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A GEOMETRIC INEQUALITY WITH APPLICATIONS TO THE KAKEYA PROBLEM IN THREE DIMENSIONS

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1 Introduction

It was shown by Besicovitch [F] that there are sets in $\mathbb{R}^d$ with measure zero that contain unit line segments in every direction. In [Fe2], C. Fefferman used Besicovitch sets to show that the ball-multiplier is bounded only on L^2. Moreover, ideas originating in Fefferman's work lead to alternate proofs, cf. [Fe3], [Có1,2], of the optimal boundedness result for Bochner–Riesz means, established originally by Carleson–Sjölin [CS], as well as for the restriction problem in $\mathbb{R}^2$, which had been solved earlier by Fefferman and Stein [Fe1]. It turns out that the crucial property of planar Besicovitch sets in this context is that they have maximal Hausdorff dimension. It is conjectured that Besicovitch sets $E \subset \mathbb{R}^d$ with $d \geq 3$ have dimension equal to d. It is easy to show that $\dim(E) \geq (d+1)/2$. This was first improved by Bourgain [Bo], who showed, e.g., for $d = 3$ that $\dim(E) \geq 7/3$. A further improvement was then achieved by Wolff [W1], who proved that $\dim(E) \geq (d+2)/2$ in all dimensions. Both these results where based in part on "bush-type" arguments. More precisely, Bourgain's argument used the observation that tubes of thickness δ with 10δ-separated directions which intersect at some point x_0 have to be disjoint outside a ball of radius $1/2$ centered at x_0. The improvement in [W1] is obtained by considering families of tubes intersecting a line.

In this paper we present a different geometric approach that leads to a nontrivial estimate for Besicovitch sets in $\mathbb{R}^3$ – in fact Bourgain's 7/3 bound. Our method is analogous to [KW] and will combine a geometric

The idea of bounding the total number of incidences between lines using geometric statements about the common transversals of three or four lines is due to Thomas Wolff. Moreover, I thank him for several helpful discussions and financial support at the California Institute of Technology. I am grateful to the referee for constructive criticism. This paper was written partly at the Institute for Advanced Study where the author was supported by the National Science Foundation, DMS 9304580.

inequality with a well-known extremal graph theory fact. The main ideas of our proof are as follows. Let E^δ be a δ-neighborhood of a Besicovitch set $E \subset \mathbb{R}^3$. If $\dim(E) \geq p$ then for any $\epsilon > 0$

$$|E^\delta| \geq \delta^{3-p+\epsilon} \tag{1.1}$$

provided δ is sufficiently small. Here we restrict ourselves to giving a heuristic argument for the weaker statement (1.1). Let $\{e_j\}_{j=1}^N \subset S^2$ be a maximally δ-separated sequence. By assumption, there are tubes $T_1^\delta, T_2^\delta, \dots, T_N^\delta \subset E^\delta$ of dimensions $1 \times \delta \times \delta$ such that T_j^δ points in direction e_j. It is clear that (1.1) holds if, say, for at least $N/2$ values of j

$$\left|\left\{x \in T_j^\delta : \sum_{i=1}^N \chi_{T_i^\delta}(x) < \delta^{-3+p-\epsilon}\right\}\right| > \frac{1}{2}|T_j^\delta| \tag{1.2}$$

(this is the concept of multiplicity from [W1]). Indeed, (1.2) implies

$$|E^\delta| \geq \delta^{3-p+\epsilon} \int_{\{E^\delta : \sum_{i=1}^N \chi_{T_i^\delta} < \delta^{-3+p-\epsilon}\}} \sum_{j=1}^N \chi_{T_j^\delta}(x)\, dx \geq \delta^{3-p+\epsilon} \frac{N}{2} \frac{\delta^2}{2},$$

which is (1.1) since $N \sim \delta^{-2}$. To prove (1.2) with a suitable p, we shall use the following simple geometric obstruction that limits the number of incidences between three fixed δ-tubes and all others. Suppose we are given lines l_1, l_2, l_3 in general position. Since a line in $\mathbb{R}^3$ is given by four parameters and incidence between lines is described by a single equation, the set of lines

$$\{l \subset \mathbb{R}^3 : l \cap l_i \neq \emptyset, \;\; i = 1, 2, 3\} \tag{1.3}$$

is a one parameter family. In particular, if tubes $T_{j_1}^\delta, T_{j_2}^\delta, T_{j_3}^\delta$ are in general position then they can have at most δ^{-1} common transversals among the $T_1^\delta, T_2^\delta, \dots, T_N^\delta$. Now consider the matrix $A = \{a_{ij}\}_{i,j=1}^N$ where $a_{ij} = 1$ or 0 depending on whether or not T_i^δ and T_j^δ intersect. If our tubes are in sufficiently general position then the geometric obstruction discussed above rules out submatrices of A of size $\delta^{-1} \times 3$ all of whose entries are equal to one. By Hölder's inequality

$$\sum_{i,j=1}^N a_{ij} \leq \left(\sum_{j=1}^N \left(\sum_{i=1}^N a_{ij} \right)^3 \right)^{1/3} N^{2/3}$$

$$\leq C\Bigg(\sum_{j=1}^N \sum_{1 \leq i_1 < i_2 < i_3 \leq N} a_{i_1 j} a_{i_2 j} a_{i_3 j} + \sum_{j=1}^N \sum_{1 \leq i_1 < i_2 \leq N} a_{i_1 j} a_{i_2 j}$$

$$
+ \sum_{i,j=1}^{N} a_{ij} \Big)^{1/3} N^{2/3}
$$

$$
\leq C(\delta^{-1}N^3 + N^3 + N^2)^{1/3} N^{2/3} \leq C\delta^{-11/3}\,.
$$

In particular, a typical tube can intersect at most $\delta^{-5/3}$ of the tubes $T_1^\delta, T_2^\delta, \ldots, T_N^\delta$. Therefore, most points of a typical tube can be contained in at most $\delta^{-2/3}$ tubes, which corresponds to $p = 7/3$ in (1.2). The bound derived in (1.4), which is known to be sharp, is a special case of a well-known result from extremal graph theory, see chapter VI in [B], especially Theorem 2.2.

The main difficulty with this heuristic argument is to quantify "general position". In fact, it is possible that all tubes intersect $T_{j_1}^\delta, T_{j_2}^\delta, T_{j_3}^\delta$. This is the case, for instance, if $T_{j_1}^\delta, T_{j_2}^\delta, T_{j_3}^\delta$ lie on a suitable quadratic surface and are distance $\delta^{1/2}$ apart, see Example 2.10 below.

The main purpose of this paper is to prove a sharp bound (up to $|\log\delta|$-factors) on the measure of the set of directions of all lines that come δ-close to three given lines, see Proposition 2.8. Using this inequality we give a rigorous version of the argument outlined above in section 3.

2 The Geometric Estimates

It is well-known, see [So], that the set of lines (1.3) is a (perhaps degenerate) quadratic surface in $\mathbb{R}^3$. If l_1, l_2, l_3 are pairwise skew, i.e., no two are coplanar, then this quadric is nondegenerate, and is therefore, up to Euclidean motions, a hyperbolic paraboloid or a one-sheeted hyperboloid. In this section we consider the set of lines that come δ-close to three given ones. We make no direct use of the aforementioned fact about quadrics. Rather, we introduce a (most likely standard) set of coordinates on the four-dimensional Grassmann manifold of all lines which make a bounded angle with the z-axis and intersect a small ball around 0. Distances and angles between lines can easily be expressed in terms of their coordinates. The desired estimates are then obtained by elementary geometric arguments.

Definition 2.1. Let $X = B(0,1) \subset \mathbb{R}^4$. $l = (x, y, \bar{x}, \bar{y}) \in X$ is the line passing through the points $a = (x, y, -1)$ and $b = (\bar{x}, \bar{y}, 1)$ with direction $\Delta(l) = b - a$. We refer to $(x, y, \bar{x}, \bar{y}) = (w, \bar{w})$ as the coordinates of l. For any set of lines $\mathcal{L}$ the four-dimensional Lebesgue measure of the set of coordinates of all lines in $\mathcal{L}$ is denoted by $|\mathcal{L}|$. l^δ is the δ-neighborhood of l. $m \lesssim n$ and $m \ll n$ mean $m < Cn$ for some absolute constant and

$m < C^{-1}n$ for some sufficiently large absolute constant, respectively. If both $m \lesssim n$ and $n \lesssim m$ then $m \sim n$.

The following lemma expresses distances and angles in terms of these coordinates. It also allows us to assume that a particular line is equal to the z-axis. Let $B_0 = B(0, 1/4) \subset \mathbb{R}^3$.

LEMMA 2.2. *Let $l_1 = (w_1, \bar{w}_1)$, $l_2 = (w_2, \bar{w}_2) \in X$ be nonparallel. Then*

i. $l_1 \cap l_2 \neq \emptyset \iff w_1 - w_2 \wedge \bar{w}_1 - \bar{w}_2 = 0$

ii. *Suppose $l_1^\delta \cap l_2^\delta \cap B_0 \neq \emptyset$ and let $\alpha > \delta$. Then*

$$\sphericalangle(l_1, l_2) \sim \alpha \iff |w_1 - w_2| \sim |\bar{w}_1 - \bar{w}_2| \sim \alpha$$

iii. *Suppose $\sphericalangle(l_1, l_2) = \alpha$. Then*

$$l_1^\delta \cap l_2^\delta \neq \emptyset \iff \big|(w_1 - w_2) \wedge (\bar{w}_1 - \bar{w}_2)\big| \lesssim \delta\alpha$$

iv. *Let $l_0 = (\xi, \eta, \bar{\xi}, \bar{\eta}) \in X$ and define a linear map on $\mathbb{R}^3$ by*

$$T(x, y, z) = \left(x - \frac{z+1}{2}\bar{\xi} + \frac{z-1}{2}\xi, y - \frac{z+1}{2}\bar{\eta} + \frac{z-1}{2}\eta, z\right).$$

Then T maps the line l_0 onto the z-axis, it distorts lengths and angles in $\mathbb{R}^3$ by an absolutely bounded factor and it preserves the measure of sets of lines, i.e., if $\mathcal{L} \subset X$, then $|T(\mathcal{L})| = |\mathcal{L}|$.

Proof. Since l_1 and l_2 are nonparallel, they intersect iff they lie in a common plane. This is equivalent to $b_2 - b_1 \parallel a_2 - a_1$, see Figure 1, which in turn is equivalent to $w_1 - w_2 \parallel \bar{w}_1 - \bar{w}_2$, as claimed in the first statement. The second statement is clear from Figure 1. For iii note that by elementary geometry the minimal separation of l_1 and l_2 is

$$\frac{|\langle (b_1 - a_1) \times (b_2 - a_2), a_1 - a_2 \rangle|}{|(b_1 - a_1) \times (b_2 - a_2)|} \sim \frac{|(w_1 - w_2) \wedge (\bar{w}_1 - \bar{w}_2)|}{\sphericalangle(l_1, l_2)}.$$

Finally, T and T^{-1} are uniformly bounded, so angles and lengths are essentially preserved. The other statements follow from the fact that a line $l = (x, y, \bar{x}, \bar{y})$ is mapped onto $T(l) = (x - \xi, y - \eta, \bar{x} - \bar{\xi}, \bar{y} - \bar{\eta})$. □

The following auxiliary lemma is a quantitative version of a simple geometric fact, cf. Figure 2.

LEMMA 2.3. *Suppose $l_1, l_2, \bar{l}_1, \bar{l}_2 \in X$ satisfy*

$$l_i^\delta \cap \bar{l}_j^\delta \neq \emptyset \quad \text{and} \quad \operatorname{dist}(\bar{l}_i^\delta \cap l_1^\delta, \bar{l}_i^\delta \cap l_2^\delta) < \lambda \quad \text{for} \quad i, j = 1, 2$$

with some $\lambda \gg \delta$. Let $\theta = \sphericalangle(l_1, l_2)$. Then

$$\operatorname{dist}(l_1^\delta \cap \bar{l}_1^\delta, l_1^\delta \cap \bar{l}_2^\delta) \lesssim \frac{\lambda}{\lambda + \theta}.$$

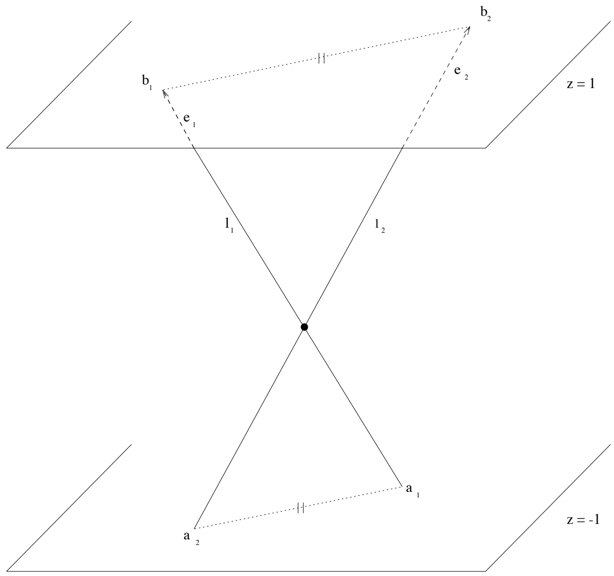

Figure 1: Characterization of incidence

Proof. It suffices to consider the case $\theta > \lambda$. Let $p_{ij} \in l_i^\delta \cap \bar{l}_j^\delta$. Since $\lambda \gg \delta$ we may assume that $l_i \cap \bar{l}_j = \{p_{ij}\}$ for $i,j = 1,2$. This changes the hypotheses or conclusion of the lemma by at most a multiplicative constant. Apply Lemma 2.2, iv, with $l_0 = l_1$ so that l_1 becomes the z-axis. Let l_1' be as in Figure 2 and denote by q and q' the projections of p_{21} onto l_1 and l_1', respectively. Clearly, $|p_{21} - q| \lesssim \lambda$ and $|q - q'| = \operatorname{dist}(l_1, l_1') \lesssim \lambda$. Consequently,

$$|p_{22} - p_{21}| \sin\theta = |q' - p_{21}| \leq |q' - q| + |p_{21} - q| \lesssim \lambda\,,$$

and therefore by symmetry $|p_{22} - p_{21}| + |p_{12} - p_{11}| \lesssim \lambda/\theta$, as claimed. $\square$

Lemma 2.4 bounds the measure of the set of all lines that come δ-close to two given ones. In our main estimate involving three lines this bound will be used in case the three lines are close to a common plane.

Lemma 2.4. *Fix $l_1, l_2 \in X$ and $\alpha, \lambda \in (\delta, 1)$. Let $\theta = \sphericalangle(l_1, l_2)$ and define*

$$\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2)} = \big\{ l \in X : l^\delta \cap l_j^\delta \cap B_0 \neq \emptyset,\ \sphericalangle(l, l_j) \in [\alpha, 2\alpha] \ \ \textit{for} \ \ j = 1,2\,, \\ \operatorname{dist}(l^\delta \cap l_1^\delta, l^\delta \cap l_2^\delta) \in [\lambda, 2\lambda] \big\}\,.$$

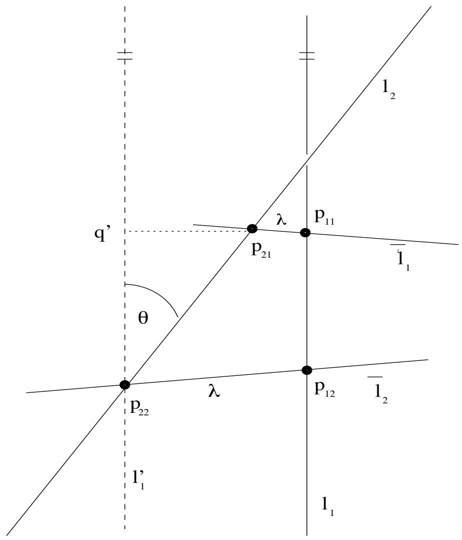

Figure 2: Lines $\bar{l}_1$ and $\bar{l}_2$ intersecting l_1 and l_2 as in Lemma 2.3

If $\lambda \gg \delta/\alpha$ *then*

$$|\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2)}| \lesssim \delta^2\alpha^2 \left(\lambda + \frac{\theta}{\alpha}\right)^{-1} . \tag{2.1}$$

Proof. Apply Lemma 2.2, iv, with $l_0 = l_1$ so that l_1 becomes the z-axis. Consider the change of variables Φ in $\mathbb{R}^3$ given by

$$\Phi : (x, y, z) \mapsto (\rho x, \rho y, z) .$$

By Lemma 2.2, ii, Φ takes $\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2)}$ into X provided $\rho \ll \alpha^{-1}$. Furthermore, δ, α, and θ are rescaled into $\rho\delta, \rho\alpha$, and $\rho\theta$, respectively (note that $\theta \lesssim \alpha$ by the triangle inequality). If α is small, λ is comparable to the distance of $l^\delta \cap l_1^\delta$ and $l^\delta \cap l_2^\delta$ in the z-direction. Thus λ remains essentially unchanged under this rescaling. Since $|\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2)}|$ scales like ρ^4, we conclude that it suffices to show (2.1) with $\alpha = 1$. This observation will be used repeatedly in what follows.

It is convenient to assume that l and l_1 are incident rather than δ-incident. Thus we introduce the auxiliary set

$$\mathcal{L} = \big\{ l \in X : l \cap l_1 \cap B_0 \neq \emptyset,\, l^\delta \cap l_2^\delta \cap B_0 \neq \emptyset,\, \sphericalangle(l, l_j) \in [1,2] \;\text{ for }\; j = 1, 2\,,$$
$$\operatorname{dist}(l^\delta \cap l_1^\delta, l^\delta \cap l_2^\delta) \in [\lambda, 2\lambda] \big\} .$$

$\lambda \gg \delta$ implies that $\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2)}$ is contained in a $C\delta$-neighborhood of $\mathcal{L}$ (translate l_1 until it intersects l). Hence it suffices to show that the three-dimensional Hausdorff measure of $\mathcal{L}$ satisfies

$$\mathcal{H}^3(\mathcal{L}) \lesssim \frac{\delta}{\lambda+\theta}\,. \tag{2.2}$$

Let $l = (w, \bar{w}) \in \mathcal{L}$ and $l \cap l_1 = (0,0,\tau)$. Then

$$\tfrac{1}{2}(1+\tau)\bar{w} + \tfrac{1}{2}(1-\tau)w = 0 \tag{2.3}$$

since the expression on the left-hand side is the (x,y)-component of the point on l with z-coordinate τ. Furthermore, by Lemma 2.2, iii, $l^\delta \cap l_2^\delta \cap B_0 \neq \emptyset$ implies

$$\big|(w-w_2)\wedge(\bar{w}-\bar{w}_2)\big| \lesssim \delta\,. \tag{2.4}$$

Since $\tau \in (-1/2, 1/2)$ the map from (w,τ) to $(w,\bar{w})$ satisfying (2.3) is a smooth parametrization of $\mathcal{L}$. Therefore it suffices to bound the volume of the set of (w,τ) parameters of all lines in $\mathcal{L}$. To this end (2.4) is rewritten as follows. By (2.3)

$$\begin{aligned}\delta &\gtrsim \big|(1+\tau)(\bar{w}-\bar{w}_2)\wedge(w-w_2)\big| = \big|(-(1-\tau)w-(1+\tau)\bar{w}_2)\wedge(w-w_2)\big| \\ &= \big|(-(1-\tau)w_2-(1+\tau)\bar{w}_2)\wedge(w-w_2)\big|\,. \end{aligned}\tag{2.5}$$

Fixing τ we conclude that the set of w satisfying (2.4) and $|w-w_2| \sim 1$ is contained in a strip in $\mathbb{R}^2$ of width $\delta|(1-\tau)w_2+(1+\tau)\bar{w}_2|^{-1}$ intersected with the unit disc. Hence the measure of all (w,τ) parametrizing $\mathcal{L}$ is bounded by

$$\int_J \frac{\delta}{|(1-\tau)w_2+(1+\tau)\bar{w}_2|}\,d\tau\,. \tag{2.6}$$

Here J is the set of τ for which there exists $l \in \mathcal{L}$ with $l \cap l_1 = (0,0,\tau)$. Fix such a τ and l. Translating l_2 one obtains $l_2' = (w_2', \bar{w}_2')$ such that $l \cap l_2' \neq \emptyset$ and $|w_2 - w_2'| + |\bar{w}_2 - \bar{w}_2'| \lesssim \delta$. $\lambda \gg \delta$ and $\sphericalangle(l_2', l) \sim 1$ imply

$$\mathrm{dist}\big(l_2' \cap \{z=\tau\}, l \cap \{z=\tau\}\big) \sim \mathrm{dist}(l^\delta \cap l_1^\delta, l^\delta \cap l_2^\delta)\,,$$

see Figure 3. Clearly, cf. (2.3),

$$\begin{aligned} l_2' \cap \{z=\tau\} &= \big(\tfrac{1}{2}(1-\tau)w_2' + \tfrac{1}{2}(1+\tau)\bar{w}_2', \tau\big) \\ l \cap \{z=\tau\} &= (0,0,\tau) \end{aligned}$$

and thus

$$\begin{aligned} \big|\tfrac{1}{2}(1-\tau)w_2 + \tfrac{1}{2}(1+\tau)\bar{w}_2\big| &= \big|\tfrac{1}{2}(1-\tau)w_2' + \tfrac{1}{2}(1+\tau)\bar{w}_2'\big| + O(\delta) \\ &\sim \mathrm{dist}(l^\delta \cap l_2^\delta, l^\delta \cap l_1^\delta) \gtrsim \lambda\,. \end{aligned}\tag{2.7}$$

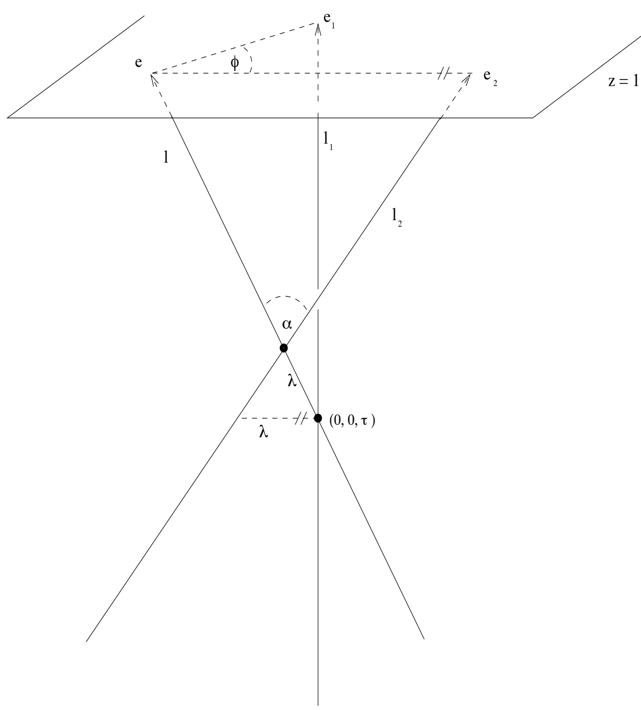

Figure 3: Lines l_1 and l_2 intersecting l as in Lemmas 2.4 and 2.5

Because Lemma 2.3 implies that J is contained in an interval of length $\frac{\lambda}{\theta+\lambda}$, we finally conclude that (2.6) $\lesssim \delta(\lambda+\theta)^{-1}$, which is (2.2). □

For the Kakeya problem it is more important to bound the measure of the set of directions of all lines in $\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2)}$ rather than the measure of $\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2)}$ itself. However, it is easy to pass from the latter to the former, provided one has a lower bound on the two-dimensional measure of all lines in $\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2)}$ with a fixed direction. This is carried out in the following lemma by determining how far a line can be translated inside $\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2)}$. Recall that $\Delta(l) = (\bar{w}-w, 2)$. If $l^\delta \cap l_j^\delta \neq \emptyset$, we let l_j' henceforth be as in the previous proof, i.e., $l_j \parallel l_j'$, $l \cap l_j' \neq \emptyset$ and $|w_j - w_j'| + |\bar{w}_j - \bar{w}_j'| \lesssim \delta$. Finally, we define $l \vee l_j'$ to be the plane spanned by l and l_j'.

Lemma 2.5. *Fix lines $l_1, l_2 \in X$ and $\alpha, \lambda, \phi_0 \in (\delta, 1)$. Let $\theta = \sphericalangle(l_1, l_2) > \delta$ and define*

$\mathcal{D}_{\alpha,\lambda}^{(l_1,l_2)}(\phi_0) = \big\{e : \exists\, l \in X$ *with direction* $e = \Delta(l)$ *so that* $l^\delta \cap l_j^\delta \cap B_0 \neq \emptyset\,,$

$$\sphericalangle(l,l_j)\in[\alpha,2\alpha] \ \ \textit{for} \ \ j=1,2,\ \operatorname{dist}(l^\delta\cap l_1^\delta, l^\delta\cap l_2^\delta)>\lambda\,,$$
$$\sphericalangle(l\vee l_1', l\vee l_2')\le\phi_0\big\}\,.$$

If $\lambda\gg\delta/\alpha$, then

$$\big|\mathcal{D}_{\alpha,\lambda}^{(l_1,l_2)}(\phi_0)\big|\lesssim\alpha^2\big(\lambda+\tfrac{\theta}{\alpha}\big)^{-1}\min\big(\phi_0,\tfrac{\theta}{\alpha}\big)|\log\delta|\,.$$

Proof. As before, it suffices to consider the case $\alpha=1$. Since (see Figure 3)

$$\sphericalangle(l\vee l_1', l\vee l_2')\sim\big|(e_1-e)\wedge(e_2-e)\big|\le|e_1-e_2|\sim\sphericalangle(l_1,l_2)=\theta \tag{2.8}$$

we may assume that $\phi_0\le\theta$. Let

$$\mathcal{L}(\phi)=\big\{l\in X: l^\delta\cap l_j^\delta\cap B_0\ne\emptyset\,,\sphericalangle(l,l_j)\sim1 \ \ \text{for} \ \ j=1,2\,,$$
$$\operatorname{dist}(l^\delta\cap l_1^\delta, l^\delta\cap l_2^\delta)>\lambda\,,\sphericalangle(l\vee l_1', l\vee l_2')\in(\phi-\delta,2\phi]\big\}\,.$$

Clearly,

$$\mathcal{D}_{1,\lambda}^{(l_1,l_2)}(\phi_0)\subset\bigcup_{\delta\le\phi\le\phi_0}\Delta(\mathcal{L}(\phi))\,, \tag{2.9}$$

where ϕ is taken to be dyadic. Let $B_1=B(0,1/2)\subset\mathbb{R}^3$ and fix some $\phi\in(\delta,\phi_0)$. In order to facilitate the translation mentioned above we define the auxiliary set

$$\mathcal{L}'(\phi)=\big\{l\in X: l^{10\delta}\cap l_j^{10\delta}\cap B_1\ne\emptyset\,,\ \sphericalangle(l,l_j)\sim1 \ \ \text{for} \ \ j=1,2\,,$$
$$\operatorname{dist}(l^{10\delta}\cap l_1^{10\delta}, l^{10\delta}\cap l_2^{10\delta})>\lambda/2\,,\ \sphericalangle(l\vee l_1', l\vee l_2')\in(\phi-\delta,2\phi]\big\}\,. \tag{2.10}$$

Replacing λ with $2^j\lambda$, $j\in[0,|\log\delta|]$, in Lemma 2.4 and summing yields

$$|\mathcal{L}'(\phi)|\lesssim\delta^2(\lambda+\theta)^{-1}|\log\delta|\,. \tag{2.11}$$

Using the coordinates (w,Δ) on X rather than $(w,\bar w)$ one obtains from Fubini's theorem

$$\int_{\Delta(\mathcal{L}(\phi))}\mathcal{H}^2\big(\{l\in\mathcal{L}'(\phi):\Delta(l)=e\}\big)\,de\le|\mathcal{L}'(\phi)|\,. \tag{2.12}$$

We claim that for any $e\in\Delta(\mathcal{L}(\phi))$

$$\mathcal{H}^2\big(\{l\in\mathcal{L}'(\phi):\Delta(l)=e\}\big)\gtrsim\tfrac{\delta^2}{\phi}\,. \tag{2.13}$$

To see this let $l\in\mathcal{L}(\phi)$ and $e=\Delta(l)$. Choose any $l'\in l\vee l_1'$ parallel to l with $\operatorname{dist}(l,l')\ll\delta/\phi$. Then $\sphericalangle(l\vee l_1', l\vee l_2')\lesssim\phi$ implies that $\operatorname{dist}(l',l_2')<\delta$. Furthermore,

$$\operatorname{dist}\big((l')^{10\delta}\cap l_1^{10\delta},(l')^{10\delta}\cap l_2^{10\delta}\big)>\lambda/2\,,$$

provided l' lies in the correct half plane of $l \vee l_1'$ relative to l (in one of the half planes the distance between l_1 and l_2 increases). Thus l' and its δ-translates belong to $\mathcal{L}'(\phi)$, which proves (2.13). We conclude from (2.11), (2.12), and (2.13) that for any $\phi \geq \delta$

$$\big|\Delta(\mathcal{L}(\phi))\big| \lesssim (\lambda+\theta)^{-1}\phi|\log\delta|\,, \tag{2.14}$$

and the lemma follows from (2.9). □

The following two lemmas are analogues of Lemmas 2.4 and 2.5 for the case of three lines. For further details in the arguments below the reader is referred to the proofs of those lemmas.

LEMMA 2.6. *Fix lines $l_1, l_2, l_3 \in X$, $\lambda \in (\delta, 1)$, and angles $\alpha \in (\delta, 1)$, $\vec{\phi} = (\phi_{12}, \phi_{13}, \phi_{23}) \in (\delta, \pi/2)^3$. Let $L \geq 2$ be some constant. Define*

$$\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2,l_3)}(\vec{\phi}) = \big\{l{\in}X : l^\delta{\cap}l_j^\delta{\cap}B_0 \neq \emptyset\,,\ \sphericalangle(l,l_j){\in}[\alpha,2\alpha]\,,\ \mathrm{dist}(l^\delta{\cap}l_i^\delta, l^\delta{\cap}l_j^\delta) \in [\lambda, L\lambda]\,,\ \sphericalangle(l\vee l_i', l\vee l_j') \in [\phi_{ij}/2, \phi_{ij}] \ \textit{ for }\ 1 \leq i \neq j \leq 3\big\}\,. \tag{2.15}$$

Let $\theta = \max \sphericalangle(l_i, l_j)$ and $\phi = \max \phi_{ij}$. If $\lambda \gg \delta/\alpha$ and $\phi \gg L\delta/\lambda\alpha$, then

$$\big|\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2,l_3)}(\vec{\phi})\big| \lesssim \alpha\delta^3\, L\lambda^{-1}\big(\lambda+\tfrac{\theta}{\alpha}\big)^{-1}\phi^{-1}\,. \tag{2.16}$$

Proof. By the triangle inequality we may assume that $\phi_{23} \sim \phi$ and $\theta_{12} \sim \theta$. We let l_1 be the z-axis and $\alpha = 1$. $\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2,l_3)}(\vec{\phi})$ is contained in a $C\delta$-neighborhood of the auxiliary set

$$\mathcal{L} = \big\{l{\in}X : l\cap l_1 \cap B_0 \neq \emptyset\,,\ l^\delta \cap l_j^\delta \cap B_0 \neq \emptyset\,,\ \sphericalangle(l,l_j) \in [1,2]\,,\ \mathrm{dist}(l^\delta \cap l_i^\delta, l^\delta \cap l_j^\delta) \in [\lambda, L\lambda]\,,\ \sphericalangle(l\vee l_1', l\vee l_2') \in [\phi_{ij}/2, \phi_{ij}] \ \text{ for }\ 1 \leq i \neq j \leq 3\big\}\,.$$

With (w,τ) being the parameters from the proof of Lemma 2.4, it will suffice to show that the three-dimensional measure of the set of (w,τ) parameters of $\mathcal{L}$ is bounded by $\delta^2 L\lambda^{-1}(\lambda+\theta)^{-1}\phi^{-1}$. Let J be the set of all τ such that there exists $l \in \mathcal{L}$ with $l \cap l_1 = (0,0,\tau)$. Fix such a τ and l. Let $v_j = \frac{1}{2}(1-\tau)w_j + \frac{1}{2}(1+\tau)\bar{w}_j$ for $j = 2,3$. In view of (2.5) the set of all w such that (w,τ) corresponds to some line in $\mathcal{L}$ is contained in the intersection of the strips

$$\big|v_j \wedge (w - w_j)\big| \lesssim \delta\,, \qquad j = 2,3\,. \tag{2.17}$$

The area given by (2.17) is bounded by $\delta^2|v_2 \wedge v_3|^{-1}$. We claim that $\inf_{\tau\in J}|v_2 \wedge v_3| \gtrsim \lambda^2\phi$. Define $v_j' = \frac{1}{2}(1-\tau)w_j' + \frac{1}{2}(1+\tau)\bar{w}_j'$. Since $l_j' \cap \{z=\tau\} = \{(v_j',\tau)\}$ and $l \cap \{z=\tau\} = \{(0,0,\tau)\}$ it follows that $v_j' \parallel e_j - e$ for $j = 2,3$, see Figure 3. By (2.7) therefore

$$|v_2' \wedge v_3'| \sim |v_2'||v_3'|\sphericalangle(l\vee l_2', l\vee l_3') \sim |v_2'||v_3'|\phi_{23} \gtrsim \lambda^2\phi\,.$$

Note also that (2.7) implies $|v_2'| + |v_3'| \lesssim L\lambda$ and thus

$$|v_2 \wedge v_3| \ge |v_2' \wedge v_3'| - \delta\, O\big(|v_2'| + |v_3'|\big) \gtrsim \lambda^2\phi - L\delta\lambda \gtrsim \lambda^2\phi\,.$$

The last step uses $\phi\lambda \gg L\delta$. By Lemma 2.3 J is contained in an interval of length $\lesssim L\lambda(\lambda+\theta)^{-1}$ and Fubini's theorem implies that the volume of the set of (w,τ) parameters of $\mathcal{L}$ is bounded by $\delta^2|v_2\wedge v_3|^{-1}|J| \lesssim \delta^2\lambda^{-2}\phi^{-1}L\lambda(\lambda+\theta)^{-1}$. □

As in the case of two lines, one can pass from an estimate on $|\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2,l_3)}(\vec\phi)|$ to an estimate on the measure of the set of directions given by all lines in $\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2,l_3)}(\vec\phi)$.

Lemma 2.7. *Fix $l_1,l_2,l_3 \in X$, some $\lambda \in (\delta,1)$, and angles $\alpha\in(\delta,1)$, $\vec\phi = (\phi_{12},\phi_{13},\phi_{23}) \in (\delta,\pi/2)^3$. Let $L\ge 2$ be some constant and let $\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2,l_3)}(\vec\phi)$ be defined as in (2.15). If $\lambda \gg \delta/\alpha$ and $\phi \gg L\delta/\lambda\alpha$ then*

$$\big|\Delta(\mathcal{L}_{\alpha,\lambda}^{(l_1,l_2,l_3)}(\vec\phi))\big| \lesssim \alpha\delta\, L\lambda^{-1}\big(\lambda + \tfrac{\theta}{\alpha}\big)^{-1}.$$

Proof. As usual we assume that l_1 is the z-axis and that $\alpha = 1$. Using coordinates (w,Δ) rather than $(w,\bar w)$ on the space of lines one obtains from Fubini's theorem and Lemma 2.6

$$\begin{aligned}\int_{\Delta(\mathcal{L}_{1,\lambda}^{(l_1,l_2,l_3)}(\vec\phi))} \mathcal{H}^2\big(\{l\in\mathcal{L} : \Delta(l) = e\}\big)de &\le \big|\mathcal{L}_{1,\lambda}^{(l_1,l_2,l_3)}(\vec\phi)\big| \\ &\lesssim \delta^3 L\lambda^{-1}(\lambda+\theta)^{-1}\phi^{-1}\,.\end{aligned} \tag{2.18}$$

Given $e\in\Delta(\mathcal{L}_{1,\lambda}^{(l_1,l_2,l_3)}(\vec\phi))$ we claim that

$$\mathcal{H}^2\big(\{l\in\mathcal{L}_{1,\lambda}^{(l_1,l_2,l_3)}(\vec\phi) : \Delta(l) = e\}\big) \gtrsim \tfrac{\delta^2}{\phi} \tag{2.19}$$

(strictly speaking, one should introduce an auxiliary set as in (2.10), but we skip those details). Indeed, suppose that $l\in\mathcal{L}_{1,\lambda}^{(l_1,l_2,l_3)}(\vec\phi)$, $e=\Delta(l)$. Since $\sphericalangle(l\vee l_i', l\vee l_j') \sim \phi_{ij} \le \phi$, any line $l'\in l\vee l_1'$ parallel to l so that $\mathrm{dist}(l,l') \ll \delta/\phi$ satisfies $\mathrm{dist}(l',l_j') < \delta$ for $j=1,2,3$. Furthermore, it follows from

$$\mathrm{dist}(l^\delta\cap l_i^\delta, l^\delta\cap l_j^\delta) \in [\lambda, L\lambda]\,,\quad \sphericalangle(l_j,l)\sim 1 \ \text{ and }\ \tfrac{\delta}{\phi} \ll \lambda$$

that $\mathrm{dist}((l')^\delta\cap l_i^\delta, (l')^\delta\cap l_j^\delta) \in [\lambda/2, 2L\lambda]$. We conclude that l' and all its δ-translates belong to $\mathcal{L}_{1,\lambda}^{(l_1,l_2,l_3)}(\vec\phi)$. This proves (2.19) and the lemma follows from (2.18). □

The following proposition is the main result of this paper. It is a quantitative version of the fact that (1.3) is a one-parameter family. Note that in contrast to Lemma 2.7 one does not need to specify the angles between the planes $l \vee l'_i$ and $l \vee l'_j$ in Proposition 2.8. This is important for applications to the Kakeya problem below, see section 3.

PROPOSITION 2.8. *Fix lines $l_1, l_2, l_3 \in X$, and $\alpha, \lambda \in (\delta, 1)$, and some constant $L \geq 2$. Let $\theta = \max \sphericalangle(l_i, l_j) > \delta$ and define*

$$\mathcal{D}^{(l_1,l_2,l_3)}_{\alpha,\lambda} = \big\{ e : \exists\, l \in X \ \ \textit{with direction } e \textit{ so that } \ l^\delta \cap l_j^\delta \cap B_0 \neq \emptyset\,, \\ \sphericalangle(l, l_j) \in [\alpha, 2\alpha]\,,\ \mathrm{dist}(l^\delta \cap l_i^\delta, l^\delta \cap l_j^\delta) \in [\lambda, L\lambda]\,,\ \textit{for}\ \ 1 \leq i \neq j \leq 3 \big\}\,.$$

If $\lambda \gg \delta/\alpha$ then

$$|\mathcal{D}^{(l_1,l_2,l_3)}_{\alpha,\lambda}| \lesssim \alpha\delta\lambda^{-1}\big(\lambda + \tfrac{\theta}{\alpha}\big)^{-1} \min\big(L, \theta\tfrac{\lambda}{\delta}\big) |\log\delta|^3\,. \tag{2.20}$$

Proof. We may assume that $\theta = \sphericalangle(l_1, l_2)$, that l_1 is the z-axis, and that $\alpha = 1$. Let $\mathcal{L}^{(l_1,l_2,l_3)}_{1,\lambda}(\vec{\phi})$ be the set defined in Lemma 2.6.

Then, with ϕ_{12} etc. dyadic numbers $\in [\delta, \pi/2]$,

$$\mathcal{D}^{(l_1,l_2,l_3)}_{1,\lambda} \subset \bigcup_{L\frac{\delta}{\lambda} \ll \max\phi_{ij}} \Delta\big(\mathcal{L}^{(l_1,l_2,l_3)}_{1,\lambda}(\vec{\phi})\big) \cup \Delta\Big(\bigcup_{\max\phi_{ij} \lesssim L\frac{\delta}{\lambda}} \mathcal{L}^{(l_1,l_2,l_3)}_{1,\lambda}(\vec{\phi})\Big)\,. \tag{2.21}$$

We bound the measure of the first union of (2.21) by Lemma 2.7. If $\theta \lesssim L\frac{\delta}{\lambda}$ the first union in (2.21) is empty, since we always have $\phi_{ij} \lesssim \theta$, see (2.8). Thus

$$\Big| \bigcup_{L\frac{\delta}{\lambda} \ll \max\phi_{ij}} \Delta\big(\mathcal{L}^{(l_1,l_2,l_3)}_{1,\lambda}(\vec{\phi})\big) \Big| \lesssim \delta\, L\lambda^{-1}(\lambda + \theta)^{-1} \min\big(1, \theta\tfrac{\lambda}{L\delta}\big) |\log\delta|^3\,. \tag{2.22}$$

The measure of the second union can be estimated via Lemma 2.5, simply by discarding l_3. Indeed, clearly

$$\Delta\Big(\bigcup_{\max\phi_{ij} \lesssim L\frac{\delta}{\lambda}} \mathcal{L}^{(l_1,l_2,l_3)}_{1,\lambda}(\vec{\phi})\Big) \subset \mathcal{D}^{(l_1,l_2)}_{1,\lambda}\big(L\tfrac{\delta}{\lambda}\big)\,,$$

and since $\sphericalangle(l_1, l_2) = \theta > \delta$ the desired estimate follows from Lemma 2.5. □

REMARK 2.9. Inequality (2.20) turns out to be sharp if $L \sim 1$, at least up to logarithmic factors. The constant L was introduced only for technical reasons. In fact, in the applications to the Kakeya problem in section 3 we shall have $L < \delta^{-\epsilon}$. Note also that the condition $\lambda \gg \delta/\alpha$ cannot be relaxed because

$$\mathrm{diam}({l'}^\delta \cap l^\delta) \sim \tfrac{\delta}{\alpha}$$

for any l, l' intersecting at angle α.

Example 2.10. Consider the following hyperbolic paraboloid. Let l_1 be the x-axis and define

$$l_2 = \{(t\cos\theta, -\lambda, t\sin\theta) : t \in \mathbb{R}\}$$
$$l_3 = \{(t\cos\theta, \lambda, -t\sin\theta) : t \in \mathbb{R}\}.$$

Here $\lambda, \theta \in (\delta, 1)$. The common transversals of these three lines are

$$l'_\xi = \{(\xi, t\lambda, t\xi\tan\theta) : t \in \mathbb{R}\},$$

see Figure 4. We shall bound the measure of the set

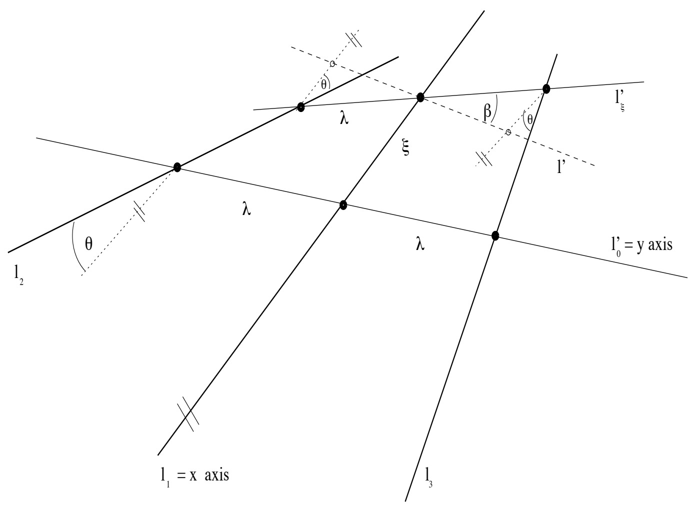

Figure 4: Hyperbolic paraboloid as in Example 2.10

$$\mathcal{D} = \{e : e = \Delta(l) \text{ with some } l \text{ satisfying } l_i^\delta \cap l^\delta \neq \emptyset,\ \mathrm{dist}(l_i^\delta \cap l^\delta, l_j^\delta \cap l^\delta) \sim \lambda\}$$

from below. Clearly, $\Delta(l'_\xi) \in \mathcal{D}$ provided $|\xi| \lesssim \frac{\lambda}{\lambda+\theta}$. Moreover, it is easy to see that

$$\left\{ \frac{\Delta(l'_\xi)}{|\Delta(l'_\xi)|} : |\xi| \lesssim \frac{\lambda}{\lambda+\theta} \right\}$$

is an arc of a great circle on S^2 of length $\sim \frac{\theta}{\lambda+\theta}$.

Now fix some ξ with $|\xi| \lesssim \frac{\lambda}{\lambda+\theta}$. Let $l' \in l_1 \vee l'_\xi$ so that $(\xi, 0, 0) \in l'$, and

$$\beta = \sphericalangle(l'_\xi, l') \leq \min\left(1, \frac{\delta}{\lambda\theta}\right),$$

see Figure 4. Then l' will be δ-close to l_1, l_2, l_3 and the intersection points remain λ-separated. Therefore $\Delta(l') \in \mathcal{D}$. Moreover, the directions of those

l' (still with fixed ξ) lie on an arc of length $\sim \min(1, \delta/\lambda\theta)$ perpendicular to the aforementioned great circle. Thus

$$|\mathcal{D}| \gtrsim \frac{\theta}{\lambda+\theta} \min\left(1, \frac{\delta}{\lambda\theta}\right) = \delta\lambda^{-1}(\lambda+\theta)^{-1} \min\left(1, \theta\frac{\lambda}{\delta}\right),$$

which agrees with (2.20) up to logarithms ($\alpha \sim L \sim 1$).

3 Application to the Kakeya Problem

Let $T^\delta(e,x) \subset \mathbb{R}^3$ denote a tube of length one and thickness δ pointing in direction $e \in S^2$ and centered at x. As usual, define the Kakeya maximal function to be

$$\mathcal{M}_\delta f(e) = \sup_{x\in\mathbb{R}^3} \delta^{-2} \int_{T^\delta(e,x)} |f(y)|\, dy.$$

Using Proposition 2.8 we show below how to obtain the bound

$$\|\mathcal{M}_\delta f\|_{L^q(S^2)} \le C_\epsilon\, \delta^{-\frac{3}{p}+1-\epsilon} \|f\|_{L^p(\mathbb{R}^3)} \tag{3.1}$$

for any $\epsilon > 0$ and with $p = 7/3$, $q = 2p' = 7/2$. Letting f be the indicator function of a fixed δ-tube one sees that $q = 2p'$ is the optimal q. In [W1] the stronger estimate (3.1) with $p = 5/2$ and $q = 2p' = 10/3$ is established. Moreover, it is a well-known conjecture that (3.1) holds with $p = q = 3$. By Lemma 2.15 in [Bo], (3.1) implies $\dim(E) \ge p$ for any Besicovitch set $E \subset \mathbb{R}^3$. In particular, we recover Bourgain's estimate $\dim(E) \ge 7/3$ from [Bo].

The following proposition establishes the equivalence of (3.1) with appropriate bounds on the multiplicity as indicated in the introduction.

Proposition 3.1. *Fix any finite $p, q > 1$. Then the following statements are equivalent.*

i. *For any $\epsilon > 0$ there exists a constant C_ϵ depending only on ϵ, p, q such that*

$$\|\mathcal{M}_\delta f\|_{L^q(S^2)} \le C_\epsilon\, \delta^{-\frac{3}{p}+1-\epsilon} \|f\|_{L^p(\mathbb{R}^3)} \tag{3.2}$$

for all $f \in L^p(\mathbb{R}^3)$.

ii. *For any collection $\mathcal{T}$ of tubes $T_1^\delta, T_2^\delta, \dots, T_N^\delta$ with δ-separated directions and any $\epsilon > 0$ there exists a subcollection $\mathcal{T}' \subset \mathcal{T}$ so that $\mathrm{card}(\mathcal{T}') \ge N/2$ and*

$$\left|\left\{x \in T^\delta : \sum_{j=1}^N \chi_{T_j^\delta}(x) > \lambda^{1-p}\delta^{p-3-\epsilon}(\delta^2 N)^{1-\frac{p}{q}}\right\}\right| < \lambda |T^\delta| \tag{3.3}$$

for all $T^\delta \in \mathcal{T}'$ and all $\lambda \in (0,1]$ provided δ is sufficiently small, i.e., $\delta \in (0, \delta_0(\epsilon, p, q)]$.

Proof. That (3.2) follows from the second statement is exactly the low multiplicity case in [W1]. First note that it suffices to show (3.2) with $f = \chi_E$, and any measurable $E \subset \mathbb{R}^3$. Fix $\lambda \in (0,1]$ and pick a maximally δ-separated set $\{e_j\}_{j=1}^N$ in $F = \{e \in S^2 : \mathcal{M}_\delta \chi_E(e) > \lambda\}$. Clearly, $|F| \lesssim \delta^2 N$ and there exist points $x_j \in \mathbb{R}^3$ such that

$$\big|T^\delta(e_j, x_j) \cap E\big| > \lambda \big|T^\delta(e_j, x_j)\big| \tag{3.4}$$

for $j = 1, 2, \ldots, N$. Let

$$\mu = \lambda^{1-p} \delta^{p-3-\epsilon} (\delta^2 N)^{1-\frac{p}{q}} . \tag{3.5}$$

Applying the second statement of the proposition with $\lambda/2$ instead of λ and with $\mathcal{T} = \{T^\delta(e_j, x_j)\}_{j=1}^N$ one concludes that

$$\begin{aligned} |E| &\gtrsim \frac{1}{\mu} \int_{\{E : \sum_{j=1}^N \chi_{T_j^\delta} \le \mu\}} \sum_{T^\delta \in \mathcal{T}'} \chi_{T^\delta} \ge \frac{1}{\mu} \mathrm{card}(\mathcal{T}') \frac{\lambda}{2} \delta^2 \\ &\gtrsim \delta^{3+\epsilon-p} \lambda^p (\delta^2 N)^{p/q} , \end{aligned} \tag{3.6}$$

for sufficiently small δ, which is the same as

$$\lambda \big|\{\mathcal{M}_\delta(\chi_E) > \lambda\}\big|^{\frac{1}{q}} \lesssim \lambda |F|^{\frac{1}{q}} \le C_\epsilon \, \delta^{-\frac{3}{p}+1-\frac{\epsilon}{p}} |E|^{\frac{1}{p}} .$$

To obtain the second statement from the first we invoke an argument that seems to originate in [S]. Fix any small $\epsilon > 0$ and assume that there exists $\mathcal{T}' \subset \mathcal{T}$ with $\mathrm{card}(\mathcal{T}') \ge \mathrm{card}(\mathcal{T})/2 = N/2$ and such that for all $T^\delta \in \mathcal{T}'$

$$\left| \left\{ x \in T^\delta : \sum_{j=1}^N \chi_{T_j^\delta}(x) > \lambda^{1-p} \delta^{p-3-\epsilon} (\delta^2 N)^{1-\frac{p}{q}} \right\} \right| > \lambda |T^\delta| \tag{3.7}$$

with some $\lambda \in (0,1]$ depending on T^δ. First note that necessarily $\lambda > \delta$. This follows easily from (3.7) for small δ since $\delta^2 N \le 1$ and $p > 1$. Applying the pigeon-hole principle one concludes that there exist $\mathcal{T}'' \subset \mathcal{T}$ with $\mathrm{card}(\mathcal{T}'') \ge \frac{1}{2} |\log \delta|^{-1} N$ and a fixed (dyadic) $\lambda \in [\delta, 1]$ such that (3.7) holds with this choice of λ and for all $T^\delta \in \mathcal{T}''$. Let μ be as in (3.5) and define $E_1 = \big\{x \in \mathbb{R}^3 : \sum_{j=1}^N \chi_{T_j^\delta}(x) > \mu\big\}$.

Case 1: $|E_1| \le \lambda^p (\delta^2 N)^{p/q} \delta^{3-p+\epsilon}$

First note that (3.7) implies that $\mathcal{M}_{2\delta}(\chi_{E_1})(e') > \mu$ for any $e' \in S^2$ such that $|e' - e| < \delta$ where e is the direction of $T^\delta \in \mathcal{T}''$. In view of (3.2) with $f = \chi_{E_1}$ one therefore obtains

$$\lambda \big(\delta^2 \mathrm{card}(\mathcal{T}'')\big)^{\frac{1}{q}} \le C_\epsilon \, \delta^{-\frac{3}{p}+1-\frac{\epsilon}{2p}} |E_1|^{\frac{1}{p}} ,$$

which contradicts our assumption on E_1 if δ is sufficiently small.

Case 2: $|E_1| > \lambda^p(\delta^2 N)^{p/q}\delta^{3-p+\epsilon}$

In this case we use duality. Let $\{e_j\}_{j=1}^M \subset S^2$ be a maximally δ-separated set of directions. The dual statement to (3.2) is

$$\Big\| \sum_{i=1}^M a_j \chi_{T^\delta(e_i,y_i)} \Big\|_{p'} \le C_\epsilon \, \delta^{\frac{2}{q'}-\frac{3}{p}+1-\frac{\epsilon}{2p}} \Big(\sum_{j=1}^M |a_j|^{q'} \Big)^{1/q'} \tag{3.8}$$

for any choice of $\{y_j\} \subset \mathbb{R}^3$ and reals a_j. To apply (3.8) we may of course assume that the $\{e_j\}$ and $\{y_j\}$ are chosen such that the tubes in $\mathcal{T}$ are among the $\{T^\delta(e_i, y_i)\}_{i=1}^M$. Now choose $a_i = 1$ or 0 depending on whether or not $T^\delta \in \mathcal{T}$. Thus

$$\mu |E_1|^{\frac{1}{p'}} \le C_\epsilon \, \delta^{\frac{2}{q'}-\frac{3}{p}+1-\frac{\epsilon}{2p}} N^{\frac{1}{q'}} \,.$$

In view of the definition of μ this implies

$$|E_1| \le C_\epsilon \, \delta^{\epsilon p'/2p} \lambda^p \delta^{3-p+\epsilon} (\delta^2 N)^{p/q} \,,$$

which contradicts the assumption of Case 2 for small δ. We conclude that $\mathcal{T}'$ with the stated properties cannot exist and we are done. $\square$

Note that Wolff's result [W1] implies the multiplicity estimate with $p = 5/2$ and $q = 2p' = 10/3$. The purpose of the following proposition is to show how to derive (3.3) with $p = 7/3$ and $q = 2p' = 7/2$ from Proposition 2.8. Arguments of this type originate in [KW].

Proposition 3.2. *For any collection $\mathcal{T}$ of tubes $T_1^\delta, T_2^\delta, \dots, T_N^\delta$ with δ-separated directions and any $\epsilon > 0$ there exists a subcollection $\mathcal{T}' \subset \mathcal{T}$ so that* $\mathrm{card}(\mathcal{T}') \ge N/2$ *and*

$$\left| \left\{ x \in T^\delta : \sum_{j=1}^N \chi_{T_j^\delta}(x) > \lambda^{-\frac{4}{3}} \delta^{-\frac{2}{3}-\epsilon} (\delta^2 N)^{\frac{1}{3}} \right\} \right| < \lambda |T^\delta|$$

for all $T^\delta \in \mathcal{T}'$ and all $\lambda \in (0,1]$ provided δ is sufficiently small.

Proof. Suppose this fails for some fixed $\epsilon > 0$. Then there exists $\mathcal{T}' \subset \mathcal{T}$, with $\mathrm{card}(\mathcal{T}') \ge \frac{1}{2}\mathrm{card}(\mathcal{T}) = \frac{1}{2}N$ and such that for all $T^\delta \in \mathcal{T}'$

$$\left| \left\{ x \in T^\delta : \sum_{j=1}^N \chi_{T_j^\delta}(x) > \lambda^{-\frac{4}{3}} \delta^{-\frac{2}{3}-\epsilon} (\delta^2 N)^{\frac{1}{3}} \right\} \right| > \lambda |T^\delta| \tag{3.9}$$

with some λ depending on T^δ. As in the previous proof, it follows from the pigeon-hole principle that there exist $\mathcal{T}'' \subset \mathcal{T}'$ with $\mathrm{card}(\mathcal{T}'') \ge \frac{1}{4}|\log\delta|^{-1} N$ and a fixed $\lambda \in [\delta, 1]$ such that (3.9) holds for all $T^\delta \in \mathcal{T}''$. Define

$$\mu = \lambda^{-\frac{4}{3}} \delta^{-\frac{2}{3}-\frac{2\epsilon}{3}} (\delta^2 N)^{\frac{1}{3}} \,. \tag{3.10}$$

Fix any $T^\delta \in \mathcal{T}''$. Clearly, for every x in the set on the left-hand side of (3.9) there exists an angle $\alpha \in [\delta, 1]$ depending on x such that

$$\begin{aligned}&\mathrm{card}\big(\{j \in \{1,2\ldots,N\} : x \in T_j^\delta \text{ and } \sphericalangle(T_j^\delta, T^\delta) \in (\alpha, 2\alpha]\}\big)\\&\qquad > \tfrac{1}{2}|\log\delta|^{-1}\lambda^{-\frac{4}{3}}\delta^{-\frac{2}{3}-\epsilon}(\delta^2 N)^{\frac{1}{3}} > \mu\,.\end{aligned}$$

Applying the pigeon-hole principle again, conclude that for some choice of α depending only on T^δ we have

$$\Big|\Big\{x \in T^\delta : \sum_{j:\sphericalangle(T_j^\delta, T^\delta)\in(\alpha,2\alpha]} \chi_{T_j^\delta}(x) > \mu\Big\}\Big| > \tfrac{1}{2}|\log\delta|^{-1}\lambda|T^\delta|\,. \tag{3.11}$$

Finally, a further application of the pigeon-hole principle shows that there exist a fixed $\alpha \in [\delta, 1]$ and a set of tubes $\mathcal{T}_1 \subset \mathcal{T}''$ with $\mathrm{card}(\mathcal{T}_1) > \frac{1}{2}|\log\delta|^{-1}\mathrm{card}(\mathcal{T}'')$ such that (3.11) holds for all $T^\delta \in \mathcal{T}_1$. Let $\bar\lambda = |\log\delta|^{-1}\lambda$.

Case 1: $\bar\lambda \le \delta^{-\epsilon}\frac{\delta}{\alpha}$

In view of (3.11)

$$\begin{aligned}\mu\bar\lambda\delta^2 &\le \int_{T^\delta} \sum_{j:\sphericalangle(T_j^\delta,T^\delta)\le 2\alpha} \chi_{T_j^\delta}(x)dx\\ &\le \sum_{j:\sphericalangle(T_j^\delta,T^\delta)\le 2\alpha} |T^\delta \cap T_j^\delta| \lesssim \min\left(N, \tfrac{\alpha^2}{\delta^2}\right)\tfrac{\delta^3}{\alpha}\,.\end{aligned}$$

Hence

$$\mu \lesssim \bar\lambda^{-\frac{4}{3}}\left(\tfrac{\delta}{\alpha}\right)^{\frac{4}{3}}\delta^{-2-\frac{\epsilon}{3}}\min(\delta^2 N, \alpha^2) \lesssim |\log\delta|^{\frac{4}{3}}\lambda^{-\frac{4}{3}}\delta^{-\frac{2}{3}-\frac{\epsilon}{3}}(\delta^2 N)^{\frac{1}{3}}\,,$$

which contradicts (3.10).

Case 2: $\bar\lambda > \delta^{-\epsilon}\frac{\delta}{\alpha}$

Consider the set

$$\begin{aligned}Q_1 = \big\{&(T^\delta, T_1^\delta, T_2^\delta, T_3^\delta) : T^\delta \in \mathcal{T}_1\,,\ T^\delta \cap T_i^\delta \ne \emptyset\,,\\ &\sphericalangle(T^\delta, T_i^\delta) \in (\alpha, 2\alpha]\,,\ i = 1,2,3\big\}\,.\end{aligned}$$

Fix any $T^\delta \in \mathcal{T}_1$ and let $R_1, R_2, \ldots, R_m$ be non-overlapping congruent subtubes of T^δ of dimensions $\delta \times \delta \times 5\frac{\delta}{\alpha}$, which we call rectangles. Note that if $T_j^\delta \cap R_k \ne \emptyset$ and $\sphericalangle(T_j^\delta, T^\delta) \in (\alpha, 2\alpha]$ then T_j^δ can intersect at most the two neighboring rectangles of R_k. In view of (3.11) the number of rectangles R_k such that

$$R_k \cap \Big\{x \in T^\delta : \sum_{j:\sphericalangle(T_j^\delta,T^\delta)\in(\alpha,2\alpha]} \chi_{T_j^\delta}(x) > \mu\Big\} \ne \emptyset \tag{3.12}$$

has to be $\gtrsim \bar{\lambda}\frac{\alpha}{\delta} > \delta^{-\epsilon}$. We shall refer to these rectangles as good rectangles. As usual, one bounds the multiplicity μ by estimating the cardinality of Q_1 in two different ways. Firstly, (3.12) implies that for a given T^δ there are many choices of triples $(T_1^\delta, T_2^\delta, T_3^\delta)$ which gives a lower bound on $\mathrm{card}(Q_1)$. Secondly, an upper bound is obtained by fixing some triple and then limiting the number of possible choices of T^δ by Proposition 2.8. However, in order to apply Proposition 2.8 one needs to know that the distances of the intersection points are approximately equal. In order to achieve this, we divide every $T^\delta \in \mathcal{T}_1$ into congruent subtubes $S_1, S_2, \dots, S_m$ of thickness δ and height $\bar{\lambda}$. Clearly, $m \sim \bar{\lambda}^{-1} \lesssim \delta^\epsilon \frac{\alpha}{\delta}$. For any $T^\delta \in \mathcal{T}_1$ the pigeon hole principle implies that there exists an integer $M = M(T^\delta) \in \{1, 2, \dots, m\}$ such that

$$\mathrm{card}\Bigg(\{j \in \{1,2,\dots,m\} : S_j \ \text{ contains at least } \ \frac{\bar{\lambda}\alpha}{2\delta M}|\log\delta|^{-1} \\ \text{good rectangles } \ R_k\}\Bigg) \geq M\,. \tag{3.13}$$

Furthermore, there exist $\mathcal{T}_0 \subset \mathcal{T}_1$ and a fixed integer $\nu \geq 0$ so that

$$\mathrm{card}(\mathcal{T}_0) \geq \tfrac{1}{2}|\log\delta|^{-1}\mathrm{card}(\mathcal{T}_1) \gtrsim |\log\delta|^{-3}N \quad\text{and}\quad M(T^\delta) \in [2^\nu, 2^{\nu+1}]$$

for all $T^\delta \in \mathcal{T}_0$.

Assume that $2^\nu > A_\epsilon^2\bar{\lambda}^{-\epsilon/5}$ with some sufficiently large A_ϵ depending only on ϵ. Then by Lemma 5.3 in [W2] there exists $\sigma > A_\epsilon\bar{\lambda}$ such that there are at least $(2^\nu\bar{\lambda}^{\epsilon/5})^3$ triples $(S_{j_1}, S_{j_2}, S_{j_3})$ with the property that $\mathrm{dist}(S_{j_l}, S_{j_k}) \in [\sigma, A_\epsilon\sigma]$, $1 \leq l < k \leq 3$. Therefore, in view of (3.13), there are at least

$$(2^\nu\bar{\lambda}^{\epsilon/5})^3 \cdot \left(\frac{\bar{\lambda}\alpha}{2^{\nu+2}\delta}|\log\delta|^{-1}\right)^3 \gtrsim \left(\bar{\lambda}\frac{\alpha}{\delta}\right)^3\delta^\epsilon \tag{3.14}$$

many triples $(R_{k_1}, R_{k_2}, R_{k_3})$ of good rectangles so that

$$\mathrm{dist}(R_{k_i}, R_{k_j}) \in [\sigma, 2A_\epsilon\sigma] \quad \text{for} \quad 1 \leq i < j \leq 3\,. \tag{3.15}$$

Now suppose that $2^\nu \leq A_\epsilon\bar{\lambda}^{-\epsilon/5}$. By (3.13) there has to be at least one subtube, say S_1, such that

$$\mathrm{card}\big(\{k : R_k \subset S_1\}\big) \geq \bar{\lambda}\tfrac{\alpha}{\delta}\,\delta^{\epsilon/4}\,,$$

provided δ is sufficiently small. Clearly, this implies that there are at least $(\bar{\lambda}\frac{\alpha}{\delta}\delta^{\epsilon/4})^3$ many triples $(R_{k_1}, R_{k_2}, R_{k_3})$ of good rectangles so that $\mathrm{dist}(R_{k_i}, R_{k_j}) \in [\bar{\lambda}\delta^{\epsilon/4}/10, \bar{\lambda}]$ for $i \neq j$. Combining this with (3.14) and

(3.15) we finally conclude that there exists $\mathcal{T}_0 \subset \mathcal{T}_1$ of cardinality $\mathrm{card}(\mathcal{T}_0) \gtrsim |\log\delta|^{-3}N$ so that for every $T^\delta \in \mathcal{T}_0$ there are at least

$$\big(\bar\lambda\tfrac{\alpha}{\delta}\big)^3\delta^\epsilon \tag{3.16}$$

triples $(R_{k_1}, R_{k_2}, R_{k_3})$ of good rectangles with the property that

$$\mathrm{dist}(R_{k_i}, R_{k_j}) \in [\sigma, L\sigma] \tag{3.17}$$

where $\sigma \ge \bar\lambda\delta^{\epsilon/4}/10$ and $L \le 10\delta^{-\epsilon/4}$. With σ and L as in (3.17) define

$$Q_0 = \big\{(T^\delta, T_1^\delta, T_2^\delta, T_3^\delta) \in Q_1 : T^\delta \in \mathcal{T}_0\,,\ \mathrm{dist}(T^\delta \cap T_i^\delta, T^\delta \cap T_j^\delta) \in [\sigma, L\sigma] \\ \text{for }\ 1 \le i < j \le 3\big\}\,.$$

To bound the cardinality of Q_0 from below, note that if $T_j^\delta \cap R_k \neq \emptyset$ and $\sphericalangle(T_j^\delta, T^\delta) \in (\alpha, 2\alpha]$ then T_j^δ can intersect at most the two neighboring rectangles of R_k. Consequently, given $T^\delta \in \mathcal{T}_0$, (3.16) and (3.12) imply

$$\mathrm{card}\big(\{(T_1^\delta, T_2^\delta, T_3^\delta) : (T^\delta, T_1^\delta, T_2^\delta, T_3^\delta) \in Q_0\}\big) \gtrsim \big(\mu\bar\lambda\tfrac{\alpha}{\delta}\big)^3\delta^\epsilon$$

and thus

$$\mathrm{card}(Q_0) \gtrsim \mathrm{card}(\mathcal{T}_0)\big(\mu\bar\lambda\tfrac{\alpha}{\delta}\big)^3\delta^\epsilon\,.$$

As usual, one bounds the cardinality of Q from above by choosing first $(T_1^\delta, T_2^\delta, T_3^\delta)$ and then T^δ. In order to apply Proposition 2.8 in this context, we need to consider a slightly smaller set $Q_0' \subset Q_0$. Clearly, there exists an angle $\theta \in [\delta, 1]$ such that the set

$$Q_0' = \Big\{(T^\delta, T_1^\delta, T_2^\delta, T_3^\delta) \in Q_0 : \max_{1\le i<j\le 3} \sphericalangle(T_i^\delta, T_j^\delta) \in (\theta, 2\theta]\Big\}$$

satisfies

$$\mathrm{card}(Q_0') \gtrsim |\log\delta|^{-1}\mathrm{card}(Q_0) \gtrsim N\big(\mu\bar\lambda\tfrac{\alpha}{\delta}\big)^3\delta^\epsilon|\log\delta|^{-4}\,. \tag{3.18}$$

On the other hand,

$$\mathrm{card}(Q_0') \lesssim \tfrac{\alpha}{\delta}\sigma^{-1}\tfrac{\alpha}{\theta}L|\log\delta|^3 \cdot N\min\big(N, \tfrac{\theta^2}{\delta^2}\big)^2\,. \tag{3.19}$$

Indeed, there are N possibilities for choosing T_1^δ. Once T_1^δ is fixed the directions of T_2^δ and T_3^δ have to lie in a θ-cap on S^2 centered at the direction of T_1^δ. This leads to the last factor in (3.19). Finally, the number of choices of T^δ is controlled by Proposition 2.8. Comparing (3.18) with (3.19) one finds that

$$\mu \lesssim \lambda^{-4/3}\delta^{-2/3}(\delta^2N)^{1/3}\big(\tfrac{\theta}{\alpha}\big)^{1/3}\delta^{-\epsilon/2}|\log\delta|^4\,.$$

Since $\theta \lesssim \alpha$ by the triangle inequality, this contradicts (3.10). $\square$

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