SMOOTHNESS OF PROJECTIONS, BERNOULLI CONVOLUTIONS, AND THE DIMENSION OF EXCEPTIONS

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Abstract. Erdős (1939, 1940) studied the distribution ν_λ of the random series $\sum_0^\infty \pm\lambda^n$, and showed that ν_λ is singular for infinitely many $\lambda \in (1/2, 1)$, and absolutely continuous for a.e. λ in a small interval $(1-\delta, 1)$. Solomyak (1995) proved a conjecture made by Garsia (1962) that ν_λ is absolutely continuous for a.e. $\lambda \in (1/2, 1)$. In order to sharpen this result, we have developed a general method that can be used to estimate the Hausdorff dimension of exceptional parameters in several contexts. In particular, we prove:

- For any $\lambda_0 > 1/2$, the set of $\lambda \in [\lambda_0, 1)$ such that ν_λ is singular has Hausdorff dimension strictly less than 1.
- For any Borel set $A \subset \mathbb{R}^d$ with Hausdorff dimension $\dim A > (d+1)/2$, there are points $x \in A$ such that the pinned distance set $\{|x-y| \,:\, y \in A\}$ has positive Lebesgue measure. Moreover, the set of x where this fails has Hausdorff dimension at most $d+1-\dim A$.
- Let $\mathcal{K}_\lambda$ denote the middle-α Cantor set for $\alpha = 1-2\lambda$ and let $K \subset \mathbb{R}$ be any compact set. Peres and Solomyak (1998) showed that for a.e. $\lambda \in (\lambda_0, 1/2)$ such that $\dim K + \dim \mathcal{K}_\lambda > 1$, the sum $K + \mathcal{K}_\lambda$ has positive length; we show that the set of exceptional λ in this statement has Hausdorff dimension at most $2 - \dim K - \dim \mathcal{K}_{\lambda_0}$.
- For any Borel set $E \subset \mathbb{R}^d$ with $\dim E > 2$, almost all orthogonal projections of E onto lines through the origin have nonempty interior, and the exceptional set of lines where this fails has dimension at most $d+1-\dim E$.
- If μ is a Borel probability measure on $\mathbb{R}^d$ with correlation dimension greater than $m + 2\gamma$, then for a "prevalent" set of C^1 maps $f : \mathbb{R}^d \to \mathbb{R}^m$ (in the sense described by Hunt, Sauer and Yorke (1992)), the image of μ under f has a density with at least γ fractional derivatives in $L^2(\mathbb{R}^m)$.

1. Introduction

The distribution ν_λ of the random series $\sum_0^\infty \pm\lambda^n$, where the signs are chosen independently with probability $\frac{1}{2}$, has been studied by many authors since the two seminal papers by Erdős in 1939 and 1940. It is immediate that ν_λ is singular for $\lambda < \frac{1}{2}$. In [6], Erdős showed that ν_λ is singular for infinitely many $\lambda \in (\frac{1}{2}, 1)$, namely those λ such that λ^{-1} is a Pisot number, and in [7] he proved that ν_λ is absolutely continuous for a.e. $\lambda \in (1-\delta, 1)$ where $\delta < 0.01$. It had been observed before by Jessen and Wintner [21] that ν_λ is either absolutely continuous or singular with respect to Lebesgue measure for any choice of λ. Works of Alexander and Yorke [2], Przytycki and Urbański [37], and Ledrappier [27] showed the importance of these distributions in several problems in dynamical systems. Solomyak [39] proved a conjecture made by Garsia in 1962, that ν_λ is absolutely continuous for a.e. $\lambda \in (1/2, 1)$; in fact, he showed that ν_λ has density in L^2 for a.e. $\lambda \in (1/2, 1)$. Peres and Solomyak [34] then gave a simpler proof of this. A survey of the results obtained on Bernoulli convolutions from the 1930's to 1999 is presented in [33].

In the present paper, we show that the density of ν_λ has a fractional derivative in L^2 for a.e. $\lambda \in (1/2, 1)$, and obtain a bound on the integrated Sobolev norm. We then use this bound to establish that for any closed interval $I \subset (1/2, 1]$, the set of $\lambda \in I$ such that ν_λ is singular, has Hausdorff dimension less than 1 (Theorem 5.4 and Corollary 5.6). In order to prove these results, we were led to develop a general method, which can be applied to improve previous results on Hausdorff dimensions of sums of Cantor sets, distance sets,

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and self-similar sets with deleted digits. (An exposition of our method for the case of Bernoulli convolutions is in [33]). Heuristically, the measures ν_λ may be viewed as nonlinear projections of the uniform measure on sequence space. Indeed, the analysis of ν_λ in [39] and [34] employed techniques developed by Kaufman [23] and Mattilla [30] to study orthogonal projections of sets and measures in Euclidean space.

The following (informally stated) principles were first discovered by Marstrand [28], Kaufman [23, 24] and Mattila [29, 30] in the study of orthogonal projections, and then extended to a variety of other settings by Falconer [12], Hu and Taylor [16], Pollicott and Simon [36], Solomyak [39, 40, 41], Peres and Solomyak [34, 35] and Sauer and Yorke [38].

- If an $m-\epsilon$ dimensional measure is "projected" to m dimensional space, then "typically" its dimension is preserved.
- If an $m+\epsilon$ dimensional measure is "projected" to m dimensional space, then "typically" the projected measure is absolutely continuous, with a density in L^2.

In making these principles precise, the appropriate notion of dimension of measures must be used (Correlation or information dimension, but not Minkowski or packing dimension; see Jarvenpää [20] and [38]), and the parametrized family of generalized projections considered must be sufficiently rich (at least m dimensional). A more delicate requirement is a certain "transversality condition" (e.g., Lemma 3.11 in [31]). For Bernoulli convolutions and other self-similar measures, this condition involves bounds on the double zeros of certain power series, and it was first made explicit by Pollicott and Simon [36]. Transversality is crucial in the works of Solomyak [39, 40, 41], Peres and Solomyak [34, 35], and in the present paper.

The general principles that underly our work can be informally stated as follows:

- If an $m-\epsilon$ dimensional measure μ is "projected" to an m dimensional space, then the set of parameters where the projection of μ has lower dimension than μ itself, has "codimension" at least ϵ.
- If an $m+\epsilon$ dimensional measure is "projected" to m dimensional space, then "typically" the projected measure has a density with a fractional derivative of order $\epsilon/2$ in L^2, and the set of parameters where the projected measure is singular has "codimension" at least ϵ.

The notion of *Sobolev dimension* of a measure, defined in (2.3) below, yields a unification of these two principles. For orthogonal projections, these strengthened principles have been stated precisely, and established as theorems, by Kaufman [24], Falconer [9] and Mattila [29], except that they did not consider fractional derivatives. Their proofs are based on averaging with respect to an appropriate Frostman measure on parameter space.

In the setting of Bernoulli convolutions, this averaging approach is not powerful enough to establish the last principle, so we were forced to develop a different technique.

Our methods apply to several concrete problems, which we now describe. Throughout the paper, dim (without subscripts) will mean **Hausdorff dimension**.

- **Orthogonal projections.** To motivate the general formulations, we start by clarifying the important results of Falconer [9] on exceptional directions for projections in $\mathbb{R}^d$ (see Sections 2 and 6). As a corollary, we find that for any Borel set $E \subset \mathbb{R}^d$ with $\dim E > 2$ and for a.e. direction θ in the sphere S^{d-1}, the orthogonal projection $\mathrm{proj}_\theta(E)$ of E to the line through 0 and θ, has nonempty interior. More precisely (see corollary 6.2),
$$\dim\{\theta \in S^{d-1} \ :\ \mathrm{proj}_\theta(E) \ \text{ has empty interior}\} \le d+1-\dim E.$$
- **Symmetric Bernoulli Convolutions.** Let ν_λ be the distribution of the random series $\sum_{n=0}^\infty \pm\lambda^n$, so ν_λ has the Fourier transform $\widehat{\nu_\lambda}(\xi) = \prod_{n=0}^\infty \cos(\lambda^n \xi)$. Here we sharpen the result of Solomyak [39] that ν_λ is absolutely continuous for a.e. $\lambda \in (1/2, 1)$, by showing that the $2,\gamma$–Sobolev norm satisfies
$$\|\nu_\lambda\|_{2,\gamma}^2 = \int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\xi)|^2 |\xi|^{2\gamma}\, d\xi < \infty$$

for a.e. $\lambda < 0.649$ such that $\lambda^{1+2\gamma} > 1/2$. Moreover, for some constant $C > 0$, and all small $\epsilon > 0$, we have

$$\dim\left\{\lambda \in (1/2+\epsilon, 1) \,:\, \nu_\lambda \text{ is singular}\right\} \quad < \quad 1 - C\epsilon\,.$$

- **Asymmetric Bernoulli Convolutions.** Let ν_λ^p be the distribution of the sum $\sum_{n=0}^{\infty} \pm\lambda^n$ where the signs are chosen randomly and independently with probabilities $(p, 1-p)$. Peres and Solomyak [35] showed that ν_λ^p is absolutely continuous for a.e. $\lambda \in (p^p(1-p)^{1-p}, 1)$, but ν_λ^p has a density in L^2 for a.e. $\lambda \in (p^2+(1-p)^2, 1)$, and not for any smaller λ. The Pisot numbers still provide infinitely many examples of singular ν_λ^p. In the present paper, we show that

$$\int_{-\infty}^{\infty} |\widehat{\nu_\lambda^p}(\xi)|^2 |\xi|^{2\gamma}\, d\xi < \infty$$

for a.e. $\lambda < 0.649$ such that $\lambda^{1+2\gamma} > p^2 + (1-p)^2$. Moreover, if $p \in (1/3, 2/3)$, then for some constant $C > 0$, and all $\epsilon > 0$,

$$\dim\left\{\lambda \in (p^2+(1-p)^2+\epsilon, 1) \,:\, \widehat{\nu_\lambda^p} \notin L^2\right\} < 1 - C\epsilon\,,$$

and

$$\dim\left\{\lambda \in (p^p(1-p)^{1-p}+\epsilon, 1) \,:\, \nu_\lambda^p \text{ is singular}\right\} < 1 - C\epsilon\,.$$

- **Intersections of sets with spheres** Suppose $E \subset \mathbb{R}^d$ is a Borel set with $d \geq 2$. Let $S_0 \subset \mathbb{R}^d$ be a strictly convex, closed C^∞–hypersurface surrounding the origin. Then

$$\dim\left\{x \in \mathbb{R}^d \,:\, (x + rS_0) \cap E = \emptyset \text{ for a.e. } r > 0\right\} \leq 1 + d - \dim E.$$

This statement fails if S_0 is the boundary of a cube or more generally, if S_0 contains a piece of a hyperplane. If S_0 is the sphere S^{d-1}, then the estimate above reduces to a strengthened form of Falconer's result in [10] about distance sets: For any Borel set $A \subset \mathbb{R}^d$ with Hausdorff dimension $\dim A > (d+1)/2$, there are points $x \in A$ such that the **pinned distance set** $\{|x-y| \,:\, y \in A\}$ has positive Lebesgue measure. Moreover, the set of $x \in \mathbb{R}^d$ where this fails has Hausdorff dimension at most $d + 1 - \dim A$ (see Corollary 8.4). Variants involving the Hausdorff dimension of the radii, as well as estimates where x is restricted to a hyperplane, can be found in section 8.2. We also obtain results if S_0 is assumed to be only in $C^{2,\delta}$ for some $1 > \delta > 0$.
- **Sums of Cantor sets.** Let

$$\mathcal{K}_\lambda = \left\{(1-\lambda)\sum_{n=0}^{\infty} a_n \lambda^n \,:\, a_n \in \{0,1\}\right\}$$

be the middle-α Cantor set for $\alpha = 1 - 2\lambda$. Let $K \subset \mathbb{R}$ be any compact set. Peres and Solomyak [35] showed that $\mathcal{H}^1(K + \mathcal{K}_\lambda) > 0$ for almost every $\lambda \in (0, \frac{1}{2})$ such that $\dim K + \dim \mathcal{K}_\lambda > 1$, and that $\dim(K+\mathcal{K}_\lambda) = \dim K + \dim \mathcal{K}_\lambda$ for a.e. $\lambda \in (0, 1/2)$ such that $\dim K + \dim \mathcal{K}_\lambda < 1$. In Theorem 5.12 we prove that

$$\dim\left\{\lambda \in (\lambda_0, 1/2) \,:\, \mathcal{H}^1(K + \mathcal{K}_\lambda) = 0\right\} \leq 2 - (\dim K + \dim \mathcal{K}_{\lambda_0}),$$

and

$$\dim\left\{\lambda \in (0, \lambda_1) \,:\, \dim(K + \mathcal{K}_\lambda) < \dim K + \dim \mathcal{K}_\lambda\right\} \leq \dim K + \dim \mathcal{K}_{\lambda_1}.$$

We also establish similar estimates for Cantor sets whose symbols belong to the Hölder space $C^{1,\delta}$ for some $\delta > 0$.
- **Keane–Smorodinsky** $\{0, 1, 3\}$**–problem.** The Hausdorff dimension of the set

$$\Lambda(\lambda) = \left\{\sum_{i=0}^{\infty} a_i \lambda^i \,:\, a_i \in \{0, 1, 3\}\right\}$$

has been studied by several authors, since it is perhaps the simplest example of a parametrized family of self-similar sets with overlap. For $\lambda \le \frac{1}{4}$, it is easy to see that $\Lambda(\lambda)$ is self–similar with Hausdorff dimension $\frac{\log 3}{-\log\lambda}$. Keane, Smorodinsky, and Solomyak [26] proved that $\Lambda(\lambda) = [0, 3/(1-\lambda)]$ if $\lambda > 2/5$ and that $\dim \Lambda(\lambda) < 1$ for infinitely many $\lambda \in (\frac{1}{3}, \frac{2}{5})$. Pollicott and Simon [36] showed that for a.e. $\lambda \in (\frac{1}{4}, \frac{1}{3})$ one still has $\dim \Lambda(\lambda) = \frac{\log 3}{-\log\lambda}$ and they found a dense subset of $(\frac{1}{4}, \frac{1}{3})$ where the dimension is strictly less than that fraction. Solomyak [39] finally established that for a.e. $\lambda \in (\frac{1}{3}, \frac{2}{5})$, the set $\Lambda(\lambda)$ has positive Lebesgue measure. In Theorem 5.10, we establish that

$$\dim\left\{\lambda \in (\lambda_0, 2/5) \;:\; \mathcal{H}^1(\Lambda(\lambda)) = 0\right\} \;\le\; 2 - \frac{\log 3}{-\log\lambda_0} \quad \text{for any } \lambda_0 > \tfrac{1}{3},$$

$$\dim\left\{\lambda \in (1/2, \lambda_1) \;:\; \dim\,\Lambda(\lambda) < \frac{\log 3}{-\log\lambda}\right\} \;\le\; \frac{\log 3}{-\log\lambda_1} \quad \text{for any } \lambda_1 > \tfrac{1}{4}.$$

The latter inequality improves an estimate from [36].

- **Self–similar sets in the plane.** Consider the self–similar sets

$$C^S_\lambda = \left\{\sum_{n=0}^{\infty} a_n\lambda^n \;:\; a_n \in S\right\}$$

where $S = \{s_1, \ldots, s_\ell\} \subset \mathbb{C}$ is a fixed set of symbols. In [41] Solomyak showed that $\mathcal{H}^2(C^S_\lambda) > 0$ for a.e. $\lambda > |l|^{-1/2}$ in a region of transversality. In Theorem 8.2 below we prove that the dimension of the exceptional set for this property has to be strictly less than 2.

- **Typical C^1 maps trade off dimension for smoothness.** Hunt, Sauer and Yorke [18] defined a Borel set A in a Banach space X to be **prevalent** if some Borel probability measure ν on X satisfies $\nu(A + x) = 1$ for all $x \in X$. Sauer and Yorke [38] and Hunt and Kaloshin [17] showed that if μ is a Borel probability measure on $\mathbb{R}^d$ with correlation dimension at most m, then the set $A_\mu(d, m)$ of C^1 maps $f : \mathbb{R}^d \to \mathbb{R}^m$ that map μ to a measure with the same correlation dimension is prevalent. To prove this they showed that the definition of prevalence holds with ν equal to Lebesgue measure on the set of $m \times d$ real matrices with entries bounded by 1 in absolute value (these matrices are considered as linear maps from $\mathbb{R}^d$ to $\mathbb{R}^m$). This result can be obtained as a special case of Theorem 7.3, which also yields the following complement (see Remark 7.4): *Let μ be a Borel probability measure on $\mathbb{R}^d$ with correlation dimension greater than $m + 2\gamma$. Then for a prevalent set of C^1 maps $f : \mathbb{R}^d \to \mathbb{R}^m$ the image of μ under f has a density with at least γ fractional derivatives in $L^2(\mathbb{R}^m)$.* Consequently, if $E \subset \mathbb{R}^d$ has $\dim E > m$ (respectively, $\dim E > 2m$), then for a prevalent set of maps $f : \mathbb{R}^d \to \mathbb{R}^m$, the image $f(E)$ has positive Lebesgue measure (respectively, nonempty interior) in $\mathbb{R}^m$.

All these examples are special cases of our main result that is formulated in the following general setting. Given a measure μ on a compact metric space (Ω, d) and a family of maps $\Pi_\lambda : \Omega \to \mathbb{R}^m$ parameterized smoothly by $\lambda \in \mathbb{R}^n$ with $n \ge m$, we show that for a.e. λ the projection of μ under Π_λ has as much "smoothness" as μ and that the Hausdorff dimension of the exceptional parameters decreases with the loss of smoothness. This requires the family Π_λ to satisfy some nondegeneracy assumption. In this paper we impose the aforementioned transversality condition. In the case of Bernoulli convolutions, $\Pi_\lambda(\omega) = \sum_{n=0}^{\infty} \omega_n\lambda^n$ where $\omega \in \Omega = \{-1, +1\}^{\mathbb{N}}$ ($\mathbb{N}$ denotes the nonnegative integers). Here transversality means that the power series $\Pi_\lambda(\omega) - \Pi_\lambda(\tau)$ do not have double zeros. Solomyak showed in [39] that this is the case if $\lambda < 0.649$. Extending our smoothness results to the entire unit interval then requires arguments that are special to Bernoulli convolutions, namely breaking the series up into various subseries.

The paper is organized as follows. In section 2, which is still mainly expository, we discuss transversality and state the general projection theorem in one dimension. Our estimate on the Hausdorff dimension of the set of λ for which ν_λ is a singular measure does not rely on Frostman's lemma. Instead we use the following fact: Let $\{h_j\}_{j=0}^{\infty}$ be a family of nonnegative C^{∞}–functions on $[0,1]$ whose derivatives grow at most exponentially in j. If $\int_0^1 \sum_0^{\infty} R^j h_j(x)\,dx < \infty$, then one can bound the dimension of the set of $x \in [0,1]$ for which $\sum_0^{\infty} r^j h_j(x) = \infty$ for any $1 \le r < R$. This is proved in section 3, see Lemma 3.1. In section 3 we also deal with the case where $\{h_j\}_{j=0}^{\infty}$ is assumed to have only finitely many derivatives. This is applied later to analyze sums of Cantor sets with symbols in the Hölder space $C^{1,\delta}$, and to the geometric problem about intersections of sets with dilates of convex surfaces. It turns out that it suffices to assume some finite degree of smoothness on the surfaces. For Bernoulli convolutions, however, the C^{∞} case suffices.

In Section 4 we prove the general projection theorem in case $m = n = 1$. Our methods rely on the dyadic decomposition of frequency space (i.e., the Littlewood-Paley decomposition from harmonic analysis), which is recalled at the beginning of section 4. To make the connection with Lemma 3.1, which we sketched in the previous paragraph, let $h_j(\lambda) = 2^{-j}\int_{|\xi|\sim 2^j} |\widehat{\nu_\lambda}(\xi)|^2\,d\xi$, where ν_λ is the distribution of $\sum \pm\lambda^n$, say. Then $\int_J \sum_{j=0}^{\infty} 2^{j(1+2\gamma)} h_j(\lambda)\,d\lambda$ is controlled by the square of the $2,\gamma$–Sobolev norm of ν_λ averaged in λ, whereas $\sum_{j=0}^{\infty} 2^j h_j(\lambda) = \infty$ characterizes those $\lambda \in J$ so that ν_λ does not have an L^2–density.
Section 4 is split into two subsections. In the first subsection we discuss the case where the projections have infinitely many derivatives with respect to the parameter, whereas the second subsection deals with the case where the dependence on the parameter has only some finite degree of smoothness.

Applications of the one–dimensional scheme are discussed in section 5. We start with classical Bernoulli convolutions, then consider asymmetric Bernoulli convolutions, the $\{0,1,3\}$–problem, and finally sums of Cantor sets. In all of these applications the main new results concern smoothness of the densities of certain measures and the dimension of the set of those parameters for which some generic property fails.

Section 6 discusses Orthogonal projections in Euclidean space. We give a short proof of the projection theorem in this special case that is simpler than the available proofs in the literature. Moreover, we obtain sufficient conditions for sets to have a.e. projection with nonempty interior. This section also serves to motivate the statement of the general projection theorem in higher dimensions, which is proved in section 7. In section 8 this result is applied to self–similar sets in the complex plane and to distance sets in arbitrary dimensions. We conclude the paper by stating some unsolved problems related to our work.

2. A general scheme for families of projections

To motivate this section we review some simple facts about orthogonal projections in Euclidean space.

Definition 2.1. Let μ be a finite measure on $\mathbb{R}^n$, $n \ge 2$, with compact support. For any $\theta \in S^{n-1}$ its projection ν_θ onto the line $\{t\theta \,:\, t \in \mathbb{R}\}$ is given by $\int f\,d\nu_\theta = \int f((x\cdot\theta)\theta)\,d\mu(x)$ for any continuous f. For any $\alpha \in (0,n)$ the α–energy of μ is defined to be

$$\mathcal{E}_\alpha(\mu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}.$$

We measure smoothness of measures ν in $\mathbb{R}^n$ in terms of the homogeneous Sobolev norm

$$\|\nu\|_{2,\gamma}^2 = \int_{\mathbb{R}^n} |\hat{\nu}(\xi)|^2 |\xi|^{2\gamma}\,d\xi.$$

Finiteness of $\|\nu\|_{2,\gamma}$ for some $\gamma > 0$, means that ν has γ (fractional) derivatives in L^2.

The following proposition relates the smoothness of the projected measures to the energy of the original measure. If $a, b > 0$, then the notation $a \asymp b$ means that $C^{-1}a < b < Ca$ for some absolute constant C.

Proposition 2.2. *Let μ be a finite measure on $\mathbb{R}^n$, $n \ge 2$, with compact support and let ν_θ be as in Definition 2.1. Suppose $0 < 1 + 2\gamma < n$. Then $\int_{S^{n-1}} \|\nu_\theta\|_{2,\gamma}^2\,d\theta \asymp \mathcal{E}_{1+2\gamma}(\mu)$.*

Proof. The projected measures ν_θ clearly satisfy $\widehat{\nu_\theta}(\xi) = \hat{\mu}((\xi\cdot\theta)\theta)$. Hence

$$\begin{aligned}\int_{S^{n-1}} \|\nu_\theta\|_{2,\gamma}^2\, d\theta &= \int_{S^{n-1}}\int_{-\infty}^{\infty} |\widehat{\nu_\theta}(t\theta)|^2\, |t|^{2\gamma}\, dt\, d\theta = \int_{S^{n-1}}\int_{-\infty}^{\infty} |\hat{\mu}(t\theta)|^2 |t|^{2\gamma}\, dt\, d\theta \\ &= 2\int_{\mathbb{R}^n} |\hat{\mu}(\xi)|^2 |\xi|^{1+2\gamma-n}\, d\xi = c_{\gamma,n}\, \mathcal{E}_{1+2\gamma}(\mu).\end{aligned}$$

The final equality follows from Plancherel's Theorem and the definition of energy; see Lemma 12.12 in [31]. $\square$

The main purpose of this section is to study regularity properties of projected measures in a more general setting. Moreover, we bound the dimension of the set of those parameters for which the projected measure has less regularity than generically. In case of projections in the plane the latter question was considered by Kaufman [24] and Falconer [9], see also section 6. For the exceptional set in Proposition 2.2 with $\gamma = 0$, Falconer found the estimate

$$\dim\{\theta\in S^1 \,:\, \widehat{\nu_\theta}\notin L^2(\mathbb{R})\} \le 2-\alpha$$

if $\mathcal{E}_\alpha(\mu) < \infty$. However, his method does not apply to Bernoulli convolutions or other examples considered in this paper. Our approach is based on the Littlewood–Paley decomposition from harmonic analysis, which provides a simple characterization of various degrees of smoothness, see section 4. Moreover, the analogue of Falconer's bound is obtained without Frostman measures.

Definition 2.3. For any finite measure ν on $\mathbb{R}^n$, let its **Sobolev dimension** be defined as

$$\dim_s(\nu) = \sup\{\alpha\in\mathbb{R} \,:\, \int_{\mathbb{R}^n} |\hat{\nu}(\xi)|^2 (1+|\xi|)^{\alpha-n}\, d\xi < \infty\}. \tag{2.1}$$

Clearly, if $0<\dim_s(\nu)<n$ then $\dim_s(\nu) = \sup\{\alpha \,:\, \mathcal{E}_\alpha(\nu)<\infty\}$. In particular, if a Borel set $E\subset\mathbb{R}^n$ supports a probability measure μ with $\dim_s(\mu)\le n$, then $\dim E \ge \dim_s(\mu)$. If $\dim_s(\nu) < n$, then $\dim_s(\nu)$ is also known as the **correlation dimension** of the measure ν. If $\dim_s(\nu) = \sigma > n$, then ν is absolutely continuous and its density has fractional derivatives of order $(\sigma-n)/2$ in $L^2(\mathbb{R}^n)$. Throughout this paper dim (without sub– or superscripts) will mean Hausdorff dimension. The following definition introduces the general framework we will be working with.

Definition 2.4. Let (Ω, d) be a compact metric space, $J\subset\mathbb{R}$ an open interval, and let $\Pi : J\times\Omega\to\mathbb{R}$ be a continuous map. We assume that for any compact $I\subset J$ and $\ell = 0,1,\ldots$ there exists a constant $C_{\ell,I}$ such that

$$\Big|\frac{d^\ell}{d\lambda^\ell}\Pi(\lambda,\omega)\Big| \le C_{\ell,I} \tag{2.2}$$

for all $\lambda\in I$ and $\omega\in\Omega$. Given any finite measure μ on Ω let $\nu_\lambda = \mu\circ\Pi_\lambda^{-1}$, where $\Pi_\lambda(\cdot) = \Pi(\lambda,\cdot)$. The α–energy of μ is defined as $\mathcal{E}_\alpha(\mu) = \int_\Omega\int_\Omega \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^\alpha}$.

Informally, one can think of $\Pi_\lambda(\cdot) = \Pi(\lambda,\cdot)$ as a family of projections parameterized by λ. We state two results on the smoothness of a typical ν_λ in terms of $\mathcal{E}_\alpha(\mu)$ assuming certain transversality conditions.

Definition 2.5. Let Ω, J, and Π be as in Definition 2.4. For any distinct $\omega_1,\omega_2\in\Omega$ and $\lambda\in J$ let

$$\Phi_\lambda(\omega_1,\omega_2) = \frac{\Pi(\lambda,\omega_1)-\Pi(\lambda,\omega_2)}{d(\omega_1,\omega_2)}.$$

J is an **interval of strong transversality for** Π if there exist positive constants C, C_ℓ so that for all $\lambda\in J$ and $\omega_1,\omega_2\in\Omega$

((i)) $|\frac{d^\ell}{d\lambda^\ell}\Phi_\lambda(\omega_1,\omega_2)| \le C_\ell\, |\frac{d}{d\lambda}\Phi_\lambda(\omega_1,\omega_2)|^\ell$ for $\ell = 2,3,\ldots$

((ii)) $|\frac{d}{d\lambda}\Phi_\lambda| \ge C$.

Theorem 2.6. *Let J and Π be as in Definition 2.4 and suppose that J is an interval of strong transversality for Π. Let μ be a finite positive measure on Ω with finite α–energy for some $\alpha > 0$. Then for any compact $I \subset J$*

$$\int_I |\widehat{\nu_\lambda}(\xi)|^2 \, d\lambda \le C_\alpha |\xi|^{-\alpha} \mathcal{E}_\alpha(\mu). \tag{2.3}$$

Moreover, $\dim_s(\nu_\lambda) \ge \alpha$ for a.e. $\lambda \in J$. More precisely, for any $\sigma \in (0, \alpha]$,

$$\dim \{\lambda \in J \,:\, \dim_s(\nu_\lambda) < \sigma\} \le \sigma + \min(1-\alpha, 0). \tag{2.4}$$

The condition of strong transversality turns out to be too restrictive for most applications. Since the cosine function has vanishing derivative at $0, \pi$ it fails for projections onto lines. More importantly, (2.3) fails for Bernoulli convolutions because of Pisot numbers, see Lemma 5.7. The correct notion of transversality in the context of Bernoulli convolutions turns out to be the following one.

Definition 2.7. Let Φ_λ be as in Definition 2.5. For any $\beta \in [0, 1)$ we say that J is an **interval of transversality of order β for Π** if there exists a constant C_β so that for all $\lambda \in J$ and $\omega_1, \omega_2 \in \Omega$ the condition $|\Phi_\lambda(\omega_1, \omega_2)| \le C_\beta \, d(\omega_1, \omega_2)^\beta$ implies

$$\Big|\frac{d}{d\lambda}\Phi_\lambda\Big| \ge C_\beta \, d(\omega_1, \omega_2)^\beta. \tag{2.5}$$

In addition, we say that Π_λ is **L–regular on J** for some positive integer L or $L = \infty$, if under the same condition and for some constants $C_{\beta,\ell}$,

$$\Big|\frac{d^\ell}{d\lambda^\ell}\Phi_\lambda(\omega_1, \omega_2)\Big| \le C_{\beta,\ell} \, d(\omega_1, \omega_2)^{-\beta\ell} \quad \text{for} \quad \ell = 1, 2, \ldots, L. \tag{2.6}$$

Our main result in this section is the following theorem.

Theorem 2.8. *Let Ω, J, Π be as in Definition 2.4 and suppose that J is an interval of transversality of order β for Π for some $\beta \in [0, 1)$ and that Π_λ is ∞–regular on J. Let μ be a finite positive measure on Ω with finite α–energy for some $\alpha > 0$. Then for any compact $I \subset J$*

$$\int_I \|\nu_\lambda\|_{2,\gamma}^2 \, d\lambda \le C_\gamma \, \mathcal{E}_\alpha(\mu) \quad \text{if} \quad 0 < (1+2\gamma)(1+a_0\beta) \le \alpha, \tag{2.7}$$

where a_0 is some absolute constant. Moreover, for any $\sigma \in (0, \alpha]$,

$$\dim \{\lambda \in J \,:\, \dim_s(\nu_\lambda) \le \sigma\} \le 1 + \sigma - \frac{\alpha}{1 + a_0\beta}, \tag{2.8}$$

and for any $\sigma \in (0, \alpha - 3\beta]$,

$$\dim \{\lambda \in J \,:\, \dim_s(\nu_\lambda) < \sigma\} \le \sigma. \tag{2.9}$$

In all our applications of this theorem we will be able to take β arbitrarily small. Moreover, the geometric problems we consider in this paper satisfy transversality with $\beta = 0$, see sections 6 and 8.2. Some of these geometric estimates are known to be optimal. Since they are covered by our general theory, it follows that the corresponding cases of Theorem 2.8 are sharp, see section 6 for more discussion. However, we do not know whether Theorem 2.8 is optimal in all cases.

The basic idea behind the proof of (2.7) is that for any finite measure ν on $\mathbb{R}$ and $\gamma \in (-1/2, 1)$

$$\|\nu\|_{2,\gamma} \asymp \Big\| \Big(\sum_{j=-\infty}^{\infty} 2^{2j(\gamma+1)} |\nu(x - 2^{-j}, x) - \nu(x, x + 2^{-j})|^2 \Big)^{\frac{1}{2}} \Big\|_{L^2(dx)} \tag{2.10}$$

as one can easily check by applying Plancherel's theorem to the right–hand side. However, it is difficult to work with the square–function in (2.10) because of the singularities of the kernel $\chi_{[-1,0]} - \chi_{[0,1]}$. As is standard in harmonic analysis, one circumvents this difficulty by considering a smoother kernel. The most convenient form is given in terms of the Littlewood–Paley decomposition, see Lemma 4.1.

Theorem 2.8 has a simple corollary concerning the dimension of the exceptional set with respect to absolute continuity without making any assumptions on the energy of μ. The hypothesis will be in terms of the upper and lower information dimension of μ. Recall that the lower pointwise dimension is given by

$$\pi_\mu^-(\omega) = \liminf_{r\to 0+} \frac{\log \mu(B(\omega,r))}{\log r}$$

and set

$$\overline{\dim}_I(\mu) = \left\|\pi_\mu^-\right\|_{L^\infty(d\mu)} \quad\text{and}\quad \underline{\dim}_I(\mu) = \mathrm{essinf}_{d\mu(\omega)}\left\{\pi_\mu^-(\omega)\right\}.$$

Notice that $\mathcal{E}_\alpha(\mu) < \infty$ implies $\underline{\dim}_I(\mu) \geq \alpha$, but the converse is clearly false.

Corollary 2.9. *Suppose J is an interval of transversality of order β for Π and that Π_λ is ∞–regular on J. Then*

$$\dim\{\lambda\in J \,:\, \nu_\lambda \ \text{ is singular}\} \quad\leq\quad 2 - \frac{\overline{\dim}_I(\mu)}{1+a_0\beta} \tag{2.11}$$

$$\dim\{\lambda\in J \,:\, \nu_\lambda \ \text{ is not absolutely continuous}\} \quad\leq\quad 2 - \frac{\underline{\dim}_I(\mu)}{1+a_0\beta}, \tag{2.12}$$

where a_0 is the constant from Theorem 2.8.

Proof. Let $\alpha = \overline{\dim}_I(\mu)$ and fix some $\tilde\alpha < \alpha$ and $\epsilon > 0$. Then

$$\tilde\Omega = \{\omega \,:\, \liminf_{r\to 0+} \frac{\log \mu(B(\omega,r))}{\log r} > \tilde\alpha\}$$

satisfies $\mu(\tilde\Omega) > 0$. By Egoroff's theorem there exists $\Omega_\epsilon \subset \tilde\Omega$ so that $\mu(\tilde\Omega\setminus\Omega_\epsilon) < \epsilon$ and

$$\liminf_{r\to 0+} \frac{\log \mu(B(\omega,r))}{\log r} \geq \tilde\alpha \quad\text{uniformly on}\quad \Omega_\epsilon.$$

Let $\mu^\epsilon = \mu \restriction \Omega_\epsilon$ and $\nu_\lambda^{(\epsilon)} = \mu^\epsilon \circ \Pi_\lambda^{-1}$. By definition, $\mathcal{E}_{\tilde\alpha-\epsilon}(\mu^\epsilon) < \infty$, and thus by Theorem 2.8,

$$\dim\{\lambda\in J \,:\, \dim_s(\nu_\lambda^{(\epsilon)}) \leq \sigma\} \leq 1 + \sigma - (\tilde\alpha - \epsilon)(1+a_0\beta)^{-1}.$$

In particular,

$$\dim\{\lambda\in J \,:\, \nu_\lambda^{(\epsilon)} \ \text{ is singular}\} \leq 2 - (\tilde\alpha-\epsilon)(1+a_0\beta)^{-1}. \tag{2.13}$$

Since

$$\{\lambda\in J \,:\, \nu_\lambda \ \text{ is singular}\} \subset \limsup_{j\to\infty}\{\lambda\in J \,:\, \nu_\lambda^{(2^{-j})} \ \text{ is singular}\}$$

(2.11) follows by letting $\tilde\alpha\to\alpha$ and $\epsilon\to 0+$ in (2.13). The proof of (2.12) is very similar and is omitted. □

3. Exceptional parameters for convergence of power series

3.1. **The C^∞ case.** The dimension bounds (2.4) in case $\alpha \leq 1$ and (2.9) are analogous to Kaufman's result for Orthogonal projections, and involve Frostman measures as in his original proof. However, if $\alpha > 1$, estimates (2.4) and (2.8) are derived from the smoothness bounds by means of the following lemma, which we state for parameters in $\mathbb{R}^n$ for an arbitrary $n \geq 1$. Recall that the length of a multi-index $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{N}^n$ is defined to be $|\eta| = \eta_1 + \ldots + \eta_n$ and that $\partial^\eta = \frac{\partial^{|\eta|}}{(\partial\lambda_1)^{\eta_1}\cdot\ldots\cdot(\partial\lambda_n)^{\eta_n}}$ where $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n)$. For applications to Bernoulli convolutions it suffices to read the following lemma with $n = 1$.

Lemma 3.1. *Let $Q \subset \mathbb{R}^n$ be a nonempty open set. Suppose $\{h_j\}_0^\infty \in C^\infty(Q)$ satisfy*

$$\sup_{j\geq 0} A^{-j|\eta|}\big\|\partial^\eta h_j\big\|_\infty \leq C_\eta \text{ for all multi–indices } \eta \in \mathbb{N}^n \text{ and } \sup_{j\geq 0}\int_Q R^j |h_j(\boldsymbol{\lambda})|\,d\boldsymbol{\lambda} \leq C_* < \infty\,, \tag{3.1}$$

where $A > 1$. Suppose that $R > r \geq 1$.

((i)) *If $A^n < \frac{R}{r}$, then $\sum_{j=0}^\infty r^j |h_j(\boldsymbol{\lambda})| < \infty$ for all $\boldsymbol{\lambda} \in Q$.*

((ii)) *If $A^\alpha \leq \frac{R}{r} \leq A^n$, then*

$$\dim\Big\{\boldsymbol{\lambda} \in Q \;:\; \sum_{j=0}^\infty r^j |h_j(\boldsymbol{\lambda})| = \infty\Big\} \leq n - \alpha\,. \tag{3.2}$$

Proof. Fix some $0 < r < R$ with $A^n \geq \frac{R}{r}$. It suffices to prove (3.2) for any compact cube $Q' \subset Q$. Fix such a Q' and define $E_j = \big\{\boldsymbol{\lambda} \in Q' \;:\; |h_j(\boldsymbol{\lambda})| > \frac{1}{j^2} r^{-j}\big\}$. Then

$$\Big\{\boldsymbol{\lambda} \in Q' \;:\; \sum_{j=0}^\infty r^j |h_j(\boldsymbol{\lambda})| = \infty\Big\} \subset \limsup_{j\to\infty} E_j. \tag{3.3}$$

We will estimate the $(n-\alpha)$–Hausdorff measure of $\limsup E_j$ by covering each E_j with cubes of side length $\simeq A^{-j}$. The idea is that any point in E_j has a neighborhood of size $\simeq A^{-j}$ on which the average of $|h_j|$ is at least $C\frac{1}{j^2}r^{-j}$. More precisely, for any positive integer N,

$$\left|\sum_{i=0}^N \binom{N}{i}(-1)^i h_j(\boldsymbol{\lambda} + iy)\right| \leq |y|^N \sup_{|\eta|=N}\big\|\partial^\eta h_j\big\|_\infty \leq C_N\big(|y|A^j\big)^N. \tag{3.4}$$

For any $j = 0, 1, \ldots$ and $\boldsymbol{\lambda}_0 \in Q'$ let

$$I_{j,N}(\boldsymbol{\lambda}_0) = \int_{[-NL_{j,N}\,, NL_{j,N}\,]^n} |h_j(\boldsymbol{\lambda}_0 + \boldsymbol{\lambda})|\,d\boldsymbol{\lambda}$$

where $L_{j,N} > 0$ will be determined below. In view of (3.4), with some dimensional constant b_n,

$$\begin{aligned} \frac{b_n\,C_N}{N+n} L_{j,N}^{N+n}\,A^{jN} \quad &\geq \quad \int_{[-L_{j,N}\,, L_{j,N}\,]^n}\Big|\sum_{i=0}^N \binom{N}{i}(-1)^i h_j(\boldsymbol{\lambda}_0 + iy)\Big|\,dy \\ &\geq \quad (2L_{j,N}\,)^n\,|h_j(\boldsymbol{\lambda}_0)| - \sum_{i=1}^N \binom{N}{i}\frac{1}{i^n} I_{j,N}(\boldsymbol{\lambda}_0). \end{aligned} \tag{3.5}$$

In particular, setting $L_{j,N} = A^{-j}(C'_N\,j^2 r^j)^{-\frac{1}{N}}$ with a suitable constant C'_N, one has

$$\frac{1}{2}(j^2 r^j)^{-1}(2L_{j,N}\,)^n \geq (2L_{j,N}\,)^n |h_j(\boldsymbol{\lambda}_0)| - 2^N I_{j,N}(\boldsymbol{\lambda}_0)$$

and therefore, for any $\boldsymbol{\lambda}_0 \in E_j$,

$$I_{j,N}(\boldsymbol{\lambda}_0) \geq 2^{-N} j^{-2}\,r^{-j}(2L_{j,N}\,)^n, \tag{3.6}$$

if j is sufficiently large (depending on $\operatorname{dist}(Q', \partial Q)$). Fix some positive integer N and let $\{U_{ij} \;:\; i = 1, \ldots, M_{j,N}\}$ be a covering of E_j with disjoint cubes of side length $2NL_{j,N}$. Pick any $\boldsymbol{\lambda}_{ij} \in U_{ij} \cap E_j$. Setting $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_{ij}$ in (3.6) and summing over $i = 1, \ldots, M_{j,N}$ yields

$$M_{j,N}\;2^{-N} j^{-2}\,r^{-j}(2L_{j,N}\,)^n \leq 2n\int_Q |h_j(\boldsymbol{\lambda})|\,d\boldsymbol{\lambda} \leq 2nC_*\,R^{-j}.$$

Therefore, by the definition of $L_{j,N}$,

$$M_{j,N} \leq \tilde{C}_N\;j^{2(1+\frac{n}{N})}\big(A^n r^{1+\frac{n}{N}}/R\big)^j. \tag{3.7}$$

Let $\alpha \in (0,1)$ satisfy $A^\alpha r^{\frac{\alpha}{N}} < \frac{R}{r}$. In view of (3.7),

$$\begin{aligned}\mathcal{H}^{n-\alpha}(\limsup E_j) &\le \tilde{C}_N \lim_{k\to\infty} \sum_{j=k}^{\infty} M_{j,N}\,(2NL_{j,N}\,)^{n-\alpha} \\ &= \tilde{C}_N \lim_{k\to\infty} \sum_{j=k}^{\infty} \Big(\frac{r^{1+\frac{\alpha}{N}}}{R} A^\alpha\Big)^j j^{2(1+\frac{\alpha}{N})} = 0.\end{aligned}$$

Thus (3.2) follows from (3.3) by letting $N \to \infty$.

To prove assertion (i) assume that $A^n < \frac{R}{r}$ and let N be a fixed positive integer such that $r^{\frac{n}{N}}\, A^n < \frac{R}{r}$. Let $\boldsymbol{\lambda}_0 \in E_j$ for some sufficiently large j. It follows from (3.6) that

$$C_* \ge R^j\, I_{j,N}(\boldsymbol{\lambda}_0) \ge \tilde{C}_N\, j^{-2(1+\frac{n}{N})} \Big(\frac{R}{Ar^{1+\frac{n}{N}}}\Big)^j,$$

which is a contradiction for large j. Thus $\limsup_{j\to\infty} E_j = \emptyset$ and (i) follows from (3.3). □

3.2. **The case of limited regularity.** Suppose $Q \subset \mathbb{R}^n$ and $\{h_j\}_0^\infty \subset C^L(Q)$ for some positive integer L. Here C^L denotes the space of functions which are L times continuously differentiable. Under the assumption (3.1) for $|\eta| \le L$ the previous proof with $N = L$ yields (3.2) provided $A^\alpha r^{\frac{\alpha}{L}} \le \frac{R}{r} \le A^n r^{\frac{n}{L}}$. Below we show how to weaken this condition on α under a slightly different assumption. For a positive integer L and $\delta \in [0,1)$ we let $C^{L,\delta}$ denote the space of functions which are L times continuously differentiable and such that the derivatives of order L satisfy a Hölder condition of order δ (we will write $C^{0,\delta} = C^\delta$).

Lemma 3.2. *Let $Q \subset \mathbb{R}^n$ be an open set and suppose $\{h_j\}_0^\infty$ are nonnegative and uniformly bounded functions satisfying $\sup_{j\ge 0} R^j \int_Q h_j(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} < C_* < \infty$. Let $H_j = \sum_{i=0}^j B^{i-j} h_i$ for some fixed $B \ge A > 1$.*

((i)) *Let $1 \le r < R$ and $r \le B$. Assume that $\{h_j\}_0^\infty \subset C^1(Q)$ and*

$$|\nabla H_j(\boldsymbol{\lambda})| \le C_1 A^j H_j(\boldsymbol{\lambda}) \quad \text{for all} \quad j = 0,1,2\ldots,\ \boldsymbol{\lambda} \in Q.$$

If $A^n < \frac{R}{r}$, then $\sum_{j=0}^\infty r^j h_j < \infty$ on Q. Otherwise,

$$\dim\Big\{\boldsymbol{\lambda} \in Q \,:\, \sum_{j=0}^\infty r^j h_j(\boldsymbol{\lambda}) = \infty\Big\} \le n - \alpha \tag{3.8}$$

provided $A^\alpha \le \frac{R}{r}$.

((ii)) *Let $B < r < R$. Assume that, for some integer $L \ge 1$ and $\delta \in [0,1)$, one has $\{h_j\}_0^\infty \subset C^{L,\delta}(Q)$ and*

$$\begin{aligned}&|\nabla H_j(\boldsymbol{\lambda})| \le C_1 A^j H_j(\boldsymbol{\lambda}) \quad \text{for all} \quad j = 0,1,2\ldots,\ \text{and}\ \boldsymbol{\lambda} \in Q\,,\\ &\sup_{|\eta|=L} \big|\partial^\eta H_j(\boldsymbol{\lambda}_1) - \partial^\eta H_j(\boldsymbol{\lambda}_2)\big| \le C_{L,\delta}\, |\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2|^\delta\, A^{j(L+\delta)} \big[H_j(\boldsymbol{\lambda}_1) + H_j(\boldsymbol{\lambda}_2)\big].\end{aligned} \tag{3.9}$$

If $A^n \big(\frac{r}{B}\big)^{\frac{n}{L+\delta}} < \frac{R}{r}$, then $\sum_{j=0}^\infty r^j h_j < \infty$ on Q. Otherwise,

$$\dim\Big\{\boldsymbol{\lambda} \in Q \,:\, \sum_{j=0}^\infty r^j h_j(\boldsymbol{\lambda}) = \infty\Big\} \le n - \alpha \tag{3.10}$$

provided $A^\alpha \big(\frac{r}{B}\big)^{\frac{n}{L+\delta}} \le \frac{R}{r}$.

Proof. It suffices to prove these statements for an arbitrary cube $Q' \subset Q$. We start with the proof of (3.8). Fix $0 < r_0 < r_1 \le B$ and $T \in (0,\infty)$ and let

$$E_j^{(T)} = \Big\{\boldsymbol{\lambda} \in Q' \,:\, h_j(\boldsymbol{\lambda}) > j^{-2}\, r_1^{-j},\ \sum_{i=0}^\infty r_0^i\, h_i(\boldsymbol{\lambda}) \le T\Big\} \tag{3.11}$$

for $j = 0, 1, \ldots$. Clearly,

$$\Big\{\boldsymbol{\lambda} \in Q' \;:\; \sum_{i=0}^{\infty} r_1^i\, h_i(\boldsymbol{\lambda}) = \infty,\;\; \sum_{i=0}^{\infty} r_0^i\, h_i(\boldsymbol{\lambda}) < \infty\Big\} \subset \bigcup_{T=1}^{\infty} \limsup_{j\to\infty} E_j^{(T)}. \tag{3.12}$$

Fix some positive integer T and let $\{U_{ij}\}_{j=1}^{M_j}$ be a covering of $E_j^{(T)}$ with disjoint cubes of side length $W_j < A^{-j}$ for $j = 0, 1, \ldots$. To make an appropriate choice of W_j we bound the variation of h_j on cubes. By the bound on $|\nabla H_j|$,

$$\Big|\frac{d}{dt} H_j(\boldsymbol{\lambda} + te)\Big| \le C_1\, A^j H_j(\boldsymbol{\lambda} + te)$$

for any unit vector e and $\boldsymbol{\lambda},\ \boldsymbol{\lambda} + te \in Q'$. By Gronwall's inequality (see, e.g., [14], Chap. 10), $H_j(\boldsymbol{\lambda}) \le H_j(\boldsymbol{\lambda}_0) \exp\big(C_1\, A^j |\boldsymbol{\lambda} - \boldsymbol{\lambda}_0|\big)$ for any $\boldsymbol{\lambda},\ \boldsymbol{\lambda}_0 \in Q'$. Hence for all $j = 0, 1, \ldots$

$$|H_j(\boldsymbol{\lambda}) - H_j(\boldsymbol{\lambda}_0)| \le C\, A^j |\boldsymbol{\lambda} - \boldsymbol{\lambda}_0|\, H_j(\boldsymbol{\lambda}_0) \tag{3.13}$$

provided $|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0| < A^{-j}$. Since $r_0 < B$,

$$\sum_{j=0}^{\infty} r_0^j\, H_j(\boldsymbol{\lambda}_0) \le \sum_{j=0}^{\infty} \Big(\frac{r_0}{B}\Big)^j \sum_{i=0}^{j} \Big(\frac{B}{r_0}\Big)^i r_0^i\, h_i(\boldsymbol{\lambda}_0) \le \big(1 - r_0/B\big)^{-1} \sum_{i=0}^{\infty} r_0^i\, h_i(\boldsymbol{\lambda}_0).$$

Combining this with (3.13) yields

$$\begin{aligned} |h_j(\boldsymbol{\lambda}) - h_j(\boldsymbol{\lambda}_0)| &= B^{-j}\Big|B^j[H_j(\boldsymbol{\lambda}) - H_j(\boldsymbol{\lambda}_0)] - B^{j-1}[H_{j-1}(\boldsymbol{\lambda}) - H_{j-1}(\boldsymbol{\lambda}_0)]\Big| \\ &\le C\big(1 - r_0/B\big)^{-1} |\boldsymbol{\lambda} - \boldsymbol{\lambda}_0| \Big(\frac{A}{r_0}\Big)^j \sum_{i=0}^{\infty} r_0^i\, h_i(\boldsymbol{\lambda}_0) \end{aligned} \tag{3.14}$$

if $|\boldsymbol{\lambda} - \boldsymbol{\lambda}_0| < A^{-j}$. Applying this to any $\boldsymbol{\lambda}_0 \in U_{ij} \cap E_j^{(T)}$ and $\boldsymbol{\lambda} \in U_{ij}$ one concludes that

$$h_j(\boldsymbol{\lambda}) \ge \frac{1}{j^2} r_1^{-j} - C\, T \big(1 - r_0/B\big)^{-1} \Big(\frac{A}{r_0}\Big)^j |\boldsymbol{\lambda} - \boldsymbol{\lambda}_0| \ge \frac{1}{2j^2} r_1^{-j}$$

if $W_j = \tilde{C}\, \frac{1}{j^2} \big(\frac{r_0}{r_1}\big)^j A^{-j}$ where $\tilde{C}$ depends only on T, B, r_0. Therefore,

$$C_*\, R^{-j} \ge \int_Q h_j(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} \ge \int_{\bigcup_{i=1}^{M_j} U_{ij}} \frac{1}{2j^2}\, r_1^{-j}\, d\boldsymbol{\lambda} = \frac{1}{2j^2} M_j W_j^n\, r_1^{-j} \tag{3.15}$$

and thus, by choice of W_j,

$$M_j \le 2C_*\, j^{2(1+n)} \Big(\frac{r_1}{R} A^n\Big)^j \Big(\frac{r_1}{r_0}\Big)^{jn}.$$

Hence

$$\begin{aligned} \mathcal{H}^{n-\alpha}\big(\limsup_{j\to\infty} E_j^{(T)}\big) &\le \lim_{k\to\infty} \sum_{j=k}^{\infty} M_j W_j^{n-\alpha} \\ &\le \lim_{k\to\infty} \sum_{j=k}^{\infty} 2C_*\, j^{2(1+n)} \Big(\frac{r_1}{R} A^n\Big)^j \Big(\frac{r_1}{r_0}\Big)^{jn} \tilde{C}^{n-\alpha}\, j^{-2(n-\alpha)} \Big(\frac{r_0}{r_1}\Big)^{j(n-\alpha)} A^{-j(n-\alpha)} \\ &= C \lim_{k\to\infty} \sum_{j=k}^{\infty} j^{2(1+\alpha)} \Big(\frac{r_1}{R} A^\alpha\Big)^j \Big(\frac{r_1}{r_0}\Big)^{j\alpha} = 0 \end{aligned}$$

provided $A^\alpha < \frac{R}{r_1} \big(\frac{r_0}{r_1}\big)^\alpha$.

In view of (3.12) therefore

$$\dim\Big\{\boldsymbol{\lambda}\in Q\;:\;\sum_{i=0}^{\infty} r_1^i\, h_i(\boldsymbol{\lambda})=\infty,\;\;\sum_{i=0}^{\infty} r_0^i\, h_i(\boldsymbol{\lambda})<\infty\Big\}\le n-\alpha \tag{3.16}$$

if $A^\alpha\le\frac{R}{r_1}\big(\frac{r_0}{r_1}\big)^\alpha$ and $0<r_0<r_1\le A$. To prove (3.8), choose $r=\rho_m>\rho_{m-1}>\ldots>\rho_2>1>\rho_1$. Then

$$\Big\{\boldsymbol{\lambda}\in Q\;:\;\sum_{i=0}^{\infty} r^i\, h_i(\boldsymbol{\lambda})=\infty\Big\}=\bigcup_{k=2}^{m}\Big\{\boldsymbol{\lambda}\in Q\;:\;\sum_{i=0}^{\infty}\rho_k^i\, h_i(\boldsymbol{\lambda})=\infty,\;\;\sum_{i=0}^{\infty}\rho_{k-1}^i\, h_i(\boldsymbol{\lambda})<\infty\Big\} \tag{3.17}$$

and (3.16) imply that

$$\dim\Big\{\boldsymbol{\lambda}\in Q\;:\;\sum_{i=0}^{\infty} r^i\, h_i(\boldsymbol{\lambda})=\infty\Big\}\le n-\alpha$$

if $A^\alpha\le\frac{R}{r}\min_{i=2,\ldots,m}\big(\frac{\rho_{i-1}}{\rho_i}\big)^\alpha$. Letting $\max\frac{\rho_i}{\rho_{i-1}}\to 1$ finishes the proof.

If $A^n<\frac{R}{r_1}\Big(\frac{r_0}{r_1}\Big)^n$, then (3.15) with $M_j=1$ leads to a contradiction if j is sufficiently large. Subdividing as in (3.17) therefore shows that the left–hand side of (3.17) is empty if $\frac{R}{r}>A^n$, and the proof of (i) is complete.

For the proof of (ii) let $r>r_0=B$ and $T\in(0,\infty)$, and define $E_j^{(T)}$ as in (3.11). The definition of H_k implies that $\sup_{k\ge 0}B^kH_k(\boldsymbol{\lambda}_0)\le T$ for all $\boldsymbol{\lambda}_0\in E_j^{(T)}$ and all j. By (3.13) therefore

$$|H_k(\boldsymbol{\lambda})-H_k(\boldsymbol{\lambda}_0)|\le C\,T|\boldsymbol{\lambda}-\boldsymbol{\lambda}_0|\;\text{ and thus }\;H_k(\boldsymbol{\lambda})\le CT\,B^{-k}\;\text{ provided }\;|\boldsymbol{\lambda}_0-\boldsymbol{\lambda}|<B^{-k} \tag{3.18}$$

for all $\boldsymbol{\lambda}_0\in E_j^{(T)}$, and all j and k. Let N be the smallest integer $\ge L+\delta$. In view of the definition of H_j, the Hölder estimate (3.9) on $\partial^\eta H_j$ for $|\eta|=L$, and (3.18) with $k=j$ and $k=j-1$,

$$\begin{aligned}\Big|\sum_{i=0}^{N}\binom{N}{i}(-1)^i h_j(\boldsymbol{\lambda}_0+iy)\Big| &= B^{-j}\Big|\sum_{i=0}^{N}\binom{N}{i}(-1)^i\big[B^jH_j(\boldsymbol{\lambda}_0+iy)-B^{j-1}H_{j-1}(\boldsymbol{\lambda}_0+iy)\big]\Big|\\ &\le C_{L,\delta}\left(A^j\,|y|\right)^{L+\delta}\big(\max_{0\le t\le N}H_j(\boldsymbol{\lambda}_0+ty)+\max_{0\le t\le N}H_{j-1}(\boldsymbol{\lambda}_0+ty)\big)\\ &\le C_{L,\delta}\,T\left(A^j\,|y|\right)^{L+\delta}B^{-j}\le\frac{1}{2j^2}\,r^{-j} \end{aligned} \tag{3.19}$$

if $\boldsymbol{\lambda}_0\in E_j^{(T)}$ and $|y|\le C\,j^{-\frac{2}{L+\delta}}\,A^{-j}\big(\frac{B}{r}\big)^{\frac{j}{L+\delta}}=W_j$ where $C=C(L,\delta,T)$. Now let $\{U_{ij}\}_{i=1}^{M_j}$ be a covering of $E_j^{(T)}$ with cubes of side length $2NW_j$ and fix some $\boldsymbol{\lambda}_{ij}\in U_{ij}\cap E_j^{(T)}$ for all i and j. Integrating (3.19) with $\boldsymbol{\lambda}_0=\boldsymbol{\lambda}_{ij}$ over $[\boldsymbol{\lambda}_{ij}-W_j,\boldsymbol{\lambda}_{ij}+W_j]^n$ yields

$$\frac{1}{2j^2}\,r^{-j}W_j^n\ge h_j(\boldsymbol{\lambda}_{ij})W_j^n-\sum_{i=1}^{N}\binom{N}{i}\frac{1}{i^n}\int_{[-NW_j,NW_j]}h_j(\boldsymbol{\lambda}_{ij}+\boldsymbol{\lambda})\,d\boldsymbol{\lambda}$$

and thus

$$\int_{[-NW_j,NW_j]}h_j(\boldsymbol{\lambda}_{ij}+\boldsymbol{\lambda})\,d\boldsymbol{\lambda}\ge 2^{-N}\big(2j^2\,r^j\big)^{-1}W_j^n. \tag{3.20}$$

Summing (3.20) over $i=1,\ldots,M_j$ implies in view of our assumption $C_*R^{-j}\ge\int_Q h_j(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}$ that

$$2nC_*\,R^{-j}\ge 2n\int_Q h_j(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}\ge\int_{\bigcup_{i=1}^{M_j}U_{ij}}\frac{1}{2^{N+1}j^2}\,r^{-j}\,d\boldsymbol{\lambda}=\frac{1}{2^{N+1}j^2}M_j(2NW_j)^n\,r^{-j}. \tag{3.21}$$

By definition of W_j, therefore

$$M_j\le C\,j^{\frac{2n}{L+\delta}}\Big(\frac{r}{R}\Big)^jA^{nj}\Big(\frac{r}{B}\Big)^{\frac{jn}{L+\delta}}.$$

Hence

$$\mathcal{H}^{n-\alpha}\big(\limsup_{j\to\infty} E_j^{(T)}\big) = 0 \ \text{ if } \ A^\alpha\Big(\frac{r}{B}\Big)^{\frac{\alpha}{L+\delta}} < \frac{R}{r} \tag{3.22}$$

for all T. Since for any $r > B$, the set $\Big\{\boldsymbol{\lambda} \in Q \,:\, \sum_{i=0}^{\infty} r^i\, h_i(\boldsymbol{\lambda}) = \infty\Big\}$ is contained in

$$\Big\{\boldsymbol{\lambda} \in Q \,:\, \sum_{i=0}^{\infty} r^i\, h_i(\boldsymbol{\lambda}) = \infty,\ \sum_{i=0}^{\infty} B^i\, h_i(\boldsymbol{\lambda}) < \infty\Big\} \cup \Big\{\boldsymbol{\lambda} \in Q \,:\, \sum_{i=0}^{\infty} B^i\, h_i(\boldsymbol{\lambda}) = \infty\Big\}, \tag{3.23}$$

(3.10) follows from (3.12), (3.22), and (3.8). To apply (3.8) to (3.23) one needs to check that $A^\alpha\big(\frac{r}{B}\big)^{\frac{\alpha}{L+\delta}} < \frac{R}{r}$ implies that $A^\alpha \le \frac{R}{B}$. However, this is an immediate consequence of $B < r$.

Finally, if $A^n\big(\frac{r}{B}\big)^{\frac{\alpha}{L+\delta}} < \frac{R}{r}$, then (3.21) with $M_j = 1$ shows that $E_j^{(T)} = \emptyset$ if j is sufficiently large and thus $\sum r^j h_j < \infty$ on Q, as claimed. ☐

4. Proofs of the general projection theorems

4.1. **The C^∞ case.** To estimate Sobolev norms it will be convenient to decompose frequency space dyadically. For the sake of completeness we first recall the construction of such a Littlewood–Paley decomposition, see Stein [43] and Frazier, Jawerth, Weiss [13]. $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of smooth functions all of whose derivatives decay faster than any power. It is a basic property of the Fourier transform that it preserves $\mathcal{S}$.

Lemma 4.1. *There exists $\psi \in \mathcal{S}(\mathbb{R}^m)$ so that $\hat{\psi} \ge 0$,*

$$\operatorname{supp}(\hat{\psi}) \subset \{\xi \in \mathbb{R}^m \,:\, 1 \le |\xi| \le 4\}, \quad \textit{and} \quad \sum_{j=-\infty}^{\infty} \hat{\psi}(2^{-j}\xi) = 1 \ \ \textit{if } \xi \ne 0. \tag{4.1}$$

Moreover, given any finite measure ν on $\mathbb{R}^m$ and any $\gamma \in \mathbb{R}$

$$\|\nu\|_{2,\gamma}^2 \asymp \sum_{j=-\infty}^{\infty} 2^{2j\gamma} \int_{\mathbb{R}^m} (\psi_{2^{-j}} * \nu)(x)\, d\nu(x) \tag{4.2}$$

where $\psi_{2^{-j}}(x) = 2^{jm}\psi(2^j x)$.

Proof. Choose $\phi \in \mathcal{S}(\mathbb{R}^m)$ with $\hat{\phi} \ge 0$, $\hat{\phi}(\xi) = 1$ for $|\xi| \le 1$ and $\hat{\phi}(\xi) = 0$ for $|\xi| > 2$. Define ψ via $\hat{\psi}(\xi) = \hat{\phi}(\xi/2) - \hat{\phi}(\xi)$. It is clear that $\hat{\psi}(\xi) \ge 0$ and that $\hat{\psi}(\xi) = 0$ if $|\xi| < 1$ or $|\xi| > 4$. (4.1) holds since the sum telescopes. Moreover, it is clear from (4.1) that there exists some constant C_γ depending only on γ so that for any $\xi \ne 0$

$$C_\gamma^{-1}|\xi|^{2\gamma} \le \sum_{j=-\infty}^{\infty} 2^{2j\gamma}\hat{\psi}(2^{-j}\xi) \le C_\gamma|\xi|^{2\gamma}.$$

Since $\widehat{\psi_{2^{-j}}}(\xi) = \hat{\psi}(2^{-j}\xi)$, Plancherel's theorem implies

$$\int_{\mathbb{R}^m} (\psi_{2^{-j}} * \nu)(x)\, d\nu(x) = \int \hat{\psi}(2^{-j}\xi)|\hat{\nu}(\xi)|^2\, d\xi$$

and (4.2) follows. Notice that (4.2) is closely related to (2.10). ☐

The following proposition shows how to obtain a dimension bound from a suitable Sobolev estimate. This will depend on Lemma 3.1.

Proposition 4.2. *Let J and ν_λ be as in Definition 2.4. If $\int_I \|\nu_\lambda\|_{2,\gamma}^2\, d\lambda < \infty$ with some $I \subset J$ and $\gamma > -1/2$ then for all $\sigma \in [0 \wedge 2\gamma, 1 + 2\gamma)$ one has* $\dim\{\lambda \in I \,:\, \dim_s(\nu_\lambda) \le \sigma\} \le \sigma - 2\gamma$.

Proof. Let ψ be the Littlewood–Paley function from Lemma 4.1. For any $j = 0, 1, 2, \ldots$ define

$$h_j(\lambda) = 2^{-j} \int \widehat{\psi_{2^{-j}}}(\xi) |\widehat{\nu_\lambda}(\xi)|^2 \, d\xi = \int_\Omega \int_\Omega \psi\Big(2^j \big[\Pi(\lambda, \omega_1) - \Pi(\lambda, \omega_2)\big]\Big) \, d\mu(\omega_1) d\mu(\omega_2). \tag{4.3}$$

Lemma 4.1 implies that

$$C \, \|\nu_\lambda\|_{2,\gamma}^2 \geq \sum_{j=0}^{\infty} 2^{j(1+2\gamma)} h_j(\lambda). \tag{4.4}$$

Moreover, for any $\epsilon > 0$,

$$\{\lambda \in I \, : \, \dim_s(\nu_\lambda) \leq \sigma\} \subset \big\{\lambda \in I \, : \, \sum_{j=0}^{\infty} (2^{\sigma+\epsilon})^j h_j(\lambda) = \infty\big\} \tag{4.5}$$

by (2.1) and Lemma 4.1. Since (2.2) implies that for any $j, \ell = 0, 1, \ldots$

$$|h_j^{(\ell)}(\lambda)| = \left| \frac{d^\ell}{d\lambda^\ell} \int_\Omega \int_\Omega \psi\Big(2^j \big[\Pi(\lambda, \omega_1) - \Pi(\lambda, \omega_2)\big]\Big) \, d\mu(\omega_1) d\mu(\omega_2) \right| \leq C_\ell \, 2^{j\ell} \sum_{i=0}^{l} \big\| \frac{d^i}{d\lambda^i} \Pi \big\|_{L^\infty(I \times \Omega)} \leq C_\ell \, 2^{j\ell},$$

the proposition follows from Lemma 3.1 with $n = 1$, $A = 2$, $r = 2^{\sigma+\epsilon}$, and $R = 2^{2\gamma+1}$ as $\epsilon \to 0+$. $\square$

Proposition 4.4 below deals with the case $\alpha \leq 1$ in Theorems 2.6 and 2.8, see (2.4) and (2.8). In that case it is more efficient to rely on Frostman measures than on the previous proposition. The idea of using Frostman measures appears repeatedly in geometric measure theory, see [31]. First we prove a simple technical lemma that describes the location of the zeros of Φ_λ.

Lemma 4.3. *Let J and Π be as in Definition 2.4. Suppose $J = (\lambda_0, \lambda_1)$ is an interval of transversality of order β for Π and that Π_λ is 1–regular on J. Let C_β be the constant from Definition 2.7,* (2.5) *and set $r = d(\omega_1, \omega_2)$. Then*

$$\{\lambda \in J \, : \, |\Phi_\lambda| < C_\beta \, r^\beta\} = \bigcup_{j=1}^{N_\beta} I_j, \tag{4.6}$$

where I_j are disjoint open intervals of length $C^{-1} r^{2\beta} \leq |I_j| \leq C$, and $N_\beta \leq C r^{2\beta}$. With the possible exception of the at most two intervals touching ∂J each I_j contains a unique zero $\overline{\lambda}_j$ of Φ_λ. If $\lambda_0 \in \overline{I_1}$ and I_1 does not contain a zero of Φ_λ, then we let $\overline{\lambda}_1 = \lambda_0$ and similarly with λ_1. All constants depend only on β and the constants in Definition 2.7.

Proof. Let the intervals I_j be defined by (4.6). Since $C_\beta \, r^\beta \leq |\frac{d}{d\lambda} \Phi_\lambda| \leq C_{\beta,1} \, r^{-\beta}$ on each I_j by Definition 2.7, their lengths are as claimed. The other statements are now clear. $\square$

Proposition 4.4. *Let Ω, J, and Π be as in Definition 2.4. Assume that J is an interval of transversality of order β for Π, that Π_λ is 1–regular on J, and that the measure μ on Ω has finite α–energy for some $\alpha \in (0, 1]$. Then $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$ satisfies*

$$\mathcal{H}^\sigma(\{\lambda \in J \, : \, \dim_s(\nu_\lambda) < \sigma\}) = 0 \tag{4.7}$$

for any $\sigma \in (0, \alpha - 3\beta]$. If J is an interval of strong transversality then (4.7) *holds for all $\sigma \in (0, \alpha]$.*

Proof. Suppose (4.7) fails for some σ. By outer regularity of Hausdorff measure there exists $\epsilon_0 > 0$ so that

$$\mathcal{H}^\sigma(\{\lambda \in J \, : \, \dim_s(\nu_\lambda) \leq \sigma - \epsilon_0\}) > 0.$$

By Frostman's lemma there exists a nonzero measure ρ on J so that $\rho(I) \leq |I|^\sigma$ for all intervals I and

$$\rho(\{\lambda \in J \, : \, \dim_s(\nu_\lambda) > \sigma - \epsilon_0\}) = 0. \tag{4.8}$$

Frostman's lemma can be applied since $\{\lambda \in J \,:\, \dim_s(\nu_\lambda) > \kappa\}$ is an $\mathcal{F}_\sigma$–set for any $\kappa > 0$. Indeed,

$$\{\lambda \in J \,:\, \dim_s(\nu_\lambda) > \kappa\} = \bigcup_{\delta>0} \bigcup_{M=1}^{\infty} \{\lambda \in J \,:\, \int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\xi)|^2\, |\xi|^{\kappa+\delta-1}\, d\xi \le M\}. \tag{4.9}$$

Since

$$|\widehat{\nu_\lambda}(\xi) - \widehat{\nu_{\lambda_0}}(\xi)| \;\le\; \int_\Omega \left| e^{i\xi \Pi_\lambda(\omega)} - e^{i\xi\Pi_{\lambda_0}(\omega)} \right| d\mu(\omega) \le |\xi| \int_\Omega |\Pi_\lambda(\omega) - \Pi_{\lambda_0}(\omega)|\, d\mu(\omega) \le C|\xi||\lambda-\lambda_0|,$$

the sets on the right–hand side of (4.9) are closed, as claimed. Thus for small $\epsilon > 0$ and with $r = d(\omega_1,\omega_2)$

$$\begin{aligned}
&\int_J \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\nu_\lambda(x) d\nu_\lambda(y)}{|x-y|^{\sigma-\epsilon}}\, d\rho(\lambda) = \int_J\int_\Omega\int_\Omega \frac{d\mu(\omega_1)d\mu(\omega_2)}{|\Pi(\lambda,\omega_1)-\Pi(\lambda,\omega_2)|^{\sigma-\epsilon}}\, d\rho(\lambda)\\
&= \int_\Omega\int_\Omega\int_J \chi_{[|\Phi_\lambda| \ge C_\beta\, r^\beta]} |\Phi_\lambda(\omega_1,\omega_2)|^{-(\sigma-\epsilon)}\, d\rho(\lambda) \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{\sigma-\epsilon}}\\
&\quad + \int_\Omega\int_\Omega\int_J \chi_{[|\Phi_\lambda| \le C_\beta\, r^\beta]} |\Phi_\lambda(\omega_1,\omega_2)|^{-(\sigma-\epsilon)}\, d\rho(\lambda) \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{\sigma-\epsilon}} \qquad (4.10)\\
&\le C \int_\Omega\int_\Omega \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{(\sigma-\epsilon)(1+\beta)}} + \int_\Omega\int_\Omega \sum_{j=1}^{N_\beta} C \int \frac{d\rho(\lambda)}{|\lambda-\overline{\lambda}_j|^{\sigma-\epsilon}} \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{(\sigma-\epsilon)(1+\beta)}}\\
&\le C \int_\Omega\int_\Omega \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{\alpha}}, \qquad (4.11)
\end{aligned}$$

since $\sigma + 3\beta \le \alpha$. The sum is obtained by applying Lemma 4.3 to (4.10). However, (4.11) contradicts (4.8) if $\epsilon < \epsilon_0$. The easier case of strong transversality is implicit in the above and is omitted. $\square$

Proof of Theorem 2.6. Fix some compact $I \subset J$ and let $\rho \in C^\infty(\mathbb{R})$ be nonnegative so that $\rho = 1$ on I and $\mathrm{supp}(\rho) \subset J$. Fix distinct $\omega_1, \omega_2 \in \Omega$ and let $r = d(\omega_1,\omega_2)$. Then for any nonnegative integer N

$$\int_{-\infty}^{\infty} e^{i\xi[\Pi(\lambda,\omega_1)-\Pi(\lambda,\omega_2)]}\rho(\lambda)\, d\lambda = \int_{-\infty}^{\infty} e^{i\xi r \Phi_\lambda(\omega_1,\omega_2)}\rho(\lambda)\, d\lambda = \int_{-\infty}^{\infty} e^{i\xi r\Phi_\lambda(\omega_1,\omega_2)} L^N(\rho)(\lambda)\, d\lambda \tag{4.12}$$

where L is the differential operator $L(\cdot) = \frac{d}{d\lambda}((-i\xi r \Phi'_\lambda(\omega_1,\omega_2))^{-1}\, \cdot\,)$. It follows from Definition 2.5 that $|L^k(f)| \le C_k\, |\xi r|^{-k} \sum_{i=0}^k \|f^{(i)}\|_\infty$ for all k. In particular, applying (4.12) with $N = 0$ and $N \ge \alpha$ yields

$$\left| \int_{-\infty}^{\infty} e^{i\xi[\Pi(\lambda,\omega_1)-\Pi(\lambda,\omega_2)]}\rho(\lambda)\, d\lambda \right| \le C_\alpha\, (|\xi| d(\omega_1,\omega_2))^{-\alpha}.$$

Therefore

$$\int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\xi)|^2 \rho(\lambda)\, d\lambda = \int_\Omega\int_\Omega\int_{-\infty}^{\infty} e^{i\xi[\Pi(\lambda,\omega_1)-\Pi(\lambda,\omega_2)]}\rho(\lambda)\, d\lambda\, d\mu(\omega_1)d\mu(\omega_2) \le C|\xi|^{-\alpha} \int_\Omega\int_\Omega \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^\alpha}.$$

This proves (2.3) and also that $\int_I \|\nu_\lambda\|^2_{2,(\alpha-1)/2-\epsilon}\, d\lambda < \infty$ for any small $\epsilon > 0$. The dimension bound (2.4) in case $\alpha \ge 1$ thus follows from Proposition 4.2, whereas (2.4) with $\alpha \le 1$ is covered by Proposition 4.4. $\square$

In view of Propositions 4.2 and 4.4, Theorem 2.8 will follow from the Sobolev estimate (2.7). This will be proved using the characterization (4.2). The main technical statement is Lemma 4.6 below. The following routine calculus lemma will be needed in the proof of that lemma.

Lemma 4.5. *Let L be a positive integer and suppose $h \in C^L(I)$ satisfies $h' \ne 0$ on some interval $I \subset \mathbb{R}$. Let H denote the inverse of h. Then for any positive integer $\ell \le L$*

$$H^{(\ell)} = \sum_{t=0}^{\ell-1} \sum_{\vec{\ell},\vec{p}} a_{\vec{\ell},\vec{p}}\, (h'\circ H)^{-(\ell+p_1+\ldots+p_t)} (h^{(\ell_1)}\circ H)^{p_1} \cdot \ldots \cdot (h^{(\ell_t)}\circ H)^{p_t} \tag{4.13}$$

with some integer coefficients $a_{\vec{\ell},\vec{p}}$. The inner sum runs over vectors $\vec{\ell} = (\ell_1, \ldots, \ell_t)$ and $\vec{p} = (p_1, \ldots, p_t)$ with integer coordinates. Moreover, if $\ell \geq 2$ one has $a_{\vec{\ell},\vec{p}} = 0$ unless $\ell \geq \ell_i \geq 2$ for all i, $\min p_j \geq 1$, and $\ell_1 p_1 + \ldots + \ell_t p_t \leq 2(\ell - 1)$.

Proof. If $\ell = 1$ then (4.13) holds with $t = 0$. The general case now follows easily by induction. □

Lemma 4.6. *Let J and Π be as in Definition 2.4. Assume that J is an interval of transversality of order β for Π and that Π_λ is ∞–regular on J. Suppose $\rho \in C^\infty(\mathbb{R})$ is supported on J. Let ψ be the Littlewood–Paley function from Lemma 4.1. Then for any distinct $\omega_1, \omega_2 \in \Omega$, any integer j, and any positive integer q,*

$$\left| \int_{-\infty}^{\infty} \rho(\lambda) \psi\big(2^j [\Pi(\lambda, \omega_1) - \Pi(\lambda, \omega_2)]\big) \, d\lambda \right| \leq C_q \, (1 + 2^j d(\omega_1, \omega_2)^{1 + a_0 \beta})^{-q} \tag{4.14}$$

where C_q depends only on q, ρ, and β and a_0 is some absolute constant.

Proof. Fix distinct $\omega_1, \omega_2 \in \Omega$ and j, q as above. We may assume that $2^j r > 1$ where $r = d(\omega_1, \omega_2)$. Let $\phi \in C^\infty$ be nonnegative with $\phi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\mathrm{supp}(\phi) \subset [-1, 1]$. Then

$$\begin{aligned} &\int_{-\infty}^{\infty} \rho(\lambda) \psi\big(2^j [\Pi(\lambda, \omega_1) - \Pi(\lambda, \omega_2)]\big) \, d\lambda \\ &= \int \rho(\lambda) \psi(2^j r \Phi_\lambda(\omega_1, \omega_2)) \phi(C_\beta^{-1} \, r^{-\beta} \Phi_\lambda) \, d\lambda + \int \rho(\lambda) \psi(2^j r \Phi_\lambda(\omega_1, \omega_2)) [1 - \phi(C_\beta^{-1} \, r^{-\beta} \Phi_\lambda)] \, d\lambda. \end{aligned} \tag{4.15}$$

Here C_β is the constant from Definition 2.7. By the rapid decay of ψ

$$\left| \int \rho(\lambda) \psi\big(2^j r \Phi_\lambda(\omega_1, \omega_2)\big) [1 - \phi(C_\beta^{-1} \, r^{-\beta} \Phi_\lambda)] \, d\lambda \right| \leq C_{q,\beta} \int |\rho(\lambda)| (1 + 2^j r^{1+\beta})^{-q} \, d\lambda \leq C_{q,\beta} (1 + 2^j r^{1+\beta})^{-q}.$$

Thus it suffices to estimate the first integral in (4.15). By Lemma 4.3 there exists $\chi \in C^\infty(\mathbb{R})$ depending only on β so that $\mathrm{supp}(\chi)$ is compact, $\chi = 1$ on a neighborhood of the origin, and such that

$$\chi\big(r^{-2\beta}(\lambda - \overline{\lambda}_j)\big) = \chi\big(r^{-2\beta}(\lambda - \overline{\lambda}_j)\big) \phi(C_\beta^{-1} \, r^{-\beta} \Phi_\lambda),$$

where these functions have disjoint supports for distinct j. We can therefore write the first integral in (4.15) as follows.

$$\begin{aligned} \int \rho(\lambda) \psi(2^j r \Phi_\lambda) \phi(C_\beta^{-1} \, r^{-\beta} \Phi_\lambda) \, d\lambda &= \sum_{i=1}^{N_\beta} \int \rho(\lambda) \chi\big(r^{-2\beta}(\lambda - \overline{\lambda}_j)\big) \psi(2^j r \Phi_\lambda) \, d\lambda \\ + \int \rho(\lambda) \Big[1 - \sum_{i=1}^{N_\beta} \chi\big(r^{-2\beta}(\lambda - \overline{\lambda}_j)\big)\Big] \psi(2^j r \Phi_\lambda) \phi(C_\beta^{-1} \, r^{-\beta} \Phi_\lambda) \, d\lambda &= \sum_{i=1}^{N_\beta} A^{(j)} + B. \end{aligned}$$

To estimate the second term B notice that $|\Phi_\lambda| \geq C \, r^{3\beta}$ on the support of the integrand. The rapid decay of ψ therefore implies

$$|B| \leq C_{q,\beta} (1 + 2^j r^{1+3\beta})^{-q}.$$

For simplicity, we assume henceforth that $i = 1$ and set $\overline{\lambda} = \overline{\lambda}_1$. By choice of χ one has $|\frac{d}{d\lambda} \Phi_\lambda| \geq C_\beta \, r^\beta$ on the support of the integrand of $A^{(1)}$. In order to change variables in $A^{(1)}$ we define H via

$$\Phi_\lambda = u \Longleftrightarrow \lambda = \overline{\lambda} + H(u) \ \text{ provided } \chi\big(r^{-2\beta}(\lambda - \overline{\lambda})\big) \neq 0. \tag{4.16}$$

Let $F(u) = \rho(\overline{\lambda} + H(u))\chi(r^{-2\beta}H(u))H'(u)$. Thus

$$
\begin{aligned}
A^{(1)} &= \int F(u)\psi(2^j ru)\,du \\
&= \int_{|u|<(2^jr)^{-\frac{1}{2}}} \psi(2^jru)\left\{\sum_{\ell=0}^{2(q-1)} \frac{F^{(\ell)}(0)}{\ell!}u^\ell + O(F^{(2q-1)}(u)u^{2q-1})\right\} du \qquad (4.17)\\
&\quad + \int_{|u|>(2^jr)^{-\frac{1}{2}}} O\Big((2^jr|u|)^{-2q-1}\Big)|F(u)|\,du \;= A_1^{(1)} + A_2^{(1)}.
\end{aligned}
$$

$|H'(u)| \le C_\beta\, r^{-\beta}$ by the choice of χ and thus $|F(u)| \le C\, r^{-\beta}$. In particular, $|A_2^{(1)}| \le C_{\beta,q}(2^jr)^{-q-1}r^{-\beta}$. Since ψ has vanishing moments of all orders

$$
A_1^{(1)} = -\int_{|u|>(2^jr)^{-\frac{1}{2}}} \psi(2^jru)\sum_{\ell=0}^{2(q-1)} \frac{F^{(\ell)}(0)}{\ell!}u^\ell\,du + O\Big(\|F^{(2q-1)}\|_\infty (2^jr)^{-q}\Big).
$$

It remains to estimate $F^{(\ell)}(u)$. Suppose λ and u are related via (4.16). By Lemma 4.5 (with $L=\infty$) and (2.6) in Definition 2.7,

$$
|H^{(\ell)}(u)| \le C_{\beta,\ell}\sum_{t=0}^{\ell-1}\sum_{\vec{\ell},\vec{p}} |a_{\vec{\ell},\vec{p}}|\,|\Phi_\lambda'|^{-(\ell+p_1+\ldots+p_t)}\, r^{-\beta(\ell_1p_1+\ldots+\ell_tp_t)} \le C_{\beta,\ell}\, r^{-\beta(4\ell-3)}.
$$

Therefore, by Leibnitz's rule,

$$
\|F^{(\ell)}\|_\infty \le C_{\beta,\ell}\big(r^{-2\beta\ell}r^{-\beta(4\ell-3)} + r^{-\beta(4l+1)}\big) \le C_{\beta,\ell}\, r^{-6\beta\ell}.
$$

Thus

$$
|A_1^{(1)}| \le C_{\beta,q}\sum_{\ell=0}^{2(q-1)} r^{-6\beta\ell}\int_{(2^jr)^{-\frac{1}{2}}}^\infty (2^jru)^{-2q-\ell-1}u^\ell\,du + O\big(r^{-6(2q-1)\beta}(2^jr)^{-q}\big) \le C_{\beta,q}\, r^{-6(2q-1)\beta}(2^jr)^{-q}.
$$

Since $N_\beta \le C_\beta\, r^{-2\beta}$, one finally obtains (4.14) with $a_0 = 12$. $\square$

Proof of (2.7). Fix some γ with $0 < (1+2\gamma)(1+a_0\beta) \le \alpha$. Let ρ be a smooth nonnegative function on the line so that $\mathrm{supp}(\rho) \subset J$. Fix any $q > 1+2\gamma$. In view of (4.2), the definition of ν_λ, and Lemma 4.6

$$
\begin{aligned}
\int_{-\infty}^\infty \|\nu_\lambda\|_{2,\gamma}^2\,\rho(\lambda)\,d\lambda &\asymp \int \sum_{j=-\infty}^\infty 2^{2j\gamma}\int_{-\infty}^\infty (\psi_{2^{-j}} * \nu_\lambda)(x)\,d\nu_\lambda(x)\;\rho(\lambda)\,d\lambda \\
&\le \int_\Omega\int_\Omega \sum_{j=-\infty}^\infty 2^{j(1+2\gamma)}\left|\int \psi\big(2^j[\Pi(\lambda,\omega_1)-\Pi(\lambda,\omega_2)]\big)\rho(\lambda)\,d\lambda\right| d\mu(\omega_1)d\mu(\omega_2) \\
&\le C_{\beta,\gamma}\int_\Omega\int_\Omega \sum_{-\infty}^\infty 2^{j(1+2\gamma)}(1+2^jd(\omega_1,\omega_2)^{1+a_0\beta})^{-q}\,d\mu(\omega_1)d\mu(\omega_2) \qquad (4.18)\\
&\le C_{\beta,\gamma}\int_\Omega\int_\Omega \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{(1+a_0\beta)(1+2\gamma)}} \le C_{\beta,\gamma}\,\mathcal{E}_\alpha(\mu) < \infty,
\end{aligned}
$$

as claimed. $\square$

4.2. **The case of limited regularity.** We now discuss the case where $\Pi : J\times\Omega \to \mathbb{R}$ only has a finite degree of smoothness as a mapping $\lambda \mapsto \Pi_\lambda$.

Definition 4.7. Let Ω, J be as in Definition 2.4. Suppose $\Pi : J\times\Omega \to \mathbb{R}$ is continuous and let L be a positive integer and $\delta\in[0,1)$. We write $\Pi_\lambda \in C^{L,\delta}(J)$ if the following conditions are satisfied: Given any compact $I\subset J$ we assume that the bounds (2.2) hold for all $\ell = 0,1,\ldots,L$ and that

$$\Big|\frac{d^L}{d\lambda^L}\Pi(\lambda_1,\omega) - \frac{d^L}{d\lambda^L}\Pi(\lambda_2,\omega)\Big| \le C_{\delta,I}|\lambda_1-\lambda_2|^\delta \quad \text{for all} \quad \lambda_1,\lambda_2\in I \quad \text{and} \quad \omega\in\Omega \tag{4.19}$$

with some suitable constant $C_{I,\delta}$.

The following definition is very similar to Definition 2.7, only here we also allow for the slightly more general case of Hölder continuity.

Definition 4.8. Suppose $\Pi_\lambda \in C^{L,\delta}(J)$ as in 4.7 and let $\beta\in(0,1)$. We say that J is an **interval of transversality of order** β **for** Π if there exists a constant C_β so that for all $\lambda_1,\lambda_2\in J$ and $\omega_1,\omega_2\in\Omega$ the condition

$$|\Phi_{\lambda_1}(\omega_1,\omega_2)| + |\Phi_{\lambda_2}(\omega_1,\omega_2)| \le C_\beta\, d(\omega_1,\omega_2)^\beta$$

implies that

$$\Big|\frac{d}{d\lambda}\Phi_{\lambda_1}\Big| \ge C_\beta\, d(\omega_1,\omega_2)^\beta. \tag{4.20}$$

In addition, we say that Π_λ is L,δ–**regular on** J if under the same condition, with some constants $C_{\beta,\ell}, C_{\beta,L,\delta}$,

$$\begin{aligned} \Big|\frac{d^\ell}{d\lambda^\ell}\Phi_{\lambda_1}(\omega_1,\omega_2)\Big| &\le C_{\beta,\ell}\, d(\omega_1,\omega_2)^{-\beta\ell} \quad \text{for} \quad \ell=1,2,\ldots,L \quad \text{and} \\ \Big|\frac{d^L}{d\lambda^L}\Phi_{\lambda_1}(\omega_1,\omega_2) - \frac{d^L}{d\lambda^L}\Phi_{\lambda_2}(\omega_1,\omega_2)\Big| &\le C_{\beta,L,\delta}\,|\lambda_1-\lambda_2|^\delta\, d(\omega_1,\omega_2)^{-\beta(L+\delta)}. \end{aligned} \tag{4.21}$$

Under these conditions one has the following analogue of Theorem 2.8. Since Lemma 4.3 and Proposition 4.4 only require that Π_λ is 1–regular on J, we restrict ourselves to a discussion of the Sobolev estimate (2.7) and the dimension bound (2.8) from Theorem 2.8. Notice that the following statements reduce to those estimates if $L=\infty$.

Theorem 4.9. *Let $\Pi : J\times\Omega\to\mathbb{R}$ be as in Definition 4.7. Assume that J is an interval of transversality of order β for some $\beta\in[0,1)$ in the sense of Definition 4.8 and that Π_λ is L,δ–regular on J with $L+\delta>1$. Let μ be a finite positive measure on Ω with finite α–energy for some $\alpha>1$. Then on any compact $I\subset J$ the family of measures $\nu_\lambda = \mu\circ\Pi_\lambda^{-1}$ satisfies*

$$\int_I \|\nu_\lambda\|_{2,\gamma}^2\, d\lambda \le C_\gamma\, \mathcal{E}_\alpha(\mu) \quad \textit{provided} \quad 0<1+2\gamma<L+\delta \quad \textit{and} \quad 1+2\gamma \le \frac{\alpha}{1+a_0\beta}, \tag{4.22}$$

where a_0 is some absolute constant. Moreover, if $\sigma\in(0,1]$, then

$$\dim\{\lambda\in J \,:\, \dim_s(\nu_\lambda)\le\sigma\} \le 1+\sigma-\min\Big(L+\delta, \frac{\alpha}{1+a_0\beta}\Big). \tag{4.23}$$

If $1<\sigma\le\alpha$, then

$$\dim\{\lambda\in J \,:\, \dim_s(\nu_\lambda)\le\sigma\} \le 1-\Big(\min\big(L+\delta, \frac{\alpha}{1+a_0\beta}\big)-\sigma\Big)\Big(1+\frac{\sigma-1}{L+\delta}\Big)^{-1}. \tag{4.24}$$

Finally, if $\sigma\in(0,\alpha-3\beta]$, then

$$\dim\{\lambda\in J \,:\, \dim_s(\nu_\lambda)<\sigma\} \le \sigma.$$

As already mentioned above, the final statement holds since Proposition 4.4 requires only C^1 parameterizations. As in the C^∞–case, we will exploit the Sobolev bound in order to obtain the dimension estimates (4.23) and (4.24). This will be accomplished by means of the following proposition which is based on Lemma 3.2, cf. Proposition 4.2.

Proposition 4.10. *Let Ω, J and $\Pi : J \times \Omega \to \mathbb{R}$ be as in Definition 4.7 for some positive integer L and $\delta \in [0,1)$. Suppose ν_λ as in 4.7 satisfies $\int_I \|\nu_\lambda\|^2_{2,\gamma}\, d\lambda < \infty$ with some $I \subset J$ and $\gamma > -1/2$.*

((i)) *If $\sigma \in [0 \wedge 2\gamma, 1 \wedge (1+2\gamma)]$, then*

$$\dim \big\{\lambda \in I \,:\, \dim_s(\nu_\lambda) \le \sigma\big\} \le \sigma - 2\gamma. \tag{4.25}$$

((ii)) *If $\sigma \in [1, 1+2\gamma)$, then*

$$\dim \big\{\lambda \in I \,:\, \dim_s(\nu_\lambda) \le \sigma\big\} \le 1 - (1+2\gamma-\sigma)\Big(1 + \frac{\sigma-1}{L+\delta}\Big)^{-1}. \tag{4.26}$$

Proof. Let $\{h_j\}_{j=1}^\infty$ be defined as in (4.3) and set $h_0 = 1$. We will verify the hypotheses of Lemma 3.2 with $A = 2$ and $H_j = \sum_{i=0}^j 2^{i-j} h_i$. Recall that the Littlewood–Paley function ψ is given by $\hat\psi(\xi) = \hat\phi(\xi/2) - \hat\phi(\xi)$ where $\hat\phi \ge 0$, $\hat\phi(\xi) = 1$ for $|\xi| \le 1$ and $\hat\phi(\xi) = 0$ for $|\xi| > 2$; see the proof of Lemma 4.1. In particular,

$$\begin{aligned} 2^j H_j(\lambda) &= 1 + \int_{-\infty}^{\infty} \big[\hat\phi(2^{-j-1}\xi) - \hat\phi(\xi)\big] |\widehat{\nu_\lambda}(\xi)|^2\, d\xi \\ &= 1 + \int_\Omega\int_\Omega \Big\{ 2^{j+1}\phi\Big(2^{j+1}\big[\Pi_\lambda(\omega) - \Pi_\lambda(\tau)\big]\Big) - \phi\Big(\big[\Pi_\lambda(\omega) - \Pi_\lambda(\tau)\big]\Big)\Big\}\, d\mu(\omega)\, d\mu(\tau). \end{aligned} \tag{4.27}$$

Let $\chi \in \mathcal{S}(\mathbb{R})$ satisfy $|\phi'| \le \chi$ (for example, set $\chi(x) = C\int_{-\infty}^\infty \max_{|y-z|\le 1} |\phi'(z)|\, \eta(x-y)\, dy$ where $\eta \in C^\infty$ is a standard bump function). Thus

$$\begin{aligned} |H_j'(\lambda)| &\le C + C\, 2^{j+1} \int_\Omega\int_\Omega \chi\Big(2^{j+1}\big[\Pi_\lambda(\omega) - \Pi_\lambda(\tau)\big]\Big)\Big(|\Pi_\lambda'(\omega)| + |\Pi_\lambda'(\tau)|\Big)\, d\mu(\omega)\, d\mu(\tau) \\ &\le C + C\int_{-\infty}^\infty \hat\chi(2^{-j-1}\xi) |\widehat{\nu_\lambda}(\xi)|^2\, d\xi \le C\, 2^j H_j(\lambda). \end{aligned} \tag{4.28}$$

To obtain the final inequality in (4.28) notice that

$$\hat\chi(2^{-j-1}\xi) \le \|\hat\chi\|_\infty\, \hat\phi(2^{-j-1}\xi) \quad \text{if} \quad |\xi| \le 2^{j+1}.$$

Hence (4.28) follows from (4.27) and the rapid decay of $\hat\chi$. In view of (4.4) and (4.5), inequality (4.25) follows from (3.8) in Lemma 3.2 with $n = 1$, $A = B = 2$, $R = 2^{1+2\gamma}$, and $r = 2^{\sigma+\epsilon}$ as $\epsilon \to 0+$.

Estimate (4.26) follows from Lemma 3.2 (ii) with the same choice of parameters. It remains to verify the Hölder estimate (3.9). For simplicity we will only carry this out for $L = 1$. The general case is only slightly more involved technically. Let $\Pi_\lambda(\omega,\tau) = \Pi_\lambda(\omega) - \Pi_\lambda(\tau)$. H_j as defined above clearly satisfies

$$|H_j'(\lambda_1) - H_j'(\lambda_2)| \le C + 2^{j+1}\int_\Omega\int_\Omega \big|\phi'\big(2^{j+1}\Pi_{\lambda_1}(\omega,\tau)\big)\big|\ \big|\Pi_{\lambda_1}'(\omega,\tau) - \Pi_{\lambda_2}'(\omega,\tau)\big|\, d\mu(\omega)\, d\mu(\tau) \tag{4.29}$$

$$+ 2^{j+1}\int_\Omega\int_\Omega \big|\phi'\big(2^{j+1}\Pi_{\lambda_1}(\omega,\tau)\big) - \phi'\big(2^{j+1}\Pi_{\lambda_2}(\omega,\tau)\big)\big|\ \big|\Pi_{\lambda_2}'(\omega,\tau)\big|\, d\mu(\omega)\, d\mu(\tau). \tag{4.30}$$

In view of (4.19) the same argument as in the first part of the proof shows that right–hand side of (4.29) is bounded by $C\,2^j H_j(\lambda_1)|\lambda_1 - \lambda_2|^\delta$. If $|\lambda_1 - \lambda_2| > 2^{-j}$, then the integral in (4.30) can be again bounded by

$$C\,2^j\big(H_j(\lambda_1) + H_j(\lambda_2)\big) \le C\,2^{j(1+\delta)}|\lambda_1 - \lambda_2|^\delta\big(H_j(\lambda_1) + H_j(\lambda_2)\big).$$

It remains to estimate (4.30) if $|\lambda_1-\lambda_2|\le 2^{-j}$. As above one constructs $\chi_1\in\mathcal{S}$ such that $|\phi''|\le\chi_1$ and $C\chi_1(x)\ge\max_{|x-y|\le 1}\chi_1(y)$ for all $x\in\mathbb{R}$. The integral in (4.30) is therefore bounded by

$$\begin{aligned}
&C\,2^{2j}\int_\Omega\int_\Omega\int_0^1\Big|\phi''\big(2^{j+1}\Pi_{\lambda_1}(\omega,\tau)+2^{j+1}t(\Pi_{\lambda_2}-\Pi_{\lambda_1})(\omega,\tau)\big)\Big|\,dt\,\big|\big(\Pi_{\lambda_2}-\Pi_{\lambda_1}\big)(\omega,\tau)\big|\,d\mu(\omega)\,d\mu(\tau)\\
&\le 2^{2j}\,|\lambda_2-\lambda_1|\int_\Omega\int_\Omega\chi_1\big(2^{j+1}\Pi_{\lambda_1}(\omega,\tau)\big)\,d\mu(\omega)\,d\mu(\tau)=C\,2^j|\lambda_2-\lambda_1|\int_{-\infty}^{\infty}\widehat{\chi_1}(2^{-j-1}\xi)|\widehat{\nu_\lambda}(\xi)|^2\,d\xi \qquad (4.31)\\
&\le C\,2^{2j}|\lambda_2-\lambda_1|\,H_j(\lambda_1)\le C\,2^{j(1+\delta)}|\lambda_2-\lambda_1|^\delta\,H_j(\lambda_1).
\end{aligned}$$

To pass to (4.31) one uses that $2^j|\Pi_{\lambda_1}(\omega,\tau)-\Pi_{\lambda_2}(\omega,\tau)|\le C2^j|\lambda_2-\lambda_1|\le C$ and the properties of χ_1. □

Proof of Theorem 4.9. By Proposition 4.10 the dimension bounds (4.23) and (4.24) follow from (4.25) and (4.26), respectively, if (4.22) holds. (4.18) shows that this is the case as soon as the inequality (4.14) is valid for some (real) $q>1+2\gamma$. In fact, inspection of the proof of Lemma 4.6 reveals that under the conditions of Definition 4.8 one has (4.14) for any $q<L+\delta$. More precisely, the only term in the proof of Lemma 4.6 that does not involve the rapid decay of ψ is the O–part of $A_1^{(1)}$, see (4.17). Thus the proof carries over unchanged up to (4.16). Now fix a small $\epsilon>0$ and some $M>0$. Since $F\in C^{L-1,\delta}$, as can be seen from the definition of F, Definition 4.7 and Lemma 4.5,

$$\begin{aligned}
A^{(1)}&=\int F(u)\psi(2^jru)\,du\\
&=\int_{|u|<(2^jr)^{-1+\epsilon}}\psi(2^jru)\left\{\sum_{\ell=0}^{L-1}\frac{F^{(\ell)}(0)}{\ell!}u^\ell+\int_0^1\big[F^{(L-1)}(tu)-F^{(L-1)}(0)\big]\frac{(1-t)^{L-1}}{(L-1)!}\,dt\,u^{L-1}\right\}du \qquad (4.32)\\
&+\int_{|u|>(2^jr)^{-1+\epsilon}}O\Big((2^jr|u|)^{-M}\Big)|F(u)|\,du\;=A_1^{(1)}+A_2^{(1)}.
\end{aligned}$$

Taking $M=M(\epsilon,q)$ large enough one obtains $|A_2^{(1)}|\le C_{\beta,q}(2^jr)^{-q}r^{-\beta}$ for any $q>0$. Estimating the contribution of the Taylor polynomial in (4.32) again involves the rapid decay of ψ. To bound the error term in (4.32) one uses the Hölder condition on $F^{(L-1)}$.

$$\begin{aligned}
&\Big|\int_{|u|<(2^jr)^{-1+\epsilon}}\psi(2^jru)\int_0^1\big[F^{(L-1)}(tu)-F^{(L-1)}(0)\big]\frac{(1-t)^{L-1}}{(L-1)!}\,dt\,u^{L-1}\,du\Big|\\
\le\;&C_{\epsilon,L}\,r^{-CL\beta}\int_{|u|<(2^jr)^{-1+\epsilon}}|u|^{L-1+\delta}\,du\le C_{\epsilon,L}(2^jr^{1+C\beta})^{-(1-\epsilon)(L+\delta)}.
\end{aligned}$$

Since $\epsilon>0$ is arbitrary, (4.14) holds for any $q<L+\delta$, as claimed. □

5. Applications

5.1. **Classical Bernoulli convolutions.** Recall that ν_λ is the distribution of $\sum_0^\infty\pm\lambda^n$. Here we show that the density of ν_λ has fractional derivatives in L^2 for almost all $\lambda\in(\frac12,1)$ and we estimate the dimension of those λ so that ν_λ is singular with respect to Lebesgue measure. First we recall the definition of δ–transversality from [34].

Definition 5.1. Let $\delta>0$. We say that $J\subset\mathbb{R}$ is an interval of δ–transversality for the class of power series

$$g(x)=1+\sum_{n=1}^\infty b_nx^n,\quad\text{with}\quad b_n\in\{-1,0,1\} \qquad (5.1)$$

if $g(x)<\delta$ implies $g'(x)<-\delta$ for any $x\in J$.

Lemma 5.3 establishes the connection between Definitions 2.7 and 5.1. A useful criterion for checking δ–transversality was found in [34]. A power series $h(x)$ is called a $(*)$–function if for some $k \geq 1$ and $a_k \in [-1, 1]$,

$$h(x) = 1 - \sum_{i=1}^{k-1} x^i + a_k x^k + \sum_{i=k+1}^{\infty} x^i.$$

In [39] Solomyak showed that among all convex combinations of series of the form (5.1), the power series with the smallest double zero must be a $(*)$–function. The following lemma from [34] bypasses this fact and reduces the search for intervals of transversality to finding a suitable $(*)$–function.

Lemma 5.2. *Suppose that a $(*)$–function h satisfies*

$$h(x_0) > \delta \quad \textit{and} \quad h'(x_0) < -\delta$$

for some $x_0 \in (0, 1)$ and $\delta \in (0, 1)$. Then $[0, x_0]$ is an interval of δ–transversality for the class of power series (5.1) .

In [34] a particular $(*)$–function was found that satisfies $h(2^{-2/3}) > 0.07$ and $h(2^{-2/3}) < -0.09$, so transversality in the sense of 5.1 holds on $[0, 2^{-2/3}]$ by this lemma. On the other hand, in [39] Solomyak proved that there is a power series of the form (5.1) with a double zero at roughly 0.68, whereas $2^{-2/3} \simeq 0.63$. We will return to this issue below.

It is easy to see that Bernoulli convolutions are a special case of the general results in section 2. Indeed, let $\Omega = \{-1, +1\}^{\mathbb{N}}$ be equipped with the product measure $\mu = \prod_0^\infty (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)$. For any distinct $\omega, \tau \in \Omega$ we define

$$|\omega \wedge \tau| = \min\{i \geq 0 \;:\; \omega_i \neq \tau_i\}.$$

Fix some interval $J = (\lambda_0, \lambda_1) \subset (0, 1)$ and define $\Pi : J \times \Omega \to \mathbb{R}$ via $\Pi_\lambda(\omega) = \sum_{n=0}^\infty \omega_n \lambda^n$. The metric on Ω (depending on J) is given by $d(\omega, \tau) = \lambda_1^{|\omega \wedge \tau|}$. One checks that $\mathcal{E}_\alpha(\mu) < \infty$ if and only if $\lambda_1^\alpha > \frac{1}{2}$.

Lemma 5.3. *Suppose $J = (\lambda_0, \lambda_1)$ is an interval of δ–transversality. Then J is an interval of transversality of order β for Π if $\lambda_0 > \lambda_1^{1+\beta}$.*

Proof. Since $d(\omega_1, \omega_2) = \lambda_1^{|\omega_1 \wedge \omega_2|}$, $\Phi_\lambda(\omega_1, \omega_2) = 2\frac{\lambda^k}{\lambda_1^k} g(\lambda)$ with $k = |\omega_1 \wedge \omega_2|$ and a power series g of the form (5.1). Therefore,

$$|\Phi_\lambda^{(\ell)}| \leq C_\ell \left(k^\ell \frac{\lambda^{k-\ell}}{\lambda_1^k} \|g\|_\infty + \frac{\lambda^k}{\lambda_1^k} \|g^{(\ell)}\|_\infty \right) \leq C_\ell (k^\ell \lambda_0^{-\ell} (1 - \lambda_1)^{-1} + (1 - \lambda_1)^{-\ell - 1}) \leq C_{\ell,\beta}\, \lambda_1^{-\beta l k},$$

since $\lambda_1 < 1$ and $\beta > 0$. Thus condition (2.6) in 2.7 holds. To check (2.5) assume $|\Phi_\lambda| \leq \delta b_\beta \lambda_1^{\beta k}$ where the constant $b_\beta \in (0, 1)$ will be determined below. Then

$$2 \left(\frac{\lambda_0}{\lambda_1} \right)^k |g(\lambda)| \leq 2 \left(\frac{\lambda}{\lambda_1} \right)^k |g(\lambda)| \leq \delta b_\beta \lambda_1^{\beta k}$$

implies $|g(\lambda)| \leq \frac{1}{2} \delta b_\beta (\lambda_0^{-1} \lambda_1^{1+\beta})^k$. Hence

$$|\Phi_\lambda'| \geq 2\big(\lambda_0 \lambda_1^{-1}\big)^k \big(|g'(\lambda)| - \frac{k}{\lambda_0} |g(\lambda)|\big) \geq 2\lambda_1^{\beta k} \left(\delta - \delta b_\beta \frac{k}{2\lambda_0} (\lambda_0^{-1} \lambda_1^{1+\beta})^k \right) \geq \delta \lambda_1^{\beta k}$$

provided $b_\beta \leq \frac{1}{2} \big[1 + \sup_{k \geq 0} \frac{k}{2\lambda_0} (\lambda_0^{-1} \lambda_1^{1+\beta})^k \big]^{-1}$. Thus condition (2.5) in Definition 2.7 holds with $C_\beta = \delta b_\beta$. ∎

Theorem 2.8 now implies the following theorem.

Theorem 5.4. *Suppose* $J = [\lambda_0, \lambda_0'] \subset (\frac{1}{2}, 1)$ *is an interval of* δ*–transversality for the power series (5.1). Then* $\dim_s(\nu_\lambda) \ge \frac{\log 2}{-\log\lambda}$ *for a.e.* $\lambda \in J$. *Furthermore,*

$$\dim\{\lambda \in J \,:\, \widehat{\nu_\lambda} \notin L^2(\mathbb{R})\} \le 2 - \frac{\log 2}{-\log\lambda_0}.$$

Proof. Fix any small $\beta > 0$ and partition J into subintervals $J_i = [\lambda_i, \lambda_{i+1}]$ for $i = 0, \ldots, m$ so that $\lambda_i \ge (1+\beta)\lambda_{i+1}^{1+\beta}$. By the previous lemma all J_i are intervals of transversality of order β for Π. Notice that the metric on Ω depends on i, in fact $d_i(\omega_1, \omega_2) = \lambda_{i+1}^{|\omega_1\wedge\omega_2|}$. In particular, μ has finite α_i–energy with respect to d_i if $\lambda_{i+1}^{\alpha_i} > \frac{1}{2}$. Theorem 2.8 therefore implies that

$$\dim_s(\nu_\lambda) \ge \frac{\log 2}{-\log\lambda_{i+1}}(1+a_0\beta)^{-1} \quad \text{for a.e. } \lambda \in J_i$$

and

$$\dim\{\lambda \in J_i \,:\, \widehat{\nu_\lambda} \notin L^2(\mathbb{R})\} \quad \le \quad 2 - \frac{\log 2}{-\log\lambda_{i+1}}(1+a_0\beta)^{-1} \le 2 - \frac{\log 2}{-\log\lambda_0}(1+a_0\beta)^{-1}.$$

Letting $\beta \to 0+$ finishes the proof. □

It is well–known that for $0 < \lambda < \frac{1}{2}$ the support of ν_λ is a Cantor set of dimension $\frac{\log 2}{-\log\lambda}$. In fact, ν_λ is a Frostman measure on that set which implies that $\dim_s(\nu_\lambda) = \frac{\log 2}{-\log\lambda}$ for $0 < \lambda < \frac{1}{2}$. In [39] Solomyak showed that the first double zero for a power series of the form (5.1) lies in the interval $[0.649, 0.683]$. In particular, the previous theorem will apply only up to some point in this interval. It seems natural to conjecture that $\dim_s(\nu_\lambda) \ge \frac{\log 2}{-\log\lambda}$ for a.e. $\lambda \in (\frac{1}{2}, 1)$, but our methods do not yield this estimate. Nevertheless, one can show that ν_λ has some smoothness for a.e. $\lambda \in (\frac{1}{2}, 1)$. This follows from Theorem 5.4 by "thinning and convolving", see [39] and [34]. As one expects, the number of derivatives tends to ∞ as $\lambda \to 1$.

Lemma 5.5. *For any* $\epsilon > 0$ *there exists a* $\gamma = \gamma(\epsilon) > 0$ *so that*

$$\int_{\frac{1}{2}+\epsilon}^{\frac{1}{\sqrt{2}}} \|\nu_\lambda\|_{2,\gamma}^2 \, d\lambda < \infty. \tag{5.2}$$

Furthermore, there exists some positive constant γ_0 *so that*

$$\int_{2^{-2^{-k}}}^{2^{-2^{-k-1}}} \left(\|\nu_\lambda\|_{2,2^k\gamma_0}^{2^{-k+1}} + \|\nu_\lambda\|_{2,\gamma_0}^2\right) \, d\lambda < \infty \quad \textit{for } k = 1, 2, \ldots. \tag{5.3}$$

Proof. As mentioned above, $[0, \lambda_1]$ is an interval of transversality for the power series (5.1) for some $\lambda_1 > 2^{-2/3}$. Fix any $\lambda_0 \in (\frac{1}{2}, 2^{-2/3}]$. Partitioning the interval $[\lambda_0, \lambda_1]$ as in the proof of Theorem 5.4, one obtains from (2.7) that

$$\int_{\lambda_0}^{\lambda_1} \|\nu_\lambda\|_{2,\gamma}^2 \, d\lambda < \infty \quad \text{provided} \quad \lambda_0^{1+2\gamma} > \frac{1}{2}. \tag{5.4}$$

To go beyond $2^{-2/3}$ we remove every third term from the original series. More precisely, let $\widetilde{\Pi_\lambda}(\omega) = \sum_{3\not| n} \omega_n \lambda^n$ and denote the distribution of this series by $\widetilde{\nu_\lambda}$. It was shown in [39] and [34] that the class of power series (5.1) that satisfy either $b_{3j+1} = 0$ for all $j \ge 0$ or b_{3j+2} for all $j \ge 0$ have $[0, \lambda_3]$ as an interval of δ–transversality for some $\lambda_3 > 1/\sqrt{2}$. As for the full series one easily deduces from Theorem 2.8 that

$$\int_{\lambda_2}^{\lambda_3} \|\widetilde{\nu_\lambda}\|_{2,\gamma}^2 \, d\lambda < \infty \quad \text{provided} \quad \lambda_2^{1+2\gamma} > 2^{-2/3}. \tag{5.5}$$

Since $|\widehat{\nu_\lambda}| \le |\widehat{\widehat{\nu_\lambda}}|$ and $\lambda_1 > 2^{-2/3}$, (5.2) follows from (5.4) and (5.5). Moreover, we have shown that there exists some $\ell_0 \in (2^{-1/2}, 2^{-1/4})$ and a $\gamma_0 > 0$ so that $\int_{\ell_0^2}^{\ell_0} \|\nu_\lambda\|_{2,\gamma_0}^2 \, d\lambda < \infty$. Using $\widehat{\nu_\lambda}(\xi) = \widehat{\nu_{\lambda^2}}(\xi)\widehat{\nu_{\lambda^2}}(\lambda\xi)$ we conclude that

$$\begin{aligned} \int_{\ell_0}^{\ell_0^{\frac{1}{2}}} \|\nu_\lambda\|_{2,2\gamma_0} \, d\lambda &= \int_{\ell_0}^{\ell_0^{\frac{1}{2}}} \left(\int_{-\infty}^{\infty} |\widehat{\nu_{\lambda^2}}(\xi)\widehat{\nu_{\lambda^2}}(\lambda\xi)|^2 \, |\xi|^{4\gamma_0} \, d\xi \right)^{\frac{1}{2}} d\lambda \\ &\le C \int_{\ell_0^2}^{\ell_0} \left(\int |\widehat{\nu_\lambda}(\xi)|^4 \, |\xi|^{4\gamma_0} \, d\xi \right)^{\frac{1}{2}} d\lambda \qquad (5.6) \\ &\le C \int_{\ell_0^2}^{\ell_0} \int |\widehat{\nu_\lambda}(\xi)|^2 \, |\xi|^{2\gamma_0} \, d\xi \, d\lambda < \infty. \qquad (5.7) \end{aligned}$$

(5.6) follows from Cauchy–Schwarz and a change of variables. To pass from (5.6) to (5.7) one basically observes that the inner integral in (5.6) behaves like a sum over $\xi \in \mathbb{Z}$. More precisely, one has $\nu_\lambda = \nu_\lambda \chi$ for some $\chi \in C^\infty(\mathbb{R})$ with compact support. Consequently,

$$\begin{aligned} & \int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\xi)|^4 |\xi|^{4\gamma_0} \, d\xi = \int_{-\infty}^{\infty} |\widehat{\nu_\lambda} * \hat{\chi}|^4(\xi) |\xi|^{4\gamma_0} \, d\xi \\ = & \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\xi - \eta)||\hat{\chi}(\eta)| \, d\eta \right)^4 |\xi|^{4\gamma_0} \, d\xi = \sum_{\ell \in \mathbb{Z}} \int_{\ell}^{\ell+1} \left(\int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\xi - \eta)||\hat{\chi}(\eta)| \, d\eta \right)^4 |\xi|^{4\gamma_0} \, d\xi \\ = & \int_0^1 \sum_{\ell \in \mathbb{Z}} \left(\int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\xi + \ell - \eta)||\hat{\chi}(\eta)| \, d\eta \right)^4 |\xi + \ell|^{4\gamma_0} \, d\xi \\ \le & \int_0^1 \left(\sum_{\ell \in \mathbb{Z}} \left(\int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\xi + \ell - \eta)||\hat{\chi}(\eta)| \, d\eta \right)^2 |\xi + \ell|^{2\gamma_0} \right)^2 d\xi \\ \le & \|\hat{\chi}\|_{L^1}^2 \int_0^1 \left(\sum_{\ell \in \mathbb{Z}} \int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\eta)|^2 |\hat{\chi}(\xi + \ell - \eta)||\xi + \ell|^{2\gamma_0} \, d\eta \right)^2 d\xi \\ \le & C \left(\int_{-\infty}^{\infty} |\widehat{\nu_\lambda}(\eta)|^2 (1 + |\eta|)^{2\gamma_0} \, d\eta \right)^2 < \infty. \end{aligned}$$

Finally, since $|\widehat{\nu_\lambda}(\xi)| \le 1$,

$$\int_{\ell_0}^{\ell_0^{\frac{1}{2}}} \|\nu_\lambda\|_{2,\gamma_0}^2 \, d\lambda = \int_{\ell_0}^{\ell_0^{\frac{1}{2}}} \int |\widehat{\nu_{\lambda^2}}(\xi)\widehat{\nu_{\lambda^2}}(\lambda\xi)|^2 \, |\xi|^{2\gamma_0} \, d\xi \, d\lambda \le C \int_{\ell_0^2}^{\ell_0} \int |\widehat{\nu_\lambda}(\xi)|^2 \, |\xi|^{2\gamma_0} \, d\xi \, d\lambda < \infty.$$

We have shown that (5.3) holds for $k = 1$. The general case now follows easily by induction along the same lines. ☐

Corollary 5.6. *Let $S(\lambda_0) = \operatorname{essinf}_{\lambda \in [\lambda_0, 1]} \dim_s(\nu_\lambda)$. Then $S(\lambda_0) > 1$ for any $\lambda_0 > \frac{1}{2}$ and $S(\lambda_0) \to \infty$ as $\lambda_0 \to 1$. Furthermore, for any $\lambda_0 > \frac{1}{2}$ there exists $\epsilon(\lambda_0) > 0$ such that*

$$\dim \{\lambda \in (\lambda_0, 1) : \nu_\lambda \text{ does not have } L^2\text{–density}\} < 1 - \epsilon(\lambda_0).$$

Proof. This follows immediately from the previous lemma and Proposition 4.2. ☐

As observed by Kahane [22], Erdős's argument yields that $\epsilon(\lambda_0) \to 1$ as $\lambda_0 \to 1-$. To conclude the discussion of classical Bernoulli convolutions we prove that (2.3) fails.

Lemma 5.7. *Suppose $\frac{1}{\lambda_0}$ is a Pisot number. Then*

$$\limsup_{|\xi|\to\infty} |\xi| \int_J |\widehat{\nu_\lambda}(\xi)|^2 \, d\lambda > 0$$

for any interval J containing λ_0.

Proof. Recall that $\limsup_{|\xi|\to\infty} |\widehat{\nu_{\lambda_0}}(\xi)| > 0$, see [6]. Also,

$$|\widehat{\nu_\lambda}(\xi) - \widehat{\nu_{\lambda_0}}(\xi)| \quad \leq \quad \int_\Omega \left| e^{i\xi\Pi_\lambda(\omega)} - e^{i\xi\Pi_{\lambda_0}(\omega)} \right| d\mu(\omega) \leq |\xi| \int_\Omega |\Pi_\lambda(\omega) - \Pi_{\lambda_0}(\omega)| \, d\mu(\omega) \leq C|\xi||\lambda - \lambda_0|.$$

Therefore, if C is a sufficiently large constant,

$$\limsup_{|\xi|\to\infty} |\xi| \int_{|\lambda-\lambda_0|\leq (C|\xi|)^{-1}} |\widehat{\nu_\lambda}(\xi)|^2 \, d\lambda > 0,$$

as claimed. □

5.2. **Asymmetric Bernoulli convolutions.** In [35] Peres and Solomyak studied the distribution ν_λ^p of the sum $\sum \pm\lambda^n$ where the signs are chosen independently with probabilities $(p, 1-p)$. For $p \in [1/3, 2/3]$ they showed that ν_λ^p is absolutely continuous for a.e. $\lambda \in \left[p^p(1-p)^{1-p}, 1\right)$ and has L^q–density for a.e. $\lambda \in \left[(p^q + (1-p)^q)^{\frac{1}{q-1}}, 1\right)$, where $q \in (1, 2]$. The results from section 2 give bounds on the dimension of the exceptional parameters for the absolutely continuous case and if $q = 2$. As in the symmetric case, $\Omega = \{-1, 1\}^{\mathbb{N}}$. Here μ is the product measure that assigns weights p and $1-p$ to $+1$ and -1, respectively.

Theorem 5.8. *Let $p \in [1/3, 2/3]$. For all $\lambda_0 \in (p^2 + (1-p)^2, 1]$ there exists $\epsilon = \epsilon(\lambda_0)$ such that*

$$\dim\left\{\lambda \in (\lambda_0, 1) \,:\, \nu_\lambda^p \text{ does not have } L^2\text{-density}\right\} < 1 - \epsilon(\lambda_0) \tag{5.8}$$

and such that for all $\lambda_0 \in (p^p(1-p)^{1-p}, 1]$

$$\dim\left\{\lambda \in (\lambda_0, 1) \,:\, \nu_\lambda^p \text{ is singular}\right\} < 1 - \epsilon(\lambda_0). \tag{5.9}$$

Proof. Since $[0, 0.649]$ is an interval of transversality for the power series (5.1), Theorem 2.8 implies via Lemma 5.3 and a subdivision of the interval as in the proof of Theorem 5.4 that

$$\dim\left\{\lambda \in (\lambda_0, 0.649) \,:\, \nu_\lambda^p \text{ does not have } L^2\text{-density}\right\} < 1$$

for any $\lambda_0 > p^2 + (1-p)^2$. To extend this up to 1, we follow [35]. The measure $\nu_\lambda^p * \nu_\lambda^p$ is the distribution of the random series $\sum_0^\infty a_n\lambda^n$ where $a_n \in \{-2, 0, 2\}$ with probabilities $p^2, 2p(1-p), (1-p)^2$. By Lemma 5.2 and an explicit choice of $(*)$–function Peres and Solomyak showed that $[0, 0.5]$ is an interval of transversality for the power series (5.1) with $b_n \in \{-4, -2, 0, 2, 4\}$. Since $(0.649)^2 > \frac{11}{27} = \max_{p\in[1/3,2/3]}(p^4 + (2p(1-p))^2 + (1-p)^4)$ Theorem 2.8 yields in the same fashion as before that

$$\dim\left\{\lambda \in (0.649, 1/\sqrt{2}) \,:\, \widehat{\nu_\lambda^p * \nu_\lambda^p} \notin L^2(\mathbb{R})\right\} = \dim\left\{\lambda \in ((0.649)^2, 0.5) \,:\, \widehat{\nu_\lambda^p} \notin L^4(\mathbb{R})\right\} < 1.$$

Splitting the random power series $\sum \pm\lambda^n$ into odd and even indices one obtains

$$\widehat{\nu_\lambda^p}(\xi) = \widehat{\nu_{\lambda^2}^p}(\xi)\widehat{\nu_{\lambda^2}^p}(\lambda\xi). \tag{5.10}$$

Therefore,

$$\dim\left\{\lambda \in (0.649, 1/\sqrt{2}) \,:\, \widehat{\nu_\lambda^p} \notin L^2(\mathbb{R})\right\} < 1.$$

The measure $\nu_\lambda^p * \nu_\lambda^p * \nu_\lambda^p$ is the distribution of the random series $\sum_0^\infty a_n\lambda^n$ where $a_n \in \{-3, -1, 1, 3\}$ with probabilities $p^3, 3p^2(1-p), 3p(1-p)^2, (1-p)^3$. It was shown in [35] that $[0, 0.415]$ is an interval of transversality for the corresponding power series. Since $2^{-3/2} > \frac{245}{729} = \max_{p\in[1/3,2/3]}\bigl(p^6 + (3p^2(1-p))^2 + (3p(1-p)^2)^2 + (1-p)^6\bigr)$, Theorem 2.8 again implies that

$$\dim\bigl\{\lambda \in (2^{-3/2}, 0.415) \,:\, \widehat{\nu_\lambda^p} \notin L^6(\mathbb{R})\bigr\} < 1.$$

Splitting the original series modulo 3 one concludes that

$$\dim\bigl\{\lambda \in \bigl(1/\sqrt{2}, (0.415)^{\frac{1}{3}}\bigr) \,:\, \widehat{\nu_\lambda^p} \notin L^2(\mathbb{R})\bigr\} < 1.$$

By this procedure we have estimated the dimension of λ inside the interval $[\lambda_0, (0.415)^{\frac{1}{3}}]$ for which ν_λ does not have L^2–density. Since $\max_{p\in[1/3,2/3]}\bigl(p^2 + (1-p)^2\bigr) = \frac{5}{9}$ and $(0.415)^{\frac{1}{3}} > \sqrt{\frac{5}{9}}$, inequality (5.8) follows from this estimate by applying (5.10) repeatedly.

By the strong law of large numbers the lower pointwise dimension of μ is

$$\pi_\mu^-(\omega) = -(p\log p + (1-p)\log(1-p)) \quad \mu\text{–a.e.}$$

It therefore follows immediately from Corollary 2.9 that

$$\dim\bigl\{\lambda \in (\lambda_0, 0.649) \,:\, \nu_\lambda^p \text{ is singular}\bigr\} < 1$$

for any $\lambda_0 > \exp\bigl[-(p\log p + (1-p)\log(1-p))\bigr]$. Since (5.8) is a stronger assertion on the interval $[0.649, 1]$, we finally obtain (5.9). □

5.3. $\{0,1,3\}$–**problem.** This problem concerns the Hausdorff dimension of the set

$$\Lambda(\lambda) = \Bigl\{\sum_{i=0}^{\infty} a_i\lambda^i \,:\, a_i \in \{0,1,3\}\Bigr\}.$$

Pollicott and Simon [36] showed that for a.e. $\lambda \in (\frac{1}{4}, \frac{1}{3})$ one has $\dim\Lambda(\lambda) = \frac{\log 3}{-\log\lambda}$ and Solomyak [39] established that for a.e. $\lambda \in (\frac{1}{3}, \frac{2}{5})$ the set $\Lambda(\lambda)$ has positive Lebesgue measure. In this section we estimate the Hausdorff dimension of the exceptional set of λ with respect to these properties.

It is again clear that the general results in section 2 apply to this case. Indeed, let $\Omega = \{0,1,3\}^{\mathbb{N}}$ be equipped with uniform product measure. For any interval $J = (\lambda_0, \lambda_1) \subset (0,1)$ define the projections Π and the metric d on Ω as in section 5.1. In [39] Solomyak showed that the smallest double zero of the class of power series

$$g(x) = a + \sum_{n=1}^{\infty} b_n\lambda^n \quad \text{with} \quad |b_n| \le 1, \quad |a| \ge \frac{1}{3} \tag{5.11}$$

occurs in the interval $[0.418, 0.437]$, see also Corollary 5.2 in [35]. In particular, a simple compactness argument shows that $[0, \frac{2}{5}]$ is an interval of δ–transversality for the class (5.11) for some $\delta > 0$ in the sense that $|g(\lambda)| < \delta$ implies $|g'(\lambda)| > \delta$ for any λ in that interval. As in Lemma 5.3, we establish the connection with Definition 2.7.

Lemma 5.9. *Suppose $J = (\lambda_0, \lambda_1)$ is an interval of δ–transversality for the class of power series* (5.11)*. Then J is an interval of transversality of order β for Π provided $\lambda_0 > \lambda_1^{1+\beta}$.*

Proof. It suffices to notice that $\Phi_\lambda(\omega_1, \omega_2) = 3(\lambda\lambda_1^{-1})^{|\omega_1\wedge\omega_2|}g(\lambda)$ where g is of the form (5.11). Otherwise the proof is identical with that of Lemma 5.3. □

Theorem 2.8 now easily leads to the following result. Let $\Delta(\lambda) = \frac{\log 3}{-\log\lambda}$.

Theorem 5.10. *For any $\lambda_0 \in (0.25, 0.4)$*

$$\dim\bigl\{\lambda \in (0.25, \lambda_0) \,:\, \dim\Lambda(\lambda) < \Delta(\lambda)\bigr\} \le \Delta(\lambda_0) \quad \textit{and} \tag{5.12}$$

$$\dim\bigl\{\lambda \in (\lambda_0, 0.4) \,:\, \mathcal{H}^1(\Lambda(\lambda)) = 0\bigr\} \le 2 - \Delta(\lambda_0). \tag{5.13}$$

Proof. Suppose $\lambda_0 \in (\frac{1}{4}, \frac{1}{3})$. Fix any $\beta \in (0,1)$. As in the proof of Theorem 5.4 we write $[\frac{1}{4}, \lambda_0] = \bigcup_{i=0}^m J_i$ with $J_i = [\lambda_{i+1}, \lambda_i]$ and $(1+\beta)\lambda_i^{1+\beta} \le \lambda_{i+1}$. Applying Lemma 5.9 to each J_i one concludes from Theorem 2.8 that

$$\dim\{\lambda \in J_i \;:\; \dim_s(\nu_\lambda) < \Delta(\lambda_i) - 3\beta\} \le \Delta(\lambda_i) \le \Delta(\lambda_0).$$

(5.12) follows by letting $\beta \to 0+$.

If $\lambda_0 \in (\frac{1}{3}, \frac{2}{5})$, one partitions $[\lambda_0, \frac{2}{5}]$ in the same fashion. Theorem 2.8 then implies

$$\dim\{\lambda \in J_i \;:\; \nu_\lambda \text{ is singular}\} \le 2 - \frac{\Delta(\lambda_{i+1})}{1+a_0\beta} \le 2 - \frac{\Delta(\lambda_0)}{1+a_0\beta}.$$

(5.13) now follows again by letting $\beta \to 0+$. □

5.4. **Sums of Cantor sets.** Following [35] we consider homogeneous self–similar sets in $\mathbb{R}$

$$\mathcal{C}_\lambda = \Big\{\sum_{n=0}^\infty s_n(\lambda)\lambda^n \;:\; s_n \in \mathcal{D},\ \lambda \in J\Big\}$$

where $\mathcal{D} = \{s_1, s_2, \ldots, s_m\}$ is a set of $C^1(J)$ functions and $J = [\lambda_0, \lambda_1] \subset (0,1)$. As in [35] we assume the strong separation condition

$$\min_{1\le i<j\le m}\ \min_{\lambda_0\le\lambda\le\lambda_1} \operatorname{dist}(s_i(\lambda) + \lambda\mathcal{C}_\lambda, s_j(\lambda) + \lambda\mathcal{C}_\lambda) > h_0 > 0. \tag{5.14}$$

This condition implies $\dim \mathcal{C}_\lambda = \frac{\log m}{-\log\lambda} < 1$, so $\lambda < m^{-1}$. The usual middle–α Cantor sets are clearly special cases of the $\mathcal{C}_\lambda$ and we refer the reader to [35] for more motivation and background. In this section we will be concerned with the problem of estimating the dimension of the sum of two Cantor sets, or more generally, of the sum of any compact set with one of the $\mathcal{C}_\lambda$. For technical reasons, we will split $\mathcal{C}_\lambda$ into the disjoint union of its cylinder sets of a certain fixed length. More precisely, let $M_0 = \max_{i\ne j}\|s_i - s_j\|_{L^\infty(J)}$ and $M_1 = \max_{i\ne j}\|s_i' - s_j'\|_{L^\infty(J)}$ and define $k_0 = \lfloor \frac{2(M_0+M_1)\lambda_1}{h_0(1-\lambda_1)^2} \rfloor$. Then every $\mathcal{C}_\lambda$ is the union of m^{k_0} congruent subsets that are obtained by fixing the first k_0 symbols s_n. Slightly abusing notation we will use $\mathcal{C}_\lambda$ to denote one of these subsets. Since we are only concerned about questions relating to dimension, this makes no difference.

To see that the case at hand is again a special case of the results in section 2, fix a compact set $K \subset \mathbb{R}$. Define $\Omega = K \times \{1, 2, \ldots, m\}^{\mathbb{N}}$ and denote a typical point of Ω by (x, ω). Let $\Pi_\lambda(x,\omega) = x + \sum_{n=0}^\infty s_{\omega_n}(\lambda)\lambda^n$ be the projections $\Pi : J \times \Omega \to \mathbb{R}$. Let ρ be a Frostman measure on K. Define the measure μ on Ω to be $\mu = \rho \times \mu_0$, where μ_0 is the uniform product measure on the sequence space $\{1, 2, \ldots, m\}^{\mathbb{N}}$. Finally, we let

$$d((x,\omega),(y,\tau)) = |x-y| + \lambda_1^{|\omega\wedge\tau|}$$

be the metric on Ω. By definition, $\Pi : J \times \Omega \to \mathbb{R}$ is continuous and C^1 in λ. In the following lemma we show that the conditions in Definition 2.7 are satisfied.

Lemma 5.11. *Suppose $s_1, s_2, \ldots, s_m \in C^{L,\delta}(J)$ for some positive integer L and some $\delta \in [0,1)$. Then $J = [\lambda_0, \lambda_1]$ is an interval of transversality of order β for Π (in the sense of Definition 4.8) provided $\lambda_1^{1+\beta} > \lambda_0$.*

Proof. Fix some (x,ω) and $(y,\tau) \in \Omega$. Let $k = |\omega \wedge \tau|$ and $r = d((x,\omega),(y,\tau))$. Notice that $k > k_0$ by our assumption above. Then

$$\Phi_\lambda((x,\omega),(y,\tau)) = \frac{x - y + \lambda^k g(\lambda)}{|x-y| + \lambda_1^k}$$

where $|g(\lambda)| > h_0$ by (5.14). Now suppose

$$|\Phi_\lambda| \le C_\beta\,(|x-y| + \lambda_1^k)^\beta = C_\beta\, r^\beta \tag{5.15}$$

where the constant C_β is to be determined. In fact, C_β can be chosen so that $|x-y| \le C_1\lambda_1^k$ with some constant C_1 that depends only on $\mathcal{D}$ and λ_1. More precisely, suppose that

$$|x-y| \ge \frac{2\lambda_1^k}{1-\lambda_1} \max_{i\neq j} \|d_i - d_j\|_{L^\infty(J)} = \frac{2\lambda_1^k}{1-\lambda_1} M_0.$$

Then

$$|\Phi_\lambda| \ge \frac{|x-y| - \lambda_1^k M_0(1-\lambda_1)^{-1}}{|x-y| + \lambda_1^k} \ge C$$

which contradicts (5.15) if C_β is sufficiently small. Consequently, $r \asymp \lambda_1^k$ and $|\Phi_\lambda| \le C_\beta\, \lambda_1^{k\beta}$. It is then easy to see that $|\Phi_\lambda^{(\ell)}| \le C_{\beta,\ell}\, \lambda_1^{-\beta k}$ for suitable constants $C_{\beta,\ell}$ and all $\ell = 1, 2, \ldots, L$ and also that a Hölder condition of order δ on $\Phi_\lambda^{(L)}$ holds if $\delta > 0$. Hence (4.21) in Definition 4.8 is satisfied. Moreover,

$$|\Phi_\lambda'| = \frac{\lambda^k}{r} \left| \frac{k}{\lambda} g(\lambda) + g'(\lambda) \right| \ge C(\lambda_0\lambda_1^{-1})^k \left| \frac{kh_0}{\lambda_1} - \frac{M_0+M_1}{(1-\lambda_1)^2} \right| \ge C\, \lambda_1^{k\beta} \ge C_\beta\, r^\beta$$

if C_β is sufficiently small, since $\lambda_0 > \lambda_1^{1+\beta}$ and $k > k_0$. Thus (4.20) also holds. □

Under the strong separation assumption (5.14) Peres and Solomyak showed in [35] that $\dim(K + \mathcal{C}_\lambda) = \dim K + \dim \mathcal{C}_\lambda$ for a.e. λ such that $\dim K + \dim \mathcal{C}_\lambda < 1$. Furthermore, they also showed that $K + \mathcal{C}_\lambda$ has positive Lebesgue measure for a.e. λ such that $\dim K + \dim \mathcal{C}_\lambda > 1$. The following theorem estimates the dimension of the exceptional values of λ in these statements. This will be a simple consequence of Theorem 4.9.

Theorem 5.12. *Suppose $K \subset \mathbb{R}$ is compact and the sets $\mathcal{C}_\lambda$ satisfy the strong separation condition* (5.14) *on $J = [\lambda_0, \lambda_0'] \subset (0,1)$. Then*

$$\dim \left\{ \lambda \in J \,:\, \dim(K + \mathcal{C}_\lambda) < \dim K + \dim \mathcal{C}_\lambda \right\} \le \dim K + \dim \mathcal{C}_{\lambda_0'}. \tag{5.16}$$

If the symbols $s_1, \ldots, s_m$ are in $C^{L,\delta}$ with $L + \delta > 1$, then

$$\dim \left\{ \lambda \in J \,:\, \mathcal{H}^1(K + \mathcal{C}_\lambda) = 0 \right\} \le 2 - \min(\dim K + \dim \mathcal{C}_{\lambda_0}, L + \delta). \tag{5.17}$$

Proof. Fix some $\epsilon > 0$ and let ρ be a Frostman measure on K with exponent $\dim K - \epsilon$. Let the measure μ on Ω be define as above. Fix $\beta \in (0,1)$ and partition $J = \bigcup_{n=1}^N J_i$ where each $J_i = [\lambda_i, \lambda_{i+1}]$ satisfies $\lambda_i \ge (1+\beta)\lambda_{i+1}^{1+\beta}$. It is easy to see that μ has finite α_i–energy with respect to the metric $d_i((x,\omega),(y,\tau)) = |x-y| + \lambda_{i+1}^{|\omega\wedge\tau|}$ on Ω if $\alpha_i < \dim K - \epsilon + \dim \mathcal{C}_{\lambda_{i+1}}$. Indeed, setting $\sigma_i = \dim \mathcal{C}_{\lambda_{i+1}}$ which is the same as $\lambda_{i+1}^{\sigma_i} m = 1$, one obtains

$$\begin{aligned} \mathcal{E}_{\alpha_i}(\mu) &= \int_\Omega \int_\Omega \int_K \int_K \frac{d\rho(x)d\rho(y)d\mu(\omega)d\mu(\tau)}{(|x-y| + \lambda_{i+1}^{|\omega\wedge\tau|})^{\alpha_i}} \le \sum_{k=0}^\infty m^{-k} \int_K \int_K \frac{d\rho(x)d\rho(y)}{(\lambda_{i+1}^k + |x-y|)^{\alpha_i}} \\ &\le C \int_K \int_K \frac{d\rho(x)d\rho(y)}{|x-y|^{\alpha_i - \sigma_i}} < \infty \end{aligned}$$

provided $0 < \alpha_i - \sigma_i < \dim K - \epsilon$. Therefore, by Lemma 5.11 and (2.9),

$$\begin{aligned} \dim \left\{ \lambda \in J_i \,:\, \dim(K + \mathcal{C}_\lambda) < \dim K - \epsilon + \sigma_i - 3\beta \right\} &\le \dim \left\{ \lambda \in J_i \,:\, \dim_s(\nu_\lambda) < \dim K - \epsilon + \sigma_i - 3\beta \right\} \\ &\le \dim K - \epsilon + \dim \mathcal{C}_{\lambda_{i+1}} \le \dim K - \epsilon + \dim \mathcal{C}_{\lambda_0'}, \end{aligned}$$

and (5.16) follows by letting $\beta \to 0+$ and then $\epsilon \to 0+$. The second statement (5.17) follows in a similar fashion from 4.23. □

6. Orthogonal projections onto planes in Euclidean space

It is easy to see that Theorem 2.8 covers various classical results on the projection of planar sets onto lines, see [28], [23], [24], [9], [31]. Let $\Omega = \mathbb{R}^2$ and $\Pi(\theta, x) = \mathrm{proj}_\theta(x) = (x \cdot \theta)\theta$, where we consider $J \subset S^1$ as an interval of angles. Here proj_θ denotes the projection onto the line $\{t\theta \;:\; t \in \mathbb{R}\}$. Note that $\Phi_\theta(x, y) = \cos(\sphericalangle(\theta, x - y))$ for any $x \neq y$ so that Definition 2.7 holds with $\beta = 0$. Suppose $E \subset \mathbb{R}^2$ is a Borel set. Applying Theorem 2.8 to a suitable Frostman measure supported on a compact subset of E one therefore obtains Kaufman's [24] and Falconer's [9] theorems in the plane, i.e.,

$$\dim \{\theta \in S^1 \;:\; \dim \mathrm{proj}_\theta(E) \leq t\} \leq t \quad \text{and} \quad \dim \{\theta \in S^1 \;:\; \mathcal{H}^1(\mathrm{proj}_\theta(E)) = 0\} \leq 2 - \dim E.$$

Kaufman and Mattila [25] proved that the first bound is optimal. As noted by Falconer [9], their example can be easily modified to show that the second bound is sharp, too. See also Falconer [11], Theorem 8.17. In particular, this implies that the estimates stated in Theorem 2.8 are optimal if $\beta = 0$, $\alpha \leq 2$, and $\sigma \leq 1$. Moreover, it is easy to see that in this case (2.7) holds with $\beta = 0$ and equality, see also Proposition 2.2. It turns out that the calculation from the proof of that proposition can be generalized to higher dimensions. $G(d, k)$ denotes the Grassmann manifold of all k–planes in $\mathbb{R}^d$ passing through the origin. Recall that $\dim G(d, k) = k(d - k)$, see [31]. For any finite measure μ on $\mathbb{R}^d$ we define its projections ν_π onto any k–plane π through the origin as usual, i.e., given any continuous f

$$\int f \, d\nu_\pi = \int f(\mathrm{proj}_\pi(x)) \, d\mu(x).$$

Proposition 6.1. *Let $d \geq 2$ and k be positive integers. Suppose μ is a compactly supported measure in $\mathbb{R}^d$. Then*

$$\dim \{\pi \in G(d, k) \;:\; \dim_s(\nu_\pi) < \sigma\} \leq k(d - k) + \sigma - \dim_s(\mu). \tag{6.1}$$

Proof. Let $\alpha = \dim_s(\mu)$. Suppose (6.1) fails. By Frostman's lemma there exists a nonzero positive measure ρ so that for some $\epsilon > 0$

$$\rho(B(\pi, r)) \leq r^{k(d-k)+\sigma-\alpha+\epsilon} \text{ for any ball } B(\pi, r) \subset G(d, k) \text{ and } \rho(\{\pi \in G(d, k) \;:\; \dim_s(\mu_\pi) \geq \sigma\}) = 0. \tag{6.2}$$

The measurability hypothesis in Frostman's lemma can be verified as in Proposition 4.4 and we skip the details. We may assume that $\mathrm{supp}(\mu) \subset B(0, 1)$. Fix a $\phi \in \mathcal{S}$ so that $\phi = 1$ on $B(0, 1)$. Hence $\mu = \phi\mu$ and thus $\hat{\mu} = \hat{\phi} * \hat{\mu}$. Clearly, $\widehat{\nu_\pi}(\xi) = \hat{\mu}(\mathrm{proj}_\pi(\xi))$ and $\mathrm{supp}(\nu_\pi) = \mathrm{proj}_\pi(\mathrm{supp}(\mu))$. Also, $|\hat{\mu}|^2 \leq \|\hat{\phi}\|_{L^1} \, |\hat{\phi}| * |\hat{\mu}|^2$ by Cauchy–Schwarz. Therefore,

$$\int_{G(d,k)} \int_\pi |\widehat{\nu_\pi}(\xi)|^2 (1 + |\xi|)^{\sigma-k} \, d\mathcal{H}^k(\xi) \, d\rho(\pi) \leq C \int_{G(d,k)} \int_\pi (|\hat{\phi}| * |\hat{\mu}|^2)(\xi)(1 + |\xi|)^{\sigma-k} \, d\mathcal{H}^k(\xi) \, d\rho(\pi) \tag{6.3}$$

$$\leq C_N \int_{\mathbb{R}^d} |\hat{\mu}(\eta)|^2 \int_{G(d,k)} \int_\pi (1 + |\xi - \eta|)^{-N} (1 + |\xi|)^{\sigma-k} \, d\mathcal{H}^k(\xi) \, d\rho(\pi) \, d\eta \tag{6.4}$$

$$\leq C_{\alpha,d} \int_{\mathbb{R}^d} |\hat{\mu}(\eta)|^2 (1 + |\eta|)^{\alpha-d-\epsilon} \, d\eta < \infty. \tag{6.5}$$

Here N is a sufficiently large integer. To pass from (6.4) to (6.5) first notice that for any $r < |\eta|$ the set $\{\pi \in G(d, k) \;:\; \mathrm{dist}(\pi, \eta) \leq r\}$ is a $(r|\eta|^{-1})$–neighborhood of a smooth manifold of codimension $d - k$ in $G(d, k)$ and is therefore contained in the union of no more than $\asymp (|\eta| r^{-1})^{(k-1)(d-k)}$ balls of radius $r|\eta|^{-1}$. Thus, by (6.2)

$$\rho(\{\pi \in G(d, k) : \mathrm{dist}(\pi, \eta) \leq r\}) \leq C(|\eta| r^{-1})^{k-\sigma+\alpha-d-\epsilon}.$$

Considering the contributions from the dyadic shells $\{\xi \in \pi \ : \ 2^j \le |\xi - \eta| \le 2^{j+1}\}$ separately, the inner integrals in (6.4) can therefore be estimated as follows.

$$\int_{G(d,k)} \int_\pi (1+|\xi-\eta|)^{-N}(1+|\xi|)^{\sigma-k}\, d\mathcal{H}^k(\xi)\, d\rho(\pi) \le C(1+|\eta|)^{\sigma-k}\rho(\{\pi \in G(d,k) : \mathrm{dist}(\pi,\eta) \le 1\})$$
$$+C \sum_{1 \le 2^j < \frac{1}{2}|\eta|} 2^{-j(N-k)}(1+|\eta|)^{\sigma-k}\rho(\{\pi \in G(d,k) : \mathrm{dist}(\pi,\eta) \le 2^j\})$$
$$+C \sum_{2^j \ge \frac{1}{2}|\eta|} 2^{-jN} \int_{|\xi| \le 2^{j+1}} (1+|\xi|)^{\sigma-k}\, d\mathcal{H}^k(\xi) \le C(1+|\eta|)^{\alpha-d-\epsilon},$$

if N is sufficiently large, as claimed. However, the finiteness of (6.3) contradicts (6.2) and the proposition follows. □

This proposition has the following simple corollary.

Corollary 6.2. *Suppose $E \subset \mathbb{R}^d$ is a Borel set with $\dim E > 2k$. Then for a.e. $\pi \in G(d,k)$ the projection of E onto π has nonempty interior. More precisely,*

$$\dim\{\pi \in G(d,k) \ : \ \mathrm{proj}_\pi(E) \ \text{ has empty interior}\} \le k(d-k) + 2k - \dim E.$$

Proof. Let μ be a Frostman probability measure on E, i.e., μ is supported on a compact subset of E and $\dim_s(\mu) = \dim E$. If $\nu_\pi = \mathrm{proj}_\pi(\mu)$ has a continuous density its support has nonempty interior and therefore so does $\mathrm{proj}_\pi(E)$. Applying Cauchy–Schwarz to $\int_{\mathbb{R}^k} |\hat{f}(\xi)|\, d\xi$ shows that for any $f \in L^2(\mathbb{R}^k)$

$$\|f\|_{2,\frac{1}{2}(k+\epsilon)} < \infty \Longrightarrow f \in C(\mathbb{R}^k),$$

see Stein [42], chapter 5, for more precise estimates. The corollary therefore follows by letting $\sigma \to 2k+$ in (6.1). □

The following example, described to us by Pertti Mattila, shows that there are Borel sets E of dimension two in $\mathbb{R}^3$, so that $\dim\{\theta \in S^2 \ : \ \mathrm{proj}_\theta(E) \text{ contains an interval}\} \le 1$. Take a Besicovitch set A in $\mathbb{R}^2$, i.e., a set of measure zero that contains a line in every direction, see [31], Theorem 18.11. Define $E = \mathbb{R}^2 \setminus \cup_{r \in \mathbb{Q}^2}(r + A)$ as a subset of the (x_1, x_2)–coordinate plane of $\mathbb{R}^3$. It clearly has dimension two but its projections onto lines which do not lie in the (x_1, x_2)–plane do not contain intervals. If $k > 1$ we do not know of an example of a $2k$–dimensional set E with the property that $\gamma_{d,k}(\{\pi \in G(d,k) \ : \ \mathrm{proj}_\pi(E) \ \text{ has empty interior}\}) > 0$ where $\gamma_{d,k}$ denotes Haar measure on $G(d,k)$. It seems unlikely that the Sobolev embedding theorem would give the sharp bound for $k > 1$.

On the other hand, it is easy to see that Corollary 6.2 allows one to recover Falconer's theorem [8] about the nonexistence of Besicovitch (d,k)–sets for $k > d/2$. Recall that a (d,k)–set is defined to be a set of measure zero that contains some translate of every k–plane. Suppose that $A \subset \mathbb{R}^d$ is such a set for some choice of $d \ge 3$ and k. Then $E = \mathbb{R}^d \setminus \cup_{r \in \mathbb{Q}^d}(A + r)$ has dimension d but the projection of E onto any $(d-k)$–dimensional plane has empty interior. This contradicts Corollary 6.2 if $d > 2(d-k)$, as claimed. However, Bourgain [3] proved a much stronger result, namely that (d,k)–sets do not exist for $k + 2^{k-1} \ge d$. It seems therefore that the case $k > 1$ in Corollary 6.2 is not optimal.

7. The general projection theorems in higher dimensions

In this section we obtain a higher–dimensional version of Theorem 4.9. The proof will closely resemble that of Theorem 4.9 in section 4. In particular, we will need to prove higher–dimensional versions of various technical lemmas. We first introduce some terminology.

Definition 7.1. Let (Ω, d) be a compact metric space, $Q \subset \mathbb{R}^n$ an open connected set, and $\Pi : Q \times \Omega \to \mathbb{R}^m$ a continuous map with $n \geq m$. The length of a multi-index $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{N}^n$ is defined to be $|\eta| = \eta_1 + \ldots + \eta_n$ and $\partial^\eta = \frac{\partial^{|\eta|}}{(\partial\lambda_1)^{\eta_1} \cdot \ldots \cdot (\partial\lambda_n)^{\eta_n}}$ where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let L be a positive integer and $\delta \in [0,1)$. We assume that for any compact $Q' \subset Q$ and any multi-index $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{N}^n$ there exist constants $C_{\eta,Q'}, C_{\delta,Q'}$ such that

$$\left|\partial^\eta \Pi(\boldsymbol{\lambda}_1, \omega)\right| \leq C_{\eta,Q'} \quad \text{provided} \quad |\eta| \leq L \text{ and } \sup_{|\eta|=L} \left|\partial^\eta \Pi(\boldsymbol{\lambda}_1, \omega) - \partial^\eta \Pi(\boldsymbol{\lambda}_2, \omega)\right| \leq C_{\delta,Q'} |\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2|^\delta$$

for all $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in Q$ and $\omega \in \Omega$. We denote these conditions by $\Pi_{\boldsymbol{\lambda}} \in C^{L,\delta}(Q)$. Given any finite measure μ on Ω let $\nu_{\boldsymbol{\lambda}} = \mu \circ \Pi_{\boldsymbol{\lambda}}^{-1}$ for all $\boldsymbol{\lambda} \in Q$, where $\Pi_{\boldsymbol{\lambda}}(\cdot) = \Pi(\boldsymbol{\lambda}, \cdot)$. The α–energy of μ is $\mathcal{E}_\alpha(\mu) = \int_\Omega \int_\Omega \frac{d\mu(\omega_1) d\mu(\omega_2)}{d(\omega_1,\omega_2)^\alpha}$.

As in the previous sections we will need a transversality condition.

Definition 7.2. Let $\Pi_{\boldsymbol{\lambda}} \in C^{L,\delta}$ be as in Definition 7.1 for some positive integer L and some $\delta \in [0,1)$. Let

$$\Phi_{\boldsymbol{\lambda}}(\omega_1, \omega_2) = \frac{\Pi(\boldsymbol{\lambda}, \omega_1) - \Pi(\boldsymbol{\lambda}, \omega_2)}{d(\omega_1, \omega_2)}.$$

For any $\beta \in [0,1)$ we say that Q is a **region of transversality of order** β **for** Π if there exists a constant C_β so that for all $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2 \in Q$ and distinct $\omega_1, \omega_2 \in \Omega$ the condition $|\Phi_{\boldsymbol{\lambda}_1}(\omega_1, \omega_2)| + |\Phi_{\boldsymbol{\lambda}_2}(\omega_1, \omega_2)| \leq C_\beta\, d(\omega_1, \omega_2)^\beta$ implies that

$$\det[D\Phi_{\boldsymbol{\lambda}_1} (D\Phi_{\boldsymbol{\lambda}_1})^t] \geq C_\beta^2\, d(\omega_1, \omega_2)^{2\beta}. \tag{7.1}$$

In addition, we say that $\Pi_{\boldsymbol{\lambda}}$ is L, δ–**regular on** Q if under the same condition, for some constants $C_{\beta,\eta}$, $C_{\beta,L,\delta}$,

$$\begin{aligned} &|\partial^\eta \Phi_{\boldsymbol{\lambda}_1}(\omega_1, \omega_2)| \leq C_{\beta,\eta}\, d(\omega_1, \omega_2)^{-\beta|\eta|} \quad \text{for any nonzero multi-index } \eta \text{ of length } \ |\eta| \leq L \\ &\sup_{|\eta|=L} \left|\partial^\eta \Phi_{\boldsymbol{\lambda}_1}(\omega_1, \omega_2) - \partial^\eta \Phi_{\boldsymbol{\lambda}_2}(\omega_1, \omega_2)\right| \leq C_{\beta,L,\delta}\, |\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2|^\delta\, d(\omega_1, \omega_2)^{-\beta(L+\delta)}. \end{aligned} \tag{7.2}$$

$D\Phi_{\boldsymbol{\lambda}}$ above denotes the Jacobi matrix of all first derivatives of $\Phi_{\boldsymbol{\lambda}}$ with respect to $\boldsymbol{\lambda}$ (the number of rows of $D\Phi_{\boldsymbol{\lambda}}$ is m). Our main result in this section is the following theorem.

Theorem 7.3. *Let $\Pi_{\boldsymbol{\lambda}} \in C^{L,\delta}$ be as in Definition 7.1 with $L + \delta > 1$. Assume that $Q \subset \mathbb{R}^n$ is a region of transversality of order β for Π and that $\Pi_{\boldsymbol{\lambda}}$ is L, δ–regular on Q in the sense of Definition 7.2. Suppose μ is a finite positive measure on Ω with finite α–energy for some $\alpha > 0$ and let $\nu_{\boldsymbol{\lambda}} = \mu \circ \Pi_{\boldsymbol{\lambda}}^{-1}$ be the projection of μ onto $\mathbb{R}^m$ under Π for any $\boldsymbol{\lambda} \in Q$. Then for any compact $Q' \subset Q$*

$$\int_{Q'} \|\nu_{\boldsymbol{\lambda}}\|_{2,\gamma}^2\, d\boldsymbol{\lambda} \leq C_\gamma\, \mathcal{E}_\alpha(\mu) \quad \textit{provided} \quad 0 < (m + 2\gamma)(1 + a_0\beta) \leq \alpha \quad \textit{and} \quad 2\gamma < L + \delta - 1. \tag{7.3}$$

Moreover, if $\sigma \in (0, \alpha \wedge m]$, then

$$\dim\{\boldsymbol{\lambda} \in Q \ : \ \dim_s(\nu_{\boldsymbol{\lambda}}) \leq \sigma\} \leq n + \sigma - \min\Big(\frac{\alpha}{1 + a_0\beta}, L + \delta\Big), \tag{7.4}$$

if $\sigma \in (m, \alpha]$, then

$$\dim\{\boldsymbol{\lambda} \in Q \ : \ \dim_s(\nu_{\boldsymbol{\lambda}}) \leq \sigma\} \leq n - \Big(\min\Big(\frac{\alpha}{1 + a_0\beta}, L + \delta\Big) - \sigma\Big)\Big(1 + \frac{\sigma - m}{L + \delta}\Big)^{-1}. \tag{7.5}$$

Finally, if $\sigma \in (0, \alpha - a_0\beta]$, then

$$\dim\{\boldsymbol{\lambda} \in Q \ : \ \dim_s(\nu_{\boldsymbol{\lambda}}) < \sigma\} \leq n + \sigma - m. \tag{7.6}$$

The constant a_0 depends only on m, n and δ.

It is easy to check that projections onto planes in $\mathbb{R}^d$ satisfy Definition 7.2 with $Q \subset G(d,k)$ a coordinate chart, $\Pi_{\boldsymbol{\lambda}}$ the Euclidean projection from $\Omega = \operatorname{supp}(\mu)$ onto the plane given by $\boldsymbol{\lambda}$, and $L = \infty$, $\beta = 0$. Hence Proposition 6.1 follows from Theorem 7.3 with $n = k(d-k)$ and $m = k$. If $\beta = 0$ and $L = \infty$, Proposition 2.2 shows that (7.3) is sharp.

Remark 7.4. (7.3) implies the following statement from the introduction:
Let μ be a Borel probability measure on $\mathbb{R}^d$ with correlation dimension greater than $m + 2\gamma$. Then for a prevalent set of C^1 maps $f : \mathbb{R}^d \to \mathbb{R}^m$ the image of μ under f has a density with at least γ fractional derivatives in $L^2(\mathbb{R}^m)$.
Let $A = \{f \in C^1(\mathbb{R}^d, \mathbb{R}^m) \ : \ \mu \circ f^{-1} \text{ has a density in } L^{2,\gamma}(\mathbb{R}^m)\}$ and let $\mathcal{L}$ be Lebesgue measure on the matrices $M^{d\times m}(\mathbb{R})$ with entries in $[-1/2, 1/2]$ (we identify linear maps $\mathbb{R}^d \to \mathbb{R}^m$ with their matrices). We need to check that $\mathcal{L}(f + A) = 0$ for any $f \in C^1(\mathbb{R}^d, \mathbb{R}^m)$. Fix some $f \in C^1(\mathbb{R}^d, \mathbb{R}^m)$. To apply Theorem 7.3, we may assume that $\operatorname{supp}(\mu) = \Omega$ is compact. Let $Q = [-1, 1]^{dm} \subset M^{d\times m}(\mathbb{R})$ and $\Pi_{\boldsymbol{\lambda}}(\omega) = -f(\omega) + \boldsymbol{\lambda}\omega$. Since $\mathcal{E}_{m+2\gamma}(\mu) < \infty$ by assumption, one has

$$\int \|\nu_{\boldsymbol{\lambda}}\|_{2,\gamma}^2 \, d\mathcal{L}(\boldsymbol{\lambda}) < \infty$$

as desired, provided transversality holds with $\beta = 0$ and regularity with $L = \infty$. Regularity is obvious. To check transversality, notice that

$$D_{\boldsymbol{\lambda}}\Phi_{\boldsymbol{\lambda}}(\omega, \tau) \cdot \boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_0 \cdot \frac{\omega - \tau}{|\omega - \tau|}$$

for all $\boldsymbol{\lambda}_0 \in M^{d\times m}(\mathbb{R})$. In particular, the linear map $\boldsymbol{\lambda}_0 \mapsto D_{\boldsymbol{\lambda}}\Phi_{\boldsymbol{\lambda}}(\omega, \tau) \cdot \boldsymbol{\lambda}_0$ is surjective and therefore $\operatorname{rank}\big[D_{\boldsymbol{\lambda}}\Phi_{\boldsymbol{\lambda}}(\omega, \tau)\big] = m$. In fact, estimate (7.1) holds with $\beta = 0$.

The following proposition is a higher–dimensional analogue of Proposition 4.10.

Proposition 7.5. *Let $\Pi_{\boldsymbol{\lambda}} \in C^{L,\delta}(Q)$ be as in Definition 7.1 for some positive integer L and $\delta \in [0, 1)$. Suppose the family of measure $\{\nu_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\in Q}$ on $\mathbb{R}^m$ as given by 7.1 satisfies $\int_{Q'} \|\nu_{\boldsymbol{\lambda}}\|_{2,\gamma}^2 \, d\boldsymbol{\lambda} < \infty$ with some $Q' \subset Q$ and $\gamma > -m/2$.*

((i)) *If $\sigma \in [0 \wedge 2\gamma, m \wedge (m + 2\gamma)]$, then*

$$\dim\big\{\boldsymbol{\lambda} \in Q' \ : \ \dim_s(\nu_{\boldsymbol{\lambda}}) \le \sigma\big\} \le n + \sigma - (m + 2\gamma). \tag{7.7}$$

((ii)) *If $\sigma \in [m, m + 2\gamma)$, then*

$$\dim\big\{\boldsymbol{\lambda} \in Q' \ : \ \dim_s(\nu_{\boldsymbol{\lambda}}) \le \sigma\big\} \le n - (m + 2\gamma - \sigma)\Big(1 + \frac{\sigma - m}{L + \delta}\Big)^{-1}. \tag{7.8}$$

Proof. Let ψ be the Littlewood–Paley function from Lemma 4.1. For any $j = 1, 2, \ldots$ we define

$$h_j(\boldsymbol{\lambda}) = 2^{-jm}\int \widehat{\psi_{2^{-j}}}(\xi)|\widehat{\nu_{\boldsymbol{\lambda}}}(\xi)|^2 \, d\xi = \int_\Omega\int_\Omega \psi\Big(2^j\left[\Pi(\boldsymbol{\lambda}, \omega_1) - \Pi(\boldsymbol{\lambda}, \omega_2)\right]\Big) \, d\mu(\omega_1) d\mu(\omega_2),$$

whereas $h_0 = 1$. Let $H_j = \sum_{i=0}^{j} 2^{m(j-i)}h_i$. As in the proof of Proposition 4.10 one checks that these functions satisfy the hypotheses of Lemma 3.2 with $A = 2$ and $B = 2^m$. In fact, up to replacing 2 with 2^m in various places the argument remains the same and we omit the details. In view of (4.4) and (4.5) the proposition now follows from Lemma 3.1 with $r = 2^{\sigma+\epsilon}$ and $R = 2^{2\gamma+m}$ as $\epsilon \to 0+$. ☐

Roughly speaking, transversality means that $\Phi_{\boldsymbol{\lambda}}$ can be inverted where it is small. However, one needs to interpret this statement more carefully in the higher dimensional case. Firstly, we are dealing with maps from $\mathbb{R}^n \to \mathbb{R}^m$ with $n \ge m$. Secondly, even if $m = n$ one cannot expect to break up a region of transversality into disjoint cubes on which either $\Phi_{\boldsymbol{\lambda}}$ is large or invertible as in the one–dimensional case, see Lemma 4.3. And thirdly, any decomposition of Q has to provide estimates involving the parameters in Definition 7.2. The following quantitative version of the inverse function theorem will be used for that purpose. It is undoubtedly well–known, but we provide a proof for the sake of completeness.

Lemma 7.6. *Suppose $f : U \subset \mathbb{R}^m \to \mathbb{R}^m$ is C^1 and that $\|Df(x) - I\| \le \frac{1}{2}$ on $B(x_0, r) \subset U$ for some $r > 0$. Then $f : B(x_0, r/3) \to f(B(x_0, r/3))$ is a diffeomorphism and $f(B(x_0, \rho)) \supset B(f(x_0), \rho/2)$ for any $0 < \rho \le r$.*

Proof. W.l.o.g. $x_0 = f(x_0) = 0$. Define $T_y(x) = y - f(x) + x$ and fix some $\rho \in (0, r]$. We claim that $T_y : B(0, \rho) \to B(0, \rho)$ is a contraction provided $|y| \le \rho/2$. Indeed, if $x, x' \in B(0, \rho)$,

$$|T_y(x)| \le \frac{1}{2}\rho + \int_0^1 \big\|I - Df(sx)\big\|\, ds\, |x| \le \rho \quad \text{and}$$

$$|T_y(x) - T_y(x')| = |x - x' - (f(x) - f(x'))| \le \int_0^1 \big\|I - Df(x' + s(x - x'))\big\|\, ds\, |x - x'| \le \frac{1}{2}|x - x'|.$$

Hence, for any $y \in B(0, \rho/2)$ there is a unique $x \in B(0, \rho)$ so that $f(x) = y$. In particular, f is one-to-one on $B(0, \rho)$ if $f(B(0, \rho)) \subset B(0, r/2)$. Since

$$|f(x)| \le \int_0^1 \big\|Df(sx)\big\|\, ds\, |x| \le \frac{3}{2}|x|,$$

it suffices to take $\rho = r/3$, as claimed. □

The following lemma is a precise statement to the effect that $\Phi_{\boldsymbol{\lambda}}$ is invertible where it is small. This will be true only locally. However, one can cover the set of small values of $\Phi_{\boldsymbol{\lambda}}$ by a collection of balls $\{B_j\}$ whose size and number can be controlled and on each of which $\Phi_{\boldsymbol{\lambda}}$ can be inverted in the following sense: there exist m coordinate directions (depending on B_j) so that the restriction of $\Phi_{\boldsymbol{\lambda}}$ to the intersection of B_j with any hyperplane in those directions is invertible. Moreover, one has uniform bounds on the derivatives of the inverse.

Lemma 7.7. *Let $\Pi_{\boldsymbol{\lambda}} \in C^{1,\delta}(Q)$ for some $0 < \delta < 1$. Suppose that $Q \subset \mathbb{R}^n$ is a region of transversality of order β for Π and that $\Pi_{\boldsymbol{\lambda}}$ is $1, \delta$–regular on Q, see Definition 7.2. Let $U \subset Q$ be open and bounded with $\mathrm{dist}(U, \partial Q) > 0$. Then there exist constants C_0, C_1, C_2 depending only on β, n, m, U, Q so that for any distinct $\omega_1, \omega_2 \in \Omega$ there exist $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_N \in Q$ with the following properties:*
Writing $r = d(\omega_1, \omega_2)$ for simplicity, one has

$$\{\boldsymbol{\lambda} \in U \;:\; |\Phi_{\boldsymbol{\lambda}}| \le C_0 r^{b_0\beta}\} \quad \subset \quad \bigcup_{j=1}^N B(\boldsymbol{\lambda}_j, C_1 r^{b_0\beta}) \tag{7.9}$$

$$\bigcup_{j=1}^N B(\boldsymbol{\lambda}_j, 2C_1 r^{b_0\beta}) \quad \subset \quad \{\boldsymbol{\lambda} \in Q \;:\; |\Phi_{\boldsymbol{\lambda}}| \le C_\beta\, r^\beta\}, \tag{7.10}$$

where $b_0 = (2 + m)\delta^{-1}$ and $N \le C_2 r^{-b_0 n}$. C_β is the constant from Definition 7.2. Moreover, on each ball $B_j = B(\boldsymbol{\lambda}_j, 2C_1 r^{b_0\beta})$ one can select $n - m$ coordinate directions $1 \le i_1 < \ldots < i_{n-m} \le n$ so that for any choice of $\vec{y} = (y_1, \ldots, y_{n-m})$

$$F_{\vec{y}} = \Phi_{\boldsymbol{\lambda}} \restriction \{\boldsymbol{\lambda} \in B_j \;:\; \lambda_{i_1} = y_1, \ldots, \lambda_{i_{n-m}} = y_{n-m}\}$$

is a diffeomorphism satisfying $|\det(DF_{\vec{y}})^{-1}| \le C_2 r^{-\beta}$ and

$$\|(DF_{\vec{y}})^{-1}\| \le C_2 r^{-m\beta}. \tag{7.11}$$

Proof. Fix distinct $\omega_1, \omega_2 \in \Omega$. By Definition 7.2 one can choose C_0 and C_3 so that

$$E = \{\boldsymbol{\lambda} \in U \;:\; |\Phi_{\boldsymbol{\lambda}}| \le C_0 r^{(2+m)\beta/\delta}\} \quad \text{satisfies}$$

$$E' = \{\boldsymbol{\lambda} \in Q \;:\; \mathrm{dist}(E, \boldsymbol{\lambda}) \le C_3 r^{(2+m)\beta/\delta}\} \subset \{\boldsymbol{\lambda} \in Q \;:\; |\Phi_{\boldsymbol{\lambda}}| \le C_\beta\, r^\beta\}. \tag{7.12}$$

Now fix any $\boldsymbol{\lambda}_0 \in E$. By Definition 7.2, (7.1) and the Cauchy–Binet formula there exist m coordinate directions, say the first m, so that the determinant of the first $m \times m$–minor of $D\Phi_{\boldsymbol{\lambda}_0}$ is bounded below by Cr^{β}. Moreover, by (7.12) and the Hölder bounds on $D\Phi_{\boldsymbol{\lambda}}$ in (7.2), this continues to hold on all of $B(\boldsymbol{\lambda}_0, C_4 r^{(2+m)\beta/\delta})$ for an appropriate choice of C_4. Let $s_m(\boldsymbol{\lambda}) \geq s_{m-1}(\boldsymbol{\lambda}) \geq \ldots \geq s_1(\boldsymbol{\lambda}) > 0$ be the singular values of the first $m \times m$–minor of $D\Phi_{\boldsymbol{\lambda}}$. We have shown that

$$s_m(\boldsymbol{\lambda}) \leq Cr^{-\beta} \ \text{ and } \ (s_m s_{m-1} \cdot \ldots \cdot s_1)(\boldsymbol{\lambda}) \geq Cr^{\beta}$$

on $B(\boldsymbol{\lambda}_0, C_4 r^{b_0\beta})$. Thus $s_1 \geq Cr^{m\beta}$ on $B(\boldsymbol{\lambda}_0, C_4 r^{b_0\beta})$. As above, let

$$F_{\vec{y}}(\lambda_1, \ldots, \lambda_m) = \Phi_{\boldsymbol{\lambda}} \restriction \{\boldsymbol{\lambda} \in B(\boldsymbol{\lambda}_0, C_4 r^{b_0\beta}) \;:\; \lambda_{m+1} = y_1, \ldots, \lambda_n = y_{n-m}\}$$

for any choice of $\vec{y} = (y_1, \ldots, y_{n-m})$. By the lower bound on the singular values, $\|(DF_{\vec{y}})^{-1}\| \leq Cr^{-m\beta}$ on the domain of $F_{\vec{y}}$. In particular, with $\boldsymbol{\lambda}' = (\lambda_1, \ldots, \lambda_m)$,

$$\|(DF_{\vec{y}}(\boldsymbol{\lambda}'_0))^{-1} \circ DF_{\vec{y}}(\boldsymbol{\lambda}') - I_{m\times m}\| \leq Cr^{-(m+1+\delta)\beta}|\boldsymbol{\lambda}' - \boldsymbol{\lambda}'_0|^{\delta}.$$

In view of Lemma 7.6 there exists a constant C_1 so that $F_{\vec{y}}$ is a diffeomorphism on

$$\{\boldsymbol{\lambda} \in B(\boldsymbol{\lambda}_0, 2C_1 r^{b_0\beta}) \;:\; \lambda_{m+1} = y_1, \ldots, \lambda_n = y_{n-m}\}$$

for any choice of $\vec{y}$. The lemma follows by applying Wiener's covering lemma to a covering of E with balls of size $\frac{1}{3}C_1 r^{b_0\beta}$. $\square$

We now prove the higher–dimensional version of Proposition 4.10.

Proposition 7.8. *Let $\Pi_{\boldsymbol{\lambda}} \in C^{1,\delta}(Q)$ for some $0 < \delta < 1$. Suppose that $Q \subset \mathbb{R}^n$ is a region of transversality of order β for Π and that $\Pi_{\boldsymbol{\lambda}}$ is $1,\delta$–regular on Q, see Definition 7.2. Assume that μ has finite α–energy for some $\alpha \in (0, m]$. Then*

$$\mathcal{H}^{\sigma+n-m}\big(\{\boldsymbol{\lambda} \in Q \;:\; \dim_s(\nu_{\boldsymbol{\lambda}}) < \sigma\}\big) = 0 \tag{7.13}$$

for any $\sigma \in (0, \alpha - a_0\beta]$ with a constant a_0 depending only on m, n, and δ.

Proof. It suffices to prove (7.13) for any subcube $Q' \subset Q$ with $\mathrm{dist}(Q', \partial Q) > 0$. As in the proof of Proposition 4.4 one assumes that (7.13) fails. By Frostman's lemma there exists a nonzero measure ρ on Q' so that $\rho(V) \leq |\mathrm{diam}(V)|^{\sigma+n-m}$ for all Borel sets $V \subset Q'$ and

$$\rho(\{\boldsymbol{\lambda} \in Q' \;:\; \dim_s(\nu_{\boldsymbol{\lambda}}) > \sigma - \epsilon_0\}) = 0 \tag{7.14}$$

for an appropriate choice of $\epsilon_0 > 0$. Fix any small $\epsilon > 0$ and let $r = d(\omega_1, \omega_2)$. In view of Lemma 7.7,

$$\begin{aligned}
&\int_{Q'}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{d\nu_{\boldsymbol{\lambda}}(x)d\nu_{\boldsymbol{\lambda}}(y)}{|x-y|^{\sigma-\epsilon}}\,d\rho(\boldsymbol{\lambda}) = \int_{Q'}\int_{\Omega}\int_{\Omega} \frac{d\mu(\omega_1)d\mu(\omega_2)}{|\Pi(\boldsymbol{\lambda},\omega_1) - \Pi(\boldsymbol{\lambda},\omega_2)|^{\sigma-\epsilon}}\,d\rho(\boldsymbol{\lambda}) \\
=\;& \int_{\Omega}\int_{\Omega}\int_{Q'} \chi_{[|\Phi_{\boldsymbol{\lambda}}| \geq C_0 r^{b_0\beta}]}|\Phi_{\boldsymbol{\lambda}}(\omega_1,\omega_2)|^{-(\sigma-\epsilon)}\,d\rho(\boldsymbol{\lambda})\frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{\sigma-\epsilon}} \\
&+ \int_{\Omega}\int_{\Omega}\int_{Q'} \chi_{[|\Phi_{\boldsymbol{\lambda}}| \leq C_0 r^{b_0\beta}]}|\Phi_{\boldsymbol{\lambda}}(\omega_1,\omega_2)|^{-(\sigma-\epsilon)}\,d\rho(\boldsymbol{\lambda})\frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{\sigma-\epsilon}} &&(7.15) \\
\leq\;& C\int_{\Omega}\int_{\Omega} \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{(\sigma-\epsilon)(1+b_0\beta)}} + C\int_{\Omega}\int_{\Omega}\sum_{j=1}^{N}\int_{B(\boldsymbol{\lambda}_j, C_1 r^{b_0\beta})} \frac{d\rho(\boldsymbol{\lambda})}{|\Phi_{\boldsymbol{\lambda}}|^{\sigma-\epsilon}}\frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{\sigma-\epsilon}} &&(7.16) \\
\leq\;& C\int_{\Omega}\int_{\Omega} \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{\alpha}}. &&(7.17)
\end{aligned}$$

To pass from (7.15) to (7.16) first notice that

$$\int_{B(\boldsymbol{\lambda}_j, C_1 r^{b_0\beta})} \frac{d\rho(\boldsymbol{\lambda})}{|\Phi_{\boldsymbol{\lambda}}|^{\sigma-\epsilon}} \leq \sum_{i=-\infty}^{\infty} 2^{i(\sigma-\epsilon)}\rho(\{\boldsymbol{\lambda} \in B(\boldsymbol{\lambda}_j, C_1 r^{b_0\beta}) : |\Phi_{\boldsymbol{\lambda}}| \leq 2^{-i}\}).$$

To bound the measure on the right–hand side one uses (7.11) and (7.2). In fact, those estimates imply that the set $\{\boldsymbol{\lambda} \in B(\boldsymbol{\lambda}_j, C_1 r^{b_0\beta}) : |\Phi_{\boldsymbol{\lambda}}| \le 2^{-i}\}$ is the union of $\asymp \Big(1 + \frac{r^{b_0\beta}}{2^{-i} r^{\beta}}\Big)^{n-m}$ many balls of diameter no larger than $C\min(r^{b_0\beta}, 2^{-i} r^{-m\beta})$. Thus

$$\int_{B(\boldsymbol{\lambda}_j, C_1 r^{b_0\beta})} \frac{d\rho(\boldsymbol{\lambda})}{|\Phi_{\boldsymbol{\lambda}}|^{\sigma-\epsilon}} \quad\le\quad C \sum_{i=-\infty}^{\infty} 2^{i(\sigma-\epsilon)} \big(\min(r^{b_0\beta}, 2^{-i} r^{-m\beta})\big)^{\sigma+n-m} \big(1 + r^{(b_0-1)\beta} 2^i\big)^{n-m} \asymp r^{-\tilde{C}\beta}$$

where $\tilde{C} = \tilde{C}(m,n,\sigma,\delta)$. Inserting this bound into (7.16) and using the bound on N given by Lemma 7.7 implies (7.17) for an appropriate choice of a_0. However, (7.17) contradicts (7.14) if $\epsilon < \epsilon_0$. $\square$

In view of Propositions 7.5 and 7.8, Theorem 7.3 will follow once the Sobolev estimate (7.3) is established. As in section 4, this will require a technical lemma about averages involving Littlewood–Paley functions, see Lemmas 4.6 and 7.10. The proof of that lemma will use the following calculus fact.

Lemma 7.9. *Suppose $h = (h_1, \ldots, h_m) : U \subset \mathbb{R}^m \to \mathbb{R}^m$ is a C^L diffeomorphism on the open set U and let H denote the inverse of h. For any nonzero multi-index $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{N}^m$ with $|\eta| \le L$,*

$$\partial^\eta H = \sum_{t=0}^{|\eta|-1} \sum_{p} \sum_{\sigma_1,\ldots,\sigma_p,\vec{\ell}} \big(Dh \circ H\big)^{-|\eta|-p}\, \vec{v}_{\eta,p}(\sigma_1,\ldots,\sigma_p,\vec{\ell}) \prod_{i=1}^{p} \big(\partial^{\sigma_i} h_{\ell_i}\big) \circ H \tag{7.18}$$

where the third sum runs over multi-indices $\sigma_i \in \mathbb{N}^m$ and $\vec{\ell} \in \{1, 2, \ldots, m\}^p$. If $|\eta| \ge 2$ the vectors $\vec{v}_{\eta,p}(\sigma_1, \ldots, \sigma_p, \vec{\ell}) \in \mathbb{R}^m$ vanish unless $|\eta| \ge |\sigma_i| \ge 2$ for all i and $\sum_{i=1}^p |\sigma_i| \le 2(|\eta| - 1)$.

Proof. If $\eta = (1, 0, \ldots, 0) = e_1$, one clearly has $\partial^\eta H = \big(Dh \circ H\big)e_1$. Hence the Lemma holds with $|\eta| = 1$. For the inductive step one differentiates (7.18):

$$\begin{aligned}
\partial^{\eta+e_1} H \quad = \quad & \sum_{t=0}^{|\eta|-1} \sum_{p} \sum_{\sigma_1,\ldots,\sigma_p,\vec{\ell}} \Big\{ -(|\eta|+p)\big(Dh \circ H\big)^{-|\eta|-p-1} \partial^{e_1}\big(Dh \circ H\big) \\
& \vec{v}_{\eta,p}(\sigma_1,\ldots,\sigma_p,\vec{\ell}) \prod_{i=1}^{p} \big(\partial^{\sigma_i} h_{\ell_i}\big) \circ H + \big(Dh \circ H\big)^{-|\eta|-p} \vec{v}_{\eta,p}(\sigma_1,\ldots,\sigma_p,\vec{\ell}) \\
& \sum_{j=1}^{p} \Big(\prod_{i=1,\, i\ne j}^{p} \big(\partial^{\sigma_i} h_{\ell_i}\big) \circ H \Big) \partial^{e_1}\big((\partial^{\sigma_j} h_{\ell_j}) \circ H\big) \Big\}.
\end{aligned}$$

Since for any multi-index σ

$$\partial^{e_1}\Big(\big(\partial^\sigma h\big) \circ H\Big) = \big(Dh \circ H\big)^{-1} \sum_{|\tau|=|\sigma|+1} a_\tau \big(\partial^\tau h\big) \circ H$$

with suitable constant vectors $a_\tau \in \mathbb{R}^m$, (7.18) holds for all multi-indices η' with $|\eta'| = |\eta| + 1$. Also notice that $\sum_{i=1}^p |\sigma_i|$ has increased by at most two, as claimed. $\square$

Lemma 7.10. *Let $\Pi_{\boldsymbol{\lambda}} \in C^{L,\delta}(Q)$ for some positive integer L and some $\delta \in [0,1)$. Suppose that $Q \subset \mathbb{R}^n$ is a region of transversality of order β for Π and that $\Pi_{\boldsymbol{\lambda}}$ is L,δ–regular on Q, see Definition 7.2. Let $\rho \in C^\infty(\mathbb{R}^n)$ be supported inside Q. If ψ is the Littlewood–Paley function in $\mathbb{R}^m$ from Lemma 4.1, then for any distinct $\omega_1, \omega_2 \in \Omega$, any integer j, and any $0 \le q < L + \delta + m - 1$,*

$$\left| \int \rho(\boldsymbol{\lambda}) \psi\big(2^j [\Pi(\boldsymbol{\lambda}, \omega_1) - \Pi(\boldsymbol{\lambda}, \omega_2)]\big)\, d\boldsymbol{\lambda} \right| \le C_q\, (1 + 2^j d(\omega_1, \omega_2)^{1+a_0\beta})^{-q} \tag{7.19}$$

where C_q depends only on $q, m, n, \rho, \beta, L, \delta$ and a_0 only depends on m, n, and L, δ.

Proof. Fix distinct $\omega_1, \omega_2 \in \Omega$ and j, q as above. We may assume that $2^j r > 1$ where $r = d(\omega_1, \omega_2)$. Let $\phi \in C^\infty$ be nonnegative with $\phi = 1$ on $[-1, 1]$ and $\text{supp}(\phi) \subset [-2, 2]$. Then

$$\begin{aligned}
&\int \rho(\boldsymbol{\lambda})\psi\big(2^j[\Pi(\boldsymbol{\lambda}, \omega_1) - \Pi(\boldsymbol{\lambda}, \omega_2)]\big)\, d\boldsymbol{\lambda} \\
&= \int \rho(\boldsymbol{\lambda})\psi(2^j r\Phi_{\boldsymbol{\lambda}}(\omega_1, \omega_2))\phi(C_\beta^{-1}\, r^{-\beta}\Phi_{\boldsymbol{\lambda}})\, d\boldsymbol{\lambda} + \int \rho(\boldsymbol{\lambda})\psi(2^j r\Phi_{\boldsymbol{\lambda}}(\omega_1, \omega_2))\big[1 - \phi(C_\beta^{-1}\, r^{-\beta}\Phi_{\boldsymbol{\lambda}})\big]\, d\boldsymbol{\lambda}. \qquad (7.20)
\end{aligned}$$

Here C_β is the constant from Definition 7.2. By the rapid decay of ψ

$$\left|\int \rho(\boldsymbol{\lambda})\psi(2^j r\Phi_{\boldsymbol{\lambda}}(\omega_1, \omega_2))\big[1 - \phi(C_\beta^{-1}\, r^{-\beta}\Phi_{\boldsymbol{\lambda}})\big]\, d\boldsymbol{\lambda}\right| \le C_{q,\beta}\int |\rho(\boldsymbol{\lambda})|(1 + 2^j r^{1+\beta})^{-q}\, d\boldsymbol{\lambda} \le C_{q,\beta}(1 + 2^j r^{1+\beta})^{-q}.$$

Thus it suffices to estimate the first integral in (7.20). To this end we introduce a partition of unity subordinate to the cover given by Lemma 7.7 with $U = \{Q\ :\ \rho > 0\}$. More precisely, by the standard construction of a partition of unity there exist $\chi_j \in C^\infty(\mathbb{R}^n)$ so that $\text{supp}(\chi_j) \subset B_j = B(\boldsymbol{\lambda}_j, 2C_1 r^{b_0\beta})$ for $j = 1, \ldots, N$ and so that $\sum_{i=1}^N \chi_j = 1$ on $\{\boldsymbol{\lambda} \in \text{supp}(\rho)\ :\ |\Phi_{\boldsymbol{\lambda}}| \le C_0 r^{b_0\beta}\}$, see (7.9). Moreover,

$$\sup_j \|\partial^\eta \chi_j\|_\infty \le C_\eta\, r^{-|\eta| b_0\beta} \qquad (7.21)$$

for all multi-indices $\eta \in \mathbb{N}^n$. By (7.10), $\chi_j(\boldsymbol{\lambda}) = \chi_j(\boldsymbol{\lambda})\phi(C_\beta^{-1}\, r^{-\beta}\Phi_{\boldsymbol{\lambda}})$ for all j. We can therefore write the first integral in (7.20) as follows.

$$\begin{aligned}
\int \rho(\boldsymbol{\lambda})\psi(2^j r\Phi_{\boldsymbol{\lambda}})\phi(C_\beta^{-1}\, r^{-\beta}\Phi_{\boldsymbol{\lambda}})\, d\boldsymbol{\lambda} &= \sum_{i=1}^N \int \rho(\boldsymbol{\lambda})\chi_j(\boldsymbol{\lambda})\psi(2^j r\Phi_{\boldsymbol{\lambda}})\, d\boldsymbol{\lambda} \\
+ \int \rho(\boldsymbol{\lambda})\big[1 - \sum_{i=1}^N \chi_j(\boldsymbol{\lambda})\big]\psi(2^j r\Phi_{\boldsymbol{\lambda}})\phi(C_\beta^{-1}\, r^{-\beta}\Phi_{\boldsymbol{\lambda}})\, d\boldsymbol{\lambda} &= \sum_{i=1}^N A^{(j)} + B.
\end{aligned}$$

To estimate the second term B notice that $|\Phi_{\boldsymbol{\lambda}}| \ge C_0 r^{b_0\beta}$ on the support of the integrand. The rapid decay of ψ therefore implies $|B| \le C_q(1 + 2^j r^{1+b_0\beta})^{-q}$. For simplicity, we assume henceforth that $i = 1$. By Lemma 7.7 the map $\boldsymbol{\lambda}' \mapsto \Phi_{(\boldsymbol{\lambda}', \boldsymbol{\lambda}'')} = F_{\boldsymbol{\lambda}''}(\boldsymbol{\lambda}')$ is a diffeomorphism on $\{\boldsymbol{\lambda}' \in \mathbb{R}^m\ :\ (\boldsymbol{\lambda}', \boldsymbol{\lambda}'') \in B_j\}$, where $\boldsymbol{\lambda}' = (\lambda_1, \ldots, \lambda_m)$ and $\boldsymbol{\lambda}'' = (\lambda_{m+1}, \ldots, \lambda_n)$ (possibly after a permutation of the coordinates). Let $H_{\boldsymbol{\lambda}''}$ denote the inverse of $F_{\boldsymbol{\lambda}''}$. Assuming as we may that $\rho(\boldsymbol{\lambda}) = \rho_1(\boldsymbol{\lambda}')\rho_2(\boldsymbol{\lambda}'')$ we let

$$G_{\boldsymbol{\lambda}''}(u) = \rho_1(H_{\boldsymbol{\lambda}''}(u))\chi_1(H_{\boldsymbol{\lambda}''}(u))\big|\det\big(DH_{\boldsymbol{\lambda}''}(u)\big)\big|.$$

Clearly, $G_{\boldsymbol{\lambda}''} \in C^{L-1,\delta}$. Since $q < L + \delta + m - 1$, there exists an integer M so that $q - m - \delta < M \le L - 1$. Fix $\epsilon > 0$ such that $(M + m + \delta)(1 - \epsilon) > q$ and rewrite $A^{(1)}$ as follows.

$$\begin{aligned}
A^{(1)} &= \int_{\mathbb{R}^{n-m}} \rho_2(\boldsymbol{\lambda}'')\int_{\mathbb{R}^m} G_{\boldsymbol{\lambda}''}(u)\psi(2^j ru)\, du\, d\boldsymbol{\lambda}'' \\
&= \int_{\mathbb{R}^{n-m}} \rho_2(\boldsymbol{\lambda}'')\int_{|u|<(2^j r)^{-1+\epsilon}} \psi(2^j ru)\left\{\sum_{|\eta|\le M} \frac{\partial^\eta G_{\boldsymbol{\lambda}''}(0)}{\eta!} u^\eta\right. \\
&\qquad \left. + \sum_{|\eta|=M} \int_0^1 \big[\partial^\eta G_{\boldsymbol{\lambda}''}(tu) - \partial^\eta G_{\boldsymbol{\lambda}''}(0)\big](1-t)^M\, dt\, \frac{u^\eta}{\eta!}\right\} du\, d\boldsymbol{\lambda}'' \\
&\quad + \int_{\mathbb{R}^{n-m}} \rho_2(\boldsymbol{\lambda}'')\int_{|u|>(2^j r)^{-1+\epsilon}} O\big((2^j r|u|)^{-2q-m}\big)|G_{\boldsymbol{\lambda}''}(u)|\, du\, d\boldsymbol{\lambda}'' \\
&= A_1^{(1)} + A_2^{(1)}.
\end{aligned}$$

By Lemma 7.7, $\|G_{\boldsymbol\lambda''}\|_\infty \le Cr^{-\beta}$. In particular, $|A_2^{(1)}| \le C_q(2^jr)^{-q-m}r^{-\beta}$. Since ψ has vanishing moments of all orders

$$
\begin{aligned}
A_1^{(1)} &= -\int_{\mathbb{R}^{n-m}} \rho_2(\boldsymbol\lambda'')\int_{|u|>(2^jr)^{-1+\epsilon}} \psi(2^jru)\sum_{|\eta|\le M}\frac{\partial^\eta G_{\boldsymbol\lambda''}(0)}{\eta!}u^\eta\,du\,d\boldsymbol\lambda'' \\
&\quad + \int_{\mathbb{R}^{n-m}} \rho_2(\boldsymbol\lambda'')O\Big(\sup_{|\eta|=M}\|\partial^\eta G_{\boldsymbol\lambda''}\|_{C^\delta}\;(2^jr)^{-(M+m+\delta)(1-\epsilon)}\Big)\,d\boldsymbol\lambda''.
\end{aligned}
\tag{7.22}
$$

It remains to estimate $\partial^\eta G_{\boldsymbol\lambda''}(u)$, which can be done uniformly in $\boldsymbol\lambda''$. Fix any $\boldsymbol\lambda''$ and let $u = F_{\boldsymbol\lambda''}(\boldsymbol\lambda')$. By Lemma 7.9, (7.2) in Definition 7.2, and Lemma 7.7,

$$
\begin{aligned}
|\partial^\eta H_{\boldsymbol\lambda''}(u)| &\le C_\eta \sum_{t=0}^{|\eta|-1}\sum_p\sum_{\sigma_1,\dots,\sigma_p,\vec\ell} |\vec v_{\eta,p}(\sigma_1,\dots,\sigma_p,\vec\ell)|\;\|\big(DF_{\boldsymbol\lambda''}\big)^{-1}\|^{|\eta|+p}\; r^{-2\beta(|\eta|-1)} \\
&\le C_\eta\, r^{-2|\eta|(m+1)\beta}.
\end{aligned}
$$

Therefore, by Leibnitz's rule and (7.21), $\|\partial^\eta G_{\boldsymbol\lambda''}\|_\infty \le C_\eta r^{-C|\eta|\beta}$ where $C = C(b_0,m)$. Plugging this into (7.22) and exploiting the rapid decay of ψ one obtains $|A_1^{(1)}| \le C_q\, r^{-a_0M\beta}(2^jr)^{-(M+m+\delta)(1-\epsilon)}$. Since $N \le C\,r^{-b_0n\beta}$ and $(M+m+\delta)(1-\epsilon) > q$, this finally yields (7.19). $\square$

Proof of (7.3). Fix some γ with $0 < (m+2\gamma)(1+a_0\beta) \le \alpha$ and $2\gamma < L+\delta-1$. Let ρ be a smooth nonnegative function on $\mathbb{R}^n$ so that $\mathrm{supp}(\rho) \subset Q$ and fix some $q \in (m+2\gamma, L+\delta+m-1)$. In view of (4.2), the definition of $\nu_{\boldsymbol\lambda}$, and Lemma 7.10

$$
\begin{aligned}
\int \|\nu_{\boldsymbol\lambda}\|_{2,\gamma}^2\,\rho(\boldsymbol\lambda)\,d\boldsymbol\lambda &\asymp \int \sum_{j=-\infty}^{\infty} 2^{2j\gamma}\int_{-\infty}^{\infty}(\psi_{2^{-j}} * \nu_{\boldsymbol\lambda})(x)\,d\nu_{\boldsymbol\lambda}(x)\;\rho(\boldsymbol\lambda)\,d\boldsymbol\lambda \\
&\le \int_\Omega\int_\Omega \sum_{j=-\infty}^{\infty} 2^{j(m+2\gamma)}\left|\int \psi\Big(2^j[\Pi(\boldsymbol\lambda,\omega_1)-\Pi(\boldsymbol\lambda,\omega_2)]\Big)\rho(\boldsymbol\lambda)\,d\boldsymbol\lambda\right| d\mu(\omega_1)d\mu(\omega_2) \\
&\le C_\gamma\int_\Omega\int_\Omega\sum_{-\infty}^{\infty} 2^{j(m+2\gamma)}(1+2^jd(\omega_1,\omega_2)^{1+a_0\beta})^{-q}\,d\mu(\omega_1)d\mu(\omega_2) \\
&\le C_\gamma\int_\Omega\int_\Omega \frac{d\mu(\omega_1)d\mu(\omega_2)}{d(\omega_1,\omega_2)^{(1+a_0\beta)(m+2\gamma)}} \le C\,\mathcal{E}_\alpha(\mu) < \infty,
\end{aligned}
$$

as claimed. $\square$

8. Applications of the higher–dimensional projection theorems

8.1. Self–similar sets in the complex plane. In [41] Solomyak considered sets of the form

$$
C^S_{\boldsymbol\lambda} = \Big\{\sum_{n=0}^{\infty} a_n\boldsymbol\lambda^n \;:\; a_n \in S\Big\}
\tag{8.1}
$$

where $S = \{s_1,\dots,s_\ell\}$ is a set of digits and $\boldsymbol\lambda \in \mathbb{D} = \{z \in \mathbb{C} \;:\; |z| < 1\}$ ($\boldsymbol\lambda^n$ of course denotes complex powers). Let $\Gamma = S - S$ and

$$
\begin{aligned}
\mathcal{B}_\Gamma &= \Big\{\sum_{n=0}^{\infty} b_n\boldsymbol\lambda^n \;:\; b_n \in \Gamma\Big\} \\
\mathcal{M}_\Gamma &= \{\boldsymbol\lambda \in \mathbb{D} \;:\; \text{there exists a nonzero } \; f \in \mathcal{B}_\Gamma \;\text{ with }\; f(\boldsymbol\lambda) = 0\} \\
\widetilde{\mathcal{M}}_\Gamma &= \{\boldsymbol\lambda \in \mathbb{D} \;:\; \text{there exists a nonzero } \; f \in \mathcal{B}_\Gamma \;\text{ with }\; f(\boldsymbol\lambda) = f'(\boldsymbol\lambda) = 0\}.
\end{aligned}
$$

If $\boldsymbol{\lambda} \in \mathbb{D} \setminus \mathcal{M}_\Gamma$ it is easy to see that $C^S_{\boldsymbol{\lambda}}$ satisfies a strong separation condition as in (5.14) and that therefore $\dim C^S_{\boldsymbol{\lambda}} = \frac{\log \ell}{-\log|\boldsymbol{\lambda}|}$. The case $\boldsymbol{\lambda} \in \mathcal{M}_\Gamma$ was studied in [41]assuming the transversality condition $\boldsymbol{\lambda} \notin \widetilde{\mathcal{M}}_\Gamma$. It was shown there that for any fixed choice of digit set $S \subset \mathbb{C}$ a.e. $\boldsymbol{\lambda} \in \mathcal{M}_\Gamma \setminus \widetilde{\mathcal{M}}_\Gamma$ satisfies

$$\dim C^S_{\boldsymbol{\lambda}} = \frac{\log \ell}{-\log|\boldsymbol{\lambda}|} \text{ if } |\boldsymbol{\lambda}| < \ell^{-1/2} \quad \text{and} \quad \mathcal{H}^2(C^S_{\boldsymbol{\lambda}}) > 0 \text{ if } |\boldsymbol{\lambda}| > \ell^{-1/2}.$$

It is straightforward to see that Theorem 7.3 applies to this case. First we verify that conditions (7.2), (7.1) in Definition 7.2 are satisfied with $L = \infty$. This is very similar to Lemma 5.3.

To specialize from section 7 let $0 < R_1 < R_2 < 1$ and

$$Q \subset \{\boldsymbol{\lambda} \in \mathbb{D} \,:\, \boldsymbol{\lambda} \notin \widetilde{\mathcal{M}}_\Gamma, \ \ R_1 < |\boldsymbol{\lambda}| < R_2\}$$

be a fixed closed cube. As observed in [41],$\widetilde{\mathcal{M}}$ is relatively closed in $\mathbb{D}$ since $\mathcal{B}_\Gamma$ is a normal family. With this choice of Q let $\Omega = \{1, \dots, l\}^{\mathbb{N}}$ be equipped with the metric $d(\omega, \tau) = R_2^{|\omega\wedge\tau|}$ and uniform product measure μ. Finally, let $\Pi(\boldsymbol{\lambda}, \omega) = \sum_{n=0}^{\infty} s_{\omega_n} \boldsymbol{\lambda}^n$.

Lemma 8.1. *Let $R_2^{1+\beta} > R_1$. Then Q is a region of transversality of order 2β for Π in the sense of Definition 7.2.*

Proof. There exists $\delta = \delta(Q) \in (0, 1)$ such that for any $g \in \mathcal{B}_\Gamma$ and $\boldsymbol{\lambda} \in Q$

$$|g(\boldsymbol{\lambda})| < \delta \quad \text{implies} \quad |g'(\boldsymbol{\lambda})| > \delta.$$

This follows since $\mathcal{B}_\Gamma$ is a normal family. Clearly, $\Phi_{\boldsymbol{\lambda}}(\omega, \tau) = (\boldsymbol{\lambda} R_2^{-1})^{|\omega\wedge\tau|} g(\boldsymbol{\lambda})$ with some $g = g_{\omega,\tau} \in \mathcal{B}_\Gamma$. Fix $\omega, \tau \in \Omega$ and let $k = |\omega \wedge \tau|$. Therefore,

$$\begin{aligned} |\partial^\eta \Phi_{\boldsymbol{\lambda}}| &\le C_\eta \Big(k^{|\eta|} |\boldsymbol{\lambda}|^{k-|\eta|} R_2^{-k} \|g\|_\infty + |\boldsymbol{\lambda}|^k R_2^{-k} \sup_{|\sigma|=|\eta|} \|\partial^\sigma g\|_\infty \Big) \\ &\le C_\eta \big(k^{|\eta|} R_1^{-|\eta|} (1-R_2)^{-1} + (1-R_2)^{-|\eta|-1} \big) \le C_{\eta,\beta}\, R_2^{-\beta|\eta|k}, \end{aligned}$$

since $R_2 < 1$ and $\beta > 0$. Thus condition (7.2) in 7.2 holds with $L = \infty$. To check (7.1) assume $|\Phi_{\boldsymbol{\lambda}}| \le \delta b_\beta R_2^{\beta k}$ where the constant $b_\beta \in (0, 1)$ will be determined below. Then

$$\big(R_1 R_2^{-1}\big)^k |g(\boldsymbol{\lambda})| \le \big(|\boldsymbol{\lambda}| R_2^{-1}\big)^k |g(\boldsymbol{\lambda})| \le \delta b_\beta R_2^{\beta k}$$

implies $|g(\boldsymbol{\lambda})| \le \delta b_\beta \big(R_1^{-1} R_2^{1+\beta}\big)^k$. Hence (here $\Phi'_{\boldsymbol{\lambda}}$ denotes the complex derivative)

$$|\Phi'_{\boldsymbol{\lambda}}| \ge \big(R_1 R_2^{-1}\big)^k \big(|g'(\boldsymbol{\lambda})| - R_1^{-1} k |g(\boldsymbol{\lambda})|\big) \ge R_2^{\beta k} \Big(\delta - \delta b_\beta R_1^{-1} k \big(R_1^{-1} R_2^{1+\beta}\big)^k \Big) \ge \delta R_2^{\beta k}/2 \ge \delta b_\beta R_2^{\beta k}$$

if $b_\beta \le \frac{1}{2} \Big[1 + \sup_{k \ge 0} \frac{k}{R_1} \big(R_1^{-1} R_2^{1+\beta}\big)^k \Big]^{-1}$. Since the Cauchy–Riemann equations imply that $\det[D\Phi_{\boldsymbol{\lambda}}] = |\Phi'_{\boldsymbol{\lambda}}|^2$, condition (7.1) in Definition 7.2 holds with 2β instead of β and $C_\beta = (\delta b_\beta)^2$. □

Theorem 8.2. *Let $0 < r < R < 1$. Then*

$$\begin{aligned} \dim \big\{ \boldsymbol{\lambda} \in \mathcal{M}_\Gamma \setminus \widetilde{\mathcal{M}}_\Gamma \,:\, r < |\boldsymbol{\lambda}| < R, \quad \mathcal{H}^2(C^S_{\boldsymbol{\lambda}}) = 0 \big\} &\le 4 - \frac{\log \ell}{-\log r} \\ \dim \Big\{ \boldsymbol{\lambda} \in \mathcal{M}_\Gamma \setminus \widetilde{\mathcal{M}}_\Gamma \,:\, r < |\boldsymbol{\lambda}| < R, \quad \dim C^S_{\boldsymbol{\lambda}} < \frac{\log \ell}{-\log|\boldsymbol{\lambda}|} \Big\} &\le \frac{\log \ell}{-\log R}. \end{aligned}$$

Proof. Fix any $\beta \in (0, 1)$. There exist countably many cubes $Q_j \subset \mathbb{C}$ so that

$$\{\boldsymbol{\lambda} \in \mathbb{D} \setminus \widetilde{\mathcal{M}}_\Gamma \,:\, r < |\boldsymbol{\lambda}| < R\} = \bigcup_j Q_j$$

and $Q_j \subset \{r_j < |\boldsymbol{\lambda}| < R_j\}$ with $r_j > R_j^{1+\beta}$. By the previous lemma each Q_j is a region of transversality of order 2β for Π. It is easy to check that with the choice of metric $d_j(\omega,\tau) = R_j^{|\omega\wedge\tau|}$ one has $\mathcal{E}_\alpha(\mu) < \infty$ if $R_j^\alpha l > 1$. Since $\mathrm{supp}(\nu_{\boldsymbol{\lambda}}) = C^S_{\boldsymbol{\lambda}}$, the theorem now follows by applying Theorem 7.3 and letting $\beta \to 0+$. $\square$

8.2. **Intersections of sets with spheres.** Let $S_0 \subset \mathbb{R}^d$, $d \geq 2$, be a closed, strictly convex C^∞–hypersurface surrounding the origin (the differentiability assumption will be relaxed later). Thus, $S_0 = \{\rho(u)u \,:\, u \in S^{d-1}\}$ for some function $\rho : S^{d-1} \to (0,\infty)$ in C^∞. Define $f \in C^\infty(\mathbb{R}^d \setminus \{0\})$ to be $f(x) = \frac{|x|}{\rho(x/|x|)}$. In particular, f is homogeneous of degree one and $\{x \in \mathbb{R}^d \,:\, f(x) = r\} = rS_0$. Moreover, by the assumption of strict convexity there exists $\kappa > 0$ so that for any $x \neq 0$ and any tangent vector v of S_0

$$|\langle D^2 f(x)v, v\rangle| \geq \kappa \frac{|v|^2}{|x|}, \tag{8.2}$$

whereas $D^2 f(x)x = 0$ by homogeneity.

For any Borel set $E \subset \mathbb{R}^d$ and $\sigma < 1$ let

$$B_\sigma = B_\sigma(E) = \{x \in \mathbb{R}^d \,:\, \dim\, f(-x+E) < \sigma\}.$$

For $\sigma = 1$ we set

$$B_1 = B_1(E) = \{x \in \mathbb{R}^d \,:\, \mathcal{H}^1(f(-x+E)) = 0\}.$$

Our main result in this section concerns the dimension of B_σ.

Theorem 8.3. *Let S_0 be a strictly convex, C^∞–hypersurface surrounding the origin and define f and B_σ as above. If $E \subset \mathbb{R}^d$ is a Borel set and $\sigma \in [0, 1 \wedge \dim E]$, then*

$$\dim\, B_\sigma \leq d + \sigma - \max(1, \dim E). \tag{8.3}$$

Furthermore, for any hyperplane $H \subset \mathbb{R}^d$,

$$\dim(B_\sigma \cap H) \leq d - 1 + \sigma - \max(1, \dim E). \tag{8.4}$$

Proof. (8.3) follows from (8.4) by classical theorems of Marstrand and Mattila, see Theorem 10.10 in [31]. For the sake of illustration, however, we begin by showing directly that (8.3) follows from Theorem 7.3. In fact, we first consider the case $S_0 = S^{d-1}$ because it is very easy to check the conditions in Definition 7.2 for Euclidean spheres. In this special case, it is more convenient to work with the square of the Euclidean norm, i.e., $f(x) = |x|^2$. Define $\Pi : \mathbb{R}^d \times E \to \mathbb{R}$ to be $\Pi(\boldsymbol{\lambda}, x) = |x - \boldsymbol{\lambda}|^2$ and let $\Phi_{\boldsymbol{\lambda}}(x,y) = |x-y|^{-1}[|x-\boldsymbol{\lambda}|^2 - |y-\boldsymbol{\lambda}|^2] = \langle \frac{x-y}{|x-y|}, x+y-2\boldsymbol{\lambda}\rangle$. Therefore, $\big|\partial^\eta \Phi_{\boldsymbol{\lambda}}(x,y)\big| \leq 2$ for any nonzero multi–index η, and condition (7.2) holds with $\beta = 0$. Moreover, $\big|D\Phi_{\boldsymbol{\lambda}}(x,y)\big| = 2\big|\frac{x-y}{|x-y|}\big| = 2$, and (7.1) is also satisfied with $\beta = 0$. Hence Definition 7.2 holds with $Q = \mathbb{R}^d$, $\Omega \subset E$ an arbitrary compact set, $L = \infty$, and $\beta = 0$. For any $\epsilon > 0$ one can choose a Frostman measure μ on Ω with exponent $> \dim E - \epsilon$. Since $f(-\boldsymbol{\lambda} + E) \supset \mathrm{supp}(\nu_{\boldsymbol{\lambda}})$, (8.3) with $S_0 = S^{d-1}$ follows from Theorem 7.3 as $\epsilon \to 0+$.

For general S_0 as described above, fix a small $\epsilon > 0$. Dilating, if necessary, one has $\dim(E \setminus Q_0) > \dim E - \epsilon$ for any cube Q_0 of side length 2. Now fix a cube $Q \subset \mathbb{R}^d$ of side length 1. To bound $\dim(B_\sigma \cap Q)$ we may therefore assume that $E \subset 2rQ \setminus (rQ)$ for some $r \geq 2$. Thus, if $|v| = 1$, $x \in E$, and $\boldsymbol{\lambda} \in Q$, (8.2) implies that

$$|\langle \nabla f(x-\boldsymbol{\lambda}), v\rangle| = 0 \Longrightarrow |D^2 f(x-\boldsymbol{\lambda})v| \geq \rho \tag{8.5}$$

for some $\rho > 0$ depending on r and κ. Now define $\Pi_{\boldsymbol{\lambda}} : E \to \mathbb{R}$ to be $\Pi_{\boldsymbol{\lambda}}(x) = f(x - \boldsymbol{\lambda})$ and let $\Phi_{\boldsymbol{\lambda}}(x,y) = |x-y|^{-1}[f(x-\boldsymbol{\lambda}) - f(y-\boldsymbol{\lambda})]$. It is clear that condition (7.2) holds with $\beta = 0$. To check (7.1) notice that

$$\Phi_{\boldsymbol{\lambda}}(x,y) = |x-y|^{-1}[f(x-\boldsymbol{\lambda}) - f(y-\boldsymbol{\lambda})] = \langle \nabla f(x-\boldsymbol{\lambda}), v\rangle + O(|x-y|)$$

where $v = \frac{x-y}{|x-y|}$. Moreover,

$$D\Phi_{\boldsymbol{\lambda}}(x,y) = -D^2 f(x-\boldsymbol{\lambda})v + O(|x-y|) \tag{8.6}$$

where D denotes differentiation with respect to $\boldsymbol{\lambda}$. In view of (8.5) there exists some small constant C_0 such that

$$|\Phi_{\boldsymbol{\lambda}}(x,y)| < C_0\,\rho \Longrightarrow |D\Phi_{\boldsymbol{\lambda}}(x,y)| > \rho/2 \tag{8.7}$$

provided $\boldsymbol{\lambda} \in Q$ and $x, y \in E$ satisfy $|x-y| < C_0\,\rho$. Now let $\Omega = E \cap B(x_0, C_0\,\rho/2)$ be equipped with the usual Euclidean metric where x_0 is arbitrary. We have shown that Definition 7.2 is satisfied with $\beta = 0$ with this choice of Ω, Q, and $\Pi_{\boldsymbol{\lambda}}$. Therefore, selecting x_0 so that $\dim(E \cap B(x_0, C_0\,\rho)) > \dim E - \epsilon$ and letting μ be an appropriate Frostman measure we conclude that (8.3) follows from (7.4) and (7.6) with $L = \infty$, $\beta = 0$, $n = d$, $m = 1$, and $\alpha = \dim E - \epsilon$ as $\epsilon \to 0+$.

The proof of (8.4) proceeds by induction in the dimension d. First we need to verify that transversality holds in all dimensions. Fix a hyperplane $H \subset \mathbb{R}^d$. W.l.o.g. $0 \in H$. Let $e_1, \ldots, e_{d-1}$ be a basis in H and define $\Pi_{\boldsymbol{\lambda}}(x) = f(x - \sum_{i=1}^{d-1} \lambda_i e_i)$, $\Phi_{\boldsymbol{\lambda}}(x,y) = |x-y|^{-1}\big[\Pi_{\boldsymbol{\lambda}}(x) - \Pi_{\boldsymbol{\lambda}}(y)\big]$ for any $\boldsymbol{\lambda} \in \mathbb{R}^{d-1}$, and distinct $x, y \in \mathbb{R}^d$. Clearly, with $p = x - \lambda_1 e_1 - \ldots - \lambda_{d-1} e_{d-1}$,

$$\Phi_{\boldsymbol{\lambda}}(x,y) = \big\langle \nabla f(p), v\big\rangle + O(|x-y|) \tag{8.8}$$

$$D\Phi_{\boldsymbol{\lambda}}(x,y) = -\Big(\big\langle D^2 f(p)v, e_1\big\rangle, \ldots, \big\langle D^2 f(p)v, e_{d-1}\big\rangle\Big) + O(|x-y|), \tag{8.9}$$

where $v = \frac{x-y}{|x-y|}$ and D denotes differentiation with respect to $\boldsymbol{\lambda}$. We claim that for any $v \in S^{d-1}$ and any $z \notin H$

$$\langle \nabla f(z), v\rangle = 0 \Longrightarrow \sum_{i=1}^{d-1} \big|\big\langle D^2 f(z)v, e_i\big\rangle\big|^2 > 0. \tag{8.10}$$

In fact, we will show that (8.10) can only fail if $z \in H$. More precisely, suppose the sum in (8.10) vanishes for some $z \in \mathbb{R}^d$, $z \neq 0$ and some $v \in S^{d-1}$. W.l.o.g. $f(z) = 1$. Then the functionals $u \mapsto \big\langle D^2 f(z)u, e_i\big\rangle$, $i = 1, \ldots, d-1$, on the tangent space to S at z are linearly dependent. Thus

$$\sum_{i=1}^{d-1} \xi_i D^2 f(z) e_i \parallel \nabla f(z) \tag{8.11}$$

for some $(\xi_1, \ldots, \xi_{d-1}) \neq (0, \ldots, 0)$. Let $e = \sum_{i=1}^{d-1} \xi_i e_i$. Clearly, $e \neq 0$. It is easy to verify that (8.11) implies that $\big\langle D^2 f(z)e, e\big\rangle = 0$. First notice that by strict convexity of S, there is a parameterization $z + te = r(t)\gamma(t)$ for small t where γ is a smooth curve in S with $\gamma(0) = z$ and r is a smooth positive function with $r(0) = 1$. Let $\partial_e f = \langle \nabla f, e\rangle$ be the directional derivative. Since $\partial_e f$ is homogeneous of degree zero,

$$\big\langle D^2 f(z)e, e\big\rangle = \frac{d^2}{dt^2} f(z+te)\Big|_{t=0} = \frac{d}{dt}\partial_e f(\gamma(t))\Big|_{t=0} = \Big\langle D^2 f(z)e, \frac{d}{dt}\gamma(0)\Big\rangle = 0,$$

by (8.11). In view of (8.2) and $D^2 f(z)z = 0$, we conclude that $e \parallel z$ and thus $z \in H$, as claimed. To check transversality, we will need the following quantitative version of (8.10), cf. (8.5). For any $\rho_0 > 0$ there exists $\rho_1 > 0$ depending on κ from (8.2) so that for any $z \in \mathbb{R}^d$ with $f(z) = 1$ and $\mathrm{dist}(z, H) > \rho_0$ one has

$$|\langle \nabla f(z), v\rangle| < \rho_1 \Longrightarrow \sum_{i=1}^{d-1} \big|\big\langle D^2 f(z)v, e_i\big\rangle\big|^2 > \rho_1. \tag{8.12}$$

Indeed, by (8.10) and compactness,

$$\min_{z\in S_0:\mathrm{dist}(z,H)>\rho_0}\ \min_{v\in T_z S_0:|v|=1} \sum_{i=1}^{d-1} \big|\big\langle D^2 f(z)v, e_i\big\rangle\big|^2 > \rho_2 > 0 \tag{8.13}$$

($T_z S_0$ denotes the tangent space to S_0 at z). If $|\langle \nabla f(z), v\rangle| < \rho_1$, then there exists $\tilde v \in T_z S_0$, $|\tilde v| = 1$, so that $|v - \tilde v| \le C\frac{\rho_1}{|\nabla f(z)|} \le C\rho_1$ and (8.12) follows from (8.13) provided ρ_1 is small compared to ρ_2.

We start by proving (8.4) for $d = 2$. Fix some $e \in S^1$ and let $\ell \subset \mathbb{R}^2$ be the line through the origin in direction e. If $\dim(E \cap \ell) = \dim E$, then

$$\dim \{r > 0 \,:\, (x + rS) \cap E \neq \emptyset\} = \dim E$$

for all $x \in \ell$ and (8.4) holds with $H = \ell$. Otherwise, for any $\epsilon > 0$ there exists $\rho_0 > 0$ and $R > 0$ so that $\dim(E \cap B(0,R) \setminus \ell^{\rho_0}) > \dim E - \epsilon$ where ℓ^{ρ_0} denotes a ρ_0–neighborhood of ℓ. Now fix such $\epsilon > 0$ and $\rho_0 > 0$ and let $J \subset \ell$ be an interval of length one. In view of (8.8), (8.9), and (8.12) there exist $\rho > 0$ and C_0 (depending on J, κ, ρ_0, R) so that

$$|\Phi_{\boldsymbol{\lambda}}(x,y)| < \rho \Longrightarrow |D\Phi_{\boldsymbol{\lambda}}(x,y)|^2 > \rho$$

for any $x, y \in B(0,R) \setminus \ell^{\rho_0}$ satisfying $|x - y| < C_0\rho$. Let $\Omega = E \cap B(x_0, C_0\rho/2) \setminus \ell^{\rho_0}$ where $x_0 \in B(0,R)$ is selected so that $\dim \Omega > \dim E - \epsilon$. We conclude that with this choice of Ω and Π, J satisfies Definition 2.7 with $L = \infty$ and $\beta = 0$. Since $f\big(-\sum_{i=0}^{d-1} \lambda_i e_i + E\big) \supset \mathrm{supp}(\nu_{\boldsymbol{\lambda}})$, (8.4) therefore follows from Theorem 2.8 with $\alpha = \dim E - \epsilon$ as $\epsilon \to 0+$.

Now suppose that (8.4) holds up to $d-1$ for some $d \geq 3$ and fix a hyperplane $H \subset \mathbb{R}^d$ passing through the origin. If $\dim(E \cap H) = \dim E$, then (8.4) follows from (8.3) in dimension $d-1$ (recall that the latter estimate follows from (8.4) in dimension $d-1$). Otherwise, for any $\epsilon > 0$ there exists $\rho_0 > 0$ so that $\dim(E \setminus H^{\rho_0}) > \dim E - \epsilon$ where H^{ρ_0} denotes a ρ_0–neighborhood of H. By the same argument we gave for $d = 2$, one concludes from (8.8), (8.9), and (8.12) that Definition 7.2 holds for any bounded cube $Q \subset H$ and a suitable choice of Ω. Hence (8.4) follows from Theorem 7.3 with $L = \infty$, $\beta = 0$, $n = d-1$, $m = 1$, and $\alpha = \dim E - \epsilon$ as $\epsilon \to 0+$. $\square$

Suppose $E \subset \ell \subset \mathbb{R}^d$ for some line ℓ is a set of dimension 1 but with $\mathcal{H}^1(E) = 0$. Then it is easy to see that for all $x \in \mathbb{R}^d$, $(x + rS^{d-1}) \cap E = \emptyset$ for a.e. r. This implies that there are no nontrivial bounds for $\sigma = 1$ and $\dim(E) \leq 1$ in Theorem 8.3. In [10] Falconer showed that for any Borel set $A \subset \mathbb{R}^d$ with $\dim A > (d+1)/2$ the distance set $D(A) = \{|x - y| \,:\, x, y \in A\}$ has positive measure. In [32] Mattila and Sjölin showed that under the same assumption $D(A)$ has nonempty interior. On the other hand, Bourgain [4] proved that $D(A)$ has positive measure if $\dim A > (d+1)/2 - \epsilon_d$ for some $\epsilon_d > 0$ if $d = 2, 3$. (8.3) implies the following sharpened version of Falconer's theorem concerning "pinned" distance sets $D_x(A) = \{|x - y| \,:\, y \in A\}$:

Corollary 8.4. *Suppose $A \subset \mathbb{R}^d$ with $\dim A > (d+1)/2$. Then*

$$\dim \{x \in \mathbb{R}^d \,:\, D_x(A) \ \text{ has Lebesgue measure zero}\} \leq d + 1 - \dim A < (d+1)/2.$$

Proof. Apply (8.3) to $f(x) = |x|$ and $E = A$. $\square$

Similarly, we obtain a sufficient condition for pinned distance sets to contain an interval. Using (7.4) with $\sigma > 2$ in the proof of Theorem 8.3, yields the following statement:

Let $A \subset \mathbb{R}^d$ with $d \geq 3$ satisfy $\dim A > (d+2)/2$. Then

$$\dim \{x \in \mathbb{R}^d \,:\, D_x(A) \ \text{ has empty interior}\} \leq d + 2 - \dim A < (d+2)/2.$$

We now discuss the case where $S_0 \in C^{k,\delta}$ for some positive integer k and $\delta \in [0,1)$.

Theorem 8.5. *Let S_0 be a strictly convex, $C^{k,\delta}$–hypersurface surrounding the origin with $k + \delta > 2$ and define f and B_σ as above. If $E \subset \mathbb{R}^d$ is a Borel set and $\sigma \in [0, 1 \wedge \dim E]$, then*

$$\dim B_\sigma \leq d + \sigma - \max\big(1, \min(\dim E, k + \delta - 1)\big). \tag{8.14}$$

Furthermore, for any hyperplane $H \subset \mathbb{R}^d$,

$$\dim(B_\sigma \cap H) \leq d - 1 + \sigma - \max\big(1, \min(\dim E, k + \delta - 1)\big). \tag{8.15}$$

In particular, if $k + \delta \geq \dim E + 1$, then (8.3) and (8.4) remain valid.

Proof. Since $f \in C^{L,\delta}$ by assumption, (8.6), (8.8), and (8.9) show that $\Phi_{\boldsymbol{\lambda}} \in C^{k-1,\delta}$ in the sense of Definition 7.1. The proof of Theorem 8.3 therefore yields that $\Phi_{\boldsymbol{\lambda}}$ as defined above satisfies Definition 7.2 with $L = k-1$ and $\beta = 0$ and as before, Theorem 7.3 implies (8.14) and (8.15). □

Finally, we give some examples to show that Theorem 8.3 can fail under weaker assumptions. Suppose $S_0 = \partial[-1,1]^d$ is the boundary of the unit cube. Let $K \subset [0,1]$ be a Cantor set and let $E = K \times [-1,1]^{d-1}$. It is easy to see that $B_\sigma(E) \supset [3,\infty) \times [-1,1]^{d-1}$ for any $\dim K < \sigma \le 1$. The bound in (8.3) would therefore require that $d \le 1 + \sigma - \dim K$, which is clearly false. A similar example shows that (8.3) fails if S_0 contains an arbitrarily small piece of a hyperplane.

In view of (8.4) one might ask whether

$$\dim(B_1 \cap \pi) \le 1 + \dim \pi - \dim E \tag{8.16}$$

for any plane $\pi \subset \mathbb{R}^d$. It turns out that this fails if $\dim \pi \le d-2$ even if $S_0 = S^{d-1}$. For simplicity, let $d = 3$ and let

$$E = \{(0, r\cos\theta, r\sin\theta) \,:\, \theta \in [0,2\pi],\ r \in K\} \tag{8.17}$$

where $K \subset [0,1]$ is a Cantor set. Clearly, for any $x \in \mathbb{R}$, $\dim\{r > 0 \,:\, (x,0,0) + rS^2 \in E\} = \dim K$, whereas (8.16) would require that $\dim(B_1 \cap \{(x,0,0) \,:\, x \in \mathbb{R}\}) \le 2 - \dim E = 1 - \dim K$.

9. Questions and Comments

9.1. **Bernoulli Convolutions.**

- ((i)) For which intervals $J \subset (1/2,1)$ is $\int_J \|\widehat{\nu_\lambda}\|_2^4\, d\lambda < \infty$?
- ((ii)) Is the distribution of the zeros of $\Pi_\lambda(\omega) - \Pi_\lambda(\tau)$ on an interval of transversality absolutely continuous?

9.2. **Geometric problems.**

- ((i)) According to Corollary 6.2, a.e. projection of a Borel set $E \subset \mathbb{R}^d$ with $\dim E > 2k$ has nonempty interior. Can $2k$ be replaced by a smaller threshold if $k > 1$?
- ((ii)) Suppose that μ is a finite planar measure satisfying the Frostman condition $\mu(B(x,r)) \le r^\alpha$ for all $x \in \mathbb{R}^2$ and $r > 0$. Let ν_θ be the projection of μ onto a line in direction θ. For $p \in [1, 2/(2-\alpha))$, the Sobolev embedding theorem implies that the projected measures ν_θ have a density in L^p for a.e. $\theta \in S^1$. Can the threshold $2/(2-\alpha)$ be increased?
- ((iii)) Is Theorem 8.3 optimal? Heuristically speaking, estimates for distance sets are related to the Erdős problem of bounding the minimum number $g_d(n)$ of different distances determined by n points in R^d. It is known that $C_\epsilon{}^{-1} n^{4/5-\epsilon} < g_2(n) < C\,\frac{n}{\sqrt{\log n}}$, see [5] and the discussion in [1], chapter 12. The analogy between cardinality and dimension suggests that $\dim(E) > 5/4 \Longrightarrow \mathcal{H}^1(D(E)) > 0$. Generally speaking, however, it is presently unclear how to make such a deduction. Nevertheless, T. Wolff [44, 45] has successfully applied sophisticated methods from combinatorial geometry to obtain Hausdorff dimension estimates; in particular, he showed that a Borel set in the plane that contains a circle of every radius must have Hausdorff dimension 2. For pinned distance sets as in Corollary 8.4, the relevant combinatorial problem seems to be estimating the minimum number of distinct distances between n points and one "typical" point. More precisely, it is known, see Corollary 12.11 in [1], that for any n points $p_1, p_2, \ldots, p_n \in \mathbb{R}^2$, one has $\mathrm{card}\big\{|p_i - p_j| \;:\; j = 1,2,\ldots,n\big\} \ge C\, n^{3/4}$ for at least $n/2$ choices of i (this appears to be the best known bound. In particular, the authors of [5] point out that their method does not show that there exists a point from which there are $n^{4/5-\epsilon}$ different distances). The analogy between cardinality and dimension alluded to above then suggests that $\dim(E) > 4/3 \Longrightarrow \mathcal{H}^1(D_x(E)) > 0$ for most points $x \in E$. Note that Corollary 8.4 requires $\dim(E) > 3/2$ for the same conclusion in $\mathbb{R}^2$. In the recent preprint [46]

Wolff shows that $\dim(E) > 4/3$ implies that $\mathcal{H}^1(D(E)) > 0$. His methods are in the spirit of [4] and involve Bochner–Riesz type arguments.

((iv)) Does Theorem 8.3 hold if $S_0 \in C^{k,\delta}$ with $k + \delta < \dim E + 1$, cf. Theorem 8.5? In particular, is Theorem 8.3 valid if S_0 is only assumed to be a continuous, strictly convex hypersurface surrounding the origin?

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