On the Formation of Gaps in the Spectrum of Schrödinger Operators with Quasi-Periodic Potentials

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Dedicated to Barry Simon on the occasion of his 60th birthday

Abstract. In this article we review some recent developments in the theory of Schrödinger operators with quasi-periodic potentials on the discrete line. We focus in particular on the work by the authors on the formation of a dense set of gaps in the spectrum of such operators with general analytic potentials, provided the Lyapunov exponent is positive.

Contents

1. Introduction

In this brief review we consider the class of operators defined as

$$(H_\theta\psi)(n) = -\psi_{n+1} - \psi_{n-1} + \lambda V(\theta + n\omega)\psi_n \tag{1}$$

with $n \in \mathbb{Z}$ and $\omega \in \mathbb{T}$ Diophantine or, in the continuum,

$$(H_\theta\psi)(x) = -\psi''(x) + \lambda V(\theta + x\omega)\psi(x) \tag{2}$$

with $x \in \mathbb{R}$ and $\omega \in \mathbb{T}^\nu$, $\nu \geq 2$, Diophantine. In both cases we shall assume that V is a real-analytic function on a suitable torus. An important special case is the almost Mathieu operator (also known as Harper's operator) for which $V(\theta) = \cos(2\pi\theta)$. Although these operators H_θ depend on a parameter, it is a basic fact that their

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spectrum $\sigma(H_\theta)$ and its spectral parts, i.e., the absolutely continuous (a.c.), singular continuous (s.c.), and pure point (p.p.) parts, are *deterministic*. This means that there are fixed compact sets which equal these respective parts of the spectrum for a.e. θ. In the case of (1) this follows from the ergodic theorem and the identity $H_{T\theta} = U^{-1} \circ H_\theta \circ U$, where $T\theta = \theta + \omega \mod \mathbb{Z}$ and U is the left shift.

We now recall some of the advances in this field:

- *Dinaburg–Sinai* (1975): For (2) and a large set of large energies, there are Bloch–Floquet waves $\psi_{\theta,E}(x) = a_{\theta,E}(x)e^{ik(\theta,E)x}$ where $a_{\theta,E}$ is quasi-periodic, $H_\theta\psi_{\theta,E} = E\psi_{\theta,E}$. Their argument is perturbative off the periodic case and they use KAM and Floquet theory.
- *Avron–Simon, Craig–Simon* (1981–1984): Discovered some of the most basic general properties of Schrödinger operators with almost periodic potentials. For example, they established purely s.c. spectrum for Liouville rotation numbers and the existence of Cantor spectrum for limit-periodic potentials. Furthermore, they gave a rigorous derivation of the Thouless formula which provides a relation between the Lyapunov exponent $L(E)$ and the integrated density of states $k(E)$, viz.,

$$L(E) = \int \log|E - E'|\, dk(E').$$

  Here the Lyapunov exponent is defined to be

$$L(E) = \lim_{N\to\infty} N^{-1} \int_0^1 \log\big\|M_N(\theta,\omega,E)\big\|\, d\theta$$

  where

$$M_N(\theta,\omega,E) = \prod_{j=N}^{1} \begin{bmatrix} \lambda V(T^j\theta) - E & -1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

  and the IDS is defined as the limiting distribution of the eigenvalues, viz.,

$$k(E) = \lim_{N\to\infty} \frac{1}{2N+1} \#\{j \,:\, E_j^{(N)}(x,\omega) \le E\}$$

  where $E_j^{(N)}(x,\omega)$ are the eigenvalues of $H_{[-N,N]}(x,\omega)$. They observed that the IDS is log-Hölder continuous as a consequence of the Thouless formula. They also gave an exact argument for Aubry duality for the almost Mathieu case.
- *Johnson–Moser* (1982) and *Herman* (1983): Introduced a rotation number $\alpha(\lambda) := \lim_{x\to\infty} x^{-1} \arg(\phi + i\phi')(x,\lambda)$ for solutions of $H_\theta\phi(\cdot,\lambda) = \lambda\phi(\cdot,\lambda)$ with H_θ as in (2) (or more generally, with almost periodic potential). They also proved continuity of α and related the intervals of constancy of α to gaps in the spectrum; more precisely, the value of $2\alpha(\lambda)$ belongs to the frequency module of the almost periodic potential. Moreover, α is the same as the IDS $k(E)$ up to a factor of π.
- *Avron–van Mouche–Simon* (1990): For the almost Mathieu case and in the limit over periodic approximants, they established that $|\sigma(H_\theta)| = |4 - 2|\lambda||$ if $|\lambda| \neq 2$. *Last* then obtained the full theorem without the limit over periodic approximants and also for $|\lambda| = 2$. Thus, for that case he showed that the spectrum has measure zero. In particular, the spectrum is nowhere dense.

- *Eliasson* (1992): Again for the case of (2), Floquet–Bloch solutions exist for a.e. large E and $\sigma(H_\theta)$ is purely a.c.. More precisely, he established reducibility: the Schrödinger cocycle is conjugate to a constant one for a.e. energy. Moreover, for generic potential the spectrum is a Cantor set.
- *Jitomirskaya, Last, Simon* (1994, 1997): Showed that the p.p. and s.c. parts of the spectrum are not constant; rather, only a.e. constant, see above. Moreover, the a.c. part is constant in the phase θ (this also due to Kotani; see the following bullet). There is an open conjecture due to Simon that the union of the s.c. and p.p. spectrum is constant.
- *Kotani* (1984–1997): Showed that the closure of the set of positivity of the Lyapunov exponent equals the essential support of the a.c. part of the spectral measure of an ergodic operator (2) (*Simon* extended these results to the discrete case (1))—thus proving the converse of the *Ishii–Pastur* theorem. Furthermore, Kotani introduced a natural notion of "determinacy" of a ergodic potential V_ω and showed that the Lyapunov exponent is always positive (and thus there is no a.c. spectrum) unless V_ω is deterministic in his sense. This body of techniques, which is based on the Weyl–Titchmarsh m-function, has become known as "Kotani theory."
- *Gordon–Jitomirskaya–Last–Simon* (1997): Refined the $|\lambda| = 2$ analysis for the almost Mathieu operator and established purely s.c. spectrum.
- *Krikorian* and *Avila–Krikorian* (2000–2004): Recent work on reducibility; they established a dichotomy between *nonuniform hyperbolicity* (i.e., continuous Oseledts splitting) and *reducibility* for quasi-periodic Schrödinger cocycles with analytic potentials; their results hold for a.e. ω and a.e. energy. They obtain further that for the almost Mathieu operator with $|\lambda| = 2$ and Diophantine ω, the spectrum is measure zero and purely s.c. for a.e. phase. This completes the analysis of Last (see above).

An important question about the operator class (1) and (2) concerns **Anderson localization (AL)**. This means that the spectrum is pure point with exponentially decreasing eigenfunctions.

- *Fröhlich–Spencer–Wittwer, Sinai* (late 1980s): Established (AL) for cosine-like potentials and large disorder. Their arguments are perturbative, treating the operators as perturbations of the multiplication operator given by the potential.
- *Jitomirskaya* (1999): For almost Mathieu, $|\lambda| > 2$, as well as Diophantine ω, proved (AL). By a lower bound on the Lyapunov exponent due to *Avron–Simon*, as well as *Herman*, $|\lambda| > 2$ implies that $\inf_E L(E) > 0$. Moreover, by Aubry duality and the Ishii–Pastur theorem, this fails when $|\lambda| \leq 2$. Hence, this result shows that (AL) holds precisely when $L(E) > 0$ for all energies.
- *Bourgain, Goldstein* (2000): Established (AL) for analytic potentials, positive Lyapunov exponents, and almost all ω. They introduced large deviation theorems and the use of semi-algebraic sets (for the elimination of double resonances) into this field.

Finally, we turn to another important question, namely: **When is the spectrum a Cantor set?** This simply means that it is nowhere dense (a more refined version of this statement is whether all gaps allowed by the gap labeling theorem are open). So far, this question had apparently only been considered for the *almost*

Mathieu case. We now list a series of results that answer this question affirmatively in the almost Mathieu case.

- *Bellissard–Simon* (1982): For a dense G_δ set of λ, ω.
- *Choi–Elliott–Yui* (1990): Liouville case, $|\lambda| = 2$ using methods of C^* algebras; their proof also applies to all $\lambda \neq 0$.
- *Puig* (2003): $|\lambda| \neq 2$, ω Diophantine. His argument proceeds by contradiction. He shows that eigenvalues of H with phase 0 are necessarily the endpoints of open gaps.
- *Avila–Jitomirskaya* (2005): For $|\lambda| \neq 0$, bridge the gap between the Liouville and Diophantine cases. Thus, they show that whenever the potential is nonperiodic (i.e., for irrational ω) there is a dense set of gaps. Thus, they provide a solution of the so-called Ten Martini problem. As in Puig's argument, the proof again proceeds by contradiction.
- The case $|\lambda| = 2$ is covered by the previous reference, but had also been settled earlier by *Last* and *Avila–Krikorian* (see above).

2. The IDS and Cantor Spectrum for General Potentials

We now discuss some recent work by the authors. We will in general emphasize ideas over technical correctness. For a review which is much more technical by design, we refer to reader to [**29**]. However, this reference does not contain any material about gaps. Rather, it reviews the long paper [**27**] which develops the machinery needed for the formation of gaps in [**28**]. This paper is intended as an exposition of the main ideas needed to pass from [**27**] to [**28**].

We remark that unless indicated otherwise, Diophantine henceforth means that, for all $n \geq 1$,

$$\|n\omega\| \geq \frac{c}{n(\log n)^a}$$

with $c > 0$ and $a > 1$. The following is proved in [**26, 27, 28**].

THEOREM 1. *Consider* (1) *with* $\lambda = 1$. *Let* $V : \mathbb{T} \to \mathbb{R}$ *be analytic and suppose that*

$$\inf_{E,\omega} L(E, \omega) > 0.$$

Then the following properties hold:

(1) *If* ω *is Diophantine, then the IDS is Hölder continuous. If* V *is in a small* L^∞ *neighborhood of a trigonometric polynomial of degree* k*, then the IDS is Hölder* $(2k + \varepsilon)^{-1}$*-continuous for all* $\varepsilon > 0$.
(2) *For a.e.* ω*, the following holds: Off a set of Hausdorff dimension zero the IDS is Lipschitz continuous.*
(3) *For a.e.* ω*, the IDS is absolutely continuous.*
(4) *For a.e.* ω*, the spectrum is a Cantor set.*

Further results include:

- estimates on the distribution of the zeros of
$$\det(H_{[-N,N]}(z, \omega) - E)$$
in the z-plane
- a quantitative separation property of the eigenvalues of $H_{[-N,N]}(z, \omega)$
- a constructive, finite-volume mechanism for the development of Anderson localization

- a constructive finite-volume mechanism for the formation of gaps based on resonances

We now give an example of some finite-volume statements which imply, and are finer than, the corresponding statements concerning the IDS in the previous theorem; they are obtained in [**27**] and address the problem of bounding the expected number of eigenvalues at a finite scale falling into a small interval.

THEOREM 2. *Let $E_j^{(N)}(x,\omega)$ be the eigenvalues of $H_{[-N,N]}(x,\omega)$. Then*

- *For any $\eta > N^{-1+\delta}$, $\delta > 0$, and ω Diophantine, there is the following bound on the expected number of eigenvalues falling into small intervals: for arbitrary E, $\kappa > 0$ and large N,*

$$\int_0^1 \#\{j \,:\, |E_j^{(N)}(x,\omega) - E| < \eta\}\, dx \leq N\eta^{\frac{1}{2k}-\kappa}$$

  where k is the degree of the underlying trigonometric polynomial.
- *Let $\varepsilon > 0$ be arbitrary but fixed. If $\omega \notin \Omega(\varepsilon)$, $E \notin \mathcal{E}_\omega(\varepsilon)$, then*

$$\int_0^1 \#\{j \,:\, |E_j^{(N)}(x,\omega) - E| < \eta\}\, dx < \exp((\log \varepsilon^{-1})^A)\, N\eta$$

  where $|\Omega(\varepsilon)| + \sup_{\omega \notin \Omega(\varepsilon)} |\mathcal{E}_\omega(\varepsilon)| < \varepsilon$.

According to Yakov Sinai, *"Anderson localization is a game of resonances".*

A *resonance* here means the following: *Let $\Lambda, \Lambda' \subset \mathbb{Z}$ be intervals, typically of comparable length and separated by more than their length. Then the Hamiltonians $H_\Lambda(x,\omega)$ and $H_{\Lambda'}(x,\omega)$ are said to be in resonance provided their spectra are very close.*

In order for this concept to be useful, it needs to be quantified. Note that for any Λ, one can of course find $\Lambda' = \Lambda + k$ (a translate) for which the distance between the spectra of $H_\Lambda(x,\omega)$ and $H_{\Lambda'}(x,\omega)$ is arbitrarily small. This follows simply from recurrence of the rotation map. Therefore, the point will be to apply this definition inside a fixed box Λ_0 of a given size and then look for (much) smaller $\Lambda, \Lambda' \subset \Lambda_0$ which produce resonances (where the notion of distance between the spectra needs to be adjusted to the length of Λ_0—typically $\ll |\Lambda_0|^{-C}$ with C large). Of particular importance is to keep track of how many smaller intervals Λ' there are for a given $\Lambda \subset \Lambda_0$ which produce resonances with Λ and also satisfy $\mathrm{dist}(\Lambda', \Lambda) \gg |\Lambda|$. If there is exactly one such Λ', then one speaks of a *double resonance*, otherwise of *triple or higher order resonance*. It is particularly important to eliminate the occurrence of such higher order resonances for Sinai [**47**], Fröhlich–Spencer–Wittwer [**23**], Bourgain [**7**], as well as the authors' argument for gap formation [**28**]. In the former three references this is accomplished perturbatively, by assuming that the potential has no more than two monotonicity intervals; in [**28**] we proceed nonperturbatively and no assumption other than analyticity is made on the potential—in order to prevent triple resonances it is necessary to eliminate some small set of ω and E (this process of elimination is essentially the one of Chan [**12**]).

We remark that it is also customary to speak of eigenfunctions of $H_\Lambda(x,\omega)$ and $H_{\Lambda'}(x,\omega)$ to be in resonance. This simply means that these eigenfunctions

correspond to close eigenvalues. At the larger scale, such eigenfunctions "merge" to form a two-humped eigenfunction of $H_{\Lambda_0}(x,\omega)$, cf. Figure 9.

To illustrate this concept, let us recall the KAM-type scheme used in Fröhlich–Spencer–Wittwer's as well as Sinai's perturbative approach to localization. In it, one starts off with singletons Λ, Λ': consider the 2×2 matrix

$$A(x) = \begin{bmatrix} V(x) & \lambda^{-1} \\ \lambda^{-1} & V(x+\omega) \end{bmatrix}$$

where $V(x_0) = V(x_0+\omega)$ and $V'(x_0) < 0$, $V'(x_0+\omega) > 0$. Its eigenvalues $E^{\pm}(x)$ for x close to x_0 form two branches with

$$E^{\pm}(x_0) = V(x_0) \pm \lambda^{-1}.$$

Figure 1 shows how the two separate branches of the potential (which are the diagonal entries of $A(x)$) separate when they yield the eigenvalues of $A(x)$. The dashed line is supposed to have length exactly equal to ω. The projections of the intersection points of this dashed line with the graphs are x_0 and $x_0+\omega$, respectively.

If V has two monotonicity intervals and ω is Diophantine, then it is easy to see that this 2×2 block determines the invertibility of the $N \times N$ Hamiltonian close to x_0 provided λ is large (however, this largeness depends on N). More precisely, it follows from the Feshbach formula that

$$\begin{aligned} &\det\left(H_N(x,\omega) - E\right) \\ &= \begin{vmatrix} \lambda V(x) - E & -1 & 0 & \cdots & \cdots & 0 \\ -1 & \lambda V(x+\omega) - E & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & & & -1 \\ 0 & \cdots\cdots\cdots\cdots & 0 & -1 & & \lambda V(x+(N-1)\omega) - E \end{vmatrix} \\ &= F(x,\lambda,E)\det A(x) \end{aligned}$$

where $F(x,\lambda,E) \neq 0$ for all x close to x_0. It turns out that this fact can be considered as the 0-order step in a KAM scheme. To pass to bigger scales one again uses the Feshbach formula using the previous scale as the information needed to invert the larger blocks.

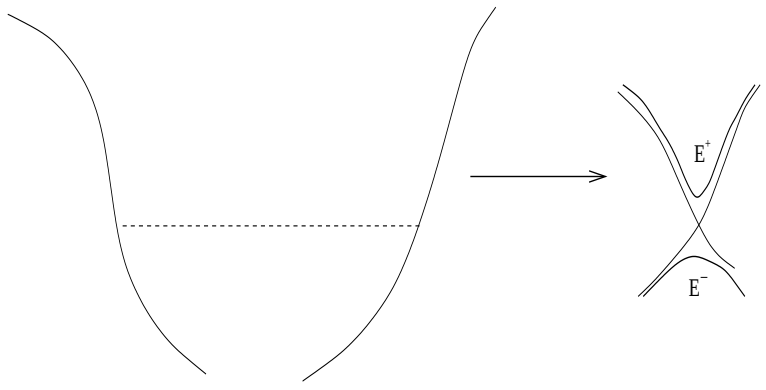

FIGURE 1. A double resonance

Sinai (1987) and Fröhlich–Spencer–Wittwer (1990) carried out versions of this scheme. Later, Bourgain (2001) obtained in this way Hölder $\frac{1}{2}-\varepsilon$ continuity of the IDS for cosine-like potentials and large disorder. The point of the exponent $\frac{1}{2}$ here is the following: the obstruction to invertibility of a finite volume Hamiltonian always arises in the form of a second order polynomial. In the proof of Theorems 1 and 2, which are of course nonperturbative, it will also be of crucial importance to obtain such fixed-degree control. Note that the degree of these polynomial will vary with the degree k of the potential (for $k=1$, it will again be quadratic and more generally, be of degree $2k$; hence the Hölder exponent $(2k)^{-1}$ of the IDS). To extract these polynomials of fixed degree we use a Weierstrass preparation theorem. Finally, we note that in the free case the IDS is no better than Hölder $\frac{1}{2}$-continuous at the edges of the spectrum.

3. A Finite-Volume Mechanism for Anderson Localization

Because of the self-adjointness of H and the analyticity of V, the solutions of

$$0 = \det\left(H_{[1,N]}(x,\omega) - E\right).$$

are real-analytic functions (which we refer to as *Rellich functions*)

$$E_1(x,\omega) < E_2(x,\omega) < \cdots < E_N(x,\omega)$$

One cannot have equality here since the eigenvalues of the Dirichlet problem are simple. Figure 2 displays the graphs of such Rellich functions for the periodic Mathieu operator with periodic boundary conditions. The interval is $[1, N=12]$ where the latter number arises as denominator of a convergent of $\omega=\sqrt{2}$. In Figure 3 the same is plotted for the almost Mathieu operator with $\omega=\sqrt{2}$ and Dirichlet conditions at the boundary. Although the graphs appear to intersect at many points, they actually separate at these points, as can be seen in a much finer resolution. Finally, Figure 4 shows the Rellich functions for a potential given by a third degree polynomial again with the same ω but the next larger denominator in the sequence of convergents.

At this point it seems natural to ask some **basic quantitative questions**:

- What is the width of the strip around $\mathbb{R}$ to which $E_j(z,\omega)$ can be analytically continued?
- What is the size of the separation between the $E_j(x,\omega)$?
- What is a reasonable lower bound on the absolute values of the slopes of the $E_j(x,\omega)$ away from critical points?

The answer turns out to be e^{-N^δ} provided $\omega \in \mathbb{T}\setminus\Omega_N$, $E_j(x,\omega) \notin \mathcal{E}_N(\omega)$. Here Ω_N, $\mathcal{E}_N(\omega)$ have small measure and complexity (the latter refers to the number of connected components a set needs to have to cover $\mathcal{E}_N(\omega)$ without increasing the measure significantly—by a multiplicative constant, say). More precisely, we have the bounds

$$|\Omega_N| < \exp(-(\log N)^{2A}), \quad \mathrm{compl}(\Omega_N) < \exp((\log N)^A)$$

and similarly for $\mathcal{E}_N(\omega)$. Note that these bounds reflect that the "bad sets" have Hausdorff dimension zero.

Central to the separation of the $E_j(x,\omega)$ is a *finite-volume understanding of (AL)*, which we now describe: Let $H_{[-N,N]}(x,\omega)\psi = E\psi$ with $\|\psi\|_2 = 1$. We seek a window $\Lambda_0 \subset [-N,N]$ so that $\|\psi\|_{\ell^2(\Lambda_0)} = 0.999$, say, and $|\Lambda_0| \ll N^\varepsilon$. Consider

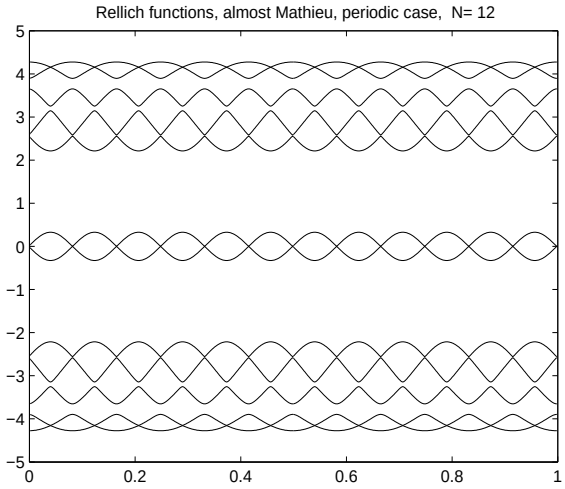

FIGURE 2. Rellich functions in the periodic case

all intervals $\Lambda \subset [-N, N]$ with $|\Lambda| = n \sim (\log N)^C$: there exists such an interval Λ so that with $f_\Lambda(x, \omega, E) = \det(H_\Lambda(x, \omega) - E)$,

$$\log |f_\Lambda(x, \omega, E)| < |\Lambda| L(\omega, E) - |\Lambda|^{\frac{1}{2}}.$$

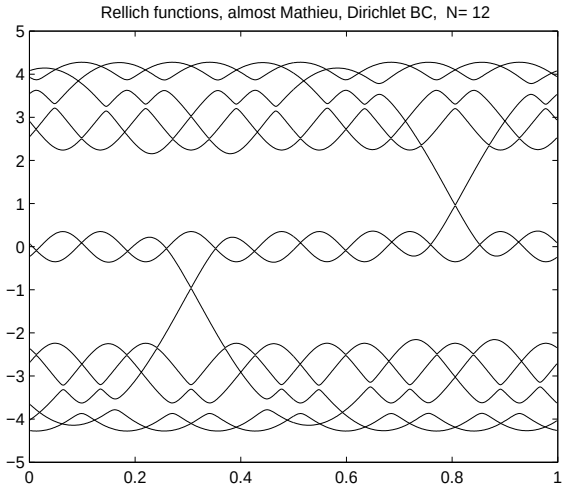

FIGURE 3. Rellich functions in the aperiodic case I

Otherwise each Green's function $G_\Lambda(x,\omega,E)$ would exhibit exponential off-diagonal decay. Indeed, by Cramer's rule,

$$\Big|\big(H_{[-N,N]}(x,\omega)-E\big)^{-1}(k,m)\Big| = \frac{\big|f_{[-N,k]}\big(e(x),\omega,E\big)\big|\,\big|f_{[m+1,N]}\big(e(x),\omega,E\big)\big|}{\big|f_{[-N,N]}\big(e(x),\omega,E\big)\big|}\ .$$

In [**27**] the following uniform upper bounds are proved:

$$\log\big|f_{[-N,k]}\big(e(x),\omega,E\big)\big| + \log\big|f_{[m+1,N]}\big(e(x),\omega,E\big)\big| \le 2NL(\omega,E) + (\log N)^{A_1}$$

for any $x\in\mathbb{T}$. Therefore,

$$\big\|\big(H_{[-N,N]}(x,\omega)-E\big)^{-1}\big\| \le N^2\,\frac{\exp\big(2NL(\omega,E)+(\log N)^A\big)}{\big|f_{[-N,N]}\big(e(x),\omega,E\big)\big|}$$

for any $x\in\mathbb{T}$. Moreover, one obtains the aforementioned off-diagonal decay in this way. This latter property, however, would force ψ to be very small everywhere in contradiction to the fact that $\|\psi\|_2=1$.

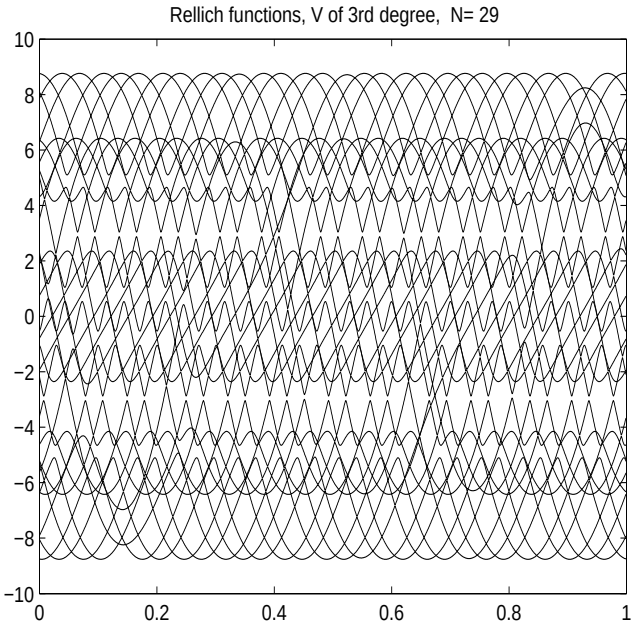

FIGURE 4. Rellich functions in the aperiodic case II

We shall now outline three main points related to the phenomenon of Anderson localization and the property of separation of the eigenvalues:

- **Point 1**: By eliminating bad ω and E (i.e., $\omega\notin\Omega_N$, $E\notin\mathcal{E}_N(\omega)$) we can ensure that

$$\log|f_{\Lambda'}(x,\omega,E)| \ge |\Lambda'|L(\omega,E) - |\Lambda'|^{\frac{1}{2}} \tag{4}$$

for all $|\Lambda'|\sim|\Lambda|$, $\operatorname{dist}(\Lambda',\Lambda)>N^\varepsilon$. This will be obtained by means of a reduction to a statement about close zeros of two determinants; see below.

- **Point 2**: From the avalanche principle (see Lemma 1 below), conclude that (with $L(E,\omega) > \gamma > 0$)
$$|\psi(n)| \leq \exp(-\gamma \mathrm{dist}(n, \Lambda_0) + N^{\varepsilon}).$$
This step can be considered to be in the spirit of the multi-scale analysis of Fröhlich–Spencer, albeit with the avalanche principle instead of the resolvent identity.
- **Point 3**: Suppose
$$H_{[-N,N]}(x,\omega)\psi = E\psi, \quad H_{[-N,N]}(x,\omega)\tilde{\psi} = \tilde{E}\tilde{\psi}$$
with $\|\psi\|_2 = \|\tilde{\psi}\|_2 = 1$ and $0 < |E - \tilde{E}| < e^{-N^{\delta}}$, and $\omega \notin \Omega_N$, $E, \tilde{E} \notin \mathcal{E}_N(\omega)$. Then provided $\varepsilon \ll \delta$,
$$\|\psi - \tilde{\psi}\|_2 < |E - \tilde{E}| e^{N^{\varepsilon}} \ll 1.$$
Clearly, this is a contradiction to $\psi \perp \tilde{\psi}$ (note that this step requires self-adjointness of the Hamiltonian). So
$$|E - \tilde{E}| > e^{-N^{\delta}}.$$
This step requires knowing that (4) holds, not just localization; in fact, we need to use that $\psi(n) = \mu(x,\omega,E) \cdot f_{[-N,n-1]}(x,\omega,E)$ for all $-N \leq n \leq N$. It is not surprising that ψ and $\tilde{\psi}$ are close if E and $\tilde{E}$ are close; indeed, this can be thought of as continuity in E or differentiability in E. Of course, the differentiation has to be carried out on determinants of a smaller scale and not the large-scale determinants. This is achieved by means of the avalanche principle and the fact that the small-scale Hamiltonians are not in resonance with those inside the window of localization.

We now recall the Avalanche Principle (AP) from [**26**] and [**27**]. It is a purely deterministic statement.

LEMMA 1 (Avalanche Principle). *Let $A_1, \ldots, A_n$ be 2×2 matrices whose determinants satisfy*
$$\max_{1 \leq j \leq n} |\det A_j| \leq 1.$$
Suppose that
$$\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n$$
$$\max_{1 \leq j < n} [\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1} A_j\|] < \tfrac{1}{2} \log \mu.$$
Then
$$\left| \log \|A_n \cdot \ldots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1} A_j\| \right| < C \frac{n}{\mu}$$
with some absolute constant C.

4. Elimination of Bad Phases and Energies

We now return to Point 1 from above in an attempt to explain some of the underlying issues, especially the need for elimination of bad ω and E. In order to obtain our window of localization, we need to exclude resonances. The latter here

refers to the situation where for some energy E there exists a phase x_0 for which we have two small determinants:

$$\begin{aligned} \log|f_\Lambda(x_0,\omega,E)| &< |\Lambda|L(\omega,E) - |\Lambda|^{\frac{1}{2}} \\ \log|f_{\Lambda'}(x_0,\omega,E)| &< |\Lambda'|L(\omega,E) - |\Lambda'|^{\frac{1}{2}} \end{aligned} \tag{5}$$

where $\Lambda, \Lambda' \subset [-N, N]$, $\operatorname{dist}(\Lambda, \Lambda') \gg |\Lambda| = |\Lambda'| = n$. This can of course happen; indeed, if we formally "eliminate" the phase x from these inequalities, then we obtain a condition on (ω, E) that describes a set of bad parameters that need to be removed. Using large deviation theorems for the determinants (see (9) below), as well as Jensen's formula (7) from complex analysis, one can show that (5) implies the following: there exist z_1, z_2 with $|z_1 - z_2| + \sum_j |\mathrm{Im} z_j| < e^{-n^{\frac{1}{4}}}$ and

$$f_{[1,n]}(z_1,\omega,E) = f_{[1,n]}(z_2 + t\omega,\omega,E) = 0. \tag{6}$$

Hence, we are required to exclude close zeros of two such determinants which we do by means of the method of resultants (we will return to this issue below). We now recall Jensen's formula from complex analysis[1]: if f is analytic on $|z - z_0| \leq r$, then

$$\int_0^1 \log|f(z_0 + re^{2\pi i\theta})|\, d\theta = \log|f(z_0)| + \sum_{\zeta : f(\zeta)=0} \log\frac{r}{|\zeta - z_0|} \tag{7}$$

In order to use this to pass from (5) to (6) we simply need to show that the sum on the right-hand side cannot vanish for $r = e^{-n^{\frac{1}{4}}}$; or, in other words, that the difference

$$\int_0^1 \log|f_{[1,n]}(x_0 + re^{2\pi i\theta},\omega,E)|\, d\theta - \log|f_{[1,n]}(x_0,\omega,E)| \neq 0.$$

In view of (5) this is accomplished by showing that with $e^{-n} < r \leq n^{-1}$, the integral satisfies

$$\int_0^1 \log|f_{[1,n]}(x_0 + re^{2\pi i\theta},\omega,E)|\, d\theta = nL(E,\omega) + O((\log n)^C)\, \log\frac{1}{r}.$$

This in turn follows from the following estimate, which we call *large deviation theorem (LDT) for the determinants*; see [**27**]:

$$|\{x \in \mathbb{T} :\ \log|f_{[1,n]}(x + iy,\omega,E)| < nL(\omega,E) - h\}| < e^{-ch/(\log n)^C} \tag{8}$$

uniformly in $|E| \leq C$ and $|y| \leq n^{-1}$. Figure 5 illustrates this bound for the case of $\log\|M_N(x,\omega,E)\|$ instead of the determinant, with M_N as in (3), $N = 100$ and $V = \cos$. The picture displays the self-similar nature of this function together with its subharmonic features: there are large deviations in the direction of small values, but for large values the function looks relatively "flat." We remark that the (LDT) for M_N goes back to [**9**] and [**26**], whereas the case of the determinant was established in [**27**].

[1]We use this device in [**27**] repeatedly. However, since it is hard to work with a fixed z_0 we are forced to average over this point as well; this is the origin of the double Jensen averages in that paper. Here, however, it suffices to freeze z_0. See also the review [**29**] for these matters.

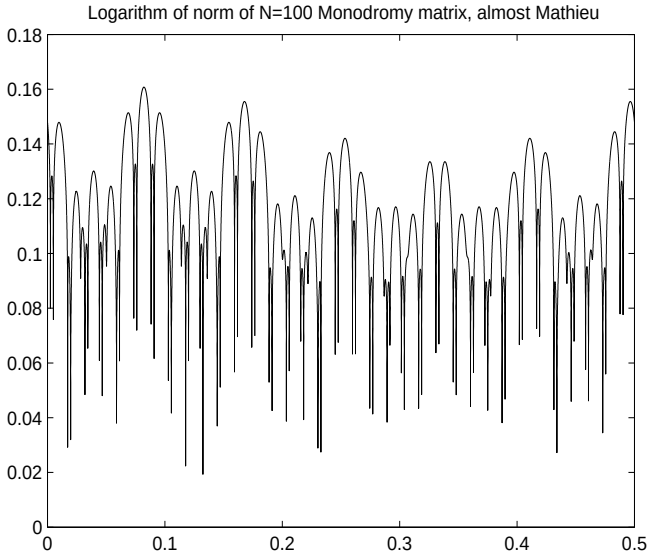

FIGURE 5. The logarithm of a monodromy matrix

By the John–Nirenberg inequality (see Garnett [**25**]), the bound (8) is equivalent with the statements that

$$\| \log |f_{[1,n]}(\cdot + iy, \omega, E)| \|_{\mathrm{BMO}} \leq (\log n)^C \tag{9}$$

and

$$\int_0^1 \log |f_{[1,n]}(x + iy, \omega, E)| \, dx = nL(E, \omega) + (\log n)^C \tag{10}$$

where (10) is uniform in $|y| \leq n^{-1}$. We remark that these estimates imply that

$$\sup_{x \in \mathbb{T}} \log |f_{\Lambda'}(x, \omega, E)| < nL(\omega, E) + (\log n)^C \tag{11}$$

via the sub-mean property of subharmonic functions. Another immediate consequence via Jensen's formula is the following bound

$$\#\{\zeta \in D(x_0, n^{-1}) \, : \, f_{[1,n]}(\zeta, \omega, E) = 0\} \leq (\log n)^C \tag{12}$$

for all x_0, E and Diophantine ω. These four facts (9)–(12) are basic to our entire analysis; see [**27**] and [**29**]. We remark that the large deviation estimate, and thus the BMO bound (9), are a reflection of that fact that the zeros of $f_{[1,n]}(\cdot, \omega, E)$ are uniformly distributed.

To see this, consider the following classical result of *Erdös–Turan*: Let $\{\zeta_j\}_{j=1}^N \subset \mathbb{T}$ be a collection of N points on the circle. Consider the polynomial

$$P(z) = \prod_{j=1}^N (z - \zeta_j).$$

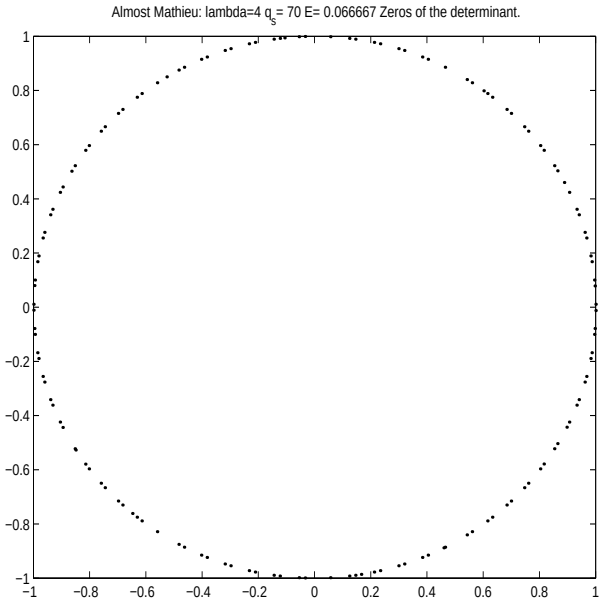

FIGURE 6. Zeros I

Then

$$\sup_{z\in\mathbb{T}}|P(z)| < e^{\tau} \Rightarrow \|\log|P(e(\cdot))|\|_{\mathrm{BMO}} \sim D_N(\{\zeta_j\}) < \sqrt{N\tau}$$

where

$$D_N(\{\zeta_j\}) = \sup_{I\subset\mathbb{T}} \big|\#\{j \,:\, \zeta_j \in I \mod 1\} - N|I|\big|$$

is the usual discrepancy. There is the following analogue for *subharmonic functions*: Let $u : \mathcal{A} \to \mathbb{R}$ be subharmonic with $\mathbb{T} \subset \mathcal{A} \subset \mathbb{C}$ an annulus of width $\frac{1}{2}$, say. Then the Riesz representation theorem (see Levin [**43**]) yields that

$$u(z) = \int \log|z-\zeta|\, d\mu(\zeta) + h(z)$$

with $\mu \geq 0$ and h harmonic. Suppose $\mu(\mathbb{C}) \leq N$. Then the analogue of the result of Erdös–Turan is the following:

$$\|u(e(\cdot))\|_{\mathrm{BMO}} \lesssim \sqrt{N[\sup_{x\in\mathbb{T}} u(e(x)) - \langle u\rangle]}\,.$$

However, this is insufficient for our purposes (it is inconsistent in the sense that the supremum bound (11) does not imply the BMO bound (9)). Luckily, it can be improved (see Bourgain–Goldstein–Schlag [**10**] and [**8, 29**]): Write $u - \langle u\rangle = u_0 + u_1$ on $\mathbb{T}$. Then

$$\|u(e(\cdot))\|_{\mathrm{BMO}} \lesssim \|u_0\|_\infty + \sqrt{N\|u_1\|_1}$$

It is easy to check that this bound *is* consistent with our estimates.

In order to obtain the estimates for the determinants which we just described, we need to reveal their "almost-invariance" under the shift. This can be done by

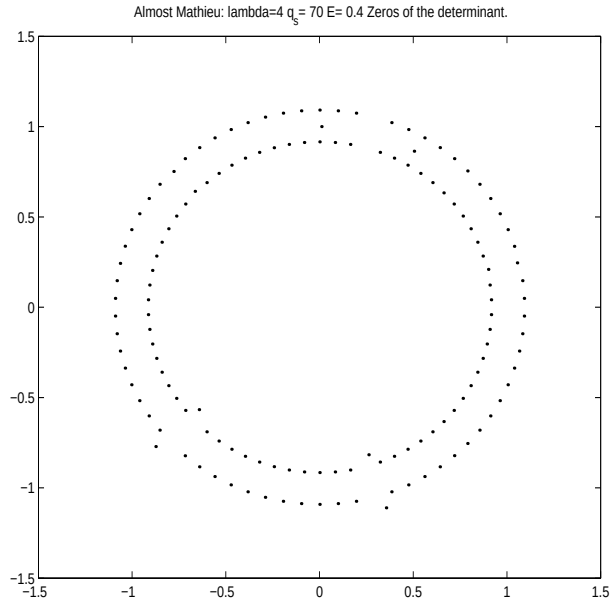

FIGURE 7. Zeros II

means of a "factorization of the determinant" via the avalanche principle. Indeed, write

$$\begin{bmatrix} f_N(x,\omega,E) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \prod_{j=N}^{1} \begin{bmatrix} V(T^j x) - E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \prod_{k=n}^{1} A_k(x,\omega,E)$$

where each A_k is the product of about $(\log N)^C$ many factors. One needs to use the large deviation theorems for the determinants and monodromies on the small scale to conclude that the conditions of the avalanche principle hold. This requires the removal of a set of phases $x \in \mathbb{T}$ of measure $< \exp(-(\log N)^B)$. For the remaining good phases we conclude that:

$$\log |f_N(x,\omega,E)| = -\sum_{j=2}^{n-1} \log \|A_j\| + \sum_{j=1}^{n-1} \log \|A_{j+1} A_j\| + O(N^{-1000}).$$

Note that this resembles an ergodic average since most of the A_j can be chosen to be shifts in the phase of a fixed one. This is what we mean by "self-similar structure" of the determinants f_N. In Figures 6 and 7, we display two sets of zeros of the determinants in the almost Mathieu case. The first one is for an energy in the spectrum, whereas the second is for an energy outside of the spectrum. Observe that the zeros look approximately evenly distributed with the exception of a few "errant" ones. Loosely speaking, these are related to nonlocalized states in the same way that the "errant" segments of Rellich graphs crossing what appears to be a gap in Figures 3 and 4 correspond to nonlocalized states (see the gap containing energies $[-2,-1] \cup [1,2]$ in Figure 3 as well as that around energy $E = 3$ in Figure 4).

Recall that we have only discussed so far how to pass from (5) to (6) but not how to exclude the latter. As indicated above, for this we need to eliminate close zeros. The following is proved in [**27**]:

LEMMA 2. *There exists* $\Omega_n \subset \mathbb{T}$, $|\Omega_n| < e^{-n^{\frac{1}{10}}}$, *so that for all* $\omega \notin \Omega_n$, $t > e^{(\log n)^C}$ *there exists* $\mathcal{E}_{n,t,\omega} \subset \mathbb{C}$, $|\mathcal{E}_{n,t,\omega}| < e^{-n^{\frac{1}{10}}}$ *so that*

$$f_{[1,n]}(z_1,\omega,E) = f_{[1,n]}(z_2+t\omega,\omega,E) = 0 \;\; \Rightarrow |z_1-z_2| > e^{-n^{\frac{1}{4}}}$$
$$\inf_{x\in\mathbb{T}} \operatorname{dist}\big(\operatorname{spec}(H_{[1,n]}(x,\omega)) \setminus \mathcal{E}_{n,t,\omega}, \operatorname{spec}(H_{[1,n]}(x+t\omega,\omega))\big) > e^{-n^{\frac{3}{4}}}$$

This is only one of several statements one can prove in this direction; more precisely, the powers of n can be lowered to n^ε with $\varepsilon > 0$ arbitrary. Further, we remark that one can also remove x instead of E by a Wegner estimate; see [**27**] and [**29**]. Finally, and crucially, we are also able to bound the complexity of the bad sets Ω_n and $\mathcal{E}_{n,t,\omega}$.

For the proof, we need to use the resultant of two polynomials $f(z), g(z)$. It is defined as

$$\operatorname{Res}(f,g) = \prod_{j,k}(z_j - \zeta_k)$$

where $f(z_j) = 0$, $g(\zeta_k) = 0$. One can show that it is a *polynomial* in the coefficients of f, g (see [**39**]). Suppose

$$f(z,\omega,E) = z^\nu + a_{\nu-1}(\omega,E)z^{\nu-1} + \cdots + a_1(\omega,E)z + a_0(\omega,E)$$
$$g(z,\omega,E) = z^\mu + b_{\mu-1}(\omega,E)z^{\mu-1} + \cdots + b_1(\omega,E)z + b_0(\omega,E)$$

with a_j, b_k analytic in ω, E. It follows that Res is also analytic in ω, E. This allows one to use analytical methods to estimate Res from below, at least for most ω, E. Clearly, if we are able to bound the resultant from below, then we are also able to give a quantitative estimate on the separation of the zeros. Figure 8 depicts the algebraic curve $\operatorname{Res}(\omega, E) = 0$ in the (ω, E)-plane. For most values of ω there will be only finitely many E-values on this curve, but we need to remove those ω for which there is a "flank"; this refers to the vertical or near vertical segments of the curve that would lead to a large set in energy for which the resultant is too small.

On a more technical, albeit crucial, point we remark that we cannot use resultants on the full determinants because of their large degree. Rather, we apply the Weierstrass preparation theorem on $f_{[1,n]}(z,\omega,E)$ before applying the resultants locally in ω: the factors which we pull out have very small degree $= (\log n)^C$ because of the control over zeros provided by (12). For more on this topic, see [**27**] or [**29**].

We now list some important consequences of the finite-volume Anderson localization and the quantitative separation of the eigenvalues:

- Use a Sard-type argument to conclude that the slopes of the Rellich functions $E_j^{(N)}(x,\omega)$ off a bad energy set $\mathcal{E}_N(\omega)$ are bounded below by e^{-N^δ} in absolute value. The Sard theorem is needed to remove the critical values of the Rellich functions. Particular care needs to be taken concerning the complexity of the resulting set of energies. Of course, it is essentially used in [**27**] that the Rellich functions are solutions of algebraic equations of controlled degrees.

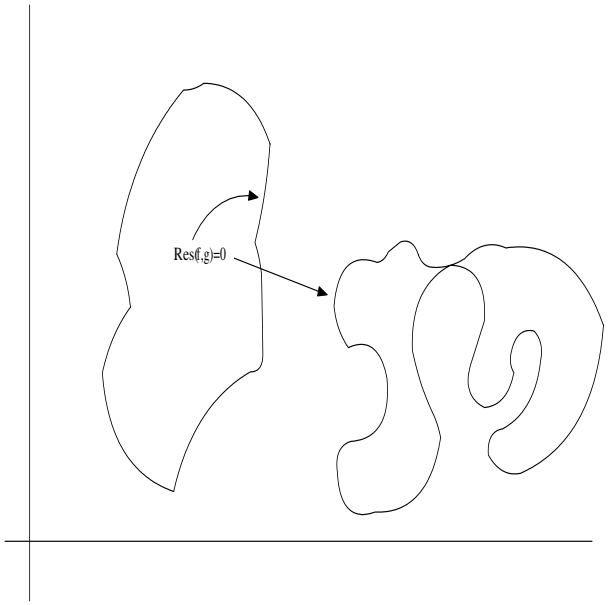

FIGURE 8. Zero set of the resultant

- Use (AL) to obtain almost shift-invariance of the Rellich graphs. Provided $E_j^{(N)}(x,\omega) \notin \mathcal{E}_N(\omega)$ and for most choices of $m \in [-N,N]$, there exists ℓ such that

$$|E_j^{(N)}(x,\omega) - E_\ell^{(N)}(x+m\omega,\omega)| < e^{-N^{\frac{1}{2}}}.$$

  This follows from the following fact: if $H_{[-N,N]}(x,\omega)\psi = E\psi$ with ψ localized, then $\psi(n+m)$ is almost an eigenfunction of $H_{[-N,N]}(x+m\omega,\omega)$. This self-similarity of the Rellich graphs is basic to the formation of gaps.
- Gaps (in some finite volume) are formed by the interaction of two localized eigenfunctions with separate supports but close eigenvalues; moreover, their respective Rellich graph segments should have opposite slopes. The following figure describes this schematically. The separate bumps on the left (which depict eigenfunctions on a smaller scale) combine to form an eigenfunction on a larger scale. The curves below depict the Rellich graphs of the two small-scale eigenfunctions which then produce the separated arcs on the right as Rellich graphs of the larger scale (cf. Figure 1) in that regard. An exact way of formulating this requires the notion of a *double resonance*. Roughly speaking, this means that there are exactly two windows of localization in our finite-volume scheme of Anderson localization. To avoid a third or more windows requires elimination of triple resonances as in Chan [**12**].

We say that (x_0, E_0) is a *point of double resonance* for $H_{[-N,N]}(x_0,\omega)$ provided there exist $\Lambda_1 = [N_1', N_1'']$, $\Lambda_2 = [N_2', N_2'']$ as shown in Figure 10 so that for $j = 1, 2$,

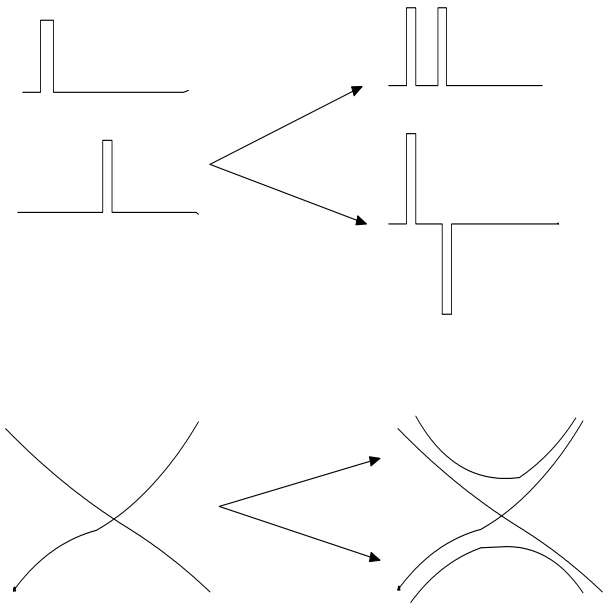

FIGURE 9. A schematic description of a double resonance

and with $\rho = e^{-N^\varepsilon}$,

$$\begin{aligned} \operatorname{spec} H_{\Lambda_j}(x_0,\omega) \cap (E_0-\rho, E_0+\rho) &= \{E_0\} \\ \#\{z \in D(e(x_0),\rho) \,:\, f_{\Lambda_j}(z,\omega,E_0) = 0\} &= 1 \end{aligned}$$

where $|\Lambda_1| \sim |\Lambda_2| \sim N^{2\varepsilon}$ and which are separated from the boundaries of $[-N,N]$. Finally, we need to avoid triple or higher order resonances: for all $\Lambda \subset [-N,N]$ separated from $\Lambda_1 \cup \Lambda_2$ by an amount $\gg N^{2\varepsilon}$, we have

$$\operatorname{spec} H_{\Lambda}(x_0,\omega) \cap (E_0-\rho, E_0+\rho) = \emptyset.$$

The importance of this notion of a point of double resonance lies with the fact that it captures the nonperturbative essence of Figure 1. Recall that there we were able to extract a second degree polynomial from the characteristic polynomial of a finite-volume Hamiltonian, i.e., from the determinant $f_N(\cdot,\omega,E)$. This was done perturbatively by means of the Feshbach formula. Here we have to proceed differently—in fact, this extraction of a quadratic factor is accomplished by means of (a quantitative version of) the Weierstrass preparation theorem.

More precisely, if (x_0,E_0) is a point of double resonance, then one can show by means of the (AP) (see "factorization of a determinant" from above) that for all $x \in I := (x_0-\rho, x_0+\rho)$ there are exactly two zeros in both the z and E variables locally around the points we are considering:

$$\begin{aligned} \#\big[\operatorname{spec} H_{[-N,N]}(x,\omega) \cap (E_0-\rho, E_0+\rho)\big] &= 2 \\ \#\big\{z \in D(e(x_0),\rho) \,:\, f_N(z,\omega,E_0) = 0\big\} &= 2 \end{aligned} \tag{13}$$

and the corresponding Rellich functions are separated (without any E removal!):

$$E^+(x,\omega) - E^-(x,\omega) > e^{-N^\delta} \quad \forall\, x \in I.$$

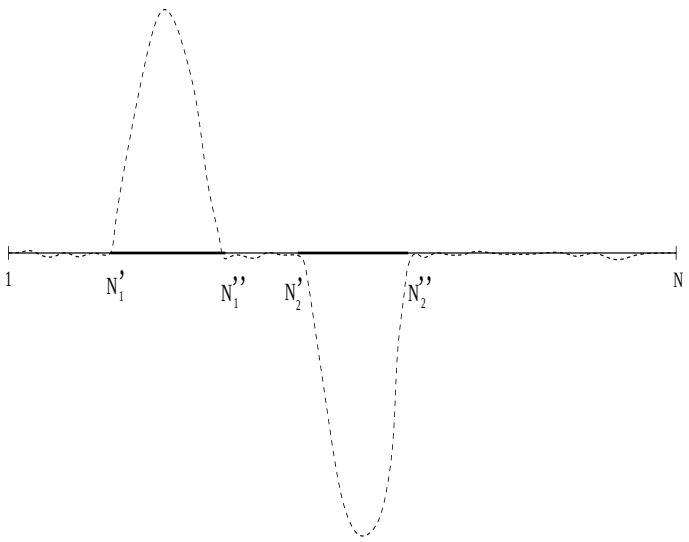

FIGURE 10. The eigenfunction corresponding to a double resonance

It is obviously essential here that we are not forced to eliminate the energy E, as otherwise we would be eliminating the gap which we are trying to construct. The fact that we can obtain separation of the eigenvalues here without eliminating an interval around E relies on the properties of a double resonance; the features needed to obtain the separation property have been included in the definition, see the discussion of Point 3 above. To obtain the zero count (13), we use the avalanche principle (see the "factorization of a determinant" from above) and the Jensen formula. Heuristically, this is a variant of the obvious fact that the number of zeros of any polynomial is the sum of the numbers of the zeros of all factors in a factorization of the polynomial. Here we of course do not have an exact factorization, but only an additive one for the logarithm that holds for most phases and up to a small error. But since the Jensen formula is based on averages and the number of zeros is integer valued, we can afford to make small errors—they produce small errors after averaging and do not affect the zero count.

Finally, in view of Figures 1 and 9, we require that the Rellich functions of the two small-scale windows Λ_1 and Λ_2 which attain the energy E have slopes of opposite signs. The appearance of such slopes is a consequence of the 1-periodicity of the continuous Rellich graphs and the fact that we are working a priori on intervals of energies on which the slopes of the Rellich functions do not vanish. Hence, it is important to realize that our construction does involve the elimination of energies and can thus, in its present form, not capture something that occurs on the entire spectrum (in particular, the argument in [**28**] shows that the gaps are dense but not much more beyond that).

If we do have graph segments of opposite slopes as in Figure 9, then we do get much more, namely, the desired gap between the branches $E^{\pm}$ as in Figure 1 and Figure 9:

$$\min_I E^+(x) > \max_I E^-(x) + e^{-N^{2\delta}}. \tag{14}$$

To see this, we need to be able to show that locally in z the determinant of the large scale at energy E behaves like a *quadratic polynomial*; the same is needed also in the E variable. As mentioned before, we use the *Weierstrass preparation theorem* to extract these quadratic polynomials based on the zero count (13).

As mentioned above, the definition of a double resonance point is tailored to our needs in the sense that it produces the exact zero count of two, and thus allows for the Weierstrass preparation argument. The question arises how to find points of double resonance—obviously, they are essential for our gap construction. The fact that there have to be at least two small-scale Hamiltonians which are in resonance is relatively simple and can basically be deduced from the Rellich graphs at the small scale. Much trickier is to ensure that there cannot be more than two resonances—this requires the elimination of ω via an implicit function type argument. For this we crucially need to have some information on the nondegeneracy of certain derivatives, which itself is a consequence of the quantitative nonvanishing of the slopes of the Rellich graphs; recall that we remove energies for this purpose. The elimination of triple resonances employed in [**28**] is a variant of that introduced by Chan [**12**].

Returning to (14), we have finally arrived at a gap between two large-scale Rellich functions locally around x_0; we call this construct a pre-gap. Using (AL) we can now move this pre-gap around in x by shifting the phase. This is due to the aforementioned "self-similarity" or "almost shift-invariance" of the Rellich graphs.

It remains to show that this pre-gap is not destroyed when passing to larger scales. Among other things, we need to insure that the energies which we remove at the next scale $\bar{N}$ are much smaller in measure than the size of any pre-gap at a previous scale N; however, the latter is at least $e^{-N^{2\delta}}$, whereas the former is at most $\exp\big(-(\log \bar{N})^C\big)$. Hence, we define scales $N_s = \exp(N_{s-1}^{\delta_1})$ with $\delta_1 > 0$ sufficiently small. This ensures that we can safely remove "bad" energies of scale N_{s+1} inside the pre-gap at scale N_s. We can therefore repeat the pre-gap construction at scale N_{s+1} inside the pre-gap of the previous scale provided, of course, Rellich graphs of scale N_{s+1} enter that pre-gap (the case where they do not is easier).

It remains to show that this process has to terminate, i.e., one needs to find a mechanism that will ensure that after some number of steps, no more pre-gaps can form inside a pre-gap of the previous scale. At that point the pre-gap will become a gap of the infinite-volume operator. It turns out that a pre-gap cannot be filled in more than k times if the underlying potential function V has degree k. This involves a counting argument involving complex zeros. The point there is that every pre-gap at scale N_s and locally around (x_0, E_0) corresponds to a pair of complex zeros of

$$\det(H_{[-N_s,N_s]}(z,\omega) - E_0)$$

in the z variable. This pair of zeros lies off the unit circle and close to the point $e(x_0)$. Moreover, their separation from the circle is basically proportional to the size of the gap. By the aforementioned shifting procedure of pre-gaps (which is based on the almost shift-invariance of the Rellich graphs and finite-volume (AL)), each such pair generates almost N_s further pairs. If there was a sequence of consecutive scales producing pre-gaps, then one can show that this would lead to an accumulation of zeros which ultimately violates some degree considerations. This process is somewhat involved and we refer the reader to Lemma 2.24 and Section 9 of [**28**].

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