Asymptotic stability of N-soliton states of NLS

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Abstract

The asymptotic stability and asymptotic completeness of NLS solitons is proved, for small perturbations of arbitrary number of non-colliding solitons.

1 Introduction

The nonlinear Schrödinger equation

$$i\,\frac{\partial\psi}{\partial t} \;=\; -\triangle\psi \;+\; \beta(|\psi|^2)\psi, \qquad x\in\mathbb{R}^n \tag{NLS}$$

has in general (exponentially) localized solutions in space, provided the nonlinearity has a negative (attractive) part. This is due to a remarkable cancellation of the dispersive effect of the linear part with the focusing caused by the attractive nonlinearity. To find such solutions, we look for time periodic solutions $\psi \,\equiv\, e^{i\omega t}\,\phi_\omega(x)$. It follows that ϕ_ω, if it exists, is a nonzero solution of the problem

$$-\omega\phi_\omega \;=\; -\triangle\phi_\omega \;+\; \beta(|\phi_\omega|^2)\phi_\omega. \tag{ENLS}$$

We shall refer to such solutions as *nonlinear eigenfunctions.* In general, for ϕ_ω to be localized (at least as L^2 function) we need $\omega>0$.

The general existence theory for this elliptic problem has been studied in great detail, see the work of Coffman [Cof], Strauss [Str], and Berestycki, Lions [BL].

It is easy to see that if $\psi_\omega(t,x)=e^{i\omega t}\phi_\omega(x)$ is a solution of NLS, then for any vector $\vec{a}\in\mathbb{R}^n$ the function $\phi_\omega(t,x-\vec{a})$ is also a solution. More generally, NLS is invariant under Galilean transformations

$$\mathfrak{g}_{\vec{v},D}(t) := e^{-i\vec{v}\cdot x-\frac{1}{2}|v|^2t}e^{i(t\vec{v}+D)p}, \tag{1.1}$$

and therefore we can construct solutions from ϕ_ω which are moving with arbitrary velocity $\vec{v}$. As a result we obtain a family of exponentially localized solutions

$$\psi_{\vec{v},\gamma,D,\omega} = e^{i\vec{v}\cdot x-i\frac{1}{2}(|v|^2-\omega)t+i\gamma}\phi_\omega(x-\vec{v}t-D)$$

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parametrized by a constant $(2n+2)$–dimensional vector $(\vec{v}, \gamma, D, \omega)$, which are known as solitons.

Solving the initial value problem for NLS requires understanding of two fundamental questions. The first is the existence of global in time solutions. Due to the focusing character of the nonlinear term the global existence theory ought to be based on the L^2 conservation law. By the results of Kato [Kato] and Tsutsumi [1] we can construct unique global solutions for any L^2 initial data under the assumption that the nonlinearity β satisfies the condition $|\beta(s)| \lesssim (1+|s|)^q$ with $q < \frac{2}{n}$.

The second problem is that of the asymptotic behavior of solutions as $t \to +\infty$. Guided by the completely integrable models in one dimension we expect to have solutions with the asymptotic profile of N independently moving solitons:

$$\psi(t,x) \approx \sum_{k=1}^{N} \psi_{\vec{v}_k,\gamma_k,D_k,\omega_k}(t,x). \tag{1.2}$$

Moreover, given such a solution $\psi(t,x)$ we expect that initial data $\psi(0,x) + R$ for a suitably small perturbation R leads to a solution $\tilde{\psi}$ with $\|\psi(t,\cdot) - \tilde{\psi}(t,\cdot)\| \to 0$ as $t \to \infty$ in an appropriate norm. The latter property is known as asymptotic stability. In the context of NLS one needs to modify (1.2) since one needs to make $(\vec{v}_k, \gamma_k, D_k, \omega_k)$ time-dependent. This fact was already observed in the context of orbital stability, see Weinstein [We1].

In this paper we give an affirmative answer to the question of existence and asymptotic stability of solutions with N-soliton profiles under the assumption of weak mutual interaction between the solitons. A superposition of N spatially separated moving solitons gives only an approximate solution of NLS. Our goal is to show that the initial data

$$\psi_0(x) = \sum_{k=1}^{N} e^{i\vec{v}_k(0)\cdot x + i\gamma_k(0)} \phi_{\omega_k(0)}(x - D_k(0)) + R_0, \tag{1.3}$$

give rise to an asymptotically stable solution with the profile of N independent solitons, with perhaps different parameters $(\vec{v}_k, \gamma_k, D_k, \omega_k) \neq (\vec{v}_k(0), \gamma_k(0), D_k(0), \omega_k(0))$.

The function R_0 is a perturbation satisfying a smallness assumption on its $L^1 \cap L^2$ norm together with its derivatives. An important aspect of our main result is the assumption that the solitons are weakly interacting. This condition can be enforced in two ways. Firstly, one can assume that the initial shifts $D_k(0)$ and the initial velocities v_k are chosen to model the case of non-colliding solitons:

$$|D_k + \vec{v}_k t - D_{k'} - \vec{v}_{k'} t| \geq L + ct, \qquad k \neq k' \tag{1.4}$$

for some sufficiently large constant L.

Alternatively, one can assume that the relative initial velocities of the solitons are large, i.e.,

$$\min_{j \neq k} |\vec{v}_j - \vec{v}_k| > L. \tag{1.5}$$

For the most part, we give details only for the case of (1.4) and leave the simple modifications required by (1.5) to the reader. Note that (1.5) does not rule out that the solitons collide. However, in view of (1.5) the time of interaction is of size L^{-1}, and therefore the overall interaction remains weak.

We shall also require certain spectral assumptions on the nonlinear eigenstates ϕ_ω which will be explained below.

We believe that the methods we use may be applied to other classes of equations with solitary type solutions and other symmetry groups (e.g. Lorentz instead of Galilean), if and whenever certain linear L^p decay estimates can be verified for the linearized operators around one such soliton. A detailed analysis of such L^p estimates for NLS was recently given in [RSS].

To explain our results we recall the precise notions of stability. Suppose we take the initial data of NLS to be an exact nonlinear eigenstate ϕ_ω, plus a small perturbation R_0. What is then the expected behavior of the solution? If the solution $\psi(t)$ stays near the soliton $\psi_\omega(t) = e^{i\omega t}\phi_\omega(x)$ up to a phase and translation for all times (in H^1 norm) we say that the soliton ψ_ω is *orbitally stable*. If, as time goes to infinity, the solution in fact converges in L^2 to a nearby soliton plus radiation[1] we say that the solution is *asymptotically stable*.

Orbital stability of one soliton solutions has been subject of extensive work in the last 20 years. The first results date back to the work of Cazenave and Cazenave-Lions on logarithmic and monomial nonlinearities. The general case has been treated in the defining works of Shatah-Strauss [ShSt], Weinstein [We1] and [We2], and Grillakis-Shatah-Strauss [GSS1]. The general phenomena that has emerged from their results is that the orbital stability is essentially controlled by the sign of the quanitity $\partial_\omega \|\phi_\omega\|_{L^2}$ (stable, if positive, and unstable, if negative). However, all there results addressed orbital stability of special class of solitons generated by ground states: positive, radial solutions of the equations ENLS of lowest energy.

In [BL] Berestycki-Lions proved the existence of a ground state in three or more dimensions for any $\omega \neq 0$ under the conditions that the nonlinearity β verifies $\lim\limits_{s\to+\infty} \beta(s^2)s^{-\frac{4}{n-2}} \geq 0$ and such that there exists $0 < s_0 < \infty$, with $G(s_0) > 0$, for $G(s) \equiv -2\int_0^s (\beta(\tau^2)s + \omega\tau)d\tau$. In fact, in their work ground states are found as minimizers of the constrianed variational problem:

$$\inf J[u] = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 \; : \; \int_{\mathbb{R}^n} G(u) = 1 \right\}. \tag{1.6}$$

The question of uniqueness of a ground state has been studied in [McS], [Kw], [McL].

The asymptotic stability of *one* ground state soliton solutions of NLS and other equations was first shown for NLS with an extra attractive potential term in [SW1], [SW2], and [PW], for one NLS soliton in dimension one in [BP1] and in dimensions $n \geq 3$ in [Cu]; for NLS-Hartree see [FTY].

While the arguments for orbital stability were essentially based on Lyapunov type analysis and relied only on some limited information about the spectrum of the associated linear problem, the proofs of asymptotic stability required much more detailed properties of the related linearized systems. In particular, it led to the need to impose additional spectral assumptions on the linear operators associated with a soliton.

To describe the problem of linear stability (and spectral theory) we linearize NLS around a soliton

[1] a function with asymptotic behavior $e^{it\triangle}f$. Alternatively, we can and will replace the L^2 by the L^∞ convergence. In the latter topology the contribution of the radiation can be ignored.

$w = e^{i\theta}\phi_\omega$, using the ansatz

$$\psi = e^{i\theta}\Big(\phi_\omega(x - \vec{v}t) + R(t, x - \vec{v}t)\Big) \tag{1.7}$$

The resulting "linear" operator acting on R, also has a term containing $\overline{R}$. After complexifying the space to $(R, \overline{R})$ we are left with a matrix non self-adjoint operator of the type

$$\mathcal{H} = \begin{pmatrix} L_+ & W(x) \\ -W(x) & -L_+ \end{pmatrix} \tag{1.8}$$

acting on $L^2 \times L^2$. Here

$$L_+ = -\triangle + \omega + \beta(\phi_\omega^2) + \beta'(\phi_\omega^2)\phi_\omega^2, \qquad W(x) = \beta'(\phi_\omega^2)\phi_\omega^2$$

The operator L_+ is a self-adjoint perturbation of $-\triangle$ by an exponentially localized function, and W is an exponentially localized potential. Moreover, centering the perturbation R around the soliton as in (1.7) ensures that $\mathcal{H}$ is *time-independent*. Note that by non self-adjointness of $\mathcal{H}$ one can no longer guarantee that $\sup_t \|e^{it\mathcal{H}}\|_{2\to 2} < \infty$ where $U(t) = e^{it\mathcal{H}}$ is given by $i\partial_t U + \mathcal{H}U = 0$, $U(0) = \mathrm{Id}$. In fact, the operator $\mathcal{H}$ has a zero root space $\mathcal{N} := \bigcup_{\ell\geq 1} \ker \mathcal{H}^\ell$ of dimension at least $2n+2$ containing the eigenfunction ϕ_ω as well as the elements generated from ϕ_ω by infinitesimal symmetries of the problem. We decompose

$$L^2 \times L^2 = \mathcal{N} + \mathcal{N}^{*\perp}$$

and let P denote the projector on the second term in this decomposition (here $\mathcal{N}^* = \bigcup_{\ell\geq 1}\ker(\mathcal{H}^*)^\ell$). It is easy to see that $\|U(t)f\|_2$ grows polynomially for some $f \in \mathcal{N}$. On the other hand, it is known from work of Weinstein [We1] that

$$\sup_t \| \, U(t)P\psi \, \|_{L^2} < \infty \tag{LS}$$

under certain conditions on the nonlinearity and provided ϕ_ω is the (positive) ground state of ENLS. Generally speaking, we refer to the property (LS) as *linear stability.*

In the case when $\mathcal{H}$ has one negative eigenvalue and ϕ_ω is the unique ground state, one can show that the condition $\sigma(\mathcal{H}) \subset \mathbb{R}$ is equivalent to the orbital stability condition $\partial_\omega \|\phi_\omega\|_{L^2} > 0$ (see [Gr], [BP1]). Although due to the lack of self-adjointness this is not sufficient for linear stability, additional arguments show that (LS) in fact holds just under the above conditions (see [We1], [GSS1]).

Linear stability of ground states has been considered for a large class of NLS in the work of Weinstein [We1], [We2], and Shatah, Strauss, Grillakis , [ShSt], [GSS1], [GSS2], [Gr], see also [SuSu], [Stu]. However, unconditional results were established only in the case of monomial nonlinearities $\beta(s) = s^p$. Moreover, linear stability has been shown to be essentially equivalent to the orbital stability (except in the case of an L^2 critical nonlinearity $\beta(s) = s^{2/n}$).

Linear stability plays an essential role in the results on asymptotic stability. Moreover, the proofs of asymptotic stability of one-soliton solutions required even more stringent assumptions on the structure of the spectrum of $\mathcal{H}$, see Buslaev, Perelman [BP1], and Cuccagna [Cu]. This can be linked to the fact that on the linearized level asymptotic stability requires dispersive estimates of the type

$$\|U(t)P\psi_0\|_{L^\infty} \lesssim t^{-\frac{n}{2}}\|\psi_0\|_{L^1} \tag{1.9}$$

[Cu] or similar L^2-weighted decay estimates. To prove such estimates one needs to impose additional *spectral conditions* such as: absence of the discrete spectrum for $\mathcal{H}$ on the subspace $\mathcal{N}^\perp$, absence of embedded eigenvalues, and absence of resonances at the edges of the continuous spectrum. The dispersive estimates for such matrix Hamiltonians in dimensions $n \geq 3$ were proved by Cuccagna by an extension of the method introduced by Yajima in the scalar case [Ya1]. In that approach, the decay estimates follows as a consequence of the proof of the $L^p \to L^p$, $\forall p \in [1,\infty]$ boundedness of the wave operators. In our recent work [RSS] we suggested a perhaps more straightforward approach for proving such estimates which instead relies on construction of the analytic extension of the resolvent of $\mathcal{H}$ and goes back to the work of Rauch [Rau] in the scalar case. This method, however, requires that all potential terms in the Hamiltonian $\mathcal{H}$ are exponentially localized functions, which perfectly fits the problem at hand. We refer to Hamiltonians verifying the required spectral assumptions (as well as the linear stability condition) as *admissible*. The proof of the dispersive estimates crucially relies on the time independence of the Hamiltonian $\mathcal{H}$, which was ensured by the choice of the ansatz.

In our work we choose the initial data of the form (1.3), with ϕ_{ω_k} verifying the elliptic problem ENLS, satisfying the separation condition (1.4) and study the time asymptotic behavior of the corresponding solution of the time-dependent NLS.

As in the study of asymptotic stablity of one-soliton solutions the first objective is the analysis of the linearized problem. It is natural to use the following ansatz for the solution:

$$\psi(t,x) = w + R(t,x) = \sum_{k=1}^N w_k + R(t,x), \qquad w_k(t,x) := e^{i\theta_k}\phi_{\omega_k}(x - \vec{v}_k t - D_k) \tag{1.10}$$

where θ_k are the phases associated with k-th soliton and $\vec{v}_k, D_k, \omega_k$ are its parameters[2]. We substitute this ansatz into the equation (NLS) and retain only those terms which are linear in $R, \bar{R}$. The resulting linear problem for the unknown $(R, \bar{R})$ is

$$i\partial_t U + \mathcal{H}(t)U = 0. \tag{1.11}$$

It contains a *time-dependent* complex Hamiltonian

$$\mathcal{H}(t) = \begin{pmatrix} L_+ & W(t,x) \\ -\overline{W}(t,x) & -L_+ \end{pmatrix}$$

where

$$L_+ = -\triangle + \beta(|w|^2) + \beta'(|w|^2)|w|^2, \qquad W(t,x) = \beta(|w|^2)w^2.$$

Because of the smallness assumption on the initial perturbation R_0 we can assume the the parameters of the final asymptotic profile will lie in a small neighborhood of the initial parameters

[2]which in the true ansatz are to be made time-dependent

$(\vec{v}_k(0), \gamma_k(0), D_k(0), \omega_k(0))$. The separation condition on the initial parameters and the exponential localization of functions ϕ_{ω_k} guarantees that w is decomposed into a sum of functions of essentially disjoint support. We thus can replace the Hamiltonian $\mathcal{H}(t)$ with

$$\mathcal{H}(t) = \begin{pmatrix} -\triangle & 0 \\ 0 & \triangle \end{pmatrix} + \sum_{k=1}^{N} \begin{pmatrix} U_k(x - \vec{v}_k t - D_k) & W_k(x - \vec{v}_k t - D_k) \\ -\overline{W}_k(x - \vec{v}_k t - D_k) & -U_k(x - \vec{v}_k t - D_k) \end{pmatrix} \tag{1.12}$$

where

$$U_k = \beta(\phi^2_{\omega_k}(x)) + \beta'(\phi^2_{\omega_k}(x))\phi^2_{\omega_k}(x))^2,$$
$$W_k = e^{2i\theta_k}\beta(\phi^2_{\omega_k}(x))\phi^2_{\omega_k}(x)$$

This Hamiltonian belongs to the class of the so called *matrix charge transfer Hamiltonians.* Each of the Hamiltonians

$$\mathcal{H}_k(t) = \begin{pmatrix} -\triangle & 0 \\ 0 & \triangle \end{pmatrix} + \begin{pmatrix} U_k(x - \vec{v}_k t - D_k) & W_k(x - \vec{v}_k t - D_k) \\ -\overline{W}_k(x - \vec{v}_k t - D_k) & -U_k(x - \vec{v}_k t - D_k) \end{pmatrix} = H_0 + V_k(x - \vec{v}_k t)$$

represents a linearization around the kth soliton. Moreover, a specially chosen Galilei transform of the type (1.1) maps a solution of the linear problem $i\partial_t U + \mathcal{H}_k(t)U = 0$ into a solution of the problem $i\partial_t U + \mathcal{H}_k U = 0$ with a time-independent Hamiltonian $\mathcal{H}_k$ of the form (1.8). Here $H_0 = \mathrm{diag}(-\triangle, \triangle)$ and V_k is an exponentially localized complex matrix potential.

Once again, at the linear level, the heart of the problem of asymptotic stability of N-solitons are the dispersive estimates for the soutions of the equation (1.11) with a matrix charge transfer Hamiltonian $\mathcal{H}(t)$. Observe that in contrast to the one soliton case, the linearized Hamiltonian is time-dependent.

Due to the separation condition the problem (1.11) admits "travelling" bound states generated by the discrete spectrum of each of the Hamitonians $\mathcal{H}_k$. These bound states are formed by eigenfunctions and elements of the root space of $\mathcal{H}_k$ boosted by the Gallilei transform corresponding to the parameters $\vec{v}_k, D_k$. The paper [RSS] establishes dispersive estimates

$$\|U(t)\psi_0\|_{L^2+L^\infty} \lesssim (1+t)^{-\frac{n}{2}} \|\psi_0\|_{L^1\cap L^2} \tag{1.13}$$

for the solutions of the linear time-dependent Schrödinger equation (1.11) with a matrix charge transfer Hamiltonian $\mathcal{H}(t)$ of the type 1.12 in dimensions $d \geq 3$. These estimates hold under the assumption that each of the time-independent Hamiltonians $\mathcal{H}_k$ is admissible (requiring spectral assumptions and the linear stability condition) and that the solution $U(t)\psi_0$ is asymptotically orthogonal to all travelling bound states of $\mathcal{H}_k(t)$. The latter means that for all $k = 1, \ldots, N$

$$\|P_b(\mathcal{H}_k, t)U(t)\psi_0\|_{L^2} \to 0, \quad \text{as} \quad t \to +\infty$$

with $P_b(\mathcal{H}_k, t)$ denoting the time-dependent projection on the k-th subspace of travelling bound states, i.e., it is the conjugation of the spectral projection of the stationary operator $\mathcal{H}_k$ onto its bound states by suitable Galilei transforms. The estimate (1.13) is our main linear estimate. However, to ensure that the perturbation $R(t, x)$ in the decomposition (1.10) is asymptotically orthogonal to the subspace of travelling bound states and thus decays in the linear approximation with the rate of $t^{-\frac{n}{2}}$, we need to make the soliton parameters $\sigma_k = (\vec{v}_k, D_k, \gamma_k, \omega_k)$ to be time-dependent. This in turn makes it necessary to change the form of $w_k(t,x)$, for example, $x - \vec{v}_k t - D_k$ becomes

$$x - \int_0^t \vec{v}_k(s)\, ds - D_k(t).$$

The resulting *nonlinear* problem for R, after the complexification $Z = (R, \bar{R})$, takes the form

$$i\partial_t Z + H(t, \sigma(t))Z = \dot{\sigma} \cdot \partial_\sigma w + \mathcal{N}(Z, w), \tag{1.14}$$

where $H(t, \sigma(t))$ is a time-dependent Hamiltonian. It is of the matrix charge transfer type provided $\sigma(t)$ =const. The term $\partial_\sigma w$ denotes the derivative of the solitary approximation $w(t, x; \sigma(t)) = \sum_{k=1}^N w_k(t, x; \sigma(t))$ with respect to its parameters σ_k, the term $\dot{\sigma}$ denotes the time derivative of the soliton parameters σ_k, and $\mathcal{N}(Z, w)$ is a nonlinear term in Z. We introduce the notion of an admissible path $\sigma(t)$ in the space of parameters and a *reference* Hamiltonian $H(t, \sigma)$ at infinity corresponding to the matrix charge transfer Hamiltonian with fixed constants $\sigma_k = \sigma_k(t = \infty) = (\vec{v}_k, D_k, \gamma_k, \omega_k)$. At each time t the solution $Z(t)$ is required to be orthogonal to the traveling bound states of the charge transfer Hamiltonians $H(t, \sigma = \sigma(t))$ obtained by fixing the parameters $\sigma_k = \sigma_k(t)$ at a given time t. This leads to the so called *modulation equations* for σ_k, which couple the PDE (1.14) for Z with an ODE for the modulation parameters $\sigma_k(t)$. To impose the orthogonality condition we first need to verify that it is satisfied initially. Using standard arguments, see e.g. [BP1], one can ensure this property by modifying the soliton parameters $\sigma(0)$ slightly in the decomposition of the initial data ψ_0,

$$\psi_0(x) = \sum_{j=1}^N w_j(0, x; \sigma(0)) + R(0, x).$$

We later justify the orthogonality condition at any positive time by showing that it is propagated. To handle the nonlinear equation (1.14) we introduce the Banach spaces $\mathcal{X}_s$ and $\mathcal{Y}_s$ of functions of (t, x)

$$\begin{aligned} \|f\|_{\mathcal{X}_s} &= \sup_{t\ge 0} \Big(\|\psi(t,\cdot)\|_{H^s} + (1+t)^{\frac{n}{2}} \sum_{k=0}^s \|\nabla^k f(t,\cdot)\|_{L^2+L^\infty} \Big) \\ \|F\|_{\mathcal{Y}_s} &= \sup_{t\ge 0} \sum_{k=0}^s \Big(\int_0^t \|\nabla^k F(\tau,\cdot)\|_{L^1}\, d\tau + (1+t)^{\frac{n}{2}+1} \|\nabla^k F(t,\cdot)\|_{L^2} \Big). \end{aligned} \tag{1.15}$$

The space $\mathcal{X}_s$ is designed to control the solution $Z(t)$ itself, while the space $\mathcal{Y}_s$ takes care of the nonlinear terms appearing as inhomogeneous terms in the equation for Z. We rewrite the equation (1.14) replacing the Hamiltonian $H(t, \sigma(t))$ with the reference charge transfer Hamiltonian $H(t, \sigma)$. The solution operator of the corresponding linear problem

$$i\partial_t U + H(t, \sigma)U = F \tag{1.16}$$

maps $\mathcal{Y}_s \to \mathcal{X}_s$ for any integer s uniformly in σ, provided that the solution is asymptotically orthogonal to all traveling bound state of $H(t, \sigma)$. We then show that the nonlinearity F arising from the inhomogeneous terms in the equation Z, written relative to the reference Hamiltonian $H(t, \sigma)$, maps $\mathcal{X}_s \to \mathcal{Y}_s$

for any $s > [\frac{n}{2}] + 1$. Given the smallness asumption on the initial data for $Z = (R, \bar{R})$ this allows one to conclude the desired properties of R. The modulation equations for $\sigma(t)$ are then used in turn to control the path $\sigma(t)$. In particular, we show that there exist final values of the parameters $\sigma = \sigma(\infty)$, thus justifying the introduction of the reference Hamiltonian at infinity. The estimates for the inhomogeneous problem (1.16) phrased in terms of the mapping between $\mathcal{Y}_s$ and $\mathcal{X}_s$ are essentially the linear dispersive estimates for the time-dependent Schrödinger equation with a matrix charge transfer Hamiltonian $\mathcal{H}(t) = H(t, \sigma)$ proved in [RSS]. As it was mentioned before, these estimates only require the admissibility of the corresponding individual time-indepedent matrix Hamlitonians $\mathcal{H}_k$ (representing the linearization around each individual soliton). The admissibility conditions can be somewhat loosely divided into two categories:

1) Conditions related to linear stability: $\sup_t \|e^{it\mathcal{H}_k} f\| < \infty$

2) Spectral assumptions on $\mathcal{H}_k$ (absence of the embedded eigenvalues, resonances, etc.).

Remark 1.1. We also require the absence of "spurious" eiegnevalues. This means that we assume that all of the discrete spectrum of $\mathcal{H}_k$ is generated purely by the nonlinear eigenfunction ϕ_{ω_k}, i.e., it is described the generalized 0 eigenspace space of $\mathcal{H}_k$ which has the precise dimension $(2n+2)$ (coinciding with the dimension of the parameter space of σ_k). This is motivated by the requirement that the solution Z has to be orthogonal to all traveling bound states of $\mathcal{H}_k$ for the dispersive estimates to hold, which can only be achieved by the choice of the parameters σ_k. In general, it is believed that the states corresponding to the spurious eigenvalues decay in time but the mechanism of this decay is purely nonlinear. We do not pursue this issue here.

The key ingredient in establishing the linear stability is known to be the monotonicity condition

$$\partial_\omega \|\phi_\omega\|_{L^2} > 0. \tag{1.17}$$

The known examples of when the monotonicity condition can be verified are limited to the case of the ground states ϕ corresponding to the monomial subcritical nonlinearities

$$\beta(s^2) = s^{p-1}, \quad 1 < p < 1 + \frac{4}{n}, \qquad \forall \omega \neq 0 \tag{1.18}$$

and the nonlinearity of the mixed type (see [Sh])

$$\beta(s^2) = s^2 - s^4, \quad \omega????$$

In this paper we find a new class of nonlinearities satisfying condition (1.17). These nonlinearities lie "near" the subcritical monomials of (1.18) but vanish much faster near $s = 0$. More precisely we consider functions

$$\beta_\theta(s^2) = s^{p-1} \frac{s^{3-p}}{\theta + s^{3-p}} \tag{1.19}$$

with a constant $\theta > 0$ and prove that given a sufficiently small neighborhood U in the space of parameters ω there exists a sufficiently small value θ_0 such that for all $\theta < \theta_0$ and all $\omega \in U$ the ground state of β_θ corresponding to α satisfies the monotonicity condition.

We note that the higher rate of vanishing of $\beta(s^2)$ at $s = 0$ is important for asymptotic stability. In particular, it should be mentioned that if the power p in the monomial example is too low ($p < 1 + \frac{2}{n}$) even the scattering theory (asymptotic stability of a trivial 0 solution) fails.

We now describe the structure of the paper.

Section 2. contains the statement of the main result together wtih the definitions of some of the fundamental objects used in the proof. The latter include the definition of an ansatz $\psi = w_\sigma + R$, separation condition on the initial data ensuring that the solitons are only weakly interacting, the notion of an admissible parameter path $\sigma(t)$, and the spectral assumptions. It also contains the first discussion of the linearized Hamiltonians appearing in the later sections. We note here that traditionally the results on the asymptotic stability require the smallness assumption on the initial data in weighted Sobolev space. Our result uses instead the Sobolev space based on the intersection of $L^1 \cap L^2(\mathbb{R}^n)$, which has a distinct advantage of being translation invariant.

Section 3. gives a detailed description of the linearization of the equation (1.11) around an N-solition profile w_σ and introduces the notion of the reference charge transfer Hamiltonian at infinity.

Section 4. describes the structure of the nullspaces of the Hamiltonians $H_j(\sigma)$, associated with the linearization on each individual nonlinear eigenfunction ϕ_j.

Section 5. recalls the dispersive estimates for solutions of the time-dependent Schrödinger equation with a charge transfer Hamiltonian.

Section 6. derives a system of ODE's for the modulation parameters $\sigma(t) = (\sigma_1, \ldots, \sigma_N)$ with $\sigma_k = (\vec{v}_k, D_k, \gamma_k, \alpha_k)$ by requiring the complexified perturbation $Z = (R, \bar{R})$ to be orthogonal to the the unstable manifold comprised of the elements of the nullspaces of $H_j(\sigma(t))$.

Section 7. gives the bootstrap assumption on the size of the perturbation Z and the admissible path $\sigma(t)$ and provides estimates on the difference between the linearized Hamiltonian $H(\sigma(t))$ and the reference Hamiltonian $H(\sigma, t)$ at infinity.

Section 8. provides the solution of the modulation equations for the path $\sigma(t)$.

Section 9. solves the nonlinear equation for the complexified perturbation Z in the space $\mathcal{X}_s$ for $s > [\frac{n}{2}] + 1$. This includes algebra estimates designated to show that the $\mathcal{Y}_s$-norm of the nonlinear terms in the equation for Z can be controlled by the $\mathcal{X}_s$-norm of Z itself.

Section 10. discusses the existence for the coupled PDE-ODE system for Z and $\sigma(t)$.

Section 11. returns to the detailed discussion of the associated linear problems and proves some of the assertions made in the first part of the paper. We give the precise definition of an admissible Hamiltonian and start the investigation of their spectral properties. In particular, in Section 11.2 we describe the spectrum of an admissible Hamitonian and prove exponential decay of the elements of its generalized eigenspaces. In Section 11.3 we specialize to the admissible Hamitonians arising from linearization around a nonlinear eigenfunction ϕ. We introduce and discuss the associated self-adjoint operators L_+ and L_- and show that the admissibility conditions on our Hamiltonian (excluding the assumptions on absence of the embedded spectrum and resonances) can be reduced to the monotonicity condition (1.17), and the statement that the null space of L_- is spanned by ϕ while L_+ has a unique negative eigenvalue with the corresponding null space spanned by $\partial_{x_j}\phi$. Some of the arguments in this section follow those of [We1], [BP1].

Section 12. establishes the desired properties of the operators L_+ and L_- for a particular class of nonlinearities (namely those used in our main result) in the case when the nonlinear eigenfunction ϕ is a ground state.

For the most part of the paper (until Section 12) we do not specify the nature of nonlinear eigenstates ϕ_{ω_k} of the elliptic problem (ENLS). In particular, we do not require them to be ground states. Instead, we choose to formulate a more general conditional result dependent upon verification of certain precise properties of the linearized operators $\mathcal{H}_k$ associated with each ϕ_{ω_k}. It is only in Section 12 that we verify some of these assumptions in the case when ϕ_{ω_k} are ground states. The reason for choosing this approach is to emphasize the method which allows us to handle weak interactions of (non-colliding) solitons generated by ϕ_{ω_k}, provided that certain properties of each *individual* eigenstate ϕ_{ω_k} hold. We believe that our method will have an even wider range of applications than described here.

2 Statement of results

Consider the NLS

$$i\partial_t\psi + \frac{1}{2}\triangle\psi + \beta(|\psi|^2)\psi = 0 \tag{2.1}$$

in $\mathbb{R}^n$, $n \geq 3$, with initial data

$$\psi_0(x) = \sum_{j=1}^{N} w_j(0,x) + R_0(x). \tag{2.2}$$

Here $w_j(0,x)$ are nonlinear eigenfunctions generating the solitons

$$w_j(t,x) = w(t,x;\sigma_j(0)) \quad = \quad e^{i\theta_j(t,x)}\phi(x - x_j(t), \alpha_j(0)) \tag{2.3}$$

$$\theta_j(t,x) \quad = \quad v_j(0)\cdot x - \frac{1}{2}(|v_j(0)|^2 - \alpha_j^2(0))t + \gamma_j(0) \tag{2.4}$$

$$x_j(t) \quad = \quad v_j(0)t + D_j(0). \tag{2.5}$$

and $\phi = \phi(\cdot,\alpha)$ is a solution of

$$\frac{1}{2}\triangle\phi - \frac{\alpha^2}{2}\phi + \beta(|\phi|^2)\phi = 0. \tag{2.6}$$

The solitons w_j as in (2.3) satisfy (2.1) with arbitrary constant parameters $\sigma_j(0) = (v_j(0), D_j(0), \gamma_j(0), \alpha_j(0))$. We assume that the nonlinearity β satisfies for all integers $\ell \geq 0$

$$|\beta^{(\ell)}(s)| \lesssim s^{\left(\frac{p-1}{2}-\ell\right)_+} \quad \text{for } 0 \leq s \leq 1 \tag{2.7}$$

$$|\beta^{(\ell)}(s)| \lesssim s^{\left(\frac{q-1}{2}-\ell\right)_+} \quad \text{for } s \geq 1 \tag{2.8}$$

where $p \geq 2 + \frac{2}{n}$, $q < 1 + \frac{4}{n}$. The main results of our paper are the following theorems.

Theorem 2.1. *Let β be as in* (1.19). *Let*

$$\psi_0(x) = \sum_{j=1}^{N} w_j(0,x) + R_0(x),$$

see (2.2) *be initial data for NLS that satisfy the separation condition* (2.17). *Moreover, assume that w_j are defined in terms of ground states (positive radial solutions) of ENLS. Finally, suppose that for all $\sigma \in \mathbb{R}^{N(2n+2)}$ with $|\sigma - \sigma(0)| < c$, the linearized operators $H_j(\sigma)$ from* (2.24) *have zero as their only eigenvalue, no resonances, and no imbedded eigenvalues in their continuous spectrum. Then there exists a positive ϵ such that for R_0 satisfying the smallness assumption*

$$\sum_{k=0}^{s} \|\nabla^k R_0\|_{L^1 \cap L^2} < \epsilon \tag{2.9}$$

for some integer $s > \frac{n}{2}$, there exists an admissible path $\sigma(t)$ with the limiting value σ^∞ such that

$$\Big\|\psi(t,x) - \sum_{j=1}^{N} w_j(t,x;\sigma_j(t))\Big\|_{L^\infty_x} \lesssim (1+t)^{-\frac{n}{2}}$$

as $t \to \infty$. Moreover, there exists $u_0 \in L^2$ so that

$$\Big\|\psi(t,\cdot) - \sum_{j=1}^{N} w_j(t,x;\sigma^\infty) - e^{i\frac{t}{2}\triangle} u_0\Big\|_{L^2} \to 0$$

as $t \to \infty$.

Our methods also allow treating initial conditions that are defined in terms of nonlinear eigenfunctions which are not ground states as well as more general nonlinearities β. However, in contrast to the case of a ground state and the nonlinearities (1.19), in this case we cannot verify the convexity conditions as well as various conditions related to the spectrum of the linearized operators $H_j(\sigma)$. Therefore, we need to include them into the hypotheses.

Theorem 2.2. *Let β be as in* (2.7) *and* (2.8). *Impose the separation and convexity condition see* (2.17) *and* (2.16), *the spectral assumption from Definition 2.3. Suppose ψ is the solution of* (2.1) *with initial condition* (2.2) *where the w_j are generated by nonlinear eigenfunctions of ENLS (without assuming that they are ground states). Then there exists a positive ϵ such that for R_0 satisfying the smallness assumption* (2.9) *the conclusion of the previous theorem holds.*

In the remainder of this section we shall discuss the theorem and its assumptions in more detail.

The soliton profiles $w_j(t,x;\sigma(t))$: In addition to the one-soliton solutions $w_j(t,x;\sigma_j)$ with constant σ_j introduced above, we need functions

$$w_j(t,x;\sigma_j(t)) = e^{i\theta_j(t,x;\sigma(t))}\phi(x - x_j(t;\sigma(t)), \alpha_j(t)). \tag{2.10}$$

The phase $\theta_j(t, x; \sigma(t))$ and the path $x_j(t; \sigma(t))$ are defined in terms of the time-dependent parameters $\sigma_j(t) = (v_j(t), D_j(t), \gamma_j(t), \alpha_j(t))$ as follows:

$$\theta_j(t, x; \sigma(t)) = v_j(t) \cdot x - \int_0^t \frac{1}{2}(|v_j|^2 - \alpha_j^2)(s)\, ds + \gamma_j(t) \tag{2.11}$$

$$x_j(t; \sigma(t)) = \int_0^t v_j(s)\, ds + D_j(t). \tag{2.12}$$

Henceforth, the functions $w_j(t, x; \sigma(t))$ or simply $w_j(\sigma(t))$, correspond to the soliton moving along the time-dependent curve $\sigma(t)$ in the parameter space according to (2.11) and (2.12), while $w_j(\sigma)$ is the true soliton moving along the straight line determined by an arbitrary constant σ as in (2.4) and (2.5).

Admissible paths $\sigma(t)$: We collect the individual parameter curves $\sigma_j(t)$ from above into a single curve $\sigma(t) := (\sigma_1(t), \ldots, \sigma_N(t)) \subset \mathbb{R}^{(2n+2)N}$. Given the initial value $\sigma(0)$ we introduce the set of *admissible curves* $\sigma(t)$ as those C^1 curves that remain in a small neighborhood of $\sigma(0)$ for all times and converge to their final value $\sigma(\infty) = \lim_{t\to+\infty} \sigma(t)$. We shall also impose the condition that for an admissible curve $\sigma(t)$

$$\int_0^\infty \int_s^\infty |\dot{v}_j(\tau) \cdot v_j(\tau) - \dot{\alpha}(\tau)_j \alpha_j(\tau)|\, d\tau\, ds < \infty, \qquad \int_0^\infty \int_s^\infty |\dot{v}_j(\tau)|\, d\tau\, ds < \infty \tag{2.13}$$

for all $1 \le j \le N$. Given an admissible curve $\sigma(t)$ we define the constant vector σ^∞ in the following fashion:

$$v_j^\infty = v_j(\infty), \qquad D_j^\infty = D_j(\infty) - \int_0^\infty \int_s^\infty \dot{v}_j(\tau)\, d\tau\, ds, \tag{2.14}$$

$$\gamma_j^\infty = \gamma_j(\infty) + \int_0^\infty \int_s^\infty (\dot{v}_j(\tau) \cdot v_j(\tau) - \dot{\alpha}_j(\tau)\alpha_j(\tau))\, d\tau\, ds, \qquad \alpha_j^\infty = \alpha_j(\infty). \tag{2.15}$$

Convexity condition: We impose the *convexity condition*

$$\langle \partial_\alpha \phi(\cdot; \alpha), \phi(\cdot; \alpha) \rangle > 0, \qquad \forall \alpha : \min_{j=1,..,N} |\alpha - \alpha_j(0)| < c \tag{2.16}$$

for some positive constant c. As noted in the Introduction, the convexity condition is closely connected with the issue of orbital stability of the individual solitons $w_j(t, x; \sigma_j(0))$.

Separation conditions: Our theorem handles the case of so-called weakly interacting solitons. This means that the initial positions $D_j(0)$ and initial velocities $v_j(0)$ are such that for all $t \ge 0$ one has the (physical) *separation condition*

$$|D_j(0) + v_j(0)t - D_\ell(0) - v_\ell(0)t| \ge L + ct, \qquad \forall j \ne \ell = 1, \ldots, N \tag{2.17}$$

with some sufficiently large constant L and a positive constant c. Another assumption under which our theorems hold equally well is the condition of *large relative velocities* of the solitons. This means that

$$\min_{j\ne\ell} |v_j - v_\ell| > L \tag{2.18}$$

for some large L.

Let $\alpha_{\min} = \min_{1\le j\le k} \alpha_j(0) - c$. It will be understood henceforth that

$$\alpha_{\min} L \ge |\log \epsilon|. \tag{2.19}$$

The small constant ϵ appears in our theorem as a measure of smallness of the initial perturbation R_0.

Spectral assumptions: We will write $w_j(\sigma(t)) = w_j(t,x;\sigma(t)) = e^{i\theta_j(\sigma(t))}\phi_j(\sigma(t))$ where $\phi_j(\sigma(t)) = \phi(t,x;\sigma_j(t)) = \phi(x - x_j(t);\alpha_j(t))$. Linearizing the equation (2.1) around the state $w = \sum_{j=1}^N w_j,\ \psi = w + R$ one obtains the following system of equations for $Z = \binom{R}{\bar{R}}$:

$$i\partial_t Z + H(t,\sigma(t))Z \quad = \quad F. \tag{2.20}$$

Here $H(t,\sigma(t))$ is the time-dependent matrix Hamiltonian

$$H(t,\sigma(t)) = H_0 + \tag{2.21}$$
$$\sum_{j=1}^N \begin{pmatrix} \beta(|w_j(\sigma(t))|^2) + \beta'(|w_j(\sigma(t))|^2)|w_j(\sigma(t))|^2 & \beta'(|w_j(\sigma(t))|^2)w_j^2(\sigma(t)) \\ -\beta'(|w_j(\sigma(t))|^2)\bar{w}_j^2(\sigma(t)) & -\beta(|w_j(\sigma(t))|^2) - \beta'(|w_j(\sigma(t))|^2)|w_j(\sigma(t))|^2 \end{pmatrix}$$

$$= H_0 + \sum_{j=1}^N V_j(t,x;\sigma(t)), \tag{2.22}$$

$$H_0 = \begin{pmatrix} \frac{1}{2}\triangle & 0 \\ 0 & -\frac{1}{2}\triangle \end{pmatrix}$$

with complex matrix time-dependent potentials $V_j(t,x;\sigma(t))$ dependent on w_j and $\sigma(t)$. The right-hand side F in (2.20) depends on $\dot{\sigma}$, w, and nonlinearly on Z. For a given constant parameter vector σ we shall introduce the Hamiltonian $H(t,\sigma)$

$$H(t,\sigma) = H_0 + \tag{2.23}$$
$$\sum_{j=1}^N \begin{pmatrix} \beta(|w_j(\sigma)|^2) + \beta'(|w_j(\sigma)|^2)|w_j(\sigma)|^2 & \beta'(|w_j(\sigma)|^2)w_j^2(\sigma) \\ -\beta'(|w_j(\sigma)|^2)\bar{w}_j^2(\sigma) & -\beta(|w_j(\sigma)|^2) - \beta'(|w_j(\sigma)|^2)|w_j(\sigma)|^2 \end{pmatrix}.$$

We refer to Hamiltonians of the form (2.23) as *matrix charge transfer* Hamiltonians. They are discussed in more detail in Section 11, as well as in [RSS]. Recall that $w_j(\sigma)$ denotes the soliton moving along the straight line determined by the constant parameters σ_j. The proof of our theorem relies on dispersive estimates for matrix charge transfer Hamiltonians that were obtained in [RSS], see also Section 11 below. For these estimates to hold, one needs to impose certain *spectral conditions* on the stationary Hamiltonians

$$H_j(\sigma) := \begin{pmatrix} \frac{1}{2}\triangle - \frac{\alpha^2}{2} + \beta(\phi_j(\sigma)^2) + \beta'(\phi_j(\sigma)^2)\phi_j(\sigma)^2 & \beta'(\phi_j(\sigma)^2)\phi_j^2(\sigma) \\ -\beta'(\phi_j(\sigma)^2)\phi_j^2(\sigma) & -\frac{1}{2}\triangle + \frac{\alpha^2}{2} - \beta(\phi_j(\sigma)^2) - \beta'(\phi_j(\sigma)^2)\phi_j(\sigma)^2 \end{pmatrix} \tag{2.24}$$

where $\phi_j(\sigma) = \phi(x,\alpha_j)$, see (2.6). These Hamiltonians arise from the matrix charge transfer problem by applying a Galilei transform to the j^{th} matrix potential in (2.23) so that this potential becomes stationary (strictly speaking, this also requires a modulation which leads to the spectral shift $\frac{\alpha^2}{2}$ in (2.24)). We impose the spectral assumption as described by the following definition.

Definition 2.3. *We say that the* spectral assumption *holds, provided for all* $\sigma \in \mathbb{R}^{N(2n+2)}$ *with* $|\sigma - \sigma(0)| < c$ *one has*

- 0 *is the only point of the discrete spectrum of* $H_j(\sigma)$ *and the dimension of the corresponding root space is* $2n+2$,
- *each of the* $H_j(\sigma)$ *is admissible in the sense of Definition 11.1 below and the stability condition*

$$\sup_t \|e^{itH_j(\sigma)} P_s\|_{2\to 2} < \infty,$$

  see (11.3)*, holds (here* P_s *is the projection onto the scattering states associated with* H_j*, see* (11.2)*).*

While the second condition is known to hold generically in an appropriate sense, see Section 11, the first condition is more restrictive and not believed to hold generically.

3 Reduction to the matrix charge transfer model

For the sake of simplicity we consider the case of two solitons, i.e., $N = 2$. Setting

$$w_1(\sigma(t)) + w_2(\sigma(t)) = w, \qquad \psi = w + R, \tag{3.1}$$

where w_j are as in (2.10), (2.11), and (2.12), we derive from (2.1) that

$$\begin{aligned} & i\partial_t R + \frac{1}{2}\triangle R + (\beta(|w|^2) + \beta'(|w|^2)|w|^2)R + \beta'(|w|^2)w^2\,\bar{R} \\ & = -(i\partial_t w + \frac{1}{2}\triangle w + \beta(|w|^2)w) + O(|w|^{p-2}|R|^2) + O(|R|^p), \end{aligned} \tag{3.2}$$

using the assumptions (2.7), (2.8) on the nonlinearity β. Observe that

$$\begin{aligned} i\partial_t w + \frac{1}{2}\triangle w + \beta(|w|^2)w \quad &= \quad -\sum_{j=1}^{2} \Big[\big(\dot{v}_j(t)\cdot x + \dot{\gamma}_j(t)\big) w_j(\sigma(t)) + ie^{i\theta_j(\sigma(t))}\nabla\phi_j(\sigma(t))\cdot \dot{D}_j(t) \\ & \qquad -ie^{i\theta_j(\sigma(t))}\partial_\alpha\phi_j(\sigma(t))\dot{\alpha}_j(t)\Big] + O(w_1 w_2). \end{aligned} \tag{3.3}$$

In view of (3.1) one has

$$\begin{aligned} & i\partial_t R + \frac{1}{2}\triangle R + (\beta(|w|^2) + \beta'(|w|^2)|w|^2)R + \beta'(|w|^2)w^2\,\bar{R} \\ = \;& i\partial_t R + \frac{1}{2}\triangle R + \sum_{j=1}^{2}\Big[\beta(|w_j(\sigma(t))|^2) + \beta'(|w_j(\sigma(t))|^2)|w_j(\sigma(t))|^2\Big]R + \sum_{j=1}^{2}\beta'(|w_j(\sigma(t))|^2)w_j(\sigma(t))^2\,\bar{R} \\ & + O(w_1(\sigma(t))w_2(\sigma(t)))R. \end{aligned}$$

Rewriting the equation (3.2) as a system for $Z = (R, \bar{R})$ therefore leads to

$$i\partial_t Z + H(\sigma(t))Z \;=\; \dot{\Sigma}W(\sigma(t)) + O(w_1w_2)Z + O(w_1w_2) + O(|w|^{p-2}|Z|^2) + O(|Z|^p). \tag{3.4}$$

Here $H(\sigma(t))$ is the time-dependent matrix Hamiltonian from (2.21) and

$$\dot{\Sigma} W(\sigma(t)) = \begin{pmatrix} f \\ -\bar{f} \end{pmatrix} \tag{3.5}$$

where

$$f = \sum_{j=1}^{2} \Big[(\dot{v}_j(t) \cdot x + \dot{\gamma}_j(t)) w_j(\sigma(t)) + i e^{i\theta_j(\sigma(t))} \nabla \phi_j(\sigma(t)) \cdot \dot{D}_j(t) - i e^{i\theta_j(\sigma(t))} \partial_\alpha \phi_j(\sigma(t)) \dot{\alpha}_j(t) \Big] \tag{3.6}$$

We shall assume that $\sigma(t)$ is an admissible path with initial values $\sigma(0)$ in the sense of (2.13). Given an admissible curve $\sigma(t)$ we introduce the *reference Hamiltonian* $H(t, \sigma^\infty)$ "at infinity"

$$\begin{aligned} &H(t, \sigma^\infty) = H_0 + \\ &\sum_{j=1}^{2} \begin{pmatrix} \beta(|w_j(\sigma^\infty)|^2) + \beta'(|w_j(\sigma^\infty)|^2)|w_j(\sigma^\infty)|^2 & \beta'(|w_j(\sigma^\infty)|^2) w_j^2(\sigma^\infty) \\ -\beta'(|w_j(\sigma^\infty)|^2) \bar{w}_j^2(\sigma^\infty) & -\beta(|w_j(\sigma^\infty)|^2) - \beta'(|w_j(\sigma^\infty)|^2)|w_j(\sigma^\infty)|^2 \end{pmatrix} \end{aligned} \tag{3.7}$$

where $\sigma^\infty = (\sigma_1^\infty, \ldots, \sigma_N^\infty)$, $\sigma_j^\infty = (v_j^\infty, D_j^\infty, \gamma_j^\infty, \alpha_j^\infty)$ is the constant vector determined by the curve $\sigma(t)$ as in (2.14) and (2.15):

$$\begin{aligned} v_j^\infty &= v_j(\infty), \qquad D_j^\infty = D_j(\infty) - \int_0^\infty \int_s^\infty \dot{v}_j(\tau)\, d\tau\, ds, \\ \gamma_j^\infty &= \gamma_j(\infty) + \int_0^\infty \int_s^\infty (\dot{v}_j(\tau) \cdot v_j(\tau) - \dot{\alpha}_j(\tau) \alpha_j(\tau))\, d\tau\, ds, \qquad \alpha_j^\infty = \alpha_j(\infty). \end{aligned}$$

Recall that $w_j(\sigma^\infty)$ is the soliton moving along the straight line determined by the constant parameters σ_j^∞. For $j = 1, \ldots, N$ we introduce the Hamiltonians

$$H_j(t, \sigma^\infty) = H_0 + \begin{pmatrix} \beta(|w_j(\sigma^\infty)|^2) + \beta'(|w_j(\sigma^\infty)|^2)|w_j(\sigma^\infty)|^2 & \beta'(|w_j(\sigma^\infty)|^2) w_j^2(\sigma^\infty) \\ -\beta'(|w_j(\sigma^\infty)|^2) \bar{w}_j^2(\sigma^\infty) & -\beta(|w_j(\sigma^\infty)|^2) - \beta'(|w_j(\sigma^\infty)|^2)|w_j(\sigma^\infty)|^2 \end{pmatrix} \tag{3.8}$$

together with their stationary counterparts $H_j(\sigma^\infty)$ as in (2.24) with $\sigma = \sigma^\infty$.

The following lemma relates the evolutions corresponding to the Hamiltonians $H_j(t, \sigma)$ and $H_j(\sigma)$ for an arbitrary σ by means of a modified Gallilean transformation.

Lemma 3.1. *Let $U_j(t, \sigma)$ be the solution operator of the equation*

$$\begin{aligned} &i \partial_t U_j(t, \sigma) + H_j(t, \sigma) U_j(t, \sigma) = 0, \\ &U_j(0, \sigma) = I \end{aligned} \tag{3.9}$$

and $e^{itH_j(\sigma)}$ be the corresponding propagator for the time-independent matrix Hamiltonian $H_j(\sigma)$. Then

$$U_j(t, \sigma) = \mathcal{G}^*_{v_j, D_j}(t) \mathcal{M}_j^*(t, \sigma) e^{itH_j(\sigma)} \mathcal{M}_j(0, \sigma) \mathcal{G}_{v_j, D_j}(0) \tag{3.10}$$

where $\mathcal{G}_{v_j, D_j}(t)$ is the diagonal matrix Galilean transformation

$$\mathcal{G}_{v_j, D_j}(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{g}_{v_j, D_j}(t) f_1 \\ \overline{\mathfrak{g}_{v_j, D_j}(t) \bar{f}_2} \end{pmatrix} \tag{3.11}$$

and

$$\mathcal{M}_j(t,\sigma) = \begin{pmatrix} e^{-i\frac{\alpha_j^2}{2}t - i(v_j\cdot D_j+\gamma_j)} & 0 \\ 0 & e^{i\frac{\alpha_j^2}{2}t+i(v_j\cdot D_j+\gamma_j)} \end{pmatrix}. \tag{3.12}$$

Proof. By definition,

$$\begin{aligned} i\dot{U}_j &= i\dot{\mathcal{G}}^*_{v_j,D_j}(t)\mathcal{M}_j^*(t)e^{itH_j(\sigma)}\mathcal{M}_j(0)\mathcal{G}_{v_j,D_j}(0) + \mathcal{G}^*_{v_j,D_j}(t)i\dot{\mathcal{M}}_j^*(t)e^{itH_j(\sigma)}\mathcal{M}_j(0)\mathcal{G}_{v_j,D_j}(0) \\ &\quad -\mathcal{G}^*_{v_j,D_j}(t)\mathcal{M}_j^*(t)H_j(\sigma)e^{itH_j(\sigma)}\mathcal{M}_j(0)\mathcal{G}_{v_j,D_j}(0). \end{aligned} \tag{3.13}$$

Clearly,

$$i\dot{\mathcal{M}}_j^*(t) = \begin{pmatrix} -\frac{\alpha_j^2}{2}\, e^{i\frac{\alpha_j^2}{2}t+i(v_j\cdot D_j+\gamma_j)} & 0 \\ 0 & \frac{\alpha_j^2}{2}\, e^{-i\frac{\alpha_j^2}{2}t-i(v_j\cdot D_j+\gamma_j)} \end{pmatrix},$$

whereas one checks that

$$i\dot{\mathcal{G}}^*_{v_j,D_j}(t)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = i\begin{pmatrix} \dot{\mathfrak{g}}^*_{v_j,D_j} f_1 \\ \overline{\dot{\mathfrak{g}}^*_{v_j,D_j}\bar{f}_2} \end{pmatrix} = \begin{pmatrix} -(v_j^2/2 - v_j\cdot p)\, \mathfrak{g}^*_{v_j,D_j} f_1 \\ (v_j^2/2 + v_j\cdot p)\, \overline{\mathfrak{g}^*_{v_j,D_j}\bar{f}_2} \end{pmatrix}.$$

Finally, we need to move $H_j(\sigma)$ to the left in (3.13). We consider the differential operator separately from the matrix potential, i.e.,

$$\begin{aligned} H_j(t,\sigma) &= H_0 + \begin{pmatrix} U_j(x-x_j(t)) & e^{2i\theta_j(t,x)}W_j(x-x_j(t)) \\ -e^{-2i\theta_j(t,x)}W_j(x-x_j(t)) & -U_j(x-x_j(t)) \end{pmatrix} \\ H_j(\sigma) &= H_0 + \begin{pmatrix} -\frac{\alpha_j^2}{2} & 0 \\ 0 & \frac{\alpha_j^2}{2} \end{pmatrix} + \begin{pmatrix} U_j & W_j \\ -W_j & -U_j \end{pmatrix} \end{aligned} \tag{3.14}$$

where $x_j(t), \theta_j(t)$ are as in (2.5), (2.4), and $U_j = \beta(\phi_j(\sigma)^2) + \beta'(\phi_j(\sigma)^2)\phi_j(\sigma)^2$, $W_j = \beta'(\phi_j(\sigma)^2)\phi_j^2(\sigma)$. Note that, on the one hand, $\mathcal{M}_j$ commutes with all matrices in (3.14) that do not involve U_j, W_j. On the other hand, one has

$$\begin{aligned} &H_0\mathcal{G}^*_{v_j,D_j}(t)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \mathcal{G}^*_{v_j,D_j}(t)\left[H_0 + \begin{pmatrix} -\frac{\alpha_j^2}{2} & 0 \\ 0 & \frac{\alpha_j^2}{2} \end{pmatrix}\right]\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha_j^2\, \mathfrak{g}^*_{v_j,D_j}(t) f_1 \\ -\frac{1}{2}\alpha_j^2\, \overline{\mathfrak{g}^*_{v_j,D_j}(t)\bar{f}_2} \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} e^{i\frac{v_j^2}{2}t} \triangle\Big(e^{ix\cdot v_j}\, e^{-iv_j\cdot(D_j+tv_j)}\, f_1(x-tv_j-D_j)\Big) - e^{i\frac{v_j^2}{2}t}\, e^{ix\cdot v_j}\, e^{-iv_j\cdot(D_j+tv_j)} \triangle f_1(x-tv_j-D_j) \\ -e^{-i\frac{v_j^2}{2}t} \triangle\Big(e^{-ix\cdot v_j}\, e^{iv_j\cdot(D_j+tv_j)}\, f_2(x-tv_j-D_j)\Big) + e^{-i\frac{v_j^2}{2}t}\, e^{-ix\cdot v_j}\, e^{iv_j\cdot(D_j+tv_j)} \triangle f_2(x-tv_j-D_j) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(v_j^2+\alpha_j^2)\, \mathfrak{g}^*_{v_j,D_j}(t)f_1 - v_j\cdot p\, \mathfrak{g}^*_{v_j,D_j}(t)f_1 \\ -\frac{1}{2}(v_j^2+\alpha_j^2)\, \overline{\mathfrak{g}^*_{v_j,D_j}(t)\bar{f}_2} - v_j\cdot p\, \overline{\mathfrak{g}^*_{v_j,D_j}(t)\bar{f}_2} \end{pmatrix}. \end{aligned}$$

Finally, we need to deal with the matrix potentials. Write $\mathcal{M}(t) := \mathcal{M}_j(t,\sigma) = \begin{pmatrix} e^{-i\omega(t)/2} & 0 \\ 0 & e^{i\omega(t)/2} \end{pmatrix}$

and set $\rho = t|\vec{v}_j|^2 + 2x\cdot\vec{v}_j$. Then (omitting the index j for simplicity)

$$
\begin{aligned}
&\mathcal{M}(t)\mathcal{G}_{\vec{v},D}(t)\begin{pmatrix} U(\cdot-\vec{v}t-D) & e^{2i\theta}W(\cdot-\vec{v}t-D) \\ -e^{-2i\theta}W(\cdot-\vec{v}t-D) & -U(\cdot-\vec{v}t-D)\end{pmatrix}\begin{pmatrix} f_1 \\ f_2\end{pmatrix} \\
=\ &\begin{pmatrix} e^{-i\omega(t)/2} & 0 \\ 0 & e^{i\omega(t)/2}\end{pmatrix}\begin{pmatrix} \mathfrak{g}_{\vec{v},D}(t)U(\cdot-\vec{v}t-D)f_1 + \mathfrak{g}_{\vec{v},D}(t)e^{2i\theta}W(\cdot-\vec{v}t-D)f_2 \\ \overline{-\mathfrak{g}_{\vec{v},D}(t)e^{2i\theta}W(\cdot-\vec{v}t-D)\overline{f_1} - \mathfrak{g}_{\vec{v},D}(t)U(\cdot-\vec{v}t-D)\overline{f_2}}\end{pmatrix} \\
=\ &\begin{pmatrix} U\mathfrak{g}_{\vec{v},D}(t)(e^{-i\omega(t)/2}f_1) + We^{-i(v^2t+2x\cdot\vec{v})}e^{i(2\theta(t,\cdot+t\vec{v}+D)-\omega)}\overline{\mathfrak{g}_{\vec{v},D}(t)\overline{e^{i\omega(t)/2}f_2}} \\ -We^{i(v^2t+2x\cdot\vec{v})}e^{i(\omega-2\theta(t,\cdot+t\vec{v}+D))}\ \mathfrak{g}_{\vec{v},D}(t)(e^{-i\omega(t)/2}f_1) - U\ \overline{\mathfrak{g}_{\vec{v},D}(t)\overline{e^{i\omega(t)/2}f_2}}\end{pmatrix} \\
=\ &\begin{pmatrix} U & e^{i(2\theta(t,\cdot+t\vec{v}+D)-\omega-\rho)}W \\ -e^{-i(2\theta(t,\cdot+t\vec{v}+D)-\omega-\rho)}W & -U\end{pmatrix}\begin{pmatrix} e^{-i\omega(t)/2} & 0 \\ 0 & e^{i\omega(t)/2}\end{pmatrix}\begin{pmatrix} \mathfrak{g}_{\vec{v}}(t)f_1 \\ \overline{\mathfrak{g}_{\vec{v},D}(t)\overline{f_2}}\end{pmatrix}.
\end{aligned}
$$

Now $2\theta(t,\cdot+t\vec{v}+D)-\rho-\omega = 2\vec{v}\cdot x + (|\vec{v}\,|^2+\alpha^2)t + 2\gamma + 2\vec{v}\cdot D - t|\vec{v}\,|^2 - 2x\cdot\vec{v} - \omega = 0$ by definition of ω, i.e., $\omega = \alpha^2 t + 2\gamma + 2\vec{v}\cdot D$. Adding these expressions shows that

$$i\dot{U}_j(t,\sigma) + H_j(t,\sigma)U_j(t,\sigma) = 0,$$

as claimed. □

4 The root spaces of $H_j(\sigma)$ and $H_j^*(\sigma)$

In view of Section 11 below (see in particular Definition 11.1 as well as (11.2)) we will need to understand the generalized eigenspaces of the stationary operators $H_j(\sigma)$ from (13.1). By our spectral assumption, see Definition 2.3 above, only generalized eigenspaces at 0 are allowed. We denote these spaces by $\mathcal{N}_j(\sigma)$ and refer to them as root spaces. Thus, $\mathcal{N}_j(\sigma) = \ker\big(H_j(\sigma)^2\big)$ and by (11.2) one has the direct (but not orthogonal) decomposition

$$L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3) = \mathcal{N}_j^*(\sigma)^\perp + \mathcal{N}_j(\sigma),$$

where $\mathcal{N}_j^*(\sigma) = \ker\big(H_j^*(\sigma)^2\big)$. The (nonorthogonal) projection onto $\mathcal{N}_j^*(\sigma)^\perp$ associated with this decomposition is denoted by $P_j(\sigma)$. While the evolution $e^{itH_j(\sigma)}$ is *unbounded* on L^2 as $t\to\infty$, it is known in many cases that it remains bounded on $\mathrm{Ran}(P_j(\sigma))$. In Section 11 this is referred to as the *linear stability assumption.*

Proposition 4.1. *Impose the hypotheses of Theorem 2.1 and let $H_j(\sigma)$ be as in (2.24). Then*

- *The nullspace $\mathcal{N}_j^*(\sigma)$ of $H_j^*(\sigma)$ is given by the following vector valued $2n+2$ functions $\xi_j^m(x;\sigma)$, $m = 1,\ldots,2n+2$:*

$$
\begin{aligned}
&\xi_j^m(x;\sigma) = \begin{pmatrix} u_j^m(x;\sigma) \\ \bar{u}_j^m(x;\sigma)\end{pmatrix}, \\
&u_j^1(x;\sigma) = \phi_j(x;\sigma), && H_j^*(\sigma)\xi_j^1(\cdot;\sigma) = 0, \\
&u_j^2(x;\sigma) = i\frac{2}{\alpha_j}\partial_\alpha\phi_j(x;\sigma), && H_j^*(\sigma)\xi_j^2(\cdot;\sigma) = -i\xi_j^1(\cdot;\sigma), \\
&u_j^m(x;\sigma) = i\partial_{x_{m-2}}\phi_j(x;\sigma), && H_j^*(\sigma)\xi_j^m(\cdot;\sigma) = 0, \quad m = 3,..,n+2, \\
&u_j^m(x;\sigma) = x_{m-n-2}\phi_j(x;\sigma), && H_j^*(\sigma)\xi_j^m(\cdot;\sigma) = -2i\xi_j^{m-n}(\cdot;\sigma), \quad m = n+3,..,2n+2
\end{aligned}
$$

- *Let*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then J is an isomorphism between the nullspaces of $H_j^*(\sigma)$ and $H_j(\sigma)$. In particular, the nullspace of $H_j(\sigma)$ has a basis $\{J\xi_j^m(\cdot;\sigma) \mid 1 \le m \le 2n+2\}$. Moreover, $\mathcal{N}_j^*(\sigma)$ is spanned by

$$J\,\partial_{\sigma_r} W_j(t,x;\sigma) \quad \textit{for} \quad 1 \le r \le 2n+2,$$

where $W_j(t,x;\sigma) = \binom{w_j(t,x;\sigma)}{\bar{w}_j(t,x;\sigma)}$.

- *One has the linear stability property*

$$\sup_t \left\| e^{itH_j(\sigma)} P_j(\sigma) \right\|_{2\to 2} < \infty$$

where $P_j(\sigma)$ is the projection onto $\mathcal{N}_j^*(\sigma)^\perp$ as introduced above.

Proof. □

For the case of monomial, subcritical nonlinearities these results go back to Weinstein's work on modulational stability [We1].

5 Estimates for the linearized problem

In (3.4) we obtained the system

(5.1)
$$i\partial_t Z + H(t,\sigma^\infty)Z = \big(H(\sigma(t)) - H(t,\sigma^\infty)\big)Z + \dot{\Sigma} W(\sigma(t)) + O(w_1 w_2)Z + O(w_1 w_2) + O(|w|^{p-2}|Z|^2) + O(|Z|^p),$$

The point of rewriting (3.4) in this form is to be able to use the dispersive estimates that were obtained in [RSS] for (perturbed) matrix charge transfer Hamiltonians, see also Sections 11 and 12.

Theorem 5.1. *Let $Z(t,x)$ solve the equation*

$$\begin{aligned} i\partial_t Z + H(t,\sigma)Z &= F, \\ Z(0,\cdot) &= Z_0(\cdot) \end{aligned} \tag{5.2}$$

where the matrix charge transfer Hamiltonian $H(t,\sigma)$ satisfies the conditions of Definition 11.2. Assume that Z satisfies

$$\|(\mathrm{Id} - P_j(\sigma))\mathcal{M}_j(\sigma,t)\mathcal{G}_{v_j,D_j}(t)Z(t,\cdot)\|_{L^2} \le B(1+t)^{-\frac{n}{2}}, \quad \forall j = 1,\ldots,k, \tag{5.3}$$

with some positive constant B, where $\mathcal{M}_j(\sigma,t)$ and $\mathcal{G}_{v_j,D_j}(t)$ are as in Lemma 3.1. Then Z verifies the following decay estimate

$$\|Z(t)\|_{L^2+L^\infty} \lesssim (1+t)^{-\frac{n}{2}} \Big(\|Z_0\|_{L^1\cap L^2} + \|F\| + B \Big) \tag{5.4}$$

for $t > 0$ with

$$\|\!|F|\!\|:=\sup_{t\geq 0}\Big[\int_0^t \|F(s)\|_{L^1}\,ds+(1+t)^{\frac{n}{2}+1}\|F(t)\|_{L^2}\Big].$$

In addition, we also have the L^2 estimate

$$\|Z(t)\|_{L^2}\lesssim \|Z_0\|_{L^1\cap L^2}+\|\!|F|\!\|+B \tag{5.6}$$

For the proof see [RSS] and Section 11 below. In particular, note that (5.3) is related to the characterization of scattering states in Definition 11.4.

In the applications the inhomogeneous term F is a nonlinear expression which depends on Z. Therefore, in addition to the estimates (5.4) and (5.6) we shall need corresponding estimates for the derivatives of Z.

For an integer $s \geq 0$ we define Banach spaces $\mathcal{X}_s$ and $\mathcal{Y}_s$ of functions of (t,x)

$$\|\psi\|_{\mathcal{X}_s} = \sup_{t\geq 0}\Big(\|\psi(t,\cdot)\|_{H^s}+(1+t)^{\frac{n}{2}}\sum_{k=0}^{s}\|\nabla^k\psi(t,\cdot)\|_{L^2+L^\infty}\Big) \tag{5.7}$$

$$\|F\|_{\mathcal{Y}_s} = \sup_{t\geq 0}\sum_{k=0}^{s}\Big(\int_0^t \|\nabla^k F(\tau,\cdot)\|_{L^1}\,d\tau+(1+t)^{\frac{n}{2}+1}\|\nabla^k F(t,\cdot)\|_{L^2}\Big) \tag{5.8}$$

The generalization of the estimates of Theorem 5.1 is given by the following theorem (see section 12, in particular Proposition 12.3, for the proof).

Theorem 5.2. *Under assumptions of Theorem 5.1 we have that for any integer $s \geq 0$*

$$\|Z\|_{\mathcal{X}_s}\lesssim \sum_{k=0}^{s}\|\nabla^k Z(0,\cdot)\|_{L^1\cap L^2}+\|F\|_{\mathcal{Y}_s}+B \tag{5.9}$$

We apply Theorem 5.2 to the equation (5.1). This will, in particular, lead to our main result, i.e., that $\|Z(t)\|_\infty \lesssim t^{-\frac{n}{2}}$ as $t\to\infty$. We need to ensure that Z is a scattering solution relative to each of the channels of the charge transfer Hamiltonian $H(t,\sigma)$, in the sense of the estimate (5.3). Analogous to Buslaev, Perelman [BP1] this will be accomplished by an appropriate choice of the path $\sigma(t)$, to be made in the following section.

In order to prove existence of solutions Z and σ we require another version of Theorem 5.2, which follows easily from that result.

Remark 5.3. It is easy to see that a time-localized version of the previous theorem also holds. Indeed, let

$$\|\psi\|_{\mathcal{X}_s(T)} = \sup_{0\leq t\leq T}\Big(\|\psi(t,\cdot)\|_{H^s}+(1+t)^{\frac{n}{2}}\sum_{k=0}^{s}\|\nabla^k\psi(t,\cdot)\|_{L^2+L^\infty}\Big) \tag{5.10}$$

$$\|F\|_{\mathcal{Y}_s(T)} = \sup_{0\leq t\leq T}\sum_{k=0}^{s}\Big(\int_0^t \|\nabla^k F(\tau,\cdot)\|_{L^1}\,d\tau+(1+t)^{\frac{n}{2}+1}\|\nabla^k F(t,\cdot)\|_{L^2}\Big). \tag{5.11}$$

Then assuming (5.3) for $0 \le t \le T$ with a constant B_T, one has

$$\|Z\|_{\mathcal{X}_s(T)} \lesssim \sum_{k=0}^{s} \|\nabla^k Z(0,\cdot)\|_{L^1\cap L^2} + \|F\|_{\mathcal{Y}_s(T)} + B_T \tag{5.12}$$

Corollary 5.4. *Let $\tilde{\xi}_j^m(t,x)$, $1 \le m \le 2n+2, 1 \le j \le N$, be a collection of smooth functions such that*

$$\sup_{t\ge 0} \|\mathcal{M}_j(\sigma,t)\mathcal{G}_{v_j,D_j}(t)\tilde{\xi}_j^m(t,x) - \xi_j^m(\cdot;\sigma)\|_{L^1\cap L^2} \le \delta \tag{5.13}$$

for some small $\delta > 0$ and some given σ. Let Z be a solution of

$$i\partial_t Z + H(t,\sigma)Z = V(t,x)Z + F, \tag{5.14}$$

$$\langle Z(t), \tilde{\xi}_j^m(t,\cdot)\rangle = 0 \tag{5.15}$$

for all $t \ge 0$, where $V(t,x)$ is a smooth function that satisfies $\sup_{|\gamma|\le s} \|\partial_x^\gamma V(t,\cdot)\|_{L^1\cap L^\infty} < \delta(1+t)^{-1}$, with a nonnegative integer s for all $t > 0$. Then

$$\|Z\|_{\mathcal{X}_s} \lesssim \sum_{k=0}^{s} \|\nabla^k Z(0,\cdot)\|_{L^1\cap L^2} + \|F\|_{\mathcal{Y}_s}. \tag{5.16}$$

Proof. By (5.15),

$$\begin{aligned}
\big|\langle \mathcal{M}_j(\sigma,t)\mathcal{G}_{v_j,D_j}(t)Z(t), \xi_j^m(\cdot;\sigma)\rangle\big| &\le \big|\langle \mathcal{M}_j(\sigma,t)\mathcal{G}_{v_j,D_j}(t)Z(t), -\mathcal{M}_j(\sigma,t)\mathcal{G}_{v_j,D_j}(t)\tilde{\xi}_j^m(t,\cdot) + \xi_j^m(\cdot;\sigma)\rangle\big| \\
&\le \delta\|Z(t)\|_{2+\infty}.
\end{aligned}$$

Therefore, for all $0 \le t \le T$,

$$\begin{aligned}
&\big|\langle \mathcal{M}_j(\sigma,t)\mathcal{G}_{v_j,D_j}(t)Z(t), \xi_j^m(t,\cdot;\sigma)\rangle\big| \le \delta(1+t)^{-\frac{n}{2}} \sup_{0\le\tau\le T}(1+\tau)^{\frac{n}{2}}\|Z(\tau)\|_{2+\infty} \\
&\le \delta(1+t)^{-\frac{n}{2}}\|Z\|_{\mathcal{X}_s(T)} =: (1+t)^{-\frac{n}{2}} B_T.
\end{aligned}$$

Hence, using (5.12) one sees that

$$\begin{aligned}
\|Z\|_{\mathcal{X}_s(T)} &\lesssim \sum_{k=0}^{s} \|\nabla^k Z(0,\cdot)\|_{L^1\cap L^2} + \|F\|_{\mathcal{Y}_s(T)} + \|VZ\|_{\mathcal{Y}_s(T)} + B_T \\
&\lesssim \sum_{k=0}^{s} \|\nabla^k Z(0,\cdot)\|_{L^1\cap L^2} + \|F\|_{\mathcal{Y}_s(T)} + \delta\|Z\|_{\mathcal{X}_s(T)},
\end{aligned}$$

and the desired conclusion follows. □

6 Modulation equations

In their analysis of the stability relative to one soliton, Buslaev and Perelman [BP1], [BP2], and Cuccagna [Cu] derive the equations for $\dot{\sigma}$ by imposing an orthogonality condition on the perturbation Z for all times. More precisely, they make the ansatz

$$\psi = e^{i\theta(t,\sigma(t))}(w(\sigma(t)) + R) \tag{6.1}$$

where $e^{i\theta(t,\sigma(t))}w(\sigma(t))$ is a single soliton evolving along a nonlinear set of parameters. The removal of the phase from the perturbation R leads to an equation which is simply the translation of the equation involving the stationary Hamiltonian (2.24) to the point $vt + D$. This in turn makes it very easy to formulate the orthogonality conditions: At time t, the function $R(\cdot + vt + D)$ in (6.1) needs to be perpendicular to all elements of the generalized eigenspaces of all $H_j(\sigma)^*$ as in (2.24), where σ is equal to the parameters $\sigma(t)$ at time t.

In the multi-soliton case the removal of the phases by means of this ansatz is not available, since distinct solitons carry distinct phases. As already indicated above, we work with the representation

$$\psi(t) = \sum_{j=1}^{N} w_j(t, \sigma(t)) + R,$$

which forces us to formulate the orthogonality condition in terms of a set of functions that is moving along with the $w_j(t, \sigma(t))$. We now define these functions.

Definition 6.1. *Let $\sigma(t)$ be an admissible path and define $\theta_j(t, x; \sigma(t))$ and $x_j(t; \sigma(t))$ as in* (2.11) *and* (2.12)*. Also, set $\phi_j(t, x; \sigma(t)) = \phi(x - x_j(t; \sigma(t)); \alpha_j(t))$. Then we let*

$$\xi_j^m(t, x; \sigma(t)) = \begin{pmatrix} u_j^m(t, x; \sigma(t)) \\ \bar{u}_j^m(t, x; \sigma(t)) \end{pmatrix}$$

with

$$\begin{aligned}
(6.2) \quad u_j^1(t, x; \sigma(t)) &= w_j(t, x; \sigma(t)) = e^{i\theta_j(t,x;\sigma(t))}\, \phi_j(t, x; \sigma(t)) \\
u_j^2(t, x; \sigma(t)) &= \frac{2i}{\alpha_j}\, e^{i\theta_j(t,x;\sigma(t))}\, \partial_\alpha \phi_j(t, x; \sigma(t)) \\
u_j^m(t, x; \sigma(t)) &= ie^{i\theta_j(t,x;\sigma(t))}\, \partial_{x_{m-2}} \phi_j(t, x; \sigma(t)) \quad \textit{for} \quad 3 \le m \le n+2 \\
u_j^m(t, x; \sigma(t)) &= e^{i\theta_j(t,x;\sigma(t))}\, (x^{m-n-2} - x_j^{m-n-2}(t; \sigma(t)))\phi_j(t, x; \sigma(t)), \quad \textit{for} \quad n+3 \le m \le 2n+2.
\end{aligned}$$

The following proposition should be thought of as a time-dependent version of Proposition 4.1. More precisely, if σ is a *fixed* set of parameters, then one can define an alternate set of vectors, $\tilde{\xi}_j^m$, say, by applying appropriate Gallilei transforms to the stationary vectors in Proposition 4.1. For example, take some ξ_j^m so that $H_j^*(\sigma)\xi_j^m = 0$. Then the corresponding $\tilde{\xi}_j^m$ satisfies

$$i\partial_t \tilde{\xi}_j^m + H_j(t, \sigma)\tilde{\xi}_j^m = 0,$$

with $H_j(t, \sigma)$ as in (3.8). Naturally, one would therefore expect that

$$i\partial_t \xi_j^m + H(\sigma(t))\xi_j^m = O(\dot{\sigma}_j) + O(e^{-ct}),$$

where $H(\sigma(t))$ is as in (2.21) (the exponentially decaying term appears because of interactions between solitons). The following proposition shows that this indeed holds, but as in [Cu] we will work with a modified set of parameters $\tilde{\sigma}_j(t) = (v_j(t), D_j(t), \alpha_j(t), \tilde{\gamma}_j(t))$ where

$$\dot{\tilde{\gamma}}_j(t) = \dot{\gamma}_j(t) + \frac{1}{2} \sum_{m=1}^{n} \dot{v}_j^m(t) x_j^m(t, \sigma(t)). \tag{6.3}$$

The point of this modification is that the $\dot\Sigma W(\sigma(t))$ term in (5.1) and (3.4) can be rewritten as

$$\begin{aligned}\dot\Sigma W(\sigma(t)) &= \sum_{j=1}^{k}\Big[\dot{\tilde\gamma}_j(t)J\xi_j^1(t,x;\sigma(t))-\frac{\alpha_j}{2}\dot\alpha_j(t)J\xi_j^2(t,x;\sigma(t))\Big]+\\ &\sum_{j=1}^{k}\sum_{m=1}^{n}\Big[\dot D_j^m(t)J\xi_j^{m+2}(t,x;\sigma(t))+\frac{1}{2}\dot v_j^m(t)J\xi_j^{m+n+2}(t,x;\sigma(t))\Big],\end{aligned}\tag{6.4}$$

where ξ_j^m are as in Definition 6.1. This is of course due to the fact that passing to $\tilde\gamma_j$ allows us to change from x to $x-x_j(t;\sigma(t))$ in (3.6).

Proposition 6.2. *Let $\sigma(t)$ be an admissible path and define $\xi_j^m(t,x;\sigma(t))$ as in Definition 6.1. Then*

$$i\partial_t\xi_j^1+H_j^*(\sigma(t))\xi_j^1 = O\big(\dot{\tilde\sigma}(|\phi_j|+|D\phi_j|)\big)\tag{6.5}$$
$$i\partial_t\xi_j^2+H_j^*(\sigma(t))\xi_j^2 = i\xi_j^1+O\big(\dot{\tilde\sigma}(|\phi_j|+|D\phi_j|+|D^2\phi_j|)\big)\tag{6.6}$$
$$i\partial_t\xi_j^m+H_j^*(\sigma(t))\xi_j^m = O\big(\dot{\tilde\sigma}(|\phi_j|+|D\phi_j|+|D^2\phi_j|)\big)\ \ \textit{for}\ \ 3\le m\le n+2\tag{6.7}$$
$$i\partial_t\xi_j^m+H_j^*(\sigma(t))\xi_j^m = -2i\xi_j^{m-n}+O\big(\dot{\tilde\sigma}(|\phi_j|+|D\phi_j|+|D^2\phi_j|)\big)\ \textit{for}\ \ n+3\le m\le 2n+2.\tag{6.8}$$

Here D refers to either spatial derivatives ∂_{x^ℓ} or derivatives ∂_α. Moreover, as in Definition 6.1, the function ϕ_j needs to be evaluated at $x-x_j(t;\sigma(t))$, $\alpha_j(t)$.

Proof. This is verified by direct differentiation of the functions in Definition 6.1. □

The following proposition collects the modulation equations for the path $\sigma(t)$ that are obtained by taking scalar products of (2.21) with the functions ξ_j^m from Definition 6.1. This will of course use (6.4). The modulation equations are derived from the orthogonality assumptions, see (6.9) below. Observe that these assumptions need not be satisfied at $t=0$. Nevertheless, as in Buslaev and Perelman [BP1], one shows by means of the implicit function theorem that one can replace the initial decomposition (2.2) by a nearby one which does satisfy the orthogonality condition. This uses the smallness of the initial perturbation R_0, as well as the separation conditions (2.17) or (2.18). The details can be found in Section 10, see Lemma 10.1. In that section it is also shown that, conversely, given the modulation of Proposition 6.3 the orthogonality condition will propagate if satisfied initially.

Proposition 6.3. *Let Z satisfy the system* (3.4). *Suppose that for all $t\ge 0$,*

$$\langle Z(t),\xi_j^m(t,\cdot;\sigma(t))\rangle=0\ \ \textit{for all}\ \ j,m\tag{6.9}$$

where ξ_j^m is as in Definition 6.1. Then the path $\tilde\sigma(t):=(v_j(t),D_j(t),\tilde\gamma_j(t),\alpha_j(t))$, $j=1,..,n$ satisfies *the following system of equations with matrix potentials* $V_r(t,x;\sigma(t))$ *as in* (2.22)*:*

$$\begin{aligned}
-2i\dot\alpha_j(t)\Big\langle\phi_j(\sigma(t)),\,\partial_\alpha\phi_j(\sigma(t))\Big\rangle+O(\dot{\tilde\sigma}\|Z(t)\|_{L^2+L^\infty}) &= \sum_{r\neq j}\Big\langle V_r(t,\cdot;\sigma(t))Z,\,\xi_j^1(t,\cdot;\sigma(t))\Big\rangle+\\
&\Big\langle\Big(O(w_1w_2)Z+O(w_1w_2)+O(|w|^{p-2}|Z|^2)+O(|Z|^p)\Big),\,\xi_j^1(t,\cdot;\sigma(t))\Big\rangle, \qquad (6.10)\\
2i\dot{\tilde\gamma}_j(t)\Big\langle\phi_j(\sigma(t)),\,\partial_\alpha\phi_j(\sigma(t))\Big\rangle+O(\dot{\tilde\sigma}\|Z(t)\|_{L^2+L^\infty}) &= \sum_{r\neq j}\Big\langle V_r(t,\cdot;\sigma(t))Z,\,\xi_j^2(t,\cdot;\sigma(t))\Big\rangle+\\
&\Big\langle\Big(O(w_1w_2)Z+O(w_1w_2)+O(|w|^{p-2}|Z|^2)+O(|Z|^p)\Big),\,\xi_j^2(t,\cdot;\sigma(t))\Big\rangle,\\
\dot v_j^m(t)\|\phi_j(\sigma(t))\|_2^2+O(\dot{\tilde\sigma}\|Z(t)\|_{L^2+L^\infty}) &= \sum_{r\neq j}\Big\langle V_r(t,\cdot;\sigma(t))Z,\,\xi_j^{m+2}(t,\cdot;\sigma(t))\Big\rangle+\\
&\Big\langle\Big(O(w_1w_2)Z+O(w_1w_2)+O(|w|^{p-2}|Z|^2)+O(|Z|^p)\Big),\,\xi_j^{m+2}(t,\cdot;\sigma(t))\Big\rangle,\\
\dot D_j^m(t)\|\phi_j(\sigma(t))\|_2^2+O(\dot{\tilde\sigma}\|Z(t)\|_{L^2+L^\infty}) &= \sum_{r\neq j}\Big\langle V_r(t,\cdot;\sigma(t))Z,\,\xi_j^{n+m+2}(t,\cdot;\sigma(t))\Big\rangle+\\
&\Big\langle\Big(O(w_1w_2)Z+O(w_1w_2)+O(|w|^{p-2}|Z|^2)+O(|Z|^p)\Big),\,\xi_j^{n+m+2}(t,\cdot;\sigma(t))\Big\rangle.
\end{aligned}$$

Proof. Differentiating (6.9) yields

$$\langle i\partial_t Z,\xi_j^m(t,\cdot;\sigma(t))\rangle=\langle Z,i\partial_t\xi_j^m(t,\cdot;\sigma(t))\rangle.$$

Taking scalar products of (3.4) thus leads to

$$\langle Z,i\partial_t\xi_j^m\rangle+\langle Z,H^*(\sigma(t))\xi_j^m\rangle=\Big\langle\dot\Sigma W(\sigma(t)),\xi_j^m\Big\rangle+\Big\langle O(w_1w_2)Z+O(w_1w_2)+O(|w|^{p-2}|Z|^2)+O(|Z|^p),\xi_j^m\Big\rangle.$$

In view of the explicit expressions (6.2) one has

$$\begin{aligned}
\langle J\xi_j^2(t,\cdot;\sigma(t)),\,\xi_j^1(t,\cdot;\sigma(t))\rangle &= -2i\langle\phi_j(\sigma(t)),\,\partial_\alpha\phi_j(\sigma(t))\rangle\\
\langle J\xi_j^m(t,\cdot;\sigma(t)),\,\xi_j^1(t,\cdot;\sigma(t))\rangle &= 0 \ \text{ for } \ m\neq 2\\
\langle J\xi_j^m(t,\cdot;\sigma(t)),\,\xi_j^2(t,\cdot;\sigma(t))\rangle &= 0 \ \text{ for } \ m\neq 1\\
\langle J\xi_j^{m+2}(t,\cdot;\sigma(t)),\,\xi_j^{m+n+2}(t,\cdot;\sigma(t))\rangle &= -2i\|\phi_j(\sigma(t))\|_2^2 \ \text{ for } \ 3\le m\le n+2.
\end{aligned}$$

Therefore, the proposition follows by taking inner products in (6.4). Note that the terms containing $\dot{\tilde\sigma}\|Z(t)\|_{L^2+L^\infty}$ appear from Proposition 6.2. □

7 Bootstrap assumptions

The proof of our main theorem relies on the bootstrap assumptions on the admissible path $\sigma(t)$ and the size of the perturbation $Z(t,x)=\binom{R(t,x)}{\bar R(t,x)}$. in the norms of the spaces $\mathcal{X}_s$ defined in (5.7).

Bootstrap assumptions

There exists a small constant $\delta = \delta(\epsilon)$ dependent on the size of the initial data R_0 and the initial separation of the solitons $w_j(0, x; \sigma(0))$, see (2.17), and a sufficiently large constant C_0 such that for some integer $s > \frac{n}{2}$

$$|\dot{\tilde{\sigma}}(t)| \leq \delta^2 (1+t)^{-n}, \quad \forall t \geq 0, \tag{7.1}$$

$$\|Z\|_{\mathcal{X}_s} \leq \delta C_0^{-1} \tag{7.2}$$

Remark 7.1. The bootstrap assumption (7.1) together with the definition (6.3) implies that

$$|\dot{\gamma}(t)| \leq \delta^2 (1+t)^{-n+1} \tag{7.3}$$

Remark 7.2. The bootstrap assumption (7.2) together with Lemma 9.4 implies that

$$\|Z(t)\|_{L^\infty} \lesssim \delta C_0^{-1} (1+t)^{-\frac{n}{2}}, \tag{7.4}$$

$$\|Z(t)\|_{H^s} \lesssim \delta C_0^{-1} \tag{7.5}$$

The bootstrap assumption (7.1) strengthens the notion of the admissible path. In particular, it allows us to estimate the deviation between the path $x_j(t; \sigma(t))$ corresponding to the path $\sigma(t)$ and the straight line $x_j(t, \sigma^\infty)$ determined by the constant parameter σ^∞ which was defined from $\sigma(t)$ in (2.14) and (2.15). This estimate will play an important role in our analysis.

Lemma 7.3. *Let $\sigma(t)$ be an admissible path satisfying the bootstrap assumption* (7.1) *and let σ^∞ be a constant parameter vector as in* (2.14) *and* (2.15)*. Then*

$$|x_j(t; \sigma(t)) - tv_j^\infty - D_j^\infty| \lesssim \delta^2 (1+t)^{-n+2} \tag{7.6}$$

Proof. By our choice of v_j^∞ and D_j^∞ one has that

$$|x_j(t; \sigma(t)) - tv_j^\infty - D_j^\infty| \lesssim \int_t^\infty \int_s^\infty |\dot{v}_j(\tau)| \, d\tau + \int_t^\infty |\dot{D}_j(s)| \, ds$$

and the lemma follows from (7.1). □

We then have the following corollary. To formulate it, we need the localizing functions

$$\begin{aligned} \chi_0(x) &= \exp\Big(-\frac{1}{2}\alpha_{\min}(1+|x|^2)^{\frac{1}{2}}\Big) \\ \chi(t, x; \sigma^\infty) &= \sum_{j=1}^k \chi_0(x - x_j(t; \sigma^\infty)). \end{aligned} \tag{7.7}$$

Here $\alpha_{\min} > 0$ satisfies $\inf_{t \geq 0, 1 \leq j \leq k} \alpha_j(t) > \alpha_{\min}$ for any admissible path $\sigma(t)$ starting at σ_0. The exponent $\alpha_{\min}$ arises because of the decay rate of the ground state of (2.6).

Corollary 7.4. *Let $\sigma(t)$ be an admissible path satisfying the bootstrap assumption* (7.1)*. With the parameters σ^∞ as in* (2.14) *and* (2.15) *one has*

$$\Big| H(t, \sigma^\infty) - H(\sigma(t)) \Big| \lesssim \delta^2 (1+t)^{2-n} \, \chi(t, x; \sigma^\infty), \tag{7.8}$$

where $H(t, \sigma^\infty)$ and $H(\sigma(t))$ are the Hamiltonians from (3.7) *and* (2.21)*.*

Proof. The difference

$$H(t,\sigma^\infty) - H(\sigma(t))$$

is a sum of matrix valued potentials that are exponentially localized around the solitons $w_j(\sigma(t))$ or $w_j(t,\sigma^\infty)$, respectively. By the previous lemma, we can assume that all the potentials are localized near the straight path $x(t,\sigma^\infty) = (x_1(t;\sigma^\infty),\ldots,x_N(t;\sigma^\infty))$. Since $v_j^\infty = v_j(\infty), \alpha_j^\infty = \alpha_j(\infty)$,

(7.9)

$$\Big|[H(t,\sigma^\infty) - H(\sigma(t))]\Big| \lesssim \sum_{j=1}^k \Big|tv_j^\infty + D_j^\infty - \int_0^t v_j(s)\,ds - D_j(t)\Big|\chi_0(x - x_j(t;\sigma^\infty))$$

(7.10)

$$+\sum_{j=1}^k \Big|\frac{1}{2}\int_t^\infty \dot v_j(s)\cdot x\,ds - \frac{1}{2}\int_0^t\int_s^\infty (\dot v_j(s)\cdot v_j(s) - \dot\alpha_j(s)\alpha_j(s))\,ds + \gamma_j - \gamma_j(t)\Big|\chi_0(x - x_j(t;\sigma^\infty)).$$

The term (7.9) arises as the difference of two paths, whereas (7.10) is the difference of the phases, i.e.,

$$|e^{i\theta_j(t,x;\sigma^\infty)} - e^{i\theta_j(t,x;\sigma(t))}|.$$

In view of the definitions of D_j, γ_j from (2.14) and (2.15) one has

(7.11)

$$\Big|H(t,\sigma^\infty) - H(\sigma(t))\Big| \lesssim \sum_{j=1}^k \Big(\int_t^\infty\int_s^\infty |\dot v_j(\tau)|\,d\tau + \int_t^\infty |\dot D_j(s)|\,ds\Big)\chi_0(x - x_j(t,\sigma))$$

$$+\sum_{j=1}^k \Big(\int_t^\infty\int_s^\infty |\dot v_j(\tau)\cdot v_j(\tau) - \dot\alpha_j(s)\alpha_j(s)|\,d\tau\,ds + \int_t^\infty |\dot\gamma_j(s)|\,ds + \int_t^\infty |\dot v_j(s)|\,ds|x|\Big)\chi_0(x - x_j(t,\sigma^\infty))$$

$$\lesssim \delta^2(1+t)^{2-n}\,\chi(t,x;\sigma^\infty).$$

For the final inequality one uses (7.3) and the fact that

$$|x|\chi_0(x - x_j(t;\sigma^\infty)) \lesssim t.$$

The corollary follows. $\square$

8 Solving the modulation equations

Our goal is to show that the system in Proposition 6.3 has a solution $\dot{\tilde\sigma}(t)$ that satisfies the bootstrap assumptions (7.1). This requires some care, as the right-hand side in Proposition 6.3 involves the perturbation Z. We will therefore first verify that the system of modulation equations is consistent with the bootstrap assumptions (7.1) and (7.2). In what follows, we will use both paths $\tilde\sigma(t)$ and $\sigma(t)$. By definition, see (6.3),

$$\tilde\gamma_j(t) = -\int_t^\infty \Big[\dot\gamma_j(s) + \frac{1}{2}\sum_{m=1}^n \dot v_j^m(s)x_j^m(s;\sigma(s))\Big]\,ds.$$

The integration is well-defined provided $\tilde{\sigma}$ satisfies the bootstrap assumption. Indeed, in that case $|v_j(t)| \lesssim (1+t)^{-n}$ and since $|x_j(t;\sigma(t))| \lesssim 1+t$, the integral is absolutely convergent. Finally, recall the property (7.3) of the derivatives.

Lemma 8.1. *Suppose the separation and convexity conditions hold, see* (2.17) *and* (2.16)*. Let $\tilde{\sigma}, Z$ be* any *choice of functions that satisfy the bootstrap assumptions for sufficiently small $\delta > 0$. If the inhomogeneous terms of the system* (6.10) *are defined by means of these functions, then this system has a solution $\dot{\tilde{\sigma}}$ that satisfies* (7.1) *with $\delta/2$ for all times.*

Proof. By the nonlinear stability condition (2.16), the left-hand side of (6.10) is of the form $B_j(t)\dot{\tilde{\sigma}}_j(t)$ with an invertible matrix $B_j(t)$. The O-term is a harmless perturbation of the matrix given by the main terms on the left-hand side, provided δ is chosen sufficiently small. This easily follows from the smallness of Z given by (7.2). We need to verify that the right-hand side of (6.10) decays like $\delta^2(1+t)^{-n}$. We consider only the first equation in (6.10), the others being the same. The terms $\langle V_r(t,\sigma)Z, \xi_j^1(t,\cdot;\sigma(t))\rangle$ for $r \neq j$ and $w_1 w_2$ are governed by the interaction of two *different* solitons. In view of the separation condition (2.17) and the exponential localization of the solitons, we have

$$
\begin{aligned}
(8.1)\qquad |\dot{\alpha}_j(t)| &\lesssim e^{-\alpha_{\min}(L+ct)}(1 + \|Z(t)\|_{L^2+L^\infty}) + \|Z(t)\|^2_{L^2+L^\infty} + \|Z(t)\|^p_{L^2+L^\infty} \\
(8.2)\qquad &\lesssim \delta C_0^{-1}(1+t)^{-n}\Big(\epsilon + \delta C_0^{-1} + \delta^{p-1}C_0^{-(p-1)}\Big) \le \Big(\frac{\delta}{2}\Big)^2(1+t)^{-n}
\end{aligned}
$$

where we have used the estimate (7.4), the condition (2.19), $L\alpha_{\min} \ge |\log\epsilon|$, and that $p \ge 2$. □

More generally, the estimates leading up to (8.1) also yield the following result. The proof is implicit in the preceding one and is therefore omitted.

Lemma 8.2. *Let $\sigma(t)$ be an admissible path satisfying the bootstrap assumption* (7.1) *and $Z(t)$ be an arbitrary function in $\mathcal{X}_s$. Define the function $\dot{\Sigma}$ as a solution of the equation*

$$
(8.3)\qquad \langle \dot{\Sigma} W(\sigma(t)), \xi_j^m(t,\cdot;\sigma(t))\rangle = \langle G(Z(t),\sigma(t)), \xi_j^m(t,\cdot;\sigma(t))\rangle + \langle \Omega_j^m(t,\cdot;\sigma(t)), Z(t,\cdot)\rangle,
$$

where

$$
\begin{aligned}
(8.4)\quad & G(Z(t),\sigma(t)) = O(w_1(\sigma(t))w_2(\sigma(t))Z + O(w_1(\sigma(t)w_2(\sigma(t)) + O(|w(\sigma(t))|^{p-2}|Z|^2) + O(|Z|^p), \\
(8.5)\quad & \Omega_j^m(t,x;\sigma(t)) = O\big(\dot{\tilde{\sigma}}(|\phi_j| + |D\phi_j| + |D^2\phi_j|)\big) + \sum_{r\neq j} V_r(t,x;\sigma(t))\xi_j^m(t,x;\sigma(t)).
\end{aligned}
$$

Then

$$
(8.6)\qquad |\dot{\Sigma}(t)| \le (1+t)^{-n}\Big(\frac{1}{4}\delta^2 + C\|Z\|^2_{\mathcal{X}_s} + C\|Z\|^p_{\mathcal{X}_s}\Big)
$$

Remark 8.3. The functions $G(Z(t),\sigma(t))$ and $\Omega_j^m(t,x;\sigma(t))$ arise as follows. In Section 10 we will rewrite the Z equation in the form

$$
i\partial_t Z + H(\sigma(t))Z = \dot{\Sigma} W(\sigma(t)) + G(Z(t),\sigma(t)),
$$

The quantity $\Omega_j^m(t,x;\sigma(t))$ is defined via the equation

$$i\partial_t \xi_j^m(t,\cdot;\sigma(t)) + H^*(\sigma(t))\xi_j^m(t,\cdot;\sigma(t)) = \mathcal{S}_k^m \xi_j^k(t,\cdot;\sigma(t)) + \Omega_j^m(t,\cdot;\sigma(t)),$$

where the matrix $\mathcal{S}$ collects the terms $\xi_j^m(t,\cdot;\sigma(t))$ on the right-hand sides of (6.5)-(6.8). However, the previous proof does not require the explicit form of G or Ω_j^m.

9 Solving the Z equation

In this section we verify the bootstrap assumptions (7.2) for the perturbation Z. This together with the already verified bootstrap estimates for $\dot{\tilde{\sigma}}$ will also lead to the existence of the function $Z(t)$ asserted in our main result. At this point we recall the imposed orthogonality conditions (6.9)

$$\langle Z(t), \xi_j^m(t,\cdot;\sigma(t))\rangle = 0 \quad \text{for all} \quad j,m \tag{9.1}$$

with $\xi_j^m(t,x;\sigma(t))$ is as in Definition 6.1. We next rewrite the equation (3.4) for Z in the form

$$i\partial_t Z + H(t,\sigma^\infty)Z = F, \tag{9.2}$$

$$F = \big(H(t,\sigma^\infty) - H(\sigma(t))\big)Z + \dot{\Sigma}W(\sigma(t)) + O(w_1w_2)Z + O(w_1w_2) + O(|w|^{p-2}|Z|^2) + O(|Z|^p) \tag{9.3}$$

with the reference hamiltonian $H(t,\sigma^\infty)$ as defined in (3.7). To verify the bootstrap assumption (7.2) we need to apply the the dispersive estimate for the inhomogeneous charge transfer problem stated in Theorem 5.2. The following lemma shows that the orthogonality conditions (6.9) and the bootstrap assumptions (7.1) , (7.2) imply that Z is asymptotically orthogonal to the bound states of $H_j^*(\sigma^\infty)$, as required in Theorem 5.2.

Lemma 9.1. *Let Z be an arbitrary function in $\mathcal{X}_s$ satisfying the orthogonality conditions* (9.1) *with respect to an admissible path $\sigma(t)$ obeying the bootstrap assumption* (7.1)*, and so that Z verifies the bootstrap assumption* (7.2)*. Then Z is asymptotically orthogonal to the null spaces of the hamiltonians $H_j^*(\sigma^\infty)$ in the sense of* (5.3)*. In fact,*

$$\|P_{N_j}(\sigma^\infty)\mathcal{G}_{v_j^\infty,D_j^\infty}(t)Z(t,\cdot)\|_{L^2} \lesssim \delta^3(1+t)^{-\frac{n}{2}-1}, \quad \forall j=1,..,k \tag{9.4}$$

Proof. By the assumption $Z(t)$ is orthogonal to the vectors $\xi_j^m(t,x;\sigma(t))$ introduced in Definition 6.1, while (9.4) is equivalent to the estimates

$$|\langle \mathcal{G}^*_{v_j^\infty,D_j^\infty}(t)\xi_j^m(\cdot;\sigma^\infty), Z(t)\rangle| \lesssim \delta^3(1+t)^{-\frac{n}{2}-1}, \quad \forall j,m$$

Here $\xi_j^m(x;\sigma^\infty)$, defined in Proposition 4.1, refer to the elements of the null spaces $\mathcal{N}_j(\sigma^\infty)$ of the stationary hamiltonians $H_j^*(\sigma^\infty)$. The desired estimate would then follow from the bootstrap assumption (7.2), in particular (7.4), and the inequality

$$\|\mathcal{G}^*_{v_j^\infty,D_j^\infty}(t)\xi_j^m(\cdot;\sigma^\infty) - \xi_j^m(t,\cdot;\sigma(t))\|_{L^1} \lesssim \delta^2(1+t)^{-1} \tag{9.5}$$

The vectors ξ_j^m are composed of the functions derived from the bound state ϕ. In particular, $\xi_j^1 = \binom{\phi}{\phi}$. Therefore,

$$|\mathcal{G}^*_{v_j^\infty,D_j^\infty}(t)\xi_j^1(x;\sigma^\infty) - \xi_j^1(t,x;\sigma(t))| = 2|e^{i(\frac{1}{2}v_j^\infty\cdot x - \frac{1}{4}(|v_j^\infty|^2 - |\alpha_j^\infty|^2)t + \gamma_j^\infty)}\phi(x - v_j^\infty t - D_j^\infty) -$$

$$e^{i(\frac{1}{2}v_j(t)\cdot x - \frac{1}{4}\int_0^t(|v_j(\tau)|^2 - \alpha_j(\tau)^2)\,d\tau + \gamma_j(t))}\phi(x - x_j(t,x;\sigma(t)))| \tag{9.6}$$

According to Lemma 7.3, $|x_j(t,x;\sigma(t)) - v_j^\infty t - D_j^\infty| \lesssim \delta^2(1+t)^{-n+2}$. Similarly, (7.10) of Corollary 7.4 gives the estimate for the difference of the phases appearing in (9.6)

$$|e^{i\theta_j(t,x;\sigma^\infty)} - e^{i\theta_j(t,x;\sigma(t))}| \lesssim \delta^2(1+t)^{-n+2}$$

The estimate (9.5) follows immediately since $n \geq 3$. □

We now in the position to aplly Theorem 5.2 to establish the improved $\mathcal{X}_s$ estimates for $Z(t)$.

Lemma 9.2. *Let Z be a solution of the equation (9.2) satisfying the bootstrap assumption (7.2) with some sufficiently small constants δ and C_0^{-1}. We also assume (due to Lemma 8.1) that the admissible path $\sigma(t)$ obeys the estimate (7.1). Then we have the following estimate*

$$\|Z(t)\|_{\mathcal{X}_s} \leq \frac{\delta}{2}C_0^{-1} \tag{9.7}$$

Proof. Perturbation Z is a solution of the inhomogeneous charge transfer problem (9.2)

$$i\partial_t Z + H(t,\sigma^\infty)Z = F,$$

$$F := \big(H(t,\sigma^\infty) - H(\sigma(t))\big)Z + \dot\Sigma W(\sigma(t)) + O(w_1w_2)Z + O(w_1w_2) + O(|w|^{p-2}|Z|^2) + O(|Z|^p) \tag{9.8}$$

Lemma 9.1 shows that Z is asymptotically orthogonal (with the constant δ^3) to the null spaces of the hamiltonians $H_j^*(\sigma^\infty)$. Therefore, Theorem 5.2 gives the estimate

$$\|Z(\cdot)\|_{\mathcal{X}_s} \lesssim \sum_{k=0}^s \|\nabla^k Z_0\|_{L^1\cap L^2} + \|F\|_{\mathcal{Y}_s} + \delta^3 \tag{9.9}$$

with

$$\|F\|_{\mathcal{Y}_s} = \sup_{t\geq 0}\Big(\sum_{k=0}^s\int_0^t \|\nabla^k F(\tau,\cdot)\|_{L^1}\,d\tau + (1+t)^{\frac{n}{2}+1}\|F(t,\cdot)\|_{H^s}\Big) \tag{9.10}$$

By the assumptions on the initial data $\sum_{k=0}^s \|\nabla^k Z_0\|_{L^1\cap L^2} \leq \epsilon << \delta$. Therefore, to obtain the conclusion of Lemma 9.2 it would suffice to verify that

$$\|F\|_{\mathcal{Y}_s} \lesssim \delta^2 \tag{9.11}$$

with F defined as in (9.8). The estimate (9.11) relies on the following lemma and the bootstrap assumptions (7.2) on $Z(t)$. □

Lemma 9.3. *Let $\sigma(t)$ be an arbitrary admissible path satisfying the botstrap assumption (7.1) and $Z(t)$ be an arbitrary function in $\mathcal{X}_s$. Then the nonlinear expression F defined in terms of the path $\sigma(t)$ and $Z(t)$ as in (9.8) obeys the estimate*

$$\|F\|_{\mathcal{Y}_s} \lesssim \delta^2 + \|Z\|^2_{\mathcal{X}_s} + \|Z\|^p_{\mathcal{X}_s} \tag{9.12}$$

9.1 Algebra estimates

In this section we establish several simple lemmas designed to ease the task of estimating the $\mathcal{Y}_s$ norm of $F = F(Z, w, \sigma)$ in connection with the $\mathcal{X}_s$ norms of Z.

We start by formulating a version of the Sobolev estimate tailored to the use of the space $L^2 + L^\infty$.

Lemma 9.4. *Let s be a positive integer. Then for any nonnegative integer $k \le s$ and any $q \in [2, q_k]$, where*

$$\frac{1}{q_k} = \frac{1}{2} - \frac{s-k}{n}, \qquad \text{if } k > s - \frac{n}{2} \tag{9.13}$$
$$q_k = \infty, \qquad \text{if } k < s - \frac{n}{2}$$

and $q \in [2, \infty)$ if $k = s - \frac{n}{2}$ the following estimates hold true

$$\|\nabla^k f\|_{L^q + L^\infty} \lesssim \sum_{l=0}^{s} \|\nabla^l f\|_{L^2 + L^\infty} \le (1+t)^{-\frac{n}{2}} \|f\|_{\mathcal{X}_s} \tag{9.14}$$

In particular, if $s > \frac{n}{2}$

$$\|f\|_{L^\infty} \lesssim (1+t)^{-\frac{n}{2}} \|f\|_{\mathcal{X}_s} \tag{9.15}$$

Proof. By duality and density it suffices to show that

$$\|f\|_{L^1 \cap L^2} \lesssim \sum_{l=0}^{s-k} \|\nabla^l f\|_{L^1 \cap L^{q'}}$$

The L^1 estimate is trivial while the the estimate for the L^2 norm follows from the standard Sobolev embedding $W^{k-l,q'} \subset L^2$, which holds for the range of parameters (k, l, q) described in the Lemma. $\square$

Next are the estimates of the nonlinear quantities arising in (9.8) in terms of the $\mathcal{X}_s$ norm.

Lemma 9.5. *Let $\gamma(\tau)$ be a smooth function which obeys the estimates*

$$|\gamma^{(\ell)}(\tau)| \lesssim \tau^{(\frac{p-1}{2} - \ell)_+} \tag{9.16}$$

for some $p \ge 2 + \frac{2}{n}$ and all non-negative integers ℓ. Here $r_+ = r$ if $r \ge 0$ and $r_+ = 0$ if $r < 0$. Then for any $s > \frac{n}{2}$ and any non-negative integer $k \le s$

$$\|\nabla^k \Big(\gamma(|f|^2) f\Big)\|_{L^1} \lesssim (1+t)^{-1} (\|f\|_{\mathcal{X}_s}^p + \|f\|_{\mathcal{X}_s}^{2k+1+(p-1-2k)_+}), \tag{9.17}$$

$$\|\nabla^k \Big(\gamma(|f|^2) f\Big)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2}-1} (\|f\|_{\mathcal{X}_s}^p + \|f\|_{\mathcal{X}_s}^{2k+1+(p-1-2k)_+}) \tag{9.18}$$

In addition, if γ is a smooth function obeying (9.16) for some $p \ge 2$ and $\zeta(x)$ is an exponentially localized smooth function, then for any $q \in [1, 2]$

$$\|\nabla^k \Big(\zeta\, \gamma(|f|^2) f\Big)\|_{L^q} \lesssim (1+t)^{-n} (\|f\|_{\mathcal{X}_s}^p + \|f\|_{\mathcal{X}_s}^{2k+1+(p-1-2k)_+}). \tag{9.19}$$

Remark 9.6. It will become clear from the proof below that if the function γ satisfies (9.16) for some $p > 2 + \frac{2}{n}$, then the estimate (9.17) holds with a better rate of decay in t. In particular,

$$\int_0^\infty \left\|\nabla^k\Big(\gamma(|f|^2)f\Big)\right\|_{L^1} dt \lesssim \|f\|^p_{\mathcal{X}_s} + \|f\|^{2k+1}_{\mathcal{X}_s}. \tag{9.20}$$

Proof. By Leibnitz's rule

$$\nabla^k\Big(\gamma(|f|^2)f\Big) = \sum_{\ell=0}^k \sum_{m_1+\ldots+m_{2\ell+1}=k} C_{\ell\vec{m}}\gamma^{(\ell)}(|f|^2)\nabla^{m_1}f\nabla^{m_2}f\ldots\nabla^{m_{2\ell+1}}f$$

with some positive integer constants $C_{\ell\vec{m}}$ and non-negative vectors $\vec{m} = (m_1, \ldots, m_{2\ell+1})$. We may assume that $m_{2\ell+1} \geq m_{2\ell} \geq \ldots \geq m_1$. Define

$$q_{m_r} = (\frac{1}{2} - \frac{s-m_r}{n})^{-1} \tag{9.21}$$

for $m_r \geq s - \frac{n}{2}$ and $q_{m_r} = \infty$ otherwise. With the above definition the Sobolev imbeddings

$$H^s \subset W^{m_r, q_{m_r}}, \qquad W^{s,2} + W^{s,\infty} \subset W^{m_r,q_{m_r}} + W^{m_r,\infty} \tag{9.22}$$

(recall that $m_r \leq k \leq s$) [3] hold true by Lemma 9.4. Then

$$\begin{aligned} &\|\nabla^k\Big(\gamma(|f|^2)f\Big)\|_{L^1} \lesssim \|\gamma(|f|^2)|\nabla^k f|^{1-\frac{2}{n}}\|_{L^1\cap L^{\frac{n}{n-1}}}\|\nabla^k f\|^{\frac{2}{n}}_{L^2+L^\infty} + \\ &\sum_{\ell=1}^k \sum_{m_1+\ldots+m_{2\ell+1}=k} \|\gamma^{(l)}(|f|^2)\nabla^{m_1}f\ldots\nabla^{m_{2\ell}}f\|_{L^1\cap L^{q'_{m_{2\ell+1}}}}\big(\|f\|_{L^2+L^\infty} + \|\nabla^s f\|_{L^2+L^\infty}\big), \end{aligned} \tag{9.23}$$

where the final term inside the parentheses in (9.23) arises via the second imbedding in (9.22). We claim for $\ell > 0$ that there exist two sets of parameters $q^1_{m_r}$ and $q^2_{m_r}$ for $r = 1, \ldots, 2\ell$ such that

$$\begin{aligned} &2 \leq q^{1,2}_{m_r} \leq q_{m_r}, \qquad \forall r = 1, \ldots, 2\ell, \\ &\sum_{r=1}^{2\ell}\frac{1}{q^1_{m_r}} = 1, \qquad \sum_{r=1}^{2\ell}\frac{1}{q^2_{m_r}} = \frac{1}{q'_{m_{2\ell+1}}}. \end{aligned} \tag{9.24}$$

To prove the claim we let τ be the number of m_r for $r = 1, \ldots, 2\ell$ such that $m_r \geq s - \frac{n}{2}$. Observe that

$$\sum_{r=1:\, m_r \geq s-\frac{n}{2}}^{2\ell} m_r \leq k - m_{2\ell+1}$$

Therefore,

$$\begin{aligned} \sum_{r=1}^{2\ell}\frac{1}{q_{m_r}} &\leq \frac{\tau}{2} - \frac{\tau s - k + m_{2\ell+1}}{n} \leq -\tau(\frac{s}{n} - \frac{1}{2}) + \frac{k - m_{2\ell+1}}{n} \\ &\leq -\tau(\frac{s}{n} - \frac{1}{2}) - \frac{s-k}{n} + \frac{1}{2} \leq \frac{1}{2} \leq \frac{1}{q'_{m_{2\ell+1}}}. \end{aligned}$$

[3] In the case of $m_r = s - \frac{n}{2}$ the value of q_{m_r} can be set arbitrarily large, but the imbedding fails for $q_{m_r} = \infty$. However, since the following argument has some "slack", we can allow ourselves to still set $q_{m_r} = \infty$ for simplicity.

The inequality in the second line above follows since $m_{2\ell+1} \geq s - \frac{n}{2}$, which holds if $\tau > 0$. On the other hand,

$$\sum_{r=1}^{2\ell} \frac{1}{2} = \ell \geq 1$$

and the claim immediately follows, provided that $\ell > 0$. Thus using the sequence $q^1_{m_r}$ to handle the L^1 norm in (9.23) and $q^2_{m_r}$ for the $L^{q'_{m_{2\ell+1}}}$ norm, we obtain

$$\|\nabla^k\Big(\gamma(|f|^2)f\Big)\|_{L^1} \lesssim \|\gamma(|f|^2)|\nabla^k f|^{1-\frac{2}{n}}\|_{L^1\cap L^{\frac{n}{n-1}}}\|\nabla^k f\|^{\frac{2}{n}}_{L^2+L^\infty}+$$
$$\sum_{\ell=1}^{k}\sum_{m_1+\ldots+m_{2\ell+1}=k}\|\gamma^{(\ell)}(|f|^2)\|_{L^\infty}\|\nabla^{m_1}f\|_{L^{q^1_{m_1}}\cap L^{q^2_{m_1}}}\ldots\|\nabla^{m_{2\ell}}f\|_{L^{q^1_{m_{2\ell}}}\cap L^{q^2_{m_{2\ell}}}}\|\nabla^s f\|_{L^2+L^\infty}.$$

By Hölder's inequality

$$\|\gamma(|f|^2)|\nabla^k f|^{1-\frac{2}{n}}\|_{L^1\cap L^{\frac{n}{n-1}}} \lesssim \|\nabla^k f\|^{1-\frac{2}{n}}_{L^2}\|f\|^{p-1}_{L^{(p-1)\frac{2n}{n+2}}\cap L^{2(p-1)}} \lesssim \|f\|^{p-\frac{2}{n}}_{H^s}$$

provided that $p \geq 2 + \frac{2}{n}$, which is dictated by the condition that $(p-1)\frac{2n}{n+2} \geq 2$. Finally, using the property (9.24) together with the estimate (9.16) we obtain

$$\begin{aligned}\|\nabla^k\Big(\gamma(|f|^2)f\Big)\|_{L^1} &\lesssim \|f\|^{p-\frac{2}{n}}_{H^s}\|\nabla^k f\|^{\frac{2}{n}}_{L^2+L^\infty} + (\|f\|^{p-1}_{H^s} + \|f\|^{2k+(p-1-2k)_+}_{H^s})\|\nabla^s f\|_{L^2+L^\infty}\\ &\lesssim t^{-1}\Big(\|f\|^p_{\mathcal{X}_s} + \|f\|^{2k+1-(p-1-2k)_+}_{\mathcal{X}_s}\Big).\end{aligned}$$

Similarly, we estimate

$$\begin{aligned}\|\nabla^k\Big(\gamma(|f|^2)f\Big)\|_{L^2} &\lesssim \|\gamma(|f|^2)\|_{L^2\cap L^\infty}\|\nabla^k f\|_{L^2+L^\infty}+\\ &\sum_{\ell=1}^{k}\sum_{m_1+\ldots+m_{2\ell+1}=k}\|\gamma^{(\ell)}(|f|^2)\nabla^{m_1}f\ldots\nabla^{m_{2\ell}}f\|_{L^2\cap L^{\frac{2q_{m_{2\ell+1}}}{q_{m_{2\ell+1}}-2}}}\big(\|f\|_{L^2+L^\infty} + \|\nabla^s f\|_{L^2+L^\infty}\big)\end{aligned} \tag{9.25}$$

To estimate the first term in (9.25) we note that

$$\|\gamma(|f|^2)\|_{L^2\cap L^\infty} \leq \|f\|^{p-1}_{L^\infty\cap L^{2(p-1)}} \leq \|f\|^{p-2}_{L^\infty}\|f\|_{L^2\cap L^\infty} \lesssim (1+t)^{-\frac{n}{2}(p-2)}\|f\|^{p-1}_{\mathcal{X}_s},$$

where the second inequality follows from interpolating $L^{2(p-1)}$ between L^2 and L^∞ and the last inequality is a consequence of Lemma 9.4 and the definition of the space $\mathcal{X}_s$. Thus

$$\|\gamma(|f|^2)\|_{L^2\cap L^\infty}\|\nabla^k f\|_{L^2+L^\infty} \lesssim (1+t)^{-\frac{n}{2}(p-1)}\|f\|^p_{\mathcal{X}_s}. \tag{9.26}$$

Furthermore, using definition (9.21) we have that

$$\frac{2q_{m_{2\ell+1}}}{q_{m_{2\ell+1}}-2} = \frac{n}{s-m_{2\ell+1}}.$$

Then

$$\|\gamma^{(\ell)}(|f|^2)\nabla^{m_1}f\ldots\nabla^{m_{2\ell}}f\|_{L^2\cap L^{\frac{n}{s-m_{2l+1}}}} \lesssim \|\nabla^{m_1}f\|_{L^{q_{m_1}}+L^\infty}\times \tag{9.27}$$
$$\|\gamma^{(\ell)}(|f|^2)\nabla^{m_2}f\ldots\nabla^{m_{2\ell}}f\|_{L^2\cap L^{\frac{n}{s-m_{2\ell+1}}}\cap L^{\frac{2q_{m_1}}{q_{m_1}-2}}\cap L^{\frac{nq_{m_1}}{q_{m_1}(s-m_{2\ell+1})-n}}}$$

Using the definition of q_{m_1} from (9.21) and the assumption that $m_1 \le m_{2\ell+1}$ we infer that the last norm reduces to the one of the space

$$L^2\cap L^{\frac{n}{s-m_{2\ell+1}}}, \qquad \text{for } m_1 \le s-\frac{n}{2},$$
$$L^2\cap L^{\frac{n}{s-m_{2\ell+1}+(s-m_1-\frac{n}{2})}}, \qquad \text{for } m_1 > s-\frac{n}{2}$$

We now let τ be the number of m_r for $r=2,..,2\ell$ such that $m_r \ge s-\frac{n}{2}$. Observe that since $s>\frac{n}{2}$ and $s\ge k$

$$\sum_{r=2}^{2\ell}\frac{1}{q_{m_r}} \le \frac{\tau}{2} - \frac{s\tau - k + m_{2\ell+1}+m_1}{n}$$
$$= -\tau\Big(\frac{s}{n}-\frac{1}{2}\Big) + \frac{k-m_{2\ell+1}-m_1}{n}$$
$$\le \min\Big\{\frac{s-m_{2\ell+1}}{n}, \frac{(s-m_{2\ell+1}-m_1)+s-\frac{n}{2}}{n}\Big\}$$

On the other hand

$$\sum_{r=2}^{2\ell}\frac{1}{2} = \ell - \frac{1}{2} \ge \frac{1}{2}.$$

It therefore follows that there exist 2 sets of parameters $q^1_{m_r}$ and $q^2_{m_r}$ for $r=1,\ldots,2\ell$ such that

$$\begin{aligned} &2\le q^{1,2}_{m_r}\le q_{m_r}, \qquad \forall r=2,\ldots,2\ell\\ &\sum_{r=2}^{2\ell}\frac{1}{q^1_{m_r}} = 2,\\ &\sum_{r=2}^{2\ell}\frac{1}{q^2_{m_r}} = \frac{(s-m_{2\ell+1}-m_1)+s-\frac{n}{2}}{n} \ \text{ or } \ \frac{s-m_{2\ell+1}}{n}. \end{aligned} \tag{9.28}$$

In either case, with the help of Lemma 9.4, we can estimate

$$\|\gamma^{(\ell)}(|f|^2)\nabla^{m_2}f\ldots\nabla^{m_{2\ell}}f\|_{L^2\cap L^{\frac{n}{s-m_{2\ell+1}}}\cap L^{\frac{2q_{m_1}}{q_{m_1}-2}}\cap L^{\frac{nq_{m_1}}{q_{m_1}(s-m_{2\ell+1})-n}}} \lesssim \|f\|_{H^s}^{(p-1-2\ell)_++2\ell-1}$$

It therefore follows that the second term in (9.25) is

$$\lesssim (1+t)^{-n}\sum_{\ell=1}^{k}\|f\|_{\mathcal{X}_s}^{(p-1-2\ell)_++2\ell+1}. \tag{9.29}$$

Now combining this with (9.26), and using the condition that $p \geq 2 + \frac{2}{n}$, we infer that

$$\|\nabla^k \Big(\gamma(|f|^2) f\Big)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2}-1} (\|f\|_{\mathcal{X}_s}^p + \|f\|_{\mathcal{X}_s}^{2k+1+(p-1-2k)_+}).$$

The proof of (9.19) proceeds along the lines of the argument for the L^2 estimate (9.18). We first observe that since $\zeta(x)$ is an exponentially localized function, the L^q estimate for $1 \leq q \leq 2$ can be reduced to the L^2 estimate. We then note that the condition that $p \geq 2 + \frac{2}{n}$ was only used in the estimate (9.26) which now takes the form

$$\begin{aligned}\|\gamma(|f|^2)\, \zeta\|_{L^2 \cap L^\infty} \|\nabla^k f\|_{L^2+L^\infty} &\lesssim \|\gamma(|f|^2)\|_{L^\infty} \|\nabla^k f\|_{L^2+L^\infty} \\ &\lesssim \||f|^{p-1}\|_{L^\infty} \|\nabla^k f\|_{L^2+L^\infty} \\ &\lesssim (1+t)^{-\frac{n}{2}p} \|f\|_{\mathcal{X}_s}^p\end{aligned}$$

The remaining estimates already have the desired form (9.29). □

9.2 L^1 estimates

In this section and the following we prove Lemma 9.3. We start with the verification of

$$\sum_{k=0}^s \int_0^\infty \|\nabla^k F(t,\cdot)\|_{L^1}\, dt \lesssim \delta^2 + \|Z\|_{\mathcal{X}_s}^2 + \|Z\|_{\mathcal{X}_s}^p$$

with F as in (9.8). By Corollary 7.4 we have

$$\Big|H(t,\sigma^\infty) - H(\sigma(t))\Big| \lesssim \delta^2 (1+t)^{2-n}\, \chi(t,x;\sigma^\infty), \tag{9.30}$$

where $\chi(t,x;\sigma^\infty)$ is a smooth cut-off function localized around the union of the paths $x_j(t;\sigma^\infty) = v_j^\infty t + D_j^\infty$. Moreover, the spatial derivatives of the above difference also satisfy the same estimates. Using the bootstrap assumptions (7.2) we obtain

$$\begin{aligned}\sum_{k=0}^s \int_0^\infty \Big\|\nabla^k \Big(\Big[H(\tau,\sigma^\infty) - H(\sigma(\tau))\Big] Z(\tau)\Big)\Big\|_{L^1} d\tau &\lesssim \delta^2 \sum_{k=0}^s \int_0^\infty \|\nabla^k Z(\tau)\|_{L^2+L^\infty} (1+\tau)^{2-n}\, d\tau \\ &\lesssim \delta^2 \|Z\|_{\mathcal{X}_s} \int_0^\infty (1+\tau)^{2-n-\frac{n}{2}}\, d\tau \lesssim \delta^2 \|Z\|_{\mathcal{X}_s}.\end{aligned} \tag{9.31}$$

The term $\dot{\Sigma} W(\sigma(t))$ obeys the pointwise bound

$$|\dot{\Sigma} W(\sigma(t))| \lesssim \max_j |\dot{\tilde{\sigma}}_j(t)| \chi(t,x;\sigma^\infty)$$

This can be easily seen from the equation (6.4) and Lemma 7.3. The same estimate also holds for the spatial derivatives of the quantity above. Thus, with the help of the already verified estimate (7.1) we infer that

$$\sum_{k=0}^s \int_0^\infty \Big\|\nabla^k \Big(\dot{\Sigma} W(\sigma(\tau))\Big)\Big\|_{L^1} d\tau \lesssim \delta^2 \int_0^\infty (1+\tau)^{-n}\, d\tau \lesssim \delta^2 \tag{9.32}$$

The estimates for the $O(w_1w_2)Z$ and $O(w_1w_2)$ terms in (9.8) are straightforward due to the separation and the exponential localization of the solitons w_1 and w_2, e.g.,

$$\sum_{k=0}^{s}\int_0^\infty \|O\Big(\nabla^k(w_1w_2)\Big)\|_{L^1} \lesssim \int_0^\infty e^{-\alpha_{\min}(L+c\tau)}\,d\tau \le \frac{e^{-\alpha_{\min}L}}{c\alpha_{\min}} \lesssim \frac{\epsilon}{\alpha_{\min}} \lesssim \delta^2 \tag{9.33}$$

Here we have used the separation assumption (2.17) and the condition (2.19), $\alpha_{\min}L \ge |\log\epsilon|$.

The exponential localization of the multi-soliton state w, the bootstrap assumptions (7.2) and the estimate (9.19) of Lemma 9.5 yield the estimate

$$\sum_{k=0}^{s}\int_0^\infty \|O\Big(\nabla^k(|w|^{p-2}Z^2)\Big)\|_{L^1}\,d\tau \lesssim \|Z\|^2_{\mathcal{X}_s}\int_0^\infty (1+\tau)^{-n}\,d\tau \lesssim \|Z\|^2_{\mathcal{X}_s}. \tag{9.34}$$

Finally, with the help of (9.17) (more specifically using the improvement (9.20) of Remark 9.6), we obtain

$$\sum_{k=0}^{s}\int_0^\infty \|\nabla^k(Z^p(\tau))\|_{L^1}\,d\tau \lesssim \|Z\|^p_{\mathcal{X}_s} \tag{9.35}$$

9.3 L^2 estimates

In this subsection we establish the estimate

$$\|F(t,\cdot)\|_{H^s} \lesssim (1+t)^{-\frac{n}{2}-1}\Big(\delta^2 + \|Z\|^2_{\mathcal{X}_s} + \|Z\|^p_{\mathcal{X}_s}\Big).$$

The arguments follows closely those of the previous section. Using the estimates (9.30), (9.19) and the bootstrap assumptions (7.2) we obtain

$$\begin{aligned}\Big\|\Big(H(\tau,\sigma^\infty) - H(\sigma(\tau))\Big)Z(t)\Big\|_{H^s} &\lesssim \delta^2(1+t)^{2-n}\sum_{k=0}^{s}\|\nabla^k Z(t,\cdot)\|_{L^2+L^\infty}\\ &\lesssim \delta^2\|Z\|_{\mathcal{X}_s}(1+t)^{2-n-\frac{n}{2}} \lesssim \delta^2\|Z\|_{\mathcal{X}_s}(1+t)^{-\frac{n}{2}-1}\end{aligned} \tag{9.36}$$

where the last inequality follows since $n \ge 3$. Similar to (9.32)

$$\|\dot\Sigma W(\sigma(t))\|_{H^s} \lesssim \delta^2(1+t)^{-n} \lesssim \delta^2(1+t)^{-\frac{n}{2}-1} \tag{9.37}$$

The estimates for the $O(w_1w_2)Z$ and $O(w_1w_2)$ terms again follow from the separation and the exponential localization of the solitons w_1 and w_2,

$$\|O(w_1w_2)\|_{H^s} \lesssim e^{-\alpha_{\min}(L+ct)} \lesssim \delta^2(1+t)^{-\frac{n}{2}-1}. \tag{9.38}$$

The exponential localization of the multi-soliton state w together with the estimate (9.19) of Lemma 9.5 and the bootstrap assumption (7.2), also give the estimate

$$\|O(|w|^{p-2}Z^2)\|_{H^s} \lesssim \|Z\|^2_{\mathcal{X}_s}(1+t)^{-n} \lesssim \|Z\|^2_{\mathcal{X}_s}(1+t)^{-\frac{n}{2}-1}. \tag{9.39}$$

Finally, using the estimate (9.18) of Lemma 9.5, we obtain

$$\|Z^p(t)\|_{H^s} \lesssim (1+t)^{-\frac{n}{2}-1}\|Z\|^p_{\mathcal{X}_s}. \tag{9.40}$$

This completes the proof of Lemma 9.3.

10 Existence

In Lemmas 8.1 and 9.2 we established the estimates

$$|\dot{\tilde{\sigma}}| \le \frac{1}{4}\delta^2(1+t)^{-n} \tag{10.1}$$

for the admissible path $\sigma(t)$ and

$$\|Z\|_{\mathcal{X}_s} \le \delta^2 \tag{10.2}$$

for the solution $Z(t,x)$ of the nonlinear inhomogeneous matrix charge transfer problem (9.2), under the bootstrap assumptions (7.1), (7.2)

$$|\dot{\tilde{\sigma}}| \le \delta^2(1+t)^{-n}, \tag{10.3}$$

$$\|Z\|_{\mathcal{X}_s} \le \delta C_0^{-1} \tag{10.4}$$

and the condition that Z is asymptotically orthogonal to the null spaces of the Hamiltonians $H_j^*(\sigma)$ with the constant δ^3. In this section we shall show that these are sufficient to establish the existence of the desired admissible path and the perturbation R. We prove existence by iteration. We shall define a sequence of admissible paths $\sigma^{(n)}(t)$ and approximate solutions $Z^{(n)}(t)$ for $n = 1, \dots$ according to the following rules. First, we write the Z equation (9.2) and (9.3) in the form

$$i\partial_t Z + H(\sigma(t))Z = \dot{\Sigma}W(\sigma(t)) + G(Z, \sigma(t)) \tag{10.5}$$

where $G(Z,\sigma(t)) = O(w_1 w_2)Z + O(w_1 w_2) + O(|w|^{p-2}|Z|^2) + O(|Z|^p)$. Set

$$\sigma^{(1)}(t) = \sigma(0), \qquad Z^{(0)} \equiv 0$$

where $\sigma^{(1)}$ is to be understood as the constant path coinciding with the initial data $\sigma(0)$ common to all admissible paths. We now define functions $Z^1(t,x)$ and $\sigma^2(t)$ to be a solution of the following *linear* system

(10.6)
$$\begin{aligned}&i\partial_t Z^{(1)} + H(\sigma^{(1)}(t))Z^{(1)} = \dot{\Sigma}^{(2)}W(\sigma^{(1)}(t)) + G(Z^{(0)}, \sigma^{(1)}(t)),\\ &Z^{(1)}(0,x) = Z_0(x),\end{aligned}$$

(10.7)
$$\Big\langle \dot{\Sigma}^{(2)}W(\sigma^{(1)}(t)), \xi_j^m(t,\cdot;\sigma^{(1)}(t))\Big\rangle = \Big\langle G(Z^{(0)}, \sigma^{(1)}(t)), \xi_j^m(t,\cdot;\sigma^{(1)}(t))\Big\rangle + \Big\langle \Omega_j^m(t,\cdot;\sigma^{(1)}(t)), Z^{(1)}\Big\rangle.$$

Here $\Omega_j^m(t,x;\sigma^{(1)}(t))$ is defined via the equation

$$i\partial_t \xi_j^m(t,\cdot;\sigma^{(1)}(t)) + H^*(\sigma^{(1)}(t))\xi_j^m(t,\cdot;\sigma^{(1)}(t)) = \mathcal{S}_k^m \xi_j^k(t,\cdot;\sigma^{(1)}(t)) + \Omega_j^m(t,\cdot;\sigma^{(1)}(t)),$$

where the matrix $\mathcal{S}$ collects the terms $\xi_j^m(t,\cdot;\sigma^{(1)}(t))$ on the right-hand sides of (6.5)-(6.8). Thus, using Propositions 6.2 and 6.3, we have

$$\Omega_j^m(t,x;\sigma^{(1)}(t)) = O\big(\dot{\tilde{\sigma}}^{(1)}(|\phi_j| + |D\phi_j| + |D^2\phi_j|)\big) + \sum_{r\ne j} V_r(t,x;\sigma^{(1)}(t))\xi_j^m(t,x;\sigma^{(1)}(t)).$$

Observe that (10.6) arises from the nonlinear equation (10.5) by replacing $\sigma(t)$ with the already defined path $\sigma^{(1)}(t)$ as well as Z on the right-hand side with $Z^{(0)} = 0$. The equation (10.7) determining $\sigma^{(2)}(t)$ ensures that $\langle Z^{(1)}(t), \xi_j^m(t, \cdot; \sigma^{(1)}(t))\rangle = 0$. Indeed, taking scalar products of (10.6) with $\xi_j^m(t, \cdot; \sigma^{(1)}(t))$ and using Propositions 6.2 and 6.3 yields

$$\frac{d}{dt}\Xi(t; \sigma^{(1)}(t)) + \mathcal{S}\Xi(t; \sigma^{(1)}(t)) = 0,$$

where $\Xi(t; \sigma^{(1)}(t))$ is the vector of $\langle \xi_j^m(t, \cdot; \sigma^{(1)}(t)), Z^{(1)}(t, \cdot)\rangle$ and where $\mathcal{S}$ is the constant matrix defined above. As long as

$$\Xi(0; \sigma^{(1)}(0)) = \Xi(0; \sigma(0)) = 0 \tag{10.8}$$

one therefore has $\Xi(t; \sigma^{(1)}(t)) = 0$ for all $t \geq 0$. Generally speaking, (10.8) need not be satisfied. However, using the fact that the initial perturbation R_0 is small, we proceed as in [BP1] Proposition 1.3.1 to show that one can modify the initial splitting in such a way that it does hold. More precisely, one has the following lemma.

Lemma 10.1. *Let $\psi(0) = \sum_{j=1}^N w_j(0, x; \sigma(0)) + R_0$ with R_0 small as in* (2.9) *and assume either one of the separation conditions* (2.17) *or* (2.18) *as well as the convexity condition* (2.16) *in a small neighborhood $\mathcal{U}$ of $\sigma(0)$. Then there exist $\widetilde{\sigma(0)} \in \mathcal{U}$ such that in the decomposition*

$$\psi(0) = \sum_{j=1}^N w_j(0, x; \widetilde{\sigma(0)}) + \widetilde{R_0}$$

the new perturbation $\begin{pmatrix} \widetilde{R_0} \\ \overline{\widetilde{R_0}} \end{pmatrix}$ is orthogonal to the root spaces $\mathcal{N}_j^*$ of $H_j^*(\widetilde{\sigma(0)})$ and satisfies the smallness condition (2.9).

Proof. We need to solve the equation (with the solution being $\widetilde{\sigma(0)}$)

$$\Big\langle \Psi(0) - \sum_{j=1}^N W_j(0, \cdot; \sigma(0)), J\partial_{\sigma_r} W_\ell(0, \cdot; \sigma(0))\Big\rangle = 0 \quad \text{for all} \quad 1 \leq r \leq 2n+2, \tag{10.9}$$

where J is the matrix from Proposition 4.1 and Ψ, W_j are the complexified versions of ψ, w_j, respectively. One solves (10.9) by means of the implicit function theorem. Indeed, the derivative of the left-hand side of (10.9) is given by

$$\Big\langle -\sum_{j=1}^N \partial_{\sigma_k} W_j(0, \cdot; \sigma(0)), J\partial_{\sigma_r} W_\ell(0, \cdot; \sigma(0))\Big\rangle + \Big\langle \Psi(0) - \sum_{j=1}^N W_j(0, \cdot; \sigma(0)), J\partial^2_{\sigma_r\sigma_k} W_\ell(0, \cdot; \sigma(0))\Big\rangle.$$

The second term is $O(\varepsilon)$ where ε controls the size of the initial perturbation R_0 in L^1, say. On the other hand, the first term is separated from zero by virtue of the convexity condition, and either the separation condition (2.17) or the assumption of large relative velocities of the solitons (2.18). Indeed, as in Proposition 1.3.1 from [BP1] one sees that

(10.10)
$$\Big|\det\Big\{\Big\langle \partial_{\sigma_k} W_\ell(0,\cdot;\sigma(0)), J\partial_{\sigma_r} W_\ell(0,\cdot;\sigma(0))\Big\rangle\Big\}_{1\le k,\ell\le 2n+2}\Big| \gtrsim \|\phi(\cdot;\alpha_\ell(0))\|_2^4\big(\partial_\alpha\|\phi(\cdot;\alpha_\ell(0))\|_2^2\big)^2 > 0$$

uniformly in the small neighborhood of $\alpha_\ell(0)$ that we are allowing, see the convexity condition (2.16). The remaining entries of the derivative matrix, which involve inner products with W_j, W_k for $j \neq k$, are small because of either the (physical) separation condition (2.17) or the velocity condition (2.18). The latter ensures that we are taking scalar products of quantities that are almost orthogonal by virtue of the large distances of their Fourier transforms. Hence the determinant of the derivative is essentially bounded below by the product of the matrices with $j = k$, see (10.10). This proves that the derivatives are invertible, and since the original perturbation R_0 is small, the image of the diffeomorphism given by the left-hand side of (10.9) contains zero, as claimed. □

In general, we define

(10.11)
$$i\partial_t Z^{(n)} + H(\sigma^{(n)}(t))Z^{(n)} = \dot\Sigma^{(n+1)}W(\sigma^{(n)}(t)) + G(Z^{(n-1)},\sigma^{(n)}(t))$$
$$Z^{(n)}(0,x) = Z_0(x)$$

(10.12)
$$\Big\langle \dot\Sigma^{(n+1)}W(\sigma^{(n)}(t)), \xi_j^m(t,\cdot;\sigma^{(n)}(t))\Big\rangle = \Big\langle G(Z^{(n-1)},\sigma^{(n)}(t)), \xi_j^m(t,\cdot;\sigma^{(n)}(t))\Big\rangle + \Big\langle \Omega_j^m(t,\cdot;\sigma^{(n)}(t)), Z^{(n)}(t)\Big\rangle.$$

Here

$$G(Z^{(n-1)},\sigma^{(n)}(t)) = O(w_1^{(n)}w_2^{(n)}) + O(w_1^{(n)}w_2^{(n)})Z^{(n-1)} + O(|w^{(n)}|^{p-2}|Z^{(n-1)}|^2) + O(|Z^{(n-1)}|^p)$$

and $\Omega_j^m(t,x;\sigma^{(n)}(t))$ once again is defined via the equation

$$i\partial_t\xi_j^m(t,\cdot;\sigma^{(n)}(t)) + H^*(\sigma^{(n)}(t))\xi_j^m(t,\cdot;\sigma^{(n)}(t)) = \mathcal{S}_k^m\xi_j^k(t,\cdot;\sigma^{(n)}(t)) + \Omega_j^m(t,\cdot;\sigma^{(n)}(t)),$$

and has the form

$$\Omega_j^m(t,x;\sigma^{(n)}(t)) = O\big(\dot{\tilde\sigma}^{(n)}(|\phi_j| + |D\phi_j| + |D^2\phi_j|)\big) + \sum_{r\neq j} V_r(t,x;\sigma^{(n)}(t))\xi_j^m(t,x;\sigma^{(n)}(t)).$$

Observe that by the same argument as in the case of $Z^{(1)}$, the perturbation $Z^{(n)}$ is orthogonal to the functions $\xi_j^m(t,x;\sigma^{(n)}(t))$. We shall assume that solutions $Z^{(n)}, \sigma^{(n)}$ of (10.11), (10.12) have already been constructed and we now proceed to estimate them. We will return to the issue of constructing those solutions at the end of this section. We shall now assume that $\sigma^{(n)}$ and $Z^{(n-1)}$ satisfy (10.3) and (10.4) and prove the estimates (10.1) and (10.2) for $\sigma^{(n+1)}$ and $Z^{(n)}$. First we estimate $\dot{\tilde\sigma}^{(n+1)}$ in terms of $Z^{(n)}$. Observe that $\dot\sigma^{(n+1)}$ verifies the system of ODE's described in Lemma 8.2 with the function $G(Z^{(n)}(t),\sigma^{(n)}(t)$ and $\Omega_j^m(t,x;\sigma^{(n)}(t))$. Therefore,

$$|\dot{\tilde\sigma}^{(n+1)}(t)| \le (1+t)^{-n}(\frac{1}{4}\delta^2 + C\|Z^{(n)}\|_{\mathcal{X}_s}^2 + C\|Z^{(n)}\|_{\mathcal{X}_s}^p) \tag{10.13}$$

We now consider the $Z^{(n)}$ equation. First, by construction $Z^{(n)}$ is orthogonal to $\xi_j^m(t,x;\sigma^{(n)}(t))$. Observe also that $Z^{(n)}$ satisfies the equation

$$i\partial_t Z^{(n)} + H(t,\sigma^{(n)\infty})Z^{(n)} = \Big(H(t,\sigma^{(n)\infty}) - H(\sigma^{(n)}(t))\Big)Z^{(n)} + \dot\Sigma^{(n+1)}W(\sigma^{(n)}(t)) + G(Z^{(n-1)},\sigma^{(n)}(t))$$

Corollary 7.4 implies that

$$H(t,\sigma^{(n)\infty}) - H(\sigma^{(n)}(t)) = V(t,x),$$

where a smooth localized potential V has the property that $\sup_{0\le|\gamma|\le s}\|\partial^\gamma V(t,\cdot)\|_{L^1\cap L^\infty} \lesssim \delta^2(1+t)^{-n+2}$. Moreover, the calculation leading to (9.6) of Lemma 9.1 shows that

$$\|\mathcal{M}_j(\sigma,t)\mathcal{G}_{v_j,D_j}(t)\xi_j^m(t,\cdot;\sigma^{(n)}(t)) - \xi_j^m(\cdot;\sigma^{(n)\infty})\|_{L^1\cap L^2} \lesssim \delta^2(1+t)^{-n+2}$$

since $\sigma^{(n)}$ is an admissible path satisfying the bootstrap assumptions. Therefore, by the results of Corollary 5.4, taking into account smallness of the initial data $Z^{(n)}(0) = Z_0$,

$$\begin{aligned}
\|Z^{(n)}\|_{\mathcal{X}_s} &\lesssim \sum_{k=0}^s \|\nabla^k Z_0\|_{L^1\cap L^2} + \|G(Z^{(n-1)},\sigma^{(n)})\|_{\mathcal{Y}_s} + \|\dot\Sigma^{(n+1)}W(\sigma^{(n)}(t))\|_{\mathcal{Y}_s}\\
&\lesssim \delta^2 + \|Z^{(n-1)}\|^2_{\mathcal{X}_s} + \|Z^{(n-1)}\|^p_{\mathcal{X}_s} + \|Z^{(n)}\|^2_{\mathcal{X}_s} + \|Z^{(n)}\|^p_{\mathcal{X}_s}, \qquad (10.14)\\
&\lesssim \delta^2 + \|Z^{(n)}\|^2_{\mathcal{X}_s} + \|Z^{(n)}\|^p_{\mathcal{X}_s}, \qquad (10.15)
\end{aligned}$$

where the inequality leading to (10.14) follows from the estimates on the nonlinear term $G(Z^{(n-1)}(t),\sigma^{(n)}(t))$ and $\dot\Sigma^{(n+1)}W(\sigma^{(n)}(t))$ obtained in Lemma 9.3. The bound on $\dot\Sigma^{(n+1)}W(\sigma^{(n)}(t))$ also uses the inequality (10.13). To pass to (10.15) we used the assumption that $Z^{(n-1)}$ satisfies (10.2). In the same way one obtains a local in time version of equation (10.15).

$$\|Z^{(n)}\|_{\mathcal{X}_s(T)} \lesssim \delta^2 + \|Z^{(n)}\|^2_{\mathcal{X}_s(T)} + \|Z^{(n)}\|^p_{\mathcal{X}_s(T)}, \qquad (10.16)$$

which by continuity in T implies the desired estimate $\|Z^{(n)}\|_{\mathcal{X}_s} \lesssim \delta^2$.

Thus, the sequence $Z^{(n)}$ is uniformly bounded and small in the space $\mathcal{X}_s$ while $(1+t)^n\dot{\tilde\sigma}^{(n)}$ is uniformly small pointwise in time. Therefore, we can choose a convergent subsequence of the paths $\sigma^{(k)}(t)\to\sigma(t)$ and a weekly convergent in $H^s(\mathbb{R}^n)$ subsequence $Z^{(k)}\to Z$. We multiply the equation (10.11) by a smooth compactly supported function $\zeta(x)$, integrate over the entire space and pass to the limit usnig that on any compact set $Z^{(n)}\to Z$ strongly in $H^{s'}$ for any $s'<s$. In particular, since $s>\frac{n}{2}$, $Z^{(n)}\to Z$ pointwise. It will follow that Z is a solution of the equation

$$\begin{aligned}
&i\partial_t Z + H(\sigma(t))Z = \dot\Sigma W(\sigma(t)) + G(Z(t),\sigma(t)), \qquad (10.17)\\
&Z(0,x) = Z_0(x)
\end{aligned}$$

We also pass to the limit in the equation (10.12) to obtain

$$\Big\langle \dot\Sigma W(\sigma(t)), \xi_j^m(t,\cdot;\sigma(t))\Big\rangle = \Big\langle G(Z(t),\sigma(t)),\xi_j^m(t,\cdot;\sigma(t))\Big\rangle + \Big\langle \Omega_j^m(t,\cdot;\sigma(t)), Z\Big\rangle. \qquad (10.18)$$

Comparing equations (10.17) and (10.18) we conclude that

$$\langle Z(t),\xi_j^m(t,\cdot;\sigma(t))\rangle = 0$$

for all j, m. Therefore, the function $\psi = R + w_1 + w_2$ solves the original NLS and by uniqueness, say in L^2, ψ is our original solution.

To show existence of the solution $Z^{(n)}, \sigma^{(n+1)}$ of the linear system (10.11), (10.12) we first construct the solution on a small time interval. We note that the "system" (10.12) for $\dot{\sigma}^{(n+1)}$ can be resolved algebraically due to the spatial separation of the paths $\sigma_j^{(n)}(t)$. Therefore, for simplicity we can replace the system (10.11), (10.12) by the following caricature:

$$
\begin{aligned}
&i\partial_t z + \frac{1}{2}\triangle z = V(t,x)z + \omega(t)a(t,x) + g(t,x),\\
&\omega(t) = \langle z, b(t,\cdot)\rangle + f(t)
\end{aligned}
$$

Here V, a, b, f are sufficiently smooth given functions and $g(t,\cdot) \in H^s$ uniformly in t. We eliminate $\omega(t)$ and infer that

$$
i\partial_t z + \frac{1}{2}\triangle z = V(t,x)z + \langle z, c(t,\cdot)\rangle a(t,x) + F(t,x)
$$

with some new smooth functions c, a and an H^s function $F(t,\cdot)$. Using the standard energy estimates we obtain that

$$
\|z(t)\|_{H^s} \le \|z_0\|_{H^s} + C_1 t \sup_{\tau\le t}\|z(\tau)\|_{H^s} + C_2 t \sup_{\tau \le t}\|F(\tau,\cdot)\|_{H^s},
$$

where the constants C_1 and C_2 depend on V, a, b, f. Therefore, we can establish the existence of the solution on the time interval of size $\frac{1}{2}C_1^{-1}$ by means of the standard contraction argument. Then we can repeat this argument indefinitely thus constructing a global *classical* solution.

11 The linearized problem

11.1 Estimates for matrix charge transfer models

In this section we recall some of the estimates from Sections 7 and 8 from our companion paper [RSS]. First, consider the case of a system with a single matrix potential:

$$
i\partial_t \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} + \begin{pmatrix} H+U & -W \\ W & -H-U\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = 0 \tag{11.1}
$$

with U, W real-valued and $H = \frac{1}{2}\triangle - \mu$, $\mu > 0$. We say that $A := \begin{pmatrix} H+U & -W \\ W & -H-U\end{pmatrix}$ is *admissible* iff the conditions of the following Definition 11.1 hold.

Definition 11.1. *Let A be as above with U, W real-valued and exponentially decaying. The operator A on* $\mathrm{Dom}(A) = H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \subset \mathcal{H} := L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *is* admissible *provided*

- $\mathrm{spec}(A) \subset \mathbb{R}$ *and* $\mathrm{spec}(A) \cap (-\mu, \mu) = \{\omega_\ell \mid 0 \le \ell \le M\}$, *for some $M < \infty$ where $\omega_0 = 0$ and all ω_j are distinct eigenvalues. There are no eigenvalues in* $\mathrm{spec}_{ess}(A) = (-\infty, -\mu] \cup [\mu, \infty)$.
- *For $1 \le \ell \le M$, $L_\ell := \ker(A - \omega_\ell)^2 = \ker(A - \omega_\ell)$, and* $\ker(A) \subsetneq \ker(A^2) = \ker(A^3) =: L_0$. *Moreover, these spaces are finite dimensional.*

- *The ranges* $\mathrm{Ran}(A-\omega_\ell)$ *for* $1 \le \ell \le M$ *and* $\mathrm{Ran}(A^2)$ *are closed.*
- *The spaces* L_ℓ *are spanned by exponentially decreasing functions in* $\mathcal{H}$ *(say with bound* $e^{-\varepsilon_0|x|}$*).*
- *The points* $\pm\mu$ *are not resonances of* A*.*
- *All these assumptions hold as well for the adjoint* A^**. We denote the corresponding (generalized) eigenspaces by* L_ℓ^**.*

We will discuss these conditions in detail in the following Subsection 11.2. It is possible to establish some of these properties by means of "abstract" methods (for example, the exponential decay of elements of generalized eigenspaces via a variant of Agmon's argument, or the closedness of $\mathrm{Ran}(A-\omega_\ell)$ from Fredholm's theory), whereas others can be reduced to statements concerning certain semi-linear elliptic operators L_+, L_-, see (11.31) (for example, that the spectrum is real or that only 0 can have a generalized eigenspace). In a later section we will prove for a particular model that L_+, L_- have the required properties. One condition that we will not deal with in this paper is the absence of imbedded eigenvalues in the essential spectrum. This property will remain an assumption.

It is shown in [RSS], Lemma 7.2 that under these conditions there is a direct sum decomposition

$$\mathcal{H} = \sum_{j=0}^{M} L_j + \Big(\sum_{j=0}^{M} L_j^*\Big)^{\perp} \tag{11.2}$$

and we denote by P_s the induced projection onto $\Big(\sum_{j=0}^{M} L_j^*\Big)^{\perp}$. In general, P_s is non-orthogonal. The letter "s" here stands for "scattering" (subspace). It is known that $\mathrm{Ran}(P_s)$ plays the role of the scattering states for the evolution e^{itA}. Indeed, the main result from Section 7 in [RSS] is that if A is admissible and the *linear stability property*

$$\sup_t \|e^{itA}P_s\|_{2\to 2} < \infty \tag{11.3}$$

holds, then one has the dispersive bound

$$\|e^{itA}P_s\psi_0\|_{L^2+L^\infty} \lesssim |t|^{-\frac{3}{2}}\|\psi_0\|_{L^1\cap L^2} \tag{11.4}$$

(if in addition $\|\hat{V}\|_1 < \infty$, then the L^2 norm can be removed on the left-hand side). Next, we recall the notion of matrix charge transfer models from Section 8 in [RSS].

Definition 11.2. *By a* matrix charge transfer model *we mean a system*

$$\begin{aligned} & i\partial_t\vec{\psi} + \begin{pmatrix} \frac{1}{2}\triangle & 0 \\ 0 & -\frac{1}{2}\triangle \end{pmatrix}\vec{\psi} + \sum_{j=1}^{\nu} V_j(\cdot - \vec{v}_j t)\vec{\psi} = 0 \\ & \vec{\psi}|_{t=0} = \vec{\psi}_0, \end{aligned} \tag{11.5}$$

where $\vec{v}_j$ *are distinct vectors in* $\mathbb{R}^3$*, and* V_j *are matrix potentials of the form*

$$V_j(t,x) = \begin{pmatrix} U_j(x) & -e^{i\theta_j(t,x)}W_j(x) \\ e^{-i\theta_j(t,x)}W_j(x) & -U_j(x) \end{pmatrix},$$

where $\theta_j(t,x) = (|\vec{v}_j|^2 + \alpha_j^2)t + 2x \cdot \vec{v}_j + \gamma_j$, $\alpha_j, \gamma_j \in \mathbb{R}$, $\alpha_j \neq 0$. *Furthermore, we require that each*

$$H_j = \begin{pmatrix} \frac{1}{2}\triangle - \frac{1}{2}\alpha_j^2 + U_j & -W_j \\ W_j & -\frac{1}{2}\triangle + \frac{1}{2}\alpha_j^2 - U_j \end{pmatrix}$$

be admissible in the sense of Definition 11.1 and that it satisfy the linear stability condition (11.3).

It is clear that the Hamiltonian in (2.23) is of this form. As in Lemma 3.1 above one now verifies the following. The Galilei transforms $\mathcal{G}_{\vec{v}} := \mathcal{G}_{\vec{v},0}$ are defined as in (3.11), i.e.,

$$\mathcal{G}_{\vec{v}}(t)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{g}_{\vec{v},0}(t) f_1 \\ \overline{\mathfrak{g}_{\vec{v},0}(t) \bar{f_2}} \end{pmatrix}$$

where $\mathfrak{g}_{\vec{v},0}(t) = e^{-i\frac{|\vec{v}|^2}{2}t} e^{-ix\cdot\vec{v}} e^{it\vec{v}\cdot p}$.

Lemma 11.3. *Let* $\alpha \in \mathbb{R}$ *and set*

$$A := \begin{pmatrix} \frac{1}{2}\triangle - \frac{1}{2}\alpha^2 + U & -W \\ W & -\frac{1}{2}\triangle + \frac{1}{2}\alpha^2 - U \end{pmatrix}$$

with real-valued U, W. *Moreover, let* $\vec{v} \in \mathbb{R}^3$, $\theta(t,x) = (|\vec{v}|^2 + \alpha^2)t + 2x \cdot \vec{v} + \gamma$, $\gamma \in \mathbb{R}$, *and define*

$$H(t) := \begin{pmatrix} \frac{1}{2}\triangle + U(\cdot - \vec{v}t) & -e^{i\theta(t,\cdot-\vec{v}t)}W(\cdot - \vec{v}t) \\ e^{-i\theta(t,\cdot-\vec{v}t)}W(\cdot - \vec{v}t) & -\frac{1}{2}\triangle - U(\cdot - \vec{v}t) \end{pmatrix}.$$

Let $\mathcal{S}(t)$, $\mathcal{S}(0) = Id$, *denote the propagator of the system*

$$i\partial_t \mathcal{S}(t) + H(t)\mathcal{S}(t) = 0.$$

Finally, let

$$\mathcal{M}(t) = \mathcal{M}_{\alpha,\gamma}(t) = \begin{pmatrix} e^{-i\omega(t)/2} & 0 \\ 0 & e^{i\omega(t)/2} \end{pmatrix} \tag{11.6}$$

where $\omega(t) = \alpha^2 t + \gamma$. *Then*

$$\mathcal{S}(t) = \mathcal{G}_{\vec{v},0}(t)^{-1}\mathcal{M}(t)^{-1} e^{itA}\mathcal{M}(0)\mathcal{G}_{\vec{v},0}(0). \tag{11.7}$$

Proof. One has

$$i\partial_t \mathcal{M}(t)\mathcal{G}_{\vec{v}}(t)\mathcal{S}(t) = \begin{pmatrix} \frac{1}{2}\dot{\omega} & 0 \\ 0 & -\frac{1}{2}\dot{\omega} \end{pmatrix}\mathcal{M}(t)\mathcal{G}_{\vec{v}}(t)\mathcal{S}(t) + \mathcal{M}(t)i\dot{\mathcal{G}}_{\vec{v}}(t)\mathcal{S}(t) - \mathcal{M}(t)\mathcal{G}_{\vec{v}}(t)H(t)\mathcal{S}(t). \tag{11.8}$$

Let $\rho(t,x) = t|\vec{v}|^2 + 2x \cdot \vec{v}$. One now checks the following properties by differentiation:

$$\begin{aligned} \mathcal{M}(t)i\dot{\mathcal{G}}_{\vec{v}}(t) &= -\begin{pmatrix} \frac{1}{2}|\vec{v}|^2 + \vec{v}\cdot\vec{p} & 0 \\ 0 & -\frac{1}{2}|\vec{v}|^2 + \vec{v}\cdot\vec{p} \end{pmatrix}\mathcal{M}(t)\mathcal{G}_{\vec{v}}(t) \\ \mathcal{M}(t)\mathcal{G}_{\vec{v}}(t)H(t) &= \begin{pmatrix} \frac{1}{2}\triangle + U & -e^{i(\theta-\rho-\omega)}W \\ e^{-i(\theta-\rho-\omega)}W & -\frac{1}{2}\triangle - U \end{pmatrix}\mathcal{M}(t)\mathcal{G}_{\vec{v}}(t) \\ &\quad - \begin{pmatrix} \frac{1}{2}|\vec{v}|^2 + \vec{v}\cdot\vec{p} & 0 \\ 0 & -\frac{1}{2}|\vec{v}|^2 + \vec{v}\cdot\vec{p} \end{pmatrix}\mathcal{M}(t)\mathcal{G}_{\vec{v}}(t). \end{aligned} \tag{11.9}$$

The right-hand side of (11.9) arises as follows. First, the Galilei transform introduces a factor of $e^{-ix\cdot\vec{v}}$, which needs to be commuted with $\frac{1}{2}\triangle$. Since

$$\begin{aligned}\frac{1}{2}\triangle\Big(e^{-ix\cdot\vec{v}}f\Big) &= -\frac{1}{2}|\vec{v}|^2e^{-ix\cdot\vec{v}}f - e^{-ix\cdot\vec{v}}i\vec{v}\cdot\vec{\nabla}f + \frac{1}{2}e^{-ix\cdot\vec{v}}\triangle f\\ &= \frac{1}{2}|\vec{v}|^2e^{-ix\cdot\vec{v}}f - i\vec{v}\cdot\vec{\nabla}\Big(fe^{-ix\cdot\vec{v}}\Big) + \frac{1}{2}e^{-ix\cdot\vec{v}}\triangle f\\ &= \Big(\frac{1}{2}|\vec{v}|^2 + \vec{v}\cdot\vec{p}\Big)\Big(fe^{-ix\cdot\vec{v}}\Big) + \frac{1}{2}e^{-ix\cdot\vec{v}}\triangle f,\end{aligned}$$

one obtains the final term on the right-hand side of (11.9). It remains to check the terms involving the potentials (for simplicity $\theta(\cdot - t\vec{v}) = \theta(t, \cdot - \vec{v}t)$):

$$\begin{aligned}&\mathcal{M}(t)\mathcal{G}_{\vec{v}}(t)\begin{pmatrix} U(\cdot-\vec{v}t) & -e^{i\theta(t,\cdot-\vec{v}t)}W(\cdot-\vec{v}t)\\ e^{-i\theta(t,\cdot-\vec{v}t)}W(\cdot-\vec{v}t) & -U(\cdot-\vec{v}t)\end{pmatrix}\begin{pmatrix}f_1\\ f_2\end{pmatrix}\\ =\ &\begin{pmatrix} e^{-i\omega(t)/2} & 0\\ 0 & e^{i\omega(t)/2}\end{pmatrix}\begin{pmatrix}\mathfrak{g}_{\vec{v}}(t)U(\cdot-\vec{v}t)f_1 - \mathfrak{g}_{\vec{v}}(t)e^{i\theta(\cdot-\vec{v}t)}W(\cdot-\vec{v}t)f_2\\ \overline{\mathfrak{g}_{\vec{v}}(t)e^{i\theta(\cdot-\vec{v}t)}W(\cdot-\vec{v}t)\overline{f_1}} - \overline{\mathfrak{g}_{\vec{v}}(t)U(\cdot-\vec{v}t)\overline{f_2}}\end{pmatrix}\\ =\ &\begin{pmatrix}U\mathfrak{g}_{\vec{v}}(t)(e^{-i\omega(t)/2}f_1) - We^{-i(v^2t+2x\cdot\vec{v})}e^{i(\theta-\omega)}\overline{\mathfrak{g}_{\vec{v}}(t)\overline{e^{i\omega(t)/2}f_2}}\\ We^{i(v^2t+2x\cdot\vec{v})}e^{i(\omega-\theta)}\ \mathfrak{g}_{\vec{v}}(t)(e^{-i\omega(t)/2}f_1) - U\ \overline{\mathfrak{g}_{\vec{v}}(t)\overline{e^{i\omega(t)/2}f_2}}\end{pmatrix}\\ =\ &\begin{pmatrix} U & -e^{i(\theta-\omega-\rho)}W\\ e^{-i(\theta-\omega-\rho)}W & -U\end{pmatrix}\begin{pmatrix} e^{-i\omega(t)/2} & 0\\ 0 & e^{i\omega(t)/2}\end{pmatrix}\begin{pmatrix}\mathfrak{g}_{\vec{v}}(t)f_1\\ \overline{\mathfrak{g}_{\vec{v}}(t)\overline{f_2}}\end{pmatrix},\end{aligned}$$

as claimed. In view of our definitions, $\theta - \rho - \omega = 0$. Since $\dot{\omega} = \alpha^2$, the lemma follows by inserting (11.9) into (11.8). $\square$

In order to prove our main dispersive estimates for such matrix charge transfer problems we need to formulate a condition which ensures that the initial condition belongs to the stable subspace. To do so, let $P_s(H_j)$ and $P_b(H_j)$ be the projectors induced by the decomposition (11.2) for the operator H_j. Abusing terminology somewhat, we refer to $\mathrm{Ran}(P_b(H_j))$ as the *bound states* of H_j.

Definition 11.4. *Let $U(t)\vec{\psi}_0 = \vec{\psi}(t,\cdot)$ be the solution of* (11.5)*. We say that $\vec{\psi}_0$ is a* scattering state *relative to H_j if*

$$\|P_b(H_j,t)U(t)\vec{\psi}_0\|_{L^2} \to 0 \ \text{as}\ t \to +\infty.$$

Here

$$P_b(H_j,t) := \mathcal{G}_{\vec{v}_j}(t)^{-1}\mathcal{M}_j(t)^{-1}P_b(H_j)\,\mathcal{M}_j(t)\mathcal{G}_{\vec{v}_j}(t) \tag{11.10}$$

with $\mathcal{M}_j(t) = \mathcal{M}_{\alpha_j,\gamma_j}(t)$ as in (11.6)*.*

The formula (11.10) is of course motivated by (11.7). Clearly, $P_b(H_j,t)$ is the projection onto the bound states of H_j that have been translated to the position of the matrix potential $V_j(\cdot - t\vec{v}_j)$. Equivalently, one can think of it as translating the solution of (11.5) from that position to the origin, projecting onto the bound states of H_j, and then translating back.

We now formulate our decay estimate for matrix charge transfer models, see Theorem 8.6 in [RSS].

Theorem 11.5. *Consider the matrix charge transfer model as in Definition 11.2. Let $U(t)$ denote the propagator of the equation* (11.5)*. Then for any initial data $\vec{\psi}_0 \in L^1 \cap L^2$, which is a scattering state relative to each H_j in the sense of Definition 11.4, one has the decay estimates*

$$\|U(t)\vec{\psi}_0\|_{L^\infty} \lesssim \langle t\rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{L^1\cap L^2}. \tag{11.11}$$

For technical reasons, we need the estimate (11.11) for perturbed matrix charge transfer equations, as described in the following corollary. This is discussed in Remark 8.6 in [RSS].

Corollary 11.6. *Let $\vec{\psi}$ be a solution of the equation*

$$\begin{aligned} &i\partial_t\vec{\psi} + \begin{pmatrix} \frac{1}{2}\triangle & 0 \\ 0 & -\frac{1}{2}\triangle \end{pmatrix} \vec{\psi} + \sum_{j=1}^{\nu} V_j(\cdot - \vec{v}_j t)\vec{\psi} + V_0(t,x)\vec{\psi} = 0 \\ &\vec{\psi}|_{t=0} = \vec{\psi}_0, \end{aligned} \tag{11.12}$$

where everything is the same as in Definition 11.2 up to the perturbation $V_0(t,x)$ which satisfies

$$\sup_t \|V_0(t,\cdot)\|_{L^1\cap L^\infty} < \varepsilon.$$

Let $\tilde{U}(t)$ denote the propagator of the equation (11.12)*. Then for any initial data $\vec{\psi}_0 \in L^1 \cap L^2$, which is a scattering state relative to each H_j in the sense of Definition 11.4 (with $U(t)$ replaced by $\tilde{U}(t)$), one has the decay estimates*

$$\|\tilde{U}(t)\vec{\psi}_0\|_{L^\infty} \lesssim \langle t\rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{L^1\cap L^2} \tag{11.13}$$

provided ε is sufficiently small.

The corresponding inhomogeneous bound is stated in Section 11.

11.2 The spectral properties I: general arguments

In order for the linear estimates to apply, we need to impose the conditions in Definition 11.1 as well as the linear stability condition (11.3) on the operators from (2.24). The admissibility conditions of Definition 11.1 were motivated to a large extent by Buslaev and Perelman [BP1], who built on earlier work of Weinstein [We1]. We now analyse these conditions in detail. As before,

$$A := \begin{pmatrix} H+U & -W \\ W & -H-U \end{pmatrix} = B + V \tag{11.14}$$

where U, W are real-valued, $H = \frac{1}{2}\triangle - \mu$ with $\mu > 0$, and V is the matrix potential consisting of U, W. In this subsection we deal with those properties that can be dealt with by means of general arguments, that make no use of any special structure of the operator.

Lemma 11.7. *Let the matrix potential V be bounded and go to zero at infinity. Then $(A-z)^{-1}$ is a meromorphic function in $\Omega := \mathbb{C} \setminus (-\infty, -\mu] \cup [\mu, \infty)$. The poles are eigenvalues of A of finite multiplicity and* $\mathrm{Ran}(A-z)$ *is closed for all $z \in \Omega$. Finally, the complement of Ω agrees with the essential spectrum of A, i.e.,* $\mathrm{spec}_{\mathrm{ess}}(A) = (-\infty, -\mu] \cup [\mu, \infty)$.

Proof. Suppose that $z \in \Omega$. Then $B - z$ is invertible, and $A - z = \Big(1 + V(B-z)^{-1}\Big)(B-z)$. Since $V(B-z)^{-1}$ is analytic and compact in that region of z's, the analytic Fredholm theorem implies that $1 + V(B-z)^{-1}$ is invertible for all but a discrete set of z's in Ω. Furthermore, the poles are precisely eigenvalues of A of finite multiplicity. It is also a general property that the ranges $\mathrm{Ran}(1+V(B-z)^{-1})$ are closed. Indeed, if K is any compact operator on a Banach space, then it is well-known and also easy to see that $\mathrm{Ran}(I-K)$ is closed. Since $B-z$ has a bounded inverse for all $z \in \Omega$, this implies that $\mathrm{Ran}(A-z)$ is closed, as claimed. Conjugating by the matrix $P = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ leads to the Hamiltonians

$$\tilde{A} := P^{-1}AP = i\begin{pmatrix} 0 & H+V_1 \\ -H-V_2 & 0 \end{pmatrix} = \tilde{B} + V, \quad \tilde{B} = i\begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, \quad V = i\begin{pmatrix} 0 & V_1 \\ -V_2 & 0 \end{pmatrix} \tag{11.15}$$

where $V_1 = U + W$ and $V_2 = U - W$. The system (11.15) corresponds to writing a vector in terms of real and imaginary parts, whereas (11.28) corresponds to working with the solution itself and its conjugate. By means of the matrix $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ one can also write

$$\tilde{B} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} J, \quad \tilde{A} = \begin{pmatrix} H+V_1 & 0 \\ 0 & H+V_2 \end{pmatrix} J.$$

Since $\tilde{B}^* = \tilde{B}$ it follows that $\mathrm{spec}(\tilde{B}) \subset \mathbb{R}$. One checks that for $\Re z \neq 0$

$$\begin{aligned} (\tilde{B}-z)^{-1} &= (\tilde{B}+z)\begin{pmatrix} (H^2-z^2)^{-1} & 0 \\ 0 & (H^2-z^2)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (H^2-z^2)^{-1} & 0 \\ 0 & (H^2-z^2)^{-1} \end{pmatrix}(\tilde{B}+z) \qquad (11.16) \\ (\tilde{A}-z)^{-1} &= (\tilde{B}-z)^{-1} - (\tilde{B}-z)^{-1}W_1\Big[1 + W_2J(\tilde{B}-z)^{-1}W_1\Big]^{-1}W_2J(\tilde{B}-z)^{-1} \qquad (11.17) \end{aligned}$$

where W_1 and W_2 are the following matrix potentials that go to zero at infinity:

$$W_1 = \begin{pmatrix} |V_1|^{\frac12} & 0 \\ 0 & |V_2|^{\frac12} \end{pmatrix}, \quad W_2 = \begin{pmatrix} |V_1|^{\frac12}\mathrm{sign}(V_1) & 0 \\ 0 & |V_2|^{\frac12}\mathrm{sign}(V_2) \end{pmatrix}.$$

The inverse of the operator in brackets exists if $z = it$ with t large, for example. Moreover, by the assumed decay of the potential the entire operator that is being subtracted from the right-hand side is compact in that case. One is therefore in a position to apply Weyl's criterion, see Theorem XIII.14 in [RS4], whence

$$\mathrm{spec}_{\mathrm{ess}}(A) = \mathrm{spec}_{\mathrm{ess}}(\tilde{A}) = (-\infty, -\mu] \cup [\mu, \infty). \tag{11.18}$$

The identity (11.17) goes back to Grillakis [Gr]. □

Next, we need to locate possible eigenvalues of A or equivalently, $\tilde{A}$. This will not be done on the same general level, but require analysis of L_+, L_- from (11.31). But we first discuss another general property of the matrix operator A.

Lemma 11.8. *Let A be as in (11.14) with U, W continuous and W exponentially decaying, whereas U is only required to tend to zero. If $f \in \ker(A-E)^k$ for some $-\mu < E < \mu$ and some positive integer k, then f decays exponentially.*

Proof. We want to emphasize that the following result is "abstract" and does not rely on any special structure of the matrix potential or on any properties of L_+ or L_-. We will use a variant of Agmon's argument [Ag]. More precisely, suppose that for some $-\mu < E < \mu$, there are $\psi_1, \psi_2 \in H^2(\mathbb{R}^n)$ so that

$$\begin{aligned}(\triangle - \mu + U)\psi_1 - W\psi_2 &= E\psi_1 \\ W\psi_1 + (-\triangle + \mu - U)\psi_2 &= E\psi_2. \end{aligned} \tag{11.19}$$

As usual, U, W are real-valued and exponentially decaying, $\mu > 0$. Suppose $|W(x)| \lesssim e^{-b|x|}$. Then define the Agmon metrics

$$\begin{aligned}\rho_E^{\pm}(x) &= \inf_{\gamma:0\to x} L_{\mathrm{Ag}}^{\pm}(\gamma) \\ L_{\mathrm{Ag}}^{\pm}(\gamma) &= \int_0^1 \min\Big(\sqrt{(\mu \pm E - U(\gamma(t)))_+}\,,\, b/2\Big)\|\dot\gamma(t)\|\,dt \end{aligned} \tag{11.20}$$

where $\gamma(t)$ is a C^1-curve with $t \in [0,1]$, and the infimum is to be taken over such curves that connect $0, x$. These functions satisfy

$$|\nabla \rho_E^{\pm}(x)| \le \sqrt{(\mu \pm E - U(x))_+}. \tag{11.21}$$

Moreover, one has $\rho_E^{\pm}(x) \le b|x|/2$ by construction. Now fix some small $\varepsilon > 0$ and set $\omega^{\pm}(x) := e^{2(1-\varepsilon)\rho_E^{\pm}(x)}$. Our goal is to show that

$$\int \Big[\omega^+(x)|\psi_1(x)|^2 + \omega^-(x)|\psi_2(x)|^2\Big]\,dx < \infty. \tag{11.22}$$

Not only does this exponential decay in the mean suffice for our applications (cf. Section 7 in [RSS]), but it can also be improved to pointwise decay using regularity estimates for ψ_1, ψ_2. We do not elaborate on this, see for example [Ag] and Hislop, Sigal [HiSig].
Fix R arbitrary and large. For technical reasons, we set

$$\rho_{E,R}^{\pm}(x) := \min\Big(2(1-\varepsilon)\rho_E^{\pm}(x), R\Big), \quad \omega_R^{\pm}(x) := e^{\rho_{E,R}^{\pm}(x)}.$$

Notice that (11.21) remains valid in this case, and also that $\rho_E^{\pm}(x) \le \min(b|x|/2, R)$. Furthermore, by choice of E there is a smooth functions ϕ that is equal to one for large x so that

$$\mathrm{supp}(\phi) \subset \{\mu + E - U > 0\} \cap \{\mu - E - U > 0\}.$$

It will therefore suffice to prove the following modified form of (11.22):

$$\sup_R \int \Big[\omega_R^+(x)|\psi_1(x)|^2 + \omega_R^-(x)|\psi_2(x)|^2\Big]\phi^2(x)\,dx < \infty. \tag{11.23}$$

All constants in the following argument will be independent of R. By construction, there is $\delta > 0$ such that

$$\delta \int \omega_R^+(x)|\psi_1(x)|^2\phi^2(x)\,dx \le \int \omega_R^+(x)(\mu+E-U(x))|\psi_1(x)|^2\phi^2(x)\,dx \tag{11.24}$$

$$= \int \omega_R^+(x)(\triangle\psi_1 - W\psi_2)(x)\bar{\psi}_1(x)\phi^2(x)\,dx$$

$$= -\int \nabla(\omega_R^+(x)\phi^2(x))\nabla\psi_1(x)\bar{\psi}_1(x)\,dx - \int \omega_R^+(x)\phi^2(x)|\nabla\psi_1(x)|^2\,dx \tag{11.25}$$

$$-\int \omega_R^+(x)W(x)\psi_1(x)\bar{\psi}_2(x)\phi^2(x)\,dx. \tag{11.26}$$

As far as the final term (11.26) is concerned, notice that $\sup_{x,R}|\omega_R^+(x)\phi^2(x)W(x)| \lesssim 1$ by construction, whence $|(11.26)| \lesssim \|\psi_1\|_2\|\psi_2\|_2$. Furthermore, by (11.21) and Cauchy-Schwarz, the first integral in (11.25) satisfies

$$\Big|\int \nabla(\omega_R^+(x)\phi^2(x))\nabla\psi_1(x)\bar{\psi}_1(x)\,dx\Big|$$

$$\le 2(1-\varepsilon)\Big(\int \omega_R^+(x)(\mu+E-U(x))\phi^2(x)|\psi_1(x)|^2\,dx\Big)^{\frac{1}{2}}\Big(\int \omega_R^+(x)\phi(x)^2\,|\nabla\psi_1(x)|^2\,dx\Big)^{\frac{1}{2}} \tag{11.27}$$

$$+2\Big(\int \omega_R^+(x)\phi^2(x)|\nabla\psi_1(x)|^2\,dx\Big)^{\frac{1}{2}}\Big(\int \omega_R^+(x)|\nabla\phi(x)|^2\,|\psi_1(x)|^2\,dx\Big)^{\frac{1}{2}}$$

Since the first integral in (11.27) is the same as that in (11.24), inserting (11.27) into (11.25) yields after some simple manipulations

$$\begin{aligned}\varepsilon\int \omega_R^+(x)(\mu+E-U(x))|\psi_1(x)|^2\phi^2(x)\,dx \quad &\le \quad \varepsilon^{-1}\int \omega_R^+(x)|\nabla\phi(x)|^2\,|\psi_1(x)|^2\,dx \\ &\quad -\int \omega_R^+(x)\phi(x)^2W(x)\psi_2(x)\,\bar{\psi}_1(x)\,dx.\end{aligned}$$

Since $\nabla\phi$ has compact support, and by our previous considerations involving ω_R^+W, the entire right-hand side is bounded independently of R, and thus also (11.24). A symmetric argument applies to the integral with ψ_2, and (11.23), (11.22) hold. This method also shows that functions belonging to generalized eigenspaces decay exponentially. Indeed, suppose $(A-E)\vec{g} = 0$ and $(A-E)\vec{f} = \vec{g}$. Then

$$\begin{aligned}(\triangle-\mu+U)f_1 - Wf_2 &= Ef_1 + g_1 \\ Wf_1 + (-\triangle+\mu-U)f_2 &= Ef_2 + g_2\end{aligned}$$

with g_1, g_2 exponentially decaying. Decreasing the value of b in (11.20) if necessay allows one to use the same argument as before to prove (11.22) for $\vec{f}$. By induction, one then deals with all values of k as in the statement of the lemma. □

11.3 The spectral properties II: reduction to L_+, L_-

We now need to specialize A from (11.14) to the form (2.24), i.e.,

$$A = \begin{pmatrix} \frac{1}{2}\triangle - \frac{\alpha^2}{2} + \beta(\phi^2) + \beta'(\phi^2)\phi^2 & \beta'(\phi^2)\phi^2 \\ -\beta'(\phi^2)\phi^2 & -\frac{1}{2}\triangle + \frac{\alpha^2}{2} - \beta(\phi^2) - \beta'(\phi^2)\phi^2 \end{pmatrix}. \tag{11.28}$$

As shown in Section 2, these are the stationary Hamiltonians derived from the linearization of NLS, see (2.24) and Lemma 11.3. Let $\alpha > 0$ and ϕ be a nonzero solution of

$$\frac{1}{2}\triangle\phi - \frac{\alpha^2}{2}\phi + \beta(\phi^2)\phi = 0, \tag{11.29}$$

i.e., $\phi = \phi(\cdot;\alpha)$ for all $\alpha \in (\alpha_0 - c_0, \alpha_0 + c_0)$. Moreover, ϕ is smooth in both variables and

$$\|\partial_\alpha\phi\|_{H^1(\mathbb{R}^n)} + \|\partial_\alpha^2\phi\|_{H^1(\mathbb{R}^n)} < \infty. \tag{11.30}$$

Finally, we require that ϕ is sufficiently rapidly decaying. A particular case would be a ϕ which is positive and radially symmetric. Such a solution is known to exist and to be unique if $\beta(u) = |u|^\sigma$ provided $0 < \sigma < \frac{2}{d-2}$ and is referred to as the "ground state". It decays exponentially. However, in order to keep this section as general as possible, we do not require ϕ to be the ground state. Let

$$L_- := -\frac{1}{2}\triangle + \frac{\alpha^2}{2} - \beta(\phi^2), \qquad L_+ := -\frac{1}{2}\triangle + \frac{\alpha^2}{2} - \beta(\phi^2) - 2\beta'(\phi^2)\phi^2 \tag{11.31}$$

with domains $\mathrm{Dom}(L_+) = \mathrm{Dom}(L_-) = H^2(\mathbb{R}^n)$ so that

$$\tilde{A} = \begin{pmatrix} 0 & -iL_- \\ iL_+ & 0 \end{pmatrix} \tag{11.32}$$

with $\mathrm{Dom}(\tilde{A}) = H^2(\mathbb{R}^n)\times H^2(\mathbb{R}^n)$. Here $\tilde{A}$ is obtained by conjugating A with the matrix P, see (11.32). For simplicity, however, we no longer distinguish between A and $\tilde{A}$, i.e., we set $A = \tilde{A}$. The spectrum of $L_\pm$ on $[\mu,\infty)$ is purely absolutely continuous, and below $\mu = \frac{\alpha^2}{2} > 0$ there are at most a finite number of eigenvalues of finite multiplicity (by Birman-Schwinger, the assumed decay of ϕ as well as $\beta(0) = 0$). Clearly,

$$L_-\phi = 0, \quad L_+(\partial_j\phi) = 0,\ 1 \le j \le n, \quad L_+(\partial_\alpha\phi) = -\alpha\phi, \tag{11.33}$$

where the final property is formal. We now collect some crucial properties discovered by M. Weinstein.

Definition 11.9. $\boxed{\textbf{Needed properties of the scalar elliptic operators } L_+ \textbf{ and } L_- :}$
Let $\phi(\cdot;\alpha)$ be as above, in particular assume that (11.30) *holds. The kernels have the following explicit form:*

$$\ker(L_-) = \mathrm{span}\{\phi\} \quad \textit{and} \quad \ker(L_+) = \mathrm{span}\{\partial_j\phi \mid 1 \le j \le n\}.$$

The operator L_+ has a single negative eigenvalue E_1 with a unique ground state $\psi > 0$, whereas L_- is nonnegative, with 0 *as an isolated eigenvalue. Furthermore, the* convexity condition $\langle\partial_\alpha\phi(\cdot;\alpha), \phi(\cdot;\alpha)\rangle > 0$ *holds, see* (1.17).

Note that according to this definition, ϕ is the ground state of the linear operator L_- and as such is positive. In case of $\phi(\cdot;\alpha)$ being the ground state of the nonlinear problem (11.29), these properties have been shown to hold by Weinstein [We1] and [We2] in case of power nonlinearities, i.e., $\beta(u) = |u|^\sigma$, $0 < \sigma < \frac{2}{d-2}$.

The purpose of this subsection is to reduce some of the admissibility conditions from Definition 11.1 to the properties of L_+, L_- from Definition 11.9. Recall from the previous subsection that several other properties hold in greater generality. We now collect those properties that follow from Definition 11.9 into a single proposition.

Proposition 11.10. *Impose the spectral assumption on L_+ and L_- from Definition 11.9. Then*

- $\mathrm{spec}(A) \subset \mathbb{R}$*, the only eigenvalue that admits a generalized eigenspace is* 0*, and* $\mathrm{Ran}(A^2)$ *is closed.*
- *the linear stability condition holds.*
- *equality holds in the relation concerning the root spaces* (11.39) *and* (11.40)*. In particular, one has* $\ker(A^2) = \ker(A^3)$ *and* $\ker((A^*)^2) = \ker((A^*)^3)$ *and the rootspace* $\mathcal{N}(A^*)$ *has dimension* $2n + 2$.

The proof of this proposition is split into several lemmas below.

Lemma 11.11. *Impose the spectral assumption on L_+ and L_- from Definition 11.9. Then* $\mathrm{spec}(A) \subset \mathbb{R}$*, the only eigenvalue that admits a generalized eigenspace is* 0*, and* $\mathrm{Ran}(A^2)$ *is closed.*

Proof. Consider

$$A^2 = \begin{pmatrix} T^* & 0 \\ 0 & T \end{pmatrix}, \quad T = L_+ L_- \tag{11.34}$$

with domain $H^4(\mathbb{R}^n) = W^{4,2}(\mathbb{R}^n)$. Following [BP1], we first show that any eigenvalue of T, and therefore also of $\mathrm{spec}(A^2)$ is real, and then under the assumption (1.17), that it is nonnegative. Because of (11.18), the latter then implies that $\mathrm{spec}(A)$ is real, as required in Definition 11.1. Clearly, $T\phi = 0$. Let $\psi \notin \mathrm{span}\{\phi\}$, $T\psi = E\psi$. Let $\psi = \psi_1 + c\phi$, $\psi_1 \perp \phi$. Then

$$L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} L_-^{\frac{1}{2}} \psi_1 = E L_-^{\frac{1}{2}} \psi_1,$$

so that $L_-^{\frac{1}{2}}\psi_1 \neq 0$ is an eigenfunction of the symmetric operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ (with domain $H^4(\mathbb{R}^n)$), and thus E is real. Hence any eigenvalue of A can only be real or purely imaginary. Since $\phi \perp \ker(L_+)$ by our assumption concerning L_+, the function

$$g(E) := \langle (L_+ - E)^{-1}\phi, \phi\rangle$$

is well-defined on an interval of the form (E_1, E_2) for some $E_2 > 0$. Moreover,

$$g'(E) = \|(L_+ - E)^{-1}\phi\|^2 > 0$$

so that $g(E)$ is strictly increasing on the interval. Finally,

$$g(0) = -\frac{1}{\alpha}\langle \partial_\alpha \phi, \phi\rangle < 0 \tag{11.35}$$

in view of (11.33) and (1.17). Now suppose that A^2 has a negative eigenvalue. Then by the preceding, so does T, and therefore also $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$. More precisely, the argument from before implies that there is $\chi \in \ker(L_-)^\perp$, $\chi \neq 0$, so that

$$\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} \chi, \chi \rangle = \langle L_+ \psi, \psi \rangle < 0$$

with $\psi = L_-^{\frac{1}{2}} \chi$. Let $P_-^\perp$ denote the projection onto the orthogonal complement of $\ker(L_-) = \mathrm{span}(\phi)$. By the Rayleigh principle this implies that the self-adjoint operator $P_-^\perp L_+ P_-^\perp$ has a negative eigenvalue, say $E_3 < 0$. Thus $L_+ \psi = E_3 \psi + c\phi$ for some $\psi \perp \phi$. If $c = 0$, then $E_3 = E_1$ so that $\psi > 0$ as the ground state of L_+. But then $\langle \phi, \psi \rangle > 0$, which is impossible. So $c \neq 0$, and one therefore obtains

$$(L_+ - E_3)^{-1} \phi = \frac{1}{c} \psi \quad \Longrightarrow \quad g(E_3) = 0.$$

But this contradicts (11.35) by strict monotonicity of g. Thus A^2 does not have any negative eigenvalues, which implies that A does not have imaginary eigenvalues. Hence all eigenvalues of A are real, as desired.

We now turn to generalized eigenspaces. Suppose $A\psi = E\psi + \chi$, where $E \neq 0$, $(A - E)\chi = 0$ and $\chi \neq 0$. This is equivalent to saying that A has a generalized eigenspace at E. Then $\psi, \chi \in \mathrm{Dom}(A^2)$, and moreover

$$(A^2 - E^2)\chi = 0, \quad (A^2 - E^2)\psi = (A - E)\chi + 2E\chi = 2E\chi,$$

so that A^2 would have a generalized eigenspace at E, and therefore also T. Hence, suppose $T\psi = E\psi$, with $E \neq 0$, $\psi \neq 0$. If $(T - E)\chi = c\psi$ with $c \neq 0$, then

$$(L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} - E) L_-^{\frac{1}{2}} \chi_1 = c L_-^{\frac{1}{2}} \psi_1 \neq 0, \qquad (L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} - E)^2 L_-^{\frac{1}{2}} \chi_1 = c L_-^{\frac{1}{2}} (L_+ L_- - E)\psi = 0$$

where ψ_1, χ_1 denote the projections of ψ, χ onto the orthogonal complement of ϕ. But $L_-^{\frac{1}{2}} \psi_1 \neq 0$ since $E \neq 0$ and thus E would have to be a generalized eigenvalue of $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$, which is impossible. So only $E = 0$ can have a generalized eigenspace. Here we used the property that $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ is self-adjoint on its domain $H^4(\mathbb{R}^n)$. While symmetry is obvious, self-adjointness on $H^4(\mathbb{R}^n)$ requires a bit more care. Suppose $\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} f, g \rangle = \langle f, h \rangle$ for all $f \in H^4(\mathbb{R}^n)$, and some fixed $g, h \in L^2(\mathbb{R}^n)$. Taking $f \in \ker(L_-)$ shows that $P_-^\perp h = h$, i.e., that $h \in (\ker(L_-^{\frac{1}{2}}))^\perp$. By the Fredholm alternative applied to the self-adjoint operator $L_-^{\frac{1}{2}}$, one can write $h = L_-^{\frac{1}{2}} h_1$ with some $h_1 \in \mathrm{Dom}(L_-^{\frac{1}{2}}) = H^1(\mathbb{R}^n)$. Note that h_1 is defined only up to an element in $\ker(L_-^{\frac{1}{2}})$, i.e., $h_1 + c\phi$ has the same property for any constant c. Thus

$$\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} f, g \rangle = \langle f, L_-^{\frac{1}{2}} (h_1 + c\phi) \rangle = \langle L_-^{\frac{1}{2}} f, h_1 + c\phi \rangle$$

for all $f \in H^4(\mathbb{R}^n)$. Equivalently, setting $f_1 = L_-^{\frac{1}{2}} f$, one has

$$\langle L_-^{\frac{1}{2}} L_+ f_1, g \rangle = \langle f_1, h_1 + c\phi \rangle.$$

Note that the class of f_1 are all functions in $H^3(\mathbb{R}^n)$ with $f_1 \perp \phi$. We now want to remove the latter restriction, which can be achieved by a suitable choice of c. Indeed, in order to achieve

$$\langle L_-^{\frac{1}{2}} L_+(f_1 + \lambda\phi), g\rangle = \langle f_1 + \lambda\phi, h_1 + c\phi\rangle$$

for all $f_1 \in H^3(\mathbb{R}^n)$, $f_1 \perp \phi$, $\lambda \in \mathbb{C}$ one chooses c such that

$$\langle L_-^{\frac{1}{2}} L_+\phi, g\rangle = \langle \phi, h_1 + c\phi\rangle,$$

which can be done since $\langle \phi, \phi\rangle > 0$. Renaming $h_1 + c\phi$ into h_1, one thus arrives at

$$\langle L_-^{\frac{1}{2}} L_+ f_1, g\rangle = \langle f_1, h_1\rangle \quad \text{for all} \quad f_1 \in H^3(\mathbb{R}^n). \tag{11.36}$$

Recall that $h = L_-^{\frac{1}{2}} h_1$. One can now continue this procedure. Indeed, since (11.36) implies that $h_1 \perp \ker(L_+)$, one can write $h_1 = L_+ h_2 = L_+(h_2 + \sum_{j=1}^k c_j\psi_j)$, where $\ker(L_+) = \mathrm{span}\{\psi_j\}_{j=1}^k$ and $h_2 \in H^3(\mathbb{R}^n)$ (in fact, $\psi_j = \partial_j\phi$ by our assumption). As before, the constants $\{c_j\}$ are chosen in such a way that

$$\Big\langle L_-^{\frac{1}{2}}(L_+ f_1 + \sum_{j=1}^k \lambda_j\psi_j), g\Big\rangle = \Big\langle L_+ f_1 + \sum_{j=1}^k \lambda_j\psi_j, h_2 + \sum_{j=1}^k c_j\psi_j\Big\rangle$$

for all λ_j. This can be done because of the invertibility of the Gram matrix of $\{\psi_j\}_{j=1}^k$. Hence

$$\langle L_-^{\frac{1}{2}} f_2, g\rangle = \langle f_2, h_2\rangle \quad \text{for all} \quad f_2 \in H^1(\mathbb{R}^n).$$

Moreover, $h = L_-^{\frac{1}{2}} L_+ h_2$ with $h_2 \in H^3(\mathbb{R}^n)$. By the self-adjointness of $L_-^{\frac{1}{2}}$ this implies that $h_2 = L_-^{\frac{1}{2}} g$. It follows that $g \in H^4(\mathbb{R}^n)$ and $h = L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} g$ as desired.

Finally, we show that $\mathrm{Ran}(A^2)$ is closed. By (11.34) it suffices to show that the ranges of both $T = L_+L_-$ and $T^* = L_-L_+$ are closed with domain $H^4(\mathbb{R}^n)$. We will first verify that these operators are closed on this domain. Indeed, they each can be written in the form $\triangle^2 + F_1\triangle + \triangle F_2 + F_3$. Since for M large

$$\|(\triangle^2 + M + F_1\triangle + \triangle F_2 + F_3)f\|_2 \ge \|(\triangle^2 + M)f\|_2 - C\|f\|_{W^{2,2}} \ge \frac{1}{2}\|(\triangle^2 + M)f\|_2 \gtrsim \|f\|_{W^{4,2}},$$

one concludes that $T + M$, $T^* + M$ are closed, and therefore also T, T^*. Next, let P_- and P_+ be the projections onto $\ker(L_-)$ and $\ker(L_+)$, respectively. Then $L_-L_+ = L_-P_-^\perp L_+ P_+^\perp$. Now

$$P_-^\perp L_+ f = L_+ f - \|\phi\|^{-2}\langle L_+ f, \phi\rangle\phi = L_+\Big(f - \|\phi\|^{-2}\langle f, L_+\phi\rangle\tilde{\phi}\Big), \tag{11.37}$$

where we have written $\phi = L_+\tilde{\phi}$, $\tilde{\phi} \in \ker(L_+)^\perp = \mathrm{Ran}(P_+^\perp)$ by virtue of the fact that

$$\phi \in \mathrm{Ran}(L_+) = \overline{\mathrm{Ran}(L_+)} = \ker(L_+)^\perp = \mathrm{span}\{\partial_j\phi\}^\perp.$$

The last equality here is Weinstein's characterization, more precisely, our assumption on L_+. Define

$$\tilde{\phi} := P_+^\perp\tilde{\phi}_1, \quad Qf := f - \|\phi\|^{-2}\langle f, L_+\phi\rangle\tilde{\phi}, \quad \tilde{Q}f := f - \|\phi\|^{-2}\langle f, P_+^\perp L_+\phi\rangle\tilde{\phi}_1$$

so that (11.37) gives

$$P_-^\perp L_+ P_+^\perp = L_+ Q P_+^\perp = L_+ P_+^\perp \tilde{Q} \quad \Longrightarrow \quad L_- L_+ = L_- L_+ P_+^\perp \tilde{Q}.$$

In particular,

$$\begin{aligned} \|T^* f\|_2 &= \|L_- P_-^\perp L_+ P_+^\perp f\|_2 \geq c_1 \|P_-^\perp L_+ P_+^\perp f\|_2 \\ &= c_1 \|L_+ Q P_+^\perp f\|_2 = c_1 \|L_+ P_+^\perp \tilde{Q} f\|_2 \geq c_1 c_2 \|P_+^\perp \tilde{Q} f\|_2, \end{aligned} \tag{11.38}$$

where the existence of $c_1, c_2 > 0$ follows from the self-adjointness of L_-, L_+. Hence, if $T^* f_n = T^* P_+^\perp \tilde{Q} f_n \to h$ in L^2, then by (11.38) and linearity $P_+^\perp \tilde{Q} f_n \to g$ in L^2. Since T^* was shown to be closed, it follows that $h = T^* g$, and $\mathrm{Ran}(T^*)$ is closed. A similar argument shows that $\mathrm{Ran}(T)$ is closed. □

Next, we derive the *linear stability assumption* as well as the structure of the generalized eigenspaces of A and A^* from our spectral assumptions on L_-, L_+. From the spectral assumptions in Definition 11.9 as well as

$$L_- \phi = 0,\ L_-(x_j \phi) = -\partial_j \phi,\ L_+(\partial_j \phi) = 0,\ L_+(\partial_\alpha \phi) = -\alpha \phi$$

it follows that

$$\begin{aligned} \ker(A) &= \mathrm{span}\Big\{ \begin{pmatrix} 0 \\ \phi \end{pmatrix}, \begin{pmatrix} \partial_j \phi \\ 0 \end{pmatrix} \ : 1 \leq j \leq n \Big\} \\ \ker(A^*) &= \mathrm{span}\Big\{ \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_j \phi \end{pmatrix} \ : 1 \leq j \leq n \Big\} \end{aligned}$$

$$\mathcal{N}(A) := \bigcup_{k=1}^{\infty} \ker(A^k) \supset \mathrm{span}\Big\{ \begin{pmatrix} 0 \\ \phi \end{pmatrix}, \begin{pmatrix} \partial_\alpha \phi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_j \phi \end{pmatrix}, \begin{pmatrix} \partial_j \phi \\ 0 \end{pmatrix} \ : 1 \leq j \leq n \Big\} =: \mathcal{M} \tag{11.39}$$

$$\mathcal{N}(A^*) := \bigcup_{k=1}^{\infty} \ker((A^*)^k) \supset \mathrm{span}\Big\{ \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_\alpha \phi \end{pmatrix}, \begin{pmatrix} x_j \phi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_j \phi \end{pmatrix} \ : 1 \leq j \leq n \Big\} =: \mathcal{M}_*. \tag{11.40}$$

One of our goals is to show that equality holds in the last two relations. This is the same as the structure statement made in Proposition 4.1, but one needs to apply the matrix $P = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ to pass between these two representations.

Now suppose that $i\partial_t \vec{\psi} + A\vec{\psi} = 0$. This can be written as $\partial_t \vec{\psi} + JM\vec{\psi} = 0$ where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$. Therefore,

$$\frac{d}{dt} \langle \vec{\psi}, M\vec{\psi} \rangle = 2\Re \langle \partial_t \vec{\psi}, M\vec{\psi} \rangle = -2\Re \langle JM\vec{\psi}, M\vec{\psi} \rangle = 0$$

by anti-selfadjointness of J. In other words,

$$Q(\vec{\psi}) := \langle L_+ \psi_1, \psi_1 \rangle + \langle L_- \psi_2, \psi_2 \rangle$$

is constant in time if $\vec{\psi}(t) = e^{itA}\vec{\psi}(0)$ (here $\vec{\psi} = \binom{\psi_1}{\psi_2}$). Although the previous calculation bascially required classical solutions, it is clear that its natural setting are $H^1(\mathbb{R}^n)$-solutions. In that case one needs to interpret the form $Q(\vec{\psi})$ via

$$
\begin{aligned}
\langle L_+\psi_1, \psi_1\rangle &= \frac{1}{2}\|\nabla\psi_1\|_2^2 + \frac{\alpha^2}{2}\|\psi_1\|_2^2 - \langle\beta(\phi^2)\psi_1, \psi_1\rangle \\
\langle L_-\psi_2, \psi_2\rangle &= \frac{1}{2}\|\nabla\psi_2\|_2^2 + \frac{\alpha^2}{2}\|\psi_2\|_2^2 - \langle(\beta(\phi^2) + 2\beta'(\phi^2)\phi^2)\psi_2, \psi_2\rangle.
\end{aligned}
\tag{11.41}
$$

In what follows, we will tacitly make this interpretation whenever it is needed. The following lemmas are due to Weinstein [We1].

Lemma 11.12. *Impose the spectral assumptions on L_+, L_- from Definition 11.9. Then $\langle L_+f, f\rangle \geq 0$ for all $f \in H^1(\mathbb{R}^n)$, $f \perp \phi$.*

This is a special case of Lemma E.1 in [We1], and we refer the reader to that paper for the proof.

Lemma 11.13. *Impose the spectral assumptions on L_+, L_- from Definition 11.9. Then there exist constants $c = c(\alpha, \beta) > 0$ such that for all $\vec{\psi} \in H^1(\mathbb{R}^n)$,*

1. *$\langle L_-\psi_2, \psi_2\rangle \geq c\|\psi_2\|_2^2$ if $\psi_2 \perp \partial_\alpha\phi$, $\psi_2 \perp \partial_j\phi$*
2. *$\langle L_+\psi_1, \psi_1\rangle \geq c\|\psi_1\|_2^2$ if $\psi_1 \perp \phi$, $\psi_1 \perp x_j\phi$.*

The constant $c(\alpha, \beta)$ can be taken to be uniform in α in the following sense: If α_0 satisfies Definition 11.9, then there exists $\delta > 0$ so that 1. and 2. above hold for all $|\alpha - \alpha_0| < \delta$ with $c(\alpha, \beta) > \frac{1}{2}c(\alpha_0, \beta)$.

Proof. Consider the minimization problems

$$
\inf_{f\in H^1} \langle L_-f, f\rangle \qquad \text{subject to constraints} \quad \|f\|_2 = 1,\ f \perp \partial_\alpha\phi,\ f \perp \partial_j\phi \tag{11.42}
$$

$$
\inf_{g\in H^1} \langle L_+g, g\rangle \qquad \text{subject to constraints} \quad \|g\|_2 = 1,\ g \perp \phi,\ g \perp x_j\phi. \tag{11.43}
$$

As usual, one would like to establish the existence of minimizers by means of passing to weak limits in minimizing sequences. While such sequences are bounded in $H^1(\mathbb{R}^n)$, this is not enough to guarantee strong convergence in $L^2(\mathbb{R}^n)$ because some (or all) of the L^2-mass might escape to infinity. Using the fact that the quadratic forms in question are perturbations of $\frac{1}{2}\|\nabla f\|_2^2 + \frac{\alpha^2}{2}\|f\|_2^2$ by a potential that decays at infinity, one can easily exclude that *all* the L^2-mass escapes to infinity. One then proceeds to show that the remaining piece of the limit, normalized to have L^2-norm one, is a minimizer. This, however, is a simple consequence of the nonnegativity of L_- and L_+, the latter under the constraint $f \perp \phi$, see Lemma 11.12 above. This argument is presented in all details in [We1], page 478 for the case of power nonlinearities. But the same argument also applies to the general nonlinearities considered here, and we do not write it out.

Assume therefore that f_0 is a minimizer of (11.42) with $\|f_0\|_2 = 1$, $f_0 \perp \partial_\alpha\phi$, $f_0 \perp \partial_j\phi$. Then

$$
L_-f_0 = \lambda_0 f_0 + c_0\partial_\alpha\phi + \sum_{j=1}^{n} c_j\partial_j\phi \tag{11.44}
$$

for some Lagrange mulitpliers $\lambda_0, c_0, \ldots, c_n$. Clearly, λ_0 agrees with the minimum sought, and therefore it suffices to show that $\lambda_0 > 0$. If $\lambda_0 = 0$, then taking the scalar product of (11.44) with ϕ implies that $c_0 = 0$ (using that $\langle \partial_\alpha \phi, \phi \rangle > 0$). Taking scalar products with $x_k \phi$ shows that also $c_k = 0$ for $1 \le k \le n$. Thus $L_- f_0 = 0$, which would imply that $f_0 = \gamma \phi$ for some $\gamma \ne 0$. However, this is impossible because of $f_0 \perp \partial_\alpha \phi$.

Proceeding in the same manner for L_+, one arrives at the Euler-Lagrange equation

$$L_+ g_0 = \lambda_0 g_0 + c_0 \phi + \sum_{j=1}^{n} c_j x_j \phi.$$

As before, λ_0 is the minimum on the left-hand side of (11.43) and thus $\lambda_0 \ge 0$ by Lemma 11.12. If $\lambda_0 = 0$, then taking scalar products with $\partial_k \phi$ leads to $c_k = 0$ for all $1 \le k \le n$. Hence $L_+ g_0 = c_0 \phi$ which implies that

$$g_0 = -\frac{c_0}{\alpha} \partial_\alpha \phi + \sum_{\ell=1}^{n} b_\ell \partial_\ell \phi.$$

Taking scalar products of this line with ϕ and $x_j \phi$ shows that $c_0 = 0$ and $b_\ell = 0$ for all $1 \le \ell \le n$, respectively. But then $g_0 = 0$ which is impossible.

Since the constants $c(\alpha, \beta) > 0$ were obtained by contradiction, one has no control on their dependence on α. However, let $|\alpha - \alpha_0| < \delta$ be as in Definition 11.9. Suppose $\|f\|_2 = 1$ satisfies $f \perp \phi(\cdot, \alpha)$, $f \perp x_j \phi(\cdot, \alpha)$. Then there is

$$h \in \operatorname{span}\Big\{ \phi(\cdot, \alpha),\, \phi(\cdot, \alpha_0),\, x_j \phi(\cdot, \alpha),\, x_j \phi(\cdot, \alpha_0) \;:\; 1 \le j \le n \Big\}$$

so that $f + h \perp \phi(\cdot, \alpha_0)$ and $f + h \perp x_j \phi(\cdot, \alpha_0)$. Moreover, since $\|\partial_\alpha \phi\|_{H^1(\mathbb{R}^n)} + \|\partial_\alpha^2 \phi\|_{H^1(\mathbb{R}^n)} < \infty$ one can take $\|h\|_{H^1(\mathbb{R}^n)}$ as small as desired provided δ is chosen small enough. One can therefore use inequality 2. from this lemma at α_0 for $f + h$ to obtain a similar bound for f at α. $\square$

The following corollary proves the crucial *linear stability assumption* contingent upon the spectral assumptions on L_+, L_- from above (and thus, in particular, contingent upon the nonlinear stability assumption). Strictly speaking, the following corollary gives a stronger statement than (11.3), since the range of P_s is potentially smaller than needed for the stability to hold.

Corollary 11.14. *Impose the spectral assumptions on L_+, L_- from Definition 11.9. Then there exist constants $C = C(\alpha, \beta) < \infty$ so that for all $\vec{\psi}_0 \in H^1(\mathbb{R}^n)$*

$$\|e^{itA} \vec{\psi}_0\|_{H^1(\mathbb{R}^n)} \le C \|\vec{\psi}_0\|_{H^1(\mathbb{R}^n)} \quad \text{provided} \quad \vec{\psi} \in \mathcal{M}_*^\perp. \tag{11.45}$$

Here $\mathcal{M}_*$ is the A^*-invariant subspace from (11.40). Moreover, the same bound holds for $H^s(\mathbb{R}^n)$-norms for any real s with s-dependent constants (and thus in particular for $L^2(\mathbb{R}^n)$). Analogous statements hold for e^{itA^*}. Finally, the constants $C(\alpha, \beta)$ can be taken to be uniform in α in the following sense: If α_0 satisfies Definition 11.9, then there exists $\delta > 0$ so that (11.45) holds for all $|\alpha - \alpha_0| < \delta$ with $C(\alpha, \beta) < 2C(\alpha_0, \beta)$.

Proof. Let $\vec{\psi}(t) = e^{itA}\vec{\psi}_0$. By Lemma 11.13 one has

$$Q(\vec{\psi}_0) = Q(\vec{\psi}(t)) \geq c\,\|\vec{\psi}(t)\|^2_{L^2(\mathbb{R}^n)}$$

provided that $\vec{\psi}_0 \in \mathcal{M}_*^\perp$. Since clearly $Q(\vec{\psi}_0) \leq C\|\vec{\psi}_0\|^2_{H^1(\mathbb{R}^n)}$ one concludes that (11.45) holds with $L^2(\mathbb{R}^n)$ on the left-hand side. In order to pass to $H^1(\mathbb{R}^n)$ write

$$\begin{aligned} \langle L_- f, f\rangle &= (1-\varepsilon)\langle L_- f, f\rangle + \frac{\varepsilon}{2}\|\nabla f\|_2^2 + \varepsilon\frac{\alpha^2}{2}\|f\|_2^2 - \varepsilon\int_{\mathbb{R}^n}\beta(\phi^2(x))|f(x)|^2\,dx \\ &\geq \frac{\varepsilon}{2}\|\nabla f\|_2^2 + c(1-\varepsilon)\|f\|_2^2 + \varepsilon(\alpha^2/2 - \|\beta\|_\infty)\|f\|_2^2, \end{aligned} \tag{11.46}$$

where the constant c in (11.46) is the one from Lemma 11.13. Taking ε small enough, one sees that the third term can be absorbed into the second. Thus the entire right-hand side of (11.46) admits the lower bound $\frac{\varepsilon}{2}\|f\|^2_{H^1(\mathbb{R}^n)}$. The same argument applies to L_+, and (11.45) follows. The uniformity statement concerning the constants $C(\alpha,\beta)$ is an immediate consequence of the analogous statement in Lemma 11.13. To obtain (11.45) for all H^s spaces note first that

$$C_\ell^{-1}\,\|\vec{\psi}\|_{H^{2\ell}(\mathbb{R}^n)} \leq \|(A+iM)^\ell\vec{\psi}\|_2 \leq C_\ell\,\|\vec{\psi}\|_{H^{2\ell}(\mathbb{R}^n)} \tag{11.47}$$

for all integers ℓ and sufficiently large $M = M(\ell)$. Indeed, to check the lower bound for $\ell = 1$ one can use $(A+iM)^{-1} = (B+iM)^{-1}\Big[1 + V(B+iM)^{-1}\Big]^{-1}$. The inverse of the operator in brackets exists provided M is large and it is a bounded operator on $L^2(\mathbb{R}^n)$. Taking powers of this relation allows one to deal with all $\ell \geq 1$ (in our case V is C^∞ which is needed here). Since $\mathcal{M}_*$ is A^*-invariant and therefore $\mathcal{M}_*^\perp$ is A-invariant, inserting (11.47) into (11.45) allows one to pass to all odd integers s. The case of general s then follows by interpolation. Finally, since all arguments in this section apply equally well to A^* as A, the corollary follows. □

This corollary has an important implication concerning the structure of the root spaces as required in Proposition 4.1.

Corollary 11.15. *Impose the spectral assumptions on L_+, L_- from Definition 11.9. Then equality holds in the relation concerning the root spaces* (11.39) *and* (11.40). *In particular, one has* $\ker(A^2) = \ker(A^3)$ *and* $\ker((A^*)^2) = \ker((A^*)^3)$.

Proof. Suppose $\dim(\mathcal{N}(A)) > 2n+2$. Then there exists $\vec{\psi}_0 \in \mathcal{N}(A)$ such that $\vec{\psi}_0 \in \mathcal{M}_*^\perp$. This is because a system of $2n+2$ equations in $2n+3$ variables always has a nonzero solution. Since $\partial_\alpha\phi \not\perp \phi$ and $\partial_j\phi \not\perp x_j\phi$, one checks that $\vec{\psi}_0 \notin \ker(A)$. Therefore, $\vec{\psi}_0 \in \ker(A^k)\setminus\ker(A^{k-1})$ for some $k \geq 2$. Expanding e^{itA} into a series implies that $\|e^{itA}\vec{\psi}_0\|_2 > c\,t^{k-1}$ for some constant $c > 0$, which contradicts Corollary 11.14. Therefore, $\dim(\mathcal{N}(A)) \leq 2n+2$. Since moreover $\phi > 0$ and $\langle\partial_\alpha\phi, \phi\rangle > 0$ imply that the $2n+2$ vectors on the right-hand side of (11.39) are linearly independent, equality must hold as claimed. Analogously for (11.40). □

12 Generalized decay estimates for the charge transfer model

Consider the time-dependent matrix charge transfer problem

$$i\partial_t\vec{\psi} + H(\sigma,t)\vec{\psi} = F$$

where the matrix charge transfer Hamiltonian $H(\sigma,t)$ is of the form

$$H(\sigma,t) = \begin{pmatrix} \frac{1}{2}\triangle & 0 \\ 0 & -\frac{1}{2}\triangle \end{pmatrix} + \sum_{j=1}^{\nu} V_j(\cdot - \vec{v_j}t)$$

where $\vec{v}_j$ are distinct vectors in $\mathbb{R}^3$, and V_j are matrix potentials of the form

$$V_j(t,x) = \begin{pmatrix} U_j(x) & -e^{i\theta_j(t,x)}W_j(x) \\ e^{-i\theta_j(t,x)}W_j(x) & -U_j(x) \end{pmatrix},$$

where $\theta_j(t,x) = (|\vec{v}_j|^2 + \alpha_j^2)t + 2x\cdot\vec{v}_j + \gamma_j$, $\alpha_j, \gamma_j \in \mathbb{R}$, $\alpha_j \neq 0$. Our goal is to extend the dispersive estimate

$$\|\vec{\psi}(t)\|_{L^2+L^\infty} \lesssim (1+t)^{-\frac{n}{2}}\Big(\|\vec{\psi}_0\|_{L^1\cap L^2} + |||F||| + B\Big) \tag{12.1}$$

with

$$|||F||| := \sup_{t\geq 0}\int_0^t \|F(\tau)\|_{L^1}\,d\tau + (1+t)^{\frac{n}{2}+1}\|F(t,\cdot)\|_{L^2} \tag{12.2}$$

to the corresponding estimates for the derivatives of $\vec{\psi}(t)$. The estimate (12.1) holds only for the solutions $\vec{\psi}(t)$ which are scattering states, i.e., for $\vec{\psi}$ obeying the a priori condition that

$$\|P_b(H_j,t)\vec{\psi}(t)\|_{L^2} \leq B(1+t)^{-\frac{n}{2}}$$

for all $j = 1,..,\nu$. Our first lemma shows that the functions

$$\vec{\psi}_k(t) := \nabla^k\vec{\psi}(t), \qquad k \in Z_+$$

are scattering states as well.

Lemma 12.1. *The functions $\vec{\psi}_k(t)$ obey the estimates*

$$\|P_b(H_j,t)\vec{\psi}_k(t)\|_{L^2} \lesssim C_k(1+t)^{-\frac{n}{2}} \tag{12.3}$$

Proof. Let $\vec{\eta}(x)$ be an arbitrary C^∞ exponentially localized function. Then for any $y \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n}\vec{\psi}_k(t,x)\cdot\vec{\eta}(x-y)\,dx = (-1)^k\int_{\mathbb{R}^n}\vec{\psi}(t,x)\nabla^k\vec{\eta}\,(x-y)\,dx \lesssim \|\vec{\psi}(t)\|_{L^2+L^\infty}\|\nabla^k\vec{\eta}\|_{L^1\cap L^2} \lesssim (1+t)^{-\frac{n}{2}}$$

Now recall that the projection

$$P_b(H_j,t) := \mathcal{G}_{\vec{v}_j}(t)^{-1}\mathcal{M}_j(t)^{-1}P_b(H_j)\,\mathcal{M}_j(t)\mathcal{G}_{\vec{v}_j}(t)$$

with the $P_b(H_j$ is given exlicitly

$$P_b(H_j)f = \sum_{\alpha\beta} c_{\alpha\beta}u_\alpha(f,v_\beta)$$

where $c_{\alpha\beta}$ are given constants and u_α, v_β are exponentially localized functions. The result now follows. $\square$

Proposition 12.2. *The functions $\vec{\psi}_k = \nabla^k\vec{\psi}$ satisfy the L^2+L^∞ dispersive estimate*

$$\|\nabla^k\vec{\psi}(t)\|_{L^2+L^\infty} \lesssim (1+t)^{-\frac{n}{2}}\sum_{\ell=0}^{k}\Big(\|\nabla^\ell\vec{\psi}_0\|_{L^1\cap L^2} + |||\nabla^\ell F||| + B\Big) \tag{12.4}$$

Proof. We have already shown that $\nabla^k\psi$ is a scattering state. Moreover, differentiating the equation k times we obtain

$$i\partial_t\nabla^k\vec{\psi} + H(t,\sigma)\nabla^k\vec{\psi} = F_k := \sum_{\ell=0}^{k-1}G_\ell(t,x)\nabla^\ell\vec{\psi} + \nabla^k F$$

where $G_\ell(t,x)$ are smooth exponentially localized potentials uniformly bounded in time. Therefore ∇^k is a scattering state solving an inhomogeneous charge transfer problem. Using the estimate (12.1) we then have

$$\|\nabla^k\vec{\psi}(t)\|_{L^2+L^\infty} \lesssim (1+t)^{-\frac{n}{2}}\Big(\|\nabla^k\vec{\psi}_0\|_{L^1\cap L^2} + |||F_k(\tau)||| + B\Big) \tag{12.5}$$

We use that for any $p\in[1,2]$

$$\|G_\ell(t,x)\nabla^\ell\vec{\psi}\|_{L^p} \lesssim \|\nabla^\ell\vec{\psi}\|_{L^2+L^\infty}$$

Proceeding by induction on k we conclude that for any $\ell<k$

$$\begin{aligned}\int_0^t\|G_\ell(t,x)\nabla^\ell\vec{\psi}\|_{L^1} &\lesssim \int_0^t(1+\tau)^{-\frac{n}{2}}\,d\tau\sum_{m=0}^{\ell}\Big(\|\nabla^m\vec{\psi}_0\|_{L^1\cap L^2} + |||\nabla^m F|||\Big)\\ &\lesssim \sum_{m=0}^{\ell}(\|\nabla^m\vec{\psi}_0\|_{L^1\cap L^2} + |||\nabla^m F|||)\end{aligned}$$

and that

$$(1+t)^{\frac{n}{2}}\|G_\ell(t,x)\nabla^\ell\vec{\psi}(t)\|_{L^2} \lesssim \sum_{m=0}^{\ell}(\|\nabla^m\vec{\psi}_0\|_{L^1\cap L^2} + |||\nabla^m F|||)$$

The result now follows from (12.5) and the inequality

$$|||F_k(\tau)||| \le |||\nabla^k F||| + \sum_{\ell=0}^{k-1}|||G_\ell(t,x)\nabla^\ell\vec{\psi}|||$$

$\square$

We recall the definition of the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ of functions of (t,x) from (5.7), (5.8)

$$\|\psi\|_{\mathcal{X}_s} = \sup_{t\geq 0}\Big(\|\psi(t,\cdot)\|_{H^s} + (1+t)^{\frac{n}{2}}\sum_{k=0}^{s}\|\nabla^k\psi(t,\cdot)\|_{L^2+L^\infty}\Big) \tag{12.6}$$

$$\|F\|_{\mathcal{Y}_s} = \sup_{t\geq 0}\sum_{k=0}^{s}\Big(\int_0^t \|\nabla^k F(\tau,\cdot)\|_{L^1}\,d\tau + (1+t)^{\frac{n}{2}+1}\|\nabla^k F(t,\cdot)\|_{L^2}\Big) \tag{12.7}$$

We can summarize our estimates for the charge transfer model in the following proposition.

Proposition 12.3. *Let $\vec{\psi}$ be a solution of the matrix charge transfer problem*

$$i\partial_t\vec{\psi} + H(t,\sigma)\vec{\psi} = F$$

satisfying the condition that for every $j = 1,..,\nu$

$$\|P_b(H_j(\sigma,t))\vec{\psi}\|_{L^2} \lesssim B(1+t)^{-\frac{n}{2}} \tag{12.8}$$

Then for any integer $s\geq 0$

$$\|\vec{\psi}\|_{\mathcal{X}_s} \lesssim \sum_{k=0}^{s}\|\nabla^k\psi(0,\cdot)\|_{L^1\cap L^2} + \|F\|_{\mathcal{Y}_s} + B \tag{12.9}$$

13 Existence of a ground state and the nonlinear stability condition

The existence of a ground state for the problem

$$-\frac{1}{2}\triangle\phi - \beta(|\phi|^2)\phi + \frac{\alpha^2}{2}\phi = 0 \tag{13.1}$$

for $\alpha\neq 0$ had been establsihed by Berestycki and Lions under the following conditions on the function β:

1. $0 \geq \overline{\lim}_{s\to+\infty}\beta(s)s^{-\frac{2}{n-2}} \geq +\infty$
2. There exists $s_0 > 0$ such that $G(s_0) = \int_0^{s_0}\beta(s^2)s\,ds - \frac{\alpha^2}{4}s_0^2 > 0$

Moreover, in the case when the function $\beta(s)$ satisfies a stronger condition that

$$\lim_{s\to+\infty}\beta(s)s^{-\frac{2}{n-2}} = 0 \tag{13.2}$$

a ground state can be constructed from a solution of the constrained minimization problem for the following functional:

$$J[u] = \Big\{\int_{\mathbb{R}^n}|\nabla u|^2 \;:\; W[u] = \int_{\mathbb{R}^n} G(u) = 1\Big\} \tag{13.3}$$

If w is a minimum it solves the equation

$$-\frac{1}{2}\triangle w - \lambda(\beta(w^2)w - \frac{\alpha^2}{2}w) = 0$$

where the Lagrange multiplier λ is determined from the condition that $W[w] = 1$. We can then find a ground state via rescaling

$$\phi(x) = w(\lambda^{-\frac{1}{2}}x) \tag{13.4}$$

Observe that it is possible to choose w a positive spherically symmetric function. We now consider the case of the monomial subcritical nonlinearity $\beta(s) = s^{\frac{p-1}{2}}$ with $p < \frac{n+2}{n-2}$. By the results of Coffman, McLeod-Serrin, and Kwong there exists a unique positive radial solution of the equation (13.1) for $\alpha \neq 0$. Let w denote the corresponding minimumizer of the functional J.

"Uniqueness of a minimizer"

Definition 13.1. *Given $\gamma > 0$ and w the minimizer of J corresponding to the unique ground state ϕ define*

$$\begin{aligned}\theta(\gamma) = \inf \big\{\theta :\ & \textit{for any positive non-increasing radial function } u \ \textit{ with the} \\ & \textit{property that } \|u - w\|_{H^1} \geq \theta \ \textit{ we have that } J[u] \geq J[w] + \gamma\big\}\end{aligned} \tag{13.5}$$

We now make the following claim

Lemma 13.2. *Function $\theta(\gamma) \to 0$ as $\gamma \to 0$.*

Proof. We argue by contradiction. Assume that there exists a sequence $\gamma_k \to 0$, a positive constant θ, and positive radial functions u_{γ_k} such that $\|u_{\gamma_k} - w\|_{H^1} \geq \theta$ but $J[u_{\gamma_k}] < J[w] + \gamma_k$. Then the sequence $u_{\gamma_k,\theta}$ is minimizing for the functional J. This implies that

$$\|\nabla u_{\gamma_k}\|_{L^2} \to \|\nabla w\|_{L^2}$$

Using the constraint $W[u_{\gamma_k}] = 1$ it is not difficult to show that the sequence u_{γ_k} is uniformly bounded in H^1, see [BL]. Thus without loss of generality we assume that $u_{\gamma_k} \to u$ weakly in H^1 for some radial non-increasing function u. Therefore, u is another minimizer of the functional J and its rescaled version is a non-increasing radial solution of the equation (13.1). By the strong maximum principle it is positive[4] and therefore a ground state. Since the ground state is unique, after rescaling back we conclude that $u = w$. Therefore, we have constructed a sequence u_{γ_k} with the properties that

$$u_{\gamma_k} \to w \quad \text{weakly in} \quad H^1, \tag{13.6}$$

$$\nabla u_{\gamma_k} \to \nabla w \quad \text{in} \quad L^2, \tag{13.7}$$

$$\int_{\mathbb{R}^n} |u_{\gamma_k}|^{p+1} = 1 + \frac{\alpha^2}{4}\int_{\mathbb{R}^n} |u_{\gamma_k}|^2, \tag{13.8}$$

$$\|u_{\gamma_k} - w\|_{H^1} \geq \theta \tag{13.9}$$

[4]The minimizer u cannot be identically zero since one can show that the minimum is attained on the function satisfying the constraint $W[u] = 1$, see [BL].

Since $2 < p+1 \le \frac{2n}{n-2}$, conditions (13.6) and (13.7) imply that

$$\int_{\mathbb{R}^n} |u_{\gamma_k}|^{p+1} \to \int_{\mathbb{R}^n} |w|^{p+1}$$

Thus from (13.8)

$$\int_{\mathbb{R}^n} |u_{\gamma_k}|^2 \to \int_{\mathbb{R}^n} |w|^2$$

and with the help of (13.6) and (13.7) we conclude that $u_{\gamma_k} \to w$ in H^1. This contradicts (13.9). □

We now consider the ground state problem

$$-\frac{1}{2}\triangle\phi_\epsilon - \beta_\epsilon(|\phi_\epsilon|^2)\phi_\epsilon + \frac{\alpha^2}{2}\phi_\epsilon = 0 \tag{gr$_\epsilon$}$$

for the nonlinearities

$$\beta_\epsilon(s^2) = -s^{p-1}\frac{s^{3-p}}{\epsilon + s^{3-p}} \tag{13.10}$$

for any $\epsilon > 0$ and any $p \in (1,3)$. Define

$$G_\epsilon(\tau) = \int_0^\tau \beta(s^2)s\,ds - \frac{\alpha^2}{4}\tau^2, \tag{13.11}$$

$$W_\epsilon[u] = \int_{\mathbb{R}^n} G_\epsilon(u(x))\,dx \tag{13.12}$$

Lemma 13.3. *We have the following estimate*

$$|W_\epsilon[u] - W_0[u]| \lesssim \epsilon^{\frac{p-1}{2}} \int_{\mathbb{R}^n} \left(|u|^2 + |u|^{p+1}\right) \tag{13.13}$$

Proof. Estimate (13.13) immediately follows from the inequality

$$|G_\epsilon(\tau) - G_0(\tau)| \le \tau^{p+1}\frac{\epsilon}{\epsilon + \tau^{3-p}} = \epsilon\tau^{2p-4}\frac{\tau^{3-p}}{\epsilon + \tau^{3-p}} \tag{13.14}$$

since the above expression can ne bounded by

$$\min\left\{\epsilon\tau^{2p-4}, \tau^{p+1}\right\}$$

Thus using the first term for the values of $\tau \ge \epsilon^{\frac{1}{2}}$ so that $\tau^{p-3} \le \epsilon^{\frac{p-3}{2}}$ (since $p < 3$), and the second term when $\tau \le \epsilon^{\frac{1}{2}}$ so that $\tau^{p-1} \le \epsilon^{\frac{p-1}{2}}$, we obtain (13.14). □

"Continuity of ground states"

We now consider the variational problem

$$J_\epsilon[u] = \left\{\int_{\mathbb{R}^n} |\nabla u|^2 : \; W_\epsilon[u] = 1\right\} \tag{13.15}$$

Proposition 13.4. *Let ϕ be the ground state of the problem (gr_0). Then for any sufficiently small $\epsilon > 0$ there exists a positive constant $\delta' = \delta'(\epsilon) \to 0$ as $\epsilon \to 0$, and a ground state ϕ_ϵ of (gr_ϵ) such that $\|\phi_\epsilon - \phi\|_{H^1} < \delta'$.*

Proof. We start by choosing a sufficiently large constant M such that for all sufficiently small ϵ any minimizer of J_ϵ is contained in a ball $B_{M/2}$ of radius $M/2$ in the space H^1. In particular, using (13.13) we will assume that for $u \in B_M$

$$|W_\epsilon[u] - W_0[u]| \lesssim \epsilon^{\frac{p-1}{2}} \tag{13.16}$$

We now observe the following trivial property of the contraint functionals $W_\epsilon[u]$: for any $\epsilon \geq 0$ and an arbitrary $\mu \neq 0$

$$W_\epsilon[u(x)] = \mu^n W_\epsilon[u(\frac{x}{\mu})] \tag{13.17}$$

We now fix a sufficiently small of $\epsilon > 0$. Let w be the minimizer of the variational problem $J = J_0$ corresponding to the unique ground state ϕ. The function w satisfies the constraint $W_0[w] = 1$. Therefore, using the rescaling property (13.17) and (13.16) we can show that there exists $\mu = \mu(w)$ with the property that

$$W_\epsilon[w(\frac{x}{\mu})] = 1,$$

$$|\mu - 1| \leq \epsilon^{\frac{p-1}{2n}} \tag{13.18}$$

Moreover,

$$J_\epsilon[w(\frac{x}{\mu})] = \mu^{n-2} J_0[w(x)] = J_0[w] + O(\epsilon^{\frac{p-1}{2n}}) \tag{13.19}$$

We now claim that there exists a small positive $\delta = \delta(\epsilon) \to 0$ as $\epsilon \to 0$, such that for any positive non-increasing radial function u satisfying the constraint $W_\epsilon[u] = 1$ and the property that

$$\|u - w(\frac{x}{\mu})\|_{H^1} \geq \delta \tag{13.20}$$

we have

$$J_\epsilon[u] \geq J_\epsilon[w(\frac{x}{\mu})] + \epsilon^{\frac{p-1}{2n}} \tag{13.21}$$

Assume for the moment that the claim holds. Then (13.20) and (13.21) imply that J_ϵ has a minimizer in the δ neighborhood of the function $w(\frac{x}{\mu})$. We denote this minimizer by w_ϵ. Then (13.18) implies that

$$\|w_\epsilon - w\|_{H^1} \leq \|w_\epsilon - w(\frac{x}{\mu})\|_{H^1} + \|w - w(\frac{x}{\mu})\|_{H^1} \leq \delta + \|w - w(\frac{x}{\mu})\|_{H^1}$$

Observe that $\|w - w(\frac{x}{\mu})\|_{H^1} \to 0$ as $\mu \to 1$, which follows by the density argument and the fact that it is easily satisfied on functions of compact support[5]. Define the function $a_w(\epsilon)$:

$$a_w(\epsilon) := \sup_{|\mu-1| \leq \epsilon^{\frac{p-1}{2n}}} \|w - w(\frac{x}{\mu})\|_{H^1}, \qquad a_w(\epsilon) \to 0 \quad \text{as } \epsilon \to 0 \tag{13.22}$$

[5]In fact, the minimizer w is smooth and localized in space and thus one could even give the precise dependence on μ

Therefore,

$$\|w_\epsilon - w\|_{H^1} \lesssim \delta + a_w(\epsilon) \tag{13.23}$$

The functions w_ϵ, w are the solutions of the Euler-Lagrange equations

$$-\frac{1}{2}\triangle w_\epsilon - \lambda_\epsilon\big(\beta_\epsilon(w_\epsilon)^2)w_\epsilon - \frac{\alpha^2}{2}w_\epsilon\big) = 0, \tag{13.24}$$

$$-\frac{1}{2}\triangle w - \lambda\big(\beta_0(w)^2)w - \frac{\alpha^2}{2}w\big) = 0, \tag{13.25}$$

where the lagrange multipliers λ_ϵ, λ are determined from the conditions that $W_\epsilon[w_\epsilon] = W_0[w] = 1$. We multiply the equations (13.24) and (13.25) by w_ϵ and w correspondingly, integrate by parts, and subtract one from another. Using the estimate

$$\int_{\mathbb{R}^n} \big|\beta_\epsilon(w_\epsilon)^2)w_\epsilon^2 - \beta(w_\epsilon)^2)w_\epsilon^2\big| \lesssim \epsilon^{\frac{p-1}{2}},$$

which is essentially the same as the estimate (13.16), and the estimate (13.23) we obtain that

$$(\lambda - \lambda_\epsilon)\int_{\mathbb{R}^n} (\beta_0(w)^2)w^2 - \frac{\alpha^2}{2}w^2\big) = O(\delta) + O(\epsilon^{\frac{p-1}{2}}) + a_w(\epsilon) \tag{13.26}$$

Recall that $\beta_0(w^2) = w^{p-1}$. The condition that $W[w] = 1$ implies that

$$\int_{\mathbb{R}^n} \big(\frac{1}{p+1}|w|^{p+1} - \frac{\alpha^2}{4}|w|^2\big) = 1$$

Thus,

$$\int_{\mathbb{R}^n} (\beta_0(w)^2)w^2 - \frac{\alpha^2}{2}w^2\big) = 2 + \frac{p-1}{p+1}\int_{\mathbb{R}^n} |w|^{p+1} \geq 2$$

This allows us to conclude that

$$|\lambda - \lambda_\epsilon| \leq \delta + \epsilon^{\frac{p-1}{2}} + a_w(\epsilon) \tag{13.27}$$

Finally, recall that the ground states ϕ_ϵ and ϕ are obtained by the rescaling of the minimizers w_ϵ and w.

$$\phi_\epsilon(x) = w_\epsilon(\lambda_\epsilon^{-\frac{1}{2}}x), \qquad \phi(x) = w(\lambda^{-\frac{1}{2}}x)$$

Thus

$$\begin{aligned}\|\phi_\epsilon - \phi\|_{H^1} &\lesssim \|w_\epsilon(\lambda_\epsilon^{-\frac{1}{2}}x) - w(\lambda^{-\frac{1}{2}}x)\|_{H^1} \\ &= \lambda_\epsilon^{\frac{n}{2}}\|w_\epsilon(x) - w\big((\frac{\lambda_\epsilon}{\lambda})^{\frac{1}{2}}x\big)\|_{H^1} \\ &\leq \lambda_\epsilon^{\frac{n}{2}}\|w_\epsilon(x) - w(x)\|_{H^1} + \|w(x) - w\big((\frac{\lambda_\epsilon}{\lambda})^{\frac{1}{2}}x\big)\|_{H^1}\end{aligned}$$

By (13.27) the constants λ_ϵ are uniformly bounded in terms of the absolute constant λ, which depends only on w. Noreover, $\lambda_\epsilon \to \lambda$ as $\epsilon \to 0$. We appeal again to the H^1 modulus of continuity of the minimizer w and define the function [6]

$$b_w(\epsilon, \delta) := \sup_{|\mu - 1| \le \delta + \epsilon^{\frac{p-1}{2}} + a_w(\epsilon)} \|w(x) - w(\mu^{-\frac{1}{2}} x)\|_{H^1} \tag{13.28}$$

The function $b_w(\epsilon, \delta) \to 0$ as $\epsilon, \delta \to 0$. Therefore, since we have already proved in (13.23) that w_ϵ is close to w in H^1, we obtain

$$\|\phi_\epsilon - \phi\|_{H^1} \lesssim \delta + \epsilon^{\frac{p-1}{2}} + a_w(\epsilon) + b_w(\epsilon, \delta) \tag{13.29}$$

Since by the claim $\delta = \delta(\epsilon) \to 0$ as $\epsilon \to 0$ and the functions $a_w(\epsilon)$, $b_w(\epsilon, \delta)$ also have this property we obtain the desired conclusion.

It remains to prove the claim (13.20), (13.21). Let u be as in the claim, i.e., $u \in B_M$ and $W_\epsilon[u] = 1$, and

$$\|u - w(\frac{x}{\mu})\|_{H^1} \ge \delta. \tag{13.30}$$

for some δ to be chosen below. Similar to (13.18) we can find a constant $\nu = \nu(u)$ such that

$$W_0[u(\frac{x}{\nu})] = 1, \qquad J_0[u(\frac{x}{\nu})] = J_\epsilon[u] + O(\epsilon^{\frac{p-1}{2}n}), \tag{13.31}$$

$$|\nu - 1| \le \epsilon^{\frac{p-1}{2n}} \tag{13.32}$$

Using (13.18), (13.30), (13.32), and defini tion (13.22) we infer that

$$\|u(\frac{x}{\nu}) - w\|_{H^1} \ge \|u(\frac{x}{\nu}) - w(\frac{x}{\mu\nu})\|_{H^1} - \|w(\frac{x}{\nu\mu}) - w(x)\|_{H^1} \ge \nu^{\frac{n}{2}}\delta - a_w(\epsilon) \ge \delta - \epsilon^{\frac{p-1}{2n}}\delta - a_w(\epsilon) \tag{13.33}$$

We now use Lemma 13.2 for the variational problem $J = J_0$. This gives a function $\theta(\gamma)$, with the property that $\theta(\gamma) \to 0$ as $\gamma \to 0$, such that for any radial non-increasing positive v with the property that $\|v - w\|_{H^1} \ge \theta(\gamma)$ and $W_0[v] = 1$ we have $J_0[v] \ge J_0[w] + \gamma$. We set

$$\gamma = 5\epsilon^{\frac{p-1}{2n}}, \qquad \delta = \theta(\gamma) + \epsilon^{\frac{p-1}{2n}} + a_w(\epsilon)$$

It follows from Definition 13.1 of $\theta(\gamma)$ and (13.33) that with these choices, function $u(\frac{x}{\nu})$ verifies the inequality

$$J_0[u(\frac{x}{\nu})] \ge J_0[w] + 5\epsilon^{\frac{p-1}{2n}}$$

[6] One can show that $\alpha \neq 0$ the Lagrange multiplier $\lambda \neq 0$. This follows from the following argument. By interpolation for $p \le \frac{n+2}{n-2}$

$$\int w^{p+1} \le \|\nabla w\|_{L^2}^{n\frac{p-1}{2}} \|w\|_{L^2}^{p+1-n\frac{p-1}{2}}$$

Thus for $n > 2$ the power $p + 1 - n\frac{p-1}{2} < 2$ and using Cauchy-Schwarz, constraint $W[w] = 1$ and the assumption that $\alpha \neq 0$, we can show that $\|\nabla w\|_{L^2} \ge c$ for some positive constant c. Repeating argument determining the Lagrange multiplier we verify that $\lambda \neq 0$

Finally, using (13.19) and (13.31) we obtain

$$J_\epsilon[u] \geq J_\epsilon[w(\frac{x}{\mu})] + 3\epsilon^{\frac{p-1}{2n}}$$

It remains to note that the constant δ in (13.20) has been chosen

$$\delta = \theta(\epsilon^{\frac{p-1}{4n}}) + \epsilon^{\frac{p-1}{2n}} + a_w(\epsilon)$$

and by Lemma 13.2 and (13.22) it goes to zero as $\epsilon \to 0$, as claimed. $\square$

From now on we restric the values of p to the subcritical case

$$\boxed{p \leq 1 + \frac{4}{n}} \tag{13.34}$$

Recall definition of the operator L^ϵ_+ associated with the ground state ϕ_ϵ.

$$L^\epsilon_+ = -\frac{1}{2}\triangle - \beta_\epsilon(\phi_\epsilon^2) - 2\beta'_\epsilon(\phi_\epsilon^2)\phi_\epsilon^2 + \frac{\alpha^2}{2} \tag{13.35}$$

Denote

$$V_\epsilon = \beta_\epsilon(\phi_\epsilon^2) + 2\beta'_\epsilon(\phi_\epsilon^2)\phi_\epsilon^2 \tag{13.36}$$

Uisng the definition of β_ϵ we compute V_ϵ explicitly

$$V_\epsilon = 3\phi_\epsilon^{p-1}\frac{\phi_\epsilon^{3-p}}{\epsilon + \phi_\epsilon^{3-p}} - (3-p)\phi_\epsilon^{p-1}\Big(\frac{\phi_\epsilon^{3-p}}{\epsilon + \phi_\epsilon^{3-p}}\Big)^2 \tag{13.37}$$

Uniform properties of ground states ϕ_ϵ guaranteed by the Proposition 13.4 imply the following result.

Lemma 13.5. *For any $p \in (1, 1+\frac{4}{n}]$ there exists a $q = q(p)$ in the interval $q \in [\frac{n}{2}, \infty)$ such that*

$$\|V_\epsilon - V_0\|_{L^q} \to 0, \quad \epsilon \to 0 \tag{13.38}$$

Proof. We have a pointwise bound

$$|V_\epsilon - p\phi_\epsilon^{p-1}| \lesssim \phi_\epsilon^{p-1}\frac{\epsilon}{\epsilon + \phi_\epsilon^{3-p}} \leq \min\left\{\phi_\epsilon^{p-1}, \epsilon\phi_\epsilon^{2p-4}\right\} \leq \epsilon^+\phi_\epsilon^{p-1-}$$

In addition, since $\phi_\epsilon \to \phi$ in H^1 we have that $\phi_\epsilon^{p-1} \to \phi^{p-1}$ in the space $L^{\frac{2}{p-1}} \cap L^{\frac{2n}{(n-2)(p-1)}}$. Since

$$|V_\epsilon - V_0| \leq |V_\epsilon - p\phi_\epsilon^{p-1}| + p|\phi_\epsilon^{p-1} - \phi^{p-1}|$$

we obtain the desired conclusion for any q in the interval $q \in (\frac{2}{p-1}, \frac{2n}{(n-2)(p-1)}]$. The existence of the Lebesque exponent q in the desired interval now follows from the restrictions (13.34) on p. $\square$

Corollary 13.6. *The operators*

$$(L_+^\epsilon - L_+)(-\triangle + 1)^{-1} : \; L^2 \to L^2,$$
$$(-\triangle + 1)^{-1}(L_+^\epsilon - L_+) : \; L^2 \to H^1$$

with the norm converging to 0 *as* $\epsilon \to 0$.

Proof. The difference $L_+^\epsilon - L_+ = V_\epsilon - V_0$. The result now follows from Lemma 13.5, Sobolev embeddings, and Hölder inequality. $\square$

"Everything follows from perturbation theory"

Theorem 13.7. *Let ϕ_ϵ be ground states constructed in Proposition 13.4. Assume that the ground state ϕ_0 is stable then for all sufficiently small ϵ the ground states ϕ_ϵ are also stable.*

Proof. The nonlinear stability condition for the a ground state $\phi_\epsilon(\alpha)$ requires that

$$\langle \phi_\epsilon, (L_+^\epsilon)^{-1}\phi_\epsilon \rangle < 0 \tag{13.39}$$

where the operators L_+^ϵ are obtained by linearizing at ϕ_ϵ. Condition (13.39) is meaningful provided that ϕ_ϵ is othogonal to the kernel of L_+^ϵ. We start by examining the spectrum of the operator L_+. As we know (?) L^+ has a unique negative eigenvalue, the zero eigenvalue has multiplicity n and the corresponding eiegenspace is spanned by the function $\frac{\partial}{\partial x_i}\phi$. The rest of the spectrum is contained in the set $[\frac{\alpha^2}{2}, \infty)$. Therefore, in the case $\alpha \neq 0$ the spectrum $\Sigma(L_+)$ of L_+ has an isolated discrete component (in fact two components). We can construct an eigenspace projector P_0 of an isolated component of the discrete spectrum

$$P_0 = \frac{1}{2\pi i}\int_\gamma (L_+ - z)^{-1}\, dz \tag{13.40}$$

with an arbitrary curve γ encircling the desired spectral set and such that $\gamma \cap \Sigma(L_+) = 0$. Consider now the resolvent of L_+^ϵ at z such that $\mathrm{dist}(z, \Sigma(L_+)) \geq C$ for some sufficiently small constant C, which only depends on L_+. We have

$$(L_+^\epsilon - z)^{-1} = (L_+ - z)^{-1} - (L_+^\epsilon - z)^{-1}(L_+^\epsilon - L_+)(L_+ - z)^{-1} \tag{13.41}$$

It is not difficult to show that for such z

$$\|(L_+ - z)^{-1} f\|_{H^2} \lesssim \|f\|_{L^2}$$

Therefore, using Corollary 13.6 we can conclude from (13.41) that

$$\|(L_+^\epsilon - z)^{-1}\| \leq 2\|(L_+ - z)^{-1}\|$$

and thus $z \notin \Sigma(L_+^\epsilon)$. Moreover,

$$\|(L_+^\epsilon - z)^{-1} - (L_+ - z)^{-1}\| \leq c(\epsilon) \tag{13.42}$$

for any z : $\operatorname{dist}(z, \Sigma(L_+)) \geq C$. By Corollary 13.6 the constant $c(\epsilon) \to 0$ as $\epsilon \to 0$. Therefore, for the same path γ as in (13.40) we can define

$$P_\epsilon = \frac{1}{2\pi i} \int_\gamma (L_+^\epsilon - z)^{-1} \, dz \tag{13.43}$$

Moreover, for all sufficiently small $\epsilon \geq 0$ the rank of P_ϵ remains constant. Thus, for any sufficiently small ϵ the operator L_+^ϵ has a unique simple negative eigenvalue and a zero eigenspace of dimension n. Since we know that the functions $\frac{\partial}{\partial x_i}\phi_\epsilon$ are contained in that subspace, they, in fact, span it. Therefore, ϕ_ϵ is orthogonal to the kernel of L_+^ϵ and the expression (13.39) is well defined.

For any sufficiently small $\epsilon \geq 0$ we set Q_ϵ to be a projection on the orthogonal complement of the null eiegnespace of L_+^ϵ. Let $\lambda \notin \cup_\epsilon \Sigma(L_+^\epsilon)$. Define the operators

$$K_\epsilon(\lambda) := Q_0 (L_+^\epsilon - \lambda)^{-1} Q_\epsilon - (L_+ - \lambda)^{-1} Q_0 \tag{13.44}$$

It follows from (13.42) and the properties of the spectrum of L_+ that for all small $\epsilon \geq 0$ and *all* $\langle$ such that $|\langle| \leq C$

$$\|(L_+^\epsilon - \lambda)^{-1} Q_\epsilon\| \leq \frac{1}{\operatorname{dist}\big(\lambda, \Sigma(L_+^\epsilon) \setminus \{0\}\big)} \leq C' \tag{13.45}$$

for some universal constant C', determined by the operator L_+. Also note that

$$\|Q_\epsilon - Q_0\| \leq c(\epsilon) \tag{13.46}$$

This is a consequence of (13.42) and the definition

$$Q_\epsilon = I - \frac{1}{2\pi i} \int_\gamma (L_+^\epsilon - z')^{-1} \, dz'$$

with a short path γ around the origin. Using the resolvent identity

$$(L_+^\epsilon - z)^{-1} = (L_+ - z)^{-1} + (L_+ - z)^{-1} (L_+^\epsilon - L_+)(L_+^\epsilon - z)^{-1}$$

we obtain that for any $\lambda \notin \cup_\epsilon \Sigma(L_+^\epsilon)$

$$\begin{aligned} K_\epsilon(\lambda) =& Q_0 (L_+ - \lambda)^{-1} Q_\epsilon - (L_+ - \lambda)^{-1} Q_0 + Q_0 (L_+ - \lambda)^{-1} (L_+^\epsilon - L_+)(L_+^\epsilon - \lambda)^{-1} Q_\epsilon = \\ & (L_+ - \lambda)^{-1} Q_0 (Q_\epsilon - Q_0) + (L_+ - \lambda)^{-1} Q_0 (L_+^\epsilon - L_+)(L_+^\epsilon - \lambda)^{-1} Q_\epsilon \end{aligned}$$

Using Corollary 13.6, (13.45), and (13.46) we infer that for any $\lambda \leq c$ and $\lambda \notin \cup_\epsilon \Sigma(L_+^\epsilon)$

$$\|K_\epsilon(\lambda)\| \leq c(\epsilon)(1 + \|(L_+ - \lambda)^{-1} Q_0 (-\triangle + 1)\|) \leq c(\epsilon) \tag{13.47}$$

uniformly in λ. The last inequality follows since the operator norm of $(L_+ - \lambda)^{-1} Q_0 (-\triangle + 1)$ is bounded by a universal constant dependent on L_+ only. This can be seen as follows. Since V_0 is a smooth ponetial and $(L_+ - \lambda)^{-1} Q_0$ is bounded on L^2 we can replace the operator $(-\triangle + 1)$ by $(L_+ - \lambda)$ and the result follows immediately.

We now test the operator $K_\epsilon(\lambda)$ on the ground state ϕ_ϵ.

$$K_\epsilon(\lambda)\phi_\epsilon = Q_0(L_+^\epsilon - \lambda)^{-1}\phi_\epsilon - (L_+ - \lambda)^{-1}\phi + (L_+ - \lambda)^{-1}Q_0(\phi - \phi_\epsilon)$$

Coupling the above identity with ϕ.

$$\begin{aligned}\langle\phi_\epsilon, (L_+^\epsilon - \lambda)^{-1}\phi_\epsilon\rangle - \langle\phi, (L_+ - \lambda)^{-1}\phi\rangle &= \langle\phi, K_\epsilon(\lambda)\phi_\epsilon\rangle + \langle(\phi_\epsilon - \phi_0), (L_+^\epsilon - \lambda)^{-1}\phi_\epsilon\rangle \\ &+ \langle\phi, (L_+ - \lambda)^{-1}Q_0(\phi - \phi_\epsilon)\rangle = O(c(\epsilon))\end{aligned}$$

where we have used that $Q_\epsilon\phi_\epsilon = \phi_\epsilon$, the bound (13.45), and the estimate $\|\phi_\epsilon - \phi\|_{H^1}$, which follows from Proposition 13.4. The above holds uniformly for all $|\lambda| \le c$ and $\lambda \notin \cup_\epsilon\Sigma(L_+^\epsilon)$. Passing to the limit $\lambda \to 0$, say from the upper half-plane, we obtain that for all sufficiently small $\epsilon \ge 0$

$$\langle\phi_\epsilon, (L_+^\epsilon)^{-1}\phi_\epsilon\rangle = \langle\phi, L_+^{-1}\phi\rangle + O(c(\epsilon)) < 0$$

The last inequality follows since by the assumption ϕ is a stable ground state, i.e., $\langle\phi, L_+^{-1}\phi\rangle < 0$. $\square$

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