A Limiting Absorption Principle for the three-dimensional Schrödinger equation with L^p potentials

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1 Introduction

Agmon's fundamental work [Agm] establishes the bound, known as the limiting absorption principle,

$$\sup_{\lambda>\lambda_0,\,\varepsilon>0} \big\|\big(-\triangle + V - (\lambda^2 + i\varepsilon)\big)^{-1}\big\|_{L^{2,\sigma}(\mathbb{R}^d)\to L^{2,-\sigma}(\mathbb{R}^d)} < \infty \tag{1}$$

provided that $\lambda_0 > 0$, $(1+|x|)^{1+}|V(x)| \in L^\infty$ and $\sigma > \frac{1}{2}$. Here

$$L^{2,\sigma}(\mathbb{R}^d) = \{(1+|x|)^{-\sigma} f \;:\; f \in L^2(\mathbb{R}^d)\}$$

is the usual weighted L^2. The bound (1) is obtained from the same estimate for $V = 0$ by means of the resolvent identity. This bound for the free resolvent is related to the so called trace lemma, which refers to the statement that for every $f \in L^{2,\frac{1}{2}+}$ there is a restriction of $\hat{f}$ to any (compact) hypersurface, and this restriction belongs to L^2 relative to surface measure. Note that this fact does not require any curvature properties of the hypersurface - in fact, it is proved by reduction to flat surfaces. Another fundamental restriction theorem is the Stein-Tomas theorem, see [Ste]. It requires the hypersurfaces $\mathcal{S} \subset \mathbb{R}^d$ with $d \geq 2$ to have non vanishing Gaussian curvature, and states that

$$\int_{\mathcal{S}} |\hat{f}(\omega)|^2\,\sigma(d\omega) \leq C\|f\|^2_{L^p(\mathbb{R}^d)} \quad \text{where} \quad p = \frac{2d+2}{d+3}. \tag{2}$$

It is not hard to see that the related estimate for the free resolvent in $\mathbb{R}^3$ is given by

$$\|R_0(\lambda^2 + i0)\|_{\frac{4}{3}\to 4} = C\,\lambda^{-\frac{1}{2}} \quad \text{for} \quad \lambda > 0. \tag{3}$$

This fact depends on the oscillation in the resolvent, i.e., on the exponential in

$$R_0(\lambda^2 + i0)(x,y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}. \tag{4}$$

In contrast, using the denominator alone one obtains that

$$\sup_{\lambda} \|R_0(\lambda^2 + i0)\|_{\frac{6}{5}\to 6} \leq C \tag{5}$$

via fractional integration. In analogy with Agmon's work, it is natural to ask for which potentials (3) can be extended to the perturbed operators $H = -\triangle + V$. In this paper we show that this is the case for real-valued $V \in L^p(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3), p > \frac{3}{2}$, and suggest two possible extensions.

Theorem 1. *Let $V \in L^p(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3), p > \frac{3}{2}$ be real-valued. Then for every $\lambda_0 > 0$, one has*

$$\sup_{0<\varepsilon<1,\, \lambda\geq\lambda_0} \left\| (-\triangle + V - (\lambda^2 + i\varepsilon))^{-1} \right\|_{\frac{4}{3}\to 4} \leq C(\lambda_0, V)\, \lambda^{-\frac{1}{2}}. \tag{6}$$

In particular, the spectrum of $-\triangle + V$ is purely absolutely continuous on $(0, \infty)$.

This theorem is the analogue of the classical Kato-Agmon-Kuroda theorem, see [ReeSim], Theorem XIII.33. It of course requires the absence of imbedded eigenvalues. In the classical context one uses Kato's theorem for that purpose. Here we wish to use a result on the absence of imbedded eigenvalues that only requires an integrability condition on V. One such result was obtained by Ionescu and Jersion [IonJer], namely:

Theorem 2. *Let $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$. Suppose $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)$ satisfies $(-\triangle + V)u = \lambda^2 u$ where $\lambda \neq 0$ in the sense of distributions. If, moreover, $\|(1 + |x|)^{\delta - \frac{1}{2}} u\|_2 < \infty$ for some $\delta > 0$, then $u \equiv 0$.*

The weighted L^2-condition with $\delta > 0$ is natural in view of the Fourier transform of the surface measure of S^2, which is a generalized eigenfunction of the free case and decays like $(1 + |x|)^{-1}$. As far as local regularity of the potential is concerned, the requirement that $V \in L^{3/2}_{\mathrm{loc}}$ is essentially optimal. There exist examples of $V \in L^{3/2}_{\mathrm{weak}}$ for which $-\Delta + V$ admits compactly supported eigenfunctions [KoTa]. The necessary decay condition on V is less clearly delineated: Ionescu and Jerison found a smooth real-valued potential V which lies in $L^q(\mathbb{R}^3)$ for all $q > 2$ but such that for $-\triangle + V$ imbedded eigenvalues exist. Their example decays like r^{-1} in some directions, and like r^{-2} in other directions. They further conjectured that their main result (Theorem 2.1 in [IonJer]) remains valid for potentials $V \in L^2(\mathbb{R}^3)$. Recent work by Koch and Tataru appears to verify this conjecture [KoTa2], and futher refinements which allow potentials to exhibit both $L^{3/2}_{\mathrm{loc}}$ singularities and L^2 decay seem possible as well. The proof of any such conjecture would immediately increase the scope of Theorem 1, as described below.

Proposition 3. *The following inferences are valid:*
1. If the conclusion of Theorem 2 holds for all $V \in L^p(\mathbb{R}^3), \frac{3}{2} \leq p < 2$, as is suggested by [KoTa2], then the conclusion of Theorem 1 also holds for all $V \in L^p(R^3)$.
2. More generally, if the conclusion of Theorem 2 holds for some $V \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3), \frac{3}{2} < p, q < 2$, then the conclusion of Theorem 1 also holds for this V.

By Kato's theory of H-smoothing operators, see [Kat], it is well-known that the limiting absorption principle for the resolvent gives rise to estimates for the evolution e^{itH} known as smoothing estimates. This is a much studied class of bounds, see [Sjo], [Veg], [ConSau1], [ConSau2], [BenKla], [Doi], [Sim]. In fact, the Fourier transform establishes a link between the resolvent and the evolution that in a precise sense allows one to state that a certain class of estimates on the evolution is equivalent to corresponding ones for the resolvent, see [Kat]. In the free case, the $\frac{4}{3} \to 4$ bound for the resolvent corresponds to the following smoothing bound for the free evolution:

$$\sup_{\|F\|_4 \leq 1} \int_{-\infty}^{\infty} \left\| F(-\triangle)^{\frac{1}{8}} e^{it\triangle} f \right\|_2^2 dt \leq C\, \|f\|_2^2.$$

However, this bound is known, see the work of Ruiz and Vega [RuiVeg]. For the perturbed evolution, $H = -\Delta + V$, one can prove similar estimates by means of Theorem 1, but we do not pursue this here. See the work of Ionescu and the second author [IonSch] for statements of this type.

This paper is organized as follows: In Section 2 we prove the bounds on the free resolvent that are needed in order to prove Theorem 1. Our main new bounds involve $R_0(\lambda^2 + i0)$ acting on functions whose Fourier transform vanish on λS^2. In Section 3 we apply these bounds in the context of the usual resolvent identity/Fredholm alternative type arguments to deal with $-\triangle + V$. This of course requires Theorem 2. Finally, in Section 4 we return to the free resolvent and prove some end point results.

2 The free resolvent

This section develops some estimates on the free resolvent given by (4). These estimates are motivated on the one hand by the Stein-Tomas theorem (2), and on the other hand, by the applications to the perturbed operator $H = -\triangle + V$, see Theorem 2. For what follows, it will be helpful to keep in mind that for real λ,

$$[R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)]f = C(\lambda) \cdot (\widehat{\sigma_{\lambda S^2}} * f),$$

which is exactly of the form T^*T, T being the restriction operator to the sphere λS^2. Thus T^*T : $L^{\frac{4}{3}}(\mathbb{R}^3) \to L^4(\mathbb{R}^3)$ in view of (2).

We will denote by $\mathbb{H}$ the closed upper half-plane in $\mathbb{C}$, and state most of our results for $\lambda \in \mathbb{H}$. For any positive real number λ, we have the boundary identites

$$(\lambda + i0)^2 = \lambda^2 + i0 \quad \text{and} \quad (-\lambda + i0)^2 = \lambda^2 - i0,$$

therefore estimates which hold uniformly out to $\partial\mathbb{H}$ are of particular importance.

Lemma 4. *Let $\lambda \in \mathbb{H}$ be any nonzero element, and $p = \frac{4}{3}$. Then $R_0(\lambda^2) : L^p(\mathbb{R}^3) \to L^{p'}(\mathbb{R}^3)$, with operator norm bounded by $|\lambda|^{-\frac{1}{2}}$.*

As suggested above, the proof follows a complex-interpolation argument strongly reminiscent of the proof of (2). For full details see Theorem 2.3 in [KenRuiSog], which establishes this bound for a more general family of inverses of second-order differential operators.

Lemma 5. *Let $\lambda \in \mathbb{H}$ be any nonzero element. For each pair of exponents $1 < p \le \frac{4}{3}$, $3p \le q \le \frac{3p}{3-2p}$ there exist constants $C_{p,q} < \infty$ such that*

$$\|R_0(\lambda^2)f\|_{L^q} \le C_{p,q}|\lambda|^{3/p-3/q-2}\|f\|_{L^p}$$

For each exponent $\frac{4}{3} \le p < \frac{3}{2}$, $\frac{p}{3-2p} \le q \le \frac{3p}{3-2p}$ there exist constants $C_{p,q} < \infty$ such that

$$\|R_0(\lambda^2)f\|_{L^{p*}} \le C_{p,q}|\lambda|^{3/p-3/q-2}\|f\|_{L^p}$$

Proof. The case $p = \frac{4}{3}, q = 4$ is Lemma 4 above. Since $R_0(\lambda^2)$ is realized as a convolution with a kernel satisfying $|K_\lambda(x)| \le |4\pi x|^{-1}$, the cases $q = \frac{3p}{3-2p}, 1 < p < \frac{3}{2}$ are precisely the Hardy–Littlewood–Sobolev inequality. Note that the scaling exponent for λ is zero for these pairs (p, q). All intermediate cases (p, q) then follow by interpolation. At the endpoint $p = 1, q = 3$, we see that $R_0^{\pm}(\lambda^2)$ maps $L^1(\mathbb{R}^3)$ to weak-$L^3(\mathbb{R}^3)$ uniformly in λ, by considering the norm

$$\|f\|_{L^3_{\text{weak}}(\mathbb{R}^3)} = \sup_{A\subset\mathbb{R}^3,\, |A|<\infty} |A|^{-\frac{2}{3}} \int_A |f(x)|\, dx$$

which is equivalent to the usual weak-L^3 "norm" and satisfies a triangle inequality, see Lieb, Loss [LieLos], Section 4.3 The cases $q = 3p$, $1 < p < \frac{4}{3}$ follow by Marcinkiewicz interpolation, and $q = \frac{p}{3-2p}$, $\frac{4}{3} < p < \frac{3}{2}$ by duality. □

The following results deal with functions whose Fourier transform vanishes on S^2. The first lemma yields a Hölder bound for the L^2 norms of the restrictions to spheres close to S^2.

Lemma 6. *Let $1 \le p < \frac{4}{3}$ and set $\gamma = \frac{2}{p} - \frac{3}{2}$. Then for all $|\delta| < \frac{1}{2}$ one has*

$$\|\hat{f}((1+\delta)\cdot)\|_{L^2(S^2)} \lesssim |\delta|^\gamma \|f\|_{L^p(\mathbb{R}^3)} \tag{7}$$

for all $f \in L^p(\mathbb{R}^3)$ with $\hat{f} = 0$ on S^2.

Proof. Let $\sigma_{(1+\delta)S^2}$ be the normalized measure on $(1+\delta)S^2$. Then one has

$$\begin{aligned} \|\hat{f}((1+\delta)\cdot)\|^2_{L^2(S^2)} &= \langle f * \widehat{\sigma_{(1+\delta)S^2}}, f\rangle = \langle f * [\widehat{\sigma_{(1+\delta)S^2}} - \widehat{\sigma_{S^2}}], f\rangle \\ &= \sum_{j=0}^{\infty} \langle f * K_j, f\rangle \end{aligned}$$

where $K_j(x) = \left(\widehat{\sigma_{(1+\delta)S^2}} - \widehat{\sigma_{S^2}}\right)\chi_j$ and $\{\chi_j\}_{j\ge0}$ are a standard dyadic partition of unity. Since $\|\widehat{\sigma_{(1+\delta)S^2}} - \widehat{\sigma_{S^2}}\|_\infty \lesssim \delta$, it follows that

$$\|K_j\|_\infty \lesssim \begin{cases} \delta & \text{if } 2^j < \delta^{-1} \\ 2^{-j} & \text{if } 2^j \ge \delta^{-1} \end{cases}$$

Thus $\|K_j\|_\infty \lesssim \min(\delta, 2^{-j}) := \alpha_j$. Moreover,

$$\begin{aligned} \|\widehat{K_j}\|_\infty &= \left\|\left(\sigma_{(1+\delta)S^2} - \sigma_{S^2}\right) * \widehat{\chi_j}\right\|_\infty \\ &= \left|\int \widehat{\chi_j}(\xi - \eta)\,\sigma_{(1+\delta)S^2}(d\eta) - \int \widehat{\chi_j}(\xi-\eta)\,\sigma_{S^2}(d\eta)\right| \\ &= \left|\int \left[\widehat{\chi_j}(\xi - (1+\delta)\eta) - \widehat{\chi_j}(\xi-\eta)\right]\sigma_{S^2}(d\eta)\right| \\ &\lesssim \min(2^{2j}\delta, 2^j) := \beta_j. \end{aligned}$$

If $1 < p < \frac{4}{3}$, let $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$ so that $\theta > \frac{1}{2}$. Then $\|K_j * f\|_{p'} \lesssim \alpha_j^\theta \beta_j^{1-\theta}\|f\|_p$ for all $j \ge 0$. Summing over j yields the desired bound. In the case $p = 1$, the estimate $\|\widehat{\sigma_{(1+\delta)S^2}} - \widehat{\sigma_{S^2}}\|_\infty \lesssim \delta$ mentioned above suffices to show that $\|\hat{f}((1+\delta)\cdot)\|_{L^2(S^2)} \lesssim \delta^{\frac{1}{2}}$.

□

The point of the following proposition is that one can take $\delta > 0$ in (8). In the following section, this will allow us to apply Theorem 2.

Proposition 7. *Let $1 \le p < \frac{4}{3}$. Then for any $\delta < \frac{1}{2} - \frac{2}{p'}$ one has*

$$\sup_{\varepsilon>0} \left\|(1+|x|)^{\delta-\frac{1}{2}} R_0(1 \pm i\varepsilon) f\right\|_2 \lesssim \|f\|_p \tag{8}$$

for any $f \in L^p(\mathbb{R}^3)$ so that $\hat{f} = 0$ on S^2.

Proof. We first consider the case where

$$\operatorname{supp}(\hat{f}) \subset \{\xi \in \mathbb{R}^3 \,:\, \frac{1}{2} < |\xi| < 2\}. \tag{9}$$

Let χ be a smooth, radial, bump function around zero so that $\hat{\chi}$ is compactly supported. Let $R \gg 1$. Then

$$\begin{aligned}
\|\chi(\frac{\cdot}{R})R_0(1+i\varepsilon)f\|_2^2 &= R^6 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \hat{\chi}(R(\xi-\eta))\frac{\hat{f}(\eta)}{|\eta|^2-1-i\varepsilon}\,d\eta \int_{\mathbb{R}^3}\hat{\chi}(R(\xi-\tilde{\eta}))\frac{\overline{\hat{f}(\tilde{\eta})}}{|\tilde{\eta}|^2-1+i\varepsilon}\,d\tilde{\eta} \\
&= R^3 \int_{\mathbb{R}^6} \rho(R(\eta-\tilde{\eta}))\,\frac{\hat{f}(\eta)}{|\eta|^2-1-i\varepsilon}\frac{\overline{\hat{f}(\tilde{\eta})}}{|\tilde{\eta}|^2-1+i\varepsilon}\,d\eta\,d\tilde{\eta}, \qquad (10)
\end{aligned}$$

where we have set

$$\begin{aligned}
\int_{\mathbb{R}^3}\hat{\chi}(R(\xi-\eta))\hat{\chi}(R(\xi-\tilde{\eta}))\,d\xi &= R^{-3}\int_{\mathbb{R}^3}\hat{\chi}(\zeta-R\eta)\hat{\chi}(\zeta-R\tilde{\eta})\,d\zeta \\
&= R^{-3}\int_{\mathbb{R}^3}\hat{\chi}(\zeta-R(\eta-\tilde{\eta}))\hat{\chi}(\zeta)\,d\zeta \\
&=: R^{-3}\rho(R(\eta-\tilde{\eta})).
\end{aligned}$$

Note that ρ is a compactly supported smooth bump-function. Introducing polar coordinates in (10) yields uniformly in $\varepsilon \neq 0$ (recall (9))

$$\begin{aligned}
(10) &= R^3\int_{\mathbb{R}^3}\int_0^\infty\int_{S^2}\rho(R(\eta-\tilde{r}\tilde{\omega}))\frac{\hat{f}(\eta)}{|\eta|^2-1-i\varepsilon}\frac{\overline{\hat{f}(\tilde{r}\tilde{\omega})}}{|\tilde{r}\tilde{\omega}|^2-1+i\varepsilon}\,d\tilde{\omega}\tilde{r}^2\,d\tilde{r}\,d\eta \\
&\lesssim R^3\int_{\mathbb{R}^3}\int_{|\eta|-R^{-1}}^{|\eta|+R^{-1}}\int_{[S^2:|\tilde{\omega}-\frac{\eta}{|\eta|}|<R^{-1}]}\frac{|\hat{f}(\eta)|}{||\eta|-1|}\frac{|\overline{\hat{f}(\tilde{r}\tilde{\omega})}|}{|\tilde{r}-1|}\,d\tilde{\omega}\,d\tilde{r}\,d\eta \\
&\lesssim R^2\int_{\mathbb{R}^3}\frac{|\hat{f}(\eta)|}{||\eta|-1|}\int_{|\eta|-R^{-1}}^{|\eta|+R^{-1}}\left(\int_{[S^2:|\tilde{\omega}-\frac{\eta}{|\eta|}|<R^{-1}]}|\hat{f}(\tilde{r}\tilde{\omega})|^2\,d\tilde{\omega}\right)^{\frac{1}{2}}\frac{d\tilde{r}}{|\tilde{r}-1|}\,d\eta \\
&\lesssim R^2\int_0^\infty\frac{dr}{|r-1|}\int_{r-R^{-1}}^{r+R^{-1}}\frac{d\tilde{r}}{|\tilde{r}-1|}\int_{S^2}|\hat{f}(r\omega)|\left(\int_{[S^2:|\tilde{\omega}-\omega|<R^{-1}]}|\hat{f}(\tilde{r}\tilde{\omega})|^2\,d\tilde{\omega}\right)^{\frac{1}{2}}
\end{aligned}$$

and therefore also

$$\begin{aligned}
(10) &\lesssim R^2\int_0^\infty\frac{dr}{|r-1|}\int_{r-R^{-1}}^{r+R^{-1}}\frac{d\tilde{r}}{|\tilde{r}-1|}\left(\int_{S^2}|\hat{f}(r\omega)|^2\,d\omega\right)^{\frac{1}{2}}\left(\int_{S^2}\int_{[S^2:|\tilde{\omega}-\omega|<R^{-1}]}|\hat{f}(\tilde{r}\tilde{\omega})|^2\,d\tilde{\omega}\,d\omega\right)^{\frac{1}{2}} \\
&\lesssim R\int_{\frac{1}{2}}^2\frac{dr}{|r-1|}\int_{r-R^{-1}}^{r+R^{-1}}\frac{d\tilde{r}}{|\tilde{r}-1|}|1-r|^\gamma|1-\tilde{r}|^\gamma\|f\|_p^2 \\
&\lesssim R^{1-2\gamma}\|f\|_p^2 = R^{\frac{4}{p'}}\|f\|_p^2,
\end{aligned}$$

where the last two lines use (7). The lemma now follows by summing over dyadic R, at least provided (9) holds. Finally, if

$$\operatorname{supp}(\hat{f}) \subset \{\xi \in \mathbb{R}^3 \,:\, |\xi| \leq \frac{1}{2} \text{ or } |\xi| \geq 2\},$$

then one notes that

$$\sup_{\varepsilon\neq 0} \|R_0(1\pm i\varepsilon)f\|_2 \lesssim \|(1-\triangle)^{-1}f\|_2 \lesssim \|f\|_p$$

by the Sobolev imbedding theorem provided $1\leq p\leq 2$ and we are done. $\square$

In Section 4 we discuss further bounds on the free resolvent which are motivated by the previous proposition.

3 The perturbed resolvent

The goal of this section is to prove theorem 1. As in [Agm], the proof of Theorem 1 is based on the resolvent identity. This requires inverting the operator $I + R_0(\lambda^2 \pm i0)V$ on $L^4(\mathbb{R}^3)$. First, we check that this is a compact perturbation of the identity.

Lemma 8. *Let $V\in L^p(\mathbb{R}^3)$, $\frac{3}{2}\leq p\leq 2$. Then for any nonzero $\lambda\in\mathbb{H}$, the map $A(\lambda) := R_0(\lambda^2)V$ is a compact operator on $L^4(\mathbb{R}^3)$.*

Proof. Firstly, note that in view of Lemma 5 and because of $V\in L^p$, $A(\lambda)$ is bounded $L^4\to L^4$. Secondly, observe that we may assume that $V\in L^\infty$ with compact support. Indeed, replace V with $V_n = V\chi_{[|V|<n]}\chi_{[|x|<n]}$. Then $\|V-V_n\|_p\to 0$ as $n\to\infty$ implies that $\|A(\lambda)-A_n\|_{4\to 4}\to 0$ as $n\to\infty$. If we can show that $A_n := R_0(\lambda^2)V_n$ are compact as operators $L^4\to L^4$ for each n, it therefore follows that $A(\lambda)$ is also compact. So assume that V is bounded, and supported in the ball $\{|x|<R\}$. Fix λ and write $A = A(\lambda)$. We first claim that $A: L^4\to W^{2,4}$. This follows from

$$(-\triangle+1)A = (-\triangle-\lambda^2)A + (\lambda^2+1)A = V + (1+\lambda^2)A \tag{11}$$

is bounded from L^4 to L^4. Meanwhile, for $|x|>2R$ there is the uniform pointwise bound

$$|Af(x)| \lesssim \|Vf\|_1|x|^{-1} \lesssim R^{\frac{9}{4}}\|V\|_\infty\|f\|_4|x|^{-1}$$

Given $\varepsilon>0$, we may choose $R_0\sim R^9\|V\|_\infty^4\varepsilon^{-4}$ so that $\|\chi_{[|x|>R_0]}Af\|_4<\varepsilon$ for all $\|f\|_4\leq 1$.

Let $\{f_j\}_{j=1}^\infty\subset L^4(\mathbb{R}^3)$ satisfy $f_j\rightharpoonup 0$ in L^4. Since $\sup_j\|Af_j\|_{W^{2,4}(\mathbb{R}^3)}<\infty$, Rellich's compactness theorem produces a subsequence f_{j_k} so that $Af_{j_k}\to 0$ in $L^4(|x|<R_0)$. Thus

$$\limsup_{k\to\infty}\|Af_{j_k}\|_4 \leq (1+C_\lambda)\varepsilon.$$

Sending $\varepsilon\to 0$ and passing to the diagonal subsequence finishes the proof. $\square$

The following lemma establishes invertibility everywhere except on the imaginary axis.

Lemma 9. *Let $V\in L^p(\mathbb{R}^3)\cap L^{\frac{3}{2}}(\mathbb{R}^3)$, $\frac{3}{2}<p<2$ and assume that V is real-valued. Then for any nonzero $\lambda\in\mathbb{H}$, the inverse $(I+R_0(\lambda^2)V)^{-1}: L^4(\mathbb{R}^3)\to L^4(\mathbb{R}^3)$ exists.*

Proof. By the previous lemma it suffices to show that

$$f\in L^4(\mathbb{R}^3),\quad f + R_0(\lambda^2)Vf = 0 \Longrightarrow f = 0.$$

Let f be as on the left-hand side and set $g = Vf$. Then $g \in L^r$, where $r = \frac{4p}{4+p} < \frac{4}{3}$. By Lemma 5, $f = -R_0(\lambda^2)g$ therefore belongs to $L^q \cap L^4$, where $\frac{1}{q} - \frac{1}{4} = \frac{3-2p}{3p} > 0$.

This bootstrapping procedure can be repeated until it is shown that $f \in L^{r'} \cap L^4$. In fact, one can continue to the point where $f \in L^\infty$, since $R_0(\lambda^2) : L^{\frac{3}{2}-\varepsilon} \cap L^{\frac{3}{2}+\varepsilon} \mapsto L^\infty$ is a bounded operator. What is important here is that f and g exist in spaces dual to each other.

Since V is real-valued, the duality pairing

$$\langle f, g\rangle = \langle f, Vf\rangle = -\langle R_0(\lambda^2)g, g\rangle$$

shows that $\langle R_0(\lambda^2 \pm i0)g, g\rangle$ is real-valued. If $\lambda^2 \notin \mathbb{R}$, then the condition

$$\Im\langle R_0(\lambda^2)g, g\rangle = \int_{\mathbb{R}^3} \frac{\Im(\lambda^2)}{\big(|\xi|^2 - \Re(\lambda^2)\big)^2 + \Im(\lambda^2)^2} |\hat{g}(\xi)|^2 \, d\xi = 0$$

requires that $\hat{g} = 0$ almost everywhere.

On the boundary $\lambda \in \mathbb{R}$, by the Stein-Tomas theorem

$$\Im\langle R_0((\lambda + i0)^2)g, g\rangle = \lim_{\varepsilon \to 0} \Im\langle R_0((\lambda + i\varepsilon)^2)g, g\rangle = c\lambda \int_{S^2} |\hat{g}(\lambda\omega)|^2 \, \sigma(d\omega)$$

with some constant $c \neq 0$. Hence, $\hat{g} = 0$ on $|\lambda|S^2$ in the L^2 sense. Since $g \in L^r(\mathbb{R}^3)$, one concludes from Proposition 7 above that $(1+|x|)^{\delta - \frac{1}{2}} R_0(\lambda^2 \pm i0)g \in L^2(\mathbb{R}^3)$ for some $\delta > 0$. Hence also $(1+|x|)^{\delta-\frac{1}{2}} f \in L^2(\mathbb{R}^3)$ for some $\delta > 0$. Since $(-\triangle + V - \lambda^2)f = 0$ in the distributional sense, and one checks easily from (11) (remembering that $f \in L^\infty \cap L^4$) that also $f \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3) \subset W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)$, Theorem 2 implies that $f = 0$, as claimed. $\square$

The following two lemmas show that the inverses in the previous lemma have uniformly bounded norms.

Lemma 10. *Let $V \in L^p(\mathbb{R}^3)$, $\frac{3}{2} \leq p \leq 2$. The map $\lambda \mapsto R_0(\lambda^2)V$ is continuous from the domain $\mathbb{H} \setminus \{0\} \subset \mathbb{C}$ to the space of bounded operators on $L^4(\mathbb{R}^3)$.*

Proof. First suppose V is bounded and has compact support in the ball $\{|x| < R\}$. The convolution kernel associated to $R_0(\lambda^2) - R_0(\zeta^2)$ has the bounds

$$|K(x)| \lesssim \begin{cases} |\lambda - \zeta|, & \text{if } |x| < |\lambda - \zeta|^{-1} \\ |x|^{-1}, & \text{if } |x| \geq |\lambda - \zeta|^{-1} \end{cases}$$

Then for any pair $\lambda, \zeta \in \mathbb{H}$, $|\lambda - \zeta| \leq \frac{1}{2R}$, we have

$$|(R_0(\lambda^2) - R_0(\zeta^2))Vf(x)| \lesssim \begin{cases} |\lambda - \zeta| \|Vf\|_1, & \text{if } |x| < |\lambda - \zeta|^{-1} \\ |x|^{-1}\|Vf\|_1, & \text{if } |x| \geq |\lambda - \zeta|^{-1} \end{cases}$$

Thus $\|(R_0(\lambda^2) - R_0(\zeta)^2)Vf\|_4 \lesssim |\lambda - \zeta|^{1/4} R^{9/4} \|V\|_\infty \|f\|_4$.

Approximate V by compactly supported $\tilde{V} \in L^\infty$ so that $\|V - \tilde{V}\|_p < \varepsilon$. By the above calculation, Lemma 5, and the simple identity

$$(R_0(\lambda^2) - R_0(\zeta)^2)V = R_0(\lambda^2)(V - \tilde{V}) + (R_0(\lambda^2) - R_0(\zeta^2))\tilde{V} - R_0(\zeta^2)(V - \tilde{V}),$$

we see that $\limsup_{\zeta \to \lambda} \|(R_0(\lambda^2) - R_0(\zeta)^2)V\|_{4 \to 4} \lesssim |\lambda|^{(3-2p)/p}\varepsilon$. $\square$

Lemma 11. *Let V be as in the previous lemma and suppose $\lambda_0 > 0$. Then*

$$\sup_{|\Re(\lambda)|\geq\lambda_0} \left\|(I + R_0(\lambda^2)V)^{-1}\right\|_{4\to 4} < \infty. \tag{12}$$

Proof. In view of Lemma 5, there is some finite $\lambda_1 \in \mathbb{R}$ so that $\|R_0(\lambda^2)V\|_{4\to4} < \frac{1}{2}$ provided $|\lambda| > \lambda_1$. It therefore suffices to prove (12) on the compact set $\{\lambda \in \mathbb{C} : \ \lambda_0 \leq |\lambda| \leq \lambda_1, |\Re(\lambda)| \geq \lambda_0\}$. The previous two lemmas, however, show that $(I + R_0(\lambda^2)V)^{-1}$ is a continuous function of λ on this set, hence it is uniformly bounded from above. □

It is now a simple matter to prove Theorem 1.

Proof of Theorem 1. By the resolvent identity, for any $\varepsilon \neq 0$,

$$R_V(\lambda^2 + i\varepsilon) = R_0(\lambda^2 + i\varepsilon) - R_0(\lambda^2 + i\varepsilon)VR_V(\lambda^2 + i\varepsilon).$$

By Lemma 11 one therefore has

$$R_V(\lambda^2 + i\varepsilon) = (I + R_0(\lambda^2 + i\varepsilon)V)^{-1}R_0(\lambda^2 + i\varepsilon)$$

and the right-hand side is uniformly bounded for $\lambda \geq \lambda_0 \geq 0$ as well as $0 < \varepsilon \leq 1$ in the L^4 operator norm. In fact, the last factor contributes a decaying factor of $\lambda^{-\frac{1}{2}}$ as L^4 operator norm in view of Lemma 4. □

Proof of Proposition 3. There is only one point in the argument where the condition $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ is used, namely the step in Lemma 9 where we wish to make use of Theorem 2. It otherwise suffices to assume that $V \in L^p(\mathbb{R}^3)$, $\frac{3}{2} < p < 2$.

For the second claim, one observes the following consequence of Lemma 5: If $V \in L^p$, $\frac{3}{2} < p \leq 2$, and $r > 4$, then $R_0(\lambda^2 \pm 10)V : L^4 \cap L^r \mapsto L^4 \cap L^s$, where $\frac{1}{s} = \max(\frac{1}{r} + \frac{1}{p} - \frac{2}{3}, 0)$. The same is true for any $V \in L^q, p \leq q \leq 2$. This allows the bootstrapping procedure on f to continue normally, and furthermore $g = Vf$ is still an element of $L^{\frac{4}{3}-\varepsilon}$, as desired. Therefore, the only matter of concern is whether the conclusion of Theorem 2 will hold for such a potential V. □

4 Further estimates on the free resolvent

Returning to Proposition 7, we note that a sharper estimate can be made at the endpoint $p = 1$.

Proposition 12. *Let f be a function in $L^1(\mathbb{R}^3)$ such that $\hat{f} = 0$ on the unit sphere S^2. Then*

$$\sup_{\varepsilon>0} \left\|R_0(1 \pm i\varepsilon)f\right\|_2 \leq \frac{1}{\sqrt{8\pi}}\|f\|_1 \tag{13}$$

Proof. Define the trace function

$$G(\lambda) = \lambda^{-2}\left\|\hat{f}|_{\lambda S^2}\right\|_2^2 = 4\pi \iint_{\mathbb{R}^6} f(x)\frac{\sin(\lambda|x-y|)}{\lambda|x-y|}\bar{f}(y)\,dx\,dy \tag{14}$$

By inspection,

$$G(\lambda) = 2\pi \iiint_{\mathbb{R}\times\mathbb{R}^6} \frac{f(x)\bar{f}(y)}{|x-y|}\chi_{|x-y|}(\tau)e^{i\lambda\tau}\,d\tau dx dy \tag{15}$$

where $\chi_{|x-y|}$ denotes the characteristic function of the interval $\{|\tau| \le |x-y|\}$. The integrand on the right-hand side is in $L^1(\mathbb{R}^7)$, so Fubini's Theorem implies that G is the inverse Fourier transform of an L^1 function.

Using the Plancherel identity (in 3 dimensions), and noting that G is an even function,

$$\|R_0(1 \pm i\varepsilon)f\|_2^2 = \frac{1}{2(2\pi)^3}\int_{-\infty}^{\infty} G(\lambda)\frac{\lambda^2}{|\lambda^2 - (1+i\varepsilon)|^2} \tag{16}$$

For any $\varepsilon > 0$, the multiplier $M_\varepsilon(\lambda) = \frac{\lambda^2}{|\lambda^2-(1+i\varepsilon)|^2}$ is integrable, hence it has Fourier transform $\hat{M}_\varepsilon \in L^\infty(d\tau)$. By Parseval's formula, this time in one dimension,

$$\|R_0(1 \pm i\varepsilon)f\|_2^2 = \frac{1}{2(2\pi)^4}\int_{\mathbb{R}} \hat{G}(\tau)\hat{M}_\varepsilon(-\tau)\,d\tau \tag{17}$$

An explicit formula for $\hat{M}_\varepsilon(\tau)$ can be obtained via residue integrals:

$$\hat{M}_\varepsilon(\tau) = \frac{\pi}{2\varepsilon}\big(\sqrt{1+i\varepsilon}\,e^{i|\tau|\sqrt{1+i\varepsilon}} + \sqrt{1-i\varepsilon}\,e^{-i|\tau|\sqrt{1-i\varepsilon}}\big) \tag{18}$$

This, along with (15), can be immediately substituted back into equation (17).

$$\begin{aligned}\|R_0(1 \pm i\varepsilon)f\|_2^2 &= \frac{1}{8\pi\varepsilon}\iint_{\mathbb{R}^6}\int_0^{|x-y|}\frac{f(x)\bar{f}(y)}{|x-y|}\big(\sqrt{1+i\varepsilon}\,e^{i\tau\sqrt{1+i\varepsilon}} + \sqrt{1-i\varepsilon}\,e^{-i\tau\sqrt{1-i\varepsilon}}\big)\,d\tau\,dx\,dy\\ &= \frac{1}{8\pi i\varepsilon}\iint_{\mathbb{R}^6}\frac{f(x)\bar{f}(y)}{|x-y|}\big(e^{i|x-y|\sqrt{1+i\varepsilon}} - e^{-i|x-y|\sqrt{1-i\varepsilon}}\big)dx\,dy\end{aligned}$$

Boundedness of $\hat{M}_\varepsilon$ enables us to continue applying Fubini's theorem to the multiple integral. We have also simplified the expression by noting that $\hat{M}_\varepsilon$ is an even function. Recall definition (14) and subtract $\frac{1}{16\pi^2\varepsilon}G(1)$ from both sides of the equation.

$$\begin{aligned}&\|R_0(1 \pm i\varepsilon)f\|_2^2 - \frac{1}{16\pi^2\varepsilon}G(1)\\ &= \frac{1}{8\pi i\varepsilon}\iint_{\mathbb{R}^6}\frac{f(x)\bar{f}(y)}{|x-y|}\Big(\big(e^{i|x-y|\sqrt{1+i\varepsilon}} - e^{i|x-y|}\big) - \big(e^{-i|x-y|\sqrt{1-i\varepsilon}} - e^{-i|x-y|}\big)\Big)dx\,dy\\ &= \frac{1}{8\pi i\varepsilon}\iint_{\mathbb{R}^6}\frac{f(x)\bar{f}(y)}{|x-y|}K(|x-y|)\,dx\,dy\end{aligned} \tag{19}$$

where $|K(|x-y|)| \le \varepsilon|x-y|$. This leads to the conclusion

$$\big|\|R_0(1 \pm i\varepsilon)f\|_2^2 - \frac{1}{16\pi^2\varepsilon}G(1)\big| \le \frac{1}{8\pi}\|f\|_1^2$$

If f satisfies the hypothesis $\hat{f}|_{S^2} = 0$, then $G(1) = 0$. □

Corollary 13. *Let f be a function in $L^1(\mathbb{R}^3)$ such that $\hat{f} = 0$ on the unit sphere S^2. Then*

$$\|R_0(1 \pm i0)f\|_2 \le \frac{1}{\sqrt{8\pi}}\|f\|_1 \tag{20}$$

Proof. This follows immediately from (16) and monotone convergence. $\square$

The condition $\hat{f} = 0$ is crucial in Proposition 7. Indeed, recall that for $f \in L^p(\mathbb{R}^3)$ real-valued with $1 \le p \le \frac{4}{3}$ one has

$$\Im R_0(1+i0)f = c\,(\widehat{\sigma_{S^2}} * f)$$

for some constant c. This follows by writing $R_0(1+i\varepsilon)$ as a sum of its real and imaginary parts, as well as from the fact that the operation of restriction $f \mapsto \hat{f}(r\cdot)$ is continuous in $r>0$ as a map $L^p(\mathbb{R}^3) \to L^2(S^2)$. However, it is clear that for any $\delta > 0$

$$\|(1+|x|)^{\delta-\frac{1}{2}}[\widehat{\sigma_{S^2}} * f]\|_2 = \infty \tag{21}$$

even for smooth bump-functions f since the function inside the norm decays like $(1+|x|)^{\delta-\frac{3}{2}}$ which just fails to be $L^2(\mathbb{R}^3)$. The following simple lemma shows, on the other hand, that $\delta < 0$ does lead to a finite norm in (21).

Lemma 14. *For any $R \ge 1$ one has*

$$\left\| \chi_{[|x|<R]}[\widehat{\sigma_{S^2}} * f] \right\|_2 \lesssim \sqrt{R}\, \|f\|_{\frac{4}{3}}$$

for all $f \in L^{\frac{4}{3}}(\mathbb{R}^3)$.

Proof. Let ϕ be a smooth cut-off function with $\hat{\phi}$ compactly supported. Then by Plancherel, and Cauchy-Schwartz,

$$\begin{aligned}
\left\| \chi\Big(\frac{\cdot}{R}\Big)[\widehat{\sigma_{S^2}} * f] \right\|_2^2 &= R^6 \int_{\mathbb{R}^3} \left| \int_{S^2} \hat{\chi}(R(\xi-\eta))\hat{f}(\eta)\,\sigma_{S^2}(d\eta) \right|^2 d\xi \\
&\lesssim R^6 \int_{\mathbb{R}^3} \int_{S^2} |\hat{\chi}(R(\xi-\eta'))|\,d\eta' \int_{S^2} |\hat{\chi}(R(\xi-\eta))||\hat{f}(\eta)|^2\,\sigma_{S^2}(d\eta)\,d\xi \\
&\lesssim R\,\|\hat{f}\|^2_{L^2(S^2)} \lesssim R\,\|f\|^2_{\frac{4}{3}},
\end{aligned}$$

as claimed. $\square$

The previous lemma suggests that one should also have the bound

$$\sup_{\varepsilon>0} \left\| \chi_{[|x|<R]} R_0(1 \pm i\varepsilon) f \right\|_2 \lesssim \sqrt{R}\, \|f\|_{\frac{4}{3}}. \tag{22}$$

This is indeed known, see the paper by Ruiz and Vega [RuiVeg].

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