$

for all $t \ne 0$ and all real a. Observe that $J(a,t)$ is a smooth solution of the 1-dimensional Schrödinger equation

$$
\begin{aligned}
i\,\frac{\partial}{\partial t}J(a,t) - \frac{\partial^2}{\partial a^2}J(a,t) &= 0, \\
J(a,0) &= \int_{-\infty}^{\infty} e^{-ia\lambda}\phi(\lambda)\,d\lambda.
\end{aligned}
$$

By the explicit representation of the kernel of the fundamental solution

$$
J(a,t) = (-4\pi i t)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-i\frac{|a-b|^2}{4t}}\,J(b,0)\,db
$$

which implies that $J(a,t)$ obeys the standard one-dimensional decay estimate

$$
|J(a,t)| \le Ct^{-\frac{1}{2}}\|J(\cdot,0)\|_{L^1}.
$$

Since the function $J(a,0)$ is the Fourier transform of the smooth bump function ϕ, the desired estimate on $J(a,t)$ follows. $\square$

The following lemma explains to some extent why condition (2.2) is needed. Iterated integrals as in (2.21) will appear in a series expansion of the spectral resolution of $H = -\triangle + V$.

Lemma 2.5. *For any positive integer k and V as in Definition 2.1*

$$
\sup_{x_0,x_{k+1}\in\mathbb{R}^3}\int_{\mathbb{R}^{3k}} \frac{\prod_{j=1}^k |V(x_j)|}{\prod_{j=0}^k |x_j-x_{j+1}|}\sum_{\ell=0}^k |x_\ell - x_{\ell+1}|\;dx_1\ldots dx_k \le (k+1)\|V\|_{\mathcal{K}}^k. \tag{2.21}
$$

Proof. Define the operator $\mathcal{A}$ by the formula

$$
\mathcal{A}f(x) = \int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\,f(y)\,dy.
$$

Observe that the assumption (2.2) on the potential V implies that $\mathcal{A}: L^\infty \to L^\infty$ and $\|\mathcal{A}\|_{L^\infty \to L^\infty} \leq c_0$ where we have set $c_0 := \|V\|_{\mathcal{K}}$ for convenience. Denote by $<,>$ the standard L^2 pairing. In this notation the estimate (2.21) is equivalent to proving that the operators $\mathcal{B}_k$ defined as

$$\mathcal{B}_k f = \sum_{m=0}^{k} < f, \mathcal{A}^{k-m} 1 > \mathcal{A}^m 1$$

are bounded as operators from $L^1 \to L^\infty$ with the bound

$$\|\mathcal{B}_k\|_{L^1 \to L^\infty} \leq (k+1) c_0^k.$$

For arbitrary $f \in L^1$ one has

$$\begin{aligned}\|\mathcal{B}_k f\|_{L^\infty} \leq \sum_{m=0}^{k} | < f, \mathcal{A}^{k-m} 1 > | \, \|\mathcal{A}^m 1\|_{L^\infty} &\leq \sum_{m=0}^{k} \|\mathcal{A}^{k-m}\|_{L^\infty \to L^\infty} \|\mathcal{A}^m\|_{L^\infty \to L^\infty} \|f\|_{L^1} \\ &\leq \sum_{m=0}^{k} c_0^k \|f\|_{L^1} \leq (k+1) c_0^k \|f\|_{L^1},\end{aligned}$$

as claimed. □

We are now in a position to prove the main result of this section.

Theorem 2.6. *With $H = -\triangle + V$ and V satisfying the conditions in Definition 2.1 one has the bound*

$$\left\| e^{itH} \right\|_{L^1 \to L^\infty} \leq C\, t^{-\frac{3}{2}}$$

in three dimensions.

Proof. Let ψ be a smooth cut-off function as in Lemma 2.4. We will show that there is an absolute constant C such that

$$\sup_{L \geq 1} \left| \left\langle e^{itH} \psi(\sqrt{H}/L) f, g \right\rangle \right| \leq C t^{-\frac{3}{2}} \|f\|_1 \|g\|_1 \tag{2.22}$$

for any $f, g \in C_0^\infty(\mathbb{R}^3)$, which proves the theorem. It will be convenient to assume that the potential V belongs to $C_0^\infty(\mathbb{R}^3)$, in addition to satisfying (2.1) and (2.2). In case of a general potential V as in Definition 2.1, one approximates V by $V_j \in C_0^\infty$ via the usual cut-off and mollifying process. Clearly, $\|V - V_j\|_R \to 0$ as $j \to \infty$ and $\|V_j\|_{\mathcal{K}} \leq \|V\|_{\mathcal{K}} < 4\pi$. Since the spectral resolution E_V of H satisfies (recall that the spectrum of H is purely absolutely continuous)

$$E_V'(\lambda) := \frac{d}{d\lambda} E_V(\lambda) = \Im R_V(\lambda + i0), \tag{2.23}$$

one concludes from Lemma 2.2 that

$$\int \left| \langle E_V'(\lambda) f, g \rangle - \langle E_{V_j}'(\lambda) f, g \rangle \right| d\lambda \to 0$$

as $j \to \infty$. In particular, with $H_j := -\triangle + V_j$,

$$\left\langle e^{itH_j}\psi(H_j/L)f, g\right\rangle \to \left\langle e^{itH}\psi(H/L)f, g\right\rangle$$

as $j \to \infty$ for any $f, g \in C_0^\infty(\mathbb{R}^3)$. It therefore suffices to prove (2.22) under the additional assumption that $V \in C_0^\infty(\mathbb{R}^3)$. Fix such a potential V, as well as any $L \geq 1$, and real $f, g \in C_0^\infty(\mathbb{R}^3)$. Then applying (2.23), Corollary 2.3, (2.5), Lemma 2.4, and Lemma 2.5 in this order,

$$\begin{aligned}
& \sup_{L\geq 1}\Big|\left\langle e^{itH}\psi(\sqrt{H}/L)f, g\right\rangle\Big| \\
\leq\ & \sup_{L\geq 1}\Big|\int_0^\infty e^{it\lambda}\ \psi(\sqrt{\lambda}/L)\langle E'(\lambda)f, g\rangle\, d\lambda\Big| \\
=\ & \sup_{L\geq 1}\Big|\int_0^\infty e^{it\lambda}\ \psi(\sqrt{\lambda}/L)\Im\langle R_V(\lambda+i0)f, g\rangle\, d\lambda\Big| \\
=\ & \sup_{L\geq 1}\Big|\int_0^\infty e^{it\lambda}\ \psi(\sqrt{\lambda}/L)\sum_{k=0}^\infty \Im\langle R_0(\lambda+i0)(VR_0(\lambda+i0))^k\, f, g\rangle\, d\lambda\Big| \\
\leq\ & \sum_{k=0}^\infty \int_{\mathbb{R}^6} |f(x_0)||g(x_{k+1})| \int_{\mathbb{R}^{3k}} \frac{\prod_{j=1}^k |V(x_j)|}{\prod_{j=0}^k 4\pi|x_j - x_{j+1}|}\cdot \\
& \qquad \cdot \sup_{L\geq 1}\Big|\int_0^\infty e^{it\lambda}\ \psi(\sqrt{\lambda}/L)\sin\Big(\sqrt{\lambda}\sum_{\ell=0}^k |x_\ell - x_{\ell+1}|\Big)\, d\lambda\Big|\ d(x_1,\ldots,x_k)\, dx_0\, dx_{k+1} \qquad (2.24) \\
\leq\ & Ct^{-\frac{3}{2}}\sum_{k=0}^\infty \int_{\mathbb{R}^6} |f(x_0)||g(x_{k+1})| \int_{\mathbb{R}^{3k}} \frac{\prod_{j=1}^k |V(x_j)|}{(4\pi)^{k+1}\prod_{j=0}^k |x_j - x_{j+1}|}\sum_{\ell=0}^k |x_\ell - x_{\ell+1}|\ d(x_1,\ldots,x_k)\ dx_0\, dx_{k+1} \\
\leq\ & Ct^{-\frac{3}{2}}\sum_{k=0}^\infty \int_{\mathbb{R}^6} |f(x_0)||g(x_{k+1})|\ (k+1)(\|V\|_{\mathcal{K}}/4\pi)^k\ dx_0\, dx_{k+1} \\
\leq\ & Ct^{-\frac{3}{2}}\|f\|_1\|g\|_1,
\end{aligned}$$

since $\|V\|_{\mathcal{K}} < 4\pi$. In order to pass to (2.24) one uses the explicit representation of the kernel of $R_0(\lambda + i0)$, see (2.5), which leads to a k-fold integral. Next, one interchanges the order of integration in this iterated integral. This is legitimate, since the corresponding L^1-integral (i.e., with absolute values on everything) is finite (V is bounded and compactly supported). The theorem follows. $\square$

3 The high energy case in $\mathbb{R}^3$ with an ϵ loss

The purpose of this section is to prove a dispersive inequality for $e^{itH}\chi(H)P_{a.c}$ where χ is a cut-off to large energies and $P_{a.c}$ is the projection onto the absolutely continuous part of $L^2(\mathbb{R}^3)$ with respect to $H = -\triangle + V$. We will assume that V satisfies the following properties:

$$\|\!|V|\!\| := \|V\|_2 + \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\, dy < \infty. \qquad (3.1)$$

Under these conditions we will prove the following result. As usual, we let $\chi \in C^\infty$ with $\chi(\lambda) = 0$ if $\lambda \leq 1$ and $\chi(\lambda) = 1$ for $\lambda \geq 2$.

Proposition 3.1. *Let $\|\!|V|\!\| < \infty$ as in* (3.1). *Then for every $\varepsilon > 0$ there exists some positive $\lambda_0 = \lambda_0(\|\!|V|\!\|, \varepsilon)$ so that*

$$\|e^{itH}\chi(H/\lambda_0)P_{a.c.}\|_{L^1_x\to L^\infty_x} \le Ct^{-\frac{3}{2}+\varepsilon} \tag{3.2}$$

for all $t > 0$.

Previously, convergence of the Born series was guaranteed by a smallness assumption on the potential V. The following lemma will allow us to sum the Born series for large potentials in $L^2(\mathbb{R}^3)$, but only for large energies. This lemma is an immediate consequence of the Stein-Tomas theorem in the formulation due to Stein [St].

Lemma 3.2. *Let $R_0(z) = (-\triangle - z)^{-1}$ for $\Im(z) > 0$ be the resolvent of the free Laplacean. Then there is an absolute constant C so that for any $\lambda > 0$*

$$\|R_0(\lambda + i0)f\|_{L^4(\mathbb{R}^3)} \le C\lambda^{-\frac{1}{4}}\|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \tag{3.3}$$

for all $f \in \mathcal{S}$.

Proof. It is well-known that the resolvent $R_0(z) = (-\triangle - z)^{-1}$ for $\Im(z) > 0$ has the kernel

$$K_0(z)(x, y) = \frac{\exp(i\sqrt{z}|x - y|)}{4\pi|x - y|} \tag{3.4}$$

where $\Im(\sqrt{z}) > 0$. By the Stein-Tomas theorem in Stein's version [St] one has

$$\Big\| \int_{\mathbb{R}^3} \frac{\exp(i|x - y|)}{4\pi|x - y|} f(y)\, dy\Big\|_{L^4(\mathbb{R}^3)} \le C\|f\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}. \tag{3.5}$$

Passing to (3.3) only requires changing variables $x \mapsto \sqrt{\lambda}x$ and $y \mapsto \sqrt{\lambda}y$, which we skip. □

It is well-known, see Simon [Si2] Theorem A.2.9, that for any $V \in L^2_{loc}(\mathbb{R}^3)$ that satisfies the so called *Kato condition*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^3} \int_{|x-y|<r} \frac{|V(y)|}{|x - y|}\, dy = 0, \tag{3.6}$$

the operator $-\triangle + V$ with domain $C_0^\infty(\mathbb{R}^3)$ is essentially self-adjoint with $\mathrm{sp}(H) \subset [-M, \infty)$ for some $0 < M < \infty$, and that H is the generator of a semi group e^{-tH} that is bounded from L^p to L^q for any choice of $1 \le p \le q \le \infty$, see Theorem B.1.1 in [Si2]. Moreover, explicit bounds for these norms are of the form

$$\|e^{-tH}\|_{L^p_x\to L^q_x} \le C\,t^{-\gamma}\,e^{At}$$

with $\gamma = \frac{3}{2}(p^{-1} - q^{-1})$ and any $A > M$ with M as before, see (B11) in [Si2]. These bounds imply the Sobolev inequalities

$$\|(H + 2M)^{-\beta}\|_{L^p_x\to L^q_x} < \infty \;\text{ for any } 1 \le p, q \le \infty \text{ and with } \beta > \frac{3}{2}(p^{-1} - q^{-1}), \tag{3.7}$$

as can be seen from writing $(H + 2M)^{-\beta}$ as the Laplace transform of the heat-semigroup, see Theorem B.2.1. in [Si2]. Since we are assuming that $V \in L^2$, Cauchy-Schwarz implies that (3.6) holds, and thus so do all aforementioned properties. In addition, we will use the following result of Jensen and Nakamura, see [JN] Theorem 2.1: Suppose that $V \in L^2_{loc}$ satisfies (3.6). Let $g \in C_0^\infty(\mathbb{R})$ and $1 \le p \le \infty$. Then there exists a constant C such that

$$\|g(\theta H)\|_{L^p_x \to L^p_x} \le C \quad \text{uniformly in } 0 < \theta \le 1. \tag{3.8}$$

Moreover, the constant C is uniform for g ranging over bounded sets of $C_0^\infty(\mathbb{R})$. As an immediate corollary of (3.7) and (3.8) one obtains that for any $g \in C_0^\infty(\mathbb{R})$, any $1 \le p \le q \le \infty$, $\beta > \frac{3}{2}(p^{-1} - q^{-1})$, there is a constant C depending on g, V, and β, such that

$$\|g(H/\lambda_0)\|_{L^p_x \to L^q_x} \le C\lambda_0^\beta \quad \text{uniformly in } \lambda_0 \ge 1. \tag{3.9}$$

This bound is needed for the following lemma. Recall that $R_0(z)$ denotes the resolvent of the free Laplacean.

Lemma 3.3. *Let $\eta \in C_0^\infty(\mathbb{R})$ be fixed. Then for any $\lambda, \lambda_0 \ge 1$ and any nonnegative integer k one has the estimate*

$$\|\eta(H/\lambda_0) R_0(\lambda + i0)(VR_0(\lambda + i0))^k \eta(H/\lambda_0)\|_{L^1_x \to L^\infty_x} \le C\lambda_0^{\frac{3}{4}+} \lambda^{-\frac{1}{4}} (\|V\|_2 \lambda^{-\frac{1}{4}})^k$$

where the constant C only depends on g and V.

Proof. By Lemma 3.2 and Hölder's inequality,

$$\|VR_0(\lambda + i0)f\|_{L_x^{\frac{4}{3}}} \le \|R_0(\lambda + i0)f\|_{L^4_x} \|V\|_{L^2_x} \le C \|V\|_{L^2_x} \lambda^{-\frac{1}{4}} \|f\|_{L_x^{\frac{4}{3}}} \tag{3.10}$$

for any $f \in \mathcal{S}$. Hence

$$\begin{aligned}
&\|\eta(H/\lambda_0) R_0(\lambda + i0)(VR_0(\lambda + i0))^k \, \eta(H/\lambda_0)\|_{L^1_x \to L^\infty_x} \\
&\le \|\eta(H/\lambda_0) R_0(\lambda + i0)\|_{L_x^{\frac{4}{3}} \to L^\infty_x} \, \|(VR_0(\lambda + i0))^k\|_{L_x^{\frac{4}{3}} \to L_x^{\frac{4}{3}}} \, \|\eta(H/\lambda_0)\|_{L^1_x \to L_x^{\frac{4}{3}}} \\
&\le \|\eta(H/\lambda_0)\|_{L^4_x \to L^\infty_x} \|R_0(\lambda + i0)\|_{L_x^{\frac{4}{3}} \to L^4_x} \, \|VR_0(\lambda + i0)\|^k_{L_x^{\frac{4}{3}} \to L_x^{\frac{4}{3}}} \, \|\eta(H/\lambda_0)\|_{L^1_x \to L_x^{\frac{4}{3}}} \\
&\le C\lambda_0^{\frac{3}{8}+} \lambda^{-\frac{1}{4}} (\|V\|_2 \lambda^{-\frac{1}{4}})^k \, \lambda_0^{\frac{3}{8}+},
\end{aligned}$$

as claimed. ∎

Proof of Proposition 3.1. We start with a justification of the Born series expansion for high energies. Let $\lambda_0 > 0$ be chosen so that $\|V\|_{L^2_x} \lambda_0^{-\frac{1}{4}} < 1$. By (3.10), the operator $1 + VR_0(\lambda + i0)$ is invertible in $L^{\frac{4}{3}}(\mathbb{R}^3)$ provided $\lambda > \lambda_0$ and the Neumann series

$$(1 + VR_0(\lambda + i0))^{-1} = \sum_{k=0}^{\infty} (-1)^k (VR_0(\lambda + i0))^k \tag{3.11}$$

converges in $L^{\frac{4}{3}}(\mathbb{R}^3)$. Therefore, the resolvent $R_V(z) := (-\triangle + V - z)^{-1}$ satisfies

$$R_V(\lambda + i0) = R_0(\lambda + i0)(1 + VR_0(\lambda + i0))^{-1} = \sum_{k=0}^{\infty} (-1)^k R_0(\lambda + i0)(VR_0(\lambda + i0))^k$$

for all $\lambda > \lambda_0$ and is thus a bounded operator from $L^{\frac{4}{3}}(\mathbb{R}^3) \to L^4(\mathbb{R}^3)$. Furthermore, since the spectral resolution $E(\cdot)$ of $H = -\triangle + V$ satisfies $P_{a.c.}\, E(d\lambda) = \Im R_V(\lambda + i0)\, d\lambda$, one has

$$\langle P_{a.c.} E(d\lambda) f, g\rangle = \sum_{k=0}^{\infty} (-1)^k \langle \Im[R_0(\lambda+i0)(VR_0(\lambda+i0))^k] f, g\rangle \, d\lambda$$

for any $f, g \in L^{\frac{4}{3}}(\mathbb{R}^3)$. Now define $\eta(\lambda) := \chi(\lambda) - \chi(\lambda/2)$. Clearly, $\eta \in C_0^\infty(\mathbb{R})$, and also

$$\sum_{j=0}^{\infty} \eta(\lambda 2^{-j}) = \chi(\lambda) \quad \text{for all } \lambda.$$

Observe that at most three terms in this sum can be nonzero for any given λ. Now let $\tilde\eta \in C_0^\infty(0,\infty)$ have the property that $\eta\tilde\eta = 1$. Then for any $f, g \in \mathcal{S}$, one has the expansion

$$\begin{aligned}
& |\langle e^{itH}\chi(H/\lambda_0) f, \chi(H/\lambda_0) g\rangle| \\
= \;& \left| \int_0^\infty e^{it\lambda} \langle E(d\lambda)\eta(H/(2^j\lambda_0)) f, \eta(H/(2^\ell \lambda_0)) g\rangle \, d\lambda \right| \\
\le \;& \left| \sum_{\substack{j,\ell=0 \\ |j-\ell|\le 1}}^{\infty} \int_0^\infty e^{it\lambda} \langle E(d\lambda)\eta(H/(2^j\lambda_0)) f, \eta(H/(2^\ell\lambda_0)) g\rangle \, \tilde\eta(\lambda/(2^j\lambda_0))\, d\lambda \right| \\
\le \;& \sum_{k=0}^{\infty} \Bigg| \sum_{\substack{j,\ell=0 \\ |j-\ell|\le 1}}^{\infty} \int_0^\infty e^{it\lambda} \langle R_0(\lambda+i0)\,(VR_0(\lambda+i0))^k \eta(H/(2^j\lambda_0)) f, \eta(H/(2^\ell\lambda_0)) g\rangle \\
& \qquad\qquad\qquad\qquad\qquad\qquad \tilde\eta(\lambda/(2^j\lambda_0))\, d\lambda \Bigg| \qquad (3.12) \\
= \;& \sum_{k=0}^{\infty} \left| \int_0^\infty e^{it\lambda} \langle R_0(\lambda+i0)\,(VR_0(\lambda+i0))^k \chi(H/\lambda_0) f, \chi(H/\lambda_0) g\rangle \, d\lambda \right|. \qquad (3.13)
\end{aligned}$$

From the previous section one has the dispersive bounds

$$\left| \int_0^\infty e^{it\lambda} \langle R_0(\lambda+i0)\,(VR_0(\lambda+i0))^k \chi(H/\lambda_0) f, \chi(H/\lambda_0) g\rangle \, d\lambda \right| \le C\, t^{-\frac{3}{2}} \|\!| V |\!\|^k \, \|f\|_{L^1_x} \, \|g\|_{L^1_x}, \qquad (3.14)$$

$$\begin{aligned}
& \left| \int_0^\infty e^{it\lambda} \langle R_0(\lambda+i0)\,(VR_0(\lambda+i0))^k \eta(H/(2^j\lambda_0)) f, \eta(H/(2^\ell\lambda_0)) g\rangle \, \tilde\eta(\lambda/(2^j\lambda_0))\, d\lambda \right| \\
& \le C\, t^{-\frac{3}{2}} \|\!| V |\!\|^k \, \|f\|_{L^1_x} \, \|g\|_{L^1_x}, \qquad (3.15)
\end{aligned}$$

where we have also used (3.9) to remove the χ and η cutoffs. On the other hand, Lemma 3.3 shows that

$$\begin{aligned}
& \left| \int_0^\infty e^{it\lambda} \langle R_0(\lambda+i0)\,(VR_0(\lambda+i0))^k \eta(H/(2^j\lambda_0)) f, \eta(H/(2^\ell\lambda_0)) g\rangle \, \tilde\eta(\lambda/(2^j\lambda_0))\, d\lambda \right| \\
\le \;& C\, (2^j\lambda_0)^{\frac{3}{2}+} (\|V\|_{L^2_x} (2^j\lambda_0)^{-\frac{1}{4}})^k \|f\|_{L^1_x} \, \|g\|_{L^1_x}. \qquad (3.16)
\end{aligned}$$

Combining (3.15) and (3.16) yields that for any $0 < \theta < 1$

$$\sum_{k=7}^{\infty} \sum_{\substack{j,\ell=0 \\ |j-\ell|\le 1}}^{\infty} \left| \int_0^\infty e^{it\lambda} \langle R_0(\lambda+i0)\, (VR_0(\lambda+i0))^k \eta(H/(2^j\lambda_0))f, \eta(H/(2^\ell\lambda_0))g\rangle\, \tilde\eta(\lambda/(2^j\lambda_0))\, d\lambda \right|$$

$$\le C \sum_{j=0}^{\infty}\sum_{k=7}^{\infty} t^{-\frac{3}{2}(1-\theta)}\, |\!|\!| V|\!|\!|^{k(1-\theta)}\, (2^j\lambda_0)^{\theta\frac{3}{2}+}(\|V\|_{L^2_x}(2^j\lambda_0)^{-\frac14})^{\theta k}\|f\|_1\, \|g\|_1$$

$$\le C \sum_{k=7}^{\infty} t^{-\frac{3}{2}(1-\theta)}\, |\!|\!| V|\!|\!|^{k(1-\theta)}\, \lambda_0^{\theta\frac{3}{2}+}(\|V\|_2\lambda_0^{-\frac14})^{\theta k}\|f\|_{L^1_x}\, \|g\|_{L^1_x}$$

$$\le C\, t^{-\frac{3}{2}(1-\theta)}\lambda_0^{\theta\frac32+} \sum_{k=0}^{\infty} |\!|\!| V|\!|\!|^k \lambda_0^{-\frac{k}{4}\theta} \|f\|_1\, \|g\|_1 \le C\, t^{-\frac32(1-\theta)}\, \lambda_0^{\theta\frac32+}\|f\|_{L^1_x}\|g\|_{L^1_x} \tag{3.17}$$

provided $|\!|\!| V|\!|\!|\lambda_0^{-\frac{\theta}{4}} < 1$. The choice of $k\ge 7$ was made to ensure summability over j. The bound (3.17) yields the desired bounds for the terms with $k\ge 7$ in (3.12). For the remaining cases of k, one simply invokes the estimate (3.14), and the proposition follows. □

Remark 3.4. It seems clear that the condition $\|V\|_{L^2_x} < \infty$ can be weakened to a condition closer to $L^{\frac32}(\mathbb{R}^3)$. The reason for this is the "slack" in the Stein-Tomas bound that yields $\lambda^{-\frac14}$, whereas the high energies argument only requires $\lambda^{-\gamma}$ for some $\gamma > 0$. It appears that a complex interpolation argument allows one to exploit this slack, but we do not pursue this here.

4 Strichartz estimates for $(1+|x|^2)^{-1-\varepsilon}$ potentials

In this section we settle a problem posed by Journé, Soffer, Sogge [JSS] concerning Strichartz estimates for the solutions of the Schrödinger equation with potentials decaying at the rate of $|x|^{-2-\varepsilon}$ at infinity. To obtain the result we prove a more general statement relating an $L^q_tL^p_x$ estimate for the semigroup e^{itH_0} to the corresponding estimate for e^{itH} with $H = H_0 + V$. The conditions of the result involve the notion of Kato's smoothing for the multiplication operator $|V|^{\frac12}$ relative to H_0 and H. Applying the abstract result to $H_0 = -\Delta$, $H = -\Delta + V$ with V obeying the estimate $|V(x)| \le C(1+|x|^2)^{-1-\varepsilon}$ requires appealing to the Agmon-Kato-Kuroda theory on the absence of positive singular continuous spectrum for H and a separate argument that deals with the point 0 in the spectrum of H.

We start with the preliminaries. Consider a self-adjoint operator H_0 on $L^2(\mathbb{R}^n)$ with domain $\mathcal{D}(H_0)$. Let e^{itH_0} be the associated unitary semigroup, which is a solution operator for the Schrödinger equation

$$\frac1i \partial_t\psi - H_0\psi = 0, \qquad \psi|_{t=0} = \psi_0.$$

We denote by $R_0(z)$ the resolvent of H_0. For complex z with $\Im z > 0$ we have that

$$R_0(z) = \int_0^\infty e^{izt} e^{itH_0}\, dt \tag{4.1}$$

as well as the inverse: for any $\beta > 0$ and $t \geq 0$,

$$e^{-\beta t} e^{itH_0} = \int_{-\infty}^{\infty} e^{-it\lambda} R_0(\lambda + i\beta)\, d\lambda$$

Let A and B be a pair of bounded operators[1] on $L^2(\mathbb{R}^n)$ and consider a self-adjoint operator $H = H_0 + B^*A$ with domain $\mathcal{D}(H_0)$, corresponding semigroup e^{itH}, and the resolvent $R(z)$. The resolvent $R(z)$ and $R_0(z)$ for $\Im z \neq 0$ are connected via the second resolvent identity

$$R(z) = R_0(z) - R_0(z) B^* A R(z) \tag{4.2}$$

On the other hand, the semigroups e^{itH} and e^{itH_0} are related via the Duhamel formula

$$e^{itH}\psi_0 = e^{itH_0}\psi_0 - i \int_0^t e^{i(t-s)H_0} B^* A e^{isH}\psi_0\, ds. \tag{4.3}$$

which holds for any $\psi_0 \in L^2_x$. We recall that for a self-adjoint operator $\bar{H}$, an operator Γ is called $\bar{H}$-smooth in Kato's sense if for any $f \in \mathcal{D}(H_0)$

$$\|\Gamma e^{it\bar{H}} f\|_{L^2_t L^2_x} \leq C_\Gamma(\bar{H}) \|f\|_{L^2_x} \tag{4.4}$$

or equivalently, for any $f \in L^2_x$

$$\sup_{\beta > 0} \|\Gamma R_{\bar{H}}(\lambda \pm i\beta) f\|_{L^2_\lambda L^2_x} \leq C_\Gamma(\bar{H}) \|f\|_{L^2_x}. \tag{4.5}$$

We shall call $C_\Gamma(\bar{H})$ the smoothing bound of Γ relative to $\bar{H}$. Let $\Omega \subset \mathbb{R}$ and let P_Ω be a spectral projection of $\bar{H}$ associated with a set Ω. We say that Γ is $\bar{H}$-smooth on Ω if ΓP_Ω is $\bar{H}$-smooth. We denote the corresponding smoothing bound by $C_\Gamma(\bar{H}, \Omega)$. It is not difficult to show (see e.g. [RS]) that, equivalently, Γ is $\bar{H}$-smooth on Ω if

$$\sup_{\beta > 0, \lambda \in \Omega} \|\Gamma R_{\bar{H}}(\lambda \pm i\beta) f\|_{L^2_\lambda L^2_x} \leq C_\Gamma(\bar{H}, \Omega) \|f\|_{L^2_x}. \tag{4.6}$$

We now are ready to state the main result of this section.

Theorem 4.1. *Let H_0 and $H = H_0 + B^*A$ be as above. We assume that that B is H_0 smooth with a smoothing bound $C_B(H_0)$ and that for some $\Omega \subset \mathbb{R}$ the operator A is H-smooth on Ω with the smoothing bound $C_A(H, \Omega)$. Assume also that the unitary semigroup e^{itH_0} satisfies the estimate*

$$\|e^{itH_0}\psi_0\|_{L^q_t L^r_x} \leq C_{H_0} \|\psi_0\|_{L^2_x} \tag{4.7}$$

for some $q \in (2, \infty]$ and $r \in [1, \infty]$. Then the semigroup e^{itH} associated with $H = H_0 + B^*A$, restricted to the spectral set Ω, also verifies the estimate (4.7), i.e.,

$$\|e^{itH} P_\Omega \psi_0\|_{L^q_t L^r_x} \leq C_{H_0} C_B(H_0) C_A(H, \Omega) \|\psi_0\|_{L^2_x} \tag{4.8}$$

[1] The assumption of boundedness is a convenience that is sufficient for our main application below.

Proof. We start with the Duhamel formula (4.3)

$$e^{itH}\psi_0 = e^{itH_0}\psi_0 - i\int_0^t e^{i(t-s)H_0}B^*Ae^{isH}\psi_0\,ds.$$

We have the following estimate with the exponents q, r described in (4.7):

$$\|e^{itH}P_\Omega\psi_0\|_{L^q_tL^r_x} \le \|e^{itH_0}P_\Omega\psi_0\|_{L^q_tL^r_x} + \|\int_0^t e^{i(t-s)H_0}B^*Ae^{isH}P_\Omega\psi_0\,ds\|_{L^q_tL^r_x}$$

$$\le C_{H_0}\|\psi_0\|_{L^2_x} + \|\int_0^t e^{i(t-s)H_0}B^*Ae^{isH}P_\Omega\psi_0\,ds\|_{L^q_tL^r_x} \tag{4.9}$$

To handle the Duhamel term we recall the Christ-Kiselev lemma. The following version is from Sogge, Smith [SoSm]

Lemma 4.2 (CK). *Let X, Y be Banach spaces and let $K(t,s)$ be the kernel of the operator $K : L^p([0,T];X) \to L^q([0,T];Y)$. Denote by $\|K\|$ the operator norm of K. Define the lower diagonal operator $\tilde{K} : L^p([0,T];X) \to L^q([0,T];Y)$*

$$\tilde{K}f(t) = \int_0^t K(t,s)f(s)\,ds$$

Then the operator $\tilde{K}$ is bounded from $L^p([0,T];X)$ to $L^q([0,T];Y)$ and its norm $\|\tilde{K}\| \le c\|K\|$, provided that $p < q$.

We shall apply this lemma to the operator with kernel $K(t,s) = e^{i(t-s)H_0}B^*$ acting between the spaces $L^2([0,\infty);L^2_x)$ and $L^q([0,\infty);L^r_x)$. Observe that by the assumptions of Theorem 4.1, $q > 2$ and thus the condition $q > p$ in Lemma [CK] is verified.
We can rewrite the Duhamel term

$$D = \int_0^t e^{i(t-s)H_0}B^*Ae^{isH}P_\Omega\psi_0\,ds$$

in the form $D = \tilde{K}\left(Ae^{i\cdot H}P_\Omega\psi_0\right)$. Therefore,

$$\|D\|_{L^q_tL^r_x} \lesssim \|K\|_{L^2([0,\infty);L^2_x)\to L^q([0,\infty);L^r_x)}\;\|Ae^{isH}\psi_0\|_{L^2_tL^2_x} \tag{4.10}$$

We now need to estimate the norm of the operator K.

$$\|KF\|_{L^q_tL^r_x} = \|\int_0^\infty e^{i(t-s)H_0}B^*F(s)\,ds\|_{L^q_tL^r_x} = \|e^{itH_0}\int_0^\infty e^{-isH_0}B^*F(s)\,ds\|_{L^q_tL^r_x}$$

$$\le C_{H_0}\|\int_0^\infty e^{-isH_0}B^*F(s)\,ds\|_{L^2_x}.$$

The last inequality is the estimate (4.7) for e^{itH_0}. By duality

$$\begin{aligned}
\|\int_0^\infty e^{-isH_0}B^*F(s)\,ds\|_{L^2_x} &= \sup_{\|\phi\|_{L^2_x}=1} < \int_0^\infty e^{-isH_0}B^*F(s)\,ds, \phi > \\
&= \sup_{\|\phi\|_{L^2_x}=1} \int_0^\infty ds < F(s), Be^{isH_0}\phi > \\
&\le \|F\|_{L^2_tL^2_x} \sup_{\|\phi\|_{L^2_x}=1} \|Be^{isH_0}\phi\|_{L^2_tL^2_x} \le C_B(H_0)\|F\|_{L^2_tL^2_x}\|\phi\|_{L^2_x},
\end{aligned}$$

where the last inequality follows from H_0-smoothness of the operator B. Thus the operator $K(t,s) = e^{i(t-s)H_0}A$ is bounded from $L^2([0,\infty); L^2_x)$ to $L^q([0,\infty); L^r_x)$. Therefore, back to (4.10)

$$\|D\|_{L^q_tL^r_x} \le C_{H_0}C_B(H_0)\|\, Ae^{isH}P_\Omega\psi_0\|_{L^2_tL^2_x}. \tag{4.11}$$

It remains to observe that since the operator A is H-smooth on Ω, we have

$$\|\, Ae^{isH}P_\Omega\psi_0\|_{L^2_tL^2_x} \le C_A(H,\Omega)\|\psi_0\|_{L^2_x}. \tag{4.12}$$

Thus, combining (4.9), (4.11), and (4.12) we finally obtain

$$\|e^{itH}\psi_0\|_{L^q_tL^r_x} \le C_{H_0}C_B(H_0)C_A(H,\Omega)\|\psi_0\|_{L^2_x},$$

as claimed. □

We apply Theorem 4.1 in the situation where $H_0 = -\Delta$ and $H = H_0 + V(x)$. We have the following family of Strichartz estimates for the semigroup $e^{-it\Delta}$ associated with $H_0 = -\Delta$:

$$\|e^{-it\Delta}\psi_0\|_{L^q_tL^r_x} \le C\|\psi_0\|_{L^2_x}, \qquad \forall (q,r,n) \ne (2, \frac{2n}{n-2}, n), \quad \frac{2}{q} = n(\frac{1}{2} - \frac{1}{r}), \tag{4.13}$$

which hold for any $\psi_0 \in L^2(\mathbb{R}^n)$. We introduce the factorization

$$V = B^*A, \qquad B = |V|^{\frac{1}{2}}, \qquad A = |V|^{\frac{1}{2}}\mathrm{sgn}V,$$

and restrict our attention to the class of potentials satisfying the assumption that for all $x \in \mathbb{R}^n$

$$|V(x)| \le C_V(1 + |x|^2)^{-1-\varepsilon} \tag{4.14}$$

with some constants $C_V, \varepsilon > 0$. This assumption, in particular, places us in the framework of the Agmon-Kato-Kuroda and the Agmon-Kato-Simon theorems guaranteeing the absence of the positive singular continuous spectrum and positive eigenvalues. In fact, one only needs the $|x|^{-1-\varepsilon}$ decay for their results to apply. We should note that for potentials satisfying (4.14) the absence of the singular continuous spectrum was established by Ikebe [Ik].
In addition, the Weyl criterion implies that the essential spectrum of H is the half-axis $[0,\infty)$. However, without an appropriate smallness or sign assumption on V, the operator $H = -\Delta + V$ can have negative eigenvalues, thus destroying any hope to have Strichartz estimates for $e^{itH}\psi_0$ for

all initial data $\psi_0 \in L^2$. Therefore, we shall assume that the initial data are orthogonal to the eigenfunctions corresponding to the possible negative eigenvalues. We achieve this in the following simple manner. Let P be a spectral projection of H corresponding to the interval $\Omega = [0, \infty)$. Our goal is to prove Strichartz inequalities for e^{itH} restricted to the absolutely continuous spectrum of H. We now state the result.

Theorem 4.3. *Let V be a potential verifying* (4.14). *In addition, we impose the condition that the point $\lambda = 0$ in the spectrum of the operator $H = -\Delta + V$ is neither an eigenvalue nor a resonance (see the discussion below, in particular Definition 4.4). Then if P is the spectral projection of H corresponding to the interval* $[0, \infty)$ *(on which H is purely absolutely continuous),*

$$\|e^{itH} P\psi_0\|_{L^q_t L^r_x} \le C\|\psi_0\|_{L^2_x}, \qquad \forall (q, r, n),\ n \ge 3, \quad \frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right). \tag{4.15}$$

To apply Theorem (4.1) we need to verify that B is an H_0-smooth operator and that A is an H-smooth operator on $[0, \infty)$. The first condition is easy to verify since by a result of Kato [Ka] any function $f \in L^{p_1} \cap L^{p_2}$ with $1 \le p_1 < n < p_2 \le \infty$ and $n \ge 3$ is a $-\Delta$-smooth multiplication operator. Since $B = |V|^{\frac{1}{2}}$ is an L^∞ function decaying at infinity as $|x|^{-1-\varepsilon}$, it falls precisely under these conditions.
The condition that A is an H-smooth operator on $[0, \infty)$ is much more subtle. First, we can show that A is H-smooth on the interval $[\delta, \infty)$ for any $\delta > 0$. This is a consequence of the results of Agmon-Kato-Kuroda on the absence of the positive singular continuous spectrum, (see [Ag], also Theorem XIII.33 and Lemma 2 XIII.8 in [RS]). In fact, even half of the assumed decay would be sufficient to prove this. To deal with the remaining spectral interval $[0, \delta)$, according to (4.6), one needs to understand the behavior of the resolvent $R(\lambda \pm i\beta)$ of the operator H near the point $\lambda = 0$, $\beta = 0$. We introduce the following

Definition 4.4. *We say that* 0 *is a regular point of the spectrum of H if it is neither an eigenvalue nor a resonance of H, i.e., the equation $-\Delta u + V(x)u = 0$ has no solutions $u \in \cap_{\alpha > \frac{1}{2}} L^{2,-\alpha}$.*
Here, $L^{2,\alpha}$ is the weighted L^2 space of functions f such that $(1 + |x|^2)^{\frac{\alpha}{2}} f \in L^2$. The 0 eigenvalue, of course, would correspond to an L^2 solution u.

The presence of a 0 eigenvalue and most likely that of a resonance would violate the validity of the Strichartz estimates (4.15) for e^{itH}. Their appearance cannot be ruled out by merely strengthening the regularity and decay assumptions on the potential V. We therefore impose an additional condition that 0 is a regular point. There are several situations where this condition, or at least part of it, is automatically satisfied. In particular, for any non-negative potential 0 is a regular point. In addition, it is well-know (see e.g. [JK]) that 0 is not a resonance in dimensions $n \ge 5$. The behavior of the resolvent near 0 in the spectrum and even its asymptotic expansions was extensively studied in [JK], [J1], [J2], but their assumptions are too strong for our purposes.

Proposition 4.5. *Suppose V satisfies the assumption* (4.14) *and assume, in addition, that* 0 *is a regular point of the spectrum of $H = -\Delta + V$. Then the operator $A = |V|^{\frac{1}{2}} sgn(V)$ is H-smooth on* $[0, \delta)$ *for some sufficiently small $\delta > 0$.*

The first observation, which follows from (4.6), is that since the potential V decays at the rate $|x|^{-2-2\varepsilon}$ it suffices to prove the following property of the resolvent $R(\lambda \pm i\beta)$ formulated in the language of the weighted spaces $L^{2,\alpha}$:

$$\sup_{0<\beta<\delta, \lambda\in[0,\delta)} \|R(\lambda \pm i\beta) f\|_{L^{2,-1-\gamma}} \le C\|f\|_{L^{2,1+\gamma}} \tag{4.16}$$

for any $f \in L^{2,1+\gamma}$ and some sufficiently small δ and γ such that $\gamma < \varepsilon$. The restriction of the range of β to the interval $(0, \delta)$ is justified since for $\beta \ge \delta$ the resolvent $R(\lambda \pm i\beta)$ in fact maps L^2 into the Sobolev space W_2^2 with a constant dependent only on δ.
One should compare (4.16) with the standard limiting absorption principle which states that on the interval $[\delta, \infty)$ the resolvent $R(\lambda \pm i\beta)$ is a bounded map between $L^{2,\frac{1}{2}+\gamma}$ and $L^{2,-\frac{1}{2}-\gamma}$ for any $\gamma > 0$. As in that case we reduce the proof to the same estimates for the free resolvent $R_0(\lambda \pm i\beta)$. The connection is established via the resolvent identity

$$R(\lambda \pm i\beta) = R_0(\lambda \pm i\beta) - R_0(\lambda \pm i\beta) V R(\lambda \pm i\beta).$$

Thus formally we can solve for $R(\lambda \pm i\beta)$,

$$R(\lambda \pm i\beta) = (I + R_0(\lambda \pm i\beta)V)^{-1} R_0(\lambda \pm i\beta). \tag{4.17}$$

We identify the boundary value of the free resolvent at $\lambda = 0$, $\beta = 0$ as the operator with the kernel given by the Green's function (up to constants),

$$G(x, y) := R_0(0)(x, y) = \frac{1}{|x-y|^{n-2}}, \qquad n \ge 3$$

and break the proof into a series of lemmas.

Lemma 4.6. *G is a bounded map from the weighted space $L^{2,1+\gamma}$ into $L^{2,-1-\gamma}$ for any $\gamma > 0$. Moreover, for any positive $\sigma < \frac{1}{2}$ we have $G : L^{2,1+\gamma+\sigma} \to L^{2,-1-\gamma+\sigma}$.*

Lemma 4.7. *The resolvent $R_0(\lambda \pm i\beta)$ is continuous at $\lambda = 0, \beta = 0$ in the topology of the bounded operators between $L^{2,1+\gamma}$ and $L^{2,-1-\gamma}$ for any $\gamma > 0$.*

Lemma 4.8. *Under the assumptions of Proposition 4.5 the operator GV is compact as an operator on $L^{2,-1-\gamma}$ for any $\gamma > 0$, and $(I + GV)$ is invertible on $L^{2,-1-\gamma}$.*

Proof of Proposition 4.5. According to Lemma 4.8 the operator $(I+GV)$ is invertible on $L^{2,-1-\gamma}$ for any $\gamma > 0$. Therefore, by continuity of $R_0(\lambda \pm i\beta)$ at $\lambda = 0, \beta = 0$ asserted in Lemma 4.7 and the fact that V maps $L^{2,-1-\gamma}$ to $L^{2,1+\gamma}$ provided that $\gamma < \beta$, there exists a small neighborhood δ of 0 such that $(I + R_0(\lambda \pm i\beta)V)$ is uniformly invertible for all $|\lambda|, \beta < \delta$ on the space $L^{2,-1-\gamma}$. It follows that for such λ, β the resolvent $R(\lambda \pm i\beta)$ is well-defined via the identity (4.17) and acts between the spaces $L^{2,1+\gamma}$ and $L^{2,-1-\gamma}$ as desired. $\square$

Proof of Lemma 4.6. The resolvent $G = R_0(0)$ is a multiplier with symbol $|\xi|^{-2}$. Therefore, after passing to the Fourier variables, $G : L^{2,1+\gamma+\sigma} \to L^{2,-1-\gamma+\sigma}$ is equivalent to showing that multiplication by $|\xi|^{-2}$ acts between the Sobolev spaces $W_2^{1+\gamma+\sigma}$ and $W_2^{-1-\gamma+\sigma}$. We consider the end-points of the desired values of γ and σ corresponding to $\gamma = 0$ and $\sigma = 0, \frac{1}{2}$ and prove that

$$|\xi|^{-2} \,:\, W_2^{1+} \to W_2^{-1-}, \qquad |\xi|^{-2} \,:\, W_2^{\frac{3}{2}+} \to W_2^{-\frac{1}{2}-},$$

where $\pm$ represent the fact that we do not prove the end-point results themselves. Since $|\xi|^{-2}$ is smooth away from $\xi = 0$ we can consider instead the operator of multiplication by $\chi(\xi)|\xi|^{-2}$ where χ is a smooth cut-off function with support in a unit ball B. We have the standard Sobolev embeddings

$$W_2^{1+} \hookrightarrow L^{\frac{2n}{n-2}+}, \tag{4.18}$$

$$W_2^{\frac{3}{2}+} \hookrightarrow L^{\frac{2n}{n-3}} (L^\infty,\, n=3), \tag{4.19}$$

the dual version of (4.18), $L^{\frac{2n}{n+2}-} \hookrightarrow W_2^{-1-}$, and $L^{\frac{2n}{n+1}-} \hookrightarrow W_2^{-\frac{1}{2}-}$. Therefore, we shall, in fact, prove a stronger result that

$$\chi(\xi)\,|\xi|^{-2}\;:\;L^{\frac{2n}{n-2}+} \to L^{\frac{2n}{n+2}-}, \qquad \chi(\xi)\,|\xi|^{-2}\;:\;L^{\frac{2n}{n-3}+} \to L^{\frac{2n}{n+1}-},$$

Since $\frac{n-2}{2n} + \frac{2}{n} = \frac{n+2}{2n}$ and $\frac{n-3}{2n} + \frac{2}{n} = \frac{n+1}{2n}$, we have

$$\|\,\chi(\xi)|\xi|^{-2}f\|_{L^{\frac{2n}{n+2}-}} \le \|\;|\xi|^{-2}\|_{L^{\frac{n}{2}-}(B)}\|f\|_{\frac{2n}{n-2}+} \le C\|f\|_{\frac{2n}{n-2}+}$$

and

$$\|\,\chi(\xi)|\xi|^{-2}f\|_{L^{\frac{2n}{n+1}-}} \le \|\;|\xi|^{-2}\|_{L^{\frac{n}{2}-}(B)}\|f\|_{L^{\frac{2n}{n-3}}+} \le C\|f\|_{L^{\frac{2n}{n-3}}+}$$

as desired. $\square$

Proof of Lemma 4.7. The result of Lemma 4.7 is contained in Ginibre-Moulin [GM] and can be traced to the earlier work of Kato [Ka]. Here we essentially reproduce the proof in [GM].
Consider the resolvent $R(\lambda + i\beta)$. We shall prove that it is continuous (in fact, Hölder continuous) in the upper half-plane $\overline{\mathbb{C}}_+$. The same statement also holds for $R(\lambda - i\beta)$. We appeal to the representation (4.1),

$$R_0(\lambda + i\beta) = \int_0^\infty e^{i(\lambda+i\beta)t} e^{itH_0}\,dt.$$

Therefore, using the inequality $|e^{i(\lambda_2+i\beta_2)t} - e^{i(\lambda_1+i\beta_1)t}| \le \min(2, (|\lambda_2-\lambda_1| + |\beta_2-\beta_1|)t)$ and the embedding $L^{\frac{2n}{n-2}+}(\mathbb{R}^n) \hookrightarrow L^{2,-1-\gamma}(\mathbb{R}^n)$, we obtain for arbitrary $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\beta_1, \beta_2 \in \mathbb{R}_+$,

$$\begin{aligned}\|(R_0(\lambda_2+i\beta_2) - R(\lambda_1+i\beta_1))f\|_{L^{2,-1-\gamma}} &\le \int_0^\infty \|e^{itH_0}f\|_{L^{2,-1-\gamma}} \min(2, (|\lambda_2-\lambda_1| + |\beta_2-\beta_1|)t)\,dt \\ &\le \int_0^\infty \|e^{itH_0}f\|_{L^{\frac{2n}{n-2}+}} \min(2, (|\lambda_2-\lambda_1| + |\beta_2-\beta_1|)t)\,dt. \end{aligned} \tag{4.20}$$

We now recall that in addition to the Strichartz estimates (4.13) the semigroup e^{itH_0} also verifies the dispersive estimates

$$\|e^{itH_0}f\|_{L^p} \le \frac{C_p}{t^{n(\frac{1}{2}-\frac{1}{p})}}\|f\|_{L^{p'}}, \qquad \forall p \in [2,\infty].$$

Inserting this bound into (4.20) and invoking the embedding $L^{2,1+\gamma} \hookrightarrow L^{\frac{2n}{n+2}-}$, we infer that

$$\|(R_0(\lambda_2+i\beta_2)-R(\lambda_1+i\beta_1))f\|_{L^{2,-1-\gamma}} \lesssim \int_0^M \frac{(|\lambda_2-\lambda_1|+|\beta_2-\beta_1|)t)}{t^{1+}}\,dt\,\|f\|_{L^{\frac{2n}{n+2}-}} + \int_M^\infty \frac{2}{t^{1+}}\,\|f\|_{L^{\frac{2n}{n+2}-}}$$
$$\lesssim \Big((|\lambda_2-\lambda_1|+|\beta_2-\beta_1|)M^{1-}+2M^{0-}\Big)\|f\|_{L^{2,1+\gamma}}$$

for some constant $M>0$. Choosing $M=(|\lambda_2-\lambda_1|+|\beta_2-\beta_1|)^{-1}$ we finally conclude that

$$\|(R_0(\lambda_2+i\beta_2)-R(\lambda_1+i\beta_1))f\|_{L^{2,-1-\gamma}} \lesssim (|\lambda_2-\lambda_1|+|\beta_2-\beta_1|)^{0+}\|f\|_{L^{2,1+\gamma}},$$

as claimed. □

Proof of Lemma 4.8. Lemma 4.6 implies the boundedness of $G: L^{2,1+\gamma}\to L^{2,-1-\gamma}$ for any positive γ. Therefore, since $|V(x)|\le C(1+|x|^2)^{-1-\varepsilon}$, the potential V maps the space $L^{2,-1-\gamma}$ into $L^{2,1-\gamma+2\varepsilon}$. Thus, using the second conclusion of Lemma 4.6, we obtain that

$$GV: L^{2,-1-\gamma}\to L^{2,-1-\gamma+2\varepsilon}. \tag{4.21}$$

provided that $0<-\gamma+2\varepsilon<\frac{1}{2}$.

Since $-\Delta(GV)=V$ and V, of course, maps $L^{2,-1-\gamma}\to L^{2,-1-\gamma}$, we conclude that GV takes the space $L^{2,-1-\gamma}$ to H^2_{loc}. The compactness of GV on $L^{2,-1-\gamma}$ then follows from the observation above and the extra 2ε decay at infinity established in (4.21). We infer that $I+R_0V$ is a Fredholm operator on $L^{2,-1-\gamma}$ and it is therefore invertible iff its null space is empty.
Let ϕ be an $L^{2,-1-\gamma}$ solution of the equation

$$\phi+GV\phi=0 \tag{4.22}$$

First we observe that by (4.21) function ϕ, in fact, belongs to the space $L^{2,-1-\gamma+2\varepsilon}$ for some $\gamma:\ 0<\gamma<2\varepsilon$. It then follows that $V\phi\in L^{2,1-\gamma+4\varepsilon}$. Lemma 4.6 then implies that, as long as $4\varepsilon<\frac{1}{2}$, also $GV\phi\in L^{2,-1-\gamma+4\varepsilon}$ and, using (4.22) again we have that $\phi\in L^{2,-1-\gamma+4\varepsilon}$. We can continue this argument and obtain that $\phi\in L^{2,-\frac{1}{2}-\alpha}$ for any $\alpha>0$.
Applying $-\Delta$ to both sides of the equation (4.22) we conclude that ϕ is an $L^{2,-\frac{1}{2}-\alpha}$ solution of the equation

$$-\Delta\phi+V\phi=0,$$

and thus either an eigenfunction or a resonance corresponding to $\lambda=0$. Since we assumed that $\lambda=0$ is a regular point, ϕ must be identically zero and the null space of $I+R_0V$ is empty. □

5 Time dependent potentials: Reduction to oscillatory integrals

Definition 5.1. *Let Y be the normed space of measurable functions $V(t,x)$ on $\mathbb{R}^3$ that satisfy the following properties: $t\mapsto \|V(t,\cdot)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\in L^\infty(\mathbb{R})$ and for a.e. $x\in\mathbb{R}^3$ the function $t\mapsto V(t,x)$ is* *in $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions. Moreover, the Fourier transform of this distribution, which we denote by $V(\hat{\tau}, x)$, is a (complex) measure whose norm satisfies*

$$\sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\|V(\hat{\tau}, x)\|_{\mathcal{M}}}{|x-y|}\, dx < \infty. \tag{5.1}$$

The norm in Y is the sum of the expression on the left-hand side of (5.1) and the norm in $L_t^\infty(L_x^{\frac{3}{2}})$.

In what follows we study the Schrödinger equation

$$\begin{aligned} &i\partial_t\psi - \Delta\psi + V(t,x)\psi = 0, \\ &\psi|_{t=s}(x) = \psi_s(x) \end{aligned} \tag{5.2}$$

for potentials $V \in Y$ and with initial data $\psi_s \in L^2(\mathbb{R}^3)$. An interesting case is $V(t,x) = \cos(t)\, V(x)$ where $V \in L^{\frac{3}{2}}$ satisfies $\sup_y \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|}\, dx < \infty$. Because of the limited regularity of potentials in Y, we define (weak) solutions $U(t,s)\psi_s$ of (5.2) via Duhamel's formula:

$$U(t,s)\psi_s = e^{i(t-s)H_0}\psi_s + i\int_s^t e^{i(t-s_1)H_0} V(s_1,\cdot)U(s_1,s)\psi_s\, ds_1. \tag{5.3}$$

In the following lemma we show by means of Keel's and Tao's endpoint Strichartz estimate [KT] that such weak solutions exist and are unique provided the potential is small in an appropriate sense. The proof is presented only in $\mathbb{R}^3$, but it carries over to any dimension $n \geq 3$. We set

$$X = L_t^\infty(L_x^2(\mathbb{R}^3)) \cap L_t^2(L_x^6(\mathbb{R}^3)) \tag{5.4}$$

and define $H_0 = -\Delta$ to be the unperturbed Schrödinger operator with evolution e^{itH_0}.

Lemma 5.2. *Assume that the potential $V(t,x)$ satisfies the smallness assumption*

$$\|V\|_{L_t^\infty L_x^{\frac{3}{2}}} = \sup_{t\in\mathbb{R}} \left(\int_{\mathbb{R}^3} |V(t,x)|^{\frac{3}{2}}\right)^{\frac{2}{3}} < c_0 \tag{5.5}$$

for some sufficiently small constant $c_0 > 0$. Then for any $s \in \mathbb{R}$ and any $\psi_s \in L_x^2$ there exists a unique weak solution $U(t,s)\psi_s$ of (5.3) with the property that $U(\cdot,s)\psi_s \in X$ and so that $t \mapsto \langle U(t,s)\psi_s, g\rangle$ is continuous for any $g \in L^2(\mathbb{R}^3)$. Moreover, for any such g and any $t > s$,

$$\begin{aligned} \langle U(t,s)\psi_s, g\rangle = \sum_{m=0}^\infty \quad i^m \int\!\!\cdots\!\!\int_{s\le s_m\le\ldots\le s_1\le t} &\langle e^{i(t-s_1)H_0}V(s_1,\cdot)e^{i(s_1-s_2)H_0}V(s_2,\cdot)\ldots \\ & V(s_m,\cdot)e^{i(s_m-s)H_0}\,\psi_s, g\rangle\, ds_1\ldots ds_m \end{aligned} \tag{5.6}$$

where the series converges absolutely. In the strong sense, i.e., without the pairing against g, this representation holds in the sense of norm convergence in the space X (and thus can only be assumed for a.e. t).

Proof. For the purposes of this proof, we let $F = F(t,x)$ be a function of $(t,x) \in \mathbb{R}^{1+3}_{t,x}$. For simplicity, we often write $F(t)$ for the function $x \mapsto F(t,x)$. Recall the following end-point Strichartz estimates for the operator H_0 proved by Keel-Tao in any dimension $n \geq 3$, see [KT]: There exists some dimensional constant $C_1 = C_1(n)$ so that for all $f \in L^2_x$ and $F \in L^2_t L_x^{\frac{2n}{n+2}}$

$$\|e^{itH_0} f\|_{L^2_t L_x^{\frac{2n}{n-2}}} \leq C_1 \|f\|_{L^2_x}, \tag{5.7}$$

$$\|\int_s^t e^{i(t-s_1)H_0} F(s_1)\, ds_1\|_{L^2_t L_x^{\frac{2n}{n-2}}} \leq C_1 \|F\|_{L^2_t L_x^{\frac{2n}{n+2}}}. \tag{5.8}$$

Consider the operator $\mathcal{K}_s$ defined by

$$(\mathcal{K}_s F)(t,\cdot) = i \int_s^t e^{i(t-s_1)H_0} V(s_1,\cdot) F(s_1,\cdot)\, ds_1.$$

Then definition (5.3) takes the form

$$\big[(1-\mathcal{K}_s) U(\cdot,s)\psi_s\big](t) = e^{i(t-s)H_0}\psi_s. \tag{5.9}$$

Inequality (5.8) and the smallness assumption (5.5) imply that the norm of the operator $\mathcal{K}_s : L^2_t L^6_x \to L^2_t L^6_x$ satisfies

$$\|\mathcal{K}_s F\|_{L^2_t L^6_x} \leq C_1 \|V\, F\|_{L^2_t L_x^{\frac{6}{5}}} \leq C_1 \|V\|_{L^\infty_t L_x^{\frac{3}{2}}} \|F\|_{L^2_t L^6_x} \leq C_1\, c_0 \|F\|_{L^2_t L^6_x}. \tag{5.10}$$

Moreover, for any $g \in L^2(\mathbb{R}^3)$,

$$\begin{aligned}
|\langle (\mathcal{K}_s F)(t), g\rangle| &= \left| \int_s^t \langle V(s_1,\cdot)F(s_1,\cdot), e^{-i(t-s_1)H_0} g\rangle\, ds_1 \right| \\
&\leq \int_s^t \|V(s_1,\cdot)F(s_1,\cdot)\|_{\frac{6}{5}}\, \|e^{-i(t-s_1)H_0} g\|_6\, ds_1 \\
&\leq \Big(\int_s^t \|V(s_1,\cdot)F(s_1,\cdot)\|^2_{\frac{6}{5}}\, ds_1\Big)^{\frac12} \Big(\int_s^t \|e^{-i(t-s_1)H_0} g\|_6^2\, ds_1\Big)^{\frac12} \\
&\leq C_1 \Big(\int_s^t \|V\|^2_{L^\infty_t(L_x^{\frac{3}{2}})} \|F(s_1,\cdot)\|_6^2\, ds_1\Big)^{\frac12} \|g\|_2 \qquad (5.11) \\
&= C_1 \|V\|_{L^\infty_t(L_x^{\frac{3}{2}})} \|F\|_{L^2_t(L^6_x)} \|g\|_2
\end{aligned}$$

where we used (5.7) to pass to (5.11). This shows that

$$\text{ess sup}_t \|(\mathcal{K}_s F)(t)\|_2 \leq C_1\, c_0 \|F\|_{L^2_t(L^6_x)}$$

which in conjunction with (5.10) yields that

$$\|\mathcal{K}_s\|_{X\to X} \leq C_1\, c_0 < \frac{1}{2}, \tag{5.12}$$

provided c_0 is small (see (5.4) for the definition of X). Therefore, the operator $I - \mathcal{K}_s$ is invertible on the space X and $U(t,s)$ can be expressed via the Neumann series

$$U(t,s) = \big[(I-\mathcal{K}_s)^{-1} e^{i(\cdot - s)H_0}\big](t) = \sum_{m=0}^\infty \big[\mathcal{K}_s^m e^{i(\cdot - s)H_0}\big](t)$$

which converges in the norm of X. Writing out $\langle U(t,s)\psi_s, g\rangle$ explicitly leads to (5.6). Next we check that for any $F \in L^2_t(L^6_x)$ the function $t \mapsto \langle \mathcal{K}_s F, g\rangle$ is continuous for any choice of $g \in L^2$. In fact, if $t_1 < t_2$, then

$$\begin{aligned}
|\langle (\mathcal{K}_s F)(t_2), g\rangle - \langle (\mathcal{K}_s F)(t_1), g\rangle| &\leq \int_s^{t_2} |\langle V(s_1)F(s_1), (e^{-i(t_1-s_1)H_0} - e^{-i(t_2-s_1)H_0})g\rangle|\, ds_1 \\
&\quad + \int_{t_1}^{t_2} |\langle V(s_1)F(s_1), e^{-i(t_1-s_1)H_0} g\rangle|\, ds_1 \\
&\leq \|V\|_{L^\infty_t(L^{\frac{3}{2}}_x)} \|F\|_{L^2_t(L^6_x)} \|g - e^{-i(t_2-t_1)H_0} g\|_2 \\
&\quad + \|V\|_{L^\infty_t(L^{\frac{3}{2}}_x)} \Big(\int_{t_1}^{t_2} \|F(s_1)\|^2_{L^6_x}\, ds_1\Big)^{\frac{1}{2}} \|g\|_2.
\end{aligned}$$

Since the last expression tends to zero as $t_2 \to t_1$, continuity follows. Hence $(\mathcal{K}_s^m F)(t)$ is also weakly continuous in t, and thus $\mathcal{K}_s^m e^{i(\cdot - s)H_0}\psi_s$ is, too. Since $\langle U(t,s)\psi_s, g\rangle$ is a uniformly convergent series of these continuous functions, it follows that it is continuous. □

Remark 5.3. The proof of Lemma 5.2 shows that the operator $\mathcal{K}_s : L^2_t L^6_x \to w - C^0_t(L^2_x)$ maps $L^2_t L^6_x$ into the space of weakly continuous functions with values in $L^2(\mathbb{R}^3)$.

For technical reasons connected with the functional calculus in the following section it will be convenient to work with smooth potentials in Y rather than general ones. To approximate a general potential V by means of smooth ones, choose nonnegative cut-off functions $\chi \in \mathcal{S}(\mathbb{R}^3)$ and $\eta \in \mathcal{S}(\mathbb{R})$ so that χ and $\hat{\eta}$ have compact support and satisfy $\int_{\mathbb{R}^3} \chi(x)\, dx = 1$, $\int_{\mathbb{R}} \eta(t)\, dt = 1$. In addition, let $\chi = 1$ on a neighborhood of 0. For any $V \in Y$ and $R > 1$ define

$$V_R^{(1)}(t,\cdot) := V(t,\cdot)\chi(\frac{\cdot}{R}) * R^3\chi(R\cdot)$$

where the convolution is in the x-variable only. Note that $V_R^{(1)}$ is well-defined, smooth and compactly supported in x, and satisfies $\|V_R^{(1)}\|_{L^\infty_{t,x}} < \infty$ since $\|V\|_{L^\infty_t(L^{\frac{3}{2}}_x)} < \infty$. Moreover, it is standard to check that

$$\sup_{R>0} \|V_R^{(1)}\|_Y \leq \|\chi\|_\infty \|V\|_Y.$$

Indeed,

$$\|V_R^{(1)}(t,\cdot)\|_{L^{\frac{3}{2}}} \leq \|\chi\|_\infty \| |V(t,\cdot)| * R^3\chi(R\cdot)\|_{L^{\frac{3}{2}}} \leq \|\chi\|_\infty \|V(t,\cdot)\|_{L^{\frac{3}{2}}},$$

whereas with $\Gamma(x) := |x|^{-1}$ and $\mathcal{M}$ denoting measures in the τ-variable,

$$\begin{aligned}
\Big(\|V_R^{(1)}(\hat{\tau},\cdot)\|_{\mathcal{M}} * \Gamma\Big)(x) &\leq \sup_y \Big(\chi(\frac{\cdot}{R}) \|V(\hat{\tau},\cdot)\|_{\mathcal{M}} * \Gamma\Big)(y) \\
&\leq \|\chi\|_\infty \sup_y \Big(\|V(\hat{\tau},\cdot)\|_{\mathcal{M}} * \Gamma\Big)(y),
\end{aligned}$$

as claimed. To regularize in t, define

$$V_R(\cdot,x) := [V_R^{(1)}(\cdot,x) * R\eta(R\cdot)]\eta(\frac{\cdot}{R})$$

where the convolution is in the t-variable only. Again one checks that

$$\|V_R\|_Y \leq (\|\eta\|_\infty + \|\hat{\eta}\|_1)\,\|V_R^{(1)}\|_Y \leq (\|\eta\|_\infty + \|\hat{\eta}\|_1)\ \|\chi\|_\infty\,\|V\|_Y$$

for any $R > 0$. We will use that $V_R \to V$ as $R \to \infty$ in the following sense: For a.e. t one has

$$\|V_R(t,\cdot) - V(t,\cdot)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \to 0 \ \text{ as } R \to \infty. \tag{5.13}$$

Firstly, it follows from standard measure theory that for a.e. t

$$\|V_R^{(1)}(t,\cdot) - V(t,\cdot)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \to 0 \ \text{ as } \quad R \to \infty. \tag{5.14}$$

Secondly, with $\eta_R(t) := R\eta(Rt)$,

$$\begin{aligned}
& \|V_R(t,\cdot) - V_R^{(1)}(t,\cdot)\|_{\frac{3}{2}} \\
\leq\ & \|(V_R^{(1)} * \eta_R)(t,\cdot) - V_R^{(1)}(t,\cdot)\|_{\frac{3}{2}} + \|V_R^{(1)}(t,\cdot)\|_{\frac{3}{2}}|1-\eta(t/R)| \\
\leq\ & \int_{-\infty}^{\infty} \eta_R(s)\,\|V_R^{(1)}(t-s,\cdot) - V_R^{(1)}(t,\cdot)\|_{\frac{3}{2}}\,ds + |1-\eta(t/R)|\,\|\eta\|_\infty \|V\|_{L_t^\infty(L_x^{\frac{3}{2}})} \\
\leq\ & \|\chi\|_\infty \int_{-\infty}^{\infty} \eta_R(s)\,\|V(t-s,\cdot) - V(t,\cdot)\|_{\frac{3}{2}}\,ds + o(1) \\
\to\ & 0
\end{aligned} \tag{5.15}$$

for a.e. t as $R \to \infty$. The conclusion (5.15) follows from the vector-valued analogue of the Lebesgue differentiation theorem (in this case "vector-valued" means with values in $L^{\frac{3}{2}}$). In combination with (5.14) this yields (5.13).

We shall now prove the convergence of the approximate solutions $\psi_R(t,x)$ satisfying the equation

$$\begin{aligned}
& i\,\partial_t\psi_R - \Delta\psi_R + V_R(t,x)\psi_R = 0, \\
& \psi_R|_{t=s} = \psi_s
\end{aligned} \tag{5.16}$$

to the solution $\psi(t,x)$ of the original problem corresponding to the potential $V(t,x)$. Note that due to the smoothness and boundedness of the potentials V_R the $L_t^\infty L_x^2$ function ψ_R can be interpreted as a distributional solution of equation (5.16). In fact, the left hand-side of (5.16) belongs to the space $L_t^\infty H^{-2}$. In addition, ψ_R is also a Duhamel solution as in (5.3).

Lemma 5.4. *Let $U_R(t,s)$ be the propagator* (5.16)*, i.e., $U_R(t,s)\psi_s = \psi_R(t,s)$. Then for any $s,t \in \mathbb{R}$ such that $t \geq s$, and arbitrary functions $\psi_s, g \in L^2(\mathbb{R}^3)$, $\|\psi_s\|_{L_x^2} = \|g\|_{L_x^2} = 1$ we have*

$$< U_R(t,s)\psi_s, g > \to < U(t,s)\psi_s, g > \qquad \text{as} \quad R \to \infty \tag{5.17}$$

Proof. First observe that since the potential V satisfies the smallness assumption (5.5), V_R also obeys (5.5) for all $R > 0$. According to Lemma 5.2,

$$\begin{aligned}
\langle U_R(t,s)\psi_s, g\rangle = \sum_{m=0}^{\infty} \quad i^m \int \cdots \int_{s\leq s_m \leq .. \leq s_1 \leq t} & \langle e^{i(t-s_1)H_0} V_R(s_1,\cdot) e^{i(s_1-s_2)H_0} V_R(s_2,\cdot).... \\
& V_R(s_m,\cdot) e^{i(s_m-s)H_0}\,\psi_s, g\rangle\, ds_1...ds_m
\end{aligned} \tag{5.18}$$

for any $\psi_s, g \in L^2(\mathbb{R}^3)$. Equivalently, $U_R(t,s)$ can be represented by the Neumann series

$$U_R(t,s) = \big[(I - \mathcal{K}_{Rs})^{-1} e^{i(\cdot - s)H_0}\big](t) = \sum_{m=0}^{\infty} \big[\mathcal{K}_{Rs}^{\ m} e^{i(\cdot - s)H_0}\big](t)$$

which converges in the norm of the space X defined above. The operators $\mathcal{K}_{Rs} : X \to X$ are defined as

$$(\mathcal{K}_{Rs}F)(t,\cdot) = i \int_s^t e^{i(t-s_1)H_0} V_R(s_1,\cdot) F(s_1,\cdot)\, ds_1.$$

To verify the conclusion of Lemma 5.4 it suffices to show that for an arbitrary positive $\varepsilon > 0$, all positive integers $m \le m_0(\varepsilon)$, and all sufficiently large $R = R_0(\varepsilon, m_0)$

$$|\langle \big(\mathcal{K}_s^m - \mathcal{K}_{Rs}^{\ m}\big) e^{i(\cdot - s)H_0}\psi_s, g\rangle(t)| < \varepsilon (C_1 c_0)^{m-1} m \tag{5.19}$$

The positive integer $m_0(\varepsilon)$ is chosen so that $2(C_1c_0)^m \le \varepsilon$ which ensures the smallness of the "tails" of the series for $U(s,t)$ and $U_R(s,t)$.
For the bounded operators $\mathcal{K}_s, \mathcal{K}_{Rs}$ on the space X we have the following identity:

$$\mathcal{K}_{Rs}^{\ m} - \mathcal{K}_s^m = \sum_{\ell=0}^{m-1} \mathcal{K}_{Rs}^{\ell} (\mathcal{K}_s - \mathcal{K}_{Rs}) \mathcal{K}_s^{m-\ell-1} \tag{5.20}$$

We shall prove that for $\ell \in [0, m-1]$

$$|\langle \mathcal{K}_{Rs}^{\ell} (\mathcal{K}_s - \mathcal{K}_{Rs}) \mathcal{K}_s^{m-\ell-1} e^{i(\cdot - s)H_0}\psi_s, g\rangle(t)| < \varepsilon (C_1 c_0)^{m-1} \tag{5.21}$$

which immediately implies (5.19).
In view of Remark 5.3 the operator $\mathcal{K}_s$, and thus also $\mathcal{K}_{Rs}$, maps $L^2_t L^6_x \to w - C^0_t(L^2_x)$. Therefore, for an arbitrary fixed $t \ge s$ we can define the operator $\mathcal{K}_{Rs,t} : L^2_t L^6_x \to L^2_x$ via the formula

$$\mathcal{K}_{Rs,t} F = (\mathcal{K}_s F)(t)$$

In addition to the L^2_x pairing $\langle , \rangle$ we define the space-time pairing $\langle , \rangle_{t,x}$ as usual: for any pair of functions $F \in L^q_t L^p_x$ and $G \in L^{q'}_t L^{p'}_x$ with $q, p \in [1, \infty]$ let

$$\langle F, G\rangle_{t,x} = \int_{\mathbb{R}} \int_{\mathbb{R}^3} F(t,x) \overline{G(t,x)}\, dx\, dt$$

We now introduce the dual operator $\mathcal{K}_{Rs,t}^{*} : L^2_x \to L^2_t L^{\frac{6}{5}}_x$. In addition, since $\mathcal{K}_{Rs} : L^2_t L^6_x \to L^2_t L^6_x$ we also define the dual of $\mathcal{K}_{Rs}$, $\mathcal{K}_{Rs}^{*} : L^2_t L^{\frac{6}{5}}_x \to L^2_t L^{\frac{6}{5}}_x$. Therefore for $\ell \ge 1$ the left hand-side of (5.21) can be written as

$$\begin{aligned} I_{R,s,t} &:= \langle \mathcal{K}_{Rs,t} \mathcal{K}_{Rs}^{\ell-1} (\mathcal{K}_s - \mathcal{K}_{Rs}) \mathcal{K}_s^{m-\ell-1} e^{i(\cdot - s)H_0}\psi_s, g\rangle \\ &= \langle (\mathcal{K}_s - \mathcal{K}_{Rs}) \mathcal{K}_s^{m-\ell-1} e^{i(\cdot - s)H_0}\psi_s, \mathcal{K}_{Rs}^{*\ell-1} \mathcal{K}_{Rs,t}^{*} g\rangle_{t,x}. \end{aligned}$$

We assume that ε and $m_0(\varepsilon)$ are now fixed and invoke Egorov's theorem. According to (5.13)

$$\|V_R(s_1,\cdot) - V(s_1,\cdot)\|_{L^{\frac{3}{2}}_x} \to 0, \qquad \text{as} \quad R \to \infty$$

for a.e. $s_1 \in [s,t]$. Therefore, for any $\delta > 0$ there exists a set $\mathcal{B} \subset [s,t]$ such that $|\mathcal{B}| \le \delta$ and

$$\|V_R(s_1,\cdot) - V(s_1,\cdot)\|_{L_x^{\frac{3}{2}}} < \varepsilon, \qquad \forall s_1 \in [s,t] \setminus \mathcal{B}$$

and all sufficiently large $R \ge R_0(\varepsilon,\delta)$. Let $\chi_{\mathcal{B}}$ be the characteristic function of the set $\mathcal{B}$. We define operators

$$\begin{aligned}\mathcal{Y}_s &= (\mathcal{K}_s - \mathcal{K}_{Rs})\chi_{\mathcal{B}},\\ \mathcal{Z}_s &= (\mathcal{K}_s - \mathcal{K}_{Rs})(1-\chi_{\mathcal{B}}),\end{aligned}$$

It is easy to see that $\mathcal{Y}_s, \mathcal{Z}_s : L_t^2 L_x^6 \to L_t^2 L_x^6$. Moreover, $\|\mathcal{Z}_s\|_{L_t^2L_x^6 \to L_t^2 L_x^6} \le C\varepsilon$ for all $R \ge R_0(\varepsilon,\delta)$ and $\|\mathcal{Y}_s\|_{L_t^2L_x^6 \to L_t^2L_x^6} \le C_1 c_0 < \frac{1}{2}$, see (5.12). Therefore,

$$I_{R,s,t} = \langle \mathcal{K}_s^{m-\ell-1} e^{i(\cdot-s)H_0}\psi_s, \mathcal{Z}_s^* \mathcal{K}_{Rs}^{*\,\ell-1}\mathcal{K}_{Rs,t}^* g\rangle_{t,x} + \langle \chi_{\mathcal{B}} \mathcal{K}_s^{m-\ell-1} e^{i(\cdot-s)H_0}\psi_s, \mathcal{Y}_s^* \mathcal{K}_{Rs}^{*\,\ell-1}\mathcal{K}_{Rs,t}^* g\rangle_{t,x}$$

We can easily estimate the first term by

$$\|\mathcal{K}_s\|^{m-\ell-1}_{L_t^2L_x^6\to L_t^2L_x^6} \|\mathcal{Z}_s^*\|_{L_t^2L_x^{\frac{6}{5}} \to L_t^2 L_x^{\frac{6}{5}}} \|\mathcal{K}_s^*\|^{\ell-1}_{L_t^2L_x^{\frac{6}{5}} \to L_t^2L_x^{\frac{6}{5}}} \|\mathcal{K}_{s,t}^*\|_{L_x^2 \to L_t^2L_x^{\frac{6}{5}}} \|\psi_s\|_{L_x^2}\|g\|_{L_x^2} \le \varepsilon 2^{-(m-1)}.$$

For the second term we have the bound

$$\begin{aligned}&\|\chi_{\mathcal{B}}\mathcal{K}_s^{m-\ell-1}e^{i(\cdot-s)H_0}\psi_s\|_{L_t^2L_x^6}\|\mathcal{Y}_s^*\|_{L_t^2L_x^{\frac{6}{5}}\to L_t^2L_x^{\frac{6}{5}}}\|\mathcal{K}_s^*\|^{\ell-1}_{L_t^2L_x^{\frac{6}{5}}\to L_t^2L_x^{\frac{6}{5}}}\|\mathcal{K}_{s,t}^*\|_{L_x^2\to L_t^2L_x^{\frac{6}{5}}}\|\psi_s\|_{L_x^2}\|g\|_{L_x^2}\\ &\qquad \le \|\chi_{\mathcal{B}}\mathcal{K}_s^{m-\ell-1}e^{i(\cdot-s)H_0}\psi_s\|_{L_t^2L_x^6}(C_1c_0)^{\ell+1}\end{aligned}$$

Observe that

$$\|\mathcal{K}_s^{m-\ell-1}e^{i(\cdot-s)H_0}\psi_s\|_{L_t^2L_x^6} \le (C_1c_0)^{m-\ell-1}\|\psi_s\|_{L_x^2} < \infty$$

Therefore, we can chose $\delta = \delta(m_0)$ in Egorov's theorem in such a way that

$$\sum_{m=1}^{m_0(\varepsilon)}\sum_{\ell=1}^{m-1}\|\chi_{\mathcal{B}}\mathcal{K}_s^{m-\ell-1}e^{i(\cdot-s)H_0}\psi_s\|_{L_t^2L_x^6} \le \varepsilon(C_1c_0)^{m_0-\ell-1}$$

Hence we have the desired bound

$$|I_{R,s,t}| \le \varepsilon(C_1c_0)^{m-1}$$

for all $1 \le \ell \le m-1$ and $m \le m_0$. To settle the remaining case of $\ell = 0$ we observe that for $\ell = 0$

$$I_{R,s,t} = \langle(\mathcal{K}_s - \mathcal{K}_{Rs})\mathcal{K}_s^{m-1}e^{i(\cdot-s)H_0}\psi_s, g\rangle(t) = \langle\mathcal{Y}_s\mathcal{K}_s^{m-1}e^{i(\cdot-s)H_0}\psi_s, g\rangle(t) + \langle\mathcal{Z}_s\mathcal{K}_s^{m-1}e^{i(\cdot-s)H_0}\psi_s, g\rangle(t)$$

Similarly to the operator $\mathcal{K}_{Rs,t}$ we can define the operators $\mathcal{Y}_{s,t}, \mathcal{Z}_{s,t} : L_t^2L_x^6 \to L_x^2$. Moreover,

$$\|\mathcal{Z}_{s,t}\|_{L_t^2L_x^6\to L_x^2} \le C\varepsilon, \qquad \|\mathcal{Y}_{s,t}\|_{L_t^2L_x^6\to L_x^2} \le C_1c_0.$$

Thus

$$|I_{R,s,t}| \le C_1c_0\,\|\chi_{\mathcal{B}}\mathcal{K}_s^{m-1}e^{i(\cdot-s)H_0}\|_{L_t^2L_x^6} + \varepsilon(C_1c_0)^{m-1} \le 2\varepsilon(C_1c_0)^{m-1}$$

by the choice of the constant δ in Egorov's theorem. □

Since the potentials $V_R(t,x)$ are smooth in both variables, the solution operators $U_R(t,s)$ are unitary on L^2_x. Together with Lemma 5.4 we have the following

Corollary 5.5. *The L^2 norm of the solution $\psi(t,\cdot)$ of the Schrödinger equation* (5.2) *is a non-increasing function of time, i.e,*

$$\|U(t,s)\psi_s\|_{L^2_x} \le \|\psi_s\|_{L^2_x}$$

for all $t \ge s$ and arbitrary functions $\psi_s \in L^2_x$.

Lemma 5.4 also implies that we can assume henceforth that $V(t,x)$ is a smooth potential with compact support in the x-variable and the variable $\hat\tau$ of the Fourier transform relative to t. We can also assume that V satisfies the smallness assumption (5.5). We shall show that the following estimates depend only on the norm of the potential in the space Y defined in Defintion 5.1 and the smallness constant c_0.

5.1 Functional calculus

The goal of this section is to obtain the explicit representation of the integral kernels of the operators involved in the Neumann series expansion (5.6) for $U(t,s)$, as some special oscillatory integrals.

We introduce the notation

$$V(\hat\tau,\cdot) := \int e^{it\tau} V(t,\cdot)\, dt.$$

The m-th term of the series (5.6), which we denote by $\mathcal{I}_m$, can then be written in the following form[2]:

$$\langle \mathcal{I}_m(t,s)\psi_s, g\rangle = \int_{\mathbb{R}^m} d\tau_1..d\tau_m \underset{s\le s_m\le..\le s_1\le t}{\int\cdots\int} ds_1...ds_m\ \langle e^{i(t-s_1)H_0} e^{is_1\tau_1} V(\hat\tau_1,\cdot)... \\ \cdot\, e^{is_m\tau_m} V(\hat\tau_m,\cdot) e^{i(s_m-s)H_0}\psi_s,\, g\rangle. \tag{5.22}$$

The identity above is verified on arbitrary functions $\psi_s, g \in L^2_x$.
We shall also make use of the spectral representation of the operator e^{itH_0},

$$e^{itH_0} = \int_{\mathbb{R}} e^{it\lambda} dE(\lambda).$$

Here, $dE(\lambda)$ is the spectral measure associated with the operator $H_0 = -\Delta$. In dimension $n = 3$, $dE(\lambda)$ has an explicit representation as an integral operator with the kernel

$$dE(\lambda)(x,y) = \begin{cases} \frac{\sin\sqrt{\lambda}|x-y|}{4\pi|x-y|}\, d\lambda & \lambda > 0, \\ 0 & \lambda \le 0 \end{cases}$$

Recall also that the resolvent $R(z) = (H_0 - z)^{-1}$ is an analytic function with values in the space of bounded operators in $z \in \mathbb{C} \setminus \mathbb{R}_+$. In the above domain,

$$R(z) = \int_{\mathbb{R}} \frac{dE(\mu)}{\mu - z} \tag{5.23}$$

[2]Here we use the fact that $V(\hat\tau,\cdot)$ has compact support in $\hat\tau$ to interchange the integrals.

We shall use the following simplified version of the limiting absorption principle stating that $R(z) = R(\lambda + ib)$ has well-defined operator limits $R_+(\lambda)$ and $R_-(\lambda)$, for $\lambda > 0$, as $b \to 0^+$ and $b \to 0^-$ respectively. The operators $R_\pm(\lambda)$ map the space of Schwartz functions $\mathcal{S}$ into the space $C^\infty \cap L^4(\mathbb{R}^3)$.

On the real axis, the resolvent $R(\lambda)$ can be then described explicitly as the integral operators with the kernels

$$
\begin{aligned}
(5.24)\qquad & R_+(\lambda)(x,y) = \lim_{\varepsilon\to 0^+} R(\lambda + i\varepsilon)(x,y) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}, \qquad \lambda \geq 0\\
& R_-(\lambda)(x,y) = \lim_{\varepsilon\to 0^+} R(\lambda - i\varepsilon)(x,y) = \overline{R_+(\lambda)(x,y)} = \frac{e^{-i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}, \qquad \lambda \geq 0,\\
& R(\lambda)(x,y) = \overline{R(\lambda)(x,y)} = \frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi|x-y|}, \qquad \lambda < 0.
\end{aligned}
$$

In particular, we can write

$$
dE(\lambda) = \Im R(\lambda). \tag{5.25}
$$

We shall make repeated use of the following regularization:

$$
\int_a^b e^{i\alpha q} dq = \frac{e^{i(\alpha+i0)b} - e^{i(\alpha+i0)a}}{\alpha + i0} = \lim_{\varepsilon\to 0^+} \frac{e^{i(\alpha+i\varepsilon)b} - e^{i(\alpha+i\varepsilon)a}}{\alpha + i\varepsilon}.
$$

which holds true for any finite $a, b \in \mathbb{R}$ and arbitrary $\alpha \in \mathbb{R}$.

Proposition 5.6. *The function $\langle \mathcal{I}_m(t,s)\psi_s,\, g\rangle$ defined in* (5.22)*, the m-th term of the Born series* (5.6)*, admits the following representation:*

$$
\begin{aligned}
\mathcal{I}_m(t,s) = i^m \int_{\mathbb{R}^m} d\tau_1..d\tau_m e^{it(\tau_1+..+\tau_m)} \int_\lambda e^{i(t-s)\lambda} \sum_{k=0}^{m+1} \Bigg\langle \Bigg(\prod_{r=1}^{k-1} R_+(\lambda + \tau_r + .. + \tau_m) V(\hat{\tau}_r, \cdot) \Bigg)\\
(5.26)\qquad dE(\lambda + \tau_k + .. + \tau_m) \Bigg(\prod_{r=k+1}^{m+1} V(\hat{\tau}_{r-1}, \cdot) R_-(\lambda + \tau_r + .. + \tau_{m+1}) \Bigg) \psi_s,\, g \Bigg\rangle,
\end{aligned}
$$

where we formally set $\tau_{m+1} = 0$. The representation holds true with arbitrary Schwartz functions $\psi_s, g \in \mathcal{S}$.

Proof. We start by verifying that the expression on the right hand-side of (5.26) defines an absolutely convergent integral. Recall that the potential $V(\hat{\tau}, x)$ is smooth and has compact support in both variables. Therefore, the variables $\tau_1, .., \tau_m$ are restricted to a finite interval of $\mathbb{R}$. It also follows, with the help of our version of the limiting absorption principle, that the operators $V(\hat{\tau}, \cdot)R_\pm(\lambda)$ map $\mathcal{S}$ into $\mathcal{S}$ for all $\hat{\tau}, \lambda \in \mathbb{R}$. In addition, we have that

$$
dE(\lambda)f = \lambda^{-N} dE(\lambda)(\Delta)^N f
$$

for an arbitrary Schwartz function f. Hence,

$$\Big|\Big\langle\Big(\prod_{r=1}^{k-1} R_+(\lambda+\tau_r+..+\tau_m)V(\hat\tau_r,\cdot)\Big) \tag{5.27}$$

$$dE(\lambda+\tau_k+..+\tau_m)\Big(\prod_{r=k+1}^{m+1} V(\hat\tau_{r-1},\cdot)R_-(\lambda+\tau_r+..+\tau_m)\Big)\psi_s, g\Big\rangle\Big|$$

$$\le C(1+|\lambda|)^{-N} \tag{5.28}$$

for arbitrary Schwartz functions ψ_s and g with a constant C depending on ψ_s, g, and V (in particular, on the size of the support of $V(\hat\tau, x)$ in $\hat\tau$). This can be seen most easily by moving the operator in (5.27) onto g.

In what follows we shall manipulate the operator valued expressions with the tacit understanding that all equalities are to be interpreted in the weak sense. However, for ease of notation we suppress the pairing with the Schwartz functions ψ_s and g. The absolute convergence of all of integrals involved (after silent pairing with ψ_s, g) will also allow us to freely interchange the order of integrations.

We replace each of the $e^{i(s_k-s_{k-1})H_0}$ in (5.22) with its spectral representation:

$$\begin{aligned}
\mathcal{I}_m(t,s) = &\int_{\mathbb{R}^m} d\tau_1..d\tau_m \underbrace{\int\cdots\int}_{\lambda_1,..,\lambda_{m+1}} \underbrace{\int\cdots\int}_{s\le s_m\le..\le s_1\le t} ds_1...ds_m\; e^{i(t-s_1)\lambda_1}e^{is_1\tau_1}dE(\lambda_1)\,V(\hat\tau_1,\cdot)e^{i(s_1-s_2)\lambda_2} \\
&e^{is_2\tau_2}dE(\lambda_2)V(\hat\tau_2,\cdot)...e^{i(s_{m-1}-s_m)\lambda_m}e^{is_m\tau_m}dE(\lambda_m)V(\hat\tau_m,\cdot)e^{i(s_m-s)\lambda_{m+1}}dE(\lambda_{m+1}) \\
= &\int_{\mathbb{R}^m} d\tau_1..d\tau_m \underbrace{\int\cdots\int}_{\lambda_1,..,\lambda_{m+1}} \underbrace{\int\cdots\int}_{s\le s_m\le..\le s_1\le t} ds_1...ds_m\; e^{it\lambda_1}dE(\lambda_1)\,V(\hat\tau_1,\cdot)e^{is_1(\tau_1-\lambda_1+\lambda_2)}dE(\lambda_2)V(\hat\tau_2,\cdot) \\
&e^{is_2(\tau_2-\lambda_2+\lambda_3)}...\,dE(\lambda_m)V(\hat\tau_m,\cdot)e^{is_m(\tau_m-\lambda_m+\lambda_{m+1})}dE(\lambda_{m+1})\,e^{-is\lambda_{m+1}}.
\end{aligned} \tag{5.29}$$

Consider the first term

$$\mathcal{I}_1 = \int_{\mathbb{R}} d\tau_1 \int_{\lambda_1,\lambda_2}\int_s^t ds_1\; e^{it\lambda_1}dE(\lambda_1)\,V(\hat\tau_1,\cdot)e^{is_1(\tau_1-\lambda_1+\lambda_2)}dE(\lambda_2)e^{-is\lambda_2}.$$

Integrating explicitly relative to s_1 we infer that

$$\begin{aligned}
\mathcal{I}_1(t,s) &= -i\int_{\mathbb{R}} d\tau_1 \int_{\lambda_1,\lambda_2} e^{it\lambda_1}dE(\lambda_1)\,V(\hat\tau_1,\cdot)\frac{e^{it(\tau_1-\lambda_1+\lambda_2+i0)}-e^{is(\tau_1-\lambda_1+\lambda_2+i0)}}{\tau_1-\lambda_1+\lambda_2+i0}dE(\lambda_2)e^{-is\lambda_2} \\
&= -i\int_{\mathbb{R}} d\tau_1 e^{it(\tau_1+i0)}\int_{\lambda_1,\lambda_2} dE(\lambda_1)\,V(\hat\tau_1,\cdot)\frac{e^{i(t-s)\lambda_2}}{\tau_1-\lambda_1+\lambda_2+i0}dE(\lambda_2) \\
&\quad + i\int_{\mathbb{R}} d\tau_1 e^{is(\tau_1+i0)}\int_{\lambda_1,\lambda_2} dE(\lambda_1)\,V(\hat\tau_1,\cdot)\frac{e^{i(t-s)\lambda_1}}{\tau_1-\lambda_1+\lambda_2+i0}dE(\lambda_2) \\
&= i\int_{\mathbb{R}} d\tau_1 e^{it\tau_1}\int_{\lambda_2} e^{i(t-s)\lambda_2}R_+(\lambda_2+\tau_1)\,V(\hat\tau_1,\cdot)dE(\lambda_2) \\
&\quad + i\int_{\mathbb{R}} d\tau_1 e^{is\tau_1}\int_{\lambda_1} e^{i(t-s)\lambda_1}dE(\lambda_1)\,V(\hat\tau_1,\cdot)R_-(\lambda_1-\tau_1) \\
&= i\int_{\mathbb{R}} d\tau_1 e^{it\tau_1}\int_{\lambda} e^{i(t-s)\lambda}\Big(R_+(\lambda+\tau_1)\,V(\hat\tau_1,\cdot)dE(\lambda)+dE(\lambda+\tau_1)\,V(\hat\tau_1,\cdot)R_-(\lambda)\Big).
\end{aligned}$$

In the above calculation we have used the spectral representation (5.23) for the resolvent and (5.24). The proof now proceeds inductively. We shall assume that

$$\mathcal{I}_m(t,s) = i^m \int_{\mathbb{R}^m} d\tau_1 \dots d\tau_m \; e^{it(\tau_1+..+\tau_m)} \int_\lambda e^{i(t-s)\lambda} dM_m(\lambda;\tau_1,..,\tau_m), \tag{5.30}$$

where $dM_m(\lambda;\tau_1,..,\tau_m)$ is the operator valued measure [3] defined by

$$\begin{aligned} dM_m(\lambda;\tau_1,..,\tau_m) = \sum_{k=0}^{m} \Big[& R_+(\lambda+\tau_1+..+\tau_m)V(\hat{\tau}_1,\cdot)R_+(\lambda+\tau_2+..+\tau_m)V(\hat{\tau}_2,\cdot)...V(\hat{\tau}_{k-1},\cdot) \\ & dE(\lambda+\tau_k+..+\tau_m)V(\hat{\tau}_k,\cdot)R_-(\lambda+\tau_{k+1}+..+\tau_m)V(\hat{\tau}_{k+1},\cdot)...V(\hat{\tau}_m,\cdot)R_-(\lambda)\Big]. \end{aligned}$$

Formally setting $\tau_{m+1}=0$, we can also write the above expression in the following more concise form:

$$\begin{aligned} dM_m(\lambda;\tau_1,..,\tau_m) = \sum_{k=0}^{m+1} & \Big(\prod_{r=1}^{k-1} R_+(\lambda+\tau_r+..+\tau_m)V(\hat{\tau}_r,\cdot)\Big) dE(\lambda+\tau_k+..+\tau_m) \\ & \Big(\prod_{r=k+1}^{m+1} V(\hat{\tau}_{r-1},\cdot)R_-(\lambda+\tau_r+..+\tau_{m+1})\Big). \end{aligned} \tag{5.31}$$

We have already verified (5.30) for $m=1$. It remains to check that

$$\mathcal{I}_{m+1}(t,s) = i^{m+1}\int_{\mathbb{R}^{m+1}} d\tau_1..d\tau_{m+1}\, e^{it(\tau_1+..+\tau_m+\tau_{m+1})} \int_\lambda e^{i(t-s)\lambda} dM_{m+1}(\lambda;\tau_1,..,\tau_{m+1}).$$

We can deduce from (5.29) the following recursive identity:

$$\mathcal{I}_{m+1}(t,s) = \int_{\tau_1} d\tau_1 \int_{\lambda_1}\int_s^t ds_1 e^{it\lambda_1} dE(\lambda_1)V(\hat{\tau}_1,\cdot)\, e^{is_1(\tau_1-\lambda_1)}\mathcal{I}_m(s_1,s).$$

Substituting the expression for $\mathcal{I}_m$ from (5.30) we obtain

$$\begin{aligned} \mathcal{I}_{m+1}(t,s) = i^m \int_{\mathbb{R}^{m+1}} d\tau_1...d\tau_{m+1} \int_{\lambda,\lambda_1}\int_s^t & ds_1 e^{it\lambda_1} dE(\lambda_1)V(\hat{\tau}_1,\cdot) \\ & e^{is_1(\tau_1+..+\tau_{m+1}-\lambda_1)}\, e^{i(s_1-s)\lambda} dM_m(\lambda;\tau_2,..,\tau_{m+1}). \end{aligned}$$

[3] Once again we make sense of $dM_m(\lambda;\tau_1,..,\tau_m)$ only after pairing it with the Schwartz functions ψ_s and g. Then $\langle dM_m(\lambda;\tau_1,..,\tau_m)\psi_s, g\rangle$ is a finite measure relative to λ — in fact, rapidly decaying in λ, see (5.28) — which depends smoothly on $\tau_1,..,\tau_m$ and vanishes outside of a compact set in these variables.

Integrating explicitly relative to s_1 we infer that

$$\mathcal{I}_{m+1}(t,s) = -i^{m+1}\int_{\mathbb{R}^{m+1}} d\tau_1...d\tau_{m+1}\int_{\lambda,\lambda_1} e^{it\lambda_1}e^{-is\lambda}dE(\lambda_1)V(\hat\tau_1,\cdot)$$

$$\frac{e^{it(\tau_1+..+\tau_{m+1}-\lambda_1+\lambda+i0)}-e^{is(\tau_1+..+\tau_{m+1}-\lambda_1+\lambda+i0)}}{\tau_1+..+\tau_{m+1}-\lambda_1+\lambda+i0}dM_m(\lambda;\tau_2,..,\tau_{m+1})$$

$$= -i^{m+1}\int_{\mathbb{R}^{m+1}} d\tau_1...d\tau_{m+1}\ e^{it(\tau_1+..+\tau_{m+1}+i0)}\int_{\lambda,\lambda_1} e^{i(t-s)\lambda}$$

$$\frac{dE(\lambda_1)}{\tau_1+..+\tau_{m+1}-\lambda_1+\lambda+i0}V(\hat\tau_1,\cdot)dM_m(\lambda;\tau_2,..,\tau_{m+1})$$

$$+\ i^{m+1}\int_{\mathbb{R}^{m+1}} d\tau_1...d\tau_{m+1}\ e^{is(\tau_1+..+\tau_{m+1}+i0)}\int_{\lambda,\lambda_1} e^{i(t-s)\lambda_1}dE(\lambda_1)$$

$$V(\hat\tau_1,\cdot)\frac{dM_m(\lambda;\tau_2,..,\tau_{m+1})}{\tau_1+..+\tau_{m+1}-\lambda_1+\lambda+i0} = J_1+J_2.$$

According to (5.23) and (5.24)

$$\int_{\lambda_1}\frac{dE(\lambda_1)}{\tau_1+..+\tau_{m+1}-\lambda_1+\lambda+i0} = -R_+(\lambda+\tau_1+..+\tau_{m+1}).$$

Therefore,

(5.32)

$$J_1 = i^{m+1}\int_{\mathbb{R}^{m+1}} d\tau_1...d\tau_{m+1}e^{it(\tau_1+..+\tau_{m+1})}\int_{\lambda} e^{i(t-s)\lambda}R_+(\lambda+\tau_1+..+\tau_{m+1})V(\hat\tau_1,\cdot)dM_m(\lambda;\tau_2,..,\tau_{m+1}).$$

Observe that, with the convention that $\tau_{m+2}=0$,

$$R_+(\lambda+\tau_1+..+\tau_{m+1})V(\hat\tau_1,\cdot)dM_m(\lambda;\tau_2,..,\tau_{m+1}) = \sum_{k=2}^{m+2}\Big[\Big(\prod_{r=1}^{k-1}R_+(\lambda+\tau_r+..+\tau_{m+1})V(\hat\tau_r,\cdot)\Big)$$

$$dE(\lambda+\tau_k+..+\tau_{m+1})\Big(\prod_{r=k+1}^{m+2}V(\hat\tau_{r-1},\cdot)R_-(\lambda+\tau_r+..+\tau_{m+2})\Big)\Big]. \tag{5.33}$$

It remains to consider the integral J_2.

$$J_2 = i^{m+1}\int_{\mathbb{R}^{m+1}} d\tau_1...d\tau_{m+1}e^{is(\tau_1+..+\tau_{m+1}+i0)}\int_{\lambda,\lambda_1} e^{i(t-s)\lambda_1}dE(\lambda_1)V(\hat\tau_1,\cdot)\frac{dM_m(\lambda;\tau_2,..,\tau_{m+1})}{\tau_1+..+\tau_{m+1}-\lambda_1+\lambda+i0}$$

(5.34)

$$= i^{m+1}\int_{\mathbb{R}^{m+1}} d\tau_1...d\tau_{m+1}e^{it(\tau_1+..+\tau_{m+1})+is(i0)}\int_{\lambda_1} e^{i(t-s)\lambda_1}dE(\lambda_1+\tau_1+..+\tau_{m+1})$$

$$V(\hat\tau_1,\cdot)\int_{\lambda}\frac{dM_m(\lambda;\tau_2,..,\tau_{m+1})}{\lambda-\lambda_1+i0}$$

Inspection of the desired expression for $dM_{m+1}(\lambda;\tau_1,..,\tau_{m+1})$ and equations (5.32)-(5.34) suffices to verify the following formula:

$$\int_{\lambda}\frac{dM_m(\lambda;\tau_2,..,\tau_{m+1})}{\lambda-\lambda_1+i0} = \Big(\prod_{r=2}^{m+1}R_-(\lambda_1+\tau_r+..+\tau_{m+1})V(\hat\tau_r,\cdot)\Big)R_-(\lambda_1)\,.$$

This is accomplished in the following two lemmas, and we are done. $\square$

We recall definition (5.31) of the operator valued measure dM_m and prove the following more general result

Lemma 5.7. *Let $a_1, .., a_m \in \mathbb{R}$ be a sequence of arbitrary real numbers and let $A_1, .., A_m$ be arbitrary operators*[4]*. Then*

$$\int_\lambda \frac{1}{\lambda - \mu + i0} \sum_{k=1}^m \Big(\prod_{r=1}^{k-1} R_+(\lambda + a_r) A_r\Big) dE(\lambda + a_k) \Big(\prod_{r=k+1}^m A_{r-1} R_-(\lambda + a_r)\Big)$$

$$= \Big(\prod_{r=1}^{m-1} R_-(\mu + a_r) A_r\Big) R_-(\mu + a_m). \tag{5.35}$$

As before, the identity holds after pairing the above expressions with a pair of Schwartz functions ψ_s, g.

Proof. We shall write each $R_\pm(\lambda + a_r)$, for all values of $r = 1, .., m$ different from k using the spectral representation

$$R_\pm(\lambda + a_r) = \int_{\lambda_r} \frac{dE(\lambda_r + a_r)}{\lambda_r - \lambda \mp i0}.$$

We shall also rename the variable of integration λ to λ_k in each term of the sum in k. The left hand-side of (5.35) then takes the following form:

$$\underset{\lambda_1,..,\lambda_m}{\int \cdots \int} \sum_{k=1}^m \frac{1}{\lambda_k - \mu + i0} \prod_{r=1}^{k-1} \frac{1}{\lambda_r - \lambda_k - i0} \prod_{r=k+1}^m \frac{1}{\lambda_r - \lambda_k + i0} \Big(\prod_{j=1}^{m-1} dE(\lambda_j + a_j) A_j\Big) dE(\lambda_m + a_m)$$

The proof of Lemma 5.7 is finished provided that we can show that the following identity holds true:

$$\sum_{k=1}^m \frac{1}{\lambda_k - \mu + i0} \prod_{r=1}^{k-1} \frac{1}{\lambda_r - \lambda_k - i0} \prod_{r=k+1}^m \frac{1}{\lambda_r - \lambda_k + i0} = \prod_{r=1}^m \frac{1}{\lambda_r - \mu + i0}$$

In the distributional sense

$$\lim_{\varepsilon_k \to 0^+} \lim_{\varepsilon_1 \to 0^+} \ldots \lim_{\varepsilon_{k-1} \to 0^+} \lim_{\varepsilon_{k+1} \to 0^-} \ldots \lim_{\varepsilon_m \to 0^-} \frac{1}{\lambda_k - \mu + i\varepsilon_k} \prod_{r=1, r\neq k}^m \frac{1}{\lambda_r - \lambda_k - i\varepsilon_r}$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{\lambda_k - \mu + ik\varepsilon} \prod_{r=1, r\neq k}^m \frac{1}{\lambda_r - \lambda_k + i(r-k)\varepsilon}$$

Therefore, we can introduce the new variables $z_r = \lambda_r - \mu + ir\varepsilon$, $r = 1, .., m$ and prove instead the following statement. $\square$

[4]It suffices to assume that that the operators A_k, $k = 1, .., m$ map the space $C^\infty(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ into the the space $\mathcal{S}$.

Lemma 5.8. *For any pairwise distinct complex numbers $z_1, .., z_m \in \mathbb{C}$,*

$$\sum_{k=1}^{m} \frac{1}{z_k} \prod_{r=1, r\neq k}^{m} \frac{1}{z_r - z_k} = \prod_{r=1}^{m} \frac{1}{z_r}.$$

Proof. The key identity is the statement of the lemma for $m = 2$

$$\frac{1}{z_1(z_2 - z_1)} + \frac{1}{z_2(z_1 - z_2)} = \frac{1}{z_1 z_2}$$

which follows immediately by inspection. The general case then can be proved by induction. We shall assume that the identity holds true for $m-1$ and prove the result for m. We first note a simple equality

$$\frac{1}{(z_m - z_k)} = \frac{1}{(z_m - z_1)} + \frac{z_k - z_1}{(z_m - z_k)(z_m - z_1)}.$$

Therefore,

$$\sum_{k=1}^{m} \frac{1}{z_k} \prod_{r=1, r\neq k}^{m} \frac{1}{z_r - z_k} = \frac{1}{z_m - z_1} \sum_{k=1}^{m-1} \frac{1}{z_k} \prod_{r=1, r\neq k}^{m-1} \frac{1}{z_r - z_k} - \frac{1}{z_m - z_1} \sum_{k=2}^{m-1} \frac{1}{z_k} \prod_{r=2, r\neq k}^{m} \frac{1}{z_r - z_k}$$
$$+ \frac{1}{z_m} \prod_{r=1}^{m-1} \frac{1}{z_r - z_m}.$$

According to the assumption $m-1$ with $z_1, .., z_{m-1}$ the first term on the right hand-side gives $\frac{1}{(z_m - z_1) z_1 \cdots z_{m-1}}$. We also have

$$\frac{1}{z_m - z_1} \sum_{k=2}^{m-1} \frac{1}{z_k} \prod_{r=2, r\neq k}^{m} \frac{1}{z_r - z_k} = \frac{1}{z_m - z_1} \sum_{k=2}^{m} \frac{1}{z_k} \prod_{r=2, r\neq k}^{m} \frac{1}{z_r - z_k} - \frac{1}{z_m - z_1} \frac{1}{z_m} \prod_{r=2}^{m-1} \frac{1}{z_r - z_m}$$
$$= \frac{1}{(z_m - z_1) z_2 \cdots z_m} - \frac{1}{z_m} \prod_{r=1}^{m-1} \frac{1}{z_r - z_m}$$

by the $m-1$ inductive assumption for $z_2, .., z_m$. Finally,

$$\frac{1}{(z_m - z_1) z_1 \cdots z_{m-1}} - \frac{1}{(z_m - z_1) z_2 \cdots z_m} = \frac{1}{z_1 \cdots z_m},$$

as desired. □

We shall now derive the explicit representation of the integral kernel of the operator $\mathcal{I}_m(t,s)$ acting on the Schwartz functions ψ_s. We start by noting the following simple identity which holds for arbitrary real numbers $a_1, .., a_{m+1}$ with $m \geq 1$:

$$\sum_{k=1}^{m+1} e^{i(a_1 + .. + a_{k-1} - a_{k+1} - .. - a_{m+1})} \sin a_k = \sin\left(\sum_{k=1}^{m+1} a_k\right). \tag{5.36}$$

This identity can be easily proved by induction on m. Recall that

$$R_\pm(\mu)(x,y) = \frac{e^{\pm i\sqrt{\mu}|x-y|}}{4\pi|x-y|}, \quad \mu \in \mathbb{R}$$

with $\sqrt{\mu}$ defined in such a way that $\mathrm{Im}\sqrt{\mu} > 0$ for $\mathrm{Im}\mu > 0$. We have $R_+(\mu) = R_-(\mu)$ for $\mu < 0$. Also recall that the kernel of the spectral measure

$$dE(\mu)(x,y) = \begin{cases} \frac{\sin\sqrt{\mu}|x-y|}{4\pi|x-y|}\, d\mu & \mu > 0, \\ 0 & \mu \le 0 \end{cases}.$$

We return to the representation (5.26) for the $\mathcal{I}_m$. Let (with $\tau_{m+1} = 0$)

$$\tau_j + .. + \tau_{m+1} = \min_{r\in[1,m]} (\tau_r + .. + \tau_{m+1}).$$

To simplify the formulae we introduce a new operator $\mathcal{J}_m(t,s)$, implicitly dependent on $\tau_1, .., \tau_m$,

$$\mathcal{I}_m(t,s) = i^m \int_{\mathbb{R}^m} d\tau_1..d\tau_m\, e^{it(\tau_1+..+\tau_m)} e^{-i(t-s)(\tau_j+..+\tau_{m+1})} \mathcal{J}_m(t,s)(\tau_1,..,\tau_m),$$

(5.37)

$$\mathcal{J}_m(t,s) := \int_\lambda e^{i(t-s)(\lambda+\tau_j+..+\tau_{m+1})} \sum_{k=1}^{m+1} \Big(\prod_{r=1}^{k-1} R_+(\lambda+\tau_r+..+\tau_m)V(\hat{\tau}_r,\cdot)\Big) dE(\lambda+\tau_k+..+\tau_m)$$
$$\Big(\prod_{r=k+1}^{m+1} V(\hat{\tau}_{r-1},\cdot)R_-(\lambda+\tau_r+..+\tau_{m+1})\Big).$$

Define non-negative numbers σ_r, $r = 1,..,m+1$

$$\sigma_r = (\tau_r + .. + \tau_{m+1}) - (\tau_j + .. + \tau_{m+1}).$$

After a change of variables we obtain the expression

$$\mathcal{J}_m(t,s) = \int_\lambda e^{i(t-s)\lambda} \sum_{k=1}^{m+1} \Big(\prod_{r=1}^{k-1} R_+(\lambda+\sigma_r)V(\hat{\tau}_r,\cdot)\Big) dE(\lambda+\sigma_k)\Big(\prod_{r=k+1}^{m+1} V(\hat{\tau}_{r-1},\cdot)R_-(\lambda+\sigma_r)\Big).$$

Observe that due to the presence of $dE(\lambda+\sigma_k)$ the k^{th} term in the sum above vanishes for $\lambda \le -\sigma_k \le 0$. Therefore,

$$\mathcal{J}_m(t,s) = \mathcal{L}_m(t,s) + \mathcal{M}_m(t,s)$$

(5.38)

$$\mathcal{L}_m(t,s) := \int_0^\infty e^{i(t-s)\lambda} \sum_{k=1}^{m+1} \Big(\prod_{r=1}^{k-1} R_+(\lambda+\sigma_r)V(\hat{\tau}_r,\cdot)\Big) dE(\lambda+\sigma_k)\Big(\prod_{r=k+1}^{m+1} V(\hat{\tau}_{r-1},\cdot)R_-(\lambda+\sigma_r)\Big)$$

$$\mathcal{M}_m(t,s) := \int_{-\infty}^0 e^{i(t-s)\lambda} \sum_{k=1}^{m+1} \Big(\prod_{r=1}^{k-1} R_+(\lambda+\sigma_r)V(\hat{\tau}_r,\cdot)\Big) dE(\lambda+\sigma_k)\Big(\prod_{r=k+1}^{m+1} V(\hat{\tau}_{r-1},\cdot)R_-(\lambda+\sigma_r)\Big).$$

To obtain the explicit formula for the integral kernel of the operator $\mathcal{L}_m(t,s)$ we make use of the following: the parameters $\sigma_k \geq 0$, $\lambda \geq 0$ on the interval of integration, and the explicit representations for the kernels of $R_\pm(\mu)$ and $dE(\mu)$. We have

$$\mathcal{L}_m(t,s)(x,y) = \int_{\mathbb{R}^m} dx_1..dx_m \int_0^\infty d\lambda\, e^{i(t-s)\lambda} \sum_{k=1}^{m+1} \Big[e^{i(\sqrt{\lambda+\sigma_1}|x-x_1|+..+\sqrt{\lambda+\sigma_{k-1}}|x_{k-2}-x_{k-1}|)}$$
$$e^{-i(\sqrt{\lambda+\sigma_{k+1}}|x_k-x_{k+1}|-..-\sqrt{\lambda+\sigma_m}|x_{m-1}-y|)} \sin\left(\sqrt{\lambda+\sigma_k}|x_{k-1}-x_k|\right) \prod_{r=1}^m \frac{V(\hat{\tau}_r, x_r)}{4\pi|x_{r-1}-x_r|}\, \frac{1}{4\pi|x_m-y|} \Big],$$

where we set $x_0 = x$. We now recall the identity (5.36) to infer that

$$\mathcal{L}_m(t,s)(x,y) = \int_{\mathbb{R}^m} dx_1..dx_m \prod_{r=1}^m \frac{V(\hat{\tau}_r, x_r)}{4\pi|x_{r-1}-x_r|} \frac{1}{4\pi|x_m-y|} \int_0^\infty d\lambda\, e^{i(t-s)\lambda} \sin\left(\sum_{k=1}^{m+1} \sqrt{\lambda+\sigma_k}|x_{k-1}-x_k| \right).$$

Changing variables in the λ-integral and integrating by parts yield (formally)

$$\int_0^\infty d\lambda\, e^{i(t-s)\lambda} \sin\left(\sum_{k=1}^m \sqrt{\lambda+\sigma_k}|x_{k-1}-x_k| \right) = 2\int_0^\infty d\lambda\, \lambda\, e^{i(t-s)\lambda^2} \sin\left(\sum_{k=1}^m \sqrt{\lambda^2+\sigma_k}|x_{k-1}-x_k| \right)$$
$$= \frac{i}{t-s} \sum_{\ell=1}^m \int_0^\infty d\lambda\, e^{i(t-s)\lambda^2} \cos\left(\sum_{k=1}^m \sqrt{\lambda^2+\sigma_k}|x_{k-1}-x_k| \right) \frac{\lambda}{\sqrt{\lambda^2+\sigma_\ell}} |x_{\ell-1}-x_\ell| \tag{5.39}$$
$$+\text{bounday term at } 0.$$

"Formally" here refers to the fact that the integration extends to ∞ and that the boundary term vanishes at ∞. These statements can be made precise in the usual way, i.e., by means of suitable cut-offs at points tending to infinity. Therefore, finally

$$\mathcal{L}_m(t,s)(x,y) = \frac{i}{t-s} \sum_{\ell=1}^m \mathcal{L}_m^\ell(t,s)(x,y) + \text{bounday term at } 0,$$
$$\mathcal{L}_m^\ell(t,s)(x,y) := \int_{\mathbb{R}^m} dx_1 \dots dx_m \prod_{r=1}^m \frac{V(\hat{\tau}_r, x_r)}{4\pi|x_{r-1}-x_r|}\, \frac{|x_{\ell-1}-x_\ell|}{4\pi|x_m-y|} \tag{5.40}$$
$$\int_0^\infty d\lambda\, e^{i(t-s)\lambda^2} \cos\left(\sum_{k=1}^m \sqrt{\lambda^2+\sigma_k}|x_{k-1}-x_k| \right) \frac{\lambda}{\sqrt{\lambda^2+\sigma_\ell}}.$$

To describe the integral kernels of the operators $\mathcal{M}_m^k(t,s)$ we shall first order and rename the parameters σ_k, $k = 1,..,m+1$. In fact, define inductively

$$\omega_d = \max\{\sigma_k\}_{k\in[1,m+1]} \setminus \{\omega_\ell\}_{\ell\in[1,d-1]},$$

and set $k = k(c)$ and $c = c(k)$ iff $\sigma_k = \omega_c$. We shall split the interval of integration in λ in $(-\infty, 0]$ into the subintervals

$$(-\infty, -\sqrt{\omega_1}\,], [-\sqrt{\omega_{d-1}}, -\sqrt{\omega_d}\,] \ \text{ for } \ d \in [2, m+1], \quad \text{and} \quad [-\sqrt{\omega}_{m+1}, 0].$$

For $\lambda \in [-\sqrt{\omega}_{d-1}, -\sqrt{\omega}_d]$, the spectral measures $dE(\lambda + \sigma_{k(c)}) = dE(\lambda + \omega_c)$ vanish for all $c \geq d$. Therefore, with the convention that $\omega_0 = \infty$ and $\omega_{m+2} = 0$, we have

$$\mathcal{M}_m(t,s) = \sum_{d=1}^{m+2} \mathcal{M}_m^d,$$

$$\mathcal{M}_m^d(t,s) := \int_{-\sqrt{\omega_{d-1}}}^{-\sqrt{\omega_d}} e^{i(t-s)\lambda} \sum_{c=1}^{d} \Big(\prod_{r=1}^{k(c)-1} R_+(\lambda + \omega_{a(r)}) V(\hat{\tau}_r, \cdot) \Big) dE(\lambda + \omega_c) \Big(\prod_{r=k(c)+1}^{m+1} V(\hat{\tau}_{r-1}, \cdot) R_-(\lambda + \omega_{a(r)}) \Big). \tag{5.41}$$

Strictly speaking, $\mathcal{M}_m^1(t,s) = 0$ so that the sum over d starts at $d = 2$. The integral kernels of $R_\pm(\lambda + \omega_{a(r)})$ for $a(r) \leq d-1$ contribute oscillating exponential phases while for $a(r) \geq d$ they produce exponentially decaying factors. Hence,

$$\mathcal{M}_m^d(t,s)(x,y) = \int_{\mathbb{R}^m} dx_1..dx_m \prod_{r=1}^{m} \frac{V(\hat{\tau}_r, x_r)}{4\pi|x_{r-1} - x_r|} \frac{1}{4\pi|x_m - y|} \int_{-\sqrt{\omega_{d-1}}}^{-\sqrt{\omega_d}} d\lambda e^{i(t-s)\lambda}$$
$$\sum_{c=1}^{d-1} e^{i(\sqrt{\lambda+\omega_1}|x_{k(1)-1} - x_{k(1)}| + .. + \sqrt{\lambda+\omega_{c-1}}|x_{k(c)-2} - x_{k(c)-1}|)}$$
$$e^{-i(\sqrt{\lambda+\omega_{c+1}}|x_{k(c+1)-1} - x_{k(c+1)}| + .. + \sqrt{\lambda+\omega_{d-1}}|x_{k(d-1)-1} - x_{k(d-1)}|)}$$
$$\sin\big(\sqrt{\lambda+\omega_c}|x_{k(c)-1} - x_{k(c)}|\big) e^{-\sum_{a=d+1}^{m+1} \sqrt{-\omega_a - \lambda}|x_{k(a)-1} - x_{k(a)}|}.$$

Once again we recall the identity (5.36) to infer that

$$\sum_{c=1}^{d-1} e^{i\big(\sum_{a=1}^{c-1} \sqrt{\lambda+\omega_a}|x_{k(a)-1} - x_{k(a)}| - \sum_{b=c+1}^{d-1} \sqrt{\lambda+\omega_{b+1}}|x_{k(b+1)-1} - x_{k(b+1)}|\big)} \sin\big(\sqrt{\lambda+\omega_c}|x_{k(c)-1} - x_{k(c)}|\big)$$
$$= \sin\Big(\sum_{c=1}^{d-1} \sqrt{\lambda+\omega_c}|x_{k(c)-1} - x_{k(c)}|\Big).$$

Therefore,

$$\mathcal{M}_m^d(t,s)(x,y) = \int_{\mathbb{R}^m} dx_1..dx_m \prod_{r=1}^{m} \frac{V(\hat{\tau}_r, x_r)}{4\pi|x_{r-1} - x_r|} \frac{1}{4\pi|x_m - y|}$$
$$\int_{-\sqrt{\omega_{d-1}}}^{-\sqrt{\omega_d}} d\lambda\, e^{i(t-s)\lambda} \sin\Big(\sum_{c=1}^{d-1} \sqrt{\lambda+\omega_c}|x_{k(c)-1} - x_{k(c)}|\Big) e^{-\sum_{a=d}^{m+1} \sqrt{-\omega_a - \lambda}|x_{k(a)-1} - x_{k(a)}|}.$$

We would like to change variables $\lambda \to \lambda^2$ and integrate by parts relative to λ, as we did for $\mathcal{L}_m$. Denote the λ-integrand in each of the kernels $\mathcal{M}_m^d(t,s)(x,y)$ by $F_d(\lambda)$. It is not difficult to see that

$F_d(-\omega_d) = F_{d+1}(-\omega_d)$ for $d = 1,..,m+1$. Therefore, the boundary terms will cancel each other telescopically, at least all boundary terms that appear pairwise as both upper and lower limits. Note that there are exactly two boundary terms that are not of this nature, namely ω_1 and $\omega_{m+2} = 0$. The latter cancels against the boundary term at zero in (5.39), whereas the former disappears due to the fact that $\sin 0 = 0$. This allows us, in what follows, to ignore the boundary terms altogether. We now make a change of variables $\lambda \to \lambda^2 - \omega_{d-1}$. We also re-introduce the notation σ_a in the new capacity:

$$\begin{aligned} 0 \le \sigma_a &= \omega_a - \omega_{d-1}, \qquad && a = 0,..,d-1,\\ 0 \le \rho_a &= \omega_{d-1} - \omega_a, \qquad && a = d,..,m+2. \end{aligned}$$

Thus

$$\mathcal{M}_m^d(t,s)(x,y) = \int_{\mathbb{R}^m} dx_1..dx_m \prod_{r=1}^m \frac{V(\hat{\tau}_r, x_r)}{4\pi|x_{r-1}-x_r|} \frac{1}{4\pi|x_m - y|}$$
$$\int_0^{\sqrt{\rho_d}} d\lambda\; \lambda e^{i(t-s)\lambda^2} \sin\left(\sum_{c=1}^{d-1} \sqrt{\lambda^2+\sigma_c}|x_{k(c)-1} - x_{k(c)}|\right) e^{-\sum_{a=d}^{m+1} \sqrt{\rho_a - \lambda^2}|x_{k(a)-1} - x_{k(a)}|}.$$

Integrating by parts relative to λ and canceling the contribution from the boundary terms as explained above, we finally obtain

$$\mathcal{M}_m^d(t,s)(x,y) = \frac{i}{t-s}\sum_{\ell=1}^{d-1} \mathcal{M}_m^{d,\ell}(t,s)(x,y) + \frac{i}{t-s}\sum_{\ell=d}^{m+1} \widetilde{\mathcal{M}}_m^{d,\ell}(t,s)(x,y),$$

(5.42)

$$\mathcal{M}_m^{d,\ell}(t,s)(x,y) := \int_{\mathbb{R}^m} dx_1..dx_m \prod_{r=1}^m \frac{V(\hat{\tau}_r, x_r)}{4\pi|x_{r-1}-x_r|} \frac{|x_{k(l)-1} - x_{k(l)}|}{4\pi|x_m - y|}$$
$$\int_0^{\sqrt{\rho_d}} d\lambda\; e^{i(t-s)\lambda^2} \cos\left(\sum_{c=1}^{d-1} \sqrt{\lambda^2+\sigma_c}|x_{k(c)-1} - x_{k(c)}|\right) e^{-\sum_{a=d}^{m+1} \sqrt{\rho_a - \lambda^2}|x_{k(a)-1} - x_{k(a)}|} \frac{\lambda}{\sqrt{\lambda^2+\sigma_\ell}}$$

(5.43)

$$\widetilde{\mathcal{M}}_m^{d,\ell}(t,s)(x,y) := -\int_{\mathbb{R}^m} dx_1..dx_m \prod_{r=1}^m \frac{V(\hat{\tau}_r, x_r)}{4\pi|x_{r-1}-x_r|} \frac{|x_{k(l)-1} - x_{k(l)}|}{4\pi|x_m - y|}$$
$$\int_0^{\sqrt{\rho_d}} d\lambda\; e^{i(t-s)\lambda^2} \sin\left(\sum_{c=1}^{d-1} \sqrt{\lambda^2+\sigma_c}|x_{k(c)-1} - x_{k(c)}|\right) e^{-\sum_{a=d}^{m+1} \sqrt{\rho_a - \lambda^2}|x_{k(a)-1} - x_{k(a)}|} \frac{\lambda}{\sqrt{\rho_\ell - \lambda^2}}.$$

Combining (5.37)-(5.43) we can state the following

Proposition 5.9. *The integral kernel of $\mathcal{I}_m(t,s)$, the m-th term of the Born series (5.6), can be written in the following form:*

$$
\begin{aligned}
\mathcal{I}_m(t,s)(x,y) = \frac{i^{m+1}}{t-s}\int_{\mathbb{R}^m} d\tau_1..d\tau_m e^{i(\tau_1+..+\tau_m)} \\
\Big(\sum_{\ell=1}^{m}\mathcal{L}^\ell_m(t,s)(x,y)(\sigma_1,..,\sigma_m) + \quad & \sum_{d=0}^{m+2}\sum_{\ell=1}^{d-1}\mathcal{M}^{d,\ell}_m(t,s)(x,y)(\sigma_1,..,\sigma_{d-1},\rho_d,..,\rho_{m+1}) + \\
& \sum_{d=0}^{m+2}\sum_{\ell=1}^{d-1}\widetilde{\mathcal{M}}^{d,\ell}_m(t,s)(x,y)(\sigma_1,..,\sigma_{d-1},\rho_d,..,\rho_{m+1})\Big).
\end{aligned}
\tag{5.44}
$$

We interpret $\mathcal{I}_m(t,s)(x,y)$ as follows: for any pair of Schwartz functions ψ_s and g

$$\langle \mathcal{I}_m(t,s)\psi_s,\, g\rangle = \int_{\mathbb{R}^6}\mathcal{I}_m(t,s)(x,y)\,\psi_s(y)\,g(x)\,dx\,dy.$$

The functions

$$\mathcal{L}^\ell_m(t,s)(x,y),\quad \mathcal{M}^{d,\ell}_m(t,s)(x,y),\quad \widetilde{\mathcal{M}}^{d,\ell}_m(t,s)(x,y)$$

are defined in (5.40), (5.42), and (5.43) correspondingly with implicit dependence on the parameters σ_k,ρ_ℓ. The latter are positive and depend exclusively and in a linear fashion on $\tau_1,..,\tau_m$.

6 Estimates for oscillatory integrals

The purpose of this section is to prove the following lemma.

Lemma 6.1. *There exists a constant C_0 which only depends on the constant a_0 so that for any positive integer m and any $1\le k\le m$,*

$$\left|\int_0^\infty e^{\frac12 i\lambda^2}\, e^{\pm i\sum_{j=1}^m b_j\sqrt{\lambda^2+\sigma_j}}\,\frac{\lambda}{\sqrt{\lambda^2+\sigma_k}}\,d\lambda\right| \le C_0\, m^2\, b_k^{-1}\,\max_\ell b_\ell \tag{6.1}$$

for any choice of $\sigma_1\ge\sigma_2\ge\ldots\ge\sigma_m\ge 0$ and $b_j>0$.

Proof. Changing variables $u=\lambda^2$ reduces the integral in (6.1) to

$$\int_0^\infty e^{\frac12 iu}\, e^{\pm i\sum_{j=1}^m b_j\sqrt{u+\sigma_j}}\,\frac{du}{\sqrt{u+\sigma_k}}. \tag{6.2}$$

Denote the phase by $\phi_\pm(u)=\frac12 u\pm\sum_{j=1}^m b_j\sqrt{u+\sigma_j}$. Consider first $\phi(u)=\phi_+(u)$. Then

$$\phi'(u) = 1+\sum_{j=1}^m\frac{b_j}{\sqrt{u+\sigma_j}},\quad \phi''(u) = -\sum_{j=1}^m\frac{b_j}{(u+\sigma_j)^{\frac32}}. \tag{6.3}$$

In particular, $\phi'(u)\ge 1$ and $|\phi''(u)|\le u^{-1}\phi'(u)$. Let χ be a smooth non-decreasing function with $\chi(u)=0$ for $u\le 1$ and $\chi(u)=1$ for $u\ge 2$. Then

$$\left|\int_0^\infty e^{i\phi(u)}\,\frac{du}{\sqrt{u+\sigma_k}}\right| \le C + \limsup_{L\to\infty}\left|\int_0^\infty e^{i\phi(u)}\, g_L(u)\,du\right| \tag{6.4}$$

where we have set

$$g_L(u) := \chi(u)(1-\chi(u/L))\,\frac{1}{\sqrt{u+\sigma_k}}. \tag{6.5}$$

Clearly, $|g_L^{(j)}(u)| \le C_j\, u^{-j-\frac12}$ for $j=0,1$ uniformly in L. Integrating by parts once inside the integral on the right-hand side of (6.4) yields an upper bound of the form

$$\int_0^\infty \Big|\frac{d}{du}\Big[\frac{1}{\phi'(u)}g_L(u)\Big]\Big|\,du \le \int_1^\infty \Big[\frac{|\phi''(u)|}{\phi'(u)^2} + \frac{1}{u\phi'(u)}\Big]\,\frac{du}{\sqrt{u}} \lesssim \int_1^\infty u^{-\frac32}\,du \le C,$$

as claimed.

Next consider $\phi(u) := \phi_-(u)$. Then

$$\phi'(u) = 1 - \sum_{j=1}^m \frac{b_j}{\sqrt{u+\sigma_j}}, \quad \phi''(u) = \sum_{j=1}^m \frac{b_j}{(u+\sigma_j)^{\frac32}}.$$

Therefore, $\phi''(u) > 0$ and ϕ has at most one non-degenerate critical point $u_0 \ge 0$.

Fix some $A > 0$ and assume that $u_0 > 2A$. Then integration by parts yields

$$\begin{aligned}\Big|\int_0^\infty e^{i\phi(u)}\,\frac{du}{\sqrt{u+\sigma_k}}\Big| &\le \int_{u_0-A}^{u_0+A}\frac{du}{\sqrt{u+\sigma_k}} + \sum_\pm \frac{1}{|\phi'(u_0\pm A)|\sqrt{u_0\pm A+\sigma_k}} \\ &\quad + \int_0^{u_0-A}\Big|\frac{d}{du}\frac{1}{\phi'(u)\sqrt{u+\sigma_k}}\Big|\,du + \int_{u_0+A}^\infty \Big|\frac{d}{du}\frac{1}{\phi'(u)\sqrt{u+\sigma_k}}\Big|\,du + \frac{1}{|\phi'(0)|\sqrt{\sigma_k}} \qquad (6.6)\\ &\le \int_{u_0-A}^{u_0+A}\frac{du}{\sqrt{u+\sigma_k}} + \sum_\pm \frac{2}{|\phi'(u_0\pm A)|\sqrt{u_0\pm A+\sigma_k}}. \qquad (6.7)\end{aligned}$$

To pass from (6.6) to (6.7) one uses that $\phi'(u)\sqrt{u+\sigma_k}$ is strictly increasing, so that

$$\begin{aligned}\int_0^{u_0-A}\Big|\frac{d}{du}\frac{1}{\phi'(u)\sqrt{u+\sigma_k}}\Big|\,du &= \frac{1}{\phi'(0)\sqrt{\sigma_k}} - \frac{1}{\phi'(u_0-A)\sqrt{u_0-A+\sigma_k}} \\ &= -\frac{1}{|\phi'(0)|\sqrt{\sigma_k}} + \frac{1}{|\phi'(u_0-A)|\sqrt{u_0-A+\sigma_k}},\end{aligned}$$

the final inequality following from the fact that $\phi'(0) < 0$ since $0 < u_0$. A similar argument applies to the other integral in (6.6). First, one has the bound

$$\int_{u_0-A}^{u_0+A}\frac{du}{\sqrt{u+\sigma_k}} \lesssim \min\Big[\frac{A}{\sqrt{u_0+A+\sigma_k}}, \sqrt{u_0+A+\sigma_k}\Big] \le \frac{A}{\sqrt{u_0+\sigma_k}}. \tag{6.8}$$

Second,

$$|\phi'(u_0-A)| = \int_{u_0-A}^{u_0}\phi''(u)\,du \ge A\phi''(u_0). \tag{6.9}$$

Set $A = \phi''(u_0)^{-\frac12}$. Then from the preceeding,

$$\frac{1}{|\phi'(u_0-A)|\sqrt{u_0-A+\sigma_k}} \lesssim \frac{1}{A\phi''(u_0)\sqrt{u_0+\sigma_k}} \lesssim \frac{A}{\sqrt{u_0+\sigma_k}},$$

which agrees with (6.8). It remains to control the $\phi'(u_0+A)$ term in (6.7). First

$$\phi'(u_0+A) = \int_{u_0}^{u_0+A} \phi''(s)\,ds = \sum_{j=1}^m \int_{u_0}^{u_0+A} \frac{b_j}{(s+\sigma_j)^{\frac{3}{2}}}\,ds$$

$$\asymp A\sum_{j=1}^m \frac{b_j}{(u_0+\sigma_j)^{\frac{3}{2}}} = A\phi''(u_0) \tag{6.10}$$

where we used that $u_0 \geq 2A$. Thus, as in the case of $\phi'(u_0-A)$,

$$\frac{1}{|\phi'(u_0+A)|\sqrt{u_0+A+\sigma_k}} \lesssim \frac{1}{A\phi''(u_0)\sqrt{u_0+\sigma_k}} \lesssim \frac{A}{\sqrt{u_0+\sigma_k}}.$$

It remains to estimate $A = [\phi''(u_0)]^{-\frac{1}{2}}$. The critical point u_0 is determined from the equation

$$1 = \sum_{j=1}^m \frac{b_j}{\sqrt{u_0+\sigma_j}}. \tag{6.11}$$

Let $p \in [1,m]$ be such that

$$\frac{b_p}{\sqrt{u_0+\sigma_p}} = \max_{j\in[1,m]} \frac{b_j}{\sqrt{u_0+\sigma_j}}.$$

Clearly, from (6.11),

$$\frac{b_p}{\sqrt{u_0+\sigma_p}} \geq \frac{1}{m}. \tag{6.12}$$

We also have that

$$\phi''(u_0) = \sum_{j=1}^m \frac{b_j}{(u_0+\sigma_j)^{\frac{3}{2}}} \geq \frac{b_p}{(u_0+\sigma_p)^{\frac{3}{2}}} \geq \frac{1}{m}\frac{1}{u_0+\sigma_p}.$$

Thus

$$A \leq m^{\frac{1}{2}}\sqrt{u_0+\sigma_p}. \tag{6.13}$$

By the maximality of $\frac{b_p}{\sqrt{u_0+\sigma_p}}$

$$\frac{b_p}{\sqrt{u_0+\sigma_p}} \geq \frac{b_k}{\sqrt{u_0+\sigma_k}}.$$

It now follows that

$$\frac{A}{\sqrt{u_0+\sigma_k}} \leq m^{\frac{1}{2}}\frac{\sqrt{u_0+\sigma_p}}{\sqrt{u_0+\sigma_k}} \leq m^{\frac{1}{2}}\frac{b_p}{b_k}.$$

It remains to consider the case $u_0 \leq 2A$. This includes the case where u_0 does not exist, in which case we set $u_0 := 0$. Define $A' = m^{\frac{1}{2}}\sqrt{u_0+\sigma_p} \gtrsim m^{\frac{1}{2}} b_p \geq A$. Note that also $A' \lesssim m^{\frac{3}{2}} b_p$. As before, integration by parts yields

$$\begin{aligned}\left|\int_0^\infty e^{i\phi(u)}\,\frac{du}{\sqrt{u+\sigma_k}}\right| \;&\leq\; \int_0^{u_0+A'} \frac{du}{\sqrt{u+\sigma_k}} + \frac{2}{|\phi'(u_0+A')|\sqrt{u_0+A'+\sigma_k}} \\ &\lesssim\; \frac{A'}{\sqrt{A'+\sigma_k}} + \frac{1}{|\phi'(u_0+A')|\sqrt{A'+u_0+\sigma_k}}.\end{aligned} \tag{6.14}$$

The condition $u_0 \leq 2A$ together with (6.13) imply that $u_0 \leq m^{\frac{1}{2}}\sqrt{u_0+\sigma_p}$. We first consider the case $u_0 + \sigma_p \geq m$. We have

$$\begin{aligned}
\phi'(u_0 + A') &= \int_{u_0}^{u_0+A'} \phi''(s)\, ds = \sum_{j=1}^{m} \int_{u_0}^{u_0+A'} \frac{b_j}{(s+\sigma_j)^{\frac{3}{2}}}\, ds \\
&\geq mA' \frac{b_p}{(u_0 + A' + \sigma_p)^{\frac{3}{2}}}
\end{aligned}$$

The condition that $u_0 + \sigma_p \geq m$ and the definition of A' imply that $u_0 + \sigma_p \geq A'$. Thus

$$\phi'(u_0 + A') \geq mA' \frac{b_p}{(u_0+\sigma_p)^{\frac{3}{2}}} \geq A' \frac{1}{u_0+\sigma_p} = \frac{m^{\frac{1}{2}}}{\sqrt{u_0+\sigma_p}} \geq m^{-\frac{1}{2}} b_p^{-1},$$

where we used that $b_p \geq m^{-1}\sqrt{u_0+\sigma_p}$ and the definition of A'. Therefore,

$$\frac{1}{|\phi'(u_0+A')|\sqrt{A'+u_0+\sigma_k}} \leq m^{\frac{1}{2}} \frac{b_p}{\sqrt{A'+u_0+\sigma_k}} \leq m^{\frac{1}{2}} \frac{b_p}{\sqrt{u_0+\sigma_k}} \leq m^{\frac{1}{2}} \frac{b_p}{b_k},$$

where we used that $\sqrt{u_0+\sigma_k} \geq b_k$. Also

$$\frac{A'}{\sqrt{A'+\sigma_k}} \lesssim \frac{A'}{\sqrt{u_0+\sigma_k}} = m^{\frac{1}{2}} \frac{\sqrt{u_0+\sigma_p}}{\sqrt{u_0+\sigma_k}} \leq m^{\frac{1}{2}} \frac{b_p}{b_k}, \tag{6.15}$$

as desired. It remains to consider the case $u_0 + \sigma_p < m$. Here the integration by parts is as follows. Fix $B = m^4$.

$$\begin{aligned}
\left| \int_0^\infty e^{i\phi(u)} \frac{du}{\sqrt{u+\sigma_k}} \right| &\leq \int_0^B \frac{du}{\sqrt{u+\sigma_k}} + \frac{2}{|\phi'(B)|\sqrt{B+\sigma_k}} \\
&\lesssim \frac{B}{\sqrt{B+\sigma_k}} + \frac{1}{|\phi'(B)|\sqrt{B+\sigma_k}}.
\end{aligned} \tag{6.16}$$

Furthermore

$$\begin{aligned}
\phi'(B) = \int_{u_0}^{B} \phi''(s)\, ds = \frac{1}{2} \sum_{j=1}^{m} \int_{u_0}^{B} \frac{b_j}{(s+\sigma_j)^{\frac{3}{2}}}\, ds \frac{1}{2} &\geq \int_{u_0}^{B} \frac{b_p}{(s+\sigma_p)^{\frac{3}{2}}}\, ds \\
&= \frac{b_p}{\sqrt{u_0+\sigma_p}} - \frac{b_p}{\sqrt{B+\sigma_p}}
\end{aligned}$$

Since $b_p \geq m^{-1}\sqrt{u_0+\sigma_p}$ and $b_p \leq \sqrt{u_0+\sigma_p} \leq \sqrt{m}$ we obtain that

$$\phi'(B) \geq \frac{1}{m} - \frac{\sqrt{m}}{m^2} \geq \frac{1}{2m}$$

Thus

$$\left| \int_0^\infty e^{i\phi(u)} \frac{du}{\sqrt{u+\sigma_k}} \right| \leq \frac{m^4}{\sqrt{m^4+\sigma_k}} + \frac{2m}{\sqrt{m^4+\sigma_k}} \lesssim m^2.$$

This finishes the proof if $u_0 \geq 0$ exists. Finally, suppose the critical point u_0 doesn't exist. Then $\sum_{j=1}^m \frac{b_j}{\sqrt{\sigma_j}} \leq 1$. If in fact $\sum_{j=1}^m \frac{b_j}{\sqrt{\sigma_j}} \leq \frac{1}{2}$, then $\phi'(u) \geq \frac{1}{2}$ for all $u \geq 0$. This case is treated in the same way as the phase ϕ_+. If, on the other hand, $\sum_{j=1}^m \frac{b_j}{\sqrt{\sigma_j}} \geq \frac{1}{2}$, then one can define the index $p \in [1, m]$ as before. In particular, on still has the crucial property $\frac{b_p}{\sqrt{\sigma_p}} \gtrsim m^{-1}$. The reader will easily check that the previous analysis of the case $u_0 \leq 2A$ applies mutatis mutandis. $\square$

Lemma 6.2. *There exists a constant C_0 so that for any choice of $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$, $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_\ell > 0$, $b_j > 0$, and $c_i > 0$, one has*

$$\left|\int_0^{\sqrt{\rho_\ell}} e^{\frac{1}{2}i\lambda^2}\, e^{\pm i\sum_{j=1}^m b_j\sqrt{\lambda^2+\sigma_j}}\, \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i-\lambda^2}\Big)\, \frac{\lambda}{\sqrt{\lambda^2+\sigma_k}}\, d\lambda\right| \leq C_0\, m^2 b_k^{-1} \max_{1\leq j\leq m} b_j \tag{6.17}$$

for any $1 \leq k \leq m$.

Proof. As in the previous proof, we set $\phi_\pm(u) = \frac{1}{2}u \pm \sum_{j=1}^m b_j\sqrt{u+\sigma_j}$. The integral on the left-hand side of (6.17) is the same as

$$\int_0^{\rho_\ell} e^{i\phi_\pm(u)}\, \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i-u}\Big)\, \frac{du}{\sqrt{u+\sigma_k}}. \tag{6.18}$$

We first consider the easier case of $\phi(u) := \phi_+(u)$. In that case $\phi'(u) \geq \frac{1}{2}$, and $|\phi''(u)| \leq u^{-1}\phi'(u)$, see (6.3). Let $w(u) = \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i-u}\Big)$ and $g(u) = \chi(u)\chi(\rho_\ell-u)(u+\sigma_k)^{-\frac{1}{2}}$, cf. (6.5). The cut-offs at the endpoints 0 and ρ_ℓ, respectively, contribute only $O(1)$ to the integral in (6.18) and can therefore be ignored. Integrating by parts yields

$$\begin{aligned}
&\left|\int_0^{\rho_\ell} e^{i\phi(u)} w(u)\, g(u)\, du\right| \leq \int_0^{\rho_\ell} \left|\frac{d}{du}\frac{g(u)w(u)}{\phi'(u)}\right| du \\
&\leq \int_0^{\rho_\ell} \left|\frac{d}{du}\frac{g(u)}{\phi'(u)}\right| du + \int_0^{\rho_\ell} \left|w'(u)\frac{g(u)}{\phi'(u)}\right| du \\
&\leq 2\int_0^{\rho_\ell} \left|\frac{d}{du}\frac{g(u)}{\phi'(u)}\right| du \lesssim \int_0^{\rho_\ell} \Big[\frac{|g'(u)|}{\phi'(u)} + \frac{|\phi''(u)|}{\phi'(u)^2} g(u)\Big]\, du \lesssim 1 + \int_1^\infty \frac{du}{u^{3/2}} \lesssim 1.
\end{aligned} \tag{6.19}$$

To deal with the second integral in (6.19) observe that $w'(u)$ has the same sign on the interval of integration. Therefore, removing the absolute values and integrating by parts reduces it to the first integral.

Next consider $\phi(u) := \phi_-(u)$. The analysis is very similar to the corresponding case in the proof of Lemma 6.1 and we will use the notation as well as some estimates from there. Thus let $\phi'(u_0) = 0$ for some critical point $u_0 \geq 0$. Furthermore, let $A = \phi''(u_0)^{-\frac{1}{2}}$ and suppose $u_0 > 2A$ and $u_0 + A < \rho_\ell$.

Then as in (6.7),

$$
\begin{aligned}
&\left|\int_0^{\rho_\ell} e^{i\phi(u)}\ \frac{w(u)}{\sqrt{u+\sigma_k}}\,du\right| \\
&\le \int_{u_0-A}^{u_0+A} \frac{du}{\sqrt{u+\sigma_k}} + \sum_{\pm} \frac{1}{|\phi'(u_0\pm A)|\sqrt{u_0\pm A+\sigma_k}} + \int_0^{u_0-A} \left|\frac{d}{du}\frac{w(u)}{\phi'(u)\sqrt{u+\sigma_k}}\right| du \qquad (6.20)\\
&+ \int_{u_0+A}^{\rho_\ell} \left|\frac{d}{du}\frac{w(u)}{\phi'(u)\sqrt{u+\sigma_k}}\right| du + \frac{1}{|\phi'(0)|\sqrt{\sigma_k}} + \frac{1}{|\phi'(\rho_\ell)|\sqrt{\rho_\ell+\sigma_k}} \qquad (6.21)\\
&\le \int_{u_0-A}^{u_0+A} \frac{du}{\sqrt{u+\sigma_k}} + \sum_{\pm} \frac{3}{|\phi'(u_0\pm A)|\sqrt{u_0\pm A+\sigma_k}}. \qquad (6.22)
\end{aligned}
$$

To deal with the integrals involving $w(u)$ in (6.21) one uses the monotonicity of w (i.e., $w'(u) > 0$) as follows:

$$
\begin{aligned}
&\int_0^{u_0-A} \left|\frac{d}{du}\frac{w(u)}{\phi'(u)\sqrt{u+\sigma_k}}\right| du \le \int_0^{u_0-A} \frac{w'(u)}{-\phi'(u)\sqrt{u+\sigma_k}}\,du + \int_0^{u_0-A} \left|\frac{d}{du}\frac{1}{\phi'(u)\sqrt{u+\sigma_k}}\right| du \\
&\le 2\int_0^{u_0-A} \left|\frac{d}{du}\frac{1}{\phi'(u)\sqrt{u+\sigma_k}}\right| du + \frac{1}{|\phi'(u_0-A)|\sqrt{u_0-A+\sigma_k}} \\
&\le \frac{3}{|\phi'(u_0-A)|\sqrt{u_0-A+\sigma_k}} - \frac{2}{|\phi'(0)|\sqrt{\sigma_k}}.
\end{aligned}
$$

To pass to the final line we use that

$$
\begin{aligned}
&\int_0^{u_0-A} \left|\frac{d}{du}\frac{1}{\phi'(u)\sqrt{u+\sigma_k}}\right| du = \frac{1}{\phi'(0)\sqrt{\sigma_k}} - \frac{1}{\phi'(u_0-A)\sqrt{u_0-A+\sigma_k}} \\
&= -\frac{1}{|\phi'(0)|\sqrt{\sigma_k}} + \frac{1}{|\phi'(u_0-A)|\sqrt{u_0-A+\sigma_k}}
\end{aligned}
$$

by monotonicity of $\phi'(u)\sqrt{u+\sigma_k}$. A similar analysis applies on the interval $[u_0+A,\rho_\ell]$, and one therefore obtains (6.22). This, however, is the same as (6.7), and the desired bound is obtained by the same analysis. Recall that we assumed $u_0 > 2A$ and $u_0 + A \le \rho_\ell$. If $u_0 + A > \rho_\ell$, then (6.20) needs to be changed only with respect to ρ_ℓ, which becomes the upper limit instead of $u_0 + A$. The other case $u_0 < 2A$ can be treated in the exact same way as the corresponding case in Lemma 6.1. The only difference being that the integration by parts needs to be changed as in (6.22). We skip the details. □

To conclude this section, we turn to oscillatory integrals with singular weights.

Lemma 6.3. *There exists a constant C_0 so that for any choice of $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_m \ge 0$, $\rho_1 \ge \rho_2 \ge \ldots \ge \rho_\ell > 0$, $b_j > 0$, and $c_i > 0$, one has*

$$
\left|\int_0^{\sqrt{\rho_\ell}} e^{\frac12 i\lambda^2}\, e^{\pm i\sum_{j=1}^m b_j\sqrt{\lambda^2+\sigma_j}}\ \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i-\lambda^2}\Big)\ \frac{\lambda}{\sqrt{\rho_k-\lambda^2}}\,d\lambda\right| \le C_0\, m^2 c_k^{-1} \max_{j,i}(b_j+c_i) \qquad (6.23)
$$

for any $1 \le k \le m$.

Proof. We start with the elementary comment that we can assume that

$$\rho_\ell \gg 1. \tag{6.24}$$

Indeed, if (6.24) fails, then the oscillatory integral in (6.23) is

$$\lesssim \int_0^{\sqrt{\rho_\ell}} \frac{\lambda}{\sqrt{\rho_k - \lambda^2}}\, d\lambda \lesssim \int_0^{\sqrt{\rho_\ell}} \frac{\lambda}{\sqrt{\rho_\ell - \lambda^2}}\, d\lambda = \sqrt{\rho_\ell} \lesssim 1.$$

Moreover, note that c_k^{-1} is always an upper bound on the left-hand side of (6.23). For future reference we also note that one can assume that

$$c_k \le \sqrt{\rho_k}. \tag{6.25}$$

Indeed, if this condition fails, then $c_k > \sqrt{\rho_k} \ge \sqrt{\rho_\ell} \ge 1$. Thus, the left-hand side of (6.23) is $\lesssim 1$, and we are done. We now change variables $u = \lambda^2$ so that the integral in (6.23) reduces to

$$\int_0^{\rho_\ell} e^{i\phi_\pm(u)} \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i - u}\Big) \frac{du}{\sqrt{\rho_k - u}}$$

where $\phi_\pm(u) = \frac{1}{2} i\lambda^2 e^{\pm i \sum_{j=1}^m b_j \sqrt{\lambda^2+\sigma_j}}$. Recall that the phase $\phi(u) := \phi_+(u)$ satisfies $\phi'(u) \ge \frac{1}{2}$ and $|\phi''(u)| \le u^{-1}\phi'(u)$. Therefore, with $\chi(u)$ the same cut-off function as before,

$$\begin{aligned}
&\Big| \int_0^{\rho_\ell} e^{i\phi_+(u)}\, \chi(u)\chi(\rho_\ell - u) \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i - u}\Big) \frac{du}{\sqrt{\rho_k - u}} \Big| \\
&\le \int_0^{\rho_\ell} \frac{|\phi''(u)|}{\phi'(u)^2}\, \chi(u)\chi(\rho_\ell - u) \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i - u}\Big) \frac{du}{\sqrt{\rho_k - u}} \\
&+ \int_0^{\rho_\ell} \frac{1}{\phi'(u)} \Big| \frac{d}{du} \chi(u)\chi(\rho_\ell - u) \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i - u}\Big) \frac{du}{\sqrt{\rho_k - u}} \Big| \\
&\lesssim \int_1^{\rho_\ell - 1} \frac{du}{u\sqrt{\rho_\ell - u}} + \int_0^{\rho_\ell} \big(|\chi'(u)| + |\chi'(\rho_\ell - u)|\big) \frac{du}{\sqrt{\rho_\ell - u}} \\
&+ \int_0^{\rho_\ell - 1} (\rho_\ell - u)^{-\frac{3}{2}}\, du + \sum_{j=1}^{\ell} \int_0^{\rho_\ell - 1} \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i - u}\Big) \frac{c_j}{\sqrt{\rho_j - u}\sqrt{\rho_\ell - u}}\, du \\
&\lesssim \rho_\ell^{-\frac{1}{2}} \int_{\frac{1}{\rho_\ell}}^{1 - \frac{1}{\rho_\ell}} \frac{du}{u\sqrt{1-u}} + 1 + \sum_{j=1}^{\ell} \int_0^{\rho_j - 1} \exp\Big(-c_j\sqrt{\rho_j - u}\Big) \frac{c_j}{\sqrt{\rho_j - u}}\, du \lesssim \frac{\log \rho_\ell}{\sqrt{\rho_\ell}} + \ell \lesssim \ell.
\end{aligned}$$

Next, we consider the phase $\phi(u) := \phi_-(u)$. As before, we need to consider the (possible) critical point u_0 of $\phi(u)$. First, suppose that $u_0 > \frac{1}{2}\rho_\ell$. Then there exists some $1 \le p \le m$ so that $m\, b_p \gtrsim \sqrt{\sigma_p + \rho_\ell}$. By (6.24) this implies that $b_p \gtrsim m^{-1}$. One concludes from the preceding that there is an upper bound of the form

$$\lesssim m\, \frac{\max_{1\le j\le m} b_j}{c_k},$$

and we are done. So we may assume that $u_0 \leq \frac{1}{2}\rho_\ell$. As in the previous proofs, we will need to integrate by parts on intervals of the form $[0, u_0 - A]$ and $[u_0 + A, \rho_\ell]$. Here $A = \phi''(u_0)^{-\frac{1}{2}}$. If $A + u_0 > \frac{3}{4}\rho_\ell$, then it follows that $A \gtrsim \rho_\ell$. In conjunction with (6.13) and (6.24) this implies that $u_0 + \sigma_p \gtrsim m^{-1}$, and thus also $b_p \geq m^{-1}\sqrt{u_0 + \sigma_p} \gtrsim m^{-\frac{3}{2}}$, which yields the desired bound as before. Hence we can assume that $u_0 < \frac{1}{2}\rho_\ell$ and $A + u_0 < \frac{3}{4}\rho_\ell$. We now consider the case $u_0 > 2A$, and split the integration interval $[0, \rho_\ell]$ into the intervals $[0, u_0 - A]$, $[u_0 - A, u_0 + A]$, $[u_0 + A, \rho_\ell]$. Our goal is to integrate by parts as in (6.7) and (6.22). For technical reasons having to do with the monotoncity of various functions we change variables to $v = \rho_\ell - u$. Setting $\psi(v) = \phi(u)$ and $v_0 = \rho_\ell - u_0$, observe that $\psi''(v) > 0$ so that ψ' is increasing. In particular, $\psi'(v) < 0$ for $v < v_0$ and $\psi'(v) > 0$ if $v > v_0$. Thus, with $\omega(v) := \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i + \rho_\ell - v}\Big)$, integration by parts yields

$$
\begin{aligned}
&\left|\int_0^{\rho_\ell} e^{i\phi(u)} \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i - u}\Big)\,\frac{du}{\sqrt{\rho_k - u}}\right| = \left|\int_0^{\rho_\ell} e^{i\psi(v)} \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i + \rho_\ell - v}\Big)\,\frac{dv}{\sqrt{\rho_k - \rho_\ell + v}}\right| \\
&\leq \int_{v_0 - A}^{v_0 + A} \frac{dv}{\sqrt{\rho_k - \rho_\ell + v}} + \int_0^{v_0 - A} \left|\frac{d}{dv}\frac{\omega(v)}{\psi'(v)\sqrt{\rho_k - \rho_\ell + v}}\right| dv + \int_{v_0 + A}^{\rho_\ell} \left|\frac{d}{dv}\frac{\omega(v)}{\psi'(v)\sqrt{\rho_k - \rho_\ell + v}}\right| dv \\
&\quad + \sum_{\pm} \frac{\omega(v_0 \pm A)}{|\psi'(v_0 \pm A)|\sqrt{\rho_k - \rho_\ell + v_0 \pm A}} + \frac{\omega(0)}{|\psi'(0)|\sqrt{\rho_k - \rho_\ell}} + \frac{\omega(\rho_\ell)}{|\psi'(\rho_\ell)|\sqrt{\rho_k}} \qquad (6.26) \\
&\lesssim \int_{u_0 - A}^{u_0 + A} \frac{du}{\sqrt{\rho_k - u}} + \left|\int_0^{v_0 - A} \frac{\omega'(v)}{\psi'(v)\sqrt{\rho_k - \rho_\ell + v}}\,dv\right| + \left|\int_{v_0 + A}^{\rho_\ell} \frac{\omega'(v)}{\psi'(v)\sqrt{\rho_k - \rho_\ell + v}}\,dv\right| \qquad (6.27) \\
&\quad + \left|\int_0^{v_0 - A} \frac{d}{dv}\frac{1}{\psi'(v)\sqrt{\rho_k - \rho_\ell + v}}\,dv\right| + \left|\int_{v_0 + A}^{\rho_\ell} \frac{d}{dv}\frac{1}{\psi'(v)\sqrt{\rho_k - \rho_\ell + v}}\,dv\right| \qquad (6.28) \\
&\quad + \sum_{\pm} \frac{\omega(v_0 \pm A)}{|\psi'(v_0 \pm A)|\sqrt{\rho_k - \rho_\ell + v_0 \pm A}} + \frac{\omega(0)}{|\psi'(0)|\sqrt{\rho_k - \rho_\ell}} + \frac{\omega(\rho_\ell)}{|\psi'(\rho_\ell)|\sqrt{\rho_k}}. \qquad (6.29)
\end{aligned}
$$

To deal with the integrals in (6.26) involving absolute values one uses the monotonicity of the numerator and denominator. This allows one to pull out the absolute values from the integrals in (6.27) and (6.28). Recall that ψ' is increasing. In particular, $\psi'(v) < 0$ if $v < v_0$ and $\psi'(v) > 0$ if $v > v_0$. Therefore, using also that $\omega' > 0$ and $\omega \leq 1$, another integration by parts yields that

$$
\begin{aligned}
&\left|\int_0^{v_0 - A} \frac{\omega'(v)}{\psi'(v)\sqrt{\rho_k - \rho_\ell + v}}\,dv\right| = -\int_0^{v_0 - A} \frac{\omega'(v)}{\psi'(v)\sqrt{\rho_k - \rho_\ell + v}}\,dv \\
&\leq \left|\int_0^{v_0 - A} \frac{d}{dv}\frac{1}{\psi'(v)\sqrt{\rho_k - \rho_\ell + v}}\,dv\right| + \frac{\omega(0)}{\psi'(0)\sqrt{\rho_k - \rho_\ell}} - \frac{\omega(v_0 - A)}{\psi'(v_0 - A)\sqrt{\rho_k - \rho_\ell + v_0 - A}} \\
&= \frac{2\omega(0)}{\psi'(0)\sqrt{\rho_k - \rho_\ell}} - \frac{2\omega(v_0 - A)}{\psi'(v_0 - A)\sqrt{\rho_k - \rho_\ell + v_0 - A}},
\end{aligned}
$$

and similarly for the integral over $[u_0 + A, \rho_\ell]$. Inserting all these estimates back into (6.27) to (6.29) one obtains

$$
\begin{aligned}
&\left|\int_0^{\rho_\ell} e^{i\phi(u)} \exp\Big(-\sum_{i=1}^{\ell} c_i\sqrt{\rho_i - u}\Big)\,\frac{du}{\sqrt{\rho_k - u}}\right| \lesssim \frac{A}{\sqrt{\rho_k}} + \sum_{\pm} \frac{\omega(v_0 \pm A)}{|\psi'(v_0 \pm A)|\sqrt{\rho_k - \rho_\ell + v_0 \pm A}} \\
&\leq \frac{A}{\sqrt{\rho_k}} + \sum_{\pm} \frac{1}{|\phi'(u_0 \pm A)|\sqrt{\rho_k - u_0 \pm A}} \lesssim \frac{A}{\sqrt{\rho_k}} + \sum_{\pm} \frac{1}{|\phi'(u_0 \pm A)|\sqrt{\rho_k}}, \qquad (6.30)
\end{aligned}
$$

where we used that $u_0 + A < \frac{3}{4}\rho_\ell \leq \frac{3}{4}\rho_k$ in the last step. Since $\phi'' > 0$ is decreasing, one has $|\phi'(u_0 - A)| \geq A\phi''(u_0) = A^{-1}$. Since we are in the case $u_0 < 2A$, (6.10) shows that $\phi'(u_0 + A) \gtrsim A\phi''(u_0) = A^{-1}$. Hence the entire bound from (6.30) is $\lesssim \frac{A}{\sqrt{\rho_k}} \lesssim \sqrt{m}\frac{\sqrt{u_0+\sigma_p}}{\sqrt{\rho_k}} \lesssim m^{\frac{3}{2}}\frac{b_p}{c_k}$. Here we first used (6.13), then $b_p \geq m^{-1}\sqrt{u_0 + \sigma_p}$ see (6.12), and finally $\sqrt{\rho_k} \leq c_k$, see (6.25). The remaining case $u_0 < 2A$ can be dealt with in the same manner as the corresponding part of the proof of Lemma 6.1. The only difference is that we have $\sqrt{rho_k}$ in the denominator instead of $\sqrt{u_0 + \sigma_k}$. For example, the analysis leading up to (6.15) now produces $\sqrt{m}\frac{b_p}{\sqrt{\rho_k}} \lesssim \sqrt{m}\frac{b_p}{c_k}$, as desired. We skip the details. $\square$

7 Putting it all together

By combining the results of the previous three sections we are now able to prove our main result.

Theorem 7.1. *Let $V(t, x)$ be a real-valued measurable function on $\mathbb{R}^4$ such that*

$$\sup_t \|V(t, \cdot)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < c_0 \quad \text{and} \quad \sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\|V(\hat{\tau}, x)\|_{\mathcal{M}}}{|x - y|}\, dx < 4\pi \tag{7.1}$$

for some small constant $c_0 > 0$, see Definition 5.1. Then

$$\|U(t, s)\psi_s\|_\infty \leq C|t - s|^{-\frac{3}{2}}\, \|\psi_s\|_1 \quad \text{for all times } t, s \text{ and any } \psi_s \in L^1,$$

where $U(t, s)$ is the weak propagator constructed in Lemma 5.2.

Proof. Recall from Proposition 5.6 that

$$\langle U(t, s)\psi_s, g\rangle = \sum_{m=0}^{\infty} \langle \mathcal{I}_m\psi_s, g\rangle$$

for any pair $\psi_s, g \in \mathcal{S}(\mathbb{R}^3)$. Furthermore, Proposition 5.9 provides a representation of the kernel of $\mathcal{I}_m(t, s)$ in terms of three kinds of oscillatory integrals, which are defined in (5.40), (5.42), and (5.43). Suppose $t > s$. Changing variables $\lambda \mapsto \frac{\lambda}{\sqrt{t-s}}$ in each of these integrals brings out one factor of $(t - s)^{-\frac{1}{2}}$, whereas (5.44) already contains the factor $(t - s)^{-1}$. This leads to the desired power $(t - s)^{-\frac{3}{2}}$. More precisely, for the oscillatory integrals from (5.40) this process leads to

$$\begin{aligned}
&\int_0^\infty e^{i(t-s)\lambda^2} \cos\left(\sum_{k=1}^{m} \sqrt{\lambda^2 + \sigma_k}|x_{k-1} - x_k|\right) \frac{\lambda}{\sqrt{\lambda^2 + \sigma_\ell}}\, d\lambda \\
&= (t - s)^{-\frac{1}{2}} \int_0^\infty e^{i\lambda^2} \cos\left(\sum_{k=1}^{m} \sqrt{\lambda^2 + \sigma_k(t - s)}\frac{|x_{k-1} - x_k|}{t - s}\right) \frac{\lambda}{\sqrt{\lambda^2 + \sigma_\ell(t - s)}}\, d\lambda,
\end{aligned} \tag{7.3}$$

and similarly for (5.42) and (5.43). Thus the parameters σ_j and ρ_k and $|x_{i+1} - x_i|$ in these expressions are rescaled to $\sigma_j\,(t - s)$, $\rho_k\,(t - s)$, and $\frac{|x_{i+1}-x_i|}{t-s}$, respectively. We now estimate (7.3) and the analogous integrals from (5.42), and (5.43) by means of Lemma 6.1, 6.2, 6.3, respectively. Using the second bound in each of these lemmas, which is invariant under the aforementioned rescaling of the parameters, one arrives at the upper bound (setting $x = x_0$ and $y = x_{m+1}$)

$$m^{\frac{3}{2}}\, \frac{\max\limits_{0\leq j\leq m+1} |x_{j+1} - x_j|}{|x_\ell - x_{\ell-1}|}$$

in case of $\mathcal{L}_m^\ell(t,s)(x,y)$, and

$$m^{\frac{3}{2}}\ \frac{\max_{0\le j\le m+1}|x_{j+1}-x_j|}{|x_{k(\ell)}-x_{k(\ell)-1}|}$$

in case of $\mathcal{M}_m^{d,\ell}(t,s)(x,y)$, $\widetilde{\mathcal{M}}_m^{d,\ell}(t,s)(x,y)$. Inserting these bounds into the (rescaled) definitions (5.40), (5.42), and (5.43) finally leads to the estimate

$$\begin{aligned}&|\langle \mathcal{I}_m\psi_s,g\rangle|\\ \le\quad & C\,m^{\frac{7}{2}}\ |t-s|^{-\frac{3}{2}}\ \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\prod_{r=1}^m\frac{|V(\hat{\tau}_r,x_r)|}{4\pi|x_{r-1}-x_r|}\ \frac{\max_{0\le j\le m+1}|x_{j+1}-x_j|}{|x_m-y|}\ dx_1\ldots dx_m\,d\tau_1\ldots d\tau_m.\end{aligned}$$

In view of Lemma 2.5 this is no larger than

$$C\,m^{\frac{9}{2}}\ |t-s|^{-\frac{3}{2}}\ \left(\sup_{y\in\mathbb{R}^3}\int_{\mathbb{R}^3}\int\frac{|V(\hat{\tau},x)|}{4\pi|x-y|}\,d\tau\,dx\right)^m,$$

and we are done. □

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Rodnianski: Department of Mathematics, Princeton University, Fine Hall, Princeton N.J. 08544, U.S.A.
email: **irod@math.princeton.edu**

Schlag: Division of Astronomy, Mathematics, and Physics, 253-37 Caltech, Pasadena, CA 91125, U.S.A.
email: **schlag@caltech.edu**