Frequency concentration and localization lengths for the Anderson model at small disorders.

W. Schlag, C. Shubin, T. Wolff

1 Introduction

In this paper we consider the random operators of the Anderson model

$$Hu := \triangle_{\mathbb{Z}^d} u + \lambda\omega u$$

acting on $u \in \ell^2(\mathbb{Z}^d)$ with $d = 1, 2$. Here $\{\omega_n\}_{n\in\mathbb{Z}^d}$ is an i.i.d. sequence which will be assumed to satisfy $\mathbb{E}\omega_0 = 0$, $\mathbb{E}\omega_0^2 = 1$, and ω_0 bounded. By $\triangle_{\mathbb{Z}^d}$ we mean the discrete nearest neighbor Laplacian, i.e.,

$$\triangle_{\mathbb{Z}^d} u(n) = \sum_{|m-n|=1} u(m).$$

Observe that we are not subtracting $2du(n)$, which would give the true discretization of the continuous Laplacian. However, this only amounts to a translation of the spectrum. Our concern is mainly to understand localization lengths in the limit $\lambda \to 0$. More precisely, as localization has not been established for $d = 2$ at small disorders, we consider eigenfunctions of the operators restricted to the cubes $[-N, N]^d$ as $N \to \infty$ and estimate the localization lengths from below uniformly in N. Our approach is based on the analysis of the Fourier transform of eigenfunctions. We show that with probability one most eigenfunctions have the property that their Fourier transforms are concentrated on annuli of thickness about λ^2 (strictly speaking, $\lambda^{2-\eta}$, where $\eta > 0$ is arbitrary but fixed). These annuli are neighborhoods of those curves which support the Fourier transform of the corresponding eigenfunctions of the free Laplacian with the same energies. "Most eigenfunctions" here means up to a set of density $o(1)$ as $\lambda \to 0$ (with the "o"–term depending on η). The concentration to annuli of thickness λ is almost immediate, whereas harmonic analysis techniques permit the improvement to λ^2. By the uncertainty principle this frequency concentration implies that eigenfunctions cannot have most of their ℓ^2 mass on squares in $\mathbb{Z}^d$ of side length much small than λ^{-2}. In one dimension it is know that this is optimal, i.e., that the localization length is on the order of λ^{-2}. In two dimensions, however, it is suspected that if localization does occur, then the localization length should in some sense be infinite (exponential in λ^{-2}). On the other hand, it is easy to see that the frequency concentration to annuli of thickness λ^2 is optimal provided "concentration" is interpreted in a certain way. Moreover, this appears to agree with predictions from statistical mechanics. The length scale λ^{-2} on the lattice $\mathbb{Z}^2$ has been observed in several instances, in particular in connection with time dependent problems. The understanding of physicists is that diffusive behavior starts after time λ^{-2}, see Spohn [13] and Erdös, Yau [3].

Recently, Magnen, Poirot, and Rivasseau have achieved important results that are strongly related to this paper, see [11], [8], and [9]. In particular, Poirot's paper [11] contains a "multiscale analysis in phase space" which is basically the same that we use. The aforementioned works are all based on renormalization group techniques from statistical mechanics, and are primarily concerned with asymptotic expansions of the expected value of the Green's function in the disorder. These expansions are well-known to be of fundamental importance in the field. Our approach is different. The Green's function is not used, and the eigenfunctions are estimated in a more direct way. Not only do we not invoke any kind of renormalization group methods, but we also do not rely on an interpretation of the Anderson model as a matrix model, which seems to be important in [11]. Rather, we are motivated by some well-known ideas from harmonic analysis that arose in the study of Bochner-Riesz multipliers, see [2] and [4]. As our techniques appear to be somewhat more elementary than those from [11], we hope that they might be of some interest.

This paper is organized as follows. In Section 2 we prove the one-dimensional result, see Theorem 2.1. The two-dimensional result is proved in Section 3, see Theorem 3.12 and Corollary 3.13. Optimality is discussed in Subsection 3.2. As some of the details in Section 3 are almost identical with those from the one-dimensional case, we do not provide all of them but rather refer the reader to the one-dimensional proofs where they are given in full.

The third author wishes to thank Frederic Klopp, since the possibility of proving a result of this nature was suggested by their joint paper [7]. The first author is grateful to Tom Spencer for pointing out references [13] and [3], as well as for providing some physical insight into the importance of the scale λ^{-2}. He gratefully acknowledges the support of the National Science Foundation DMS-9706889 and thanks Barry Simon for making it possible to visit Caltech. The second author was supported by the grant NASA-NCC5-489.

2 Frequency concentration and localization lengths for the one-dimensional Anderson model

In this section our goal is to prove the following theorem about the one-dimensional Anderson model.

Theorem 2.1. *Consider the one-dimensional random operator*

$$(Hu)_n = u_{n+1} + u_{n-1} + \lambda\omega_n u_n \tag{2.1}$$

where $\lambda > 0$ and $\{\omega_n\}$ are i.i.d. with $\mathbb{E}\omega_n = 0$, $\mathbb{E}\omega_n^2 = 1$, and bounded. For any positive integer N let $\{u_j^{(N)}\}$ be an orthonormal basis on $\ell^2([-N,N])$ of eigenfunctions of H restricted to $[-N,N]$, i.e.,

$$(H - E_j^{(N)})u_j^{(N)} = 0 \quad on \quad [-N,N] \quad and \quad u_{-N-1}^{(N)} = u_{N+1}^{(N)} = 0.$$

Fix any small $\tau > 0$ and $\eta > 0$. Then for sufficiently small λ one has

$$\limsup_{N\to\infty} \frac{1}{N} \#\Big\{ j \;\Big|\; \|u_j^{(N)}\|_{\ell^4(\mathbb{Z})} \ge \lambda^{\frac{1}{2}-\eta} \|u_j^{(N)}\|_{\ell^2(\mathbb{Z})},\ -2+\tau < E_j^{(N)} < 2-\tau \Big\} \le \lambda^{-9} e^{-\lambda^{-2\eta}} \tag{2.2}$$

with probability one.

This says that for small λ and with high probability most eigenfunctions have localization length λ^{-2} up to logarithms, at least away from the edges of the spectrum. In one dimension this is known to be the correct order of magnitude. The proof of Theorem 2.1 rests on the fact that the Fourier transforms of eigenfunctions are basically concentrated on λ^2-intervals. Showing this is the main difficulty in the argument, and the statement about ℓ^4 norms then follows easily. Moreover, the latter implication also shows that the support of the Fourier transforms of eigenfunctions cannot be significantly smaller than λ^2. It is much easier to show that the Fourier transforms localize to λ-intervals, which would correspond to the localization length λ^{-1}. This will be done in the following subsection.

2.1 A simple perturbative result

Let $\mathbb{T} = [0, 1]$ be the circle. The Fourier transform of any $u \in \ell^2(\mathbb{Z})$ is

$$\hat{u}(\theta) = \sum_{n=-\infty}^{\infty} u_n e(n\theta)$$

where $e(\theta) = e^{2\pi i\theta}$. For any interval $I \subset \mathbb{R}$, let P_I denote the projection operator onto the energies in I. More precisely, with

$$S = S(I) = \{\theta \in \mathbb{T} \mid 2\cos(2\pi\theta) \in I\} \tag{2.3}$$

one defines $\widehat{P_I u}(\theta) = \chi_S(\theta)\hat{u}(\theta)$, where χ_S is the indicator function of the set S. If $I \subset (-2+\tau, 2-\tau)$, then the length of $S(I)$ satisfies $C^{-1}|I| \le |S(I)| \le C|I|$, with a constant C depending on τ (this will also be denoted as $|S| \asymp |I|$). For any $\lambda > 0$ it is known that Anderson localization takes place, see [6], [12]. Thus, let $\{u_k\}$ be an orthonormal basis of eigenfunctions with energies $\{E_k\}$. Let u be any one of them and $E \in (-2+\tau, 2-\tau)$ its energy. Taking Fourier transforms one obtains that

$$[(H-E)u]^{\wedge}(\theta) = (2\cos(2\pi\theta) - E)\hat{u}(\theta) + \lambda\widehat{\omega u}(\theta). \tag{2.4}$$

Cover the energy axis by disjoint intervals I_j of length $C_0\, 2^j\, \lambda$, with I_0 being centered at E. Here C_0 is some constant that will be determined below. For every $j \neq 0$ there are two copies of I_j, one to the left of E and the other to the right (see Figure 2.1, where the second copy of I_2 containing -2 is not shown). It suffices to consider $0 \le j \lesssim |\log\lambda|$ many of those intervals (throughout this paper $a \lesssim b$ means that $a \le C\, b$ for some constant C. Also, if $a \le C^{-1}\, b$ with some large C, then we write $a \ll b$). Observe that $\{I_j\}$ are chosen in such a way that $|I_j| \asymp \mathrm{dist}(I_j, E)$ for $j \neq 0$.

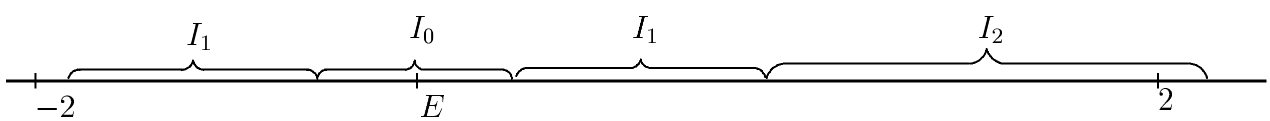

Figure 2.1: The intervals for a given energy E

Applying the projection P_{I_j} for some $j \neq 0$ to the equation (2.4) yields

$$|I_j| \|P_{I_j} u\|_2 \lesssim \lambda \|P_{I_j}(\omega u)\|_2 \lesssim \lambda \sum_k \|P_{I_j} \omega P_{I_k} u\|_2 \lesssim \|\omega\|_\infty\, \lambda \sum_k \|P_{I_k} u\|_2. \tag{2.5}$$

Here we have used the trivial bound

$$\|P_{I_j}\omega P_{I_k}\| \leq \|\omega\|_\infty \tag{2.6}$$

as well as the fact that $\sum_j P_{I_j} = \mathrm{Id}$. Consider that interval I_j that maximizes the expression $\sqrt{|I_j|}\|P_{I_j}u\|_2$. Suppose that $j \neq 0$. Then one concludes from (2.5) that

$$|I_j|\|P_{I_j}u\|_2 \lesssim \lambda \sum_k \|P_{I_k}u\|_2 \lesssim \lambda \sum_k \sqrt{\frac{|I_j|}{|I_k|}}\|P_{I_j}u\|_2$$

and thus $\sqrt{C_0\lambda} \lesssim \lambda \sum_k |I_k|^{-\frac{1}{2}} \lesssim \frac{\lambda}{\sqrt{C_0\lambda}}$. This is a contradiction for large C_0. Consequently, the maximum has to be attained for $j = 0$, which is equivalent to

$$\|P_{I_j}u\|_2 \lesssim \sqrt{\frac{|I_0|}{|I_j|}}\,\|P_{I_0}u\|_2 \quad \text{for all} \quad j. \tag{2.7}$$

Loosely speaking, this means that the frequencies of any eigenfunction u_k with energy E_k lie λ-close to those two frequencies θ_k for which $2\cos(2\pi\theta_k) = E_k$. Therefore, the localization length is at least λ^{-1}. This can also be formulated in terms of a ℓ^4 to ℓ^2 comparison by means of the following simple technical lemma.

Lemma 2.2. *Let $f, g \in L^2(\mathbb{T})$. Then*

$$\|f * g\|_{L^2(\mathbb{T})} \leq \min(|\mathrm{supp}(f)|, |\mathrm{supp}(g)|)^{\frac{1}{2}}\, \|f\|_2\|g\|_2.$$

Moreover, this bound is sharp.

Proof. Let $E = \mathrm{supp}(f)$ and $F = \mathrm{supp}(g)$. Assume that $|F| \leq |E|$. Then

$$\begin{aligned}\int_{\mathbb{T}}\left|\int_{\mathbb{T}} f(x-y)g(y)\,dy\right|^2 dx &\leq \int_{\mathbb{T}}\int_{\mathbb{T}}|f(x-y)|^2\chi_F(y)\,dy \int_{\mathbb{T}}|g(z)|^2\,dz\,dx \\ &\leq |F|\,\|f\|_2^2\,\|g\|_2^2,\end{aligned}$$

as claimed. Finally, one checks that these bounds are attained for indicator functions. □

Now suppose u is an arbitrary function satisfying (2.7). Then

$$\begin{aligned}\|u\|_4^2 &= \|u^2\|_{\ell^2(\mathbb{Z})} = \|\hat{u} * \hat{u}\|_{L^2(\mathbb{T})} \leq \sum_{j,k}\left\|\widehat{P_{I_j}u} * \widehat{P_{I_k}u}\right\|_2 \\ &\leq \sum_{j,k}|S(I_j)|^{\frac{1}{2}} \wedge |S(I_k)|^{\frac{1}{2}}\,\|P_{I_j}u\|_2\,\|P_{I_k}u\|_2 \lesssim \sum_{j,k}|I_j|^{\frac{1}{2}} \wedge |I_k|^{\frac{1}{2}}\,\frac{|I_0|}{|I_j|^{\frac{1}{2}}|I_k|^{\frac{1}{2}}}\,\|P_{I_0}u\|_2^2 \\ &\lesssim \sum_{j\leq k}\frac{|I_0|}{|I_k|^{\frac{1}{2}}}\|P_{I_0}u\|_2^2 \lesssim |I_0|^{\frac{1}{2}}\,\|P_{I_0}u\|_2^2.\end{aligned} \tag{2.8}$$

This finally implies that

$$\|u\|_{\ell^4(\mathbb{Z})} \lesssim \lambda^{\frac{1}{4}}\|u\|_{\ell^2(\mathbb{Z})}.$$

2.2 A nonperturbative improvement of the frequency concentration

This subsection is devoted to an improvement of the argument of the previous section. More precisely, our goal is to prove (2.7) with smaller intervals $\{I_j\}$ of length $|I_j| \asymp 2^j \lambda^{2-4\eta}$. The calculation leading to (2.8) then shows that $\|u\|_{\ell^4(\mathbb{Z})} \lesssim \lambda^{\frac{1}{2}-\eta}\|u\|_{\ell^2(\mathbb{Z})}$, which is basically what is claimed in Theorem 2.1. The weakest part of the previous argument is inequality (2.6). However, with probability one that bound turns out to be optimal as we shall show below. Nevertheless, a substantial improvement is possible provided one restricts ω to intervals of certain bounded sizes, see Lemma 2.8 for a more precise statement.

For any subset $\Lambda \subset \mathbb{Z}$ let R_Λ denote the restriction operator to Λ. For example,

$$R_{[-N,N]} H R_{[-N,N]}$$

is the operator H restricted to $[-N, N]$ with Dirichlet boundary conditions. Thus

$$\begin{aligned}\left[R_{[-N,N]}(H-E)R_{[-N,N]}u\right]^\wedge(\theta) &= (2\cos(2\pi\theta)-E)\hat{u}(\theta)+\lambda\widehat{\omega u}(\theta)\\ &\quad -u_{-N}e(-(N+1)\theta)-u_N e((N+1)\theta).\end{aligned} \tag{2.9}$$

In this section it will be technically necessary for us to define P_I by means of $\widehat{P_I f} = \widehat{\psi_{S(I)}}\,\hat{f}$, where ψ_S is a smooth bump function. We make this precise in the following lemma. For any interval $I \subset \mathbb{T}$ and $c > 1$ we let the <u>c-dilate</u> of I be that interval I^* with the same center as I and length $c|I|$. If $c|I|$ exceeds the length of the torus, then we set $I^* = \mathbb{T}$.

Lemma 2.3. *Let $\{J_\ell\}$ be a finite collection of pairwise disjoint intervals on $\mathbb{T}$ such that $\bigcup_\ell J_\ell = \mathbb{T}$. Fix some small constant $c > 1$ and denote the collection of c-dilates by $\{J_\ell^*\}$. Assume that the following two properties hold for some number A:*

$$\sup_\theta \sum_\ell \chi_{J_\ell^*}(\theta) \le A \tag{2.10}$$

and if $J_\ell^* \cap J_k^* \neq \emptyset$, then

$$A^{-1}|J_k^*| \le |J_\ell^*| \le A|J_k^*|. \tag{2.11}$$

Then there exist $\psi_\ell \in \ell^2(\mathbb{Z})$ with $\mathrm{supp}(\widehat{\psi_\ell}) \subset J_\ell^*$ such that $\widehat{\psi_\ell} \in C^\infty(\mathbb{T})$,

$$\sum_\ell \widehat{\psi_\ell}^{\,2} = 1, \tag{2.12}$$

and such that for every $m = 0, 1, 2, \ldots$ there exist constants $C(m, c)$ with the property that

$$|\psi_\ell(n)| \le C(m,c)\,A^2\,|J_\ell|\,[1+|J_\ell||n|]^{-m} \qquad \forall \ell. \tag{2.13}$$

Proof. Let $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on $(-\frac{1}{2}, \frac{1}{2})$, and $\mathrm{supp}(\varphi) \subset (-(1+c)/2, (1+c)/2)$. Define

$$\widehat{\psi_\ell}(\theta) = \frac{\varphi((\theta-\theta_\ell)/|J_\ell|)}{\sqrt{\sum_k \varphi^2((\theta-\theta_k)/|J_k|)}} \tag{2.14}$$

where θ_ℓ is the midpoint of J_ℓ. Denote the denominator in (2.14) by D. By assumption (2.10), no more than A terms are nonzero in the sum appearing in D. Hence,

$$1 \le D \le \sqrt{A}.$$

Moreover, (2.11) implies that if $\theta \in J_\ell$, then the derivatives satisfy

$$|D^{(s)}(\theta)| \le C_s\, A^2 |J_\ell|^{-s} \quad \text{for all} \quad s = 1, 2, 3, \ldots$$

where C_s also depends on c. Consequently,

$$\left|\widehat{\psi_\ell}^{(s)}(\theta)\right| \le C_s\, A^2\, |J_\ell|^{-s} \quad \text{for all} \quad s = 1, 2, 3, \ldots.$$

This implies (2.13), whereas (2.12) holds by construction. □

The choice of squares in (2.12) turns out to be convenient in this context, but is otherwise arbitrary. Lemma 2.3 is applied as follows: Fix some small $\tau > 0$ and let $E \in (-2+\tau, 2-\tau)$, and $0 < \ell_0 \ll \tau$ (below $\ell_0 \le \lambda$ so that the latter assumption holds for small λ). Cover the energy axis by pairwise disjoint intervals I_j of length $|I_j| = 2^j \ell_0$ where I_0 is centered at E, see Figure 2.1. We denote this partition by

$$\mathcal{I}(E). \tag{2.15}$$

It give rise to a partition $\{S(I_j)\}_j$ of $\mathbb{T}$, where each $S(I_j)$ is the union of at most two intervals, say $S(I_j) = J_j \cap J'_j$, see (2.3). Since E is separated from -2 and 2, one has

$$|J_j| = |J'_j| \asymp |I_j| \tag{2.16}$$

at least for those j with $-2, 2 \notin I_j$. In case $I_j = (a, b)$ with $a < 2 < b$ violates (2.16), one slightly shifts a to the left so that the length of I_j and that of its left neighbor changes by at most a factor of two, say, but (2.16) holds. Similarly with -2. In this way one arrives at a partition $\{J_\ell, J'_\ell\}_\ell$ of $\mathbb{T}$ with the property that adjacent intervals have comparable lengths. This property implies that the hypotheses of Lemma 2.3 are satisfied with some $c > 1$ sufficiently close to 1 and $A = 2$. Let $\{\psi_\ell\}_\ell$ be the functions given by Lemma 2.3 and set

$$\widehat{P_{I_j} u} = (\widehat{\psi_{\ell_1}} + \widehat{\psi_{\ell_2}})\hat{u} \tag{2.17}$$

where $S(I_j) = J_{\ell_1} \cup J_{\ell_2}$ (if $S(I_j)$ is a single interval, then of course only one ψ appears in (2.17)). By our construction,

$$\mathrm{supp}(\widehat{\psi_{\ell_1}}) \cap \mathrm{supp}(\widehat{\psi_{\ell_2}}) = \emptyset.$$

Therefore, (2.12) implies that

$$\sum_j P_{I_j}^2 = \mathrm{Id}_{\ell^2(\mathbb{Z})}. \tag{2.18}$$

Moreover, one has $(P_{I_j}^2 u)(n) = \sum_{n' \in \mathbb{Z}} K_{I_j}(n - n') u(n')$ with a kernel K_{I_j} satisfying

$$|K_{I_j}(n)| \;\le\; C_s\, |I_j|\, [1 + |I_j||n|]^{-s} \quad \text{for all} \quad s = 1, 2, \ldots \tag{2.19}$$

$$\mathrm{supp}\Big(\widehat{K_{I_j}}\Big) \;\subset\; S(I_j)^* \tag{2.20}$$

where $S(I_j)^*$ is a little bit larger than $S(I_j)$. Properties (2.18), (2.19), and (2.20) are going to be used repeatedly in what follows. We shall also need the following construction involving P_{I_j}. Let

$$\widetilde{P_{I_j}} := \sum_{k:|I_k|\le|I_j|} P_{I_k}^2. \tag{2.21}$$

By construction,

$$\widetilde{P_{I_j}} u = u \ \text{ if } \ \mathrm{supp}(\hat{u}) \subset \bigcup_{k:|I_k|<|I_j|} S(I_k).$$

Consequently, $\widetilde{P_{I_j}}$ is given by a kernel $\widetilde{K_{I_j}}$ that satisfies

$$|\widetilde{K_{I_j}}(n)| \le C_s\, |I_j|\, [1 + |I_j||n|]^{-s} \ \text{ for all } \ s = 1, 2, \dots \tag{2.22}$$

In addition to this smooth partition of unity on $\mathbb{T}$ we shall also require one on $\mathbb{Z}$. One possibility of doing this is given by the following lemma.

Lemma 2.4. *There exists a positive Schwartz function φ on $\mathbb{R}$ such that* $\mathrm{supp}(\hat{\varphi}) \subset (-\frac{1}{2}, \frac{1}{2})$ *and with the property that*

$$\sum_{n\in\mathbb{Z}} \varphi(x - n) = 1 \qquad \forall x \in \mathbb{R}. \tag{2.23}$$

Proof. Taking Fourier transforms in (2.23) yields

$$\hat{\varphi}(\xi) \sum_{n\in\mathbb{Z}} e(-n\xi) = \sum_{k\in\mathbb{Z}} \hat{\varphi}(k)\delta_k = \delta_0,$$

where the first equality sign is Poisson's summation formula. To ensure the second equality sign, it suffices to take $\mathrm{supp}(\hat{\varphi}) \subset (-\frac{1}{2}, \frac{1}{2})$ and to set $\hat{\varphi}(0) = 1$. To obtain positivity, start with any even Schwartz function φ_0 with $\mathrm{supp}(\widehat{\varphi_0}) \subset (-\frac{1}{4}, \frac{1}{4})$ and $\widehat{\varphi_0}(0) = 1$. Since φ_0^2 extends to an entire function in $\mathbb{C}$, one has

$$\mathrm{mes}[\varphi_0^2 = 0] = 0.$$

Therefore $\varphi := \varphi_0^2 * \varphi_0^2 > 0$ everywhere, whereas

$$\hat{\varphi} = [\widehat{\varphi_0} * \widehat{\varphi_0}]^2$$

has support in $(-\frac{1}{2}, \frac{1}{2})$. Finally, observe that

$$\hat{\varphi}(0) = \Big(\int \widehat{\varphi_0}(\xi)\widehat{\varphi_0}(-\xi)\, d\xi \Big)^2 = \Big(\int \widehat{\varphi_0}(\xi)^2\, d\xi \Big)^2 > 0.$$

The second equality sign uses that $\widehat{\varphi_0}$ is even whereas positivity follows since $\widehat{\varphi_0}$ is real. Hence,

$$\sum_{n\in\mathbb{Z}} \varphi(x - n) = \hat{\varphi}(0) \qquad \forall x \in \mathbb{R}$$

by the preceding argument. Dividing by the right-hand side finishes the proof. ☐

Fix some φ as in Lemma 2.4. Given an interval $Q \subset \mathbb{Z}$ so that $0 \in Q$, define $\varphi_Q(x) = \varphi(x/|Q|)$. Then, with $L = |Q|$,

$$\sum_{t \in L\mathbb{Z}} \varphi_Q(n+t) = 1 \qquad \forall n \in \mathbb{Z}. \tag{2.24}$$

If $\mathcal{Q} = \{Q\}$ is a partition of $\mathbb{Z}$ by means of congruent intervals, we define φ_Q as follows: Let Q_0 be the unique interval from $\mathcal{Q}$ for which $0 \in \mathcal{Q}$ and define φ_{Q_0} as before. If $Q = Q_0 + t$, then set $\varphi_Q := \varphi_{Q_0}(\cdot - t)$. An equivalent formulation of (2.24) is

$$\sum_{Q \in \mathcal{Q}} \varphi_Q = 1. \tag{2.25}$$

Observe that with this definition, which we follow throughout this section,

$$\operatorname{supp}(\widehat{\varphi_Q}) \subset \Big(-\frac{1}{2|Q|}, \frac{1}{2|Q|} \Big), \quad \text{and} \quad |\varphi_Q(n)| \le C_m \left[1 + |Q|^{-1} \operatorname{dist}(n,Q)\right]^{-m} \quad \text{for all } m \ge 0 \tag{2.26}$$

where C_m are constants that do not depend on Q.
We repeatedly use Schur's lemma in this paper. For the sake of completeness, we now recall the statement. The simple proof can be found in many places and will be omitted.

Lemma 2.5. *Suppose (X, μ) is a σ-finite measure space. Let $K : X \times X \to \mathbb{C}$ be a measurable function satisfying*

$$\begin{aligned} \sup_{x \in X} \int_X |K(x,y)|\, d\mu(y) =: A &< \infty \\ \sup_{y \in X} \int_X |K(x,y)|\, d\mu(x) =: B &< \infty. \end{aligned}$$

Then for any $f \in L^2(X)$ the integral on the right-hand side of

$$Tf(x) = \int_X K(x,y) f(y)\, d\mu(y)$$

converges for a.e. x and defines an operator T for which

$$\|T\| \le \sqrt{AB}.$$

With this technical preparatory work behind us, we now turn to an entropy bound that will play an essential role in bounding the norm $\|P_I \omega_Q P_J\|$ where ω_Q denotes the restriction of $\{\omega\}$ to Q. More precisely, Lemma 2.6 limits the number of functions u, v for which one needs to estimate

$$|\langle \omega_Q P_J u, P_I v \rangle|.$$

Statements of this type are quite well-known, see for example Buslaev and Vituškin [1]. Moreover, a two dimensional version of the following lemma is in [7].

Lemma 2.6. *Let $I \subset \mathbb{T}$ and $Q \subset \mathbb{Z}$ be intervals with $|Q| \cdot |I| \geq 1$ and suppose that Q is centered at 0. Define the space*

$$X_I := \{u \in \ell^2(\mathbb{Z}) \mid \mathrm{supp}(\hat{u}) \subset I\}$$

and let $\mathcal{B}_I$ be its unit ball. Then for every $0 < \kappa < 1$ there exist functions $\{u_j\}_{j=1}^M$ in $\mathcal{B}_I$ with $M \lesssim \exp\Big(C\,\kappa^{-2}|Q||I|\Big)$, C being an absolute constant, so that for any $u \in \mathcal{B}_I$ one has

$$\|u\,\varphi_Q - u_j\,\varphi_Q\|_{\ell^2(\mathbb{Z})} \leq \kappa$$

for some $j \in \{1, 2, \ldots, M\}$.

Proof. Partition I into intervals $\{J\}$ of length $C_1^{-1}\kappa|Q|^{-1}$, where C_1 is some large constant. Let θ_J be the midpoint of J. For any $u \in X_I$ denote the conditional expectation of $f := \hat{u}$ with respect to this partition by g_f. Thus g_f is constant on each of the intervals J, and $\int_J g_f\, d\theta = \int_J f\, d\theta$. Clearly, $\|g_f\|_2 \leq \|f\|_2$ and furthermore,

$$\begin{aligned}
|\widehat{\varphi_Q} * (f - g_f)(\theta)| &= \Big|\sum_{J \subset I} \int_J \widehat{\varphi_Q}(\theta - \theta')(f - g_f)(\theta')\, d\theta'\Big| \\
&= \Big|\sum_{J \subset I} \int_J [\widehat{\varphi_Q}(\theta - \theta') - \widehat{\varphi_Q}(\theta - \theta_J)](f - g_f)(\theta')\, d\theta'\Big| \\
&\leq \int \sum_{J \subset I} \chi_J(\theta') \Big|\widehat{\varphi_Q}(\theta - \theta') - \widehat{\varphi_Q}(\theta - \theta_J)\Big|\, |(f - g_f)(\theta')|\, d\theta'. \qquad (2.27)
\end{aligned}$$

Using the derivative bound $\left\|\widehat{\varphi_Q}'\right\|_1 \lesssim |Q|$, one easily checks that

$$\sup_{\theta'} \sum_{J \subset I} \chi_J(\theta') \int \big|\widehat{\varphi_Q}(\theta - \theta') - \widehat{\varphi_Q}(\theta - \theta_J)\big|\, d\theta \lesssim |Q||J| \lesssim C_1^{-1}\kappa.$$

Similarly, for fixed $\theta \in \mathbb{T}$ one has

$$\int \sum_J \chi_J(\theta') \Big|\widehat{\varphi_Q}(\theta - \theta') - \widehat{\varphi_Q}(\theta - \theta_J)\Big|\, d\theta' \lesssim C_1^{-1}\kappa.$$

Hence, by Lemma 2.5 the L^2-norm of the quantity in (2.27) is no larger than $\kappa/3$ provided C_1 was large, i.e.,

$$\|\varphi_Q[u - (g_{\hat{u}})^\vee]\|_2 = \|\widehat{\varphi_Q} * (f - g_f)\|_2 \leq \kappa/3.$$

On the other hand, the functions $\Big\{\widehat{\varphi_Q} * g_{\hat{u}} \mid u \in \mathcal{B}_I\Big\}$ are a subset of a ball of radius about one in the d-dimensional Hilbert space $\Big\{\widehat{\varphi_Q} * g_{\hat{u}} \mid u \in X_I\Big\}$ where $d \lesssim \kappa^{-1}\,|Q||I|$. Thus there is a subset $\{u_j\} \subset \mathcal{B}_I$ of cardinality $\lesssim \kappa^{-Cd}$ such that if $u \in \mathcal{B}_I$, then

$$\min_j \left\|\widehat{\varphi_Q} * g_{\hat{u}} - \widehat{\varphi_Q} * g_{\widehat{u_j}}\right\|_2 \leq \kappa/3.$$

For this latter property see for example Pisier [10], formula (4.22). Consequently, for every $u \in \mathcal{B}_I$ there is j such that

$$\|\widehat{\varphi_Q} * (\hat{u} - \widehat{u_j})\|_2 \le \|\widehat{\varphi_Q} * (\hat{u} - g_{\hat{u}})\|_2 + \|\widehat{\varphi_Q} * (g_{\hat{u}} - g_{\widehat{u_j}})\|_2 + \|\widehat{\varphi_Q} * (\widehat{u_j} - g_{\widehat{u_j}})\|_2 \le \kappa,$$

as claimed. □

The following lemma presents some almost orthogonality statements that will be crucial in the proof of Theorem 2.1. By P_I we mean one of the operators defined in the discussion following Lemma 2.3. In particular, (2.18), (2.19), and (2.20) are satisfied. Also, recall that $\widetilde{P_I}$ is defined in (2.21). By χ_Q we always mean the indicator of Q.

Lemma 2.7. *There exist constants C_m depending only on $m = 1, 2, \dots$ and τ such that for arbitrary intervals $I \subset (-2+\tau, 2-\tau)$ and $Q, Q' \subset \mathbb{Z}$*

$$\|\chi_{Q'} P_I^2 \chi_Q\| \le C_m\, |I| \sqrt{|Q||Q'|}\, [1 + |I| \mathrm{dist}(Q, Q')]^{-m} \quad \textit{for all } m \ge 1. \tag{2.28}$$

In particular, let $\mathcal{P}_{I,L} = \{Q\}$ be a partition of $\mathbb{Z}$ into congruent intervals of length $L|I|^{-1}$, where $L \ge 1$ is a parameter. Then

$$\|P_I u\|_2^2 \lesssim \sum_{Q \in \mathcal{P}_{I,L}} \|P_I \chi_Q u\|_2^2 + C_m\, L^{-m} \|u\|_2^2 \tag{2.29}$$

for all $u \in \ell^2(\mathbb{Z})$. (2.28) and (2.29) remain correct if one or both of $\chi_Q, \chi_{Q'}$ are replaced with $\sqrt{\varphi_Q}, \sqrt{\varphi_{Q'}}$, respectively. Moreover, one can also replace P_I with $\widetilde{P_I}$ without changing the statement.

Proof. For simplicity, we restrict ourselves to the case of the indicators χ_Q and $\chi_{Q'}$. The smooth bump functions φ_Q and $\varphi_{Q'}$ represent only a small technical variation requiring the use of (2.26). By Schur's lemma, it suffices to control the L^1-norm of the kernel K of $\chi_{Q'} P_I^2 \chi_Q$. By definition,

$$K(x, y) = \chi_{Q'}(x) K_I(x - y) \chi_Q(y).$$

Recall that here K_I is a kernel satisfying (2.19). Hence

$$\sum_x |K(x,y)| \le C_m |I||Q'|\, [1 + |I| \mathrm{dist}(Q, Q')]^{-m} \quad \text{for all } m \ge 1, \tag{2.30}$$

$$\sum_y |K(x,y)| \le C_m |I||Q|\, [1 + |I| \mathrm{dist}(Q, Q')]^{-m} \quad \text{for all } m \ge 1. \tag{2.31}$$

By Lemma 2.5, the norm $\|\chi_{Q'} P_I^2 \chi_Q\|$ is controlled by the square root of the product of the terms on the right-hand side of (2.30) and (2.31), which implies (2.28).

To prove (2.29), one expands relative to the partition $\mathcal{P}_{I,L}$, i.e., $u = \sum_{Q\in\mathcal{P}_{L,I}} \chi_Q u$:

$$\begin{aligned}
\|P_I u\|_2^2 &= \sum_{Q,Q'\in\mathcal{P}_{I,L}} \langle P_I\chi_Q u, P_I\chi_{Q'}u\rangle = \sum_{\substack{Q,Q'\in\mathcal{P}_{I,L}\\ \mathrm{dist}(Q,Q')\le \mathrm{diam}(Q)}} \langle P_I\chi_Q u, P_I\chi_{Q'}u\rangle \\
&\quad + \sum_{\mathrm{dist}(Q,Q')>\mathrm{diam}(Q)} \langle \chi_Q u, \chi_Q\, P_I^2\, \chi_{Q'}u\rangle \\
&\lesssim \sum_{Q\in\mathcal{P}_{I,L}} \|P_I\,\chi_Q u\|_2^2 + \sum_{\mathrm{dist}(Q,Q')>\mathrm{diam}(Q)} C_m\, L[1+|I|\mathrm{dist}(Q,Q')]^{-m}\, \|\chi_Q u\|_2\|\chi_{Q'}u\|_2 \quad (2.32)\\
&\lesssim \sum_{Q\in\mathcal{P}_{I,L}} \|P_I\,\chi_Q u\|_2^2 + C_m\, L^{1-m}\|u\|_2^2 \quad (2.33)
\end{aligned}$$

where (2.32) follows from (2.28), and (2.33) follows from (2.32) by means of Schur's test. To obtain the statement involving $\widetilde{P_I}$, one uses (2.22) instead of (2.19). Otherwise the proof is unchanged. □

The following lemma is a key ingredient in our argument.

Lemma 2.8. *Fix small $\tau > 0$ and $\eta > 0$. Let $I, J \subset (-2+\tau, 2-\tau)$ be intervals with $0 < |I| \le |J|$. For any interval $Q \subset \mathbb{Z}$ so that $|Q| = N \ge |J|^{-1}$ there is the estimate*

$$\mathbb{P}\Big[\|P_J\,\omega_Q\,P_I\| = \|P_I\,\omega_Q\,P_J\| > C\,A\sqrt{|J|}\Big] < N\,|J|\,e^{-A^2} \quad (2.34)$$

for all $A \ge |J|^{-\eta}$ and some constant C which depends only on τ and η. Here ω_Q denotes the restriction of $\{\omega_n\}$ to Q.

Proof. Since the distribution of

$$\|P_J\,\omega_Q\,P_I\| = \|P_I\,\omega_Q\,P_J\| = \sup_{\|u\|_2=\|v\|_2=1}\Big|\langle \omega_Q P_J u, P_I v\rangle\Big|$$

does not depend on the position of Q, we may assume that Q is centered at 0 (the first equality follows by passing to adjoints). Let $\varepsilon = |J|$. Suppose that there is some $u \in \ell^2(Z)$ so that for a fixed sequence $\{\omega\}$

$$\|P_J\omega_Q P_I u\|_2^2 \ge B^2\|u\|_2^2 \ge B^2\|P_I u\|_2^2 \quad (2.35)$$

with some $B \gg \varepsilon^5$. Let $Q = \bigcup Q'$ be a partition of Q into pairwise disjoint intervals Q' of size $\varepsilon^{-1-\eta}$. By Lemma 2.7, more precisely (2.29) applied to $\omega_Q\,P_I u$ with $L = \varepsilon^{-\eta}$,

$$\|P_J\omega_Q P_I u\|_2^2 \lesssim \sum_{Q'\subset Q} \|P_J\omega_{Q'}P_I u\|_2^2 + \varepsilon^{10}\|P_I u\|_2^2$$

and

$$\|P_I u\|_2^2 \gtrsim \sum_{Q'\subset Q} \|\varphi_{Q'}P_I u\|_2^2.$$

One therefore concludes from (2.35) that there exists $Q' \subset Q$ so that

$$\|P_J \omega_{Q'} P_I u\|_2 \gtrsim B \|\varphi_{Q'} P_I u\|_2.$$

Hence,

$$\begin{aligned}
\mathbb{P}\big[\|P_J \omega_Q P_I\| \geq B\big] &\leq \sum_{Q' \subset Q} \mathbb{P}\Big[\exists\, u \,\Big|\, \|P_J \omega_{Q'} P_I u\|_2 \gtrsim B \|\varphi_{Q'} P_I u\|_2\Big] \\
\leq \quad & \sum_{Q' \subset Q} \mathbb{P}\Big[\exists\, u, v \,\Big|\, |\langle \varphi_{Q'}^{-2} \omega_{Q'} \varphi_{Q'} P_I u, \varphi_{Q'} P_J v\rangle| \gtrsim B \|\varphi_{Q'} P_I u\|_2 \|\varphi_{Q'} P_J v\|_2\Big] \\
\leq \quad & N\varepsilon\, \mathbb{P}\Big[\exists\, u \in \varphi_{Q'_0} \cdot X_{S(\tilde{I})^*},\, v \in \varphi_{Q'_0} \cdot X_{S(J)^*} \,\Big|\, |\langle \varphi_{Q'_0}^{-2} \omega_{Q'_0} u, v\rangle| \gtrsim B \|u\|_2 \|v\|_2\Big]. \qquad (2.36)
\end{aligned}$$

Here Q'_0 is the interval from the partition that contains 0, and $\tilde{I}$ is the interval with the same center as I, but length $|J|$. To pass to line (2.36), we used that the probabilities in the preceding line do not depend on the position of Q'. Moreover, we replaced I with $\tilde{I}$ in line (2.36) because

$$\operatorname{diam}(\widehat{\varphi_{Q'_0}}) \asymp |Q'_0|^{-1} \asymp \varepsilon^{1+\eta},$$

see (2.26). Let u, v be as in (2.36). Then by (2.20) the Fourier supports of u and v have measure at most $C_\tau\, |J|$. By Lemma 2.2 therefore

$$\|uv\|_{\ell^2(\mathbb{Z})} = \|\hat{u} * \hat{v}\|_{L^2(\mathbb{T})} \leq C_\tau \sqrt{|J|}\; \|u\|_2 \|v\|_2. \qquad (2.37)$$

Denote $\varphi_{Q'_0}^{-2} \omega_{Q'_0}$ by $\tilde{\omega}$. Notice that these random variables are uniformly bounded as $\varphi_{Q'_0}$ is bounded below by some positive constant on Q'_0 which does not depend on Q'_0. Combining (2.37) with the standard large deviation estimate

$$\mathbb{P}\Big[|\sum_n \tilde{\omega}_n u_n v_n| > t\Big(\sum_n |u_n v_n|^2\Big)^{\frac{1}{2}}\Big] \lesssim e^{-ct^2}$$

yields

$$\mathbb{P}\Big[|\sum_n \tilde{\omega}_n u_n v_n| > Ct\sqrt{|J|}\; \|u\|_2 \|v\|_2\Big] \lesssim e^{-ct^2} \qquad (2.38)$$

for any u, v as in (2.36), $C = C_\tau$ being some constant. By Lemma 2.6 the space

$$\mathcal{B}_I := \{\varphi_{Q'_0} w \mid w \in X_{S(\tilde{I})^*},\ \|w\|_2 \leq 1\}$$

has a C_0^{-1}-net $\mathcal{N}_1$ of size $\exp\big(C\, C_0^2 \varepsilon^{-\eta}\big)$ where C_0 is some constant to be specified below. Similarly, the space

$$\mathcal{B}_J := \{\varphi_{Q'_0} w \mid w \in X_{S(J)^*},\ \|w\|_2 \leq 1\}$$

has a C_0^{-1}-net $\mathcal{N}_2$ of size $\exp\big(C\, C_0^2 \varepsilon^{-\eta}\big)$. Now suppose that

$$|\sum_n \tilde{\omega}_n \tilde{u}_n \tilde{v}_n| \leq \frac{B}{2}$$

for all $\tilde u \in \mathcal{N}_1$ and $\tilde v \in \mathcal{N}_2$. Moreover, assume that

$$|\sum_n \tilde\omega_n u_n v_n| \le 2B \quad \text{for all} \quad u \in \mathcal{B}_I,\ v \in \mathcal{B}_J.$$

Thus, for any $u \in \mathcal{B}_I$, $v \in \mathcal{B}_J$ there are $\tilde u \in \mathcal{N}_1$ and $\tilde v \in \mathcal{N}_2$ so that

$$\begin{aligned} |\sum_n \tilde\omega_n u_n v_n| &\le |\sum_n \tilde\omega_n (u_n - \tilde u_n) v_n| + |\sum_n \tilde\omega_n (v_n - \tilde v_n)\tilde u_n| + |\sum_n \tilde\omega_n \tilde u_n \tilde v_n| \\ &\le 4B\, C_0^{-1} + \frac{B}{2} \le B \end{aligned} \tag{2.40}$$

if C_0 was chosen large enough. Therefore, by (2.38),

$$\begin{aligned} &\mathbb{P}\Big[\exists\, u \in \varphi_{Q_0'} \cdot X_{S(\tilde I)^*},\ v \in \varphi_{Q_0'} \cdot X_{S(J)^*} \,\Big|\, |\langle \tilde\omega u, v\rangle| \ge B\|u\|_2\|v\|_2\Big] \\ \le\ &\mathbb{P}\Big[\exists\, u \in \varphi_{Q_0'} \cdot X_{S(\tilde I)^*},\ v \in \varphi_{Q_0'} \cdot X_{S(J)^*} \,\Big|\, |\langle \tilde\omega u, v\rangle| \ge 2B\|u\|_2\|v\|_2\Big] \\ &\qquad + \mathbb{P}\Big[\exists\, u \in \mathcal{N}_1,\ v \in \mathcal{N}_2 \,\Big|\, |\langle \tilde\omega u, v\rangle| \ge \frac{1}{2}B\|u\|_2\|v\|_2\Big] \\ \le\ &\sum_{\ell \ge 0} \sum_{u \in \mathcal{N}_1} \sum_{v \in \mathcal{N}_2} \mathbb{P}\Big[|\langle \tilde\omega u, v\rangle| \ge 2^{\ell-1}\, B\, \|u\|_2 \|v\|_2\Big] \\ \le\ &\sum_{\ell \ge 0} \exp(C\varepsilon^{-\eta}) \exp\Big(-cB^2\, 2^{2\ell}/|J|\Big) \lesssim \exp\Big(-cB^2/|J|\Big), \end{aligned} \tag{2.41}$$

provided $B \gg \varepsilon^{-\eta/2}\sqrt{|J|}$. Combining this with (2.36) yields

$$\mathbb{P}\big[\|P_J \omega_Q P_I\| \ge B\big] \lesssim N\varepsilon\ \exp\Big(-c\,B^2/|J|\Big)$$

for those B, and the lemma follows. □

In contrast to Lemma 2.8 we would like to point out that for any choice of nonempty $I, J \subset (-2, 2)$ one has a.s. $\|P_I\, \omega\, P_J\| \asymp 1$. Indeed, assume without loss of generality that $|I| \le |J|$ and partition $\mathbb{Z}$ into congruent intervals Q of size about $|I|^{-1}$. Denote the resulting partition by $\mathcal{Q}$. Observe that for any pair Q, Q' of disjoint intervals the variables $\|P_I\, \omega_Q\, P_J\|$ and $\|P_I\, \omega_{Q'}\, P_J\|$ are independent. Since one can easily check that $\|P_I\, \omega_Q\, P_J\| \asymp 1$ with positive probability, this implies that with probability one there exists some $Q \in \mathcal{Q}$ with this property. By almost orthogonality of the operators

$$\{P_I \omega_Q P_J\}_{Q \in \mathcal{Q}}$$

one has

$$\|P_I \omega P_J\| \asymp \sup_{Q \in \mathcal{Q}} \|P_I \omega_Q P_J\|,$$

and the claim follows. Hence the restriction of ω to Q in Lemma 2.8 is essential. This argument will be carried out in full detail in Subsection 2.3 below, see Proposition 2.13.

Definition 2.9. *Fix some small $\eta > 0$ and let $\lambda > 0$ be the disorder in (2.1). For any small $\delta > 0$ let $\mathcal{Q}_\delta$ denote a partition of $\mathbb{Z}$ into disjoint, congruent dyadic intervals of size $\asymp \lambda^{-\eta}\delta^{-1}$. We shall consider only finitely many scales δ, namely those of the form $2^j\lambda^{2-4\eta}$, $0 \le j \lesssim |\log\lambda|$. These scales are called λ-admissible. We fix some collection $\mathcal{C}_\lambda$ of about λ^{-2} many intervals whose lengths are λ-admissible with the property that for any energy $E \subset (-2+\tau, 2-\tau)$ one has $\mathcal{I}(E') \subset \mathcal{C}_\lambda$ for some E' with $|E'-E| \ll \lambda^{2-4\eta}$, where $\mathcal{I}(E')$ is defined in (2.15). We refer to those intervals as λ-admissible. Let I and J be λ-admissible. For a given realization of the random sequence $\{\omega_n\}$ we say that an interval $Q \in \mathcal{Q}_{\min(|I|,|J|)}$ is (I,J)-good, provided*

$$\|P_I \omega_Q P_J\| \lesssim \lambda^{-\eta}\, |I|^{\frac{1}{2}} \vee |J|^{\frac{1}{2}}, \tag{2.42}$$

and (I,J)-bad otherwise.

If $I, J \subset (-2+\tau, 2-\tau)$, then Lemma 2.8 states that

$$\mathbb{P}\Big[Q \text{ is } (I,J)\text{-bad}\Big] \lesssim e^{-\lambda^{-2\eta}} \tag{2.43}$$

for all $Q \in \mathcal{Q}_{\min(|I|,|J|)}$. Therefore, for fixed (I,J) and with high probability any large interval $[-N,N] \subset \mathbb{Z}$ should contain no more than $Ne^{-\lambda^{-2\eta}}$ many (I,J)-bad intervals. Lemma 2.10 makes this precise.

Furthermore, it suffices to consider families of about λ^{-2} many λ-admissible intervals, see the discussion following Lemma 2.3.

Lemma 2.10. *For any λ-admissible I, J one has*

$$\#\Big\{Q \subset [-N,N] \,\Big|\, Q \in \mathcal{Q}_{\min(|I|,|J|)} \text{ is } (I,J)\text{-bad}\Big\} \lesssim e^{-\lambda^{-2\eta}}\, N. \tag{2.44}$$

up to probability at most $e^{-c_{\lambda,\eta}N}$, with some constant $c_{\lambda,\eta}$ depending on λ and η. In particular, a.s. all but finitely many N satisfy the bound (2.44) for any given I, J.

Proof. We shall use the following well-known large deviation theorem: Let Y_j be i.i.d. Bernoulli variables with $\mathbb{P}(Y_j = 0) = 1-p$ and $\mathbb{P}(Y_j = 1) = p$. Then for large M

$$\mathbb{P}\Big[\sum_{j=1}^{M} Y_j > 2Mp\Big] \le e^{-b_p M},$$

where $b_p > 0$ is a constant depending only on p. Here the $Y_j = 1$ or 0 depending on whether or not the j-th interval $Q \in \mathcal{Q}_{\min(|I|,|J|)}$ is bad or good. Observe that the Y_j are an i.i.d sequence. Because of (2.43) one can take $p = e^{-\lambda^{-2\eta}}$, and $M \ge N\lambda^2$. The final statement of the lemma follows from the first one by means of the Borel-Cantelli lemma. ∎

The next idea is that of all eigenfunctions $\{u_j^{(N)}\}$ as in Theorem 2.1 only few can have significant mass on the sparse set of bad intervals from the previous lemma.

Lemma 2.11. *Let $\{u_j\}$ be an arbitrary orthonormal sequence in $\ell^2(\mathbb{Z})$ and suppose $\mathcal{S} \subset \mathbb{Z}$ is some finite subset. Then*

$$\#\Big[j \,\Big|\, \|u_j\|_{\ell^2(\mathcal{S})} > t\Big] \le \#\mathcal{S}\, t^{-2}$$

for any $t > 0$.

Proof. Fix an $t > 0$. Define $\mathcal{J} = \{j \mid \|u_j\|_{\ell^2(\mathcal{S})} > t\}$. Let $(e_k)_n = 1$ if $n = k$ and $(e_k)_n = 0$ else. By Bessel's inequality

$$t^2\, \#\mathcal{J} \le \sum_{j \in \mathcal{J}} \|u_j\|^2_{\ell^2(\mathcal{S})} \le \sum_{k \in \mathcal{S}} \sum_j |\langle u_j, e_k\rangle|^2 \le \#\mathcal{S},$$

as claimed. □

In the following lemma we present some additional almost-orthogonality properties that will be relevant in the proof of Theorem 2.1. Let $\mathcal{P}_{I,L}$ be defined as in Lemma 2.7. It will be convenient to use the following notation: For any $Q, Q' \in \mathcal{P}_{I,L}$, one has $Q \sim Q'$ iff $Q = Q'$ or Q and Q' are nearest neighbors.

Lemma 2.12. *Assume that $\{\omega\}$ satisfies $\|\omega\|_\infty \le 1$, and let ω_Q denote the restriction of $\{\omega\}$ to any $Q \in \mathcal{P}_{I,L}$. Then for any $J \subset (-2+\tau, 2-\tau)$ satisfying $|I| \le |J|$ one has (the sums are over $Q' \in \mathcal{P}_{I,L}$)*

$$\|P_I \omega_Q P_J^2 u\|_2^2 \lesssim \sum_{Q' \sim Q} \|P_I \omega_Q P_J^2 \varphi_{Q'} u\|_2^2 + C_m\, L|J||I|^{-1} \sum_{Q' \not\sim Q} [1 + |J| \mathrm{dist}(Q, Q')]^{-m} \|\sqrt{\varphi_{Q'}} u\|_2^2 \quad (2.45)$$

for all $u \in \ell^2(\mathbb{Z})$ and $m \ge 1$. Furthermore,

$$\|P_I \omega_Q \widetilde{P_I} u\|_2^2 \lesssim \sum_{Q' \sim Q} \|P_I \omega_Q \widetilde{P_I} \varphi_{Q'} u\|_2^2 + C_m\, L \sum_{Q' \not\sim Q} [1 + |I| \mathrm{dist}(Q, Q')]^{-m} \|\sqrt{\varphi_{Q'}} u\|_2^2 \quad (2.46)$$

for all $u \in \ell^2(\mathbb{Z})$ and $m \ge 1$. Here $\widetilde{P_I}$ is the operator defined in (2.21).

Proof. As far as (2.45) is concerned, the expansion $u = \sum_{Q \in \mathcal{P}_{I,L}} \varphi_Q\, u$ yields:

$$\begin{aligned}
& \|P_I \omega_Q P_J^2 u\|_2^2 \\
\lesssim\ & \sum_{Q' \sim Q} \|P_I \omega_Q P_J^2 \varphi_{Q'} u\|_2^2 + \Big(\sum_{Q' \not\sim Q} \|\chi_Q\, P_J^2 \sqrt{\varphi_{Q'}}\| \|\sqrt{\varphi_{Q'}} u\|_2\Big)^2 && (2.47) \\
\lesssim\ & \sum_{Q' \sim Q} \|P_I \omega_Q P_J^2 \varphi_{Q'} u\|_2^2 + \sum_{Q' \not\sim Q} \|\chi_Q\, P_J^2 \sqrt{\varphi_{Q'}}\| \sum_{Q' \not\sim Q} \|\chi_Q\, P_J^2 \sqrt{\varphi_{Q'}}\| \|\sqrt{\varphi_{Q'}} u\|_2^2. && (2.48)
\end{aligned}$$

Here (2.48) follows from (2.47) by means of Cauchy-Schwarz. Now (2.45) follows from (2.48) by means of (2.28). Indeed,

$$\begin{aligned}
\sum_{Q' \not\sim Q} \|\chi_Q\, P_J^2 \sqrt{\varphi_{Q'}}\| &\le C_m \sum_{Q' \not\sim Q} L|J||I|^{-1} [1 + |J| \mathrm{dist}(Q, Q')]^{-m} \\
&\le C_m\, L|J||I|^{-1} \Big(\frac{L|J|}{|I|}\Big)^{-m} \le C_m\, L^{1-m},
\end{aligned}$$

as claimed. The proof of (2.46) is basically the same (see the last sentence in the statement of Lemma 2.7). □

We are now ready to prove Theorem 2.1. The basic approach is the same as in Subsection 2.1. The main ideas are as follows: (2.6) is replaced with Lemma 2.8. This can only be done for good intervals Q. However, Lemma 2.10 insures that with high probability the union of all bad intervals is a set of density $o(1)$ as $\lambda \to 0$. Finally, one can use Lemma 2.11 to conclude that most eigenfunctions have very little mass on this bad set. Most work in the following proof has to do with carrying out the details of the micro-local analysis. As usual, the reader might find it helpful to assume that various cut-off functions have no tails. In that case the proof becomes very simple.

Proof of Theorem 2.1. Pick some eigenfunction $u = u_j^{(N)}$ with energy $E = E_j^{(N)} \in (-2+\tau, 2-\tau)$. For E fixed, we partition $(-2,2)$ into intervals I_j where I_0 is centered close to E and $|I_j| = 2^j\lambda^{2-4\eta}$. As mentioned in Definition 2.9, it suffices to consider at most about λ^{-2} such partitions in total, which we fix beforehand. For a given E, we choose that partition for which the smallest interval is centered as close to E as possible. It follows that $\mathrm{dist}(E, I_j) \asymp |I_j|$ for $j \neq 0$, see Figure 2.1. Recall that

$$\widehat{P_{I_j}^2 f} = \widehat{K_{I_j}}\hat{f},$$

where (2.18), (2.19), and (2.20) hold. Now choose any $I = I_j$ with $j \neq 0$. Let $\delta = |I|$. Then (2.9) and (2.29) applied to ωu (with $L = \lambda^{-\eta}$ and m large) imply that

$$\begin{aligned}
\delta\|P_I u\|_2 &\lesssim \lambda\|P_I(\omega u)\|_2 + \delta(|u_{-N}| + |u_N|) \\
&\lesssim \lambda\Big(\sum_{Q\in\mathcal{Q}_\delta}\|P_I\omega_Q u\|_2^2\Big)^{\frac12} + \delta(|u_N|^2+|u_{-N}|^2)^{\frac12} + \lambda^{10}\|u\|_2. \qquad (2.49)
\end{aligned}$$

We call $Q \in \mathcal{Q}_\delta$ a <u>bad</u> interval, if it is (I,J)-bad for some choice of λ-admissible J. By Lemma 2.10 we may assume that there are at most

$$\lambda^{-4}e^{-\lambda^{-2\eta}}N \qquad (2.50)$$

many such bad intervals $Q \in \mathcal{Q}_\delta$. By (2.18),

$$\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\|P_I\omega_Q u\|_2^2 \lesssim |\log\lambda|\sum_{|I_j|>|I|}\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\|P_I\omega_Q P_{I_j}^2 u\|_2^2 + \sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\|P_I\omega_Q\widetilde{P_I}u\|_2^2, \qquad (2.51)$$

where we have used (2.21). The appearance of $|\log\lambda|$ is due to the fact that there are $\lesssim |\log\lambda|$ many intervals I_j. To deal with the first term on the right-hand side of (2.51), we invoke (2.45) with $L = \lambda^{-\eta}$ and $J = I_j$. More precisely,

$$\begin{aligned}
\sum_{|I_j|>|I|}\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\|P_I\omega_Q P_{I_j}^2 u\|_2^2 &\lesssim \sum_{|I_j|>|I|}\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\sum_{Q'\sim Q}\|P_I\omega_Q P_{I_j}\|^2\|P_{I_j}\varphi_{Q'}u\|_2^2 \qquad (2.52)\\
&+ \sum_{|I_j|>|I|}\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}} C_m L|I_j||I|^{-1}\sum_{Q'\not\sim Q}[1+|I_j|\mathrm{dist}(Q,Q')]^{-m}\|\sqrt{\varphi_{Q'}}u\|_2^2 \qquad (2.53)
\end{aligned}$$

The triple sum (2.53) is a small error term. In fact, for large m,

$$(2.53) \lesssim \sum_{|I_j|>|I|} L|I_j||I|^{-1} \sum_{Q'\in\mathcal{Q}_\delta} \Big(|I_j|L/|I|\Big)^{-m} \|\sqrt{\varphi_{Q'}}u\|_2^2$$

$$\lesssim \sum_{|I_j|>|I|} \Big(\frac{L|I_j|}{|I|}\Big)^{1-m} \|u\|_2^2 \lesssim L^{1-m}\|u\|_2^2 \lesssim \lambda^{10}\|u\|_2^2 \tag{2.54}$$

since $L = \lambda^{-\eta}$. By the definition of good intervals the sums on the right-hand side of (2.52) are no larger than

$$\sum_{|I_j|>|I|} \sum_{\substack{Q\in\mathcal{Q}_\delta \\ Q \text{ good}}} \lambda^{-2\eta}|I_j|\|P_{I_j}\varphi_Q u\|_2^2 \lesssim \sum_{|I_j|>|I|} \sum_{\substack{Q\in\mathcal{Q}_\delta \\ Q \text{ good}}} \sum_{k:|I_k|\asymp|I_j|} \lambda^{-2\eta}\,|I_j|\|P_{I_j}\varphi_Q P_{I_k}^2 u\|_2^2, \tag{2.55}$$

where we have again used (2.18) and the fact that

$$P_{I_j}\varphi_Q P_{I_k} = 0 \;\text{ if }\; |I_k| \not\asymp |I_j|,$$

see (2.20) and (2.26). One now estimates (2.55) as follows:

$$\sum_{|I_j|>|I|} \sum_{\substack{Q\in\mathcal{Q}_\delta \\ Q \text{ good}}} \sum_{k:|I_k|\asymp|I_j|} |I_j|\|P_{I_j}\varphi_Q P_{I_k}^2 u\|_2^2 \;\lesssim\; \sum_{Q\in\mathcal{Q}_\delta} \sum_{k:|I_k|\gtrsim|I|} |I_k|\|\varphi_Q P_{I_k}^2 u\|_2^2$$

$$\lesssim \sum_{k:|I_k|\gtrsim|I|} |I_k|\|P_{I_k}u\|_2^2. \tag{2.56}$$

Combining (2.54), (2.55), (2.56), yields

$$(2.52) + (2.53) \lesssim \lambda^{-2\eta} \sum_{k:|I_k|\gtrsim|I|} |I_k|\|P_{I_k}u\|_2^2 + \lambda^{10}\|u\|_2^2.$$

Therefore, the first term in (2.51) is bounded by

$$\lesssim |\log\lambda|\lambda^{-2\eta} \sum_{k:|I_k|\gtrsim|I|} |I_k|\|P_{I_k}u\|_2^2 + \lambda^9\|u\|_2^2. \tag{2.57}$$

The second term in (2.51) can be estimated similarly, but one uses (2.46) instead. In fact,

$$\sum_{\substack{Q\in\mathcal{Q}_\delta \\ Q \text{ good}}} \|P_I\omega_Q\widetilde{P_I}u\|_2^2 \lesssim \sum_{\substack{Q\in\mathcal{Q}_\delta \\ Q \text{ good}}} \sum_{Q'\sim Q} \|P_I\omega_Q\widetilde{P_I}\varphi_{Q'}u\|_2^2 \tag{2.58}$$

$$+ \sum_{\substack{Q\in\mathcal{Q}_\delta \\ Q \text{ good}}} C_m\,\lambda^{-\eta} \sum_{Q'\not\sim Q} [1+|I|\mathrm{dist}(Q,Q')]^{-m}\|\sqrt{\varphi_{Q'}}u\|_2^2. \tag{2.59}$$

As in the case of (2.53), one checks that (2.59) is no larger than $\lambda^{10}\|u\|_2^2$, cf. (2.54). Recalling the definition of $\widetilde{P_I}$, one obtains for the term in (2.58)

$$\begin{aligned}
\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\sum_{Q'\sim Q}\|P_I\omega_Q\widetilde{P_I}\varphi_{Q'}u\|_2^2 &\lesssim |\log\lambda|\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\sum_{Q'\sim Q}\sum_{j:|I_j|\le|I|}\|P_I\omega_Q P_{I_j}\|^2\|P_{I_j}\varphi_{Q'}u\|_2^2\\
&\lesssim \lambda^{-2\eta}|\log\lambda|\,|I|\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\sum_{Q'\sim Q}\sum_{j:|I_j|\le|I|}\|P_{I_j}\varphi_{Q'}u\|_2^2\\
&\lesssim \lambda^{-2\eta}|\log\lambda|\,|I|\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\sum_{Q'\sim Q}\sum_{\substack{j:|I_j|\le|I|\\ k:|I_k|\lesssim|I|}}\|P_{I_j}\varphi_{Q'}P_{I_k}^2u\|_2^2\\
&\lesssim \lambda^{-2\eta}|\log\lambda|\,|I|\sum_{k:|I_k|\lesssim|I|}\|P_{I_k}u\|_2^2.
\end{aligned}$$

Combining this with (2.57) shows that

$$\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ good}}}\|P_I\omega_Q u\|_2^2\lesssim\lambda^{-2\eta}|\log\lambda|\sum_j\max(|I_j|,|I|)\|P_{I_j}u\|_2^2+\lambda^9\|u\|_2^2. \tag{2.60}$$

By Lemma 2.11 and (2.50) all but at most

$$\lambda^{-9}e^{-\lambda^{-2\eta}}N$$

many eigenfunctions $\{u_j^{(N)}\}$ satisfy

$$\|\chi_{\bigcup_{Q \text{ bad}}Q}u\|_2^2\lesssim\lambda^5\|u\|_2^2.$$

For these eigenfunctions one therefore has

$$\sum_{\substack{Q\in\mathcal{Q}_\delta\\ Q \text{ bad}}}\|P_I\omega_Q u\|_2^2\lesssim\|\chi_{\bigcup_{Q \text{ bad}}Q}u\|_2^2\lesssim\lambda^5\|u\|_2^2. \tag{2.61}$$

Combining (2.49), (2.60), and (2.61), implies that

$$|I|\|P_Iu\|_2 \lesssim \lambda^{1-\eta}|\log\lambda|^{\frac12}\sum_j\max(|I|,|I_j|)^{\frac12}\|P_{I_j}u\|_2+\lambda^2\|u\|_2+\delta(|u_N|^2+|u_{-N}|^2)^{\frac12} \tag{2.62}$$

$$\lesssim \lambda^{1-\eta}|\log\lambda|^{\frac12}\sum_j\max(|I|,|I_j|)^{\frac12}\|P_{I_j}u\|_2. \tag{2.63}$$

To remove the last two terms in (2.62), one uses (2.18) and Lemma 2.11. Maximize the quantities $\sqrt{|I_j|}\|P_{I_j}u\|_2$. If the maximum is assumed for $j\neq0$, then the maximizing interval would have to satisfy (2.63). Since $|I_j|\ge\lambda^{2-4\eta}$ for all j, this can easily be seen to lead to a contradiction.
We have reached the following conclusion: a.s. and for all large integers N most eigenfunctions $u=u_j^{(N)}$ of the Dirichlet problem on $[-N,N]$ have the property that

$$\|P_Ju\|_2\lesssim\sqrt{\frac{|I|}{|J|}}\,\|P_Iu\|_2, \tag{2.64}$$

where I is the interval of length $\asymp \lambda^{2-4\eta}$ centered at the energy $E = E_j^{(N)}$ of $u_j^{(N)}$, and J is any of the other intervals with the property that $\mathrm{dist}(J, E) \asymp |J|$. "Most eigenfunctions" here means up to $\lambda^{-9}\, e^{-\lambda^{-2\eta}} N$ many, as $\lambda \to 0$.
Now fix a sufficiently large N and let u be any such eigenfunction. The theorem follows from (2.64) and Lemma 2.2 by means of the calculation leading up to (2.8). $\square$

2.3 The question of optimality

The order λ^2 for the concentration of the Fourier transform cannot be improved. Indeed, suppose that (2.7) holds for some eigenfunction u at energy E with intervals I_j so that $|I_j| = 2^j \lambda^{2+\eta}$ for some $\eta > 0$. Then the calculation leading up to (2.8) would show that

$$\|u\|_4^4 \lesssim \lambda^{2+\eta} \|u\|_2^4.$$

On the other hand, by the theory of Anderson localization, there is some $n_0 \in \mathbb{Z}$ such that for any $\varepsilon > 0$

$$|u(n)| \le C_\varepsilon\, e^{-(\gamma(E,\lambda)-\varepsilon)|n-n_0|}$$

where $\gamma(E, \lambda) > 0$ is the Lyapunov exponent. By the Figotin-Pastur formula, see Theorem 14.6 in [5], one has

$$\gamma(E, \lambda) \sim C(E)\lambda^2$$

as $\lambda \to 0$. Here $0 < C(E) < \infty$ if $E \in (-2 + \tau, -\tau) \cup (\tau, 2 - \tau)$. However, the preceding three inequalities are incompatible. Hence λ^2 is optimal, as claimed.
Inspection of the proof of Theorem 2.1 shows that therefore one cannot have a better estimate than

$$\|P_I \omega_Q P_J\| \lesssim \sqrt{|I|} \vee \sqrt{|J|} \tag{2.65}$$

with high probability. Here $Q \in \mathcal{Q}_{\min(|I|,|J|)}$. Observe that (2.65) is basically what Lemma 2.8 provides. In this subsection we show directly, i.e., without recourse to the proof of Theorem 2.1 that (2.65) is optimal. Moreover, we show that the left-hand side of (2.65) is a.s. of size one if $Q = \mathbb{Z}$.

Proposition 2.13. *Suppose $\{\omega_n\}$ is a sequence of i.i.d. variables with $\mathbb{E}\omega_0 = 0$ and $\mathbb{P}(\omega_0 \ne 0) > 0$. Then for any pair of nonempty intervals $I, J \subset (-2, 2)$ one has*

$$\mathbb{P}\Big[\|P_I \omega P_J\| \ge c\Big] = 1$$

where $c > 0$ is a constant that only depends on the distribution of ω_0.

Proof. Fix two intervals I, J as above and let $S(I), S(J)$ be as in (2.3). For simplicity, we shall assume that

$$\widehat{P_I u} = \chi_{S(I)} \hat{u} \quad\text{and}\quad \widehat{P_J v} = \chi_{S(J)} \hat{v} \tag{2.66}$$

rather than letting the multipliers of P_I and P_J being smooth. This latter case is just a small technical variation. Now let $\delta > 0$ be sufficiently small such that $S(I) \supset (\theta_I - \delta, \theta_I + \delta)$ and

$S(J) \supset (\theta_J - \delta, \theta_J + \delta)$ for suitable θ_I, θ_J. Let φ be as in Lemma 2.4. Pick $k \in \mathbb{Z}$ arbitrary and define

$$\begin{aligned} \hat{u}(\theta) &= \hat{\varphi}((\theta - \theta_I)/\delta)\, e(\theta k) \\ \hat{v}(\theta) &= \hat{\varphi}((\theta - \theta_J)/\delta)\, e(\theta k). \end{aligned}$$

Thus,

$$P_I u = u \quad \text{and} \quad P_J v = v$$

regardless of the choice of k. Moreover, $\|u\|_2 \asymp \sqrt{\delta}$ and $\|v\|_2 \asymp \sqrt{\delta}$ independently of k. Since

$$\langle P_J \omega P_I u, v \rangle = \sum_n \omega_n u_n \overline{v_n} = \delta^2 \sum_n \omega_n\, e((\theta_J - \theta_I)(n+k))\, \varphi^2(\delta(n+k))$$

it follows that with probability one

$$\begin{aligned} \|P_J \omega P_J\| &\gtrsim \delta \sup_k \Big| \sum_n \omega_{n-k}\, e((\theta_J - \theta_I)n) \varphi(\delta n)^2 \Big| \\ &\gtrsim \delta \sum_n \varphi(\delta n)^2 \asymp 1. \end{aligned} \tag{2.67}$$

To understand the argument leading up to (2.67) let us assume first that $\{\omega\}$ are Bernoulli variables. Then use the following two elementary facts:

$$\mathbb{P}\Big[\text{for every } N \geq 1 \text{ the sequence } \{\omega\} \text{ contains a copy of every word of length } N\Big] = 1 \tag{2.68}$$

$$\text{for all finite complex sequences } \{z_j\}_{j=1}^N \text{ one has } \sup_{\varepsilon_j = \pm 1} \Big| \sum_j \varepsilon_j z_j \Big| \geq c_1 \sum_j |z_j|. \tag{2.69}$$

The constant c_1 in (2.69) can be taken to be $\frac{1}{4\sqrt{2}}$. To obtain (2.67) in the Bernoulli case, pick a large integer N such that

$$\sum_{|n|>N} \varphi(\delta n)^2 \ll \delta^{-1}, \tag{2.70}$$

say. This is possible because of (2.26). Secondly, choose a word $\{\varepsilon_j\}_{j=-N}^N$ of ± 1 with the property that

$$\Big| \sum_{|n| \leq N} \varepsilon_n e((\theta_J - \theta_I)n) \varphi(\delta n)^2 \Big| \geq c_1 \sum_{|n| \leq N} \varphi(\delta n)^2.$$

By (2.68), with probability one, $\{\omega\}$ contains a copy of $\{\varepsilon_j\}_{j=-N}^N$, which implies (2.67) for the Bernoulli case. In case of general random variables one uses that $\mathbb{P}[\omega_0 > \gamma] > 0$ and $\mathbb{P}[\omega_0 < -\gamma] > 0$ for some $\gamma > 0$. Then (2.67) follows along the same lines as before, with a constant that also depends on γ. □

The brief argument preceding Definition 2.9 is only superficially different from the proof of Proposition 2.13. Indeed, observe that in (2.70) one can choose $N = A\delta^{-1}$ with A large. Thus the proof is indeed based on independence and almost orthogonality of the operators $P_I \omega_Q P_J$ where Q ranges over a partition at scale δ^{-1}. Next we consider the case where ω has finite support.

Proposition 2.14. *Let $I, J \subset (-2+\tau, 2-\tau)$. Assume that $|I| \le |J|$ and let $Q \subset \mathbb{Z}$ be an arbitrary interval satisfying $|Q| \asymp |I|^{-1}$. Then*

$$\|P_J \omega_Q P_I\| \gtrsim |J|^{\frac{1}{2}}$$

with probability at least $\frac{1}{2}$.

Proof. Let $\delta = |I|$, $\varepsilon = |J|$, and denote the centers of I, J by θ_I, θ_J, respectively. As already observed several times, we may assume that $Q = [-\delta^{-1}, \delta^{-1}]$. Pick a sequence $\{n_k\}_{k=1}^K \subset Q$ so that $n_{k+1} - n_k = A\varepsilon^{-1}$ and $K \asymp A^{-1}\frac{\varepsilon}{\delta}$. Here A is a large number to be fixed. Define

$$\begin{aligned} \hat{u}(\theta) &= \hat{\varphi}((\theta-\theta_I)/\delta) \\ \hat{v}(\theta) &= \sum_{k=1}^K b_k\, e(-\theta n_k)\hat{\varphi}((\theta-\theta_J)/\varepsilon), \end{aligned} \tag{2.71}$$

where $b_k = 0, \pm 1$ are random. This means that they depend on the choice of $\{\omega\}$. In what way will be explained below. Inverting the Fourier transform yields

$$\begin{aligned} u(n) &= e(n\theta_I)\delta\varphi(\delta n) \\ v(n) &= \sum_{k=1}^K b_k\, e(\theta_J(n-n_k))\varepsilon\varphi(\varepsilon(n-n_k)). \end{aligned} \tag{2.72}$$

Clearly, $\|u\|_2 \asymp \sqrt{\delta}$, whereas

$$\begin{aligned} \|v\|_2^2 &= \sum_{k=1}^K |b_k|^2 \varepsilon \|\varphi\|_2^2 + O\Big(\varepsilon \sum_{k\neq\ell} b_k \overline{b_\ell}\, [1+\varepsilon|n_k - n_\ell|]^{-2}\Big) \\ &> \frac{1}{2}\varepsilon \sum_{k=1}^K |b_k|^2\ \|\varphi\|_2^2 \asymp \varepsilon \#\{k \in \{1,\dots,K\} \mid b_k \neq 0\} \end{aligned} \tag{2.73}$$

provided A is large, see (2.26). Under the simplifying assumption (2.66) one obtains

$$\langle P_J \omega_Q P_I u, v\rangle = \sum_{n\in Q} \omega_n u_n \overline{v_n} = \varepsilon\delta \sum_{k=1}^K b_k\, e(\theta_J n_k) s_k(\omega), \tag{2.74}$$

where we have set

$$s_k(\omega) := \sum_{n \in Q} \omega_n\, e((\theta_I - \theta_J)n)\varphi(\varepsilon(n-n_k))\varphi(\delta n). \tag{2.75}$$

Clearly,

$$\mathbb{E}|s_k|^2 \asymp \varepsilon^{-1}.$$

Applying Lemma 2.15 to $s_k(\omega)$ therefore shows that

$$\mathbb{P}\Big[|s_k| \le A^{-1}\varepsilon^{-\frac{1}{2}}\Big] \lesssim A^{-1}$$

assuming, as we may, that ε is small. Define random variables X_k by setting $X_k = 1$ if $|s_k| \le A^{-1}\varepsilon^{-\frac{1}{2}}$, and $X_k = 0$ otherwise. In view of the preceding,

$$\mathbb{E} \sum_{k=1}^{K} X_k \lesssim A^{-1} K.$$

Hence

$$\mathbb{P}\Big[\text{the majority of } k \in \{1, 2, \dots, K\} \text{ satisfy } |s_k| \ge A^{-1}\varepsilon^{-\frac{1}{2}}\Big] \ge \frac{1}{2}.$$

Now define b_k in (2.71) to be $= 0$ if $X_k = 1$, and ± 1 if $X_k = 0$. Taking the supremum over all choices of ± 1 for those b_k in (2.74) shows that, with probability at least $\frac{1}{2}$,

$$\begin{aligned} \|P_J \omega_Q P_I\| &\gtrsim \|u\|_2^{-1} \varepsilon\delta \sup_{b_k = 0, \pm 1} \|v\|_2^{-1} \Big| \sum_{k=1}^{K} b_k s_k(\omega) \Big| \\ &\gtrsim A^{\frac{1}{2}} \varepsilon^{-1} \varepsilon\delta \frac{\varepsilon}{\delta A} A^{-1} \varepsilon^{-\frac{1}{2}} &(2.76) \\ &= A^{-\frac{3}{2}} \sqrt{\varepsilon}, &(2.77) \end{aligned}$$

as claimed. To pass to line (2.76) one uses that $\|u\|_2 \|v\|_2 \asymp A^{-\frac{1}{2}}\varepsilon$, see (2.73). □

The following lemma is a very weak form of a central limit type estimate. It is most likely a well-known fact from the classical probability literature, but we include it for the reader's convenience.

Lemma 2.15. *Let $\{\omega_n\}$ be an i.i.d. sequence with $\mathbb{E}\omega_0 = 0$, $\mathbb{E}\omega_0^2 = 1$, and $\mathbb{E}|\omega_0|^3 < \infty$. Then*

$$\mathbb{P}\Big[\Big| \sum_{n=1}^{N} \omega_n a_n \Big| \le \beta \Big(\sum_{n=1}^{N} |a_n|^2 \Big)^{\frac{1}{2}} \Big] \lesssim \beta \tag{2.78}$$

provided $|a_n| \le 1$ and $\big(\sum_{n=1}^{N} |a_n|^2 \big)^{-\frac{1}{2}} \lesssim \beta \le 1$.

Proof. The characteristic function f_0 of ω_0 satisfies

$$|f_0(\xi)| = |\mathbb{E} e^{i\xi\omega_0}| = 1 - \frac{\xi^2}{2} + O(\xi^3) \le e^{-\frac{\xi^2}{4}} \quad \text{provided} \quad |\xi| \le c_0$$

with a constant c_0 that depends only on the distribution of ω_0. Let $S_N = \sum_{n=1}^{N} \omega_n a_n$ and $\sigma_N^2 = \sum_{n=1}^{N} |a_n|^2$. Then

$$\Big| \mathbb{E} e^{i\xi S_N} \Big| = \prod_{k=1}^{N} |f_0(\xi a_k)| \le \exp\Big(-\frac{1}{4}\xi^2 \sigma_N^2 \Big) \tag{2.79}$$

provided $|\xi| \le c_0$. Hence,

$$\begin{aligned} \mathbb{P}\Big[|S_N| \le \beta\sigma_N\Big] &\lesssim \int \varphi\Big(\frac{S_N}{\beta\sigma_N}\Big)\, d\mathbb{P} \\ &\lesssim \int \beta\sigma_N \hat{\varphi}(\beta\sigma_N \xi) \prod_{k=1}^{N} |f_0(\xi a_k)|\, d\xi \\ &\lesssim \beta \int \sigma_N \exp\Big(-\frac{1}{4}\xi^2\sigma_N^2 \Big)\, d\xi &(2.80) \\ &\lesssim \beta, \end{aligned}$$

as claimed. To pass to (2.80), observe that the integrand on the line above vanishes if $|\xi| \gg (\beta\sigma_N)^{-1}$. Therefore, on can apply (2.79) if $(\beta\sigma_N)^{-1} \le c_0$. However, this is our assumption. $\square$

If $\omega_0 = \pm 1$ with equal probability, then one has

$$\mathbb{P}\Big[\sum_{n=1}^{N} \omega_n = 0\Big] \asymp \frac{1}{\sqrt{N}}.$$

This shows that the lower bound on β in the lemma is optimal.

3 The two-dimensional case

3.1 Frequency concentration in two dimensions

The purpose of this section is to develop an analogous approach for the two-dimensional Anderson model given by the random operators on $\ell^2(\mathbb{Z}^2)$

$$Hu = \triangle_{\mathbb{Z}^2} u + \lambda \omega u. \tag{3.1}$$

We start by establishing some simple technical facts. The following lemma is the analogue of Lemma 2.4. As the proof is almost identical with the one-dimensional case, we do not write it out.

Lemma 3.1. *There exists a positive radial Schwartz function b on the plane such that*

$$\sum_{n \in \mathbb{Z}^2} b(x - n) = 1 \qquad \forall x \in \mathbb{R}^2 \tag{3.2}$$

and so that $\operatorname{supp}(\hat{b}) \subset [-1, 1]^2$.

Fix such a function b for the rest of this section. In this section, we let $\varphi^{(M)}(x) := \min\big(1, |x|^{-M}\big)$. It will suffice to take $M = 10$ for our purposes, and we set $\varphi := \varphi^{(10)}$ for the rest of this section. For any radial function f in the plane and rectangle ρ, the function f_ρ is defined to be $f \circ A$, where A is an affine map that takes ρ onto the unit square. Since f is radial, f_ρ is well-defined. For any rectangle ρ, a dual rectangle ρ^* to ρ is defined to have the same pair of axes as ρ but reciprocal side lengths. The following lemma is from [14].

Lemma 3.2. *Assume that M is sufficiently large, and let $\varphi = \varphi^{(M)}$. Let G be a function on $\mathbb{R}^2$. If $\widehat{G}$ is supported in a rectangle ρ and if ρ^* is any dual rectangle, then*

$$\|\varphi_{\rho^*}^{-1} G\|_\infty^2 \lesssim |\rho|\, \|\varphi_{\rho^*}^{-2} G\|_2^2.$$

Taking the Fourier transform of the two-dimensional discrete Laplacian leads to the multiplier

$$m_\triangle(\theta_1, \theta_2) := 2\cos(2\pi\theta_1) + 2\cos(2\pi\theta_2).$$

The following lemma describes the level curves of this function.

Lemma 3.3. *Define*

$$\gamma(E) := \Big\{\theta \in \mathbb{T}^2 \;\Big|\; 2\cos(2\pi\theta_1) + 2\cos(2\pi\theta_2) = E\Big\}.$$

Then for any small $\tau > 0$ there exists a constant C_τ such that for all $E \in (-4+\tau, -\tau) \cup (\tau, 4-\tau)$ the curve $\gamma(E)$ is a closed curve satisfying

$$C_\tau^{-1} < \inf_{\theta\in\gamma(E)} \kappa(\theta) \le \sup_{\theta\in\gamma(E)} \kappa(\theta) < C_\tau$$

where $\kappa(\theta)$ is the curvature at the point θ. Moreover, if $I \subset (-4+\tau, -\tau) \cup (\tau, 4-\tau)$, then

$$\mathrm{mes}[\theta \in \mathbb{T}^2 \mid m_\Delta(\theta) \in I] \le C_\tau\, |I|. \tag{3.3}$$

Proof. The gradient

$$\nabla m_\Delta = 4\pi(-\sin(2\pi\theta_1), -\sin(2\pi\theta_2))$$

only vanishes at the points $(0,0), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}), (\frac{1}{2}, 0)$. Clearly, these points lie on the curves $\gamma(0)$, $\gamma(4)$, and $\gamma(-4)$. Hence

$$\inf_{\theta\in\gamma(E)} |\nabla m_\Delta(\theta)| \ge C_\tau > 0$$

for all $E \in (-4+\tau, -\tau) \cup (\tau, 4-\tau)$. This implies (3.3). By calculus,

$$\kappa = \frac{\langle D^2 m_\Delta\,(\nabla m_\Delta)^\perp, (\nabla m_\Delta)^\perp\rangle}{|\nabla m_\Delta|^3}.$$

One checks that

$$\begin{aligned} \frac{1}{(2\pi)^4}\langle D^2 m_\Delta\,(\nabla m_\Delta)^\perp, (\nabla m_\Delta)^\perp\rangle(\theta) &= 8[\cos(2\pi\theta_1)\sin^2(2\pi\theta_2) + \cos(2\pi\theta_2)\sin^2(2\pi\theta_1)] \\ &= 4m(\theta)(1 - \cos(2\pi\theta_1)\cos(2\pi\theta_2)). \end{aligned} \tag{3.4}$$

Clearly, (3.4) is nonzero provided $m \neq 0$ and

$$(\theta_1, \theta_2) \neq (0,0), \qquad (\theta_1, \theta_2) \neq (\frac{1}{2}, \frac{1}{2}),$$

and the lemma follows. ☐

From now on it will be understood that some arbitrary but fixed small τ has been chosen. Furthermore, γ will denote an arbitrary curve of the form $\gamma(E)$ as in the lemma. The ε-neighborhood of γ will be denoted by γ^ε. By $2\gamma^\varepsilon$ we mean the double of γ^ε, i.e., the annulus with twice the thickness of γ^ε and the same center curve. As in Section 2, we associate with a given $E \in (-4+\tau, -\tau) \cup (\tau, 4-\tau)$ a dyadic family of intervals $\{I_j\}$ on the energy axis, which we refer to as $\mathcal{I}(E)$, see Figure 2.1. Here I_0 is centered at E, but otherwise $\mathrm{dist}(I_j, E) \asymp |I_j|$ for $j \neq 0$. For given $\eta > 0$ and $\lambda > 0$ the lengths satisfy $|I_j| \asymp 2^j\, \lambda^{2-4\eta}$ for all j. The following lemma shows that such a dyadic family gives rise to a partition of unity on $\mathbb{T}^2$ similar to Lemma 2.3.

Lemma 3.4. *Let $\mathcal{I}(E) \subset (-4+\tau,-\tau)\cup(\tau,4-\tau)$ be some dyadic family of intervals $\{I_j\}_{j=0}^{M}$ centered at E. There exist functions $\{\widehat{g_j}\}_{j=0}^{M+1} \subset C^\infty(\mathbb{T}^2)$ with the property that*

$$\sum_{j=0}^{M+1} \widehat{g_j}^{\,2} = 1 \quad on \quad \mathbb{T}^2, \tag{3.5}$$

and such that for $0 \le j \le M$

$$\mathrm{supp}(\widehat{g_j}) \subset 2[\theta \in \mathbb{T}^2 \mid m_\Delta(\theta) \in I_j]. \tag{3.6}$$

Finally, for all $0 \le j \le M+1$,

$$|g_j(n)| \le C_m\, |I_j|\, [1 + |I_j||n|]^{-m} \tag{3.7}$$

for every $m \ge 1$ with constants C_m depending only on m and τ.

Proof. For simplicity and w.l.o.g., we assume that $E \in (\tau, 4-\tau)$. Identify the interval $[-4,4]$ with $\mathbb{T}$ and let $I_{M+1} := \mathbb{T} \setminus \bigcup_{j=0}^{M} I_j$. Clearly, $|I_{M+1}| \asymp 1$. The family of intervals $\{I_j\}_{j=0}^{M+1}$ satisfies the hypotheses of Lemma 2.3 with a constant $c > 1$ that depends on τ. Hence there exist functions $\{\psi_j\}_{j=0}^{M+1}$ with the properties (2.12) and (2.13). Define

$$\widehat{g_j}(\theta) := \widehat{\psi_j}(m_\Delta(\theta)) \;\; \text{for} \;\; 0 \le j \le M+1.$$

By construction,

$$|\nabla m_\Delta| > c_\tau > 0 \;\; \text{on} \;\; \bigcup_{j=0}^{M+1} \mathrm{supp}(\widehat{\psi_j}').$$

Hence, $\widehat{g_j} \subset C^\infty(\mathbb{T}^2)$ for all $0 \le j \le M+1$, and (3.5) and (3.6) hold. To prove (3.6), fix some j and denote the support of $\widehat{g_j}$ by γ_ε where $\varepsilon \asymp |I_j|$. Now divide $\mathbb{T}$ into $\lceil \varepsilon^{-\frac{1}{2}} \rceil$ many congruent intervals $\{J_\ell\}_\ell$ of size about $\sqrt{\varepsilon}$. Let $\widehat{\beta_\ell}$ be a smooth partition of unity as provided by Lemma 2.3 for these intervals (i.e., here $\widehat{\beta_\ell} = \widehat{\psi_\ell}^{\,2}$). Therefore,

$$\widehat{g_j}(\theta) = \sum_\ell \widehat{g_j}(\theta)\widehat{\beta_\ell}(\theta/|\theta|) =: \sum_\ell \widehat{h_\ell}. \tag{3.8}$$

Here we are identifying $\mathbb{T}^2$ with $\mathbb{R}^2$, which is justified since all $\gamma(E)$ are symmetric with respect to 0 if $E > 0$. Observe that the functions $\widehat{h_\ell}$ on the right-hand side of (3.8) are supported in $\varepsilon \times \sqrt{\varepsilon}$-rectangles which we denote by ρ_ℓ. It is a well-known fact that

$$|h_\ell(x)| \le C_M\, \varepsilon^{\frac{3}{2}}\, \varphi^{(M)}_{\rho_\ell^*}(x) \qquad \forall\, x \in \mathbb{R}^2,$$

where ρ_ℓ^* is the dual rectangle to ρ_ℓ that is centered at the origin, see [4] and [2]. Thus

$$|h_\ell(n)| \le \sum_\ell C_M\, \varepsilon^{\frac{3}{2}}\, \varphi^{(M)}_{\rho_\ell^*}(n) \le C_M\, \varepsilon\, [1 + \varepsilon\, |n|]^{-M} \tag{3.9}$$

for all $M \ge 1$, as claimed. $\square$

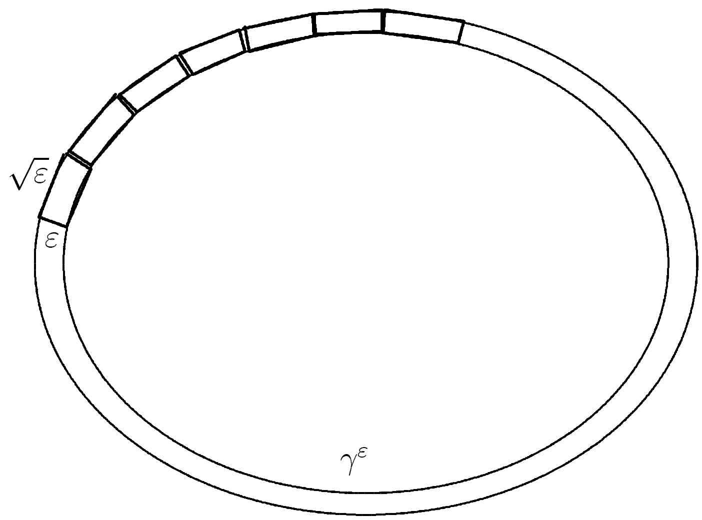

Figure 3.1: The decomposition into $\varepsilon \times \sqrt{\varepsilon}$-rectangles

Given a dyadic family $\mathcal{I}(E)$ as in Lemma 3.4, fix some ℓ and set $\varepsilon = |I_\ell|$. Let P_ε be defined as

$$\widehat{P_\varepsilon u} := \widehat{g_\ell}\, \hat{u},$$

where g_ℓ is given by Lemma 3.4. Throughout this section it will be assumed that P_ε arises in this way. For any ε and γ let

$$X_{\varepsilon,\gamma} := \left\{ u \in \ell^2(\mathbb{Z}^2) \;\middle|\; \operatorname{supp}(\hat{u}) \subset \gamma^\varepsilon \right\}.$$

As already apparent in the proof of Lemma 3.4, angular decompositions are crucial in this section. In this context it is more natural to consider the projective angle rather than the usual one. More precisely, for any nonzero vectors $\xi_1, \xi_2 \in \mathbb{R}^2$ let $\angle(\xi_1, \xi_2) \in [0, \pi]$ be the usual angle, measured counter clockwise. The projective angle $\sphericalangle(\xi_1, \xi_2)$ is defined as

$$\sphericalangle(\xi_1, \xi_2) := \min[\angle(\xi_1, \xi_2), \angle(\xi_1, -\xi_2)].$$

Observe that this angle is always in $[0, \pi/2]$. The (projective) angle between two sets $S_1, S_2 \subset \mathbb{R}^2 \setminus \{0\}$ is defined as

$$\sphericalangle(S_1, S_2) := \inf\{\sphericalangle(\xi_1, \xi_2) \mid \xi_1 \in S_1,\ \xi_2 \in S_2\}.$$

Lemma 3.5. *Let $u \in X_{\varepsilon,\gamma_1}$ and $v \in X_{\varepsilon,\gamma_2}$ be such that $\sphericalangle[\operatorname{supp}(\hat{u}), \operatorname{supp}(\hat{v})] = \alpha$. Then*

$$\|uv\|_2 \lesssim \frac{\varepsilon}{\sqrt{\alpha + \sqrt{\varepsilon}}} \|u\|_2 \|v\|_2. \tag{3.10}$$

Proof. For technical reasons, we first reduce ourselves to the case where

$$\operatorname{dist}(\operatorname{supp}(\hat{u}) + \operatorname{supp}(\hat{v}), 0) \gtrsim 1. \tag{3.11}$$

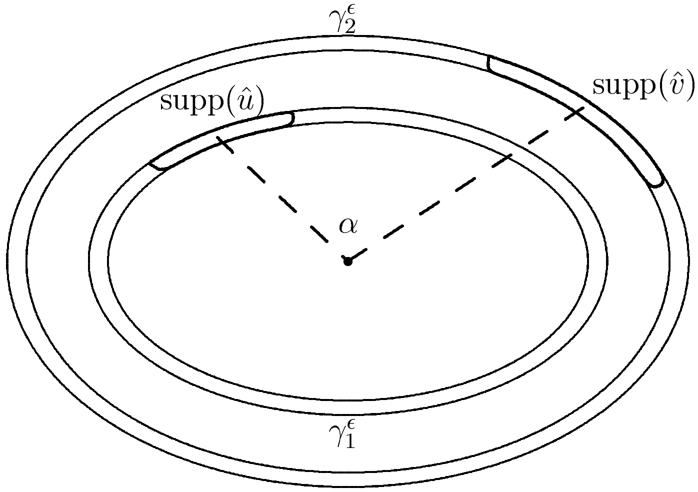

Figure 3.2: The supports of $\hat{u}$ and $\hat{v}$ with angular separation α

Firstly, decompose γ_1^ε into ten pieces $\{\Gamma_j\}_{j=1}^{10}$ each of which has angular width $\frac{2\pi}{10}$. Fix one of them, say Γ_1. Split γ_2^ε into four pieces $\{\widetilde{\Gamma}_{1k}\}_{k=1}^4$ by drawing the line that joins the midpoint of Γ_0 with the origin and the line perpendicular to it through the origin. Suppose that $\widetilde{\Gamma}_{11}$ lies opposite Γ_1. It is clear that (3.11) holds for $\chi_{\Gamma_1}\hat{u}$ and $\chi_{\widetilde{\Gamma}_{1k}}\hat{v}$ if $k = 2, 3, 4$, whereas it fails for $k = 1$. In the latter case let

$$\widehat{w}(\theta) := \overline{\chi_{\widetilde{\Gamma}_{11}}(-\theta)\hat{v}(-\theta)},$$

which is the same as

$$\overline{w_n} = [\chi_{\widetilde{\Gamma}_{11}}\hat{v}]_n^\vee.$$

On the one hand, $\chi_{\Gamma_1}\hat{u}$ and $\hat{w}$ satisfy (3.11). On the other hand, switching to w does not change the ℓ^2 norms in (3.10). Consequently, if we can prove (3.10) under the additional assumption (3.11), then summing up over the decomposition we just described leads to the result in the general case. Thus suppose that (3.11) holds. Decompose $u = \sum_\rho u_\rho$ so that $\widehat{u_\rho}$ is supported in an $\varepsilon \times \sqrt{\varepsilon}$-rectangle ρ. Similarly, $v = \sum_\sigma v_\sigma$, see Figure 3.2. Let $u_\rho^{\rho^*} = b_{\rho^*}u_\rho$ and $v_\sigma^{\sigma^*} = b_{\sigma^*}v_\sigma$. In view of (3.2),

$$u = \sum_\rho \sum_{\rho^*} u_\rho^{\rho^*}, \qquad v = \sum_\sigma \sum_{\sigma^*} v_\sigma^{\sigma^*}. \tag{3.12}$$

By construction, every $\widehat{u_\rho^{\rho^*}}$ is supported in a dilate $C\rho$ for some absolute constant C. For simplicity, we do not distinguish between ρ and $C\rho$. It is a well-known observation by C. Fefferman, see [4] and [2] that

$$\sum_{\rho,\sigma} \chi_{\rho+\sigma} \lesssim 1 \tag{3.13}$$

under the assumption (3.11). Indeed, the equivalent inequality

$$\sum_{\rho,\sigma} \chi_{\frac{1}{2}(\rho+\sigma)} \lesssim 1$$

basically states that the midpoint of a secant inside a circle determines its endpoints. This is clearly true provided the secant is not a diameter. The latter case is excluded, however, because of (3.11). We refer the reader to the aforementioned works for more details. Since

$$\operatorname{supp}(\widehat{u_\rho} * \widehat{v_\sigma}) \subset \rho + \sigma,$$

one conclude by means of Plancherel that

$$\begin{aligned}\Big\| \sum_\rho u_\rho \sum_\sigma v_\sigma \Big\|_2^2 &= \int_{\mathbb{R}^2} \Big| \sum_{\rho,\sigma} [\widehat{u_\rho} * \widehat{v_\sigma}](\xi)\, \chi_{\rho+\sigma}(\xi) \Big|^2 d\xi \\ &\le \int_{\mathbb{R}^2} \sum_{\rho,\sigma} \Big| \widehat{u_\rho} * \widehat{v_\sigma}(\xi) \Big|^2 \sum_{\rho,\sigma} \chi_{\rho+\sigma}(\xi)\, d\xi \qquad (3.14) \\ &\lesssim \sum_{\rho,\sigma} \|u_\rho v_\sigma\|_2^2. \qquad (3.15)\end{aligned}$$

To pass to line (3.14) one uses Cauchy-Schwarz, whereas (3.15) follows from (3.13).
Suppose $\rho \cap \operatorname{supp}(\hat{u}) \neq \emptyset$ and $\sigma \cap \operatorname{supp}(\hat{v}) \neq \emptyset$. Assume first that $\alpha \gg \sqrt{\varepsilon}$. Then the longer sides of ρ^* and σ^* make an angle of size about α. This implies that

$$|\rho^* \cap \sigma^*| \lesssim \varepsilon^{-1}\alpha^{-1} \text{ and also } \int_{\mathbb{R}^2} \varphi_{\rho^*}\varphi_{\sigma^*}\, dx \lesssim \varepsilon^{-1}\alpha^{-1}. \qquad (3.16)$$

On the other hand, if $\alpha \lesssim \sqrt{\varepsilon}$, then it is clear that

$$|\rho^* \cap \sigma^*| \lesssim \varepsilon^{-\frac{3}{2}} \text{ and also } \int_{\mathbb{R}^2} \varphi_{\rho^*}\varphi_{\sigma^*}\, dx \lesssim \varepsilon^{-\frac{3}{2}}.$$

Therefore,

$$\begin{aligned}\|u_\rho v_\sigma\|_2^2 &= \int_{\mathbb{R}^2} \Big| \sum_{\rho^*} u_\rho^{\rho^*} \varphi_{\rho^*}^{-1}\varphi_{\rho^*} \sum_{\sigma^*} v_\sigma^{\sigma^*} \varphi_{\sigma^*}^{-1}\varphi_{\sigma^*} \Big|^2 dx \\ &\le \int_{\mathbb{R}^2} \sum_{\rho^*,\sigma^*} |u_\rho^{\rho^*}\varphi_{\rho^*}^{-1}|^2\, |v_\sigma^{\sigma^*}\varphi_{\sigma^*}^{-1}|^2\, \varphi_{\rho^*}\varphi_{\sigma^*} \sum_{\rho^*,\sigma^*} \varphi_{\rho^*}\varphi_{\sigma^*}\, dx \qquad (3.17) \\ &\lesssim \sum_{\rho^*,\sigma^*} \|u_\rho^{\rho^*}\varphi_{\rho^*}^{-1}\|_\infty^2 \|v_\sigma^{\sigma^*}\varphi_{\sigma^*}^{-1}\|_\infty^2 \int_{\mathbb{R}^2} \varphi_{\rho^*}\varphi_{\sigma^*}\, dx \qquad (3.18) \\ &\lesssim \sum_{\rho^*,\sigma^*} \|u_\rho^{\rho^*}\varphi_{\rho^*}^{-1}\|_\infty^2 \|v_\sigma^{\sigma^*}\varphi_{\sigma^*}^{-1}\|_\infty^2 \frac{\varepsilon^{-1}}{\alpha + \sqrt{\varepsilon}} \qquad (3.19)\end{aligned}$$

where the last line invokes (3.16). To pass to line (3.18) one needs to remove the second sum in (3.17). This is possible since

$$\sum_{\rho^*} \varphi_{\rho^*} \lesssim 1,$$

which is the same as

$$\sup_{x\in\mathbb{R}^2} \sum_{n\in\mathbb{Z}^2} \varphi(x-n) \lesssim 1.$$

Hence

$$\begin{aligned}
\|uv\|^2_{\ell^2(\mathbb{Z}^2)} &= \Big\| \sum_{\rho,\sigma} u_\rho\, v_\sigma \Big\|_2^2 & (3.20)\\
&\lesssim \sum_{\rho,\sigma} \|u_\rho\, v_\sigma\|_2^2 & (3.21)\\
&= \sum_{\rho,\sigma} \Big\| \sum_{\rho^*} u_\rho^{\rho^*} \sum_{\sigma^*} v_\sigma^{\sigma^*} \Big\|_2^2 \\
&\lesssim \sum_{\rho,\sigma} \sum_{\rho^*,\sigma^*} \Big\| \varphi_{\rho^*}^{-1} u_\rho^{\rho^*} \Big\|_\infty^2 \Big\| \varphi_{\sigma^*}^{-1} v_\sigma^{\sigma^*} \Big\|_\infty^2 \frac{\varepsilon^{-1}}{\alpha+\sqrt{\varepsilon}} & (3.22)\\
&\lesssim \sum_{\rho,\sigma} \sum_{\rho^*,\sigma^*} \Big\| \varphi_{\rho^*}^{-2} u_\rho^{\rho^*} \Big\|_2^2 |\rho| \Big\| \varphi_{\sigma^*}^{-2} v_\sigma^{\sigma^*} \Big\|_2^2 |\sigma| \frac{\varepsilon^{-1}}{\alpha+\sqrt{\varepsilon}} & (3.23)\\
&\lesssim \frac{\varepsilon^2}{\alpha+\sqrt{\varepsilon}} \sum_\rho \|u_\rho\|_2^2 \sum_\sigma \|v_\sigma\|_2^2 = \frac{\varepsilon^2}{\alpha+\sqrt{\varepsilon}} \|u\|_2^2\, \|v\|_2^2. & (3.24)
\end{aligned}$$

The first equality (3.20) is given by (3.12). To pass to line (3.21), one uses (3.15). (3.22) is obtained from (3.19), whereas (3.23) follows from Lemma 3.2. Finally, to pass to line (3.24) one uses that

$$\sum_{\rho^*} \varphi_{\rho^*}^{-4} b_{\rho^*}^2 \lesssim 1.$$

By definition this is the same as

$$\sup_{x\in\mathbb{R}^2} \sum_{n\in\mathbb{Z}^2} \varphi^{-4}(x-n) b^2(x-n) \lesssim 1,$$

which holds since b is a Schwartz function. □

The importance of Lemma 3.5 lies with the following probabilistic estimate.

Lemma 3.6. *Let $\{\omega_n\}_{-\infty}^{\infty}$ be a sequence of independent mean-zero random variables with $|\omega_n| \le 1$. Suppose $u \in X_{\varepsilon,\gamma_1}$ and $v \in X_{\varepsilon,\gamma_2}$ are such that $\sphericalangle[\mathrm{supp}(\hat{u}), \mathrm{supp}(\hat{v})] = \alpha$. Then for all $s > 0$*

$$\mathbb{P}\Big[\big|\sum_n \omega_n u_n \bar{v}_n\big| > Cs\|u\|_2\, \|v\|_2 \Big] \lesssim \exp\Big(-\frac{s^2}{\varepsilon^2}\,(\alpha+\sqrt{\varepsilon}) \Big)$$

with some absolute constant $C > 0$.

Proof. By the usual subgaussian estimate

$$\mathbb{P}\Big[\big|\sum_n \omega_n u_n \bar{v}_n\big| \gg t\|uv\|_2 \Big] \lesssim e^{-t^2}$$

for all $t > 0$. In view of Lemma 3.5,

$$\|uv\|_2 \lesssim \frac{\varepsilon}{\sqrt{\alpha+\sqrt{\varepsilon}}} \|u\|_2 \|v\|_2,$$

and the lemma follows with $t = \frac{s}{\varepsilon}\sqrt{\alpha+\sqrt{\varepsilon}}$. □

As in the one-dimensional case, we shall use Lemma 3.6 in order to control the norms $\|P_\delta \omega_Q P_\varepsilon\|$ where $Q \subset \mathbb{Z}^2$ is a square of side length δ^{-1} (assuming w.l.o.g. that $\varepsilon \geq \delta$). This again requires an entropy bound that limits the number of functions one needs to test the operator $P_\delta \omega_Q P_\varepsilon$ on. Moreover, it turns out to be crucial to split the square Q into smaller squares Q' of side length ε^{-1}. This requires simple almost orthogonality arguments that are presented in Lemma 3.7, whereas the entropy estimate is given by Lemma 3.8. Those results are then used in Lemma 3.9, which is a basic result of this section.

Lemma 3.7. *Let $0 < \varepsilon < 1$ and $Q_1, Q_2 \subset \mathbb{Z}^2$ be squares. Then there exist constants C_m so that*

$$\|\chi_{Q_1} P_\varepsilon^2 \chi_{Q_2}\| \leq C_m\, \varepsilon \sqrt{|Q_1||Q_2|}\, [1 + \varepsilon \operatorname{dist}(Q_1, Q_2)]^{-m} \tag{3.25}$$

for all $m \geq 1$. In particular, let $\mathcal{Q}_{\varepsilon,L}$ be a partition of $\mathbb{Z}^2$ into squares of side length $L\varepsilon^{-1}$ where $L \geq 1$. Then

$$\|P_\varepsilon u\|_2^2 \lesssim \sum_{Q \in \mathcal{Q}_{\varepsilon,L}} \|P_\varepsilon \chi_Q u\|_2^2 + C_m\, \varepsilon^{-1} L^{-m} \|u\|_2^2 \tag{3.26}$$

for all $u \in \ell^2(\mathbb{Z}^2)$.

Proof. The kernel of $\chi_{Q_1} P_\varepsilon^2 \chi_{Q_2}$ is given by

$$K(x,y) = \chi_{Q_1}(x) K_\varepsilon(x-y) \chi_{Q_2}(y)$$

where $\widehat{K_\varepsilon} = \hat{g}^2$ is a smooth cut-off function on the annulus γ^ε. In particular, there is the estimate

$$|K_\varepsilon(x)| \leq C_m\, \varepsilon\, [1 + \varepsilon |x|]^{-m}$$

for all $m \geq 1$, see Lemma 3.4. The norm in (3.25) is now estimated by means of Schur's lemma. To obtain (3.26), one argues as usual:

$$\begin{aligned}
\|P_\varepsilon u\|_2^2 &= \sum_{Q_1, Q_2 \in \mathcal{Q}_{\varepsilon,L}} \langle P_\varepsilon \chi_{Q_1} u, P_\varepsilon \chi_{Q_2} u\rangle \\
&\leq \sum_{\substack{Q_1, Q_2 \in \mathcal{Q}_{\varepsilon,L} \\ \operatorname{dist}(Q_1,Q_2) \leq \operatorname{diam}(Q_1)}} \langle P_\varepsilon \chi_{Q_1} u, P_\varepsilon \chi_{Q_2} u\rangle + \sum_{\substack{Q_1, Q_2 \in \mathcal{Q}_{\varepsilon,L} \\ \operatorname{dist}(Q_1,Q_2) > \operatorname{diam}(Q_1)}} \|\chi_{Q_2} P_\varepsilon^2 \chi_{Q_1}\|\, \|\chi_{Q_1} u\|_2\, \|\chi_{Q_2} u\|_2 \\
&\lesssim \sum_{Q \in \mathcal{Q}_{\varepsilon,L}} \|P_\varepsilon \chi_Q u\|_2^2 + C_m\, L^2 \varepsilon^{-1} L^{-m} \|u\|_2^2. \qquad (3.27)
\end{aligned}$$

To pass to line (3.27), one uses Cauchy-Schwarz to derive the first term, whereas the second term is obtained by means of (3.25) and Schur's lemma. □

The following lemma is the analogue of Lemma 2.6 from the one-dimensional case. It will be important to control the length of the support of $\hat{u}$ for any $u \in X_{\varepsilon,\gamma}$. To this end we define

$$X_{\varepsilon,\gamma}^\alpha := \Big\{ u \in X_{\varepsilon,\gamma} \;\Big|\; \operatorname{supp}(\hat{u}) \text{ is contained in an arc of length } \alpha \Big\}.$$

Let $\mathcal{B}_{\varepsilon,\gamma}^\alpha$ denote the unit ball in $X_{\varepsilon,\gamma}^\alpha$.

Lemma 3.8. *Let $0 < \varepsilon < 1$ and $Q \subset \mathbb{Z}^2$ be a square of side length $L\varepsilon^{-1}$ centered at the origin. For every $0 < \kappa < 1$ there exist functions $\{u_j\}_{j=1}^M$ in $\mathcal{B}^\alpha_{\varepsilon,\gamma}$ with $M \le \exp\Big(C\kappa^{-3}L^2\alpha/\varepsilon\Big)$ such that for any $u \in \mathcal{B}^\alpha_{\varepsilon,\gamma}$ one has*

$$\|b_Q u - b_Q u_j\|_2 \le \kappa$$

for some j.

Proof. Partition $\mathbb{T}^2$ into small squares of side length about $C_1^{-1}\kappa\,\mathrm{diam}(Q)^{-1}$. Denote the resulting partition by $\mathcal{P}$ and the conditional expectation of any $f \in L^2(\mathbb{T}^2)$ by

$$\mathbb{E}[f|\mathcal{P}] =: g_f.$$

Thus, if $u \in X^\alpha_{\varepsilon,\gamma}$, then $g_{\hat{u}}$ is constant on each square $J \in \mathcal{P}$. Moreover,

$$g_{\hat{u}} \restriction J \neq 0$$

for at most

$$\lesssim \frac{\alpha\varepsilon}{|J|} \le \frac{C_1^2 L^2\,\alpha}{\kappa^2\varepsilon} =: d \tag{3.28}$$

many of the squares $J \in \mathcal{P}$. This implies that the dimension of the Hilbert space $\Big\{\widehat{b_Q} * g_{\hat{u}} \mid u \in X^\alpha_{\varepsilon,\gamma}\Big\}$ is at most Cd. Hence there are functions $\{u_j\}_{j=1}^M$ with $M := \kappa^{-Cd}$ such that

$$\min_j \big\|\widehat{b_Q} * g_{\hat{u}} - \widehat{b_Q} * g_{\widehat{u_j}}\big\|_2 \le \kappa/3 \tag{3.29}$$

for all $u \in \mathcal{B}^\alpha_{\varepsilon,\gamma}$. To finish the proof it suffices to show that for large C_1

$$\|\widehat{b_Q} * \hat{u} - \widehat{b_Q} * g_{\hat{u}}\|_2 \le \kappa/3$$

for any $u \in \mathcal{B}^\alpha_{\varepsilon,\gamma}$. This can be done by means of Schur's lemma as in the one-dimensional case. As the details are almost identical we skip them. ☐

Lemma 3.9. *Fix some small $\eta > 0$. Let $Q \subset \mathbb{Z}^2$ with $\mathrm{diam}(Q) = M \ge \varepsilon^{-1}$ and let $\frac{1}{2} \ge \varepsilon \ge \delta > 0$. Then*

$$\mathbb{P}\Big[\|P_\delta \omega_Q P_\varepsilon\| = \|P_\varepsilon \omega_Q P_\delta\| \ge C_\eta\, A\sqrt{\varepsilon}\Big] \lesssim M^2 \varepsilon^{\frac{3}{2}} \exp\Big(-\frac{A^2}{\sqrt{\varepsilon}|\log\varepsilon|^2}\Big)$$

for all $A \ge \varepsilon^{-\eta}$ where C_η only depends on η and τ.

Proof. Let $Q = \bigcup Q'$ be a partition into squares Q' of size $\varepsilon^{-1-\eta/2}$. By Lemma 3.7 with $L = \varepsilon^{-\eta/2}$,

$$\|P_\varepsilon \omega_Q P_\delta u\|_2^2 \lesssim \sum_{Q' \subset Q} \|P_\varepsilon \omega_{Q'} P_\delta u\|_2^2 + \varepsilon^{10}\|u\|_2^2. \tag{3.30}$$

Moreover,

$$\|u\|_2^2 \ge \sum_{Q' \subset Q} \|b_{Q'} u\|_2^2. \tag{3.31}$$

Let $B \gg \varepsilon^5$ be such that $\|P_\varepsilon \omega_Q P_\delta u\|_2 \geq B\|u\|_2$ for some u. Then (3.30) and (3.31) imply that

$$\|P_\varepsilon \omega_{Q'} P_\delta u\|_2 \gtrsim B\|b_{Q'} u\|_2 \ \text{ for some } \ Q' \subset Q.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\Big[\exists\, u \;\Big|\; \|P_\varepsilon \omega_Q P_\delta u\|_2 \gg B\|u\|_2\Big] \\
\leq\ & \sum_{Q'\subset Q} \mathbb{P}\Big[\exists\, u, v \;\Big|\; |\langle \omega_{Q'} P_\delta u, P_\varepsilon v\rangle| \geq B\|b_{Q'} u\|_2 \|v\|_2\Big] \\
\leq\ & \sum_{Q'\subset Q} \mathbb{P}\Big[\exists\, u, v \;\Big|\; |\langle b_{Q'}^{-2} \omega_{Q'}\, b_{Q'} P_\delta u, b_{Q'} P_\varepsilon v\rangle| \geq B\|b_{Q'} u\|_2 \|b_{Q'} v\|_2\Big] && (3.32)\\
\lesssim\ & M^2\varepsilon^2\, \mathbb{P}\Big[\exists\, u \in b_{Q_0'} \cdot X_{\varepsilon,\gamma_1},\ v \in b_{Q_0'} \cdot X_{\varepsilon,\gamma_2} \;\Big|\; |\langle b_{Q_0'}^{-2} \omega_{Q_0'} u, v\rangle| \geq B\|u\|_2\, \|v\|_2\Big] && (3.33)
\end{aligned}$$

where Q_0' is the square centered at the origin. To pass to line (3.32) note that $b_{Q'}$ is uniformly bounded below on Q' by a positive constant. Moreover, $\|v\|_2 \geq \|b_{Q'} v\|_2$ since $0 \leq b \leq 1$. This allows one to replace $\|v\|_2$ with $\|b_{Q'} v\|_2$. To obtain (3.33) one uses that the probabilities in (3.32) do not depend on the position of Q'. The appearance of $u \in b_{Q_0'} \cdot X_{\varepsilon,\gamma_1}$ instead of $u \in b_{Q_0'} \cdot X_{\delta,\gamma_1}$ (which can only increase (3.33) since $\varepsilon \geq \delta$) is due to the fact that

$$\operatorname{diam}(\operatorname{supp}(\widehat{b_{Q_0'}})) \asymp \operatorname{diam}(Q_0')^{-1} \asymp \varepsilon^{1+\eta/2}. \tag{3.34}$$

This implies that the two aforementioned classes of functions are basically the same. More importantly, in view of (3.34),

$$u \in b_{Q_0'} \cdot X_{\varepsilon,\gamma_1} \Longrightarrow \operatorname{supp}(\hat{u}) \subset \gamma_1^{2\varepsilon} \tag{3.35}$$

for small ε. The next step is to estimate the probability in (3.33) by means of Lemma 3.6 with the random variables

$$\widetilde{\omega} = b_{Q_0'}^{-2} \omega_{Q_0'}. \tag{3.36}$$

This requires angular separation of the supports of $\hat{u}$ and $\hat{v}$, which can be obtained as follows. As above, there is a decomposition $\gamma_1^\varepsilon = \bigcup_j R_j$ into rectangles of size $\varepsilon \times \sqrt{\varepsilon}$ and $\gamma_2^\varepsilon = \bigcup_j \widetilde{R}_j$ so that R_j and $\widetilde{R}_j$ belong to the same angular sector when viewed from the origin. We may assume that in both cases the number of rectangles appearing in the decomposition is equal to the same power of two. We now describe a partition of the set of all pairs $(R_j, \widetilde{R}_k)$ into sets

$$\mathcal{A}_\ell \subset \Big\{ (R_j, \widetilde{R}_k) \ \mid\ 2^\ell\sqrt{\varepsilon} - O(\sqrt{\varepsilon}) \lesssim \sphericalangle(R_j, \widetilde{R}_k) \lesssim 2^\ell\sqrt{\varepsilon} \Big\} \tag{3.37}$$

for $\ell = 0, 1, 2, \ldots$. In case $\ell = 0$ we let

$$\mathcal{A}_0 := \bigcup_j \Big\{ (\pm R_i, \pm\widetilde{R}_k) \ \mid\ i \in \{2j, 2j+1\},\ k \in \{2j-1, 2j, 2j+1, 2j+2\} \Big\}.$$

By $-R_j$ we mean the rectangle opposite to R_j. Figure 3.3 shows this situation for the case of one j. The idea is simply to group those rectangles together that make an angle about $\sqrt{\varepsilon}$ or less. To define the next set of pairs $\mathcal{A}_1$, one divides the set of all rectangles $\{R_j \subset \gamma_1^\varepsilon\}$ into groups of four

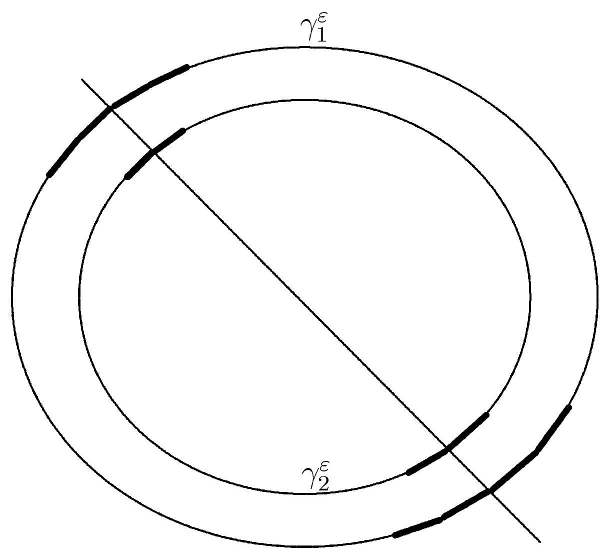

Figure 3.3: The first set of pairs $\mathcal{A}_0$

adjacent ones and also combines antipodal rectangles. One such group of eight rectangles (four on each side) is shown in Figure 3.4. They are the ones that lie on γ_1^ε close to the solid line through the origin. Fix such a group and let R_j be one of these rectangles. We then let $(R_j, \widetilde{R}_k) \in \mathcal{A}_1$ provided $\widetilde{R}_k \subset \gamma_2^\varepsilon$ belongs to one of the four sets of eight rectangles each, which are shown in Figure 3.4. Those four sets are chosen in such a way that they are adjacent to those $\widetilde{R}_i \subset \gamma_2^\varepsilon$ with the property that $(R_j, \widetilde{R}_i) \in \mathcal{A}_0$ for the R_j under consideration. Observe that by this construction

$$\sphericalangle(R_j, \widetilde{R}_k) \asymp \sqrt{\varepsilon}$$

for any pair $(R_j, \widetilde{R}_k) \in \mathcal{A}_1$.
Proceeding inductively, one defines the sets of pairs $\mathcal{A}_2$ as follows: start with 16 rectangles in γ_1^ε consisting of two groups of eight antipodal ones. Associate them with the four groups of 16 rectangles each that lie on γ_2^ε and which are adjacent to those from the previous steps $\mathcal{A}_0$ and $\mathcal{A}_1$. It is clear that this process gives rise to a partition as in (3.37). Furthermore, returning to $\langle \widetilde{\omega} u, v\rangle$, our constructions allows one to write

$$\langle \widetilde{\omega}\, u, v\rangle = \sum_{\ell} \sum_{(\rho,\sigma)\in\mathcal{A}_\ell} \langle \widetilde{\omega}\, u_\rho, v_\sigma\rangle = \sum_{\alpha}\sum_{k} \langle \widetilde{\omega}\, u_{\Gamma_k^\alpha}, v_{\widetilde{\Gamma}_k^\alpha}\rangle \tag{3.38}$$

where Γ_k^α are pairs of sectors in γ_1^ε of size about $\alpha = 2^\ell\sqrt{\varepsilon}$ and $\widetilde{\Gamma}_k^\alpha$ are unions of four sectors on γ_2^ε of similar sizes. Here we have set

$$u_{\Gamma_k^\alpha} = \sum_{\rho\in\Gamma_k^\alpha} u_\rho$$

and similarly for v. Moreover,

$$\alpha - O(\sqrt{\varepsilon}) \lesssim \sphericalangle(\Gamma_k^\alpha, \widetilde{\Gamma}_k^\alpha) \lesssim \alpha$$

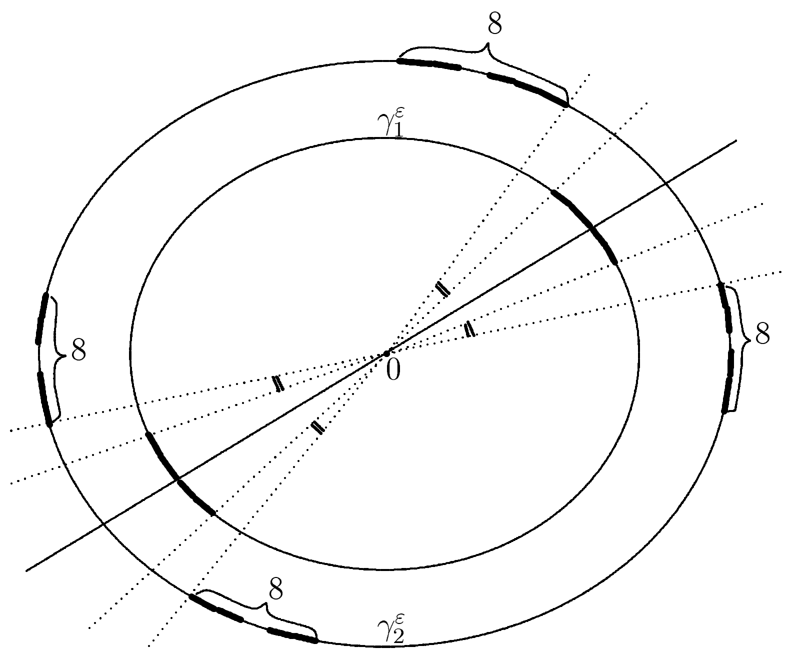

Figure 3.4: The second set of pairs $\mathcal{A}_1$

for all α and k, see Figure 3.4 where one configuration $\Gamma_k^\alpha, \widetilde{\Gamma}_k^\alpha$ for $\ell = 1$ is shown. Finally, observe that for all α

$$\|u\|_2^2 \asymp \sum_k \|u_{\Gamma_k^\alpha}\|_2^2 \quad \text{and} \quad \|v\|_2^2 \asymp \sum_k \|v_{\widetilde{\Gamma}_k^\alpha}\|_2^2.$$

Suppose that for all angles $\alpha = 2^\ell\sqrt{\varepsilon}$ with $0 \le \ell \lesssim |\log\varepsilon|$ one has

$$\big|\langle \tilde\omega\, u_{\Gamma_k^\alpha}, v_{\widetilde{\Gamma}_k^\alpha}\rangle\big| \le C_1\, \|u_{\Gamma_k^\alpha}\|_2\, \|v_{\widetilde{\Gamma}_k^\alpha}\|_2 \qquad \forall\, k, \tag{3.39}$$

where C_1 is some constant that does not depend on α, k and $\tilde\omega$ is a fixed sequence. Setting $u_{\alpha,k} = u_{\Gamma_k^\alpha}$ and $v_{\alpha,k} = v_{\widetilde{\Gamma}_k^\alpha}$ for simplicity, (3.38) therefore implies that

$$\begin{aligned}
\big|\langle \tilde\omega\, u, v\rangle\big| &\le \sum_{\substack{0\le\ell\lesssim|\log\varepsilon| \\ \alpha=2^\ell\sqrt{\varepsilon}}} \Big|\sum_k \langle \tilde\omega\, u_{\alpha,k}, v_{\alpha,k}\rangle\Big| \\
&\le \sum_{\substack{0\le\ell\lesssim|\log\varepsilon| \\ \alpha=2^\ell\sqrt{\varepsilon}}} \sum_k C_1\, \|u_{\alpha,k}\|_2 \|v_{\alpha,k}\|_2 \\
&\le \sum_{\substack{0\le\ell\lesssim|\log\varepsilon| \\ \alpha=2^\ell\sqrt{\varepsilon}}} C_1 \Big(\sum_k \|u_{\alpha,k}\|_2^2\Big)^{\frac{1}{2}} \Big(\sum_k \|v_{\alpha,k}\|_2^2\Big)^{\frac{1}{2}} \\
&\lesssim C_1\, |\log\varepsilon|\, \|u\|_2 \|v\|_2.
\end{aligned} \tag{3.40}$$

This means that if the reverse inequality holds in (3.40), then (3.39) has to be violated for some choice of α and k. Define

$$X^{\alpha,k}_{\varepsilon,\gamma_1} = \Big\{ u \in X_{\varepsilon,\gamma_1} \,\Big|\, \mathrm{supp}(\hat{u}) \subset \Gamma^{\alpha}_{k},\ \|u\|_2 \le 1 \Big\} \quad\text{and}\quad X^{\alpha,k}_{\varepsilon,\gamma_2} = \Big\{ v \in X_{\varepsilon,\gamma_2} \,\Big|\, \mathrm{supp}(\hat{v}) \subset \widetilde{\Gamma}^{\alpha}_{k},\ \|v\|_2 \le 1 \Big\}.$$

The argument between (3.39) and (3.40) allows one to introduce angular separation into (3.33). Indeed,

$$\begin{aligned} &\mathbb{P}\Big[\exists\, u \in b_{Q_0'} \cdot X_{\varepsilon,\gamma_1},\ v \in b_{Q_0'} \cdot X_{\varepsilon,\gamma_2} \,\Big|\, |\langle \tilde{\omega} u, v\rangle| \ge B\|u\|_2\,\|v\|_2\Big] \\ &\le \sum_{\substack{0\le \ell \lesssim |\log\varepsilon| \\ \alpha = 2^{\ell}\sqrt{\varepsilon}}} \sum_{k} \mathbb{P}\Big[\exists\, u \in b_{Q_0'} \cdot X^{\alpha,k}_{\varepsilon,\gamma_1},\ v \in b_{Q_0'} \cdot X^{\alpha,k}_{\varepsilon,\gamma_2} \,\Big|\, |\langle \tilde{\omega} u, v\rangle| \gtrsim \frac{B}{|\log\varepsilon|}\|u\|_2\,\|v\|_2\Big]. \end{aligned} \tag{3.42}$$

Now fix a pair α, k. By Lemma 3.8 there exists a $\frac{1}{10}$-net $\mathcal{N}_1$ of $b_{Q_0'} \cdot X^{\alpha,k}_{\varepsilon,\gamma_1}$ with

$$\#\mathcal{N}_1 \lesssim \exp\Big(\alpha\,\varepsilon^{-1-\eta}\Big), \tag{3.43}$$

and let $\mathcal{N}_2$ be the analogous net for γ_2. Set

$$C_1 = \frac{B}{|\log\varepsilon|}. \tag{3.44}$$

Then the probabilities in (3.42) can be estimated as follows:

$$\begin{aligned} &\mathbb{P}\Big[\exists\, u \in b_{Q_0'} \cdot X^{\alpha,k}_{\varepsilon,\gamma_1},\ v \in b_{Q_0'} \cdot X^{\alpha,k}_{\varepsilon,\gamma_2} \,\Big|\, |\langle \tilde{\omega}\, u, v\rangle| \ge C_1\,\|u\|_2\,\|v\|_2\Big] \\ \le\ &\mathbb{P}\Big[\exists\, u \in b_{Q_0'} \cdot X^{\alpha,k}_{\varepsilon,\gamma_1},\ v \in b_{Q_0'} \cdot X^{\alpha,k}_{\varepsilon,\gamma_2} \,\Big|\, |\langle \tilde{\omega}\, u, v\rangle| \ge 2C_1\,\|u\|_2\,\|v\|_2\Big] \\ &\quad + \mathbb{P}\Big[\exists\, u \in \mathcal{N}_1,\ v \in \mathcal{N}_2 \,\Big|\, |\langle \tilde{\omega}\, u, v\rangle| \ge \frac{1}{2}\, C_1\,\|u\|_2\,\|v\|_2\Big] \qquad (3.45) \\ \le\ &\sum_{j\ge 0} \mathbb{P}\Big[\exists\, u \in \mathcal{N}_1,\ v \in \mathcal{N}_2 \,\Big|\, |\langle \tilde{\omega}\, u, v\rangle| \ge 2^{j-1}\, C_1\,\|u\|_2\,\|v\|_2\Big] \qquad (3.46) \\ \lesssim\ &\sum_{j\ge 0} \exp\big(\alpha\varepsilon^{-1-\eta}\big)\, \exp\Big(-c\, 2^{2j} C_1^2\, \frac{\alpha}{\varepsilon^2}\Big) \lesssim \exp(\alpha\varepsilon^{-1-\eta})\, \exp\Big(-c\, C_1^2\, \frac{\alpha}{\varepsilon^2}\Big). \qquad (3.47) \end{aligned}$$

Inequality (3.45) is obtained by means of a 3ε-argument just as in the one-dimensional case, see (2.40). Line (3.46) follows by induction and to pass to (3.47) one uses Lemma 3.6 and (3.43).
The lemma now follows by combining (3.33), (3.36), (3.42), and (3.47). In fact, let $B = A\varepsilon^{\frac{1}{2}-\frac{\eta}{2}}|\log\varepsilon|$. With C_1 as in (3.44),

$$\frac{C_1^2}{\varepsilon^2}\alpha = \frac{A^2}{\varepsilon}\varepsilon^{-\eta}\alpha \gg \varepsilon^{-\eta}\frac{\alpha}{\varepsilon}$$

for large A. This implies that the entropy term in (3.47) is dominated by the probabilistic bound. One concludes therefore that

$$\mathbb{P}\Big[\exists\, u \,\Big|\, \|P_\varepsilon \omega_Q P_\delta u\|_2 \gg B\|u\|_2\Big] \lesssim M^2 \varepsilon^{\frac{3}{2}} \exp\Big(-A^2\varepsilon^{-\eta}/\sqrt{\varepsilon}\Big)$$

and after a suitable rescaling of A,

$$\mathbb{P}\Big[\,\|P_\varepsilon \omega_Q P_\delta\| \ge C_\eta\, A\sqrt{\varepsilon}\Big] \lesssim M^2 \varepsilon^{\frac{3}{2}} \exp\Big(-\frac{A^2}{\sqrt{\varepsilon}|\log\varepsilon|^2}\Big)$$

provided $A \ge \varepsilon^{-\frac{\eta}{2}}$, as claimed. □

Fix some small $\eta > 0$ and let $\lambda > 0$ be the disorder in (3.1). For any small $\delta > 0$ let $\mathcal{Q}_\delta$ denote a partition of $\mathbb{Z}^2$ into disjoint, congruent dyadic squares of side length $\asymp \lambda^{-\eta}\delta^{-1}$. Recall that we have fixed $\tau > 0$ and are considering about λ^{-2} many dyadic partitions of $\mathbb{T}^2$ by means of annuli with thickness $2^j\,\lambda^{2-4\eta}$ as given by Lemma 3.4. It will be understood from now on that P_ε denotes a projection onto one such annulus belonging to this fixed family of annuli.

Definition 3.10. *Let $\varepsilon \ge \delta$ be given. We say that $Q \in \mathcal{Q}_\delta$ is (δ,ε)-good, if*

$$\|P_\varepsilon \omega_Q P_\delta\| \le \lambda^{-\eta}\sqrt{\varepsilon}$$

holds (this of course depends on the randomness ω).

As in the one-dimensional case, Lemma 3.9 allows one to conclude that with high probability the bad squares are very sparse. This can easily be expressed in terms of densities as in Lemma 3.11 below. The proof is basically identical with the proof of Lemma 2.10.

Lemma 3.11. *With probability one*

$$\limsup_{N\to\infty} N^{-2}\#\{Q \subset [-N,N]^2 \mid Q \in \mathcal{Q}_\delta \;\; \text{is } (\delta,\varepsilon)\text{-bad}\} \lesssim e^{-\lambda^{-2\eta}}$$

for any pair $0 < \delta \le \varepsilon$.

Theorem 3.12 below is the main result of this section. It is the two-dimensional analogue of Theorem 2.1. In order to formulate it, we introduce the notion of frequency concentration. More precisely, we say that an eigenfunction u of the operator H in (3.1) (or a restricted version of it) with eigenvalue E satisfies property $\mathcal{FC}(\delta)$ if

$$\|P_\varepsilon u\|_2 \le \sqrt{\frac{\delta}{\varepsilon}}\;\|P_\delta u\|_2 \;\;\text{for all}\;\; \varepsilon. \tag{3.48}$$

Here P_δ denotes the projection onto the annulus centered at $\gamma(E)$ of thickness δ and P_ε are the projections onto other annuli of thickness ε from the family $\mathcal{I}(E)$. Clearly, (3.48) means that the Fourier transform of u is basically localized to an annulus of size δ. In the following theorem we show that a.s. all eigenfunctions on the square $[-N,N]^2$, up to a set of size $o(N^2)$, satisfy $\mathcal{FC}(\lambda^{2-4\eta})$ provided their energies lie in the usual range. As the proof is very similar to that of Theorem 2.1, we only provide a sketch. The missing details can be transfered verbatim from the proof of Theorem 2.1. We use the notation

$$\partial[-N,N]^2 = \{n \in \mathbb{Z}^2 \setminus [-N,N]^2 \mid |m-n| = 1 \;\;\text{for some}\;\; m \in [-N,N]^2\}.$$

Theorem 3.12. *Consider the two-dimensional random operator* (3.1) *where* $\lambda > 0$ *and* $\{\omega_n\}$ *are i.i.d. with* $\mathbb{E}\omega_n = 0$, $\mathbb{E}\omega_n^2 = 1$, *and bounded. For any positive integer* N *let* $\{u_j^{(N)}\}$ *be an orthonormal basis in* $\ell^2([-N,N]^2)$ *of eigenfunctions of* H *restricted to* $[-N,N]^2$, *i.e.,*

$$(H - E_j^{(N)})u_j^{(N)} = 0 \quad \text{on} \quad [-N,N]^2 \quad \text{and} \quad u^{(N)} = 0 \text{ on } \partial[-N,N]^2.$$

Fix any small $\tau > 0$ *and* $\eta > 0$. *Then for sufficiently small* λ *one has*

$$\limsup_{N\to\infty} \frac{1}{N^2} \#\Big\{ j \;\Big|\; u_j^{(N)} \ \text{ does not satisfy } \mathcal{FC}(\lambda^{2-4\eta}),\ E_j^{(N)} \in \mathcal{E}_\tau \Big\} \le \lambda^{-9} e^{-\lambda^{-2\eta}} \tag{3.49}$$

with probability one, where $\mathcal{E}_\tau = (-4+\tau, -\tau) \cup (\tau, 4-\tau)$.

Proof. Let $u = u_j^{(N)}$ be an eigenfunction with energy $E = E_j^{(N)} \in \mathcal{E}_\tau$. Denote the restriction to the square $\Lambda = [-N,N]^2$ by R_Λ. Taking Fourier transforms of

$$R_\Lambda (H - E) R_\Lambda u = 0$$

yields

$$0 = (m_\Delta(\theta) - E)\hat{u}(\theta) + \lambda \widehat{\omega u} - \sum_{n\in\partial\Lambda} u(n') e(n\theta)$$

where $n' \in \Lambda$, $|n - n'| = 1$ (we denote the set of those n' by $\partial^- \Lambda$). Let P_δ denote the projection onto some annulus of thickness δ from the dyadic family associated with E. If this annulus is not the one centered at $\gamma(E)$, the it has distance about δ from $\gamma(E)$. Hence, applying P_δ to the equation shows that

$$\begin{aligned} \delta \|P_\delta u\|_2 &\lesssim \lambda \|P_\delta(\omega u)\|_2 + \Big(\sum_{n\in\partial^-\Lambda} |u(n)|^2 \Big)^{\frac12} \\ &\lesssim \lambda \Big(\sum_{Q\in\mathcal{Q}_\delta} \|P_\delta \omega_Q u\|_2^2 \Big)^{\frac12} + \lambda^{10}\|u\|_2 + \|u \restriction_{\partial^-\Lambda}\|_2. \end{aligned} \tag{3.50}$$

The sum in (3.50) follows from 3.7. By Lemma 2.11,

$$\#\{ j \mid \|u_j^{(N)} \restriction_{\partial^-\Lambda}\|_2 \ge \lambda^{10} \|u_j^{(N)}\|_2 \} \le N\lambda^{-20}.$$

Hence, we may assume that the final term in (3.50) is no larger than $\lambda^{10}\|u\|_2$, without affecting (3.49). Furthermore, by Lemma 3.11 and Lemma 2.11, we may assume that all but

$$\lambda^{-9} e^{-\lambda^{-2\eta}} N$$

many eigenfunctions $\{u_j^{(N)}\}$ satisfy

$$\|\chi_{\bigcup_{Q \text{ bad}} Q}\, u_j^{(N)}\|_2^2 \lesssim \lambda^5 \|u_j^{(N)}\|_2^2. \tag{3.51}$$

Here Q is called <u>bad</u>, if for some choice of ε, δ it does not satisfy the condition in Definition 3.10. It therefore suffices to treat the sum over good squares in (3.50). To this end one splits u as follows:

$$\|P_\delta \omega_Q u\|_2^2 \lesssim |\log \lambda| \sum_{\varepsilon > \delta} \|P_\delta \omega_Q P_\varepsilon^2 u\|_2^2 + \|P_\delta \omega_Q \sum_{\varepsilon\le\delta} P_\varepsilon^2 u\|_2^2. \tag{3.52}$$

As in the previous section we use the notation $Q \sim Q'$ for squares $Q, Q' \in \mathcal{Q}_\delta$ satisfying

$$\mathrm{dist}(Q, Q') \leq \mathrm{diam}(Q).$$

By the two-dimensional analogue of Lemma 2.12, whose straightforward proof we leave to the reader, one obtains for $\varepsilon > \delta$ and with the sums running over squares $Q \in \mathcal{Q}_\delta$,

$$\begin{aligned}
\sum_{Q \text{ good}} \|P_\delta \omega_Q P_\varepsilon^2 u\|_2^2 &\lesssim \sum_{Q \text{ good}} \sum_{Q' \sim Q} \|P_\delta \omega_Q P_\varepsilon^2 b_{Q'} u\|_2^2 + \lambda^{10} \|u\|_2^2 \\
&\lesssim \sum_{Q \text{ good}} \sum_{Q' \sim Q} \|P_\delta \omega_Q P_\varepsilon\|^2 \|P_\varepsilon b_{Q'} u\|_2^2 + \lambda^{10} \|u\|_2^2 \\
&\lesssim \sum_{Q \text{ good}} \sum_{Q' \sim Q} \lambda^{-2\eta} \varepsilon \, \|P_\varepsilon b_{Q'} u\|_2^2 + \lambda^{10} \|u\|_2^2 \qquad (3.53) \\
&\lesssim \sum_{Q \text{ good}} \sum_{Q' \sim Q} \lambda^{-2\eta} \varepsilon \sum_{\varepsilon' \asymp \varepsilon} \|P_\varepsilon b_{Q'} P_{\varepsilon'}^2 u\|_2^2 + \lambda^{10} \|u\|_2^2 \\
&\lesssim \lambda^{-2\eta} \varepsilon \sum_{\varepsilon' \asymp \varepsilon} \|P_{\varepsilon'} u\|_2^2 + \lambda^{10} \|u\|_2^2. \qquad (3.54)
\end{aligned}$$

In addition to the two-dimensional analogue of Lemma 2.12 we used the definition of good squares to pass to line (3.53), as well as the fact that

$$P_\varepsilon b_{Q'} P_{\varepsilon'}^2 = 0$$

if $\varepsilon \not\asymp \varepsilon'$. On the other hand,

$$\begin{aligned}
\sum_{Q \text{ good}} \|P_\delta \omega_Q \sum_{\varepsilon \leq \delta} P_\varepsilon^2 u\|_2^2 &\lesssim \sum_{Q \text{ good}} \sum_{Q' \sim Q} \|P_\delta \omega_Q \sum_{\varepsilon \leq \delta} P_\varepsilon^2 b_{Q'} u\|_2^2 + \lambda^{10} \|u\|_2^2 \\
&\lesssim |\log \lambda| \sum_{\varepsilon \leq \delta} \sum_{Q \text{ good}} \sum_{Q' \sim Q} \|P_\delta \omega_Q P_\varepsilon\|^2 \|P_\varepsilon b_{Q'} u\|_2^2 + \lambda^{10} \|u\|_2^2 \\
&\lesssim |\log \lambda| \sum_{\varepsilon \leq \delta} \sum_{Q \text{ good}} \sum_{Q' \sim Q} \lambda^{-2\eta} \delta \, \|P_\varepsilon b_{Q'} u\|_2^2 + \lambda^{10} \|u\|_2^2 \\
&\lesssim |\log \lambda| \sum_{\varepsilon \leq \delta} \sum_{Q \text{ good}} \sum_{Q' \sim Q} \lambda^{-2\eta} \delta \sum_{\varepsilon' \asymp \varepsilon} \|P_\varepsilon b_{Q'} P_{\varepsilon'}^2 u\|_2^2 + \lambda^{10} \|u\|_2^2 \\
&\lesssim |\log \lambda| \sum_{\varepsilon \lesssim \delta} \lambda^{-2\eta} \delta \|P_\varepsilon u\|_2^2 + \lambda^{10} \|u\|_2^2. \qquad (3.55)
\end{aligned}$$

Summing (3.52) over good squares and using (3.54), (3.55) as well as (3.51), yields

$$\begin{aligned}
\delta \|P_\delta u\|_2 &\lesssim \lambda^{1-\eta} |\log \lambda|^{\frac{1}{2}} \sum_{\varepsilon} \sqrt{\varepsilon} \vee \sqrt{\delta} \|P_\varepsilon u\|_2 + \lambda^2 \|u\|_2 \qquad (3.56) \\
&\lesssim \lambda^{1-\eta} |\log \lambda|^{\frac{1}{2}} \sum_{\varepsilon} \sqrt{\varepsilon} \vee \sqrt{\delta} \|P_\varepsilon u\|_2. \qquad (3.57)
\end{aligned}$$

To pass to line (3.57), one uses that the first term in (3.56) dominates the second because $\varepsilon, \delta \geq \lambda^2$. The theorem now follows by the exact same maximization argument involving

$$\sqrt{\delta} \|P_\delta u\|_2$$

as in the proof of Theorem 2.1. $\square$

The following corollary combines Theorem 3.12 with an uncertainty-type argument to conclude that the localization length is about λ^{-2} in size. More precisely, we show that with probability one most eigenfunctions have the property that any square of size much smaller than $\lambda^{-2+\eta}$ can contain only a small fraction of its ℓ^2-mass.

Corollary 3.13. *Suppose u satisfies* (3.48)*. Then*

$$\sup_{x\in\mathbb{Z}^2} \|u\chi_{Q(x,R)}\|_{\ell^2(\mathbb{Z}^2)} \lesssim \sqrt{R\delta}|\log\delta|\,\|u\|_2$$

where $Q(x,R)$ is the square of side length R centered at x. In particular, with probability one, most eigenfunctions $\{u_j^{(N)}\}_j$ as described in Theorem 3.12 have the property that for $R=\lambda^{-2+\eta}$

$$\sup_{x\in\mathbb{Z}^2} \|u_j^{(N)}\chi_{Q(x,\rho R)}\|_{\ell^2(\mathbb{Z}^2)} \lesssim \sqrt{\rho}\,\|u_j^{(N)}\|_2.$$

"Most eigenfunctions" here means up to a set of density $o(1)$ as $\lambda\to 0$ (where the o depends on η).

Proof. Fix some square $Q=Q(x,R)$ and consider the operator

$$T_\varepsilon f(\theta) := \int \chi_{A_\varepsilon}(\theta')\widehat{b_Q}(\theta-\theta')f(\theta')\,d\theta'$$

on $L^2(\mathbb{T}^2)$. Here A_ε is an annulus of thickness ε and b is the bump function from Lemma 3.1. Since $\text{supp}(\widehat{b_Q})$ is a disk of radius about R^{-1}, a standard application of Schur's lemma shows that

$$\|T_\varepsilon\| \lesssim \sqrt{R\varepsilon}. \tag{3.58}$$

Indeed,

$$\sup_\theta \int \chi_{A_\varepsilon}(\theta')\widehat{b_Q}(\theta-\theta')\,d\theta' \lesssim R\varepsilon,$$

and

$$\sup_{\theta'} \int \chi_{A_\varepsilon}(\theta')\widehat{b_Q}(\theta-\theta')\,d\theta \lesssim 1$$

so that (3.58) follows from Lemma 2.5. Since $T_{C\varepsilon}(P_\varepsilon^2 u) = b_Q\,P_\varepsilon^2 u$ if C is large, the corollary follows from (3.58), (3.5), and (3.48). $\square$

3.2 Optimality in two dimensions

It is easy to see that eigenfunctions of (3.1) cannot satisfy (3.48) with $\delta \ll \lambda^2$. Thus Theorem 3.12 is optimal up to the factors $\lambda^{-\eta}$ (and perhaps also up to the removal of a small fraction of eigenfunctions). Indeed, assume that

$$\triangle_{\mathbb{Z}^2}u + \lambda\omega u = Eu.$$

For simplicity, we now let ω be Bernoulli, i.e., $\omega_0 = \pm 1$. Then

$$\begin{aligned} \lambda^2 \|u\|_2^2 &= \|(\triangle - E)u\|_2^2 = \sum_\varepsilon \|\chi_{\mathcal{A}_\varepsilon}(m_\triangle - E)\hat{u}\|_2^2 \\ &\lesssim \sum_\varepsilon \varepsilon^2 \|\chi_{\mathcal{A}_\varepsilon}\hat{u}\|_2^2 \end{aligned} \tag{3.59}$$

where $\mathcal{A}_\varepsilon$ are the annuli dyadic annuli around the curve $m_\triangle = E$ with $\varepsilon = 2^j\delta$. As above, $\mathcal{A}_\delta$ is the annulus centered around $m_\triangle = E$. We want to show that $\delta \gtrsim \lambda^2$ if (3.48) holds. Thus, suppose

$$\left\|\chi_{\mathcal{A}_\varepsilon}\hat{u}\right\|_2 \le \sqrt{\frac{\delta}{\varepsilon}}\, \left\|\chi_{\mathcal{A}_\delta}\hat{u}\right\|_2$$

for all ε. Inserting this estimate into (3.59) yields

$$\lambda^2\|u\|_2^2 \lesssim \sum_\varepsilon \varepsilon\delta \left\|\chi_{\mathcal{A}_\delta}\hat{u}\right\|_2^2 \lesssim \delta \left\|\chi_{\mathcal{A}_\delta}\hat{u}\right\|_2^2.$$

This requires $\delta \gtrsim \lambda^2$, as claimed. This argument shows that concentration to thinner annuli can only hold under a weaker requirement of "concentration" than (3.48). For example, λ^4 would require replacing $\sqrt{\delta/\varepsilon}$ in (3.48) with $(\delta/\varepsilon)^{\frac{1}{4}}$. It is important to notice, however, that this weaker inequality would no longer allow one to apply the uncertainty principle in the sense of Corollary 3.13. Indeed, one needs to balance the $\sqrt{R\varepsilon}$ in (3.58) against the rate of decay as the annuli grow thicker. In order to do this one needs at least $\sqrt{\delta/\varepsilon}$-decay. Hence any improvement of Corollary 3.13 in terms of larger negative powers of λ would have to proceed along different lines.
Since λ^2 cannot be improved, inspection of the proof of Theorem 3.12 therefore shows that the bound on $\|P_\delta\omega_Q P_\varepsilon\|$ provided by Lemma 3.9 is optimal. In the next lemma we show this directly, without any recourse to the proof.

Lemma 3.14. *Let $\varepsilon \ge \delta > 0$ and $Q \subset \mathbb{Z}^2$ be a square of side length at least δ^{-1}. Then*

$$\mathbb{E}\|P_\delta\omega_Q P_\varepsilon\|_2^2 \gtrsim \varepsilon.$$

Proof. We may assume that $Q = Q(0, \delta^{-1}/2)$. Take a smooth function u with $\|u\|_2 = 1$ and $\|P_\delta u\|_2 \gtrsim 1$ and such that $P_\delta u$ is supported in the square $Q(0, \delta^{-1}/2)$. In particular, notice that $\omega_Q P_\delta u = \omega P_\delta u$. Pick a $\sqrt{\varepsilon}$-net $\{\xi_\sigma\}_\sigma$ in γ_2^ε. Here σ are $\sqrt{\varepsilon}\times\varepsilon$-rectangles in γ_2^ε and the ξ_σ can be taken to be their centers. Consider all functions v of the form

$$v(n) = \sum_\sigma \sum_{\sigma^*} h^\sigma_{\sigma^*} b_{\sigma^*}(n) e(\xi_\sigma \cdot n) \tag{3.60}$$

where the inner sum runs over all rectangles σ^* that are dual to $\varepsilon \times \sqrt{\varepsilon}$-rectangles $\sigma \subset \gamma_2^\varepsilon$. It follows from the usual orthogonality considerations that

$$\|v\|_2^2 \lesssim \sum_{\sigma,\sigma^*} |h^\sigma_{\sigma^*}|^2 \varepsilon^{-\frac{3}{2}}.$$

Indeed, observe that the Fourier transform of the inner sum in (3.60) is supported in σ, so that these sums are orthogonal for distinct choices of σ. On the other hand, for σ fixed, one uses (3.2). Hence (with v as in (3.60))

$$\begin{aligned} \|P_\varepsilon \omega_Q P_\delta\|^2 &\gtrsim \sup_{\|v\|_2=1} |\langle \omega_Q P_\delta u, P_\varepsilon v\rangle|^2 = \sup_{\|v\|_2=1} \Big| \sum_{\sigma,\sigma^*} h^\sigma_{\sigma^*} \, \langle e(\xi_\sigma \cdot)\, \omega_Q P_\delta u, b_{\sigma^*}\rangle \Big|^2 \\ &\gtrsim \varepsilon^{\frac{3}{2}} \sum_{\sigma,\sigma^*} |\sum_n \omega_n e(\xi_\sigma \cdot n)\, (P_\delta u)(n) b_{\sigma^*}(n)|^2. \end{aligned} \tag{3.61}$$

Taking expectations in (3.61) shows that

$$\mathbb{E}\|P_\varepsilon \omega_Q P_\delta\|^2 \gtrsim \varepsilon^{\frac{3}{2}} \sum_{\sigma,\sigma^*} \sum_n |(P_\delta u)(n) b_{\sigma^*}(n)|^2 \asymp \sum_\sigma \varepsilon^{\frac{3}{2}} \sum_n |(P_\delta u)(n)|^2 \gtrsim \varepsilon \|u\|_2^2,$$

as claimed. The final inequality follows from the fact that there are about $\varepsilon^{-\frac{1}{2}}$ many rectangles σ. $\square$

References

[1] Buslaev, V. I., Vituškin, A. G. *An estimate of the length of a signal code with a finite spectrum in connection with sound transcription problems.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 867–895.

[2] Cordoba, A. *The Kakeya maximal function and the spherical summation multipliers.* Amer. J. Math. 99 (1977), 1-22.

[3] Erdös, L., Yau, H.-T. *Linear Boltzman equation as the weak coupling limit of a random Schrödinger equation.* Comm. Pure Appl. Math. 53 (2000), no. 6, 667–735.

[4] Fefferman, C. *The multiplier problem for the ball.* Ann. of Math. (2) 94 (1971), 330-336.

[5] Figotin, A., Pastur, L. *Spectra of random and almost-periodic operators.* Grundlehren der mathematischen Wissenschaften 297, Springer-Verlag, Berlin, 1992.

[6] Klein, A., Martinelli, F., Perez, J. F. *A rigorous replica trick approach to Anderson localization in one dimension.* Comm. Math. Phys. 106 (1986), no. 4, 623–633.

[7] Klopp, F., Wolff, T. *Lifshits tails for 2-dimensional random Schrödinger operators.* Preprint, 2000.

[8] Magnen, J., Poirot, G., Rivasseau, V. *Renormalization group methods and applications: first results for the weakly coupled Anderson model.* STATPHYS 20 (Paris, 1998). Phys. A 263 (1999), no. 1-4, 131–140.

[9] Magnen, J., Poirot, G., Rivasseau, V. *Ward-type identities for the two-dimensional Anderson model at weak disorder.* J. Statist. Phys. 93 (1998), no. 1-2, 331–358.

[10] Pisier, G. *The volume of convex bodies and Banach space geometry.* Cambridge Tracts in Mathematics, 94. Cambridge University Press, Cambridge, 1989.

[11] Poirot, G. *Mean Green's function of the Anderson model at weak disorder with an infra-red cut-off.* Ann. Inst. H. Poincaré Phys. Théor. 70 (1999), no. 1, 101–146.

[12] Shubin, C., Vakilian, R., Wolff, T. *Some harmonic analysis questions suggested by Anderson-Bernoulli models.* Geom. Funct. Anal. 8 (1998), no. 5, 932–964.

[13] Spohn, H. *Derivation of the transport equation for electrons moving through random impurities.* J. Statist. Phys. 17 (1977), no. 6, 385–412.

[14] Wolff, T. *Decay of Circular Means of Fourier Transforms of Measures.* IMRN, 1999, no. 10, 547–567.

Schlag: Department of Mathematics, Princeton University, Fine Hall, Princeton N.J. 08544, U.S.A.
email: **schlag@math.princeton.edu**

Shubin: Department of Mathematics, Caifornia State University at Northridge, 18111 Nordhoff Street, Northridge, C.A. 91330, U.S.A.
email: **carol.shubin@math.csun.edu**