Class notes for Ma 142
Winter/Spring 2003

01/07

Let's first study the free Schrödinger equation:

$$\begin{cases} i\partial_t \Psi + \frac{1}{2}\Delta\Psi = 0 \\ \Psi\big|_{t=0} = \Psi_0 \ . \end{cases}$$

Throughout these notes we will use the following convention for the Fourier transform (FT) and its inverse:

$$\begin{aligned} \hat{f}(\xi) &= \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx \ , \\ f(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi \ . \end{aligned}$$

After applying the FT, the equation becomes

$$\begin{cases} i\partial_t \hat{\Psi}(t,\xi) - \frac{1}{2}|\xi|^2 \hat{\Psi}(t,\xi) = 0 \\ \hat{\Psi}(0,\xi) = \hat{\Psi}_0(\xi) \ , \end{cases}$$

which can be solved as $\hat{\Psi}(t,\xi) = e^{-\frac{i}{2}t|\xi|^2}\hat{\Psi}_0(\xi)$.

Applying the inverse FT,

$$\Psi(t,x) = (2\pi)^{-d} \int e^{i(x\cdot\xi - \frac{1}{2}t|\xi|^2)} \hat{\Psi}_0(\xi)\, d\xi \ .$$

Obviously, for $t > 0$, we have the property that $\| \Psi(t,\cdot) \|_2 = \| \Psi_0 \|_2$ (use the FT).

Denote $H^\gamma = W^{\gamma,2}$ with the norm

$$\| f \|_{H^\gamma} = \left(\int (1 + |\xi|^2)^\gamma |\hat{f}(\xi)|^2 d\xi \right)^{1/2} , \quad \gamma \in \mathbb{R} \ .$$

The corresponding homogeneous norm is

$$\| f \|_{\dot{H}^\gamma} = \left(\int |\xi|^{2\gamma} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \ .$$

Upon scaling, $f(x) \to f(\lambda x)$, $\hat{f}(\xi) \to \lambda^{-d}\hat{f}(\lambda^{-1}\xi)$,

$$\begin{aligned} \| f(\lambda\,\cdot) \|_{\dot{H}^\gamma} &= \left(\int_{\mathbb{R}^d} |\xi|^{2\gamma} |\lambda^{-d}\hat{f}(\xi\lambda^{-1})|^2 d\xi \right) \\ &= (\lambda^{-2d}\lambda^d\lambda^{2\gamma})^{1/2} \| f \|_{\dot{H}^\gamma} \\ &= \lambda^{\gamma - d/2} \| f \|_{\dot{H}^\gamma} \ . \end{aligned}$$

Lemma.

$$\| \Psi(t,\cdot) \|_{H^\gamma} = \| \Psi_0 \|_{H^\gamma}$$
$$\| \Psi(t,\cdot) \|_{\dot H^\gamma} = \| \Psi_0 \|_{\dot H^\gamma}$$

Proof.

$$\begin{aligned}\| \Psi(t,\cdot) \|^2_{\dot H^\gamma} &= \int\limits_{\mathbb{R}^d} |\xi|^{2\gamma}|\ e^{-\frac{1}{2}t|\xi|^2}\hat\Psi(\xi)|^2\, d\xi \\ &= \int\limits_{\mathbb{R}^d} |\xi|^{2\gamma}|\hat\Psi_0(\xi)|^2\, d\xi \\ &= \| \Psi_0 \|^2_{\dot H^\gamma}\end{aligned}$$

and the same is true for the inhomogeneous norm after replacing $|\xi|^{2\gamma}$ by $(1+|\xi|^2)^\gamma$. □

Let us study some special solutions (for $t > 0$).

(1) Gaussians

$$\begin{aligned}\Psi_0(x) &= e^{-|x|^2/2} \\ \hat\Psi_0(\xi) &= \int\limits_{\mathbb{R}^d} e^{-ix\cdot\xi}\ e^{-|x|^2/2}\, dx \\ &= \int\limits_{\mathbb{R}^d}\ e^{-|x+i\xi|^2/2}\, dx \cdot e^{-|\xi|^2/2} \\ &= \int\limits_{\mathbb{R}^d}\ e^{-|x|^2/2}\, dx \cdot e^{-|\xi|^2/2} \\ &\qquad \text{(shift the contour)} \\ &= (2\pi)^{d/2} e^{-|\xi|^2/2}\end{aligned}$$

$$\begin{aligned}\Psi(t,x) &= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{i(x\cdot\xi-\frac{t}{2}|\xi|^2)}\ e^{-|\xi|^2/2}\, d\xi \\ &= (2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} e^{-(1+it)|\xi|^2/2}\ e^{ix\cdot\xi}\, d\xi \\ &= (1+it)^{-d/2}\ e^{-|x|^2/2(1+it)} \\ |\Psi(t,x)|^2 &= (1+t^2)^{-d/2}\ e^{-|x|^2/(1+t^2)}\end{aligned}$$

(2) Modulated gaussians

$$\begin{aligned}\Psi_0(x) &= e^{-|x|^2/2}\ e^{ix\cdot v} \\ \hat\Psi(\xi) &= (2\pi)^{d/2}\ e^{-|\xi-v|^2/2}\end{aligned}$$

$$\begin{aligned}
\Psi(t,x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - \frac{1}{2}t|\xi|^2)}\; e^{-|\xi-v|^2/2}\, d\xi \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\left(x(\xi+v) - \frac{1}{2}t|\xi+v|^2\right)}\; e^{-|\xi|^2/2}\, d\xi \\
&= e^{ix\cdot v}\; e^{-\frac{1}{2}t|v|^2} (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x-tv)\cdot\xi}\; e^{-(1+it)|\xi|^2/2}\, d\xi \\
&= e^{ix\cdot v}\; e^{-\frac{1}{2}t|v|^2} (1+it)^{-d/2}\; e^{-|x-tv|^2/2(1+it)}
\end{aligned}$$

(3) Fundamental solution. Let us derive it from scaling considerations. By translation invariance, we must have

$$\Psi(t,x) = \int_{\mathbb{R}^d} K(t,x-y)\Psi_0(y)\, dy$$

In addition, $K(0,x) = \delta_0(x)$. Consider the scaling

$$\Psi_0(x) \to \Psi_0(\lambda x) \quad , \quad \Psi(t,x) \to \Psi(\lambda^2 t, \lambda x)\ .$$

Then

$$\begin{aligned}
\Psi(\lambda^2 t, \lambda x) &= \int K(t,x-y)\Psi_0(\lambda y)\, dy \\
\Psi(t,x) &= \int \lambda^{-d} K\big(\lambda^{-2}t, \lambda^{-1}(x-y)\big)\Psi_0(y)\, dy\ .
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow K(t,x-y) = \lambda^{-d} K\big(\lambda^{-2}t, \lambda^{-1}(x-y)\big) \\
\text{or}\quad &K(t,x-y) = \lambda^{d} K(\lambda^2 t, \lambda x)\ .
\end{aligned}$$

It is reasonable to look for $\lambda^d K(\lambda^2 t, \lambda x) = t^{-d/2}\Phi\big(\frac{x^2}{t}\big)$ and let $u = x^2/t$. Since $i\partial_t K + \frac{1}{2}\Delta K = 0$, the equation for Φ is

$$\begin{aligned}
-i\,\frac{d}{2}\, t^{-\frac{d}{2}-1}\Phi(u) - it^{-\frac{d}{2}}\Phi'(u)\cdot\frac{u}{t} + \frac{1}{2}\, t^{-\frac{d}{2}}\,\mathrm{div}_x\bigg(2\,\frac{x}{t}\,\Phi'(u)\bigg) &= 0 \\
\Rightarrow\ -i\,\frac{d}{2}\,\Phi(u) - iu\Phi'(u) + d\Phi'(u) + 2u\Phi''(u) &= 0 \\
d\Big(\Phi'(u) - \frac{i}{2}\,\Phi(u)\Big) + 2u\,\frac{d}{du}\,\Big(\Phi'(u) - \frac{i}{2}\Phi(u)\Big) &= 0\ .
\end{aligned}$$

This at least admits the solution $\Phi(u) = e^{\frac{i}{2}u}$, i.e., $K(t,x) = t^{-d/2}\; e^{\frac{i}{2}\frac{|x|^2}{t}}$.

We could have guessed this result from

$$\begin{aligned}
K(t,x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - \frac{1}{2}t|\xi|^2)}\, d\xi \\
&= (2\pi)^{-d} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-(\varepsilon+it)|\xi|^2/2} |\xi|^2\, d\xi
\end{aligned}$$

$$= \lim_{\varepsilon\to 0^+} (\varepsilon + it)^{-d/2}(2\pi)^{-d/2}\, e^{-|x|^2/2(\varepsilon+it)}$$
$$= (2\pi it)^{-d/2}\, e^{-|x|^2/2it}\,.$$

No problem with the definition of this quantity, take the branch cut on the negative real t-axis. Therefore

$$\Psi(t,x) = (2\pi it)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2it}}\Psi_0(y)\,dy\,. \tag{1}$$

Theorem 1. *Let $\Psi_0 \in \mathcal{S}(\mathbb{R}^2)$. Then there exists a solution to*

$$\begin{cases} i\Psi_t + \frac{1}{2}\Delta\Psi = 0\,, \\ \Psi|_{t=0} = \Psi_0\,, \end{cases}$$

unique on $\mathcal{S}(\mathbb{R}^d)$, and given by (1).

We have obtained the *a priori* estimate $\| \Psi(t,\cdot) \|_2 = \| \Psi_0 \|_2$, for $\Psi \in \mathcal{S}$ and $t > 0$. This indicates that a natural setting is the space $L^\infty\big((0,\infty), L^2_x\big)$.

We can combine it with the dispersive estimate $\| \Psi(t,\cdot) \|_\infty \le |2\pi t|^{-d/2} \| \Psi_0 \|_1$, to obtain by interpolation

$$\| \Psi(t,\cdot) \|_{L^{p'}(\mathbb{R}^d)} \le C(d,p) t^{-d\left(\frac{1}{p}-\frac{1}{2}\right)} \| \Psi_0 \|_{L^p(\mathbb{R}^d)}\,,$$

valid for $1 \le p \le 2$ and $t \ge 0$.

In the sequel, we will consider the inhomogeneous Schrödinger equation

$$\begin{cases} i\Psi_t + \frac{1}{2}\Delta\Psi = F \quad , & F \in \mathcal{S}_{t,x} \\ \Psi|_{t=0} = \Psi_o \quad , & \Psi_o \in \mathcal{S}\,. \end{cases}$$

Its solution is explicitly given by the Duhamel formula

$$\Psi(t) = e^{\frac{i}{2}t\Delta}\Psi_0 - i\int_0^t e^{i(t-s)\Delta/2}F(s)\,ds\,.$$

The notation $e^{\frac{i}{2}t\Delta}\Psi_0$ should be understood as $\big(e^{-\frac{i}{2}t|\xi|^2}\hat{\Psi}_o\big)^\vee$.

01/09

Theorem 2. *(Strichartz estimates for the inhomogeneous Schrödinger equation)*

$$\| \Psi \|_{L^{p'}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \| \Psi_0 \|_{L^2_x(\mathbb{R}^d)} + \| F \|_{L^p_{t,x}(\mathbb{R}\times\mathbb{R}^d)}$$

for $p' = 2 + 4/d$.

Remark 3. This value of p is the only possible one, and it is necessary to have conjugate exponents p', p on the lhs and rhs respectively. This can be seen from the following scaling argument:

$$\Psi_0(x) \to \Psi_0(\lambda x)$$
$$\Psi(t,x) \to \Psi(\lambda^2 t, \lambda x)$$
$$F(t,x) \to \lambda^2 F(\lambda^2 t, \lambda x)$$

$$\| \Psi \|_{L^{p'}_{t,x}} \to \lambda^{-d/p'} \lambda^{-2/p'} \| \Psi \|_{L^{p'}}$$
$$\| \Psi_0 \|_{L^2_x} \to \lambda^{-d/2} \| \Psi_0 \|_{L^2}$$
$$\Rightarrow \frac{d+2}{p'} = \frac{d}{2} \Rightarrow p' = 2 + \frac{4}{d}$$
$$\text{and} \quad p = \left(1 - \frac{d}{2d+4}\right)^{-1} = \frac{2d+4}{d+4} \ .$$

Put q the exponent to consider in $\| F \|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)}$. Then

$$\| F \|_{L^q_{t,x}} \to \lambda^2 \lambda^{-\frac{d+2}{q}} \| F \|_{L^q_{t,x}}$$
$$\Rightarrow 2 - \frac{d+2}{q} = -\frac{d}{2} \Rightarrow q = \frac{2d+4}{d+4} = p \ .$$

Lemma 1. *Let* $\Psi(t) = e^{\frac{i}{2}t\Delta}\Psi_0$. *Then*

$$\| \Psi(t) \|_{L^{q'}_x(\mathbb{R}^d)} \lesssim |t|^{-d\left(\frac{1}{q}-\frac{1}{2}\right)} \| \Psi_0 \|_{L^q_x(\mathbb{R}^d)} \ ,$$

for all $1 \leq q \leq 2$.

Lemma 2. *(Fractional integration, Hardy-Littlewood-Sobolev).*
Let $0 < \alpha < 1$, $g \in \mathcal{S}(\mathbb{R}^d)$, *and*

$$(T_\alpha g)(t) = \int_{-\infty}^{\infty} |t-s|^{-\alpha} g(s) \, ds$$

$$\textit{Then} \quad \| T_\alpha g \|_{L^q(\mathbb{R})} \leq C \| g \|_{L^p(\mathbb{R})}$$

where $1 + \frac{1}{q} = \alpha + \frac{1}{p}$, $1 < p < q < \infty$.

[Note: one gains regularity by fractional integration. We could not expect $p = 1$ *above because the kernel would not be in* $L^q(\mathbb{R})$*, only in weak* $-L^q(\mathbb{R})$*].*

Proof. Strichartz ($T - T^*$ argument)

Put $Uf = e^{\frac{i}{2}t\Delta} f$, then $U : L^2_x(\mathbb{R}^d) \to L^\infty_t(L^2_x)$. Then the adjoint is given by

$$U^* : L^1_t(L^2_x) \to L^2_x \ ,$$

$$\begin{aligned}
\langle f, U^* G \rangle_{L^2_x} &= \langle Uf, G \rangle_{L^2_{t,x}} \\
&= \iint (e^{\frac{i}{2}t\Delta} f)(x) \bar{G}(t,x) \, dt \, dx \\
&= \int f(x) \int \overline{\left(e^{-\frac{i}{2}t\Delta} G(t,\cdot)\right)}(x) \, dt \, dx
\end{aligned}$$

$$\Rightarrow U^*G(x) = \int \big(e^{-\frac{i}{2}s\Delta}G(s)\big)(x)\,ds$$

(this indeed belongs to L^2_x by the assumptions on G).

$$(UU^*G)(t,x) = \int_{-\infty}^{\infty} \big(e^{\frac{i}{2}(t-s)\Delta}G(s)\big)(x)\,ds$$

such that $UU^* : L^1_t(L^2_x) \to L^\infty_t(L^2_x)$.

Let us show that $\| UU^*G \|_{L^{p'}_{t,x}} \leq C \cdot \| G \|_{L^p_{t,x}}$

$$\begin{aligned}\| UU^*G \|_{L^{p'}_x} &\leq \int_0^{\infty} \| e^{\frac{i}{2}(t-s)\Delta}G(s) \|_{L^{p'}_x}\ ds \\ &\leq C \cdot \int_{-\infty}^{\infty} |t-s|^{-d\left(\frac{1}{p}-\frac{1}{2}\right)} \| G(s) \|_{L^p_x}\ ds\end{aligned}$$

by lemma 1.

By lemma 2, we get precisely $\| UU^*G \|_{L^q_t(L^p_x)} \leq C \cdot \| G \|_{L^p_{t,x}}$ for q that satisfies

$$1 + \frac{1}{q} = d\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{1}{p} \Rightarrow q = 2 + \frac{4}{d} = p'$$

(and we check that $0 < \alpha = d\big(\frac{1}{p} - \frac{1}{2}\big) < 1$).

Let us show Strichartz for $F = 0$:

$$\begin{aligned}\| U^*G \|^2_{L^2_x} &= \langle UU^*G, G\rangle \\ &\leq \| UU^*G \|_{L^{p'}_{t,x}} \| G \|_{L^p_{t,x}} \\ &\leq C \cdot \| G \|^2_{L^p_{t,x}}\end{aligned}$$

so that $\quad U^* : L^p_{t,x} \to L^2_x \quad$ boundedly,

therefore $\quad U : L^2_x \to L^{p'}_{t,x} \quad$ boundedly,

$$\| U\Psi_0 \|_{L^{p'}_{t,x}} \leq C \cdot \| \Psi_0 \|_{L^2_x}\ .$$

When $F \neq 0$,

$$\Psi(t) = e^{\frac{i}{2}t\Delta}\Psi_0 - i\int_0^t e^{\frac{i}{2}(t-s)\Delta}F(s)\,ds$$

$$\begin{aligned}\| \Psi \|_{L^{p'}_{t,x}} &\leq C \cdot \| \Psi_0 \|_2 + \Big\| \int_0^t \| e^{\frac{i}{2}(t-s)\Delta}F(s) \|_{L^{p'}_x}\ ds \Big\|_{L^{p'}_t} \\ &\leq C \cdot \| \Psi_0 \|_2 + \Big\| \int_{-\infty}^{\infty} |t-s|^{-d\left(\frac{1}{p}-\frac{1}{2}\right)} \| F(s) \|_{L^p_x}\ ds \Big\|_{L^{p'}_t}\end{aligned}$$

$$\text{by lemma 1}$$
$$\leq C \cdot \left(\| \Psi_0 \|_2 + \| F \|_{L^p_{s,x}} \right) \quad \text{by lemma 2}$$

□

Remark 3. The (free) Strichartz estimate can also be derived from the Stein-Tomas restriction theorem. Let $S \subset \mathbb{R}^n$, $n = d+1$, by a hypersurface with nonvanishing gaussian curvature, and σ_s be the corresponding surface measure. Let ϕ be compactly supported on S. Then the Stein-Tomas theorem says that

$$\| \widehat{\phi \sigma_s} \|_{L^r(\mathbb{R}^n)} \leq C \cdot \| \phi \|_{L^2(\sigma_s)}$$

when $r = \frac{2n+2}{n-1}$.

In our case, $\Psi(t,x) = \int e^{i\left[x\cdot\xi - \frac{1}{2}t|\xi|^2\right]} \hat{\Psi}_0(\xi)\, d\xi$ and S is the paraboloid

$$S = \left\{ (\xi,\tau) : \tau = -\frac{1}{2}|\xi|^2 \;,\; \xi \in \mathbb{R}^d \right\}$$

$$\begin{aligned} \text{so that} \quad \Psi(t,x) &= (\phi\sigma_s)^\vee(t,x) \\ \phi(\xi,\tau) &= \hat{\Psi}_0(\xi) \\ \sigma_s(d\xi, d\tau) &= (2\pi)^d \, d\xi \end{aligned}$$

Indeed,

$$\begin{aligned} \Psi(t,x) &= (2\pi)^{-d} \int e^{i(x\cdot\xi + t\tau)} \phi(\xi,\tau)\sigma_s(d\xi,d\tau) \\ &= \int e^{i(x\cdot\xi - \frac{1}{2}t|\xi|^2)} \hat{\Psi}_0(\xi)\, d\xi \end{aligned}$$

(ϕ is the "lift" of $\hat{\Psi}_0$ to the paraboloid)

Then we have $\| (\phi\sigma_s)^\vee \|_{L^r(\mathbb{R}^n)} \leq C. \| \phi \|_{L^2(\sigma_s)}$

$$\Rightarrow \quad \| \Psi \|_{L^{2+4/d}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \leq C \cdot \| \hat{\Psi}_0 \|_{L^2_\xi} = C \cdot \| \Psi_0 \|_{L^2_x}$$

This is valid for Ψ_0 such that supp $\hat{\Psi} \subset B(0,1)$, or by scaling in $B(0,R)$ for arbitrarily large R (the constant C is independent of R). We conclude by density of the compactly supported functions.

Theorem 4. *(Kato $\frac{1}{2}$-smoothing estimate)*
For $d \geq 2$, the $\chi \geq 0$ a smooth cutoff function such that $\hat{\chi}$ is compactly supported. Then

$$\| \chi(x)(1-\Delta)^{\frac{1}{4}} e^{\frac{i}{2}t\Delta} f \|_{L^2_{t,x}} \leq C \cdot \| f \|_{L^2_x} \;.$$

Remark 5. We gain 1/2 derivative provided we cut off by χ in space. We can integrate in time because the cutoff χ implies decay in time. It does not matter whether the spatial cutoff is done before or after taking the derivatives, as long as it is smooth.

Proof. Take $\eta \in \mathcal{S}(\mathbb{R})$ so that supp $\hat{\eta} \subset (-1,1)$. After passing to the limit $\epsilon \to 0$, the l.h.s. is

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}} \left|\eta(\epsilon t)\chi(x)\big((1-\Delta)^{1/4}e^{\frac{i}{2}t\Delta}f\big)(x)\right|^2 dt\ dx \\
&= \iint \left|\eta(\epsilon t)\int \hat{\chi}(\xi-\xi')(1+|\xi|^2)^{1/4}e^{-\frac{i}{2}t|\xi'|^2}\hat{f}(\xi')\ d\xi'\right|^2 d\xi\ dt \\
&\qquad \text{(by Plancherel in } x-\xi) \\
&= \iint\left|\iint \hat{\chi}(\xi-\xi')\frac{1}{\epsilon}\hat{\eta}\Big(\frac{\tau-\tau'}{\epsilon}\Big)\big(1+|\xi'|^2\big)^{1/4}\delta\big(\tau'-\frac{1}{2}|\xi'|^2\big)\right. \\
&\qquad\qquad \left.\hat{f}(\xi')\ d\xi'\ d\tau'\right|^2 d\xi\ d\tau \\
&\qquad \text{(by Plancherel in } t-\tau) \\
&= \iint\left|\int \hat{\chi}(\xi-\xi')\frac{1}{\epsilon}\hat{\eta}\Big(\frac{\tau-\frac{1}{2}|\xi'|^2}{\epsilon}\Big)(1+|\xi'|^2)^{1/4}\hat{f}(\xi')\ d\xi'\right|^2 d\xi\ d\tau \\
&\le \iint \cdot \left(\int\left|\hat{\chi}(\xi-\xi')\frac{1}{\epsilon}\hat{\eta}\Big(\frac{\tau-\frac{1}{2}|\xi'|^2}{\epsilon}\Big)\right|\ d\xi'\right) \\
\circledast &\cdot \left(\int\left|\hat{\chi}(\xi-\xi')\frac{1}{\epsilon}\hat{\eta}\Big(\frac{\tau-\frac{1}{2}|\xi'|^2}{\epsilon}\Big)\right|(1+|\xi'|^2)^{1/2}|\hat{f}(\xi')|^2\ d\xi'\right)\ d\xi\ d\tau \\
&\qquad \text{(by Cauchy-Schwarz in } \xi')
\end{aligned}$$

The domains of integration are limited by

$$\begin{cases} -\epsilon < \tau - \frac{1}{2}|\xi'|^2 < \epsilon & (1) \\ |\xi-\xi'| < 1 & (2) \end{cases}$$

$$\begin{aligned}
\text{(2) implies that}\quad & \big||\xi|-|\xi'|\big| = O(1) \\
\text{and also that}\quad & \big||\xi|^2-|\xi'|^2\big| = \big||\xi|-|\xi'|\big|\cdot(|\xi|+|\xi'|) \\
& = O(1+|\xi|)\ .
\end{aligned}$$

Since $\tau = O(|\xi'|^2)$, the range of values for τ is $O(1+|\xi|)$.

(1) implies that $\left|\sqrt{2|\tau|}-|\xi'|\right| = O\Big(\frac{\epsilon}{1+|\xi|}\Big)$, and on the other hand $\xi' \in B(\xi,1)$, therefore the range of values taken on by ξ' is $O\Big(\frac{\epsilon}{1+|\xi|}\Big)$, the volume of the shaded area in the figure. We bound $\circledast$ by

$$\begin{aligned}
C.\int d\xi\Big(\frac{1}{\epsilon}\cdot\frac{\epsilon}{1+|\xi|}\Big)\cdot\Big(\frac{1}{\epsilon}\cdot\frac{\epsilon}{1+|\xi|}\Big)\cdot(1+|\xi|^2)^{1/2}\cdot(1+|\xi|)|\hat{f}(\xi)|^2 \\
\le C.\int |\hat{f}(\xi)|^2\ d\xi = C.\parallel f\parallel^2_{L^2_x}
\end{aligned}$$

This estimate is uniform as $\epsilon \to 0$. $\square$

Corollary 6. *For $d \ge 2$, for every $\epsilon > 0$,*

$$\parallel (1+|x|)^{-\frac{1}{2}-\epsilon}(-\Delta)^{1/4}e^{\frac{i}{2}t\Delta}f\parallel_{L^2_{t,x}} \le C_\epsilon\cdot\ \parallel f\parallel_{L^2_x}$$

Proof. Think of χ as $1_{B(0,1)}$, the indicator of the unit ball. Let us consider $(-\Delta)^{1/4}$ instead of $(1-\Delta)^{1/4}$ in the previous estimate, in order to get a homogeneous behavior under rescaling.

$$\begin{aligned}
&\| \chi(2^{-j}x)(-\Delta)^{1/4} e^{\frac{i}{2}t\Delta} f \|_{L^2_{t,x}} \qquad \text{let } x \to 2^j x \\
&= 2^{jd/2} 2^j \| \chi \cdot \big((-\Delta)^{1/4}(e^{\frac{i}{2}t\Delta} f)\big)(2^{2j}t, 2^j x) \|_{L^2_{t,x}} \\
&= 2^{jd/2} 2^j 2^{-j/2} \| \chi \cdot (-\Delta)^{1/4}\big((e^{\frac{i}{2}t\Delta} f)(2^{2j}t, 2^j x)\big) \|_{L^2_{t,x}} \\
&= 2^{jd/2} 2^{j/2} \| \chi \cdot (-\Delta)^{1/4}\big(e^{\frac{i}{2}t\Delta} f(2^j \cdot)\big)(t,x) \|_{L^2_{t,x}} \\
&\le C \cdot 2^{jd/2} 2^{j/2} \| f(2^j \cdot) \|_2 \le C \cdot 2^{j/2} \| f \|_2
\end{aligned}$$

The homogeneous Kato smoothing estimate would therefore remain valid if we replaced $\chi(x)$ by the dyadic superposition

$$\sum_{j\ge 0} 2^{-j\left(\frac{1}{2}+\varepsilon\right)} \chi(2^{-j}x) \ ,$$

which is essentially $(1+|x|)^{-\frac{1}{2}-\varepsilon}$. □

Theorem 7. *(*$1-D$ *Kato smoothing estimate).*
For $d=1$*,*

$$\sup_x \| (-\Delta)^{1/4} e^{\frac{i}{2}t\Delta} f \|_{L^2_t} \le C \cdot \| f \|_{L^2_x}$$

Proof. When $\varepsilon \to 0$, the l.h.s. is

$$\| \eta(\varepsilon t) \int_{-\infty}^{\infty} e^{i(x\xi - \frac{1}{2}t|\xi|^2)} |\xi^{1/2} \hat{f}(\xi)\, d\xi \|^2_{L^2_t}$$

$$\circledast \quad = \int \Big| \int_{-\infty}^{\infty} e^{ix\xi} \frac{1}{\varepsilon} \hat{\eta}\left(\frac{\tau - \frac{1}{2}|\xi|^2}{\varepsilon}\right) |\xi|^{1/2} \hat{f}(\xi)\, d\xi \Big|^2 \, d\tau$$

The integration in ξ is limited by $-\varepsilon < \tau - \frac{1}{2}\xi^2 < \varepsilon$
$\Rightarrow |\xi - \sqrt{2\tau} < \frac{2\varepsilon}{\sqrt{2\tau}+\xi} \simeq \frac{\varepsilon}{\sqrt{2\tau}}$
or $|\xi - \sqrt{2\tau}| \lesssim \frac{\varepsilon}{\sqrt{2\tau}}$
As $\varepsilon \to 0$,

$$\begin{aligned}
\circledast \quad &\ge C \cdot \int_{\infty}^{\infty} d\tau \left[\frac{\varepsilon}{\sqrt{2\tau}} \frac{1}{\varepsilon} \cdot (2\tau)^{1/4} \big(\hat{f}(\sqrt{2\tau}) + \hat{f}(-\sqrt{2\tau}) \big) \right]^2 \\
&\ge C \cdot \int_{-\infty}^{\infty} \left(\frac{|\hat{f}(\sqrt{2\tau})|^2}{\sqrt{2\tau}} + \frac{|\hat{f}(-\sqrt{2\tau})|^2}{\sqrt{2\tau}} \right) d\tau \\
&\ge C \cdot \| f \|_2^2 \qquad \text{uniformly in } x.
\end{aligned}$$

□

The Kato smoothing estimate is sharp, as can be seen on the example of a modulated gaussian. Consider $f(t,x) = (e^{\frac{i}{2}t\Delta} f_0)(x)$ where $f_0(x) = e^{-|x|^2/2} e^{-ix\cdot v}$. Intuitively, the time spent in $B(0,1)$ is $O(\frac{1}{|v|})$, so that the L_t^2 norm yields a factor $O\Big(\frac{1}{|v|^{1/2}}\Big)$. This compensates the factor $|v|^{1/2}$ coming from the half-derivative.

Lemma. *For every* $\delta > 0$*,*

$$\| \, \chi(-\Delta)^{1/4} e^{\frac{i}{2}t\Delta} (e^{\frac{-|x|^2}{2}} e^{ix\cdot v}) \, \|_{L^2_{x,t}} \sim 1$$
$$\text{and } \sup_v \| \, \chi(-\Delta)^{1/4+\delta} e^{\frac{i}{2}t\Delta} (e^{\frac{-|x|^2}{2}} e^{ix\cdot v}) \, \|_{L^2_{x,t}} = \infty$$

Proof. Take supp $\hat{\chi} \subset B(0,1)$.

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}^d} \Big| \chi(x) \eta(\varepsilon t) \big((-\Delta)^{1/4} \, e^{\frac{i}{2}t\Delta} f_0 \big)(x) \Big|^2 \, dx \, dt \\
&= \iint \Big| \int \hat{\chi}(\xi' - \xi) |\xi|^{1/2} e^{-\frac{i}{2}t|\xi|^2} \eta(t\varepsilon) \hat{f}_0(\xi) \, d\xi \Big|^2 \, d\xi' \, d\tau \\
&= \iint \Big| \int \hat{\chi}(\xi' - \xi) |\xi|^{1/2} \frac{1}{\varepsilon} \eta\Big(\frac{\tau - \frac{1}{2}|\xi|^2}{\varepsilon} \Big) e^{-|\xi - v|^2/2} \, d\xi \Big|^2 \, d\xi' d\eta \circledast
\end{aligned}$$

Again,

$$\begin{aligned}
&-\varepsilon < \tau - \frac{1}{2}|\xi|^2 < \varepsilon \\
\Rightarrow &\Big| |\xi| - \sqrt{2\tau} \Big| < \frac{2\varepsilon}{|\xi| + \sqrt{2\tau}}
\end{aligned}$$

As $\varepsilon \longrightarrow 0$,

$$\circledast = C \cdot \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Big| \int_{\sqrt{2\tau} S^{d-1}} \hat{\chi}(\xi' - \xi) e^{-|\xi - v|^2/2} |\xi|^{1/2} \frac{2}{|\xi| + \sqrt{2\tau}} \, d\sigma(\xi) \Big|^2 \, d\xi' \, d\tau$$

The range of values for ξ is $O(1)$ (the $(d-1)$-measure of the shell fragment corresponding to the shaded area is $O(1)$). The range of values for ξ' is $O(1) = |\text{supp } \hat{\chi}|$. The range of values for τ is

$$\begin{aligned}
&\frac{1}{2} \big| |v| - 1 \big|^2 \lesssim \tau \lesssim \varepsilon + \frac{1}{2} \big| |v| + 1 \big|^2 \\
\Rightarrow &\big| 2\tau - |v|^2 \big| \lesssim |v| \, .
\end{aligned}$$

Consequently,

$$\begin{aligned}
\circledast =& C \cdot \int \Big| \frac{(\sqrt{\tau})^{1/2}}{\sqrt{\tau}} \Big|^2 \, d\tau = C \cdot \frac{1}{|v|} \int_{|2\tau - |v|^2 \lesssim |v|} d\tau \\
=& O(1) \quad \text{uniformly in } v.
\end{aligned}$$

Had we considered $(-\Delta)^{\frac{1}{4}+\delta}$ instead, the above $1/|v|$ would become $1/|v|^{1-4\delta}$ and the estimate would not be uniform in $|v|$ anymore. □

Consider now

$$\begin{cases} i\Psi_t + \frac{1}{2}\Delta\Psi + V(t,x)\Psi = 0 \\ \Psi|_{t=0} = \Psi_0 \in H^\gamma(\mathbb{R}^d) \ . \end{cases}$$

Lemma. *Let* $\Psi_0 \in H^\gamma(\mathbb{R}^d)$*, then*

$$e^{\frac{i}{2}t\Delta}\Psi_0 \in C^\circ\big(\mathbb{R}, H^\gamma(\mathbb{R}^d)\big) \cap C^1\big(\mathbb{R}, H^{\gamma-2}(\mathbb{R}^d)\big)$$

Proof.

$$\begin{aligned} &\| e^{\frac{i}{2}t\Delta}\Psi_0 - e^{\frac{i}{2}s\Delta}\Psi_0 \|^2_{H^\gamma} \\ = \ &\| e^{\frac{i}{2}(t-s)\Delta}\Psi_0 - \Psi_0 \|^2_{H^\gamma} \\ = \ &\int (1+|\xi|^2)^\gamma |e^{-\frac{i}{2}(t-s)|\xi|^2} - 1|\hat{\Psi}_0(\xi)|^2 \, d\xi \end{aligned}$$

Since $\Psi_0 \in H^\gamma$ we can apply Lebesque's dominated convergence theorem to conclude that the integral goes to zero as $t-2 \to 0$.

On the other hand,

$$\begin{aligned} &\| \Delta e^{\frac{i}{2}t\Delta}\Psi_0 - \Delta e^{\frac{i}{2}s\Delta}\Psi_0 \|_{H^{\gamma-2}} \\ \leq \ &\| e^{\frac{i}{2}\Delta}\Psi_0 - e^{\frac{i}{2}s\Delta}\Psi_0 \|_{H^\gamma} \end{aligned}$$

□

Theorem 8. *Assume* V_{real}*,* $\sup_t \| \partial_x^\alpha V(t,\cdot) \|_{L_x^\infty} < C_\alpha$ *and* $\| \partial_x^\alpha V(t,\cdot) - \partial_x^\alpha V(s,\cdot) \|_{L_x^\infty} \to 0$ *as* $s \to t$ *for every multiindex* α*. Then for every* $k \geq 0$*, if* $\Psi_0 \in H^k(\mathbb{R}^d)$*, there exists* $\Psi \in C^\circ(\mathbb{R}, H^k) \cap C^1(\mathbb{R}, H^{k-2})$ *such that*

$$(DH) \quad \Psi(t) = e^{\frac{i}{2}t\Delta}\Psi_0 + i\int_0^t e^{\frac{i}{2}(t-s)\Delta} V(s)\Psi(s) \, ds$$

(the integral is a vector-valued Riemann integral). There is exactly one $\Psi \in C^\circ(\mathbb{R}, H^k)$ *that satisfies (DH), Moreover,*

$$\| \Psi(t) \|_{H^k(\mathbb{R}^d)} \leq C_{k,V}(1+|t|)^k$$

Proof.

(1) Local solvability on $[0,T]$. The operator

$$(A\phi)(t) = e^{\frac{i}{2}t\Delta}\Psi_0 - i\int_0^t e^{\frac{i}{2}(t-2)\Delta} V(s)\phi(s) \, ds$$

maps $X = C^\circ\big[(0,T], H^k\big)$ to itself, with the estimate

$$\begin{aligned} \| A\phi \|_X &= \sup_{0\leq t\leq T} \| A\phi(t) \|_{H^k} \\ &\leq \| \Psi_0 \|_{H^k} + \int_0^T \underbrace{\| V(s)\phi(s) \|_{H^k}} \, ds \end{aligned}$$

$$\leq C_k \cdot \sum_{|\alpha|\leq k} \| \partial^\alpha \big(V(s)\phi(s)\big) \|_L^2$$

$$\leq \| \Psi_0 \|_{H^k} + C_{V,k} \int_0^T \| \phi(s) \|_H^k \; ds$$

$$\leq \| \Psi_0 \|_{H^k} + C_{V,k} T \| \phi \|_X$$

Take $R = 2 \| \Psi_0 \|_{H^k}$ and $C_{V,k}T = 1/2$, so that $A : B_X(OR) \circlearrowleft$.

Ψ given by (DH) belongs to X in the first place because

- of the lemma for the first term
- t in the upper bound of the integral yields a continuous dependence
- Fourier transform and apply Lebesque's dominated convergence for the dependence on t of the integrand $(e^{\frac{i}{2}(t-s)\Delta})$

So we get $\| A\phi - A\tilde{\phi} \|_X \leq \frac{1}{2} \| \phi - \tilde{\phi} \|_X$, a contraction. By the fixed point theorem, we conclude that there exists $\Psi \in C^\circ\big([0,T], H^k\big)$ so that (DH) holds for $0 \leq t \leq T$.

(2) Extension to a global solution.

Take $\Psi(T)$ as initial condition and solve for $T \leq t \leq 2T$,

$$\Psi(t) = e^{\frac{i}{2}(t-T)\Delta}\Psi(T) + i \int_T^t e^{\frac{i}{2}(t-T-s)\Delta} V(s)\Psi(s) \; ds$$

(we can apply local solvability because $C_{V,k}$ does not depend on the initial condition). Then it is easy to see that (DH) can be composed, and that $\Psi(t)$, $T \leq t \leq 2T$ given by the above expression actually satisfies

$$\Psi(t) = e^{\frac{i}{2}\Delta}\Psi_0 + i \int_0^t e^{\frac{i}{2}(t-s)\Delta} V(s)\Psi(s) \; ds \; ,$$

for $T \leq t \leq 2T$.

(3) Uniqueness

Recall the Gronwall inequality. For $f(t) \geq 0$ and $K(T) \geq 0$ such that $f(t)a \leq \int_0^t K(s)f(s)\; ds$, then $f(t) \leq a \exp\big(\int_0^t K(s)\; ds\big)$.

Apply this to

$$\| \Psi(t) - \tilde{\Psi}(t) \|_{H^k} \leq \int_0^t C \cdot \| \Psi(s) - \tilde{\Psi}(s) \|_{H^k} \; ds \; ,$$

so that $\| \Psi(t) - \tilde{\Psi}(t) \|_{H^k} \leq 0 \; , \quad \Psi(t) = \tilde{\Psi}(t)$.

(4) Note that Gronwall would imply a bound on $\| \Psi \|_{H^k}$ growing exponentially in time. For V_{real}, we actually get a bound polynomial in time, as we show next.

$$\begin{aligned} \frac{d}{dt} \| \Psi(t) \|_2^2 &= \frac{d}{dt} \langle \Psi(t), \Psi(t) \rangle \\ &= 2\mathrm{Re}\langle \Psi'(t), \Psi(t) \rangle \\ &= 2\mathrm{Re}\Big\langle \frac{i}{2}\Delta\Psi(t) + iV(t)\Psi(t), \Psi(t) \Big\rangle \end{aligned}$$

$$= 2\text{Re}\, i\Big(\int -\frac{1}{2}|\nabla\Psi|^2(t,x)\,dx + \int V(t,x)|\Psi(t,x)|^2\,dx\Big)$$
$$= 0$$

Formally, let $\phi = \partial_j\Psi(t,x)$, so that

$$i\phi_t + \frac{1}{2}\Delta\phi + V\phi = -(\partial_j V)\Psi$$

$$\phi(t) = U(t,0)\partial_j\Psi_0 + i\int_0^t U(t,s)(\partial_j V)(s)\Psi(s)\,ds\ ,$$

where $U(t,s)$ is the full evolution operator.

$$\|\ \phi(t)\ \|_2 \leq \|\ \partial_j\Psi_0\ \|_2 + \int_o^t \|\ \partial_j V(s)\ \|_\infty \|\ \Psi_0\ \|_2\ \ ds$$

Adding this to $\|\ \Psi(t)\ \|_2 = \|\ \Psi_0\ \|_2$, we get $(t > 0)$

$$\|\ \Psi(t)\ \|_{H^1} \leq C_V(1+t)\ \|\ \Psi_0\ \|_{H^1}\ \ .$$

Inductively, by similar arguments,

$$\|\ \Psi(t)\ \|_{H^k} \leq C_{V,k}(1+t)^k\ \|\ \Psi_0\ \|_{H^k}\ \ .$$

□

Remark 9. The growth of the H^k norm comes from the fact that V forces Ψ to acquire higher momenta.

Theorem 10. *(Bourgain). For $d \geq 3$, V_{real}, assume $\sup_t\ \|\ \partial^K V(t,x)\ \|_{L^\infty_\alpha} < C_\alpha$, and $\sup_{|t-t_0|\leq 1}|V(t,x)|$ compactly supported in x, with the diameter of the support not depending on t_0 (sliding potential). Then, for Ψ from the previous theorem,*

$$\|\ \Psi(t)\ \|_{H^\gamma(\mathbb{R}^d)} \leq C_\varepsilon(1+|t|)^\varepsilon\ \|\ \Psi_0\ \|_{H^\gamma}\ ,$$

for every $\varepsilon > 0$.

Proof. Put $|||f\ ||| = \inf\{\|\ f_1\ \|_2 + \|\ f_2\ \|_\infty : f = f_1 + f_2\}$

Consider the splitting

$$\partial^\alpha\Psi(t) = \partial^\alpha e^{\frac{i}{2}t\Delta}\Psi_0 + i\int_0^{t-A} e^{\frac{i}{2}(t-s)\Delta}\partial^\alpha\big(V(s)\Psi(s)\big)\,ds$$
$$+\, i\int_{t-A}^t e^{\frac{i}{2}(t-s)\Delta}\partial^\alpha\big(V(s)\Psi(s)\big)\,ds$$

The first and third terms will be the L^2 piece and the second term will be the L^∞ piece of the combined $|||\cdot\ |||$ norm.

$$|||\partial^\alpha\Psi(t)\ |||\ \leq \|\ \partial^\alpha\Psi_0\ \|_2 + C\ \cdot\int_0^{t-A}|t-s|^{-d/2}\ \|\ \partial^\alpha\big(V(s)\Psi(s)\big)\ \|_{L^1}\ \ ds$$

(dispersive estimate)

$$+\sup_{\|\phi\|_2=1}\int_{t-A}^{t}\left|\left\langle e^{-\frac{i}{2}(t-s)\Delta}\phi\,,\partial^\alpha\big(V(s)\Psi(s)\big)\right\rangle\right|ds$$

(note that $|t-s|^{d/2}$ is not integrable at $s=t$, hence the cutoff $t-A$).

$$\leq\|\,\partial^\alpha\Psi_0\,\|_2+C\,\cdot\,A^{1-d/2}\sup_{0\leq s\leq t}\|\,\partial^\alpha\big(V(s)\Psi(s)\big)\,\|_{L^1}$$

$$+\sup_{\|\phi\|_2=1}\int_{t-A}^{t}\left\langle(1-\Delta)^{1/4}\chi e^{\frac{-}{i}2(t-s)\Delta}\phi\,,(1-\Delta)^{-1/4}\partial^\alpha\big(V(s)\Psi(s)\big)\right\rangle ds$$

(take $\chi=1$ on supp V)

$$\leq\|\,\partial^\alpha\Psi_0\,\|_2+C\,\cdot\,A^{1-d/2}\sup_{\substack{0\leq\beta\leq\alpha\\0\leq s\leq t}}\|\,\partial^\beta V(s)\partial^{\alpha-\beta}\Psi(s)\,\|_1$$

$$+\sup_{\|\phi\|_2=1}\left(\int_{t-A}^{t}\|\,(1-\Delta)^{1/4}\chi e^{-\frac{i}{2}(t-s)\Delta}\phi\,\|_2^2\ ds\right)^{1/2}$$

$$\cdot\left(\int_{t-A}^{t}\|\,(1-\Delta)^{-1/4}\partial^\alpha\big(V(s)\Psi(s)\big)\,\|_2^2\ ds\right)^{1/2}$$

(apply Cauchy-Schwarz once in x, once in s)

$$\leq\|\,\partial^\alpha\Psi_0\,\|_2+C\,\cdot\,A^{1-d/2}\sup_{\substack{0\leq\beta\leq\alpha\\0\leq s\leq t}}|||\partial^\beta\Psi(s)\,|||$$

($1\to2$ and $1\to\infty$ because we have a uniform control over the size of the support of V)

$+\,(C\,\cdot\,\sqrt{A})$ (by Kato,invariant by translation)

$$t\to t-s$$

$$\cdot\,\sqrt{A}\sup_{0\leq s\leq t}\ \sup_{|\alpha|m=k}\|\,(1-\Delta)^{1/4}\partial^\alpha\big(V(s)\Psi(s)\big)\|_2\ .$$

By Sobolev interpolation, there exists $\delta=\delta(k)>0$ such that $\|\,f\,\|_{H^{k-1/2}}\leq C_k\cdot\ \|\,f\,\|_{H^k}^{1-\delta}\|\,f\,\|_{L^2}^{\delta}$. So

$$\sup_{|\alpha|=k}\|\,(1-\Delta)^{-1/4}\partial^\alpha\big(V(s)\Psi(s)\big)\,\|_2$$

$$\leq C_k\cdot\ \|\,V(s)\Psi(s)\,\|_2^\delta\,\cdot\,\sup_{|\beta|\leq k}\|\,\partial^\beta V(s)\Psi(s)\,\|_2^{1-\delta}$$

$$\leq C_{V,k,\delta}\cdot\ \|\,\Psi_0\,\|_2^\delta\cdot\sup_{|\beta|\leq k}|||\partial^\beta\Psi(s)\,|||^{1-\delta}$$

(as before, localization by V allows us to use $|||\cdot\,|||$). Therefore,

$$|||\partial^\alpha\Psi(t)\,|||\leq\|\,\partial^\alpha\Psi_2\,\|_2+CA^{1-d/2}\sup_{\substack{0\leq|\beta|\leq|\alpha|\\0\leq s\leq t}}|||\partial^\beta\Psi(s)\,|||$$

$$+ CA \parallel \Psi_0 \parallel_2^\delta \sup_{\substack{0\le|\beta|\le|\alpha|\\ 0\le s\le t}} |||\partial^\beta\Psi(s)\ |||^{1-\delta}$$

Young's inequality reads $x^{1-\delta}y^\delta \le \frac{1}{1-\delta}x + \frac{1}{\delta}y$, and we will use $\left(\varepsilon^{\frac{1}{1-\delta}}x\right)^{1-\delta} \cdot \left(\varepsilon^{-1/\delta}y\right)^\delta \le \frac{\varepsilon^{\frac{1}{1-\delta}}}{1-\delta}x + \frac{\varepsilon^{-1/\delta}}{\delta}y$,

$$\sup_{\substack{|\alpha|\le k\\ 0\le s\le t}} |||\partial^\alpha\Psi(s)\ ||| \le \parallel \Psi_0 \parallel_{H^k} + C \cdot A \cdot \varepsilon^{-1/\delta} \parallel \Psi_0 \parallel_2$$

$$+ C\left(A^{1-\frac{d}{2}} + A\varepsilon^{\frac{1}{1-\delta}}\right) \sup_{|\alpha|\le k} |||\partial^\alpha\Psi(s)\ |||$$

We can take ε so small and t, A so large that $C(A^{1-d/2} + A\varepsilon^{\frac{1}{1-\delta}}) \le \frac{1}{2}$, hence

$$\sup_{\substack{|\alpha|\le K\\ 0\le s\le t}} |||\partial^\alpha\Psi(s)\ ||| \le C_{V,k} \cdot \parallel \Psi_0 \parallel_{H^k}$$

As a result,

$$\parallel \Psi(t) \parallel_{H^k} \le \parallel \Psi_0 \parallel_{H^k} + \int_0^t \parallel V(s)\Psi(s) \parallel_{H^k}\ ds$$

with

$$\parallel V(s)\Psi(s) \parallel_{H^k} \le C_k \sup_{|\alpha|\le k} \parallel \partial^\alpha V(s)\Psi(s) \parallel_2$$

$$\le C_k \sup_{|\alpha|\le k} |||\partial^\alpha\Psi(s)\ |||$$

(as we have seen, localization by V enables control by both $\parallel \cdot \parallel_2$ and $\parallel \cdot \parallel_\infty$). So

$$\parallel \Psi(t) \parallel_{H^k} \le \parallel \Psi_0 \parallel_{H^k} (1 + C_{V,k}t)\ .$$

□

Ma 142 01/16

Take $V = V(x)$, real. $H = \frac{1}{2}\Delta + V$

$$\Psi = e^{itH}\Psi_0$$

Spectral theory of $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$.
$\text{Dom}(H) = W^{2,2}(\mathbb{R}^d)$, $V \in L^\infty(\mathbb{R}^d)$

$$Hf = -\Delta f + Vf \qquad \in L^2(\mathbb{R}^d)$$

$$f, g \in \text{Dom}H : \langle Hf, g\rangle = \int(-\Delta f)\bar{g}\ dx + \int Vf\bar{g}\ dx$$

$$= \int f - \overline{\Delta g}\ dx + \int fV\bar{g} = \langle f, Hg\rangle$$

Definition.

(1) a densely defined operator A on $\mathcal{H}$ Hilbert is symmetric if

$$\langle Af, g\rangle = \langle f, Ag\rangle \qquad \forall f, g \in \text{Dom}(A)\ .$$

(2) A^* is defined as follows:

$$\begin{aligned}\mathrm{Dom}(A^*) &= \{f \in \mathcal{H} \cdot \exists h : \langle A\phi, f\rangle \\ &= \langle \phi, h\rangle \forall \phi \in \mathrm{Dom} A\}\end{aligned}$$

Set $A^* f = h$.

(3) A is closed provided the graph of A is closed as a subset of $\mathcal{H} \times \mathcal{H}$ i.e., if

$$\left.\begin{array}{lcl} f_n & \to & f \\ A f_n & \to & g \end{array}\right\} \Rightarrow f \in \text{Dom } A \text{ and } Af = g$$

Lemma.

(1) *A symmetric $\Rightarrow$ Dom $A^* \supset$ Dom A ($\Rightarrow A^*$ densely defined) and all egv of A are real*
(2) *for any A, A^* is closed*
(3) $(\ker A)^\perp = \overline{\mathrm{Ran} A^*}$ *and* $(\ker A^*)^\perp = \overline{\mathrm{Ran} A}$
(4) *If $A = A^*$, then Spec$(A) := \mathbb{C} \setminus \{z \in \mathbb{C} : (A - z)^{-1}$ exists as a bdd operator$\} \subset \mathbb{R}$ and* $\| (A - z)^{-1} \| \leq |\operatorname{Im} z|^{-1}$

Proof.

(1) Because if in Dom A one can integrate by parts.
(2) $f_n \in$ Dom A^*, $f_n \to f$
$A^* f_n \to g$.

$$\begin{array}{ccc} \langle A\phi, f_n\rangle & = & \langle \phi, A^* f_n\rangle \qquad \forall \phi \in \text{Dom } A \\ \downarrow & & \downarrow \\ \langle A\phi, f\rangle & = & \langle \phi, g\rangle \end{array}$$

so $f \in$ Dom A^* and $g = A^* f$.

(3)

$$\begin{array}{c} \langle A\phi, f\rangle = 0 \qquad \forall \phi \in \text{Dom } A \\ \shortparallel \\ \langle \phi, 0\rangle \qquad \Rightarrow \quad f \in \text{Dom } A^* \text{ and } A^* f = 0 \end{array}$$

(4) $A = A^*$

$$\begin{aligned} &\big\langle \big(A - (x + iy)\big) f, f \big\rangle \\ = &\Big| \underbrace{\langle (A - x) f, f\rangle}_{\text{real}} - iy \underbrace{\langle f, f\rangle}_{\text{real}} \Big| \geq |y| . \| f \|_2^2 \\ &\leq \| f \| . \| (A - z) f \| \\ &\Rightarrow \| (A - z) f \| \geq |y| \, \| f \| \qquad \text{ker is 0.} \end{aligned}$$

$$\begin{aligned} \overline{\mathrm{Ran}(A - z)} &= \big(\ker(A^* - \bar{z})\big)^\perp \\ &= \big(\ker(A - \bar{z})\big)^\perp = \mathcal{H} \end{aligned}$$

but still have to show Ran$(A - z)$ closed
$(A - z) f_n \to g$

$$\| (A - z) f_n - (A - z) f_m \| \geq |y| \; \| f_n - f_m \|$$
$$f_n, f_m \text{ Cauchy} \Rightarrow f_n \to f \, .$$

$$\begin{aligned} &\text{and } A = A^* \text{ is closed} \\ &\qquad \Rightarrow A - z \text{ closed} \\ &\qquad \Rightarrow f \in \text{Dom } A \text{ and } Af = g \\ &\qquad \Rightarrow \text{Ran } (A - z) \text{ closed.} \end{aligned}$$

(one-to-one, onto, bound OK). □

Example

$$A = -\frac{d^2}{dx^2} \quad \text{on} \quad L^2[0,1] \, .$$

$$\text{Dom } A = \big\{ f \in W^{2,2}([0,1]) : f(0) = f(1) = 0 \\ f'(0) = f'(1) = 0 \big\} \, .$$

symmetric, not self-adjoint.

Dom $A^* = W^{2,2}$

Spectral Theorem Let A be self-adjoint on $\mathcal{H}$. Then $\exists$ resolution of the identity:

$$\begin{array}{cccc} E: & B(\mathbb{R}) & \rightarrow & B(\mathcal{H},\mathcal{H}) \\ & \downarrow & & \downarrow \\ & \text{Borel sets} & & \text{bdd ops} \end{array}$$

(i) $E(I)$ is an orthogonal projection for each $I \subset \mathbb{R}$ interval
(ii) $E(\mathbb{R}) = Id_{\mathcal{H}}$, $E(\emptyset) = 0$
(iii) $E(I)E(J) = E(I \cap J)\ \forall I, J$
(iv) $E(\cup I_j) = \sum E(I_j)$

$$A = \int \lambda E(d\lambda)$$

in the following sense

$$\text{Dom } A = \{ f \in \mathcal{H} : \int \lambda^2 \langle E(d\lambda) f, f \rangle < \infty \}$$

$$\text{and} \quad \langle Af, g \rangle = \int \lambda \langle E(d\lambda) f, g \rangle$$

Def. if $f \in C_b^\circ(\mathbb{R})$, define $f(A) = \int f(\lambda E(d\lambda)$

Example 1.

(1) $\mathcal{H} = \mathbb{R}^n$, A hermitian matrix

$$E(I) = \sum_{j:\lambda_j \in I} P_{L_j}$$

where $\lambda_1 \leq \cdots \leq \lambda_k$ are e.v. without multiplicity and eigenspaces $L_1, \cdots, L_k$

$$E(d\lambda) = \sum_j \delta\lambda_j(d\lambda) P_{L_j}$$

(2) $e^{itA} = \int e^{it\lambda} E(d\lambda)$

$$\| e^{itA} f \|^2 = \int e^{it\lambda} \langle E(d\lambda) f, \int e^{it\mu} E(d\mu) f \rangle$$

$$= \iint e^{it(\lambda-\mu)} \langle E \underbrace{(d\mu) E(d\lambda)}_{\mu \text{ must be } = \lambda} f, f \rangle \; .$$

$\delta @ \lambda = \mu$

$$= \int \langle E(d\lambda) f, f \rangle = \langle f, f \rangle = \| f \|^2 \quad \to \text{isometry.}$$

$$e^{itA} e^{isA} f = e^{i(t+s)A} f$$

$$e^{-itA} e^{itA} = Id_{\mathcal{H}}$$

e^{itA} is unitary, and $(e^{itA})^* = (e^{itA})^{-1} = e^{-itA}$

(3) $\mathcal{H} = L^2(X, \mu)$, μ positive probability measure.

$$A = \mathcal{M}_\phi \; : \; Af = \phi f \text{ where } \phi : X \to \mathbb{R} \; .$$

Dom $A = \{ f \in L^2 = \phi f \in L^2 \}$ dense in $\mathcal{H}$

because ϕ locally finite

$\mu(|\phi| > M) \to 0$ as $M \to \infty$.

$A = A^*$, ϕ is real.

$A \subset A^*$ OK.

$$\langle Af, g \rangle = \langle f, h \rangle$$

$$\| $$

$$\int \phi f \bar{g} \mu(dx) = \int f(x) \overline{\phi g}(x) \mu(dx)$$

$$= \int f \bar{h}(x) \mu(dx)$$

$\phi g = h \in L^2$ so $g \in$ Dom A .

$A^* \subset A$ OK.

$$E(I) f = \chi_{[\phi \in I]} f$$

(4) $A = -\Delta$ on $L^2(\mathbb{R}^d)$

Dom $A = W^{2,2}(\mathbb{R}^d)$.

$\langle -\Delta f, g \rangle = \langle f, h, \rangle$ no $\int$ by parts.

$\mu = -\Delta g \in \mathcal{S}' |\langle \mu, f \rangle| \leq C. \| f \|_2$ with $C = \| h \|_2$

Hahn-Banach, μ extends to a co. fnl on L^2

$$\| \mu \| \leq \| n \|_2$$
$$\Delta g \in L^2 \ , \ g \in W^{2,2}$$
$$\left. \begin{array}{r} \Rightarrow \text{Dom}\,\Delta^* \subset \text{Dom}\,\Delta \\ + \text{ symmetric} \end{array} \right\} \Rightarrow \text{self-adj.}$$

$$\widehat{Af} = \xi^2 \hat{f}(\xi)$$
$$FT(A)(FT)^{-1} = \mathcal{M}_{\xi 2} \ . \qquad (\Delta \text{ unit eq. to mult. } \xi^2)$$
$$FT : L^2_x \to L^2_\xi \text{ unitary}$$
$$FT E_{-\Delta}(FT)^{-1} = E_{\xi 2}$$
$$E_{-\Delta}(I)f = (\chi_{[\xi^2 \in I]}\hat{f})^\vee$$
$$\Rightarrow \quad \underbrace{\widehat{e^{itA}f}}_{\substack{\text{defined} \\ \text{via spec. thm.}}} = e^{it\xi^2}\hat{f}(\xi)$$

(5) $H = -\Delta + V$, $V \in L^\infty$.
Claim: $H = H^*$ on $W^{2,2}(\mathbb{R}^d)$.
H symmetric $H \subset H^*$

$$\langle -\Delta f + Vf, g \rangle = \langle f, h \rangle$$
$$|\langle -\Delta f, g \rangle| \leq (\| h_2 \| + \| V_g \|_2) \| f \|_2$$
$$\leq \| V \|_\infty \| g \|_2$$

Hahn-Banach.
($\Delta g \in \mathcal{S}'$ satisfies an L^2 bound $\Rightarrow \Delta g \in L^2$).

Theorem. *If $V \in L^\infty$ then*

$$\sup_{+} \| \nabla e^{itH} f \|_2 \leq C \| f \|_{H^1}$$
$$\sup_{+} \| \nabla^2 e^{itH} f \|_2 \leq C. \| f \|_{H^2}$$

Proof.

$$H + C \geq Id \ .$$
$$\langle (H + C)f, f \rangle \geq \| f \|_2^2 \qquad \text{(int. by parts, positive)}$$
$$C = 1 + \| V \|_\infty$$
$$(H + C)^{-1} : L^2 \to W^{2,2}$$
$$\| (H + C)^{-1} f \|_{W^{2,2}} \leq C \| f \|_2$$
$$\| f \|_{W^{2,2}} \lesssim \| (H + C) f \|_2$$
$$\| e^{itH} f \|_{W^{2,2}} \lesssim \| (H + C) e^{itH} f \|_{L^2}$$

$$= \| e^{itH}(H+C)f \|_{L^2} \lesssim \| f \|_{W^{2,2}} \quad .$$

$\square$

Calculation of $\big(-\Delta - (\lambda + i\varepsilon)\big)^{-1}$ in $d=1$, $d=3$.

$$\widehat{\big(-\Delta - (\lambda + i\varepsilon)\big)^{-1} f} = \frac{\hat{f}(\xi)}{\xi^2 - (\lambda + i\varepsilon)}$$

$$\big(R_0(\lambda + i\varepsilon) f\big)(x) = \frac{1}{(2\pi)^d} \int \frac{e^{ix\xi}\hat{f}(\xi)}{\xi^2 - (\lambda + i\varepsilon)}\, d\xi \ .$$

$$d = 3. \qquad R_0(\lambda + i\varepsilon)(x) = \frac{1}{(2\pi)^3} \int \frac{e^{ix\xi}}{\xi^2 - (\lambda + i\varepsilon)}\, d\xi \ .$$

$$= \frac{1}{2\pi^2} \int_0^\infty \int_{S^2} e^{irx\cdot\omega} d\sigma_{S^2}(\omega) \frac{r^2}{r^2 - (\lambda + i\varepsilon)}\, dr$$

$$\int_{S^2} e^{ia\omega\cdot\vec{e}_3}\, d\sigma_{S^2}(\omega) = 2\pi \int_0^\pi e^{ia\omega\sin\theta} \sin\theta\, d\theta$$

$$= 2\pi \int_{-1}^{1} e^{iau}\, du = 4\pi \frac{\sin a}{a}$$

$$= \frac{1}{2\pi^2} \int_0^\infty \frac{\sin(r|x|}{r|x|} \frac{r^2}{r^2 - (\lambda + i\varepsilon)}\, dr$$

$$= \frac{1}{2\pi|x|} \int_0^\infty \frac{\sin(r|x|)}{r^2 - (\lambda + i\varepsilon)}\, r\, dr$$

$$= \frac{1}{4\pi|x|} \int_{-\infty}^{\infty} \quad \frac{\quad}{\quad}$$

$$= \frac{1}{16\pi|x|} \int_{-\infty}^{\infty} (e^{ir|x|} - e^{-ir|x|}) \left(\frac{1}{r - \sqrt{\lambda + i\varepsilon}} - \frac{1}{r + \sqrt{\lambda + i\varepsilon}} \right) dr$$

$$= \frac{1}{4\pi|x|} e^{i|x|\sqrt{\lambda + i\varepsilon}}$$

$$R_0(\lambda + i0)(x) = \lim_{\varepsilon \to 0} R_0(\lambda + i\varepsilon)(x)$$

$$= \frac{e^{i|x|\sqrt{\lambda}}}{4\pi|x|}\ , \quad -\infty < \lambda < \infty\ ,$$

$\sqrt{\ }$ is principal branch.

$\mathbf{d=1}$. $$R_0(\lambda+i\varepsilon)(x,y)=\frac{f_1(\lambda+i\varepsilon,x)f_2(\lambda+i\varepsilon,y)}{W\big[f_1(\lambda+i\varepsilon)f_2(\lambda+i\varepsilon)\big]}\quad \text{if } x<y$$

otherwise reverse.

Green's fn

$G_{\lambda+iz}(x,y)$

Lemma.

$$\Big(-\frac{d^2}{dx^2}-(\lambda+i\varepsilon)\Big)G_{\lambda+i\varepsilon}=Id\ .$$

$$\left.\begin{array}{rcl} f_1(\lambda+i\varepsilon,x) &=& e^{-i\sqrt{\lambda+i\epsilon}\,x} \\ f_2(\lambda+i\varepsilon,y) &=& e^{+i\sqrt{\lambda+i\epsilon}\,y} \end{array}\right\}\text{exp decaying!}$$

$$\underbrace{W[f_1,f_2]}_{\text{const. in } x}=\begin{vmatrix} f_1 f_1' \\ f_2 f_2'\end{vmatrix}=-2i\sqrt{\lambda+i\varepsilon}$$

$$\Rightarrow R_0(\lambda+i\varepsilon)=\frac{-e^{+i\sqrt{\lambda+i\varepsilon}|x-y|}}{2i\sqrt{\lambda+i\varepsilon}}\quad\text{and lim as } \varepsilon\to 0$$

Finer properties of the spectral measure.

Lemma. *A is selfadjoint on* $\mathcal{H}$

$$\mathcal{H}=\mathcal{H}_{ac}\oplus\mathcal{H}_{sc}\oplus\mathcal{H}_{pp}$$

$$\mathcal{H}_{\substack{ac\\sc\\pp}}=\big\{f\in\mathcal{H}:\langle E(\,\cdot\,)f,f\rangle \text{ is } \begin{array}{lcl} ac &\Rightarrow& \text{absolutely continuous}\\ sc &\Rightarrow& \text{singular continuous}\\ pp &\Rightarrow& \text{pure point (atomic)}\end{array}$$

Proof.

(i) If $f,g\in\mathcal{H}_{ac}$, $|S|=0$.

$$\begin{aligned}\langle E(S)(f+g),f+g\rangle &=\| E(S)f\|^2 \qquad \}=0\\ &\quad+\| E(S)g\|^2 \qquad \}=0\\ &\quad+\langle E(S)f,g\rangle+\langle E(S)g,f\rangle=0\\ &=0\end{aligned}$$

because f,g are in a.c.

$$\begin{array}{lcl} E(S)f_n=0\ , & & f_n\to f\ ,\\ \downarrow & & \\ E(S)f & \Rightarrow & \mathcal{H}_{ac} \text{ is a closed subspace.}\end{array}$$

(ii) $f,g\in\mathcal{H}_{sc}$

S_f support f;

S_f,S_g *s.t.* $|S_f|=0,|S_g|=0$

$E(S_f^c)f=0$,

$$E(S_g^c)f = 0\ .$$

$$E\big(\{\lambda\}\big)f = 0$$
$$E\big(\{\lambda\}\big)g = 0$$

$$S_{f+g} = S_f \cup S_g$$

$$\begin{aligned} E(S_{f+g}^c)(f+g) = E(S_f^c)\,&\overbrace{E(S_g^c)g}^{0} \\ + E(S_g^c)\,&\underbrace{E(S_f^c)f}_{0} = 0\ . \end{aligned}$$

$$E\big(\{\lambda\}\big)(f+g) = 0\ .$$
$$f_n \in \mathcal{H}_{sc} \to f \qquad \cup_n Sf_n =: S_f$$
$$E(S_f^c)f_n = 0 \to E(S_f^c)f = 0$$

closed subspace, too.

(iii) exercise

□

$$\begin{aligned} \langle E(A)f, f\rangle =& \langle E(S_{ac}\cap A)f, f\rangle \\ &+ \langle E(S_{sc}\cap A)f, f\rangle \\ &+ \langle E(S_{pp}\cap A)f, f\rangle \end{aligned}$$

$$S_{ac} \cap S_{sc} = \emptyset \text{ etc.}$$

$$(ac) \perp (sc \text{ and } pp): \quad f \in \mathcal{H}_{ac}\ ,\ g \in \mathcal{H}_{\text{sing}}$$
$$g = E(S_g)g$$

$$\langle f, g\rangle = \langle E\underbrace{(S_g)}_{=0} f, g\rangle = 0$$

Ma 142 01/21

Definition $E_{ac} = E \circ P_{ac}$

Ex.

(1) $\mathcal{M}_\phi = f = \phi f.$
$\phi : \mathbb{R} \to \mathbb{R}.$
E_{pp} : plateaux of ϕ
$\big(E(I)f\big)(x) = \chi_{[\phi\in I]}f(x).$

(2) $-\Delta = FT\mathcal{M}_{\xi^2}FT^{-1}$
$\phi(\xi) = \xi^2$
purely a.c. because

$$\langle E(\overset{\text{Borel}}{I^{\nearrow}})\, f, f\,\rangle = \int\limits_{[\xi^2\in I]} |\hat{f}(\xi)|^2 \, d\xi = 0 \quad \text{if } |I| = 0\,.$$

(3) $H = -\Delta - \lambda V, \quad V \in L^\infty_{\text{comp}}, \quad V \geq 0, \quad V \neq 0.$

$$f \in \mathcal{S}\,, \langle Hf, f\rangle = \int |\nabla f|^2 - \lambda \underbrace{\int V|f|^2 \, dx}_{\neq 0} < 0$$

for some $f \in \mathcal{S}$ and λ large.
$\Rightarrow$ $\text{spec}(H) \cap (-\infty, 0) \neq \emptyset$.
Fact: ess spec $(H) = (0, \infty)$
$:= sp(H) \setminus \{z \in \mathbb{C} | z \text{ is an isolated egv of finite multiplicity}\}$.
(4) $e^{itH}f = e^{itE}f \quad$ if $Hf = Ef$, no time decay.

Theorem. $(d = 1)$. *Take* $H = -\frac{d^2}{dx^2} + V$
assume $\int (1 + |x|)^3 |V(x)| \, dx < \infty$ *and* V *generic. Then*

$$\| \, e^{itH} P_{ac}(H) f \, \|_\infty \leq C.|t|^{-1/2} \, \| \, f \, \|_1$$

(contrast with bound states, here $P_{ac} = 1 - P_{\substack{\text{bound}\\ \text{states}}}$*)*

Lemma. *If* A *is self-adjoint on* $\mathcal{H}$*, then for almost every* λ

$$\langle E_{ac}(d\lambda) f, g\rangle = \left\langle \frac{1}{2i\pi}\big(R(\lambda + i0) - R(\lambda - i0)\big) f, g \right\rangle d\lambda$$

Proof. Absolutely c. measure in $d\lambda$, identify the densities = a.e.

$$\begin{aligned}
&\left\langle \frac{1}{2i\pi}\left[\big(A - (\lambda + i\varepsilon)\big)^{-1} - \big(A - (\lambda - i\varepsilon)\big)^{-1}\right] f, g \right\rangle \\
&= \int \frac{1}{2i\pi}\left(\frac{1}{t - (\lambda + i\varepsilon)} - \frac{1}{t - (\lambda - i\varepsilon)}\right) \langle E(dt) f, g\rangle \\
&= \int \frac{1}{\pi} \frac{\varepsilon}{(t - \lambda)^2 + \varepsilon^2} \langle E(dt) f, g\rangle
\end{aligned}$$

for a.e. λ $\quad \underset{\text{as } \varepsilon \to 0_+}{\longrightarrow} \quad \dfrac{\langle E_{ac}(d\lambda) f, g\rangle}{d\lambda}$

□

Lemma. *High-energy part. Let* $\chi \in C_0^\infty(\mathbb{R}_+)$,

$$\chi(x) = 1 \text{ if } \lambda \geq 2\lambda_0\,. \qquad V \in L^1$$

and $\chi(\lambda) = 0$ *if* $\lambda < \lambda_0$ *(for some large but fixed* λ_0*). (Then same estimate, but)*

$$\| \, e^{itH} \chi(H) P_{ac}(H) f \, \|_\infty \leq C.|t|^{-1/2} \, \| \, f \, \|_1$$

Proof. $\langle e^{itH}\chi(H)P_{ac}(H)f,g\rangle =$

$$= \int_0^\infty e^{it\lambda}\chi(\lambda)\frac{1}{2\pi i}\Big\langle\big(R_V(\lambda+i0)-R_V(\lambda-i0)\big)f,g\Big\rangle\, d\lambda$$

$$\begin{aligned} R_V(\lambda+i0) &= \lim_{\varepsilon\to 0^+}\left(-\frac{d^2}{dx^2}+V-(\lambda+i\varepsilon)\right)^{-1} \qquad \text{not } L^2 \text{ op} \\ &= \lim_{\varepsilon\to 0^+}\big(H_0-(\lambda+i\varepsilon)+V\big)^{-1} \\ &= R_0(\lambda+i0)-R_0(\lambda+i0)VR_0(\lambda+i0)+\cdots \\ &= \sum_{n=0}^\infty R_0(\lambda+i0)\big(-VR_0(\lambda+i0)\big)^n \Rightarrow \text{Born series} \end{aligned}$$

Neumann series for $(H_0-\lambda)^{-1}\big(1+V(H_0-\lambda)^{-1}\big)^{-1}$.

$$\left.\begin{aligned} R_0(\lambda+i0)(x) &= -\tfrac{1}{2i\sqrt{\lambda}}\, e^{i\sqrt{\lambda}|x|} \\ R_0(\lambda-i0)(x) &= \tfrac{1}{2i\sqrt{\lambda}}\, e^{-i\sqrt{\lambda}|x|} \end{aligned}\right\} \text{NB: positive and expo decaying}$$

Convergence: pick λ large enough.

Claim

$$\langle R_V(\lambda+i0)f,g\rangle = \sum_{n=0}^\infty (-1)^n\langle(R_0(\lambda+i0)\big(VR_0(\lambda+i0)\big)^n f,g\rangle$$
$$\forall\lambda>\lambda_0\ ,\ f,g\in\mathcal{S}$$

Proof.

$$\begin{aligned} &\Big|\Big\langle R_0(\lambda+i\varepsilon)\big(VR_0(\lambda+i\varepsilon)\big)^n f,g\Big\rangle\Big| \\ \le &\int\cdots\int \frac{1}{|2\sqrt{\lambda+i\varepsilon}|^{n+1}} \qquad |V(x_1)\cdots V(x_n)|dx_1\cdots dx_n \\ &\qquad\qquad |f(x_0)|\ |g(x_{n+1})|\ dx_0\ dx_{n+1} \\ \le &\left(\frac{1}{2\sqrt{\lambda}}\right)^{n+1} \|\ V\ \|_1^n \|\ f\ \|_1 \|\ g\ \|_1 \qquad \text{uniform in } \varepsilon \end{aligned}$$

uniform in ε. Conv. of the series: take $2\sqrt{\lambda} > \|\ V\ \|_1$ so $\lambda_0 = \|\ V\ \|_1^2$. ☐

$$\begin{aligned} \therefore \Big|\Big\langle e^{itH}\chi(H)P_{ac}(H)f,g\Big\rangle\Big| \le \sum_{n=0}^\infty \Big|\frac{(-1)^n}{2\pi i}\int_0^\infty e^{it\lambda}\chi(\lambda)\cdots \\ \cdots\Big(\Big\langle R_0(\lambda+i0)\big(VR_0((\lambda+i0)\big)^n f,g\Big\rangle \\ -\Big\langle R_0(\lambda+i0)\big(VR_0(\lambda+i0)\big)^n f,g\Big\rangle\Big)\ d\lambda\Big| \end{aligned}$$

The general term is $\int\limits_0^\infty e^{it\lambda}\chi(\lambda)\langle R_0(\lambda+i0)\big(VR_0(\lambda+i0)\big)^n f,g\rangle\big)\,d\lambda$

$$= \iint f(x_0)g(x_{n+1})\int\cdots\int V(x_1)\cdots V(x_n)\int\limits_0^\infty e^{it\lambda^2}2\lambda\chi(\lambda)\,.$$

$$\frac{e^{i\lambda(\sum_{j=1}^{n+1}|x_j-x_{j-1}|)}}{(-2i\lambda)^{n+1}}\,d\lambda$$

$$\int\limits_0^\infty e^{it\lambda}\chi(\lambda)\big\langle R_0(\lambda+i0)\big(VR_0(\lambda+i0)\big)^n f,g\big\rangle\,d\lambda$$

$$= \iint f(x_0)g(x_{n+1})\int\cdots\int V(x_1)\cdots V(x_n)\int\limits_0^\infty e^{it\lambda^2}2\lambda\chi(\lambda^2)\cdot\frac{e^{-i\lambda\sum|\text{—}|}}{(2i\lambda)^{n+1}}\,d\lambda$$

then $\lambda\to-\lambda$ and $\int_0^\infty-\int_0^\infty$
$\to\int_0^\infty+\int_{-\infty}^0$
conclusion $|\langle\text{—}\rangle|\leq$

$$\sum_{n=0}^\infty\frac{1}{2\pi}\underbrace{\int dx_0\int dx_{n+1}}_{\mathbb{R}^2}|f(x_0)|\,|g(x_{n+1})|$$

$$\int\cdots\int dx_1\dots dx_n\Big|V(x_1)\dots V(x_n)\Big|\cdot$$

$$\underbrace{\sup_{x_0\,,\,\dots\,,\,x_{n+1}}\Big|\int\limits_{-\infty}^\infty e^{it\lambda^2}\lambda\chi(\lambda^2)\frac{e^{i\lambda\sum\limits_{j=1}^{n+1}|x_j-x_{j-1}|}}{(2i\lambda)^{n+1}}\,d\lambda\Big|}_{\text{claim: }\leq\frac{C.|t|^{-1/2}}{(2\lambda_0)^n}\text{(then done)}}$$

Expression between $|\text{—}|$.

$$\int\limits_{-\infty}^\infty e^{i(t\lambda^2+a\lambda)}\hat\mu_n(\lambda)\,d\lambda\qquad\text{(stationary phase).}$$

$$|\quad\text{—}\quad|\leq C.|t|^{-1/2}\parallel\mu_n\parallel_1$$

(disp.est.)

$$\text{where}\quad\mu_n(x)=\frac{\left(\frac{i}{2}\right)^{n+1}}{2\pi}\int\limits_{-\infty}^\infty\frac{\chi(\lambda^2)}{\lambda^n}\,e^{-i\lambda x}\,d\lambda$$

$$\parallel\mu_n\parallel_{\text{meas}}\leq C\left(\frac{1}{2\lambda_0}\right)^n$$

□

Lemma. *(low energy) Assume V as in theorem. Let $\chi(\lambda)$ be a smooth cutoff around 0. Then*

$$\| e^{itH}\chi(H)P_{ac}f \|_\infty \leq C.|t|^{-1/2} \| f \|_1$$

First steps of the proof

$$\langle e^{itH}\chi(H)P_{ac}f, g\rangle = \frac{1}{2\pi i}\int_0^\infty e^{it\lambda^2}2\lambda\chi(\lambda^2)$$

$$\Big\langle \big(R_V(\lambda^2+i0) - R_V(\lambda^2-i0)\big)f, g\Big\rangle \, d\lambda$$

$$R_V(\lambda^2+i0)(x,y) = \frac{f_+(\lambda,y)f_-(\lambda,x)}{W\big(f_+(\lambda,\cdot), f_-(\lambda,\cdot)\big)}\chi_{\{x<y\}} + \frac{f_+(\lambda,x)f_-(\lambda,y)}{W\big(f_+(\lambda,\cdot), f_-(\lambda,\cdot)\big)}\chi_{\{y<x\}}$$

($V_{\text{generic}} \Rightarrow$ denom @ 0 is nonzero) where $f_\pm(\lambda,\cdot)$ are the Jost solutions, i.e.,

$$\begin{cases} -f_\pm'' + Vf_\pm = \lambda^2 f_\pm \\ |f_+(\lambda,x) - e^{i\lambda x}| \to 0 \quad \text{as } x\to\infty \\ |f_-(\lambda,x) - e^{-i\lambda x}| \to 0 \quad \text{as } x\to -\infty \end{cases}$$

$$R_V(\lambda^2 - i0)(x,y) = \frac{f_+(-\lambda,y)f_-(-\lambda,x)}{W\big[f_+(-\lambda,\cdot), f_-(-\lambda,\cdot)\big]} \qquad y>x$$

Conclusion:

$$\begin{aligned} &\big\langle e^{itH}\chi(H)P_{ac}(H)f, g\big\rangle \\ =&\frac{1}{\pi i}\int_{-\infty}^{\infty}\int e^{it\lambda^2}\lambda\chi(\lambda^2)\frac{f_+(\lambda,y)f_-(\lambda,x)}{W(\lambda)}\, d\lambda\ f(x)g(y) \qquad \chi_{[x<y]}dx\, dy \\ &+\frac{1}{\pi i}\int_{-\infty}^{\infty}\int e^{it\lambda^2}\lambda\chi(\lambda^2)\frac{f_+(\lambda,x)f_-(\lambda,y)}{W(\lambda)}\, d\lambda\ f(x)g(y) \qquad \chi_{[x>y]}dx\, dy \\ &+ \text{terms for } \ R_v(\lambda^2 - i0) \end{aligned}$$

then show $\int d\lambda \leq C.|t|^{-1/2}$ independent of x, y.
$\to$ show L^1 bddness of $\frac{f_+f_-}{W}$.
Existence of $f_\pm$ (var. of const. = Duhamel)

$$-f_+'' + Vf_+ = z^2 f_+\ ,\ \text{Im} z \geq 0$$

$$f_+(z,x) = e^{izx} + \int_x^\infty G_z(x,y)V(y)f_+(z,y)\, dy\ .$$

solves free equation, with inhom. $\delta(x-y)$.

$$\begin{cases} G_z(x,x) = 0\ . \\ \partial_x G_z(x,x) = -1 \\ \big(-\frac{d^2}{dx^2} - z^2\big)G_z(x,y) = 0 \quad \text{if } x<y\ . \end{cases}$$

$$\Rightarrow G_z(x,y) = \frac{\sin\big(z(x-y)\big)}{z} \text{ if } x < y$$

$$\text{Solve} \quad f_+(z,x) = e^{izx} + \int_x^\infty \frac{\sin\big(z(y-x)\big)}{z} V(y) f_+(z,y)\, dy$$

$$m_+(z,x) = e^{-izx} f_+(z,x)$$

$$\circledast \quad m_+(z,x) = 1 + \int_x^\infty \frac{e^{2iz(y-x)}-1}{2iz} V(y) m_+(z,y)\, dy\ .$$

Lemma. *Suppose $V \in L^1(\mathbb{R})$. Then for every $\Im az \geq 0$, $z \neq 0$, there exists one and only one $m_+(z,\cdot)$ that satisfies $\circledast$. Moreover, $|m_+(z,x)-1| \to 0$ as $x \to \infty$.*

Proof.

$$(Am)(x) = 1 + \int_x^\infty \frac{e^{2iz(y-x)}-1}{2iz} V(y) m(y)\, dy$$

$$A : X \circlearrowleft \qquad X = \{ f \in C\big([x,\infty)\big) : \| f \|_{L^\infty([x,\infty))} < \infty \}\ .$$

$$\| Am \|_X \leq 1 + \frac{1}{|z|} \int_{x_1}^\infty |V(y)|\, dy \, \| m \|_X$$

$$\leq 1 + \underbrace{\frac{\| V \|_{L^1(x_1,\infty)}}{|z|}}_{<\frac{1}{2} \text{ if } x_1 \text{ large enough.}} \| m \|_X =: R\ .$$

$$A : B(0,R) \to B(0,R)\ .$$

$$\| Am - A\tilde{m} \|_X \leq \underbrace{\frac{\| V \|_{L^1(x_1,\infty)}}{|z|}}_{\text{contraction.}} \| m - \tilde{m} \|_X$$

Therefore there exists one and only one m in $B_R(X)$ that satisfies $\circledast$ for $x \geq x_1$.

To extend to $x < x_1$, we can write

$$(A_2\phi)(x) = 1 + \int_{x_1}^\infty \frac{e^{2iz}(y-x)}{2iz} m_+(z,y) V(y)\, dy$$

$$+ \int_x^{x_1} \frac{e^{2iz}(y-z)}{2iz} \phi(y) V(y)\, dy\ .$$

on some $[x_2, x_1]$ this is a contraction:

Choose x_2 s.t $\frac{1}{|z|}\int_{x_2}^{x_1} |V(t)|\ dt < 1/2$. ($L^1 \Rightarrow$ sufficiently small on uniform intervals.) Extend to 0. $\square$

Ma 142 01/23

Low energy lemma

$$\begin{aligned}
&\langle e^{itH} P_{ac}(H)\chi_{\text{small}}(H)f, g\rangle \\
&= \int_0^\infty d\lambda \cdot 2\lambda e^{it\lambda^2}\chi_{\text{small}}(\lambda^2) \\
&\qquad \left\langle \frac{1}{2i\pi}\big(R_V(\lambda^2+i0) - R_V(\lambda^2-i0)\big)f, g\right\rangle .
\end{aligned}$$

$$R_V(\lambda^2+i0) = \chi_{[y\le x]}\frac{f_+(\lambda,x)f_-(\lambda,y)}{W\big[f_+(\lambda),f_-(\lambda)\big]} + \chi_{[y>x]}\cdot\frac{f_+(\lambda,y)f_-(\lambda,x)}{W\big[f_+(\lambda),f_-(\lambda)\big]} .$$

Lemma. (1) *Let* $\operatorname{Im} z > 0$, $z \neq 0$, $V \in L^1$, *then* $\exists! m_+(z,\cdot)$ *s.t.*

$$m_+(z,\cdot) = 1 + \int_x^\infty \frac{e^{2iz(y-x)}-1}{2iz}V(y)m_+(z,y)\,dy .$$

(2) *If* $\Im z \ge 0$, $V \in L^1_1 = \big\{f\in L^1(\mathbb{R}) : \int(1+|x|)|V(x)|\,dx < \infty\big\}$ *then* $\exists! m_+(z,x)$ *s.t.*

$$\circledast \qquad m_+(z,x) = 1 + \int_x^\infty D_z(y-x)V(y)m_+(z,y)\,dy$$

$$\textit{where}\;\; D_z(y-x) = \int_0^{y-x} e^{2izt}\,dt \;\textit{when}\; \lambda > 0.$$

Corollary. *Set* $f_+(z,x) = e^{izx}m_+(z,x)$
$\operatorname{Im} z \ge 0$, *then* $f_+(z,\cdot)$ *satisfies*

$$\circledast\circledast \qquad f_+(z,x) = e^{izx} + \int_x^\infty \frac{\sin\big(z(y-x)\big)}{z}V(y)f_+(z,y)\,dy$$

$$\begin{aligned}
&|f_+(z,x) - e^{izx}| \to 0 \quad \textit{as } x\to\infty. \\
&|f_+'(z,x) - ize^{izx}| \to 0 \quad \underline{\qquad\quad} \\
&f_+(z,\cdot) \in \big\{f\in C^1(\mathbb{R}) : f' \in AC_{\text{loc}}\,, \\
&\qquad\qquad\qquad f'' \in L^1_{\text{loc}}\,,\big\} \\
\textit{and} &- f_t''(z,x) + V(x)f_+(z,x) \\
&= z^2 f_+(z,x) a.e., -x .
\end{aligned}$$

Proof.

(1) ok

$$|f_+(z,x) - e^{izx}| \leq \int\limits_x^\infty \frac{2e^{\text{Im } z(y-x)}}{|-z|} |V(y)|\ |f_+(z,y)|\ dy$$

$$\leq \int\limits_x^\infty \frac{e^{\text{Im}\, z(y-x)}}{|z|} |V(y)| \cdot e^{-(\text{Im}\, z)y} |m_+(z,y)|\ dy$$

$$\leq \frac{e^{-x\,\text{Im } z}}{|z|} \underbrace{\int\limits_x^\infty |V(y)|\ dy}_{\substack{\to 0 \\ \text{as } x\to\infty}} \ \underbrace{\sup_{y\geq x} |m_+(z,y)|}_{\text{uniform boundedness}}$$

$$\to 0\ .$$

$$\circledast\ m_+(z,x) = 1 + \int\limits_x^\infty D_z(y-x)V(y)m_+(z,y)\ dy$$

$$= 1 + \int\limits_x^\infty D_z(y_1 - x)V(y)\ dy + \int\limits_x^\infty \int\limits_{y_1}^\infty D_z(y-x)D_z(y_2 - y_1)$$

$$V(y_1)V(y_2)\ dy_1\ dy_2$$

$$+ \dots$$

$$+ \int\limits_x^\infty \int\limits_{y_1} \cdots \int\limits_{y_n - 1} D \cdots D \cdots V \cdots V$$

$$\big|D_z(y-x)\big| \leq \frac{1}{|z|}$$

$$m_+(z,x) \leq 1 + \cdots \int\limits_{y_n \geq y_{n-1} \geq \cdots \geq x} \frac{1}{|z|^n} \cdot \big|V(y_1)\cdots V(y_n)\big|\ dy_1 \cdots dy_n$$

$$= 1 + \cdots + \frac{1}{n!}\frac{1}{|z|^n}\Big(\int\limits_x^\infty |V(y)|dy)\Big)^n$$

$$\leq \exp\Big(\frac{\|\ V\ \|_1}{|z|}\Big)\ .$$

Smoothness: Prove $f'(z,x) = ize^{izx} - \int_x^\infty \cos\big(z(y-x)\big)V(y)f_+(z,y)\ dy$
$(\cdots)$

$$\int\limits_{x+h} \cos\cdots \qquad \text{continuous in } h$$

because int. in L^1
because the exp cancel each other.

$$\text{then} \quad \left|\frac{1}{h}\Big(\int\limits_{x+h} \sin - \int\limits_{x} \sin\Big) - \int\limits_{x} \cos\right| <$$

$$\Big|\underbrace{\int\limits_{x}^{x+h} \frac{1}{h}}_{\substack{\text{OK}\\(1)}} + \int\limits_{x+h}^{\infty} \underbrace{\underbrace{\left(\frac{\sin - \sin}{hz} + \cos\right)}_{O(e^{\operatorname{Im} zy})}}_{\substack{\text{DCT}\\ \sigma(1)}} V f_+\Big|$$

$$\begin{aligned} \text{and} \quad f_+''(z,x) &= - z^2 e^{izx} + V(x) f_+(z,x) \\ &\quad + z^2 \int\limits_{x}^{\infty} \frac{\sin\big(z(y-x)\big)}{z} V(y) f_+(z,y)\, dy \\ &= - z^2 f_+(z,x) + V f_+(z,x) \quad \text{only a.e.} \end{aligned}$$

□

Ex. (1) $V = 0$ $\qquad f_+(\lambda, x) = e^{i\lambda x}$

$$f_-(\lambda, x) = e^{-i\lambda x}$$

$$W(\lambda) = W\big[f_+(\lambda), f_-(\lambda)\big] = \begin{vmatrix} e^{i\lambda x} & e^{-i\lambda x} \\ i\lambda e^{i\lambda x} & -i\lambda e^{-i\lambda x} \end{vmatrix} = -2i\lambda$$

vanishes at 0. (exceptional case).

(2)

$$f_+(\lambda, x) = e^{i\lambda x} \qquad \text{if } x > 1$$

$$\begin{aligned} 0 < x < 1: \qquad -f'' + f &= \lambda^2 f \\ f''(x) &= (1-\lambda^2) f(x) \\ f(x) &= \alpha(\lambda) e^{\sqrt{1-\lambda^2}x} \\ &\quad + \beta(\lambda) e^{-\sqrt{1-\lambda^2}x} \end{aligned}$$

matching $\Rightarrow \alpha, \beta$
$x < 0$: $f_+(\lambda, x) = a(\lambda) e^{i\lambda x} + b(\lambda) e^{-i\lambda x}$

matching $\Rightarrow a, b$.

$$x<0: \quad W(\lambda) = \begin{vmatrix} a(\lambda)e^{i\lambda x}+0 & a(\lambda)i\lambda e^{i\lambda x}+0 \\ e^{-i\lambda x} & -i\lambda e^{-i\lambda x} \end{vmatrix}$$
$$= -2i\lambda a(\lambda)$$
$$W(0) = \cdots = -2\big(\alpha(0)-\beta(0)\big) \neq 0 \,.$$

$$\lambda = 0 \;: \quad m_+(0,x) = f_+(0,x)$$
$$= a + bx \qquad b \neq 0 \,.$$
$$(-f_+'' = 0)$$
$$W(0) = \begin{vmatrix} f_+ & b \\ 1 & 0 \end{vmatrix} = -b \neq 0 \,.$$

but remains bdd to the right ($x>0$) so that's why we had to $y<x$, $y>x$ before ($W(0)\neq 0$).

Lemma. Im $z \geq 0$ *and* $\int\big(1+|x||V(x)|\,dx < \infty$. *Then* $|m_+(z,x)| \leq K\big(1+\max(0,-x)\big)$.

Proof. $x>0$

$$m_+(z,x) = 1 + \cdots + \int\limits_{x \leq y_1 \leq \cdots \leq y_n} D_z(y_1-x)\cdots D_z(y_n-y_{n-1})$$
$$V(y_1)\cdots V(y_n)\,dy_1\cdots dy_n$$

$$|m_+(z,x)| \leq 1 + \cdots + \iint (y_1-x)\cdots(y_n-x)|V(y_1)\cdots V(y_n)|\,dy_1\cdots dy_n \,.$$
$$\leq \exp\Big(\int_x^\infty (y-x)|V(y)|\,dy\Big) \qquad \text{OK when } x>0.$$
$$\leq \exp\Big(\int_0^\infty y|V(y)|\,dy\Big) \quad \text{for } x>0.$$

$$x<0 \;: \qquad |m_+(z,x)| \leq 1 + \int_x^\infty (y-x)|V(y)|\,|m_+(z,y)|\,dy$$
$$\leq 1 + \int_0^\infty y|V(y)|\,|m_+(z,y)|\,dy$$
$$+ \int_x^\infty (-x)|V(y)|\cdot|m_+(z,y)|\,dy \,.$$
$$\leq 1 + \underbrace{C.\int y|V(y)|}_{K} + |x|\int_x^\infty |V(y)|\,|m_+(z,y)|\,dy$$

$$M_+(z,x) = \big(K_1(1+|x|)\big)^{-1}|m_+(z,x)|$$
$$\leq 1 + \int_x^\infty |V(y)|(1+|y|)M_+(z,y))\,dy\,.$$

has a uniformly bdd solution because Volterra

$$\sup_x M_+(z,x) \leq \exp\bigg(\int_{-\infty}^{\infty} (1+|y|)|V(y)|\,dy\bigg)\,.$$

Conclusion $\qquad |m_+(z,x)| \leq K_1(1+|x|)e^{\int(1+|y|)|V(y)|\,dy}\,.$

□

Back to f_+, f_- $\lambda \neq 0$.
Reflexion transmission coefficient

$$f_+(\lambda,x) = \alpha_+(\lambda)f_-(\lambda,x) + \beta_+(\lambda)f_-(-\lambda,x)$$
$$f_-(\lambda,x) = \alpha_-(\lambda)f_+(\lambda,x) + \beta_-(\lambda)f_+(-\lambda,x)$$

$$W(\lambda) = W\big[f_+(\lambda), f_-(\lambda)\big] = \beta_+(\lambda)W\big[f_-(-\lambda), f_-(\lambda)\big]$$
$$\text{as } x\to\infty, \qquad = \beta_+(\lambda)\lim\begin{vmatrix} e^{i\lambda x} & i\lambda e^{i\lambda x} \\ e^{-i\lambda x} & -i\lambda e^{-i\lambda x}\end{vmatrix}$$
$$= -2i\lambda\beta_+(\lambda)\,, \quad \text{same for } \beta.$$

$$\beta_+(\lambda) = \beta_-(\lambda) = \beta(\lambda)$$

$$W\big[f_+(\lambda), f_-(-\lambda)\big] = \alpha_+(\lambda)W\big[f_-(\lambda), f_-(-\lambda)\big]$$
$$= 2i\lambda\alpha_+(\lambda)$$
$$W\big[f_-(\lambda), f_+(-\lambda)\big] = -2i\lambda\alpha_-(\lambda)$$

$$f_+(\lambda,x) = \alpha_+(\lambda)\big(\alpha_-(\lambda)f_+(\lambda,x) + \beta(\lambda)f_+(-\lambda,x)\big)$$
$$+\beta_+(\lambda)\big(\alpha_-(-\lambda)f_+(-\lambda,x) + \beta(-\lambda)f_+(\lambda,x)\big)$$

$$1 = \alpha_+(\lambda)\alpha_-(\lambda) + \beta(\lambda)\beta(-\lambda)$$
$$2i\lambda\beta(-\lambda) = W(-\lambda) = \overline{W(\lambda)} = 2i\lambda\overline{\beta(\lambda)}$$
$$\text{and } -2i\lambda\alpha_+(\lambda) = 2i\lambda\overline{\alpha_-(x)}$$

$$1 = -|\alpha_+(\lambda)|^2 + |\beta(\lambda)|$$
$$\lambda \neq 0 \Rightarrow |\beta(\lambda)| \geq 1$$

Conclusion $W(\lambda) \neq 0$ if $\lambda \neq 0$ for every general $V(x)$.

Lemma. $H = -\frac{d^2}{dx^2} + V$, $V \in L^\infty \cap L_1'$

Then $\forall\lambda \neq 0$, and $\lambda = 0$, $W(0) \neq 0$ (definition of generic)

$$\begin{aligned}\big\langle R_V(\lambda + i0)\phi, \psi\big\rangle &= \lim_{\varepsilon\to 0^+} \big\langle R_V(\lambda + i\varepsilon)\phi, \psi\big\rangle \\ &= \iint\limits_{x<y} \frac{f_+(\lambda, y) f_-(\lambda, x)}{W(\lambda)} \phi(x)\overline{\psi(y)}\, dx\, dy \\ &\quad + \iint\limits_{x>y} \frac{f_+(\lambda, x) f_-(\lambda, y)}{W(\lambda)} \phi(x)\overline{\psi(y)}\, dx\, dy\end{aligned}$$

Proof.

$$\begin{aligned}G(\lambda + i\varepsilon)(x, y) &= \frac{f_+(\lambda + i\varepsilon, y) f_-(\lambda + i\varepsilon, x)}{W(\lambda + i\varepsilon)} \chi_{[x<y]} \\ &\quad + \frac{f_+(\lambda - i\varepsilon, x) f_-(\lambda + i\varepsilon, y)}{W(\lambda + i\varepsilon)} \chi_{[x>y]} \ .\end{aligned}$$

$$\Rightarrow \text{show} \quad \big\langle G(\lambda + i\varepsilon)\phi, \big(H - (\lambda^2 - i\varepsilon)\big)\psi\big\rangle\cdot = \langle\phi, \psi\rangle \forall \phi, \psi \in \mathcal{S}$$

G is L^2 via Schur's lemma:

$$\begin{aligned}\| G(\lambda + i\varepsilon) \|_{L^2\to L^2} &\leq \sup_y \int |G(\lambda + i\varepsilon)(x, y)| dx \\ &\leq \sup_y \int_{-\infty}^{y} C_\varepsilon \frac{e^{\operatorname{Im} z\cdot x}}{|W(\lambda + i\varepsilon)|}\, dx \cdot e^{-\operatorname{Im} z\cdot y} \\ &\leq C\ .\end{aligned}$$

because $\| m_+ \|_\infty \leq \exp\big(\frac{1}{|z|} \| V \|_1\big)$.

$$\frac{1}{W(\lambda + i\varepsilon)} \iint\limits_{y>x} f_+(\lambda + i\varepsilon, y) f_-(\lambda + i\varepsilon, x)\phi(x)\, dx\, dy$$

$$\big(-\bar{\psi}'' + V\bar{\psi} - (\lambda^2 + i\varepsilon)\bar{\psi}\big)(y)\ .$$

$$\int\limits_y : \text{boundary term at } x.$$

the other term, $\int_y$ cancels the first $\int_y$. What remains is $\frac{\cdots}{W}$ then DCT, use linear growth bound. ∎

Ma 142 01/28

Low-energy lemma. $H = -\frac{d^2}{dx^2} + V(x)$.

Lemma. *$\chi(\lambda) = 1$ if $|\lambda| \leq \lambda_0$, $\chi(\lambda) = 0$ if $\lambda > 2\lambda_0$. Assume $V \in L^1 \cap L^\infty$, $\int_{-\infty}^{\infty}(1 + |x|)^3 V(x)|\ dx < \infty$ and V is generic $(W\big[f_+(0,\cdot), f_-(0,\cdot)\big] \neq 0)$ then $\| e^{itH} P_{ac}(H)\chi(H) \|_{1\to\infty} \leq C.|t|^{-1/2}$*

Proof. $f, g \in \mathcal{S}(\mathbb{R})$

$$\left|\langle e^{itH} P_{ac}(H)\chi(H)f, g\rangle\right| = \Big|\int_1^\infty e^{it\lambda}\chi(\lambda)\langle E_{ac}(d\lambda)f, g\rangle\Big|$$

$$\overset{\text{(Stone)}}{=} \Big|\int_0^\infty e^{it\lambda^2} 2\lambda\chi(\lambda^2)\frac{1}{2\pi i}\big\langle (R_V(\lambda^2+i0)) - (R_V(\lambda^2-i0))f, g\big\rangle\Big|$$

$\downarrow$ Resolvent = Green function

$$\leq \frac{1}{\pi}\iint_{x<y}\Big|\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda^2)\frac{f_+(\lambda,x)f_-(\lambda,y)}{W(\lambda)}\,d\lambda\Big|\,|f(x)|\,|g(y)|\,dx\,dy$$

$$+\frac{1}{\pi}\iint_{x<y}\Big|\int_{-\infty}^\infty e^{it\lambda^2}\lambda\chi(\lambda^2)\frac{f_+(\lambda,y)f_-(\lambda,x)}{W(\lambda)}\,d\lambda\Big|\,|f(x)|\,|g(y)|\,dx\,dy$$

Case 1: $y < 0 < x$

$$\Big|\int_{-\infty}^\infty e^{it\lambda^2}e^{i\lambda(x-y)}\frac{\lambda\chi(\lambda^2)}{W(\lambda)}m_+(\lambda,x)m_-(\lambda,y)\,d\lambda\Big|$$

disp. est. for free eq.

$$\leq C.|t|^{-1/2}\,\Big\|\Big[\frac{\lambda\chi(\lambda^2)}{W(\lambda)}m_+(\lambda,x)m_-(\lambda,y)\Big]^\vee(\tau)\Big\|_{L^1_\tau}$$

$$\leq C.|t|^{-1/2}\,\|\,[\qquad\underline{\qquad\qquad\qquad}]''\,\|_{L^\infty}$$

twice diff. on compact $\to$ decays like $1/\tau^2$

$$m_+(\lambda,x) = 1 + \int_x^\infty\int_0^{y-x} e^{2is\lambda}\,ds V(y)m_+(\lambda,y)\,dy$$

$$\partial_\lambda m_+(\lambda,x) = \int_y^\infty\int_0^{y-x} 2ise^{2is\lambda}\,ds\;V(\lambda)m_+(\lambda,y)\,dy$$

$$+\int_x^\infty D_\lambda(y-x)V(y)\partial_\lambda m_+(\lambda,y)\,dy$$

$$x\geq 0,\;|\partial_\lambda m_+(\lambda,x)| \leq \int_0^\infty \overbrace{(y-x)^2|V(y)|\,dy}^{<\infty}\underbrace{\sup_{y\geq 0}|m_+(\lambda,y)|}_{\leq\|V\|_{L^1_1}}$$

$$+ \int_x^\infty (y-x)|V(y)|\,|\partial_\lambda m_+(\lambda,y)|\,dy$$

$$\sup_{x\geq 0} | \quad \rule{3cm}{0.4pt} \quad | \leq K \exp\Big(\int_0^\infty |y|\,|V(y)|\,dy\Big)$$

Similarly, $\sup_{x\geq 0}|\partial_\lambda^2 m_+(\lambda,x)| \leq$ const. since $\int(1+|y|)^3 V(y)|\,dy < \infty$. Next, check that $|W''(\lambda)| \leq C$ on the supp. of $\chi(\lambda^2)$

$$\partial_\lambda W(\lambda) = \begin{vmatrix} \partial_\lambda f_+(\lambda,0) & f_+'(\lambda,0) \\ \partial_\lambda f_-(\lambda,0) & f_-'(\lambda,0) \end{vmatrix} + \begin{vmatrix} f_+(\lambda,0) & \partial_\lambda \partial x f_+(\lambda,0) \\ f_-(\lambda,0) & \partial_\lambda \partial x f_-(\lambda,0) \end{vmatrix}$$

$$f_+(\lambda,x) = e^{i\lambda x} m_+(\lambda,x)$$

$$\partial_\lambda f_+(\lambda,0) = \partial_\lambda m_+(\lambda,0) \to \text{OK, 2 moments.}$$

$$f_+', f_-' \text{ are bdd in } \lambda$$

$$\partial_\lambda f_+(\lambda,0) = i\lambda m_+(\lambda,0) + \partial_x m_+(\lambda,0)$$

$$\partial_\lambda \partial_x f_+(\lambda,0) = \underbrace{im_+(\lambda,0)}_{\text{OK}} + \underbrace{i\lambda}_{\substack{\text{OK on}\\ \text{supp } \chi}} \underbrace{\partial_\lambda m_+(\lambda,0)}_{OK} + \partial_\lambda \partial_x m_+(\lambda,0)$$

$$\partial_x m_+(\lambda,x) = \int_x^\infty D_\lambda'(y-x)V(y)m_+(\lambda,y)\,dy$$

$$\partial_\lambda \partial_x m_+(\lambda,x) = -\int_x^\infty 2i(y-x)e^{2i\lambda(y-x)}V(y)m_+(\lambda,y)\,dy$$

$$-\int_x^\infty e^{2i\lambda(y-x)}V(y)\partial_\lambda m_+(\lambda,y)\,dy\,.$$

$$\sup_{x\geq 0}|\partial_\lambda \partial_x m_+(\lambda,x)| < \infty\,.$$

second der. $\to$ use third moment.
Case 2: $y < x < 0$
 $\to$ linear growth of f_+ in x.

$$f_+(\lambda,x) = \alpha_+(\lambda)f_-(\lambda,x) + \beta_+(\lambda)f_-(-\lambda x) \quad \text{and use asympt.}$$

$$\int_{-\infty}^\infty e^{it\lambda^2}\lambda\chi(\lambda^2)\frac{f_+(\lambda,x)f_-(\lambda,y)}{W(\lambda)}\,d\lambda = \int_{-\infty}^\infty e^{it\lambda^2}e^{-i\lambda(x+y)}$$

$$\cdot \frac{\lambda\chi(\lambda)}{W(\lambda)}\alpha_+(\lambda)m_-(\lambda,x)m_-(\lambda,y)\,d\lambda$$

$$+\int_{-\infty}^\infty e^{it\lambda^2}e^{i\lambda(x-y)}\frac{\lambda\chi(\lambda^2)}{W(\lambda)}\beta_+(\lambda)m_-(-\lambda,x)m_-(\lambda,y)\,d\lambda$$

$$\text{use}\quad 2i\lambda\beta_+(\lambda) = W(\lambda)$$
$$2i\lambda\alpha_+(\lambda) = W\big[f_+(\lambda), f_-(-\lambda)\big]$$
$$\frac{\lambda\alpha_+(\lambda)}{W(\lambda)} = \frac{W\big[f_+(\lambda), f_-(-\lambda)\big]}{2iW(\lambda)} \quad \text{has second der. in } \lambda.$$

need uniform bound of second der. in x, y, etc. □

if V is not generic, you lose 1 moment. (expand $W(\lambda)$ about 0).
Wiener algebra: $f, \hat{f}$ in L^1
stable under multiplication.

Remark 1. If $|V(x)| < C.(1+|x|)^{-2-\varepsilon}$ in 1-D.

- Then $\exists$ only finitely many negative eigv.
- $\operatorname{spec}(H) \cap (0, \infty)$ is purely abs. cont.

$$P_{ac} = I - \sum_{j=1}^{N} P_{\lambda j} \quad \text{where } \lambda_N < \cdots < \lambda_1 < \lambda_0 \leq 0$$

$$|V(x)| < (1+|x|)^{-1-\varepsilon}$$
$$N \to \infty$$
(Cluster about the origin)
$$Hf = Ef\ , \quad E > 0? \qquad \text{no.}$$

Remark 2. Supp. $H = -\frac{d^2}{dx^2} + V$ does not have *any* egv. Then $P_{ac} = Id$.

$$\Rightarrow \| e^{itH} \|_{1\to\infty} \lesssim |t|^{-1/2}$$

Claim. $$\begin{cases} i\partial_t \Psi - \partial_x^2 \psi + V\Psi &= F \\ \Psi|_{t=0} &= f \end{cases}$$

then $$\| \Psi \|_{L^6_{t,x}} \lesssim \| F \|_{L^{6/5}_{t,x}} + \| f \|_{L^2_x}\ .$$

proof: identical (interp., TT^*)
replace $e^{it\Delta}$ by e^{itH}

Remark 3. If V is singular, what matters is whether H is self-adjoint. Properties of egf depends on long time $\to$ decay.

Def. and lemma $A : \mathcal{H} \to \mathcal{H}$ TFAE

1. A takes bdd sets to rel. comp. sts.
2. A takes bdd sets to sets that have finite $\varepsilon-$ nets for every $\varepsilon > 0$.
3. if $\{u_n\}$ is a bdd sequence in $\mathcal{H}$, then Au_n has a strongly conv. subsequence.
4. A takes weakly conv. seq. to strongly conv. seq.

$\Rightarrow A$ compact.

Theorem 4. *Spectral thm for compact s.a. operators. Let $A : \mathcal{H} \to \mathcal{H}$ be compact and $A = A^*$. Then $\exists$ ONB $\{\phi_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ s.t.*

$$A\phi_n = \lambda_n \phi_n \quad \textit{where } \lambda_n \in \mathbb{R}$$

and they can only accumulate at zero. (no ac., no sc.).

Thm: *Weyl's criterion. Let A, B be s.a. and bdd below. Assume $(A+M)^{-1}-(B+M)^{-1} =: K$ is compact for some numerical constant M. Then ess* spec.*(A) = ess* spec(B)

ess spec(A) = spec$(A)\setminus\{$*isolated egv with finite mult*$\}$

Proof. $\lambda \in$ ess spec(A), $\varepsilon > 0$.

claim:$\exists$ ∞-dim subspace $\mathcal{L} \subset \mathcal{H}$ so that

$$\| (A+M)^{-1}f - (\lambda+M)^{-1}f \| < \varepsilon \| f \|$$

$\forall f \in \mathcal{L}$.

Because

$$\| (A+M)^{-1}f - (\lambda+M)^{-1}f \|_2^2 = \int \left| \frac{1}{t+M} - \frac{1}{\lambda+M} \right|^2 \langle E(dt)ff$$

$$< \varepsilon^2 \| f \|^2$$

$$\text{take } \mathcal{L} = \operatorname{Ran} \underbrace{E(\lambda-\delta, \lambda+\delta)}_{P}$$

Since $\lambda \in$ ess spec$A \Rightarrow \dim \mathcal{L} = \infty$, (otherwise purely atomic in the interval).

$$\| \underbrace{PK^2Pg}_{\substack{\text{compact}\\ \text{\& s.a.}}} \| < \varepsilon^2 \| g \| \text{ for all } g \in \tilde{\mathcal{L}} \subset \mathcal{L}\ ,\ \dim \tilde{\mathcal{L}} = \infty$$

(use spectral theorem)

$$\| Kg \|^2 = \langle PK^2Pg, g \rangle < \varepsilon^2 \| g \|^2\ \forall g \in \tilde{\mathcal{L}}$$

$$\forall g \in \tilde{\mathcal{L}} \Rightarrow \| Kg \| < \varepsilon \| g \| \text{ on } \mathcal{L}.$$

$$\text{switch to } \quad \| (B+M)^{-1}g - (\lambda+M)^{-1}g \| < 2\varepsilon \| g \|\ \forall g \in \tilde{\mathcal{L}}$$

□

Example$A = -\Delta$, $B = -\Delta + V$

$$(-\Delta+M)^{-1} - (-\Delta+V+M)^{-1}$$

$$= \sum_{n=1}^{\infty} (-\Delta+M)^{-1}\big(-V(-\Delta+M)\big)^n \qquad \text{op. conv. in } L^2 \text{ if } M \text{ large}$$

$$\| (\underbrace{-\Delta}_{\substack{\text{strictly}\\ \text{post spec}}} + M)^{-1} \|_{2\to 2} \leq \frac{1}{M}$$

$$\| V(-\Delta+M)^{-1} \|_{2\to 2} \leq \frac{\| V \|_\infty}{M}$$

If $V \to 0$ at ∞, the $V(-\Delta+M)^{-1}$ is compact $\to$ then the series is compact, as is shown below.

Theorem. $V \in L^\infty$ *If $V \to 0$ at ∞, then $V(-\Delta+1)^{-1} =: K$ is compact.*

Proof. $B \subset L^2(\mathbb{R}^n)$ bdd
show $\forall \varepsilon > 0$, $K(B)$ has a finite ε-net.

$$\| V(-\Delta+1)^{-1}u - V(-\Delta+1)^{-1}v \|_2 \qquad u, v \in B_1$$
$$\underbrace{< 2 \| V \|_{L^\infty(\mathbb{R}^n \setminus B_R)}}_{\substack{<\varepsilon/2 \\ \text{(take } R \text{ large)} \\ \text{use } V \to 0 \text{ as } x \to \infty}} + \| V \underbrace{(-\Delta+1)^{-1}u - V(-\Delta+1)^{-1}}_{\substack{\text{find finite collection} \\ \text{of } v's \text{ s.t. small.}}} v \|_{L^2(B_R)}$$
$$\leq \| V \|_\infty \| (-\Delta+1)^{-1}u - (-\Delta+1)^{-1}v \|_{L^2 B_R}$$

(Arzela-Ascoli later).

$$W := \left\{ w = (-\Delta+1)^{-1}v \;\middle|\; \| v \|_2 \leq 1 \right\}$$

$$w_\varepsilon = \chi_\varepsilon \star w$$

$$\begin{aligned} \| w_\varepsilon(x) - w(x) \|^2_{L^2(B_R)} &= \int_{B_R} \chi_\varepsilon^2(y) \big(w(x-y) - w(x) \big)^2 \, dy \\ &= \int_{B_R} dy \; \chi_\varepsilon^2(y) \Big(\int_0^1 \nabla w(x-ty) y \, dt \Big)^2 \\ &\leq \int dy \; \chi_\varepsilon^2(y) |y|^2 \underbrace{| \sup_t \nabla w(x-ty) |^2}_{\leq \|\nabla w\|_{L^2} \leq C\,.} \\ &\leq C\varepsilon \qquad \text{good approximants} \end{aligned}$$

$\to$ work with mollified family, does it admit ε-net? $W_\varepsilon = \{ \chi_\varepsilon \star w | w \in W \}$.

$$\text{bddness} \quad |w_\varepsilon(x)| = |(\chi_\varepsilon \star w)(x)| \leq C \cdot \varepsilon^{-n/2} \| w \|_2 \leq C . \varepsilon^{-n/2}$$

$$\text{equicontinuity} \quad |w_\varepsilon(x) - w_\varepsilon(y)| = \left| \int \big(\chi_\varepsilon(x-z) - \chi_\varepsilon(y-z) \big) w(z) \, dz \right|$$
$$\leq C . \varepsilon^{-n/2} \varepsilon^{-1} |x-y|$$

Hence W_ε unif. bdd. and equico.
Arzela Ascoli $\Rightarrow W_\varepsilon$ is rel. comp. in $C(B_R)$.

$$\to L^\infty \varepsilon\text{-net}$$
$$\to L^2 \varepsilon\text{-net}$$

□

Ma 142 01/30
$H = -\Delta + V, \quad V \in L^\infty, V(x) \to 0$ as $|x| \to \infty$.
spec(H) looks like

Lemma. *Suppose* $V(x) \to 0$ *as* $x \to \infty$. *TFAE*

(1) $\exists \lambda < 0$ *eig of* $H = -\Delta + V$
(2) $\exists f \in W^{2,2}$ *s.t.* $\langle Hf, f \rangle < 0$.

Proof.

(1) $\Rightarrow$ (2), take f the egf
(2) $\Rightarrow$ (1), spec must extend below zero (by spectral theorem), but another theorem says this must be an egv.

□

Theorem. *Birman-Schwinger. Let* $V \subset \mathbb{R}^3$ *satisfy*

$$\iint\limits_{\mathbb{R}^3 \mathbb{R}^3} \frac{|V^-(x)|\,|V^-(y)|}{|x-y|^2}\, dx\, dy < (4\pi)^2 \, .$$

Then $-\Delta + V$ does not have any negative egv.

Proof. Assume $\exists \lambda < 0$ egv $\Rightarrow$ (lemma) $\exists f \in W^{2,2}$ s.t. $\langle Hf, f \rangle < 0$. Since $V = V^+ + V^-$,

$$\Big\langle (-\Delta + V^-) f, f, \Big\rangle < 0 \qquad \text{(only makes it worse)}$$

lemma $\Rightarrow \tilde{H} = -\Delta + V^-$ has negative bound state.

$$(-\Delta + V^-)\phi = \lambda \phi \qquad \text{for some } \phi \neq 0.$$

$$\underbrace{(-\Delta - \lambda)}_{\substack{\text{positive} \\ \text{invertible}}} \phi = -V^- \phi \in L^2$$

$$\phi = -(-\Delta - \lambda)^{-1} V^- \phi$$

$$\sqrt{|V^-|}\, \phi = \underbrace{\sqrt{|V^-|}\, (\Delta - \lambda)^{-1} \sqrt{|V^-|}}_{A}\, \underbrace{\sqrt{|V^-|}\, \phi}_{\neq 0}$$

A has egv 1, s.a. $\Rightarrow \| A \| \geq 1$

$$\text{but} \quad \| A \| \leq \| A \|_{HS} = \Big(\int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{R}^3} |\text{kernel}(A)|^2 dx\, dy \Big)^{1/2}$$

$$= \Big(\iint\limits_{\mathbb{R}^6} |V^-(y)|\, |(-\Delta - \lambda)^{-1}(y,x)|^2 |V^-(x)| dx\, dy \Big)^{1/2}$$

$$= \Big(\iint\limits_{\mathbb{R}^6} |V^-(y)|\, |V(x)| \frac{e^{-i\sqrt{\lambda}|x-y|}}{\big(4\pi |x-y|\big)^2}\, dx\, dy \Big)^{1/2} < 1 \, .$$

□

Theorem (B-S) actually says

$$\#\big\{ \lambda_j | \lambda_j \leq 0 \quad \text{is an egv of } H \big\}$$

$$\leq \frac{1}{16\pi^2} \iint\limits_{\mathbb{R}^6} \frac{|V^-(x)|\,|V^-(y)|}{|x-y|^2}\,dx\,dy$$

look into Reed-Simon, IV.

Lemma. *Suppose $V(x) \in L^1(\mathbb{R})$ and assume $\int_{-\infty}^{\infty} V(x)\,dx < 0$. Then $H = -\frac{d^2}{dx^2} + V$ has a negative bound state (in 1-D!!).*

Proof. Find f s.t. $\langle Hf, f\rangle < 0$. Take $f(x) = \phi(\varepsilon x)$ where

$$\langle Hf, f\rangle = \varepsilon^2 \int |\phi'(\varepsilon x)|^2\,dx + \int V(x)\phi(\varepsilon x)^2\,dx$$

$$= \varepsilon \parallel \phi' \parallel_2^2 + \underbrace{\int V(x)\phi(\varepsilon x)^2\,dx}_{\to \int V < 0} < 0 \quad \text{for } \varepsilon \to 0$$

□

NB: unitary equivalence of $-\Delta$ and $-\Delta + V$ for V small, i.e.,

$$\int \frac{|V(x)|\,|V(y)|}{|x-y|^2}\,dx\,dy < (4\pi)^2 .$$

NB: $\parallel V \parallel_{\mathrm{R}ollnik} = \left(\int\limits_{\mathbb{R}^6} \frac{|V(x)|\,|V(y)|}{|x-y|^2}\right)^{1/2}$ upper bound for

$\sqrt{|V|}R_0(\lambda)\sqrt{|V|}$. (Hilbert-Schmidt norm).

Theorem.

Suppose $\parallel V \parallel_K = \sup\limits_{x\in\mathbb{R}^2} \int\limits_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\,dy < 4\pi$

Then $\parallel e^{it(-\Delta+V)}(P_{ac}) \parallel_{1\to\infty} \leq C.|t|^{-3/2}$

Proof.

$$\langle e^{itH}P_{ac}f, g\rangle = \lim_{L\to\infty} \left\langle e^{itH}\chi\left(\frac{H}{L}\right)P_{ac}f, g\right\rangle$$

$$= \lim_{L\to\infty} \int\limits_0^\infty e^{it\lambda}\chi\left(\frac{\lambda}{L}\right)\frac{1}{2\pi i}\big\langle \big(R_V(\lambda+i0) - R_V(\lambda-i0)\big)f, g\big\rangle\,d\lambda$$

Formally, $R_V(\lambda+i0) = \sum\limits_{n=0}^{\infty} R_0(\lambda+i0)\big(-VR_0(\lambda-i0)\big)^n$

$$R_0(\lambda \pm i0)(x,y) = \frac{e^{\pm i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$$

$$\Big\langle \big(R_V(\lambda+i0)\big) - R_V(\lambda - i0)f, g\Big\rangle = \sum_{n=0}^{\infty} \int_{\mathbb{R}^3} dx_0 \int dx_{n+1} \cdot$$

$$\frac{1}{\pi}\int_{\mathbb{R}^{3n}} \frac{\sin(\sqrt{\lambda}\sum_{j=1}^{n+1}|x_j - x_{j-1}|)}{\prod_{j=1}^{n+1}(4\pi|x_j - x_{j-1}|)} \prod_{j=1}^{n} V(x_j) dx_1 \cdots dx_n f(x_0) g(x_{n+1})$$

$$\leq \sup_{L\geq 1} \frac{1}{\pi} \sum_{n=0}^{\infty} \left| \int_0^{\infty} 2\lambda e^{it\lambda^2} \chi\left(\frac{\lambda^2}{L}\right) \frac{\sin(\lambda \sum_{j=1}^{n}|x_j - x_{j-1}|)}{\Pi 4\pi |x_j - x_{j-1}|}\, d\lambda \right|$$

$$\prod_{j=1}^{n} |V(x_j)| \,.\, |f(x_0)|\,|g(x_{n+1})|\, dx_0\, dx_{n+1}$$

integrate by parts. gen $\sum |\text{—}|$ stationary phase or interpret as solution of the $1-D$ free eq. check init data has FT in L^1

$$\leq C. \sum_{n=0}^{\infty} \int_{\mathbb{R}^6}\int_{\mathbb{R}^{3n}} t^{-3/2} \sum_{j=1}^{n+1} |x_j - x_{j-1}| \frac{\Pi |V(x_j)|}{\Pi 4\pi |x_j - x_{j-1}|}$$

$$|f(x_0)|\,|g(x_{n+1})|\, dx_0 \cdots dx_{n+1}$$

$n = 0$ OK $\leq \| f \|_1 \| g \|_1$

$n = 1$ $\displaystyle\int \frac{|V(x_1)|}{|x_2 - x_1|\,|x_1 - x_0|}\big(|x_0 - x_1| + |x_1 - x_2|\big)|f(x_0)|\,|g(x_2)|$

$\leq 2 \| V \|_K \| f \|_1 \| g \|_1$

$n = 2$ $\displaystyle\int \frac{|V(x_2)|\,|V(x_1)|}{|x_2 - x_3|\,|x_0 - x_1|}|f(x_0)|\,|g(x_3)|$

$\leq 3 \| V \|_K^2 \| f \|_1 \| g \|_1$

take sup x_3 (free x_3) take sup x_0

$$\leq C \cdot t^{-3/2} \cdot \sum_{n=0}^{\infty} (n+1) \left(\frac{\| V \|_K}{4\pi}\right)^n \| f \|_1 \| g \|_1 \quad \text{(exercise)} .$$

□

Conv. of Born series? Does if Rollnik norm $< \infty$.
If Rollnik $< 4\pi$, unit. eq $\to$ can remove P_{ac}
Schlag-Bourgain. Scale invariance of the theory

$$f_R(x) = f(Rx)$$

$$\big((\Delta + V) f_R\big)(x) = -R^2(\Delta f)(Rx) + (V(x) f(Rx)$$
$$= R^2\big(-\Delta f + \underline{R^{-2} V(R^{-1}\cdot)} f\big)(Rx)$$

all conditions of the previous theorem must be invariant under $V \to R^{-2}V(R^{-1})$, e.g., Kato and Rollnik is OK. $L^{3/2}$ is OK and close to these conditions.

Remark 1.

$$\begin{aligned} &\| R^{-2}V(R^{-1}\cdot) \|_{L^p(\mathbb{R}^n)} \\ &= R^{-2}R^{n/p} \| V \|_{L^p} \qquad p = n/2 \end{aligned}$$

e.g., $\| V \|_R \leq C \cdot \| V \|_{L^{3/2}(\mathbb{R}^3)}$ when $V \leq 0$ or $V \geq 0$.

Proof.

$$(TV)(y) = \int_{\mathbb{R}^3} \frac{V(y)}{|x-y|^2}\, dy \qquad T : L^p \to L^q \text{ provided}$$

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{2}{3}$$

$$\begin{aligned} V \geq 0\ ,\ \| V \|_R^2 &= \langle TV, V\rangle \qquad L^{3/2} \to L^3 \\ &\leq C \cdot \| V \|_{L^{3/2}}^2 \end{aligned}$$

□

Theorem. *Perturbation argument (Schlag-Bourgain):*

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta\psi + V(t)\psi = 0 \\ \psi|_{t=0} = \psi_0 \end{cases}$$

$$\textit{Assume } \sup_t \big(\| V(t) \|_1 + \| V(t) \|_\infty \big) < \varepsilon\ .$$

$$\textit{Then} \quad \| \psi(t) \|_{L_x^\infty + L_x^2} \leq C \cdot \langle t\rangle^{-n/2} \| \psi(0) \|_{L^1 \cap L^2}$$

$$\textit{Here } \langle t\rangle = (1+t^2)^{1/2}$$

$$\| f \|_{L^2+L^\infty} = \inf\big\{ \| f_1 \|_{L^2} + \| f_2 \|_\infty \colon f = f_1 + f_2 \big\}\ .$$

Proof. Duhamel

$$\begin{aligned} \| \psi \|_{2+\infty} \leq &\text{ free part OK (use } L^2 \text{ and dispersive)} \\ &C\langle t\rangle^{-n/2} \| \psi_0 \|_{1\cap 2} \\ &+ \Big\| \int_0^{t-1} e^{\frac{i}{2}(t-s)\Delta} V(s)\psi(s)\, ds \Big\|_\infty \\ &\Big\| \int_{t-1}^{t} e^{\frac{i}{2}(t-s)\Delta} V(s)\psi(s)\, ds \Big\|_2 \end{aligned}$$

$$\leq \overbrace{C \cdot \langle t\rangle^{-n/2} \| \psi_0 \|_{1\cap 2}}^{\alpha} + \int_0^{t-1} \overbrace{C \cdot \langle t-s\rangle^{-n/2} \| V(s)\psi(s) \|_1}^{(1)}\ ds$$

$$+\int_{t-1}^{t} \| V(s)\psi(s) \|_2 \;\; ds \quad (2)$$

$$J(t) = \langle t\rangle^{n/2} \| \psi(t) \|_{2+\infty} \quad \text{show bdd by const.}$$

$$\| V(s)\psi(s) \|_{\binom{1}{2}} \leq \big(\| V(s) \|_{\binom{2}{\infty}} + \| V(s) \|_{\binom{1}{2}}\big) \| \psi(s) \|_{\binom{2+\infty}{2+\infty}}$$

$$(1) \leq \int_0^{t-1} C.\langle t-s\rangle^{-n/2}\langle s\rangle^{-n/2} J(s)\, ds \sup_s \big(\| V(s) \|_2 + \| V(s) \|_1\big)$$

$$(2) \leq \int_{t-1}^{t} \langle s\rangle^{-n/2} J(s)\, ds \sup_s \big(\| V(s) \|_2 + \| V(s) \|_\infty\big)$$

$$\sup_{0\leq t\leq T} J(t) \leq C\cdot \| \psi_0 \|_{1\cap 2} + \sup_{0\leq s\leq T} J(s)\cdot\varepsilon$$

□

Agmon bound

$$(-\Delta + V)\psi = E\psi$$
$$\psi \in W^{2,2}\ ,\ E < 0\ .$$

Theorem. *Assume $V \in C(\mathbb{R}^n)$, and $V(x) \to 0$ as $|x| \to \infty$. Then*

$$\int_{\mathbb{R}^n} e^{\alpha|x|}|\psi(x)|^2\, dx < \infty\ . \qquad \textit{for some } \alpha > 0$$

If, in addition, $\sup \| \partial^\gamma V \|_\infty < \infty$, *then* $|\psi(x)| \leq C.e^{-\frac{\alpha}{2}|x|}$

$$|\gamma| \leq \frac{n+1}{2}$$

Some motivation. $\quad -\psi'' + V(x)\psi = E\psi\ .$

$$\psi(x) \simeq \exp\Big(-\int_0^x \sqrt{\big(V(y)-E\big)_+}\, dy\Big)$$

$$\psi'(x) = -\sqrt{V(x)-E}\ \psi(x)$$

$$\psi'' = \underbrace{\big(V(x)-E\big)\psi(x)}_{\text{equation}} - \underbrace{V'(x)}_{\text{small}}\big(V(x)-E\big)^{-1/2}\psi$$

Proof. Define Agmon metric

$$P_E(x) = \inf_{\gamma:0\leftrightarrow x} \int_0^1 \sqrt{\Big(V\big(\gamma(t)\big)-E\Big)_+}\ |\dot\gamma(t)|\, dt$$

Long excursions of the curve → makes $\int$ big.

$$|\nabla P_E(x)| \le \sqrt{\big(V(x)-E\big)_+} \qquad \text{(steepest descent)}$$
$$\text{and} \quad P_E(x) \le \big(\| V \|_\infty + E\big)^{1/2} |x|$$
$$\text{introd.} \quad \omega(x) = \exp\big[\min[2(1-\varepsilon)P_E(x), N]\big]$$

Let $\phi \in C^\infty$ s.t. $\phi = 1$ for large x and *supp* $\phi \subset \{x : V(x) - E > 0\}$ □

Claim:

$$\int \omega(x)|\psi(x)|^2\phi^2(x)\, dx \le C_{\varepsilon,M,E} \| \psi \|_2^2$$

Ma 142 02/04

Theorem. *(Agmon). Let $V \in L^\infty(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n) V(x) \to 0$ as $|x| \to \infty$. Suppose $\psi \in W^{2,2}$, $-\Delta\psi + V\psi = E\psi$, $E < 0$. Then $\exists \alpha > 0$, $\int e^{\alpha|x|}|\psi(x)|^2\, dx < \infty$. Moreover, if $V \in W^{k,\infty}(\mathbb{R}^n)$ where $k > \frac{n}{2}$, then*

$$|\psi(x)| \le Ce^{-\alpha|x|/2} \qquad \forall x \in \mathbb{R}^n \ .$$

Proof. (1-D). Agmon's metric.

$$P_E(x) = \inf_{\gamma:0\leftrightarrow x} \int \Big(V\big(\gamma(t)\big) - E\Big)_+^{1/2} |\dot\gamma(t)|\, dt \qquad \text{Lipschitz}$$

$$|\nabla P_E(x)| \le \big(V(x) - E\big)_+^{1/2} \qquad \text{a.e.}$$

Let $N > 0$, set $P_E^{(N)}(x) = \min\big(P_E(x), N\big)$ → still have $|\nabla P_E^{(N)}(x)| \le \big(V(x) - E\big)_+^{1/2}$

Take $\phi \in C^\infty$, $\phi(x) = 1$ if x large. supp $\phi \subset \{x : V(x) - E > 0\}$.
$V(x) - E > \delta$ on supp ϕ cfr δ later.
Fix $\varepsilon > 0$.

$$\omega(x) = \exp\big(2(1-\varepsilon)P_E^{(N)}(x)\big) \ .$$

Claim.

$$I_N = \int \omega(x)|\psi(x)|^2\phi(x)^2\, dx \le C \text{ independent of } N \ .$$

Remark 2. Can take $\alpha < 2\sqrt{-E}$ (dist. from bound state to origin).

$$\delta > 0\ . \qquad \delta I_N \le \int \underbrace{\big(V(x) - E\big)\psi(x)}_{\Delta\psi} \omega(x)\bar\psi\phi^2\, dx$$
$$\le - \int |\nabla\psi|^2 \omega(x)\phi^2\, dx$$
$$+ 2(1-\varepsilon) \int |\nabla\psi|\, |\psi| \omega \sqrt{V(x) - E}\phi^2(x)\, dx$$
$$+2 \int |\nabla\psi|\, |\psi| \omega\phi |\nabla\phi|\, dx$$

and use estimate on $|\nabla P_E|$.

$$\begin{aligned}
&\leq -\int |\nabla\psi|^2\omega(x)\phi^2(x)\,dx \\
&\quad + (1-\varepsilon)\int |\nabla\psi|^2\omega(x)\phi^2(x)\,dx \\
&\quad + (1-\varepsilon)\int |\psi|^2\big(V(x)-E\big)\omega(x)\phi^2(x)\,dx \qquad \text{cfr. l.h.s!} \\
&\quad + \varepsilon\int |\nabla\psi|^2\omega\phi^2(x)\,dx \\
&\quad + 1/\varepsilon\int |\psi|^2\omega|\nabla\phi|^2\,dx
\end{aligned}$$

$$\text{Cond.} \qquad \delta I_n\cdot\varepsilon \leq \frac{1}{\varepsilon}\int |\psi(x)|^2\ \underbrace{\omega(x)}_{\substack{\text{bdd}\\ \text{dep on } N}}\ |\underbrace{\nabla\phi(x)|^2}_{\substack{\text{bdd}\\ \text{comp. supp.}}}\ dx$$

$$\leq \frac{1}{\varepsilon}\int\limits_{|x|\leq R_{\delta,V,E}} e^{2C_{V,E}|x|}|\psi(x)|^2\,dx < \infty \text{ independent of } N$$

δ,ε fixed small, so the claim holds true.

Claim $\Rightarrow$ concl. because $\omega(x)$ grows as $x\to\infty$.

Pointwise bound: elliptic regularity.

$$(-\Delta+V)\psi = E\psi \qquad\qquad \psi\in W^{2,2}$$

$$\rightarrow \quad \int |\nabla\psi|^2 + V|\psi|^2 = \int E|\psi|^2 < 0\ .$$

$$\int |\nabla\psi|^2 dx \leq \| V \|_\infty \int |\psi|^2\ .$$

$$-\Delta(\partial_\alpha\psi) + \partial_\alpha(V\psi) = E\partial_\alpha\psi$$

$$\int |\nabla\partial_\alpha\psi|^2\,dx \leq \int |\partial_\alpha(V\psi)|\,|\partial_\alpha\psi|\,dx$$

$$\text{Inductively,} \quad \int |\nabla\partial^\gamma\psi|^2 \leq C.\,\| V \|_{W^{k,\infty}}\cdot\| \psi \|_2^2$$

$$k = |\gamma|$$

$$\text{So} \quad \int\limits_{B_R(x_0)} e^{\alpha|x|}\,|\psi(x)|^2\,dx \sim e^{\alpha|x_0|}\int\limits_{B_R(x_0)} |\psi(x)|^2\,dx$$

majorize $|\psi(x_0)|^2$ by $⨍_{B_R(x_0)}|\psi(x)|^2\,dx$. Sobolev embedding.

$$\text{Localize} \quad \int(-\Delta\psi + V\psi)(x)\bar{\psi}(x)\phi^2(x-y)\,dx$$

$$= E\int |\psi(x)|^2\phi^2(y-x)\,dx$$

$$= \int |\nabla\psi|^2(x)\phi^2(x-y)\,dx$$

$$\text{c.s.}\quad + \int \nabla\psi(x)\bar{\psi}2\phi(x-y)\nabla\phi(x-y)\,dx$$

$$+ \int V(x)|\psi|^2(x)\phi^2(x-y)\,dx$$

$$\frac{1}{2}\int |\nabla\psi|^2(x)\phi^2(x-y)\,dx \le 4\int |\psi(x)|^2|\nabla\phi(x-y)|^2\,dx$$

$$+ \| V \|_\infty \int |\psi|^2(x)\phi^2(x-y)\,dx$$

$$+E\int |\psi|^2(x)\phi^2(x-y)\,dx$$

$$\int\limits_{|x-y|\le 1} |\nabla\psi(x)|^2\,dx \le C.\big(\| V \|_\infty + |E| + 1\big) \int\limits_{|x-y|\le 2} |\psi(x)|^2\,dx$$

more generally, $$\int\limits_{|x-y|\le 1} |\nabla\partial^\alpha\psi(x)|^2\,dx \le C_\alpha\big(\| V \|_{W^{|\alpha|\infty}} + |E| + 1\big)\cdot$$

$$\int\limits_{|x-y|\le 2} |\psi(x)|^2\,dx$$

To finish Agmon, use Sobolev embedding

$$\sup_{|x-y|\le 1} |\psi(x)| \le C_n \sum_{|\alpha|\le k} \int\limits_{|x-y|\le 1} |\partial^\alpha\psi(y)|^2\,dy\ .$$

Sobolev embedding is false for L^∞, you end up in BMO.

$$k > n/2 \quad : \quad W^{\alpha,p} \hookrightarrow L^q$$

$$\frac{1}{p} - \frac{1}{q} = \frac{|\alpha|}{n} \qquad \frac{1}{2} - \frac{1}{\infty} = \frac{|\alpha|}{n} \Rightarrow |\alpha| = \frac{n}{2}$$

$$\le C_{n,V,E} \int\limits_{|x-y|\le 2} |\psi(y)|^2\,dy$$

$$\le C_{nVE} \int\limits_{|x-y|\le 2} e^{\alpha|y|}|\psi(y)|^2\,dy\; e^{-\alpha(|x|-2)}$$

$$\le e^{-\alpha|x|}\cdot C_{nVE} \| \psi \|_2^2$$

□

Short-range scattering theory

Basic object $\Omega_{\mp} = s\text{-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$

$$\begin{cases} H_0 = -\Delta \\ H = -\Delta + V \qquad \| e^{-itH_0} f - e^{-itH} g \|_2 \to 0 \,. \end{cases}$$

s-lim means $\forall f \in L^2$, $\exists^? g \in L^2$ s.t. $\| e^{itH} e^{-itH_0} f - g \|_2 \to 0$?

Theorem. $\sup_x |V(x)|(1+|x|)^{1+\varepsilon} < \infty$, *then* $\Omega_\mp$ *exist and are isometries and* $\Omega_- e^{-itH_0} = e^{-itH}\Omega_-$ *(intertwining property).*

$$\Omega_- H_0 f = H\Omega_- f \qquad \forall f \in W^{2,2}$$

$$H \circ P_{\operatorname{Ran}\Omega_-} = \Omega_- H_0 \Omega_-^* \quad \text{on } W^{2,2}$$

$$\mathcal{H}_{ac} = L^2_{ac}(H) \supset \operatorname{Ran}\Omega_-$$

Proof. (Cook's Method)

Let's show $\int_1^\infty \| \frac{d}{dt} e^{itH} e^{-itH_0} f \|_2 \, dt < \infty$. $(*)$

$\forall f$ in a dense subspace of L^2.

$$\text{Then okay:} \quad e^{isH} e^{-isH_0} f - e^{itH} e^{-itH_0} f = \int_t^s \frac{d}{d\tau} e^{i\tau H} e^{-i\tau H_0} f \, d\tau$$

$$\| e^{isH} e^{-isH_0} f - e^{itH} e^{-itH_0} f \|_2 \le \int_t^s \| \frac{d}{d\tau} e^{i\tau H} e^{-i\tau H_0} f \ \|_2 \, d\tau$$

Cauchy seq.

$\to 0$ $\qquad\qquad$ $\to 0$ as $s, t \to \infty$

dense subset $\to L^2$ since $\sup_t \| e^{itH} e^{-itH_0} \|_{2\to 2} = 1$

$$(*) \quad \| \frac{d}{dt} \qquad \| \le \| e^{itH}(H - H_0) e^{-itH_0} f \|_2$$

$$= \| \quad V e^{-itH_0} f \|_2$$

$$\mathcal{C} = \{ f \in \mathcal{S}(\mathbb{R}^n) : \hat{f}(\xi) = 0 \quad \text{if } |\xi| < a$$

$$\text{or } |\xi| > b \text{ for some } a, b \} \text{ is dense.}$$

if $f \in \mathcal{C}$, then $(e^{-itH_0} f)(x) = \int e^{-i(t|\xi|^2 + x\cdot\xi)} \hat{f}(\xi) \, d\xi$

Stationary phase:

Critical point is: $2t\xi + x = 0$, $\xi = -\frac{x}{2t}$

Rapid decay if critical point $\notin [a, b]$.

$$|e^{-itH_0} f(x)| \le C f |t|^{-n/2}$$

Moreover, $|e^{-itH_0} f(x)| \le C_N |t|^{-N}$ if $|x| < \frac{a}{2}|t|$ or $> 4b|t|$. (non-stat. phase).

$$\therefore \| V e^{-itH_0} f \|_2 \le \| V e^{-itH_0} f \|_{L^2_x(\frac{at}{2} < |x| < 4bt)}$$

$$+ \| V e^{-itH_0} f \|_{L^2(|x| > 4bt \quad \text{or} \quad |x| < \frac{a}{2} t)}$$

$$\le C.t^{-1-\varepsilon} \underbrace{t^{n/2} t^{-n/2}}_{\substack{\text{volume shell} \\ \times \text{ size in} L^\infty}} + C.(at)^{n/2} t^{-N} + C.t^{-1-\varepsilon} \cdot 1$$

$$\leq C.t^{-1-\varepsilon} \quad \text{integrable.}$$

- Isometries. $\surd$ (OK $\forall t$, take s-lim).
- $\Omega_- e^{-itH_0} f = O_{L^2}(\varepsilon) + e^{isH} e^{-isH_0} e^{-itH_0} f$

$$= e^{-itH} e^{i(s+t)H} e^{-i(s+t)H_0} f + O_{L^2}(\varepsilon)$$
$$= e^{-itH} \Omega_- f + O_{L^2}(\varepsilon)$$

O_{L^2} not increased because unitary.

- Laplace transform

$$\Omega_- \int_0^\infty e^{-itH_0} e^{-t\varepsilon} dt = \int_0^\infty e^{-t\varepsilon} e^{-itH}\, dt \Omega_-$$
$$\Omega_-(iH_0+\varepsilon)^{-1} = \underbrace{(iH+\varepsilon)^{-1}}_{\substack{\text{bdd from} \\ L^2 \text{ to } W^{2,2}}} \Omega_-$$

$$\Omega_- : W^{2,2} \to W^{2,2}$$
$$\Omega_-^* : W^{2,2} \to W^{2,2}$$

$$\text{apply} \quad (iH+\varepsilon)\Omega_-(iH_0+\varepsilon)^{-1} = \Omega_- \quad \text{on } L^2$$
$$(iH+\varepsilon)\Omega_- = \Omega_-(iH_0+\varepsilon) \qquad \text{on } W^{2,2}$$

- $H \circ P_{\mathrm{Ran}\Omega_-} \overset{?}{=} \Omega_- H_0 \Omega_-^* \quad$ on $W^{2,2}$

Since Ω_- is an isometry,

$$Q := \Omega_-^*\Omega_- = Id_{L^2}\ , \qquad \Omega_-\Omega_-^* = P_{\mathrm{Ran}\Omega_-}$$
$$Q^* = Q \geq 0$$
$$\langle Qf, f\rangle = \| \Omega_- f \|^2 = \| f \|^2 = \langle f, f\rangle$$
$$\langle (Q-I)f, f\rangle = 0 \qquad \forall f\ , \qquad Q\text{s.a} \Rightarrow Q = I$$

$\downarrow$ spectral theorem

$$P = \Omega_-\Omega_-^*$$
$$P^2 = \Omega_- \underbrace{\Omega_-^*\Omega_-}_{=I} \Omega_-^* = P \qquad P^* = P \geq 0\ .$$

P is an orthog. proj.
$\rightarrow$ an identify on its range
$\rightarrow$ Ker $P = (\mathrm{Ran} P)^\perp$

$$\mathrm{Ker}\, P = \{f : \langle Pf, f\rangle = 0\}$$
$$= \{f : \| \Omega_-^* f \|^2 = 0\} = \mathrm{Ker}\, \Omega_-^*$$

$\Rightarrow P$ is proj. *onto* Ran Ω_-. (There is nothing more in Ker$P$ than KerΩ_-).

then take $\Omega_- H_0 f = H\Omega_- f$ and mult right by Ω_-^*.

$$E(d\lambda)P_{\mathrm{Ran}\Omega_-} = \Omega_- E_0(d\lambda)\Omega_-^*$$

$$P_{\mathrm{Ran}\Omega_-} E(d\lambda) P_{\mathrm{Ran}\Omega_-} = \Omega_- \overbrace{\Omega_-^* \Omega_-}^{I} E_0(d\lambda)\Omega_-^*$$

$$\langle E(d\lambda)Pf, Pf\rangle = \underbrace{\langle E_0(d\lambda)\Omega_-^* f, \Omega^* f\rangle}_{\text{always a.c.}}$$

$$\Rightarrow Pf \in L^2_{ac} \text{ rel. to } H$$

□

2/6/2003

Last time we showed under SR (short range condition)

$$|V(x)| \lesssim (1+|x|)^{-1-\varepsilon} \qquad \text{(SR)}$$

that $\Omega_- = \text{s-lim}_{t\to\infty} e^{itH}e^{-itH_0}$ exists and $\mathrm{Ran}\,\Omega_- \subset L^2_{ac}(\mathbb{R}^n)$

Q: When is Ran $\Omega_- = L^2_{ac}(\mathbb{R}^n)$?

Prop: Assume $n \geq 3$, (SR) $\| V \|_{L^2(\mathbb{R}^n)} < \infty$, $\| e^{-itH}P_{ac}(H) \|_{1\to\infty} \lesssim |t|^{-n/2}$ Then $\mathrm{Ran}\,\Omega_- = L^2_{ac}$.

Proof. $\tilde{\Omega} = \text{s-lim}\, e^{itH_0}e^{-itH}$ doesn't exist. So define $\tilde{\Omega} = s-\lim e^{itH_0}e^{-itH}P_{ac}(H)$. By Cook's method, it suffices to show

$$\int_1^\infty \| \underbrace{\frac{d}{dt} e^{itH_0}e^{-itH}P_{ac}(H)f}_{\|V_e^{-itH}P_{ac}(H)f\|_2} \|_2 \; dt < \infty$$

(for $f \in L^1 \cap L^2$).

$$\int_1^\infty \| Ve^{-itH}P_{ac}(H)f \|_2 \, dt \leq \int_1^\infty \| V \|_2 \| e^{-itH}P_{ac}(H)f \|_\infty \; dt$$

$$\leq \| V \|_2 \int_1^\infty c t^{-\frac{n}{2}} \| f \|_1 \; dt < \infty$$

So $\tilde{\Omega}$ exists. Then $\Omega_-\tilde{\Omega}f = e^{itH}e^{-itH_o}\tilde{\Omega}f + o_{L^2}(1) \qquad \forall t \geq T$.

Hence $\Omega_-\tilde{\Omega}f = e^{itH}e^{-itH_0}e^{itH_0}e^{-itH}P_{ac(H)}f + o_{L^2}(1) = P_{ac}(H)f + o_{L^2}(1)$

So $\Omega_-\tilde{\Omega}f = P_{ac}(H)f \; \forall f \in L^2 \Rightarrow \mathrm{Ran}\,\Omega_- = \mathrm{Ran}\,P_{ac}(H) = L^2_{ac}$ □

Remark 1. Instead of $V \in L^2$ take $V \in L^p$ so that

$$\| Ve^{-itH}P_{ac}(H)f \|_2 \leq \| V \|_p \| e^{-itH}P_{ac}(H)f \|_q \qquad \frac{1}{2} = \frac{1}{p} + \frac{1}{q}$$

$$\leq Ct^{-1-\varepsilon} \qquad \text{if} \quad 1+\varepsilon = \frac{n}{2}\left(\frac{1}{q'} - \frac{1}{q}\right)$$

$$\text{(because } \| e^{-itH} P_{ac}(H) f \|_q \leq C|t|^{-\frac{n}{2}\left(\frac{1}{q'}-\frac{1}{q}\right)} \| f \|_{q'}$$

so if $\varepsilon = 0 \Rightarrow n = p$, i.e., $V \in L^{n+\varepsilon}$ is enough. But (SR) $\subset L^{n+\varepsilon} \Rightarrow$ don't need $V \in L^2$.

Remark 2. When does the Proposition apply? We did case $n = 3$, $|V(x) \leq C_0(1+|x|)^{-2-\varepsilon}$ and c_0 small so that $\sup_x \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\, dy < 4\pi$.

2/6/2003

Last time we showed under SR (short range condition)

$$|V(x)| \lesssim (1+|x|)^{-1-\varepsilon} \qquad \text{(SR)}$$

that $\Omega_- = \text{s-lim}_{t\to\infty} e^{itH}e^{-itH_0}$ exists and $\operatorname{Ran}\Omega_- \subset L^2_{ac}(\mathbb{R}^n)$.
Question: When is $\operatorname{Ran}\Omega_- = L^2_{ac}(\mathbb{R}^n)$?

Proposition. *Assume $n \geq 3$, (SR), $\| V \|_{L^2(\mathbb{R}^n)} < \infty$, $\| e^{-itH} P_{ac}(H) \|_{1\to\infty} \lesssim |t|^{-n/2}$. Then* $\operatorname{Ran}\Omega_- = L^2_{ac}$.

Proof. $\tilde{\Omega} = \text{s-lim}\, e^{itH_0}e^{-itH}$ doesn't exist. So define $\tilde{\Omega} = \text{s-lim}\, e^{itH_0}e^{-itH}P_{ac}(H)$. By Cook's method, it suffices to show (for $f \in L^1 \cap L^2$)

$$\int_1^\infty \underbrace{\| \frac{d}{dt} e^{itH_0}e^{-itH} P_{ac}(H) f \|_2}_{\|V_e^{-itH} P_{ac}(H) f\|_2}\, dt < \infty$$

$$\begin{aligned}\int_1^\infty \| Ve^{-itH}P_{ac}(H)f \|_2\, dt &\leq \int_1^\infty \| V \|_2 \| e^{-itH}P_{ac}(H)f \|_\infty\, dt \\ &\leq \| V \|_2 \int_1^\infty ct^{-\frac{n}{2}} \| f \|_1\, dt < \infty\end{aligned}$$

So $\tilde{\Omega}$ exists. Then $\Omega_-\tilde{\Omega}f = e^{itH}e^{-itH_0}\tilde{\Omega}f + O_{L^2}(1) \qquad \forall t \geq T$.
Hence $\Omega_-\tilde{\Omega}f = e^{itH}e^{-itH_0}\ e^{itH_0}e^{-itH}f + O_{L^2}(1) = P_{ac}(H)f + O_{L^2}(1)$
So $\Omega_-\tilde{\Omega}f = P_{ac}(H)f$ for all $f \in L^2 \Rightarrow \operatorname{Ran}\Omega_- = \operatorname{Ran}P_{ac}(H) = L^2_{ac}$. ∎

Remark 1. Instead of $V \in L^2$ take $V \in L^p$ so that

$$\begin{aligned}\| Ve^{-itH}P_{ac}(H)f \|_2 &\leq \| V \|_p \| e^{-itH}P_{ac}(H)f \|_q \qquad \frac{1}{2} = \frac{1}{p} + \frac{1}{q} \\ &\leq Ct^{-1-\varepsilon} \qquad \text{if} \quad 1+\varepsilon = \frac{n}{2}\left(\frac{1}{q'} - \frac{1}{q}\right)\end{aligned}$$

$$\text{(because } \| e^{-itH} P_{ac}(H) f \|_q \leq C|t|^{-\frac{n}{2}\left(\frac{1}{q'}-\frac{1}{q}\right)} \| f \|_{q'})$$

so if $\varepsilon = 0 \Rightarrow n = p$, i.e., $V \in L^{n+\varepsilon}$ is enough. But (SR) $\subset L^{n+\varepsilon} \Rightarrow$ don't need $V \in L^2$.

Remark 2. When does the Proposition apply? We did case $n = 3$, $|V(x)| \leq C_0(1+|x|)^{-2-\varepsilon}$ and C_0 small so that $\sup_x \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|}\, dy < 4\pi$.

Classical (SR) scattering: $\dot{x} = \xi$
$\dot{\xi} = -\nabla V(x)$

$$\Sigma_{\pm} = \Big\{(y,\eta) \in \mathbb{R}^{2n} : V(y) + \frac{1}{2}|\eta|^2 > 0 \text{ and } \overline{\lim_{t\to\pm\infty}} |x(t;y,\eta)| = \infty\Big\}$$

$$\textbf{Note:} \quad \frac{d}{dt}\Big(V\big(x(t)\big) + \frac{1}{2}|\xi(t)|^2\Big) = \nabla V\big(x(t)\big)\cdot\xi - \xi\cdot\nabla V x(t) = 0$$

$\Sigma_{\text{scat}} = \Sigma_+ \cap \Sigma_-$.

Lemma. *Assume* $|\nabla V(x)| \le C_0(1+|x|)^{-1-\varepsilon}$ *and* $V(x) \to 0$ *as* $|x| \to \infty$. *Then:* $(y,\eta) \in \sum_+ \Leftrightarrow \lim_{t\to\infty} \xi(t;y,\eta)$ *exists and is* $\neq 0$.

Note This is long-range.
If $\lim_{t\to\infty}\xi(t;y,\eta)$ exists and $\neq 0$, then $|x(t;y,\eta)| \to \infty$.
By conservation of energy $V\big(x(t)\big) + \frac{1}{2}|\eta|^2 > 0$.
Conversely, suppose $(y,\eta) \in \sum_+$. Want to show $|x(t)| \ge at$ for some $a > 0$.
If so, done: $|\xi(t) - \xi(s)| \le \int_s^t |\nabla V\big(x(\tau)\big)|d\tau \le \int_s^\infty C_0 a^{-1-\varepsilon}\tau^{-1-\varepsilon}d\tau \to 0$ as $s \to \infty$.
$t \ge t_0 \Rightarrow$ particle stays inside the cone

Claim $|x(t)| \ge \max\Big(\frac{R}{2}, C_1(t-t_0)\Big)$, ($C_1$ not depending on R)

Proof. Assume the above formula holds on $t \in [t_0, T]$ and show we can extend it

$$|\xi(t) - \xi(t_0)| \le \int_{t_0}^{t_0+\frac{R}{2C_1}} C_0\Big(\frac{R}{2}\Big)^{-1-\varepsilon} dt + \int_{t_0+\frac{R}{2C_1}}^{\infty} C_0\big(C_1(t-t_0)\big)^{-1-\varepsilon} dt$$

$$\le C_0\frac{2^\varepsilon R^{-\varepsilon}}{C_1} + \frac{C_0 C_1^{-1-\varepsilon}}{\varepsilon}\Big(\frac{R}{2C_1}\Big)^{-\varepsilon} \le \frac{C_0}{C_1}2^\varepsilon\big(1+\frac{1}{\varepsilon}\big)R^{-\varepsilon}$$

$$x(t) = x(t_0) + \int_{t_0}^{t}\xi(\tau)\,d\tau = x(t_0) + (t-t_0)\xi(t_0) + \int_{t_0}^{t}\xi(\tau) - \xi(t_0)\,d\tau$$

$$|x(t)| \ge \sqrt{R^2 + (t-t_0)^2|\xi(t_0)|^2} - \int_{t_0}^{t}\frac{C_0}{C_1}\frac{2^{1+\varepsilon}}{\varepsilon}R^{-\varepsilon}\,d\tau$$

because $\alpha \ge 90°$

$$\ge \frac{R}{\sqrt{2}} + \frac{(t-t_0)|\xi(t_0)|}{\sqrt{2}} - (t-t_0)\frac{C_0}{C_1}\frac{2^{1+\varepsilon}}{\varepsilon}R^{-\varepsilon}$$

$$\geq \max\left(\frac{R}{\sqrt{2}}, (t-t_0)\Big(\underbrace{\frac{|\xi(t_0)|}{2} - \frac{C_0}{C_1}\frac{2^{1+\varepsilon}}{\varepsilon}R^{-\varepsilon}}_{\text{less than } C_1\text{, by choosing small } C_1 \text{ and then huge } R}\Big)\right) \Rightarrow \text{can extend beyond T}$$

□

Proposition: Assume (SR): $|\nabla V(x)| \lesssim (1+|x|)^{-2-\varepsilon}$ and $|D^2V(x)| \lesssim (1+|x|)^{-2-\varepsilon}$. Then $\forall (y,\eta) \in \sum_+$, $\exists (a,p) \in \mathbb{R}^{2n}$, $p \neq 0$ so that for $t \to \infty$

$$|a + tp - x(t; y, \eta)| \to 0$$
$$|p - \xi(t; y, \eta)| \to 0 \qquad \text{(and conversely)}$$

Proof. $(y,\eta) \in \sum_+ \Rightarrow \lim_{t\to\infty} \xi(t) = p \neq 0$ (we know)

$$x(t) = y + \int_0^t \xi(s)\, ds = y + tp + \int_0^t \big(\xi(s) - p\big)\, ds$$
$$= y + tp + \int_0^t \int_s^\infty \nabla V\big(x(\tau)\big)\, d\tau\, ds$$
$$= y + tp + \underbrace{\int_0^\infty \int_s^\infty \nabla V\big(x(\tau)\big)\, d\tau\, ds}_{=a-y} - \underbrace{\int_t^\infty \int_s^\infty \nabla V\big(x(\tau)\big)\, d\tau\, ds}_{\to 0} \quad \text{because (SR)}$$

Conversely: $x(t) = a + tp + \Delta x(t)$—want to find x

$$a + tp + \Delta x(t) = a + tp - \int_t^\infty ds \int_s^\infty d\tau \nabla V\big(a + \tau p + \nabla x(\tau)\big)$$

Contract in the unit ball in $C([T,\infty), \mathbb{R}^n)$

$$(Az)(t) = -\int_t^\infty ds \int_s^\infty d\tau\, \nabla V\big(a + \tau p + z(\tau)\big)$$

$$|a + tp + z(\tau)| \geq |a + tp| - 1 \geq \frac{1}{2} t|p| \text{ if } t \geq T \text{ large}$$

$$\text{So } |Az(t)| \leq C \int_t^\infty \int_s^\infty \Big(\frac{1}{2}\tau|p|\Big)^{-2-\varepsilon} d\tau\, ds \leq C_\varepsilon |p|^{-2-\varepsilon} T^{-\varepsilon} \leq 1$$

$$|Az(t) - A\tilde{z}(t)| \leq C \int_t^\infty \int_s^\infty \Big(\frac{1}{2}\tau|p|\Big)^{-2-\varepsilon} \parallel z - \tilde{z} \parallel \, d\tau\, ds \leq C|p|^{-2-\varepsilon} T^{-\varepsilon} \parallel z - \tilde{z} \parallel$$

So A is a contraction and such an x exists. □

$\Omega : \sum_{\text{free}} = \{(a,p) \in \mathbb{R}^{2n} | p \neq 0\} \to \sum_{+} \{(y,\eta)| \cdots\}$
Hence, Ω is one-to-one and onto
Moreover, $V(t)\Omega = \Omega V_0(t)$, since

$$\begin{array}{rll} V(s)V(t)\Omega(a,p) & = V(s+t)\Omega(a,p) & \to \binom{a+(s+t)p}{p} \\ & & \quad \| \\ V(s)\Omega V_0(t)(a,p) & = V(s)\Omega(a+tp,p) & \to \binom{a+tp+sp}{p} \end{array}$$

2/11/2003

Easy propagation facts

Lemma. *Let $A \in \mathcal{B}(\mathcal{H})$. Then A is compact $\Leftrightarrow \forall \varepsilon > 0 \exists \tilde{A}$ of finite rank such that $\| A - \tilde{A} \| < \varepsilon$.*

Proof. By definition A is compact $\Leftrightarrow \forall \varepsilon > 0$ $A(B)$ has a finite ε-net where $B =$ unit ball in $\mathcal{H}$.

$\Leftarrow \tilde{A}(B) \subset \underbrace{\text{Ran}\tilde{A}}_{\substack{\text{finite dimensional} \\ \text{subspace of } \mathcal{H}}}$ and $\tilde{A}(B)$ bounded $\Rightarrow \tilde{A}(B)$ compact $\Rightarrow$

$$\left.\begin{array}{l} \tilde{A}(B) \text{has a finite } \varepsilon\text{-net} \\ \text{But } \| A - \tilde{A} \| < \varepsilon \end{array}\right\} \Rightarrow A(B) \text{ has a finite } 2\varepsilon \text{ net}$$

$\Rightarrow$ Let $\{\phi_j\}$ be an orthonormal basis in $\mathcal{H}$. Let P_N be the projection on span $\{\phi_1, \phi_2, \ldots, \phi_n\}$ i.e., $P_N : \mathcal{H} \to \mathcal{H}$, $P_N f = \sum_{j=1}^{N} \langle f, \phi_j \rangle \phi_j$.

Obviously, $P_N A$ is of finite rank.

It suffices to prove that $\forall \varepsilon > 0$, $\| A - P_N A \| < \varepsilon$ if N is large. Since A is compact, it suffices to prove that $\forall K \subset \mathcal{H}$, K compact we have

$$\sup_{f \in K} \| f - P_N f \| < \varepsilon \text{ if } N \text{ is large}$$

This follows from the fact that we can consider a finite $\frac{\varepsilon}{2}$-net for K and N large enough such that $\| f - P_N f \| < \frac{\varepsilon}{2} \forall f$ in the $\frac{\varepsilon}{2}$ net. □

Lemma. *Let $H = -\Delta + V$. Then for $\forall f \in L^2_{ac}$*

$$\| \chi_{[|x| \leq R]} e^{-itH} f \|_2 \to 0 \text{ as } t \to \infty \, .$$

Proof. It suffices to check this for $f = (H+i)^{-1} g$ so show

$$\| \chi_{[|x| \leq R]} (H+i)^{-1} e^{-itH} g \|_2 \to 0 \circledast \left(\text{or } (H+\mu)^{-1}\right)$$

By the previous lemma, it suffices to show that

$$\| C e^{-itH} g \|_2 \to 0 \text{ as } t \to \infty$$

C has finite rank, without loss of generality, we can assume rank $C = 1$

$$\circledast \text{ becomes } \| \psi_1 \langle e^{-itH} g, \psi_2 \rangle \|_2 = \| \psi_1 \| \; |\langle e^{-itH} g, \psi_2 \rangle| =$$

$$= \| \psi_1 \| \int_{-\infty}^{\infty} e^{-it\lambda} \underbrace{\langle E(d\lambda) g, \Psi_2 \rangle}_{\text{a.c. measure}} \overset{t\to\infty}{\longrightarrow} 0 \text{ by Riemann-Lebesgue}$$

□

RAGE (Ruelle, Amrein, Georgescu, Enss) Theorem For any $f \in L^2_{\text{cont}} = L^2_{ac} \oplus L^2_{sc}$:

$$\frac{1}{T}\int_{-T}^{T} \| \chi_{[|x|\le R]} e^{-itH} f \|_2^2 \; dt \to 0 \text{ as } T \to \infty$$

Proof. As before, let $f = (H+\mu)^{-1}g$. Then, it suffices to show that

$$\frac{1}{T}\int_{-T}^{T} |\langle e^{-itH} g, \Psi_2\rangle|^2 \; dt \overset{T\to\infty}{\to} 0 \Leftrightarrow \frac{1}{T}\int_{-T}^{T} |\hat{\mu}_{g,\Psi_2}(t)|^2 dt \overset{T\to\infty}{\to} 0$$

Apply Wiener's theorem

$$\frac{1}{T}\int_{-T}^{T} |\hat{\mu}_{g,\Psi_2}(t)|^2 \; dt \to 2\sum_{\tau\in\mathbb{R}} |\mu_{g,\Psi_2}(\{\tau\})|^2 = 0$$

□

Theorem. *(Wiener) Let $\mu \in \mathcal{M}(R)$. Then*

$$\frac{1}{2T}\int_{-T}^{T} \left|\hat{\mu}(t)\right|^2 \; dt \to \sum_{\tau\in\mathbb{R}} \left|\mu(\{\tau\})\right|^2$$

Proof.

$$(\mu * \nu)(E) = \int_R \mu(E-t)\nu \; dt \quad \text{Let } \mu^{\#}(E) = \overline{\mu(-E)} \text{ so that } \widehat{\mu^{\#}}(\xi) = \overline{\hat{\mu}(\xi)}$$

$E < R$(usual)

$$\mu * \mu^{\#}(\{0\}) = \int_{\mathbb{R}} \mu(\{-t\})\mu^{\#}(dt) = \sum_{\substack{-t \text{ atom} \\ \text{of } \mu}} \mu(\{-t\}) \cdot \mu^{\#}(\{-t\}) = \sum_{\substack{t \text{ atom} \\ \text{of } \mu}} \left|\mu(\{t\})\right|^2$$

$$\frac{1}{2T}\int_{-T}^{T} e^{it\tau}\hat{\mu}(t)dt = \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty} e^{-it(s-\tau)}\mu(ds)\; dt \overset{Fubini}{=} \int_{-\infty}^{\infty} \frac{1}{2T} D_T(s-\tau)\mu(ds)dt =$$

$$= \int_{-\infty}^{\infty} \frac{1}{2T} D_T(s-\tau)\left[\mu(ds) - \mu(\tau)\delta_{\{\tau\}}(ds)\right] \; dt + \mu(\tau) \overset{T\to\infty}{\to} \mu(\tau)$$

(where D_T is the Dirichlet kernel).

Apply for $\mu * \mu^{\#}$, $\tau = 0$ and get $\frac{1}{2T}\int_{-T}^{T} \left|\hat{\mu}(t)\right|^2 \longmapsto \sum_{\tau\in\mathbb{R}} \left|\mu(\{\tau\})\right|^2$ □

Theorem. *(Enss) Let V be a SR potential (i.e., $|V(x)| \lesssim (1+|x|)^{-1-\varepsilon}$). Then we have asymptotic completeness, i.e.,*

$$\text{ran } \Omega_{\mp} = L^2_{ac} \textit{ and } L^2_{sc} = \{0\} \; .$$

Corollary. *If V is SR, then $\forall f \in L^2(\mathbb{R}^n)$ $\exists \{f_j\}_{j=0}^N \in L^2$ such that $Hf_j = \lambda_j f_j$ for $j \ge 1$. $\lambda_j \le 0$ and $e^{-itH}f = e^{-itH_0}f_0 + \sum_{j=1}^N e^{-it\lambda_j} f_j + O_{L^2}(1)$ as $t \to \infty$.*

Proof of the corollary. From the theorem $L^2 = L^2_{ac} + L^2_{pp} \Rightarrow$

$$\Rightarrow f = g + \sum_{j=1}^{N} f_j \quad g \in L^2_{ac} = \text{ran } \Omega_- \quad f_j \in L^2_{pp}$$

By Kato there cannot be positive eigenvalues. Thus $\mathcal{H} f_j = \lambda_j f_j$, $\lambda_j \leq 0$. Furthermore, $g_0 = \Omega_- f_0$

$$e^{-itH} f = e^{-itH}\Omega_- f_0 + \sum_{j=1}^{N} e^{-it\lambda_j} f_j$$

Since $\| e^{-itH}\Omega_- f_0 - e^{-itH_0} f_0 \|_2 \overset{t\to\infty}{\to} 0$ we are done. □

Proof of Enss's Theorem.
I. Heuristic sketch: Take $\Psi \in L^2_{\text{cont}}$ and $\Psi \perp \text{ran}\Omega_-$. Want to prove that $\Psi = 0$
Pick $\tau_n \to \infty$ increasing, such that $e^{-i\tau_n H}\Psi$ gets far away from the origin.

$$\| \chi_{[|x|\leq n]} e^{-i\tau_n H}\Psi \|_2 \to 0$$
$$\Psi_n = \chi_{[|x|\leq n]}\Psi_n + \Psi_n^{(\text{out})} + \Psi_n^{(\text{in})}$$

Let $\chi_j : \chi_j(\xi) + \chi_j(-\xi) = 1$ if $0 < a < |\xi| < b$

$$\chi_{[|x|>n]}\Psi_n = \sum \chi_{j,n}\Psi_n = \sum_j \chi_j(p)\chi_{j,n}\Psi_n + \sum_j \chi_j(-p)\chi_{j,n}\Psi_n$$
$$\chi_j(p) f = \left(\chi_j(\xi)\hat{f}\right)^\vee$$

- $\| (\Omega_- - \mathbf{I})\Psi_n^{(\text{out})} \|_2 \to 0$ as $n \to \infty$ (therefore $\| (\Omega_+ - \mathbf{I})\Psi_n^{(\text{in})} \|_2 \to 0$) $\circledast$

$$\| (\Omega_- - \mathbf{I})\Psi_n^{(\text{out})} \|_2 \leq \int_0^\infty \| \frac{d}{dt} e^{itH} e^{-itH}\Psi_n^{(\text{out})} \| \, dt =$$
$$= \int_0^\infty \| V e^{-itH_0}\Psi_n^{(\text{out})} \| \; dt \; \to 0 \text{ as } n \to \infty$$
$$\lesssim (n+t)^{-1-\varepsilon}$$

- $\langle \Psi_n^{(\text{in})}, \Psi_n \rangle = \langle \Omega_+ \Psi_n^{(\text{in})}, e^{-i\tau_n H}\Psi_n \rangle + O_{L^2}(1)$
$\uparrow$ see $\circledast$
$= \langle e^{i\tau_n H}\Omega_+\Psi_n^{(\text{in})}, \Psi_n \rangle + O_{L^2}(1) = \underbrace{\langle e^{i\tau_n H_0}\Psi_n^{(\text{in})}, \Omega_+^*\Psi_n \rangle}_{O_{L^2}(1)} + O_{L^2}(1)$

□

II. Rigorous proof. Let $\tilde{\Psi} \in L^2_{\text{cont}}$, $\tilde{\Psi} \perp \text{ran } \Omega_-$
$\| \tilde{\Psi} - \Psi \|_2$ so that $\Psi = \chi_{(a,b)}(H)\Psi$

By RAGE, pick $\tau_n \to \infty$ increasing such that $\begin{cases} \| \chi_{[|y| \le 2C_0 n]} e^{-i\tau_n H} \Psi \|_2 \to 0 \\ \int\limits_{-n}^{n} \| \chi_{[|x|\le n]} e^{-i(t+\tau_n)H} \Psi \|_2 \; dt \to 0 \end{cases}$

Let $\Psi_n := e^{-i\tau_n H}\psi$, $\Phi_n = \chi_{(a,b)}(H_0)\Psi_n$

Claim $\| \Psi_n - \Phi_n \|_2 = \| \left[\chi_{(a,b)}(H) - \chi_{(a,b)}(H_0)\right]\Psi_n \|_2$

Lemma. *Let $\phi \in L^1(\mathbb{R})$. Then $\| \big(\hat{\phi}(H) - \hat{\phi}(H_0)\big)\Psi_n \|_2 \mapsto 0$.*

Proof.

$$\|\Big(\int\limits_{-\infty}^{\infty} \phi(x)\big(e^{-ixH} - e^{-ixH_0}\big)\, dx\Big)\Psi_n\|_2$$

$$\le \int\limits_{-n}^{n} |\phi(x)| \; \| (e^{ixH_0}e^{-ixH} - I)\Psi_n \|_2 \; dx + \int\limits_{|x|>n} |\phi(x)|\, dx \cdot 2 \| \Psi_n \|$$

$$\le \int\limits_{-n}^{n} |\phi(x)| \Big| \int\limits_{0}^{x} \| V e^{-iyH}\Psi_n \|_2 \; dy \Big| dx + O(1) \text{ as } n \to \infty$$

$$\le \| \phi \|_1 \| V \|_\infty \int\limits_{-n}^{n} \| \chi_{[|x|\le n]} e^{-i(t+\tau_n)H}\psi \|_2 \, dt +$$

$$+ \| \phi \|_1 \int\limits_{-n}^{n} \| V\chi_{[|x|>n]} \|_\infty \; dt \, \| \psi \|_2 + O(1) = O(1)$$

□

2/16/2003

We are in the middle of the proof of Enss's theorem:

If V is SR (i.e., $|V|(x)| \lesssim (1+|x|)^{-1-\varepsilon}$), then $\operatorname{ran}\Omega_- = L^2_{\text{cont}}$ (in particular, $\operatorname{ran}\Omega_- = L^2_{ac}$, $L^2_{sc} = \{0\}$).

Proof. Let $\tilde{\Psi} \in L^2_{\text{cont}}$, $\tilde{\Psi} \perp \operatorname{ran}\Omega_-$.

Pick Ψ:

Now switch to $\Psi \in L^2_{\text{cont}}$ and $\chi_{(a,b)}(H)\Psi = \Psi$. Define Ψ_n via RAGE:

$$\| \chi_{[|x|\le 2c_0 n]}(H)\underbrace{e^{-i\tau_n H}\Psi}_{\Psi_n} \|_2 \to 0$$

and

$$\int_{-n}^{n} \| \chi_{[|x|\leq n]}(H)e^{-i(\tau_n+t)H}\Psi \|_2 \, dt \to 0 \, .$$

We let $\Phi_n = \chi_{(a,b)}(H_0)\Psi_k$. □

Lemma. *From last time* $\Rightarrow \| \Phi_n - \Psi_n \|_2 \to 0$ *as* $n \to \infty$.

$$\text{supp } \hat{\Phi}_n \subset \{\xi \in \mathbb{R}^d | \sqrt{200}\, a \leq |\xi| \leq b-a\} \, .$$

Phase space decomposition:

Pick $\{\vec{e}_j\}_{j=1}^{\tau} \in \mathcal{S}^{d-1}$ (we are in $\mathbb{R}^d$).

Fines decomposition: $\rho \in \mathcal{S}$, $\rho \geq 0$, $\int \rho = 1$, supp$\hat{\rho} \subset \{\xi \mid |\xi| \leq a\}$

$$1 = \sum_{\nu\in\mathbb{Z}^d} \chi_{[0,1]^d}(\cdot+\nu)$$

$$1 = \sum_{\nu\in\mathbb{Z}^d} \underbrace{\chi_{[0,1]^d}(\cdot+\nu) * \rho}_{\chi_\nu \geq 0}$$

$$\text{supp } \hat{\chi}_\nu \subset \{\xi| - |\xi| \leq a\} \, .$$

$$\begin{aligned}
\text{So} \quad \Phi_n &= \sum_{\nu\in\mathbb{Z}^d\setminus\cup_j C_{j,n}} \chi_\nu\Phi_n + \sum_{j=1}^{\tau} \sum_{\nu\in C_{j,n}\cap\mathbb{Z}^d} \chi_\nu\Phi_n \\
&= \sum_{\nu\in\mathbb{Z}^d\setminus\cup_j C_{j,n}} \chi_\nu\Phi_n + \sum_{j=1}^{\tau} \sum_{\nu\in C_{j,n}\cap\mathbb{Z}^d} \chi_j(p)\chi_\nu\Phi_n + \sum_{j=1}^{\tau} \sum_{\nu\in C_{j,n}\cap\mathbb{Z}^d} \chi_j(-p)\chi_\nu\Phi_n
\end{aligned}$$

$$\Rightarrow \Phi_n = \overbrace{\sum_{\nu\in\mathbb{Z}^d\setminus\cup_j C_{j,n}} \chi_\nu\Phi_n}^{\Phi_n^{(\text{trash})}} + \Phi_n^{(\text{out})} + \Phi_n^{(\text{in})}$$

$$p = -i\nabla$$

$$\chi_j(p)f = \big(\chi_j(\xi)\hat{f}(\xi)\big)^{\vee}$$

We need $\chi_j(\xi) + \chi_j(-\xi) = 1$ for

$$(\sqrt{200}-1)a \leq |\xi| \leq (b-a)+a$$

$$\text{and supp } \chi_j \subset \{12a \leq |\xi| \leq 2b\}$$

We need $x \cdot \xi \geq -\frac{1}{2}|x|\,|\xi|$ for $\forall x \in C_{n,j}$, $\forall \xi \in \text{supp } \chi_j$. ($\cos\sphericalangle(x_1\xi) \geq -\frac{1}{2}$).

First observation:

$$\| \sum_{\nu\in\mathbb{Z}^d\setminus\cup_{j=1}^{\tau}C_{j,n}} \chi_\nu\Phi_n \|_2 \leq \| \left(\sum_{\nu\in\mathbb{Z}^d\setminus\cup_{j=1}^{\tau}C_{j,n}} \chi_{[0,1]^d}(\cdot+\nu) * \rho \right) \Phi_n \|_2 \leq$$

$$\leq \| \; |\Phi_n|(\chi_{(|x|\leq c_o n+|d|)} * \rho) \|_2 \leq \overbrace{\| \Phi_n\chi_{\{|x|\leq 2c_o n\}} \|_2}^{O(1)} .$$

$$\cdot \| \underbrace{\chi_{(|x|\le c_o n+|d|)} * \rho \|_\infty}_{\le 1} + \underbrace{\| \Phi_n \|_2}_{\|\Psi\|_2} \| \underbrace{\chi_{(|x|\le c_o n+\sqrt{d})} * \rho \|_{L^\infty(|x|>2cn)}}_{O(1)}$$

$$\Rightarrow \longrightarrow 0 \text{ as } n \to \infty$$

Claim:

$$\| \chi_{(|x|\le n+at)} e^{-itH_0} \Phi^{(\text{out})}_{n,j} \|_2 \le C_{a,b}(n+t)^{-N} \ ,\forall N \ .$$

Side calculation:

$$\begin{aligned} \text{Take:} \quad & \nu \in C_{j,n} \quad , \ \xi \in \text{supp } \chi_j \\ \text{Then} \quad & |\nu + t\xi|^2 = |\nu|^2 + t^2|\xi|^2 + 2t\nu\cdot\xi \ge |\nu|^2 + t^2|\xi|^2 - t|\nu \parallel \xi| \\ & \ge \tfrac{1}{2}(|\nu|^2 + t^2|\xi|^2) \\ & \ge \tfrac{1}{4}(|\nu| + t|\xi|)^2 \end{aligned}$$

$$\begin{aligned} |x-\nu-t\xi| &\ge \frac{1}{2}(|\nu|+t|\xi|) - |x| \ge \frac{1}{2}(|\nu|+t|\xi|) - n - at \\ &\ge \frac{1}{2}|\nu| - \frac{|\nu|}{C_0} + 6at - at \ge \frac{1}{4}|\nu| + 5at \end{aligned}$$

$$\begin{aligned} &\| e^{-itH_0}\chi_j(p)\chi_\nu\Phi_n \|_{L^2(|\chi|\le n+at)} \\ &= \| \int e^{i(\underbrace{x\cdot\xi}_{(x-\nu)\xi+\nu\cdot\xi} - \frac{1}{2}t|\xi|^2)} \chi_j(\xi)\widehat{\chi_\nu\Phi_n}(\xi)\, d\xi \|_{L^2(|\chi|\le n+at} = (1) \end{aligned}$$

Introduce

$$L = \frac{x-\nu-t\xi}{|x-\nu-t\xi|^2}\cdot\nabla_\xi \Rightarrow L(e^{i((x-\nu)\xi - \frac{1}{2}t\xi^2)}) = e^{i((x-\nu)\xi-\frac{1}{2}t\xi)}$$

Then

$$\begin{aligned} (1) &= \| \int |(L^*)^N \chi_j(\xi)(\widehat{\chi_\nu\Phi_n)(\nu} + \cdot)(\xi)| d\xi \|_{L^2(|\chi|\le n+at)} \\ &\le C_{N,a,d}(|\nu|+t)^{-N} \sum_{|t|\le N} \| \partial^\gamma \chi_j(\xi)\chi_\nu\widehat{\Phi_n(\nu} + \cdot)(\xi) \|_{L^1_\xi} (h+t)^{\frac{d}{2}} \\ &\le C_{N,a,d_p}(|\nu|+t)^{-N+\frac{d}{2}} \left(\sum_{|\gamma|\le N} \| \partial^\gamma \chi_\nu \widehat{\Phi_n(\nu} + \cdot)(\xi) \|^2_{L^2_\xi} \right)^{\frac{1}{2}} \\ &\uparrow \text{Leibnitz \& Schwartz} \\ &\le C_{N,a,d,b}(|\nu|+t)^{-N+\frac{d}{2}} \| \underbrace{(1+|x-\nu|)^N\chi_\nu\Phi_n}_{\|\cdot\|_\infty\le \text{const. indep. of} \nu,n} \|_{L^2_x} \\ &\le C_{N,a,b,d,\rho}(|\nu|+t)^{-N+\frac{d}{2}} \underbrace{\| \Phi_n \|_{L^2}}_{\le\|\Psi\|_{L^2}} \end{aligned}$$

To get claim, sum in $|\nu| \ge n : \sum_{|\nu|\ge n}(|\nu|+t)^{\overbrace{-N+\frac{d}{2}}}_{<-d}$.

Important consequence:

$$\| (\overbrace{\Omega_- - 1}^{\int_0^\infty \frac{d}{dt}(e^{itH}e^{-itH_o})\,dt} \Phi_n^{(\text{out})} \|_2 \leq$$

$$\leq \int_0^\infty \| V e^{-itH_0} \Phi_n^{(\text{out})} \|_2 \; dt$$

$$\leq \int_0^\infty \| V \chi_{[|x| \leq n+at]} e^{-itH_0} \Phi_n^{(\text{out})} \|_2 \; dt$$

$$+ \int_0^\infty \| V \chi_{[|x| \geq n+at]} e^{-itH_0} \Phi_n^{(\text{out})} \|_2 \; dt$$

$$\leq \| V \|_\infty \int_0^\infty (n+t)^{-2} dt + \| \Psi \|_2 \int_0^\infty (n+at)^{-1-\varepsilon} \; dt \to 0 \text{ as } n \to \infty$$

Dually:

$$\| \Omega_+ - 1) \Phi_n^{(\text{in})} \|_2 \to 0 \text{ as } n \to \infty$$

$$\langle \Phi_n^{(\text{in})}, \Psi_n \rangle = \langle \Omega_+ \Phi_n^{(\text{in})}, \underset{\substack{\shortparallel \\ e^{-i\tau_n H}\Psi}}{\Psi_n} \rangle + o(1) \text{ as } n \to \infty$$

$$= \langle \Omega_+ e^{i\tau_n H_0} \Phi_n^{(\text{in})}, \Psi \rangle + o(1)$$

$$= \langle \underbrace{\chi_{(|x| \leq n + a\tau_n)} e^{-i(\tau_n)H_0} \Phi_n^{(\text{in})}}_{\| \quad \|_2 \to 0 \text{ , by similar arg to ``Claim''.}} , \Omega_+^* \Psi \rangle +$$

$$\underbrace{+\langle \chi_{(|x| \geq n + a\tau_n)} e^{i\tau_n H_0} \Phi_n^{(\text{in})} , \Omega_+^* \Psi \rangle}_{\| e^{i\tau_n H_0} \Phi_n^{(\text{in})} \|_2 \cdot \| \chi_{(|x| > n + a\tau_n)} , \Omega_+^* \Psi \|_2 \to 0 \text{ as } n \to \infty} \quad + o(1) = o(1)$$

$$\langle \Phi_n, \Psi_n \rangle = \| \Psi_n \|_2 + o(1)$$

$$\langle \Phi_n^{(\text{trash})}, \Psi_n \rangle + \langle \Phi_n^{(\text{out})}, \Psi_n \rangle + \langle \Phi_n^{(\text{in})}, \Psi_n \rangle$$

$$= o(1) + \langle \Omega_- \Phi_n^{(\text{out})}, \Psi_n \rangle$$

$$= o(1) + \langle \Omega_- e^{i\tau_n H_0} \Phi_n^{(\text{out})}, \Psi - \tilde{\Psi} \rangle \quad (\text{as } \tilde{\Psi} \perp \text{Ran } \Omega).$$

$$\Rightarrow \underset{\substack{\shortparallel \\ \| \Psi \|^2}}{\| \Psi_n \|^2} \leq o(1) + \| \Psi \| \cdot \| \Psi - \tilde{\Psi} \| \lesssim \varepsilon \text{ for large } n.$$

$$\Rightarrow \| \tilde{\Psi} \|_2 \leq \sqrt{\varepsilon} + \varepsilon \lesssim \sqrt{\varepsilon} \, .$$

Thus $\tilde{\Psi} = 0$.

Long range case:

$$|V(x)| \leq C(1+|x|)^{-\alpha}$$

(no longer require $\alpha > 1$) and :

$$|\partial^\alpha V(x)| \leq C(1+|x|)^{-\alpha-|\gamma|}$$

for all $|\gamma| \leq k$.

Theorem 3. *(Buslaev-Matveev). Assume $(LR)_k$ for $k = \left[\frac{d}{2}\right] + 2$ and $\alpha > \frac{1}{2}$. Then*

$$W_-(H_1 H_0) := \operatorname*{s-lim}_{t\to\infty} e^{itH} \exp(-itH_0 - i \int_0^t V(p\tau)d\tau)$$

exists : modified wave operator

Remarks

(1) $\exp(-i \int_0^t V(p\tau)d\tau) f(\xi) = \exp(-i \int_0^t V(\xi\tau)d\tau)\hat{f}(\xi)$

(2) If $\alpha > 1$; then $W_-(H_1 H_0) = \Omega_-(H_1 H_0)\exp(-i \int_0^\infty V(p\tau)d\tau)$ where the final operator is well-defined:

$$\int_0^\infty V(\xi\tau)d\tau = \frac{1}{|\xi|}\int_0^\infty \left(V\frac{\xi}{|\xi|}\tau\right)d\tau$$

(3) Clearly: $\| W_-(H_1 H_0) f \|_2 = \| f \|_2$

$$= e^{isH} W_-(H_1 H_0) = \operatorname*{s-lim}_{t\to\infty} e^{i(t+s)H} \exp(-i(t+s)H_0 - i \int_0^{t+s} V(p\tau)d\tau) \cdot$$

$$\cdot \exp(+isH_0 + i \int_t^{t+s} V(p\tau)d\tau)$$

Clearly:

$$\| \exp(i \int_t^{t+s} V(p\tau)d\tau) f - f \|_2 =$$

$= \| (e^{i \int_t^{t+s} V(\xi\tau)d\tau} - 1)\hat{f}(\xi) \|_2 \to 0$ by Dominate conv. theorem

$\Rightarrow e^{isH} W_-(H_1 H_0) = W_-(H_1 H_0) e^{isH_0}$

. $\square$

2/20/2003

Last time: long range potentials

$$(L-R)_k \qquad |\partial^\gamma V(x)| \leq C_\gamma (1+|x|)^{-\alpha-|\gamma|}$$
$$0 \leq |\gamma| \leq k$$

Theorem 4. *(Buslaev-Matveev)*

Assume $(L-R)_k$ for large enough k depending on the dimension d, and $\alpha > 1/2$ Then the modified wave operators

$$W_-(H_1 H_0) = \operatorname*{s-lim}_{t\to\infty} e^{itH} e^{-itH_0 - i\int_0^t V(\tau p)d\tau}$$

exists. Morever, they are isometries and have the intertwining property for e^{-itH} and e^{-itH_0}.

Remarks

(1) if $\alpha > 1$ nothing new here.

(2) motivation, classically

$$\begin{aligned} H_{cl} &= \frac{1}{2}p^2 + V(x) \approx \frac{1}{2}p^2 + V(tp) \qquad x \approx tp \\ H &\approx -\frac{1}{2}\Delta + V(tp) \qquad p = -i\nabla \\ U(t) &= e^{i(-\frac{1}{2}\Delta t + \int_0^t V(\tau p)d\tau)} \\ &= e^{i(tH_0 + \int_0^t V(\tau p)d\tau)} \end{aligned}$$

(3) Why $\alpha > \frac{1}{2}$? Take $V(x) = (1+|x|)^{-\alpha}$

$$0 < \alpha < 1$$

$$\left[\begin{array}{l} \dot{x} = \xi \\ \dot{\xi} = -\nabla V(x) \end{array}\right.$$

$$\begin{aligned} \xi(t) &= \eta + \int_0^t -\nabla V\big(x(\tau)\big)d\tau \\ &= \eta - \int_0^\infty \nabla V\big(x(\tau)\big)d\tau + \underbrace{\int_t^\infty \nabla V\big(x(\tau)\big)d\tau}_{o(t^{-\alpha})} \end{aligned}$$

$$\begin{aligned} x(t) &= t\xi_\infty + o(t^{1-\alpha}) \\ &= t\xi(t) + o(t^{1-\alpha}) \\ V(x) - V(t\xi) &= o(t^{-1-\alpha})t^{1-\alpha} = o(t^{-2\alpha}) \end{aligned}$$

this is integrable if $\alpha > 1/2$

(4) More generally: What if $\alpha \le \frac{1}{2}$? Ansatz:

$$W_-(H_1 H_0) = \operatorname*{s-lim}_{t\to\infty} e^{itH} e^{-iW(t,p)} \qquad \text{is isometry}$$

$$W(t,\xi) \in \mathbb{R}$$

$$\begin{aligned} e^{isH} W_-(H_1 H_0) e^{-isH_0} &= \operatorname*{s-lim}_{t\to\infty} e^{i(s+t)H} e^{-iW(t+s,p)} e^{i[W(t+s,p) - W(t,p) - s\frac{p^2}{2}]} \\ &= W_-(H_1 H_0) \qquad \textit{provided} \text{ we have} \end{aligned}$$

$$\lim_{t\to\infty} W(t+s,\xi) - W(t,\xi) - s\frac{\xi^2}{2} = 0$$

How to choose the function W?

Cook's method $\displaystyle\int_1^\infty \| \frac{d}{dt} e^{itH} e^{-iW(t,p)} f \|_{L^2}\, dt \overset{?}{<} \infty$

$$= \int_1^\infty \| \big(V(x) - \partial_t W(t,p)\big) e^{-iW(t,p)} f \|_{L^2_x}\, dt$$

$$= \int_1^\infty \| \int_{\mathbb{R}^d} e^{i(x\xi - W(t,\xi))} \big(V(x) - \partial_t W(t,\xi)\big) \hat{f}(\xi) d\xi \|_{L^2_x}\, dt$$

stationary phase should vanish at critical point: $x - \nabla_\xi W(t,\xi) = 0$

leads to: Hamilton-Jacobi equation

$$\frac{1}{2}\xi^2 + V\big(\nabla_\xi W(t,\xi)\big) - \partial_t W(t,\xi) = 0$$

$$\text{Characteristic ODE: } \left[\begin{array}{l} \dot{x} = \xi \\ \dot{\xi} = -\nabla V(x) \end{array}\right.$$

first approx: $W^{(0)}(t,\xi) = \dfrac{1}{2} + \xi^2$

$$W^{(1)}(t,\xi) = W^{(0)}(t,\xi) + \int_0^t V\big(\underbrace{\nabla_\xi W^{(0)}(\tau,\xi)}_{\tau\xi}\big)\, dt$$

We are working with the $W^{(1)}$ approx.

(5) Let $\alpha = 1$, $V(x) = (1+|x|)^{-1}$. Assume $W_-(H_1 H_0)$ exists. Then $(*) = \operatorname{s-lim}_{t\to-\infty} e^{-itH_0} e^{itH} P_{ac}$ does not exist.

Proof. Let $M(t) = e^{-i\int_0^t V(\tau p)\, d\tau}$. Assume $(*)$ exists.

$$\begin{aligned} g = P_{ac}A, &\langle e^{itH} e^{-itH_0} M(t) f, g\rangle \\ &= \langle M(t) f, \underbrace{e^{itH_0} e^{-itH} P_{ac} g}_{\overset{\text{strongly}}{\rightarrow} h}\rangle \\ &= \langle M(t) f, h\rangle + o(1) \text{ as } t \to \infty \\ &= \int e^{-i\int_0^t V(\tau\xi)\, d\tau} \hat{f}(\xi) \overline{\hat{h}(\xi)}\, d\xi \\ &= \int_{\mathbb{R}^d} \exp\Big(-i\frac{\log(1+t|\xi|)}{|\xi|}\Big) \hat{f}(\xi)\overline{\hat{h}(\xi)}\, d\xi \\ &\longrightarrow 0 \end{aligned}$$

claim

It suffices to check that:

$$\int_0^\infty \exp\Big(-i\frac{\log(1+t\tau)}{\tau}\Big)\phi(\tau)\, d\tau \longrightarrow 0$$

for $\phi \in C_0^\infty(0,\infty)$

$$= \int_0^\infty \exp\Big(-i\frac{\log t}{\tau}\Big) e^{-i\frac{\log \tau}{\tau}} \exp\Big(-\frac{i}{\tau}\log\Big(1+\frac{1}{t\tau}\Big)\phi(\tau)\, d\tau$$

this term tends to 0, using $\tau \to \frac{1}{\tau}$ and integration by parts. □

Proof. of Theorem: Apply Cook's method

$$\Phi(t,x,\xi) = x\cdot\xi - \frac{1}{2}t\xi^2 - \int_0^t V(\tau\xi)d\tau$$

We need to check that:

$$\infty > \int_1^\infty \| \int_{\mathbb{R}^d} e^{i\Phi(t,x,\xi)}\big(V(x) - \int_0^t V(\tau\xi)d\tau\big)\hat{f}(\xi)d\xi \|_{L^2_x}\ dt$$

Here $f \in \mathcal{S}_1$, supp $\hat{f} \subset \{0 < a < |\xi| < b\}$

Classically forbidden region.

Note:

$$\nabla_\xi \Phi = x - t\xi - \int_0^t \nabla V(\tau\xi)\tau\, d\tau$$

(1) $$|x| < \frac{a}{2}t$$

$$\begin{aligned} |\nabla_\xi \Phi(t,x,\xi)| &\geq at - \frac{a}{2}t - \int_0^t |\nabla V(\tau\xi)|\tau\, d\tau \\ &\geq \frac{a}{2}t - C\int_0^t (1+\tau a)^{-\alpha-1}\tau\, d\tau \\ &\geq \frac{at}{2} - \frac{C}{a^2}(1+ta)^{1-\alpha} \\ &\geq \frac{a}{4}t \quad \text{if} \quad t \geq t_0(f,V) \end{aligned}$$

integrate by parts N times $\Rightarrow$

$$\text{diff. operator}$$
$$L_\xi = \frac{-i\nabla_\xi \Phi}{|\nabla\Phi|^2}\cdot\nabla_\xi$$

$$\Big|\int_{\mathbb{R}^d} e^{i\Phi(t,x,\xi)}\big(V(x)-\int_0^t V(\tau\xi)d\tau\big)\hat{f}(\xi)d\xi\Big|$$
$$\geq Ct^{-N}t^{1-\alpha}$$

Side remark

$$\Big|\int_0^t \nabla V(\tau\xi)\tau\ d\tau\Big| \lesssim t^{1-a}$$

$$\Big|\partial_\xi^\gamma \int_0^t \nabla V(\tau\xi)\tau\ d\tau\Big| \lesssim t^{1-\alpha}$$

L chosen in such a way:

$Le^{i\phi} = e^{i\phi}$

Aside. Go back to

$$\|\ldots\|_{L^2(|x|<\frac{a}{2}t)} \lesssim t^{\frac{d}{2}}t^{-N+1-\alpha}$$

You need $N \geq [\frac{d}{2}]+2$

Forbidden region (2) $|x| > 2bt$

$$|\nabla\Phi(t,x,\xi)| \geq |x| - tb - Ct^{1-a} \geq \frac{1}{4}|x|$$

$$\Big|\int_{\mathbb{R}^d} e^{i\Phi(t,x,\xi)}\big(V(x)-\int_0^t V(\xi)\big)\hat{f}(\xi)d\xi\Big|$$

$$\leq \int_{a<|\xi|<b}\Big|(L_\xi^*)^N\big(V(x)-\int_0^t V(\tau\xi)\ d\tau\big)\hat{f}(\xi)d\xi\Big|$$
$$\geq (1+|x|)^{-N}t^{1-\alpha}$$

$$\|\ldots\|_{L^2(|x|>2bt)} \lesssim t^{1-\alpha}\Big(\int_{|x|>2bt}(1+|x|)^{-2N}dx\Big)^{\frac{1}{2}}$$
$$\lesssim t^{1-\alpha}t^{-N+\frac{d}{2}}\Big\langle \begin{matrix} t^{-1-\alpha} \\ t^{-\frac{1}{2}-\alpha}\end{matrix}$$

Note:

$$N = \left[\frac{d}{2}\right]+2\Big\langle \begin{matrix} \frac{d}{2}+2 \text{ if even} \\ \frac{d}{2}+2-\frac{1}{2} \text{ if odd}\end{matrix}$$

(Classically allowed region (3) important case

$$\int_1^\infty \| \int_{\mathbb{R}^d} e^{i\Phi(t,x,\xi)} \big(V(x) - \int_0^t V(\tau\xi)d\tau\big) \hat{f}(\xi) d\xi \, \|_{L^2(|x|\sim t)} \; dt$$

Intuition critical point $\xi_0(t,x)$

$$x - t\xi_0 - \int_0^t \nabla V(\tau\xi_0)\tau \; d\tau = 0$$

$$|V(x) - \int_0^t V(\tau\xi_0)d\tau| \lesssim t^{-2a}$$

$$\text{stationary phase} \Rightarrow \int_1^\infty t^{\frac{d}{2}} t^{-\frac{d}{2}} t^{-2\alpha} dt < \infty$$

Aside: Stationary phase:

$$\left| \int e^{it\Psi(\xi)} a(t,\xi) d\xi \right| \lesssim$$

$$\left(\begin{array}{l} \text{we are in a non-linear} \\ \text{case/never mind} \end{array} \right) \lesssim t^{-\frac{d}{2}} |a(t,\xi_0)|$$

where ξ_0 is the critical point.

Pick a smooth characteristic function ϕ around 0

$$\delta = t^{-\frac{1}{2}}$$

$$\int_{\mathbb{R}^d} e^{i\Phi} \underbrace{\big(V(x) - \int_0^t V(\tau\xi)d\tau\big) \hat{f}(\xi)}_{\omega} \; d\xi =$$

$$\int_{\mathbb{R}^d} e^{i\Phi} \phi\Big(\frac{\xi - \xi_0(t,x)}{\delta}\Big) \omega(x,t,\xi) \; d\xi + \int_{\mathbb{R}^d} e^{i\Phi} \left(1 - \phi\Big(\frac{\xi - \xi_0(t,x)}{\delta}\Big)\right) \omega \; d\xi$$

Justify the existence of the critical point:

$$x - t\xi - \int_0^t \nabla V(\tau\xi)\tau \; d\tau = 0$$

$$\text{solve this for fixed } t, |x| \sim t$$

$$x - t\psi(t\xi) = 0 \text{ for } a < |\xi| < b$$

$$D\psi(t,\xi) = Id + \underbrace{\frac{1}{t}\int_0^t D^2V(\tau\xi)\tau^2 d\tau}_{o(t^{-a})}$$

$$\psi(t,\xi) = \xi + \frac{1}{t}\int_0^t \nabla V(t\xi)\tau \, d\tau$$

□

PDE (Schlag) 02/25/03

From last time we need to prove that

$$\int_1^\infty \| \int e^{i\Phi(t,x,\xi)}\big(V(x) - V(t\xi)d\xi\big) \|_{L^2_x} \; dt < \infty$$

we did the "classically forbidden" region

(1) $|x| < \frac{at}{2}$
(2) $|x| > 2bt$

$$f \in \xi \;\sup \hat{f} \in \{\xi : 0 < |a| < |\xi| \subset b\}$$

We had $L = \frac{-i\nabla_\xi \Phi}{|\nabla_\xi \Phi|^2} \cdot \nabla$, $\quad L \cdot e^{i\Phi(t,x,\xi)} = e^{i\Phi(t,x,\xi)}$ then do integral by part.

$$L^* = i \;\mathrm{div}\bigg(\frac{\nabla_\xi \Phi}{|\nabla_\xi \Phi|^2} \cdot\bigg)$$

(3) critical point in $\frac{at}{2} < |x| < 2bt$

$$\nabla_\xi \Phi = 0 \quad \Phi = x\xi - \frac{t \cdot |\xi|^2}{2} - \int_0^t V(\xi\tau)d\tau$$

$$\nabla_\xi \Phi = x - t\xi - \int_0^t \nabla V(\xi\tau)\tau \, d\tau = 0$$

$$D^2_\xi \Phi = -tI_d - \int_0^t D^2V(\xi\tau)\tau^2 \, d\tau$$

t fixed as large number.

B_j = radius independent on t.

Consider the rescaled map

$$\Phi_t : \xi \to \frac{x}{t}$$

$$\frac{x}{t} = \xi + \underbrace{\frac{1}{t}\int_0^t \nabla V(\xi\tau)t\, d\tau}_{O\left(\frac{1}{t}\cdot t\cdot t^{-1-\alpha}\cdot t\right)=O(t^{-\alpha})} = \Phi_t(\xi)$$

cover the annulus with ball B;

$$D^2\Psi_t(\xi) = I_d + O(t^{-\alpha})$$

$$\text{Upshot: } \forall x \quad \frac{at}{2} < |x| < 2bt$$

$$\exists \xi_0(t,x) s.t. \nabla_\xi \Phi(t,x,\xi_0) = 0$$

$$\int e^{i\Phi}\big(V(x) - V(t\xi)\big)\hat{f}(\xi)d\xi$$

$$= \int e^{i\Phi}\left(\chi\Big(\frac{\xi-\xi_0(t,x)}{\delta}\Big) + \left(1-\chi\Big(\frac{\xi-\xi_0(t,x)}{\delta}\Big)\right)\big(V(x)-V(t\xi)\big)\hat{f}(\xi)\right)d\xi$$

$$= I + II$$

$$\chi \text{ is smooth cut of } \; B(0, t^{-\frac{1}{2}})$$

$$\text{write } \delta = t^{-\frac{1}{2}}$$

$$|I| \leq \int_{|\xi-\xi_0|<2\delta} |V(x) - V(t\xi)| d\xi \, \| \hat{f} \|_{L^\infty}$$

$$\leq \delta^n \cdot \underset{\nwarrow \nabla V}{t^{-1-\alpha}} (t^{1-\alpha} + t\delta) \leq t^{-\frac{n}{2}} - t^{-2\alpha} + t^{-\frac{n}{2}} = t^{-\alpha-\frac{1}{2}}$$

$$\lesssim t^{-\frac{n}{2}} \cdot t^{-\alpha-\frac{1}{2}}$$

$$\text{note} \quad x - t\xi = x - t\xi_0 + t(\xi_0 - \xi)$$

$$= \int_0^t \nabla V(\tau\xi_0)\tau\, d\tau + t(\xi_0 - \xi)$$

$$|x - t\xi| \leq t^{1-\alpha} + t\cdot\delta$$

$$|II| \leq \int \left|(L^*)^N\big(V(x) - V(t\xi)\big)\hat{f}(\xi)\left(1-\chi\Big(\frac{\xi-\xi_0}{\delta}\Big)\right)\right| d\xi$$

$$(L^*)^N = \sum \text{coeff} \frac{\partial^{r_1}\Phi \;\cdots\; \partial^{\partial_{N+2m}}\Phi}{|\nabla\Phi|^{2N+2m}} \partial^\beta$$

$$|\gamma_1| + - \cdot + |\gamma_{N+2m}| + |\beta| = 2N + 2m$$

$$0 \leq m \leq N$$

Proof.

$$N=1 \quad \partial_j\left(\frac{\partial_j\Phi}{|\nabla\Phi|^2}f\right)=\frac{\Delta\Phi}{|\nabla\Phi|^2}g+\frac{\partial_j\Phi\cdot\partial_j g}{|\nabla\Phi|^2}-2\frac{\partial_j\Phi\partial_{ji}\Phi}{|\nabla\Phi|^4}\cdot\partial_i\Phi g$$

$$\begin{matrix} \beta=0 & m=1 & \beta=1 \\ m=0 & \beta=0 & m=0 \\ 2=2 & 1+2+1=4 & 1+1=2 \end{matrix}$$

by induction

$$\begin{aligned}
&\partial_j\left(\frac{\partial_j\Phi}{|\nabla\Phi|^2}\cdot\frac{\partial^{\gamma 1}\Phi\cdots\partial^{\gamma_{N+2m}}\Phi}{|\nabla\Phi|^{2N+2m}}\partial^\beta g\right)\\
=&\left(\frac{\Delta\Phi}{|\nabla\Phi|^2}-\frac{2\partial_j\Phi\partial_{ji}\Phi\partial_i\Phi}{|\nabla\Phi|^4}\right)\frac{\partial^{\gamma_1}\Phi\ \cdots\ \partial^{\gamma_{N+2m}\Phi}}{|\nabla\Phi|^{2N+2m}}\partial^\beta g\\
&+\ \frac{\partial_j\Phi}{|\nabla\Phi|^2}\frac{\partial^{\gamma_{1+e_j}}\Phi\ \cdots\ \partial^{\gamma_{N+2m}}\Phi}{|\nabla\Phi|^{2N+2m}}\partial^\beta g\\
&+\frac{\partial_j\Phi}{|\nabla\Phi|^2}\frac{\partial^{\gamma_1}\Phi\ \cdots\ \partial^{\gamma_{N+2m}}\Phi}{|\nabla\Phi|^{2N+2m}}\partial^{\beta+1}g\\
&+\frac{\partial_j\Phi}{|\nabla\Phi|^2}\frac{\partial^{\gamma_1}\phi\ \cdots\ \partial^{\gamma_{N+2m}}\cdot\partial_i\Phi\partial_{ji}\Phi}{|\nabla\Phi|^{2N+2(m+1)}}\partial_g^\beta
\end{aligned}$$

check the $|\partial_1|+\ \cdots\ +|\gamma_{N+2m}|+|\beta|=2N+2m$ □

Let $\lambda=\#\{1\le j\le N+2m=|\gamma_j|:1\}$
then $\lambda+2(N+2m-\lambda)+|\beta|\le 2N+2m\Rightarrow 2m+|\beta|\le\lambda$
First case $\beta=0 \quad x-t\xi=x-t\xi_0+t(\xi_0-\xi)$

$$\begin{aligned}
&\int\limits_{|\xi-\xi_0|>\delta}\frac{t^{N+2m-\lambda}}{|\nabla\Phi|^{2N+2m-\lambda}}t^{-1\alpha}\left(t^{-1-\alpha}+t(\xi_0-\xi)\right)\,d\xi\\
\lesssim&\int\limits_{|\xi-\xi_0|<\delta}\frac{t^{N+2m-\lambda}\cdot t^{-\alpha}}{t^{N+2m-\lambda}|\xi-\xi_0|^{2N+2m-\lambda-1}}\,d\xi\lesssim t^{-N}\cdot t^{-\alpha}\delta^{-2N-2m+\lambda+1}\cdot\delta^d\\
\le& t^{-N}\delta^d\delta^{-2N+1}\cdot t^{-\alpha}\le t^{-\frac{d}{2}}t^{-\alpha-\frac{1}{2}}
\end{aligned}$$

similar for other term

Commutator methods

Proposition

$$F(x)=\sum_{|\alpha|\le N}a_\alpha x^d \quad \text{polynomial}$$

Then

$$\begin{aligned}
e^{-i\frac{t}{2}\Delta}F(x)e^{+i\frac{t}{2}\Delta}&=e^{-i\frac{|x|^2}{2t}}F(tp)e^{i\frac{|x|^2}{2t}}\\
&=F(x+tp)
\end{aligned}$$

Proof. Suffice to consider dim $d=1$

$$e^{-i\frac{t}{2}\Delta}x_1^{\alpha_1}\cdot x_d^{\alpha_d}e^{i\frac{t}{2}\Delta}=e^{-i\frac{t}{2}\partial_1^2}x_1^{\alpha_1}e^{\frac{it}{2}\partial_1^2}e^{-i\frac{t}{2}\partial_2^2}\cdots$$

$$\underbrace{e^{-i\frac{t^2}{2}\partial_d^2} x_d^{\partial_d} \cdot e^{\frac{it^2}{2}\partial_d^2}}_{(x_d+tp_d)^{\alpha_d}} \cdot$$

$$= \pi(x_j + tp_j)^{\alpha_i}$$

$$\begin{aligned}
(*) &= \left(e^{-i\frac{t}{2}\Delta} F(x) e^{i\frac{t}{2}\Delta} f\right)(x) \\
&= \iint \frac{1}{(-2\pi it)^{\frac{1}{2}}} e^{-i\frac{|x-u|^2}{2t}} F_{(u)} \frac{e^{\frac{i}{2t}|u-y|^2}}{(2\pi it)^{\frac{d}{2}}} f(y)\, dy\, du \\
&= (2\pi)^{-d} t^{-d} e^{-i\frac{|x|^2}{2t}} \iint \underbrace{e^{-\frac{i}{t}u(-x+y)} F(u)\, du}_{\hat{F}\left(\frac{y-x}{t}\right)} e^{\frac{i}{2t}|y|^2} f(y)\, dy
\end{aligned}$$

$$\left(\begin{array}{c} \text{if } F(x) = x_1^{\alpha_1} \cdots x_d^{\alpha_d} = x^\alpha \\ \\ \hat{F}(\xi) = (2\pi)^d (id_\xi)^\alpha \delta_0 \end{array}\right)$$

$$\begin{aligned}
t^{-d} \int_{R^d} \hat{F}\left(\frac{y-x}{t}\right) \psi_{(y)}\, dy &= \int_{R^d} \hat{F}\left(y - \frac{x}{t}\right) \psi_{(ty)}\, dy \\
&= (2\pi)^d \int \delta_0 \left(y - \frac{x}{t}\right) \left((-it\partial_y)^\alpha \psi\right)_{(ty)}\, dy \\
&= (2\pi)^d (tp)^\alpha \psi(x)
\end{aligned}$$

$$\text{So } (*) = \left(e^{\frac{-i|x|^2}{2t}} F(tp) e^{-i\frac{|x|^2}{2t}} f\right)(x)$$

Check:

$$e^{-i\frac{t}{2}\frac{d^2}{dx^2}} x^\lambda e^{i\frac{t}{2}\frac{d^2}{dx^2}} = (x+tp)^\lambda$$

Induction: $\lambda = 0 \quad \surd$

$$\begin{aligned}
i\frac{d}{dt}(\lambda hs) &= e^{-\frac{i}{2}t\Delta} \frac{1}{2} [\Delta, x^\lambda] e^{i\frac{t}{2}\Delta} \\
&= e^{-i\frac{t}{2}\Delta} \left(\frac{\lambda(\lambda-1)}{2} x^{\lambda-2} + \lambda x^{\lambda-1} \frac{d}{dx} e^{\frac{it}{2}\Delta} \right. \\
&= \frac{\lambda(\lambda-1)}{2} (x+tp)^{\lambda-2} + \lambda(x+tp)^{\lambda-1} \frac{d}{dx}
\end{aligned}$$

$$i\frac{d}{dt}(\gamma hs) = \frac{d}{dx}(x+tp)^{l-1} + (x+tp)\frac{d}{dx}(x+tp)^{\lambda-2} + \cdots + (x+tp)^{\lambda-1}\frac{d}{dx}$$

$$\frac{d}{dx}(x+tp)^m = m(x+tp)^{m-1} + (x+tp)^m \frac{d}{dx}$$

By induction

$$\begin{aligned}
\frac{d}{dx}(x+tp)^{m+1} &= \left(it(x+tp)\frac{d}{dx}\right)(x+tp)^m \\
&= (x+tp)^m + m(x+tp)^m + (x+tp)^{m+1}\frac{d}{dx}
\end{aligned}$$

□

Plug in

$$i\frac{d}{dt}(rhs) = \frac{l(\lambda-1)}{2}(x+tp)^{l-2} + l(x+tp)^{\lambda-1}\frac{d}{dx}$$

Corollary

$$\| \ |x-tp|e^{i\frac{\Delta}{2}t}\phi \ \|_2^2 = \| \ |x|\phi \ \|_2^2$$

Proof.

$$\langle |x-tp|^2 e^{it\frac{\Delta}{2}}\phi \ , e^{it\frac{\Delta}{2}}\phi\rangle = \langle |x|^2\phi \ , \phi\rangle$$

□

Corollary

(1) Suppose $(1+|x|)\psi_0 \in L^2$ and $\nabla\psi_0 \in L^2$ then

$$\| \ (1+|x|)e^{\frac{it}{2}\Delta}\psi_0 \ \|_2 \leq (1+|t|)$$

(2) Suppose $\sum_{|\alpha|+|\beta|\leq m} \| \ x^\alpha \partial^\beta \psi_0 \ \|_2 < \infty$ then

$$\| \ |x|^m e^{\frac{it}{2}\Delta}\psi_0 \ \|_2 \lesssim (1+|t|)^m$$

2/27/2003

Last time: Pseudo-conformal identity

$$\begin{aligned} \langle |x-tp|^2\psi(t), \psi(t)\rangle &= \text{const} \\ &= \langle |x|^2\psi(0), \psi(0)\rangle \\ \text{if} \quad i\partial_t\psi + \frac{1}{2}\Delta\psi &= 0 \end{aligned}$$

Corollary.

(*i*) *Suppose* $(1+|x|)\psi \in L^2$ *and* $\psi \in H^1$. *Then*

$$\| \ (1+|x|)e^{\frac{i}{2}t\Delta}\psi \ \|_2 \lesssim (1+|t|) \ .$$

(*ii*) *Suppose* $\sum_{|x|+|\beta|\leq m} \| \ x^\alpha\partial^\beta\psi \ \|_2 < \infty$. *Then*

$$\| \ |x|^m e^{\frac{it}{2}\Delta}\psi \ \|_2 \lesssim (1+|t|)^m$$

Proof.

$$\begin{aligned} \| \ x^\alpha e^{\frac{it}{2}\Delta}\psi \ \|_2 &= \| \ e^{-\frac{it}{2}\Delta}x^\alpha e^{i\frac{t}{2}\Delta}\psi \ \|_2 \\ &= \| \ (x+tp)^\alpha\psi \ \|_2 \\ &\leq (1+|t|)^{|\alpha|} \sum_{|\beta|+|\gamma|\leq|\alpha|} \| \ x^\beta\partial^\gamma\psi \ \|_2 \end{aligned}$$

□

Proposition Suppose $x^\alpha\phi \in L^2$ $\forall|\alpha| \leq k$. Then $\partial^\alpha e^{-i\frac{t}{2}\Delta}\phi \in L^2$ for $\forall|\alpha| \leq k$ and $t \neq 0$. (Gain of derivatives, cfr. comp. supp. init. cond. becomes analytic when $t > 0$).

Proof.

$$e^{-\frac{it}{2}\Delta}x^\alpha e^{\frac{it}{2}\Delta} = e^{-i|x|^2/2t}(tp)^\alpha e^{i|x|^2t/2t}$$

$$\begin{aligned}\infty > \| x^\alpha \phi \|_2 &= \| e^{i\frac{t}{2}\Delta} e^{-i\frac{|x|^2}{2t}}(tp)^\alpha e^{\frac{i|x|^2}{2t}} e^{-i\frac{t}{2}\Delta}\phi \|_2 \\ &= \| (tp)^\alpha e^{\frac{i|x|^2}{2t}} e^{-\frac{it}{2}\Delta}\phi \|_2 \\ &\Rightarrow \| \partial^\beta e^{-\frac{it}{2}\Delta}\phi \|_2 < \infty \text{ if } t \neq 0 \ , \ |\beta| \leq k\end{aligned}$$

□

Now $H = -\frac{1}{2}\Delta + V$.
Make operators commute through the evolution.

$$\underbrace{Ae^{-itH} = e^{-itH}A}_{\substack{\text{globally}\\ \text{(Lie Group)}}} + i\int_0^t e^{-i(t-s)H} \underbrace{[H,A]}_{\substack{\text{infinitesimally}\\ \text{(Lie Algebra)}}} e^{-isH}\, ds \ .$$

$$\begin{aligned}\text{Because } \frac{d}{dt}e^{itH}Ae^{-itH} &= e^{itH}i(HA - AH)e^{-itH} \\ &= ie^{itH}[H,A]e^{-itH}\end{aligned}$$

$$\Rightarrow e^{iTH}Ae^{-iTH} - A = i\int_0^T e^{itH}[H,A]e^{-itH}\, dt \ .$$

Lemma. *Suppose* $V \in L^\infty$, $H = -\frac{1}{2}\Delta + V$

(i) *If* $(1+|x|)\phi \in L^2$ *and* $\phi \in H^1$ *then* $\| (1+|x|)e^{-itH}\phi \|_2 \lesssim (1+|t|)$.

(ii) *If* $\sum_{|\alpha|+|\beta|\leq 2} \| x^\alpha \partial^\beta \psi \|_2 < \infty$, *then* $\| (1+|x|)^2 e^{-itH}\psi \|_2 \lesssim (1+|t|)^2$.

Proof.

(i)
$$x_j e^{-itH}\phi = e^{-itH}x_j\phi + i\int_0^t e^{-i(t-s)H}[H,x_j]e^{-isH}\phi\, ds$$

with $[H,x_j] = \left[-\frac{1}{2}\Delta + V_j, x_j\right] = \left[-\frac{1}{2}\Delta, x_j\right] = -\partial_j$

(1)
$$\| x_j e^{-itH}\phi \|_2 \leq \| x_j\phi \|_2 + \int_o^t \| \partial_j e^{-isH}\phi \| \ ds$$

Take M large enough, $\| \partial_j (H+M)^{-\frac{1}{2}} \|_{2\to 2} < \infty$ because obviously $\| \Delta(H+M)^{-1} \|_{2\to 2} < \infty$ (resolvent → gains 2 derivatives) then apply complex interpolation: we have

$$\sup_\sigma \| (-\Delta)^{1+i\sigma}(H+M)^{-1-i\sigma} \|_{2\to 2} \leq C_M$$

$$\sup_\sigma \| (-\Delta)^{i\sigma}(H+M)^{-i\sigma} \|_{2\to 2} \le C_M \ .$$

Proof.

$$\| \underbrace{(-\Delta)^{i\sigma}}_{\substack{\text{bdd from } 2\to 2\\ \text{(multipliers)}}} \underbrace{(-\Delta)(H+M)^{-1}}_{\text{OK}} \underbrace{(H+M)^{-i\sigma}}_{\substack{\text{bdd from } 2\to 2\\ \text{(multipliers)}}} \|_{2\to 2}$$

$$\le \| (-\Delta)(H+M)^{-1} \|_{2\to 2} < \infty \ .$$

□

then $T_z = (-\Delta)^z (H+M)^{-z}$, $T_{\frac{1}{2}} : L^2 \to L^2$.

$$\begin{aligned} \therefore\ (1)\ &\le \| \ |x|\phi \|_2 + \int_0^t \| \partial_j (H+M)^{-\frac{1}{2}} \|_{2\to 2} \| e^{-iHs}(H+M)^{\frac{1}{2}}\phi \|_2 \ ds \\ &\le \| \ |x|\phi \|_2 + C.t. \| \phi \|_{H^1} \end{aligned}$$

(ii) $|\alpha| = 2$

$$\begin{aligned} [H, x^\alpha] &= \left[-\frac{1}{2}\Delta, x^\alpha\right] \\ &= -\frac{1}{2}\Delta(x^\alpha) - \nabla(x^\alpha)\cdot\nabla \end{aligned}$$

$$\begin{aligned} \| x^\alpha e^{-itH}\phi \|_2 \le \| \ |x|^2\phi \|_2 &+ C \int_0^t \| \phi \|_2 \ ds \\ &+ C \max_{j,l} \int_0^t \| x_j \ \partial_e^{-isH}\phi \|_2 \ ds \end{aligned}$$

Use $\quad \| [(H+M)^{\frac{1}{2}}, x_j] \|_{2\to 2} < \infty$.

$\qquad \| [\partial_e, x_j] \|_{2\to 2} < \infty$

because $[(H+M)^{\frac{1}{2}}, x_j] = ?$

$$(H+M)^{-\frac{1}{2}} \stackrel{?}{=} C \int_0^\infty (H+M+\lambda)^{-1} \frac{d\lambda}{\sqrt{\lambda}}$$

(integr. at zero) $\left(\frac{1}{\sqrt{\lambda}}\right)$ and at ∞ $(\lambda^{-\frac{3}{2}})$

$$= C \int_0^\infty \int_{-\frac{M}{2}}^\infty \frac{1}{\tau + M + \lambda} \ E(d\tau) \frac{d\lambda}{\sqrt{\lambda}}$$

because $\| V \|_\infty < \infty$, take M large

$$= \int_{-\frac{M}{2}}^{\infty} \underbrace{C \int_0^\infty \frac{1}{1+\lambda} \frac{d\lambda}{\sqrt{\lambda}}}_{\text{take } C \ s.t. = 1} \frac{E(d\tau)}{\sqrt{\tau + M}} = (H+M)^{-\frac{1}{2}}$$

Now $$\left[(H+M)^{\frac{1}{2}}, x_j\right] = C \int_0^\infty \left[(H+M)(H+M+\lambda)^{-1}, x_j\right] \frac{d\lambda}{\sqrt{\lambda}}$$

$$- (H+M)^{\frac{1}{2}} \left[(H+M)^{-\frac{1}{2}}, x_j\right] (H+M)^{\frac{1}{2}}$$

$$= -(H+M)^{\frac{1}{2}} C \cdot \int_0^\infty \left[(H+M+\lambda)^{-1}, x_j\right] \frac{d\lambda}{\sqrt{\lambda}} \cdot (H+M)^{\frac{1}{2}}$$

$$= -(H+M)^{\frac{1}{2}} C \int_0^\infty (H+M+\lambda)^{-1} \underbrace{\left[(H+M+\lambda), x_j\right]}_{-\partial_j} (H+M+\lambda)^{-1} \frac{d\lambda}{\sqrt{\lambda}} (H+M)^{\frac{1}{2}}$$

$$\| \ \left[(H+M)^{-\frac{1}{2}}, x_j\right] (H+M)^{-\frac{1}{2}} \|_{2\to 2}$$

$$\le C \int_0^\infty \underbrace{(1)(2)(3)}_{\substack{\text{see} \\ \text{above} \\ \text{for definition}}} \cdot \| \underbrace{(H+M+\lambda)^{-\frac{1}{2}}}_{\lesssim \frac{1}{\sqrt{1+\lambda}}} \|_{2\to 2} \frac{d\lambda}{\sqrt{\lambda}} < \infty \ .$$

Then
$$\begin{aligned}
\| \partial_e x_j e^{-isH} \phi \|_2 &\le C \cdot \| (H+M)^{\frac{1}{2}} x_j e^{-isH} \phi \|_2 \\
&\le C \cdot \| x_j e^{isH} (H+M)^{\frac{1}{2}} \phi \|_2 + C \| (H+M)^{\frac{1}{2}} e^{-isH} \phi \|_2 \\
&\le C \cdot (1+|s|) \left\{ \| (1+|x|)(H+M)^{\frac{1}{2}} \phi \|_2 + \| \nabla (H+M)^{\frac{1}{2}} \phi \|_2 \right\} \\
&\qquad \text{(from part I)} \qquad + C \cdot \| (H+M)^{\frac{1}{2}} \phi \|_2 \\
&\le C \cdot (1+|s|) \sum_{|\alpha|+|\beta|\le 2} \| x^\alpha \partial^\beta \phi \|_2 \\
\Rightarrow \| x^\alpha e^{-itH} \phi \|_2 &\le (1+|t|)^2 \sum_{|\alpha|+|\beta|\le 2} \| x^\alpha \partial^\beta \phi \|_2
\end{aligned}$$

□

Lemma. *(Viriel identity)*

$$\frac{d^2}{dt^2} \int |x|^2 |\psi(t,x)|^2 dx = 2 \| \nabla \psi(t) \|_2^2$$

where $i\partial_t \psi + \frac{1}{2} \Delta \psi = 0$.

Proof.

$$\frac{d^2}{dt^2}\int |x|^2|\psi(t,x)|^2dx = \frac{d^2}{dt^2}\big\langle |x|^2 e^{\frac{i}{2}t\Delta}\psi_0, e^{\frac{i}{2}t\Delta}\psi_0\big\rangle$$
$$= \frac{d^2}{dt^2}\big\langle e^{\frac{-it}{2}\Delta}|x|^2 e^{\frac{i}{2}t\Delta}\psi_0,\psi_0\big\rangle$$
$$= \frac{d}{dt}\Big(-\frac{i}{2}\Big)\langle e^{-i\frac{t}{2}\Delta}[\Delta,|x|^2]\cdot e^{\frac{i}{2}t\Delta}\psi_0,\psi_0\rangle$$

(1) $$= \Big(-\frac{i}{2}\Big)^2\big\langle \big[\Delta,[\Delta,|x|^2]\big]\psi_0,\psi_0\big\rangle$$

but
$$[\Delta,|x|^2] = \Delta(|x|^2) + 2\nabla(|x|^2)\cdot\nabla$$
$$= 2d + 4\underbrace{x\cdot\nabla}_{\substack{\text{generator}\\ \text{of dilations}}}$$
$$\big[\Delta,[\Delta,|x|^2]\big] = 4[\Delta, x\cdot\nabla]$$
$$= 8\nabla(x\cdot\nabla)\cdot\nabla = 8\Delta\ .$$

$(1) = -2\langle\Delta\psi,\psi\rangle = 2 \parallel \nabla\psi \parallel_2^2$ □

Semilinear equations:

$$\begin{cases} \psi|_{t=\infty} = \psi_0 \\ i\psi_t + \frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi = 0 \end{cases}$$

Questions:

- well posedness: existence, uniqueness, continuous dependence on initial data? (H^s spaces)
- local vs. global well-posedness
- conservation laws (symmetries of the equation) (e.g., $\psi \to e^{i\beta}\psi$)
- blow up
- special solutions—solitons and stability around special solutions

Conservation laws and symmetries

1) $$\frac{d}{dt} \parallel \psi(t) \parallel_2^2 = 0$$

$$\frac{d}{dt}\langle\psi(t),\psi(t)\rangle = 2\mathrm{Re}\, i\langle -i\psi_t,\psi\rangle$$
$$= 2\mathrm{Re}\, i\Big\langle \frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi,\psi\Big\rangle$$
$$= 2\mathrm{Re}\, i\Big\langle\frac{1}{2}\Delta\psi,\psi\Big\rangle = 0 \qquad (V \text{ real})$$

2) $$H(t) = \int\Big[\frac{1}{2}\big|\nabla\psi(t,x)\big|^2 - V|\psi|^2 - F(|\psi|^2)\Big]\,dx$$

where $F'(u) = f(u)$.

Then $\frac{d}{dt}H(t) = 0$.

$$\dot H(t) = 2\mathrm{Re}\int\Big(\frac{1}{2}\nabla\psi_t\nabla\bar\psi - V\psi_t\bar\psi - f(|\psi|^2)\psi_t\bar\psi\Big)\,dx$$
$$= 2\mathrm{Re}\int \psi_t\underbrace{\Big(-\frac{1}{2}\Delta\bar\psi - V\bar\psi - f(|\psi|^2)\Big)}_{-i\bar\psi_t}\,dx = 0$$

3) $$X = \int x|\psi|^2\,dx\Big/\int|\psi|^2\,dx \qquad \text{center of mass}$$

$$NX = \int x|\psi|^2\,dx \qquad N = \text{charge}$$

$$N\dot X = 2\mathrm{Re}\int x\psi_t\bar\psi$$
$$= 2\mathrm{Re}\, i\int x\Big(\frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi\Big)\bar\psi\,dx$$
$$= 2\mathrm{Re}\, i\int x\Delta\psi\bar\psi\,dx$$
$$= \mathrm{Re}\, i\int x\,\mathrm{div}(\nabla\psi\cdot\bar\psi)\,dx$$
$$= -\mathrm{Re}\, i\int\nabla\psi\cdot\bar\psi\,dx = \langle p\psi,\psi\rangle = \vec P$$

where $p = -i\nabla$ s.a.

$N\dot X = \vec P$ .without potential we will have $\dot{\vec P} = 0$

3/4/2003

Last time we started doing some formal calculations concerning properties of solution of semilinear Schrödinger equation

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi = 0\ , \\ \psi|_{t=0} = \psi_0\ , \end{cases}$$

where V and f are real.

(1) L^2 conservation:

$$\frac{d}{dt}\int|\psi|^2 = 0\ .$$

(2) Let $\mathcal{H}(t) = \displaystyle\int\Big[\frac{1}{2}|\nabla\psi|^2 - V|\psi|^2 - F(|\psi|^2)\Big](t)\,dx$, where $F' = f$.

It holds $\dot{\mathcal{H}}(t) = 0$.

(3) $N\vec x = \displaystyle\int x|\psi|^2\,dx$.

$$N\dot{\vec X} = \int x\psi_t\psi\,dx + \int x\psi\bar\psi_t\,dx = 2\mathrm{Re}\int x\psi_t\bar\psi\,dx$$

$$
\begin{aligned}
&= 2\mathrm{Re}\, i \int x(-i\psi_t)\bar{\psi}\, dx \\
&= 2\mathrm{Re}\ i \int x\Big(\frac{1}{2}\Delta\psi + \tilde{V\psi} + \underbrace{f(|\psi|^2)\psi\Big)\ \overbrace{\bar{\psi}}^{\in\mathbb{R}}}_{\in\mathbb{R}}\ dx \\
&= \mathrm{Re}\ i \int x\Delta\psi\bar{\psi}\, dx \\
&= \mathrm{Re}\ i \int x\,\mathrm{div}(\nabla\psi\bar{\psi})\, dx \\
&- \mathrm{Re}\ i \int \nabla\psi\cdot\bar{\psi}\, dx = \mathrm{Re}\langle p\psi,\psi\rangle = \frac{\langle p\psi,\psi\rangle + \overline{\langle p\psi,\psi\rangle}}{2} \\
&\overset{p^*=p}{=} \langle p\psi,\psi\rangle =: \vec{P}\ \text{, the linear momentum.}
\end{aligned}
$$

Is $\vec{P}$ preserved?

$$
\dot{\vec{P}} = \langle p\psi_t,\psi\rangle + \langle p\psi,\psi_t\rangle = 2\mathrm{Re}\langle p\psi_t,\psi\rangle = 2\mathrm{Re}\ i\langle p(-i\psi_t),\psi\rangle
$$

$$
= 2\mathrm{Re}\ i\Big\langle \frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi, p\psi\Big\rangle
$$

$$
= \mathrm{Re}\ i\,\underbrace{\langle\Delta\psi,p\psi\rangle}_{\in\mathbb{R}} \quad +2\mathrm{Re}\ i\langle V\psi,p\psi\rangle \quad +2\mathrm{Re}\ i\langle f(|\psi|^2)\psi,p\psi\rangle
$$

$$
2\mathrm{Re}\ i\langle V\psi,p\psi\rangle = -2\mathrm{Re}\langle V\psi,\nabla\psi\rangle = -\langle V\psi,\nabla\psi\rangle - \langle\nabla\psi,V\psi\rangle = \langle(\nabla V - V\nabla)\psi,\psi\rangle
$$

$$
2\mathrm{Re}\ i\langle f(|\psi|^2)\psi,p\psi\rangle = -2\mathrm{Re}\int f(|\psi|^2)\psi\nabla\bar{\psi} = -\int f(|\psi|^2)\underbrace{(\psi\nabla\bar{\psi} + \bar{\psi}\nabla\psi)}_{\nabla|\psi|^2} = -\int\nabla\big(F(|\psi|^2)\big) = 0
$$

Hence, $N\ddot{\vec{X}} = \dot{\vec{P}} = \langle(\nabla V)\psi,\psi\rangle$.

Side remark:

If $f = 0$, then $\psi(t) = e^{itH}\psi_0$, where $H = \frac{1}{2}\Delta + V$ and

$$
\begin{aligned}
\frac{d}{dt}\langle A\psi,\psi\rangle &= \langle AiH\psi,\psi\rangle + \langle A\psi, iH\psi\rangle \\
&= i\langle[A,H]\psi,\psi\rangle\ .
\end{aligned}
$$

In particular, if $A = -i\nabla$, then $i\Big[-i\nabla, \frac{1}{2}\Delta + V\Big] = 0 + [\nabla, V] + (\nabla V)$.

(4) Angular momentum (we will be in $\mathbb{R}^3$)

$$\vec{M} = \frac{1}{2}i\int \vec{x}\times(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi})\,dx$$
$$= \frac{1}{2}\int \psi i\vec{x}\times\vec{\nabla}\psi\,dx + \frac{1}{2}\int \psi\,\overline{i\vec{x}\times\vec{\nabla}\psi}\,dx\,.$$

If we put $\vec{\rho} = i\vec{x}\times\vec{\nabla}$, then $\vec{M} = \mathrm{Re}\langle\psi,\vec{\rho}\psi\rangle$.

Clearly, $\vec{\rho}^* = \vec{\rho}$ (by integration by parts: $\int ix_k\partial_e\psi\bar{\phi} = \int\psi\overline{ix_k\,\partial_e}\ldots$). So, $\vec{M} = \langle\vec{\rho}\psi,\psi\rangle$ and we have

linear contribution — nonlinear contribution

$$\dot{\vec{M}} = i\langle[\rho,H]\psi,\psi\rangle + \qquad 2\mathrm{Re}\; i\langle f(|\psi|^2)\psi,\rho\psi\rangle\,.$$

$$\mathrm{Re}\,i\int f(|\psi|^2)\psi\overline{i(\vec{x}\times\vec{\nabla})_j\psi}\,dx$$
$$= \mathrm{Re}\int f(|\psi|^2)\psi\sum_{k,l=1}^{3}\varepsilon_{jkl}x_k\partial_l|\psi|^2\,dx$$
$$= \frac{1}{2}\int f(|\psi|^2)\sum_{k,l=1}^{3}\varepsilon_{jkl}x_k\partial_l|\psi|^2\,dx$$
$$= \frac{1}{2}\int\sum_{k,l=1}^{3}\varepsilon_{jkl}x_k\partial_l\big(F(|\psi|^2)\big)\,dx$$
$$= -\frac{1}{2}\int F(|\psi|^2)\varepsilon_{jkk}\,dx = 0$$

where ε is the Levi-Civita symbol.

$$[\rho,H]\phi = [i\vec{x}\times\vec{\nabla},\frac{1}{2}\Delta+V]\phi\,.$$

Since
$$\big[(\vec{x}\times\vec{\nabla})_j,\Delta\big]\phi = \sum_{k,l,m}\varepsilon_{jkl}\big(x_k\partial_{lmm}f - \partial_{mm}(x_k\partial_l f)\big)$$
$$= -\sum_{k,l}\varepsilon_{jkl}\partial_{kl}f \overset{\varepsilon_{jkl}+\varepsilon_{jlk}=0}{=} 0\,,$$

we have $[\rho,H]\phi = i\big((\vec{x}\times\vec{\nabla})V\big)f$ and $\dot{\vec{M}} = -\big\langle\big((\vec{x}\times\vec{\nabla})V\big)\psi,\psi\big\rangle$.

(5) Scaling

Let $V=0$ and $f(u)=u^{2\sigma}$.

If $\partial_t\psi + \frac{1}{2}\Delta\psi + |\psi|^{2\sigma}\psi = 0$, (∗)

then $\psi_\lambda(t,x) := \lambda^{\frac{1}{\sigma}}\psi(\lambda^2 t,\lambda x)$ and satisfies (∗).

$$\|\,\psi_\lambda(t)\,\|_{\mathring{H}^s_x} = \lambda^{\frac{1}{\sigma}}\lambda^s\lambda^{-\frac{d}{2}} \overset{?}{=} \lambda^0\,.$$

$$\sigma_{\text{critical}} = \frac{1}{-s+\frac{d}{2}} = \frac{2}{d-2s}\ .$$

Subcritical case ($\sigma < \sigma_{\text{critical}}$) — well understood
critical case — many open questions
supercritical case — young person gets Fields medal for solving.

(6) Galilei invariance

$$\mathfrak{g}_{\vec{v},y}(t) \qquad = e^{-1\frac{|v|^2}{2}t} \qquad e^{-i\vec{x}\cdot\vec{v}} \qquad e^{i(y+t\vec{v})}\cdot\vec{p}$$

we will see: ↑ translation in momentum space; ↑ translation in physical space

Operator $e^{i\vec{a}\cdot\vec{p}}$, where $\vec{a}\in\mathbb{R}^n$ (and $\vec{p}=-\vec{\nabla}$) is well defined and is unitary.
Let $U(s)=e^{si\vec{a}\cdot\vec{p}}$.
Then $U(0)=id$ and $\frac{d}{ds}(U|s|f)(x) = \big(U(s)i\vec{a}\cdot\vec{p}f\big)(x)$.
So, $\partial_s g - \vec{a}\cdot\vec{\nabla}g = 0$, $g(0,x)=f(x)$, where we denoted $g(s,x)=U(s)f(x)$.
It follows that $g(s,x)=f(x+s\vec{a})$, i.e., $\boxed{e^{i\vec{a}\cdot\vec{p}}f(x)=f(x+\vec{a})}$.
Another way of seeing this:

$$[e^{i\vec{a}\cdot\vec{p}}]^{\wedge}f(\xi) = e^{i\vec{a}\cdot\vec{\xi}}\hat{f}(\xi)\,,\ \text{so}\ (e^{i\vec{a}\cdot\vec{p}}f)(x)=f(x+\vec{a})$$

What does this tell us?
Take $f\in\xi$.

$$\mathfrak{g}_{\vec{v},y}(t)f = e^{-i\frac{|v|^2}{2}t}e^{-i\vec{x}\cdot\vec{v}}f(x+y+t\vec{v})\ .$$

classically: $x\mapsto x-y-t\vec{v}$.
And what happens in momentum space?

$$\begin{aligned}[\mathfrak{g}_{\vec{v},y}(t)f]^{\wedge}(\xi) &= e^{-i\frac{|v|^2}{2}t}[e^{-i\vec{x}\cdot\vec{v}}f(x+y+t\vec{v})]^{\wedge}(\xi)\\ &= e^{-i\frac{|v|^2}{2}t}[f(x+y+t\vec{v})]^{\wedge}(\xi+\vec{v})\\ &= e^{-i\frac{|v|^2}{2}t}+e^{i(y+t\vec{v})(\xi+\vec{v})}\vec{f}(\xi+\vec{v})\end{aligned}$$

so $p\mapsto p-\vec{v}$
Finally, we will prove

$$\boxed{\mathfrak{g}_{\vec{v},y}(t)e^{it\frac{\Delta}{2}} = e^{it\frac{\Delta}{2}}\mathfrak{g}_{\vec{v},y}(0)}$$

by proving

$$\begin{aligned}&(e^{-it\frac{\Delta}{2}}\mathfrak{g}_{\vec{v},y}(t)e^{it\frac{\Delta}{2}})^{\cdot} =\\ &= \dot{\mathfrak{g}}_{\vec{v},y}(t) + i\,\mathfrak{g}_{\vec{v},y}(t)\frac{\Delta}{2} - i\frac{\Delta}{2}\mathfrak{g}_{\vec{v},y}(t) = 0\ (*)\end{aligned}$$

(1)
$$\dot{\mathfrak{g}}_{\vec{v},y}(t) = -i\frac{|v|^2}{2}\mathfrak{g}_{\vec{v},y}(t) + i\,\vec{v}\cdot\vec{p}\mathfrak{g}_{\vec{v},y}(t)$$

$$-i\frac{\Delta}{2}(\mathfrak{g}_{\vec{v},y}(t)f(x) = i\frac{|v|^2}{2}\mathfrak{g}_{\vec{v},y}(t)f(x)$$
$$+ 2\frac{-i}{2}\big(-\vec{v}\cdot\vec{\nabla}\mathfrak{g}_{\vec{v},y}(t)f(x)\big)$$
$$+ \mathfrak{g}_{\vec{v},y}(t)\Delta f(x) \tag{2}$$

Summing (1) and (2) gives $(*)$.

Local well posedness in H^1 of

$$\left[\begin{array}{l} i\partial_t\psi + \frac{1}{2}\Delta\psi + F(\psi) = 0 \\ \psi(0) = \phi \in H^1(\mathbb{R}^d) \end{array}\right.$$

with $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, $F(0) = 0$, $|F^1(z)| \le C|z|^{p-1}$ if $|z| \le 1$, where $1 < p < \frac{d+2}{d-2}$ if $d \ge 3$ and $1 < p < \infty$ if $d = 1, 2$.

Theorem. *(Kato, '87)*

(1) *If $\phi \in H^1$, then there exists $T > 0$ and a unique*

$$\psi \in C\big([0,T], H^1\big) \cap C^1\big([0,T], H^{-1}\big) \text{ solving } (**)$$

Moreover, ψ depends continuously on ϕ in H^1.

$$\psi(t) = e^{i\frac{t}{2}\Delta\phi} + i\int_0^t e^{i\frac{t-s}{2}}\Delta F\big(\psi(s)\big)\,ds\,. \tag{$**$}$$

We will prove that this is meaningful.

regularity: *if $\phi \in H^2$, then $\psi \in C\big([0,T], H^2\big) \cap C^1\big([0,T], L^2\big)$*

and one has continuous dependence in H^2 norm.

Remarks:

Let $F(z) = |z|^{2\sigma}z$.

Then $\frac{\partial F}{\partial z} = \sigma(z\bar{z})^{\sigma-1}\bar{z}z + (z\bar{z})^{\sigma}$ and $\frac{\partial F}{\partial \bar{z}} = \sigma(z\bar{z})^{\sigma-1}z^2$ are continuous at $z = 0$ provided $\sigma > 0$.

What about subcriticality?

$$2\sigma = p - 1 < \frac{d+2}{d-2} \Rightarrow \sigma < \frac{1}{2}\Big(\frac{d+2}{d-2} - 1\Big) = \frac{2}{d-2}\,.$$

We will need more general Strichartz estimates:

Lemma. *(Strichartz estimates):*

Let $G_0(t) = e^{it\frac{\Delta}{2}}$ and

$$(Gv)(t) = \int_0^t e^{i(t-s)\frac{\Delta}{2}}v(s)\,ds\,.$$

Then

(*i*) $G_0 : L^2 \to L^r_t(L^q_x)$ *provided* $\frac{2}{r} + \frac{d}{q} = \frac{d}{2}$ *and* $2 < r < \infty$. *(Note: this includes $r = 0$, $g = \Omega$; $r = 2$ is also okay, but we won't prove it.)*

(*ii*) $G = L^1_t(L^q_x) \to L^r_t(L^q_x)$ *and*
$G : L^{p'}_t(L^{q'}_x) \to L^r_t(L^q_x)$.

(*iii*) $G : L_t^{r'}(L_x^{q'}) \to L_t^r(L_x^q)$.

Proof. Necessary conditions on r and q we get by scaling:

If $\psi_t(t) := \psi(\lambda^2, \lambda_x)$, where $\psi(t,x) = \big(e^{-it\frac{\Delta}{2}}\psi(0,\cdot)\big)(x)$, then $\| \psi_\lambda \|_{L_t^r(L_x^q)} = \lambda^{-\frac{2}{r}}\lambda^{-\frac{d}{q}} \| \psi \|_{L_t^r(L_x^q)}$ and $\| \psi_\lambda(0,x) \|_{L^2} = \lambda^{-\frac{d}{q}} \| \psi(0,x) \|_{L^2}$.

Hence $-\frac{2}{r} - \frac{d}{q} = -\frac{d}{2}$.

It holds

$$(G_0^* v)(x) = \int \big(e^{-i\frac{s}{2}\Delta} v(s)\big)(x)\, dx \text{ , so}$$

$$(G_0 G_0^* v)(t,x) = \int e^{i(t-s)\frac{\Delta}{2}} v(s) \text{ and}$$

$$\Big(\int \big(\| (G_0G_0^*v)(t) \|_{L_x^q}\big)^r\, dt\Big)^{\frac{1}{r}} \leq$$

$$\leq c\Big(\int\Big(\int |t-s|^{-\frac{d}{2}(\frac{1}{q'}-\frac{1}{q})} \| v(s) \|_{L_x^{q'}}\Big)^{r'}\, dt\Big)^{\frac{1}{r'}}$$

Now, if $1 + \frac{1}{r} = \frac{d}{2}\Big(\frac{1}{q'} - \frac{1}{q}\Big) + \frac{1}{r'}$, and $0 \leq \frac{d}{2}\Big(\frac{1}{q'} - \frac{1}{q}\Big)$, then apply fractional integration and complete the proof as we did when we first proved Strichartz estimates.

We have

$$(Gv, w) = \int \Big\langle \int_\delta^t e^{i(t+s)} v(s) ds, w(t) \Big\rangle\, dt$$

$$- \int \langle v(s), \chi \overbrace{\int \chi_{[0<s<t]} e^{-i(t-s)\frac{\Delta}{2}} w(t)\, ds}^{G^* r} \Big\rangle\, dt$$

$$\| \int_0^t e^{i(t-s)\frac{\Delta}{2}} v(s)\, ds \|_{L_x^2} \leq C \int_{-\infty}^{\infty} |t-s|^{-\frac{d}{2}(\frac{1}{q'}-\frac{1}{q})} \| v(s) \|_L^{q'}\, dx$$

and the claim follows. ☐

PDE (Schlag) 03/06/03

(SE)
$$i\partial_t\psi + \frac{1}{2}\Delta\psi + F(\psi) = 0$$

$$F \text{ is } C^1(\mathbb{R}^2, \mathbb{R}^2)$$

$$\frac{\partial F}{\partial z}, \frac{\partial F}{\partial \bar z} \text{ exist and are continuous}$$

$$F(0) = 0$$

$$|F'(z)| \equiv \Big|\frac{\partial F}{\partial z}(z)\Big| + \Big|\frac{\partial F}{\partial \bar z}(z)\Big| \leq M|z|^{p-1} \quad \text{for } |z| \geq 1$$

Growth condition

Range of possible p, $1 < p < \frac{d_+2}{d-2}$ $\circledast$

If $F(z) = |z|^{2\sigma}z$, then $\circledast \Leftrightarrow \sigma < \frac{2}{d-2}$

(DH)
$$\psi(t) = e^{i\frac{t}{2}\Delta}\psi_0 + i\int_0^t e^{i\frac{t-s}{2}\Delta}F\big(\psi(s)\big)\,ds$$
$$= (G_0\psi_0)(t) + iGF(\psi)$$

Lemma.

(i) $G_0 : L^2 \to L^r_t(L^q_x)$ $\quad \frac{2}{r} + \frac{d}{q} = \frac{d}{2}$ $\quad 2 < r \leq \infty$

which includes $L^2 \to L^\infty_t(L^2_x)$

(ii) $G : L^1_t(L^2_x) \to L^r_t(L^q_x)$ $\qquad \leftarrow$ *includes* $L^1_t(L^2_x) \to L^\infty_t(L^2_x)$

$L^{r'}_t(L^q_x) \to L^\infty_t(L^2_x)$

(iii) $G : L^{r'}_t(L^{q'}_x) \to L^r_t(L^q_x)$

REMARK: These are homogeneous estimates (scaling invariant) thus you can't improve the constant by localizing $0 \leq t \leq T$.

Lemma. *With F as above, one has $F = F_1 + F_2$*

where $|F_1'| \leq M \qquad |F_2(z) - F_2(w)| \leq M\big(|z|^{p-1} + |w|^{p-1}\big)|z-w|$

$$|F_1(z) - F_1(w)| \leq M|z-w|$$

Proof. $F_1 = \chi F$, $F_2(1-\chi)F$, where χ is smooth cut-off of radius ~ 1. □

Idea of proof of H^1 well-posedness.

Formally $\partial\psi(t) = G_0\partial\psi + iG\big(F'(\psi)\cdot\partial\psi\big)$

$$F(\psi) = |\psi|^{2\sigma}\psi = |\psi\bar{\psi}|^\sigma\psi$$
$$\partial_j F^2(\psi) = \sigma|\psi|^{2(r-1)}(\partial_j\psi\bar{\psi} + \phi\cdot\partial_j\bar{\psi})\psi + |\psi|^{2\sigma}\partial_j\psi$$
$$= (\sigma+1)|\psi|^{2\sigma}\partial_j\psi + \sigma|\psi|^{2(\sigma-1)}\psi\partial_j\bar{\psi}$$

$$\frac{d}{d\varepsilon}\Big|_0 F(z+\varepsilon w) = \frac{\partial F}{\partial z}(z)w + \frac{\partial F}{\partial \bar{z}}(z)w \overset{\text{def}}{=} F'(z)\cdot w$$
$$\partial\psi(t) = G\partial\psi_0 + G\big(F'(\psi)\cdot\partial\psi\big)$$

show
$$\sup_{0\leq t\leq T} \| \partial\psi(t) \|_2 \leq \| \partial\psi_0 \|_2 + \sup_{0\leq t\leq T} \| G\Big(F'(t)\cdot\partial\psi\Big)(t) \|_2$$
$$\leq \| \partial\psi_0 \|_2 + C \| F'(\psi)\cdot\partial\psi \|_{L^1_t(L^2_x)}$$

If F' were bounded, then

$$\| \partial\psi \|_{L^\infty(I,L^2_x)} \leq \| \partial\psi_0 \|_2 + C \| F' \|_\infty \cdot T \cdot \| \partial\psi \|_{L^\infty(I,L^2_x)}$$

Since F' isn't bounded, do the following

$$\| F_1'(\psi)\cdot\partial\psi \|_{L^1(I,L^2_x)} \lesssim T \| \partial\psi \|_{L^\infty(I,L^2_x)}$$
$$\| F_2'(\psi)\cdot\partial\psi \|_{L^{r'}(I,L^{q'}_x)} \lesssim \Big\| \| \psi(t) \|^{p-1}_{L^q_x} \cdot \| \partial\psi(t) \|_{L^q_x} \Big\|_{L^{r'}(I)} \quad **$$

Guess proper value of q':

$$\| \; |f|^{p-1} g \; \|_{q'} \le \left(\int |f|^{(p-1)q'} |g|^{q'} \, dx \right)^{\frac{1}{q'}}$$

$$\le \left(\int |f|^{(p-1)q's} \, dx \right)^{\frac{1}{sq'}} \left(|g|^{q's'} \, dx \right)^{\frac{1}{s'q'}}$$

Choose $$\left.\begin{array}{r} q's' = q \\ (p-1)q's = q \end{array}\right\} \to \frac{q'}{q} + \frac{(p-1)q'}{q} = \frac{1}{s} + \frac{1}{s'} = 1$$

$$\frac{pq'}{q} = \frac{p}{q-1} = 1 \Rightarrow \boxed{q = p+1} > 2$$

By Sobolev, $$(**) \lesssim \| \psi \|^{p-1}_{L^\infty(I, L^q_x)} \cdot \| \partial\psi \|_{L^{r'}(I, L^q_x)} \qquad H^1 \hookrightarrow L^q \;\; \frac{1}{2} - \frac{1}{q} = \frac{1}{d}$$

$$\lesssim \| \psi \|^{p-1}_{L^\infty(I, H^1_x)} \cdot \| \partial\psi \|_{L^{r'}(I, L^q_x)} \qquad p+1 = q < \frac{2d}{d-2} \quad p < \frac{d+2}{d-2}$$

$$\lesssim \| \psi \|^{p-1}_{L^\infty(I, H^1_x)} \cdot T^{\frac{1}{r'} - \frac{1}{r}} \| \partial\psi \|_{L^r(I, L^q_x)} \qquad \text{Note: } r' < 2 < r$$

Similarly, one obtains

$$\| \partial\psi \|_{L^\infty(I, L^2_x)} + \| \partial\psi \|_{L^r(I, L^q_x)}$$

$$\le \| \partial\psi_0 \|_{L^2} + T \| \partial\psi \|_{L^\infty(I, L^2_x)} + CT^{\frac{1}{r'} - \frac{1}{r}} \| \psi \|^{p-1}_{L^\infty(I, H^1)} \| \partial\psi \|_{L^r(I, L^q)}$$

Define $LHS = \mu$
Then

$$\mu \lesssim \| \partial\psi_0 \|_2 + T_\mu + CT^\alpha \mu^p \, ,$$

$$\Rightarrow \quad \mu(T) \le C \| \partial\psi_0 \|_2$$

assuming L^2 conservation and μ continuous provided $T < T^* = T^*_p(\| \psi_0 \|_{H^1})$.
Careful Proof We'll show the following

Theorem. *The equation $\psi = G_0\phi_0 + iGF(\phi)$ for every $\phi_0 \in H^1$ has a unique solution*

$$\phi \in C\big([0,T], H^1\big) \cap C^1\big([0,T], H^{-1}\big)$$

$$\textit{with} \quad T = T(\| \phi_0 \|_{H^1}) > 0 \, .$$

Moreover, the solution depends continuously on ϕ_0 in the H^1 norm.

Set $q = p+1, < \frac{2d}{d-2}$. $\frac{2}{r} + \frac{d}{q} = \frac{d}{2}$, so $r > 2$, (since $q < \frac{2d}{d-2}$)

$$\text{Set} \left[\begin{array}{lcl} X_0 & = & L^\infty(I, L^2 \cap L^q) \\ X & = & L^\infty(I, L^2_x) \cap L^r(I, L^q_x) \\ X' & = & L^1(I, L^2) + L^{r'}(I, L^{q'}) \\ \bar{X} & = & C(I, L^2) \cap L^r(I, L^2) \end{array} \right.$$

Lemma.

$$G_0 : L^2 \to X$$

$$G : X' \to X$$

Moreover: $$G_0 : L^2 \to \bar{X} \ and\ G : X' \to \bar{X}$$

Proof. First two statements are just Lemma 1.

$$\| e^{it\frac{\Delta}{2}} f - e^{is\frac{\Delta}{2}} f \|_2 = \| (e^{-i(t-s)\frac{|\xi|^2}{2}} - 1)\hat{f}(\xi) \|_{L^2(\xi)} \to 0$$

by Dominated Convergence theorem as $t - s \to 0$ □

Lemma. *$F : X_o \to X'$ continuously and boundedly.*

Moreover
$$\| F_1(v) - F_1(w) \|_{1,2} \lesssim T \| v - w \|_{\infty,2}$$
$$\| F_2(v) - F_2(w) \|_{r',q'} \lesssim T^\alpha (\| v \|_{X_0}^{p-1} + \| w \|_{X_0}^{p}) \| v - w \|_{r,q}$$

where $\alpha = \frac{1}{r'} - \frac{1}{r} > 0$,

On a ball $B_R(X_0)$, F is Lipschitz: $X_0 \to X_1'$ (and constant may be made small for fixed R_1 and small T).

Proof. $F = F_1 + F_2$

$$\left|F_1(v) - F_1(w)\right|(x,t) \lesssim \left|v(x,t) - w(x,t)\right|$$

$$\int_0^T \| F_1\big(v(t,x)\big) - F_1\big(w(t,x)\big) \|_{L_x^2} \ dt \lesssim \int_0^T \| v(x,t) - w(x,t) \|_{L_x^2} \ dt$$
$$\lesssim T \| v - w \|_{L^\infty(I, L_x^2)}$$

$$\left|F_2\big(v(x,t) - F_2\big(w(x,t)\big)\right| \lesssim \big(\left|v(x,t)\right|^{p-1} + \left|w(x,t)\right|^{p-1}\big)\left|v(x,t) - w(x,t)\right|$$

$$\| F_2(v) - F_2(x) \|_{L^r(I,L^q)} \lesssim \Big(\int_0^T \big(\| v(t) \|_{L_x^q}^{p-1} + \| w(t) \|_{L_x^q}^{p-1}\big)^{r'} \| v(t) - w(t) \|_{L_x^q}^{r'} \ dt\Big)^{\frac{1}{r'}}$$

Hölder $q = p + 1^{\nearrow}$

$$\lesssim \big(\| v \|_{X_0}^{p-1} + \| w \|_{X_o}^{p-1}\big) T^\alpha \| v - w \|_{L^r(I, L_x^q)}$$

□

Corollary. *$GF : X_o \to \bar{X}$, and on each ball $B_R(X_0)$, GF is a contraction in X norm for small T.*

Ma 142 03/11/04

$$I = [0, T], q = p + 1$$
$$X_0 = L^\infty(I, L_x^2 \cap L_x^q)$$
$$X = L^\infty(I, L^2) \cap L^r(I, L^q)$$
$$X = C(I, L^2) \cap L^r(I, L^q)$$
$$X' = L^1(I, L^2) + L^{r'}(I, L^{q'})$$

(SE)
$$i\partial_t\psi + \frac{1}{2}\Delta\psi + F(\psi) = 0$$
$$\psi|_{t=0} = \psi_0 \quad \in H^1$$
$$F(0) = 0,\ F \in C^1(\mathbb{R}^2, \mathbb{R}^2)\ ,$$
$$\left|\frac{\partial F}{\partial z}(z)\right| + \left|\frac{\partial F}{\partial \bar{z}}(z)\right| \leq M|z|^{p-1}, \quad |z| \geq 1$$
$$1 < p < \frac{d+2}{d-2}$$

(DH)
$$\psi = \underbrace{G_0}_{e^{i\frac{t}{2}\Delta}}\psi_0 + \underbrace{i\,GF(\psi)}_{\int_0^t e^{i(t-s)\frac{\Delta}{2}}F(\psi(s))\,ds}$$

$\rightarrow$ strong solution of SE?
$\psi \in C(I, H^1) \cap C^1(I, H^{-1})$
remark : $F : X_0 \rightarrow X'$
$C(I, H^1) \hookrightarrow X_0$ by Sobolev embedding
$\left(H^1 \hookrightarrow L^q = L^{p+1}, \frac{1}{2} - \frac{1}{q} \leq \frac{1}{d}\right)$
$X' \hookrightarrow L^1(I, H^{-1})$
$(H^1 \hookrightarrow L^2 \cap L^q, L^2 + L^{q'} \hookrightarrow H^{-1})$

Lemma. *$GF : X_0 \rightarrow \bar{X}$ continuously and boundedly. On each ball $B_R(X_0)$, GF is a contraction, with T suff. small (dep. on R). In fact, $\forall u, v \in B_{R(X_0)} : \|\ GF(u) - GF(v)\|_{\bar{X}} \leq C(T + T^\alpha R^{p-1})\|v - u\|_X$.*

Corollary. *Suppose $\psi \in X_0$ is a weak solution of (DH) (check meaningful). Then ψ is unique.*

Proof.

$$\text{Given } \psi_0 \in L^2,\ \text{suppose} \psi = G_0\psi_0 + iGF(\psi)$$
$$\tilde{\psi} = G_0\psi_0 + iGF(\tilde{\psi})$$

$$\psi - \tilde{\psi} = i\ GF(\psi) - i\ GF(\tilde{\psi})$$
$$\|\psi - \tilde{\psi}\|_X \leq \delta\|\psi - \tilde{\psi}\|_X \quad \text{by contraction}$$
$$\Rightarrow \psi = \tilde{\psi} \text{ on } [0, T_0] \quad \left(T_0 \text{ chosen so small that } \delta < \tfrac{1}{2}\right)$$

Repeat $[T_0, 2T_0]$ but:
Remark: not clear we can talk about $\psi(T_0)$ and $\tilde{\psi}(T_0)$. However, ψ and $\tilde{\psi} \in C(I, H^{-1})$ because of (DH) : $\quad G_0\psi_0 \in C(I, L^2)$ and

$$\int_0^t e^{i(t-s)\frac{\Delta}{2}}\underbrace{F\big(\psi(s)\big)\,ds}_{\in L^1(I, H^{-1})} \in C(I, H^{-1})$$

So talk about $\psi(T_0), \tilde{\psi}(T_0) \in H^{-1}$. Since

$$\begin{gathered}\psi, \tilde{\psi} \in L^\infty(I, L^2) \\ \Rightarrow \psi(T_0) \text{ and } \tilde{\psi}(T_0) \in L^2\end{gathered}$$

$$\begin{aligned}\big|\langle \psi(T_0), \phi\rangle\big| &= \lim_{t_j \to T_0} |\langle \psi(t_j), \phi \rangle| \quad \text{for some seq } t_j \\ &\le \|\psi\|_{L^\infty(I,L^2)} \cdot \|\phi\|_2\end{aligned}$$

Q: need to check that

$$\psi(T_0+h) = e^{it\frac{\Delta}{2}}\psi(T_0) + i \int_{T_0}^{T_0+h} e^{i(T_0+h-s)\frac{\Delta}{2}} F\big(\psi(s)\big)\, ds\ .$$

OK because

$$\begin{aligned}\psi(T_0+h) &= e^{i(T_0+h)\frac{\Delta}{2}}\psi(0) + i \int_0^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\frac{\Delta}{2}} F\big(\psi(s)\big)\, ds \\ &= e^{\frac{i}{2}h\Delta} e^{\frac{i}{2}T_0\Delta}\psi(0) + i e^{i\frac{h}{2}\Delta} \int_0^{T_0} e^{\frac{i}{2}(T_0-s)\Delta} F\big(\psi(s)\big)\, ds \\ &\quad + i \int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta} F\big(\psi(s)\big)\, ds \\ &= e^{\frac{i}{2}h\Delta}\psi(T_0) + i \int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta} F\big(\psi(s)\big)\, ds\end{aligned}$$

NB: T_0 depends only on $R(\psi \in X_0, \tilde{\psi} \in X_0)$
Or: extend from a point between $\frac{T_0}{2}$ and T_0 where ψ and $\tilde{\psi}$ agree.
NB: Because of R, cannot prove existence immediately.

$$\begin{aligned}Y &= \{v \in X, \partial_x v \in X\} \subseteq L^\infty(I, H^1) \\ \bar{Y} &= \{v \in \bar{X}, \partial_x v \in \bar{X}\} \subseteq C(I, H^1) \\ Y' &= \{v \in X', \partial_x v \in X'\}\end{aligned}$$

Sobolev embedding:

$$\begin{aligned}&Y \subset X_0\ , \\ &\bar{Y} \subset X,\ Y \subset X,\ Y' \subset X'.\end{aligned}$$

□

Lemma.

$$G_0 : H^1 \to \bar{Y}$$

$$G : Y' \to \bar{Y}$$

Proof.

$$v \in L^2\ ,\ \|G_0 v\|_{L^\infty(I,L^2)} = \|e^{\frac{i}{2}t\Delta} v\|_{L^\infty(I,L^2)}$$
$$= \|v\|_{L^2}$$

$$\left(\frac{2}{r} + \frac{d}{q} = \frac{d}{2}\right) \|G_0 v\|_{L^r(I,L^q_x)} \le C\|v\|_{L^2}$$

So $G_0 : L^2 \to \bar{X}$.

Since $\partial G_0 = G_0 \partial$, we see that $G_0 = H^1 \to \bar{Y}$ and $X' \to \bar{X}$ and $\partial G = G\partial$, we have $G : Y' \to \bar{Y}$. □

Lemma. *$F : Y \to Y'$ continuously and boundedly and* $\|F(v)\|_{Y'} \le C\left(T + T^\alpha \|v\|_Y^{p-1}\right)\|v\|_Y$.

Proof.

$$\|F(v)\|_{X'} \le C(T + T^\alpha \|v\|_{X_0}^{p-1})\|v\|_X$$
$$\le C\left(T + T^\alpha \|v\|_Y^{p-1}\right)\|v\|_Y\ .$$

$$(\tau_h)(x) = f(x+h) \qquad \left(\text{dunno } F \in C^1\right)$$

$$\left\|\frac{1}{|h|}(\tau_h - 1)F_1(v)\right\|_{L^1(I,L^2_x)} \le C\frac{1}{|h|}\|(\tau_h - 1)v\|_{L^1(I,L^2)}$$
$$\le CT\|\partial v\|_{L^\infty(I,L^2)} \qquad \text{(fundamental)}$$
$$|\langle F_1(v), \partial\phi\rangle| \le CT\|\partial v\|_{L^\infty(I,L^2)}$$
$$\text{means}\ \ \|\partial F_1(v)\|_{L^1(I,L^2)} \le CT\|\partial v\|_X$$

$$\|\partial F_2(v)\|_{L^{r'}(I,L^{q'}_x)} \le C\||v|^{p-1}\partial v\|_{L^{r'}(I,L^{q'}_x)}$$

$\left(\|\,|v|^p\,\|_{L^{q'}_x} \text{ ok because } pq' = q \Rightarrow p+1 = q\right)$

$$\le C\left(\int (\|v(t)\|_{L^q_x}^{p-1}\|\partial v(t)\|_{L^q_x})^{p'} dt\right)^{\frac{1}{r'}}$$
$$\le C\|v\|_{L^\infty(I,L^q)}^{p'} T^\alpha \|\partial v\|_{L^r(I,L^q_x)}$$
$$\le CT^\alpha \|v\|_Y^{p-1}\ \|\partial v\|_X$$

□

Proof. Let $\psi_0 \in H^1$. Then there exists $T = T(\|\psi_0\|_{H^1}) > 0$ so that there exists a unique $u \in \bar{Y}([0,T])$, with $u = G_0\psi + iGF(u)$. □

Proof. Let $\Phi(u) = G_0\psi_0 + iGF(u)$

$\Phi = B_R(Y) \circlearrowleft$ provided $R = C_1\|\psi_0\|_{H^1}$

where $C_1 =$ big abs. const.

$$\|\Phi(u)\|_Y \le C\|\psi_0\|_{H^1} + C\left(T + T^\alpha \|u\|_Y^{p-1}\right)\|u\|_Y$$

$$\leq \underbrace{C\|\psi_0\|_{H^1}}_{<R/2} + \underbrace{C(T+T^\alpha R^{p-1})}_{<1/2} R$$
$$\leq R$$

Let $E = \{B_R(Y), \|\cdot\|_X\}$. Control in stronger norm, contract in weaker norm.

$$\begin{aligned} \left\|\Phi(u)-\Phi(v)\right\|_X &\leq C(T+T^\alpha R^{p-1})\left\|u-v\right\|_X \\ &< \frac{1}{2}\left\|u-v\right\|_X . \end{aligned}$$

Fixed point is in $\bar{Y}$.
Is Y complete with respect to $\left\|\cdot\right\|_X$?
Suppose $\left\|u_n - u_m\right\|_X \to 0$ as $n, m \to \infty$.

$$\{u_n\} \subset B_R(Y)$$
$$u_n \xrightarrow{\|\cdot\|_X} u$$
$$\begin{aligned} \langle \partial u_n(t), \phi(t)\rangle = &-\langle u_n(t), \partial\phi(t)\rangle \\ &\qquad\downarrow \\ &-\langle u(t), \partial\phi(t)\rangle \end{aligned}$$

has estimate uniform on n, not involving $\partial\phi$. □

Lemma. *Let* $u = u_{\psi_0}$ *be the solution from the previous proposition. Then* $\psi_0 \to u_{\psi_0}$ *is a continuous map from* $H^1 \to \bar{Y}$ *in the following sense. Given* $\psi_0 \in H^1$*, there exists* $T = T(\psi_0) > 0$ *so that for every* $\varepsilon > 0, \exists\, \delta > 0$*, if* $\psi_1 \in H^1, \left\|\psi_0 - \psi_1\right\|_{H^1} < \delta \Rightarrow u = u_{\psi_o}$ and $v = v_{\psi_1}$ *exists in* $\bar{Y}(0,T)$ *and* $\left\|u-v\right\|_Y < \varepsilon$.

Proof. It is clear that there exists $T = T(\psi_0)$,
$\Phi(w) = \Phi_{\psi_0}(w) = G_0\psi_0 + iGF(w)$
$u = \lim_{n\to\infty} \Phi^n_{\psi_0}$ (iterate)

$$\|u - \Phi^n_{\psi_0}(\psi_0)\|_X < \delta^n \qquad \text{(not the same } \delta \text{ as above)}$$
$$\|v - \Phi^n_{\psi_1}(\psi_1)\|_X < \delta^n$$
$$\left\|u-v\right\|_X \leq 2\delta^n + \|\Phi^n_{\psi_0}(\psi) - \Phi^n_{\psi_1}(\psi_1)\|_X$$

Inductively:

$$\begin{aligned} \|\Phi_{\psi_0}(\psi_0) - \Phi_{\psi_1}(\psi_1)\|_X &\leq \left\|G(F\psi_0 - F\psi_1)\right\|_X + \|G_0(\psi_0-\psi_1)\|_X \\ &\leq C\|\psi_0-\psi_1\|_2 + C(T+T^\alpha R^{p-1})\|\psi_0-\psi_1\|_X \\ &\leq C\|\psi_0-\psi_1\|_2 + C\|\psi_0-\psi_1\|_{H^1} \\ \|\Phi^2_{\psi_0}(\psi_0) - \Phi^2_{\psi_1}(\psi_1)\|_X &\leq C\|\psi_0-\psi_1\|_2 + C(T+T^\alpha R^{p-1})\|\Phi_{\psi_0}(\psi_0) - \Phi_{\psi_1}(\psi_1)\|_X \\ &\leq \underset{\substack{\downarrow \\ \text{does not grow for } n \\ \text{(but does not matter anyway)}}}{\mathrm{C}} \|\psi_0-\psi_1\|_2 + \underset{\substack{\downarrow \\ C_1 = C_0 + \delta C_1}}{\mathrm{C_1}} \|\psi_0-\psi_1\|_2 . \end{aligned}$$

□

Ma 142 03/11/2003

$$\begin{aligned} \text{Recall} \quad I &= [0,T] \\ X_0 &= L^\infty(I; L^q_x \cap L^2_x) \\ X &= L^\infty(I; L^2) \cap L^r(I; L^q) \\ X &= C(I; L^2) \cap L^r(I; L^q) \\ X' &= L^1(I; L^2) + L^{r'}(I; L^{q'}) \end{aligned}$$

(SE)
$$\left[\begin{array}{c} i\partial_t \psi + \frac{1}{2}\Delta\psi + F(\psi) = 0 \\ \psi|_{t=0} = \psi_0 \in H^1 \\ F(0) = 0,\ F \in C^1(\mathbb{R}^2; \mathbb{R}^2), \\ \left|\frac{\partial F}{\partial z}(z)\right| + \left|\frac{\partial F}{\partial z}(z)\right| \le M|z|^{p-1}, \quad |z| \ge 1 \\ 1 < p < \frac{d+2}{d-2} \end{array} \right.$$

(DH)
$$\left[\begin{array}{l} \psi = \underbrace{G_0\psi_0}_{e^{i\frac{t}{2}\Delta}\psi_0} + \underbrace{iGF(\psi)}_{\int_0^t e^{(t-s)\frac{\Delta}{2}} F(\psi(s))ds} \end{array} \right.$$

We're trying to show that there exists a unique strong solution of (SE)

$$\psi \in C(I, H^1) \cap C^1(I; H^{-1})$$

Remark: $F : X_0 \to X'$
$C(I; H^1) \hookrightarrow X_0$ by Sobolev embedding

$$\left(H^1 \hookrightarrow L^q = L^{p+1} ; \frac{1}{2} - \frac{1}{q} \le \frac{1}{d} \right)$$

$$X' \hookrightarrow L^1 \left(I; H^{-1} \right)$$

$$\left(H^1 \hookrightarrow L^2 \cap L^q, \overset{\text{duality}}{\Longrightarrow} L^2 + L^{q'} \hookrightarrow H^{-1} \right)$$

Last time we did:

Lemma. *$GF : X_0 \to \bar{X}$ continuous and bounded and on each ball $B_R(X_0), GF$ is a contraction with T sufficiently small (depending on R):*

$$\|GF(u) - GF(v)\|_{\bar{X}} \le C(T + T^\alpha R^{p-1})\|u - v\|_X$$
$$\forall u, v \in B_R(X_0)$$

Corollary. *Suppose $\psi \in X_0$ is a weak solution of (DH). Then ψ is unique.*

Proof. Given $\psi_0 \in L^2$ suppose

$$\begin{aligned}\psi &= G_0\psi_0 + i\,GF\psi\\ \tilde\psi &= G_0\psi_0 + i\,GF\tilde\psi\\ \Rightarrow \psi - \tilde\psi &= iG\big(F(\psi) - F(\tilde\psi)\big)\\ \Rightarrow \|\psi-\tilde\psi\|_X &\le \delta\|\psi-\tilde\psi\|_X \ ; \text{ provided } T \text{ is small}\end{aligned}$$

$\Longrightarrow \psi = \tilde\psi$ on $[0,T_0]$ (with T_0 so that $\delta < \frac{1}{2}$)

To extend to any existence interval, repeat. Careful: It is not clear we can talk about $\psi(T_0)$ or $\tilde\psi(T_0)$. However $\psi, \tilde\psi \in C(I;H^{-1})$ since clearly $G_0\psi_0 \in C(I;L^2)$ and

$$\int_0^t e^{i(t-s)\frac{\Delta}{2}} \underbrace{F(\psi(s))}_{\in L^1(I;H^{-1})}\ ds \in C(I;H^{-1})$$

So we can talk about $\psi(T_0), \tilde\psi(T_0) \in H^{-1}$. But since $\psi,\tilde\psi \in L^\infty(I;L^2) \xrightarrow{?} \psi(T_0), \tilde\psi(T_0) \in L^2$!

$$|\langle \psi(T_0),\phi\rangle| \;=\; \lim_{t_j\to T_0} |\langle \psi(t_j),\phi\rangle| \le \|\psi\|_{L^\infty(I;L^2)}\|\phi\|_2$$

Question:

$$\psi(T_0+h) = e^{i\frac{h}{2}\Delta}\psi(T_o) + i\int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta}F\big(\psi(s)\big)\ ds$$

$$\begin{aligned}\psi(T_0+h) &= e^{i\frac{T_0+h}{2}\Delta}\psi(0) + i\int_0^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta}F\big(\psi(s)\big)\ ds\\ &= e^{\frac{i}{2}h\Delta}e^{\frac{i}{2}T_0\Delta}\psi(0) + e^{i\frac{h}{2}\Delta} i\int_0^{T_0} e^{\frac{i}{2}(T_0-s)\Delta}F\big(\psi(s)\big)\ ds\\ &\quad + i\int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta}F\big(\psi(s)\big)\ ds\\ &= e^{\frac{i}{2}h\Delta}\psi(T_0) + i\int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta}F\big(\psi(s)\big)\ ds\ .\end{aligned}$$

□

New spaces:

$$\begin{aligned}Y &= \{v\in X, \partial_x v \in X\}\\ \bar Y &= \big\{v\in \bar X, \partial_x v\in \bar X\big\}\\ Y' &= \{v \in X', \partial_x v\in X'\}\end{aligned}$$

Comments

$$Y \subset L^\infty(I; H^1)$$
$$\bar{Y} \subset C(I; H^1)$$

Note

$$\begin{aligned} &Y \subset X_0 \qquad \text{(by Sobolev embedding)} \\ &\bar{Y} \subset \bar{X} \\ &Y \subset X \\ &Y' \subset X' \end{aligned}$$

Lemma.

$$\begin{cases} G_0 : & H^1 & \to & \bar{Y} \\ G : & Y^1 & \to & \bar{Y} \end{cases}$$

Proof.

$$v \in L^2 \Longrightarrow \|G_0 v\|_{L^\infty(I,L^2)} = \|e^{i\frac{t}{2}\Delta} v\|_{L^\infty(I;L^2)} = \|v\|_{L^2}$$

$$\|G_0 v\|_{L^r(I;L^q_x)} \leq C\|v\|_{L^2}$$

$$\left(\frac{2}{r} + \frac{d}{q} = \frac{d}{2}\right)$$

So
$$\left.\begin{array}{l} G_0 : L^2 \longrightarrow \bar{X} \\ [G_0, \partial_x] = 0 \end{array}\right\} \Rightarrow G_0 : H' \longrightarrow \bar{Y}$$

Since
$$\left.\begin{array}{l} G : X' \longrightarrow \bar{X} \\ [G, \partial_x] = 0 \end{array}\right\} \Rightarrow G : Y' \longrightarrow \bar{Y}$$

□

Lemma. *$F : Y \longrightarrow Y'$ continuously and boundedly and*

$$\|F(v)\|_{Y'} \leq C\left(T + T^\alpha \|v\|_Y^{p-1}\right)\|v\|_Y$$

Proof.

$$\begin{aligned} \|F(v)\|_{X'} &\leq C\left(T + T^\alpha \|v\|_{X_0}^{p-1}\right)\|v\|_X \\ &\leq C\left(T + T^\alpha \|v\|_Y^{p-1}\right)\|v\|_Y \\ &\uparrow \\ &\text{Sobolev embedding} \end{aligned}$$

Let $(\tau_h f)(x) = f(x+h)$

$$\begin{aligned} \left\|\frac{1}{h}(\tau_h - 1)F_1(v)\right\|_{L^1(I;L^2_x)} &\leq C\frac{1}{|h|}\left\|(\tau_h - 1)v\right\|_{L^1(I;L^2)} \\ &\leq CT\|\partial_x v\|_{L^\infty(I;L^2)} \\ \Rightarrow \left|\langle F_1(v), \partial\phi\rangle\right| &\leq CT\|\partial v\|_{L^\infty(I;L^2)} \end{aligned}$$

Thus
$$\|\partial F_1(v)\|_{L^1(I;L^2)} \leq CT\|\partial v\|_X$$

$$
\begin{aligned}
\left\|\partial F_2(v)\right\|_{L^{r'}(I;L_x^{q'})} &\le C\left\| \, |v|^{p-1}\partial v\right\|_{L^{r'}(I;L_x^{q'})} \\
&\le C\left(\int \left(\left\|v(t)\right\|_{L_x^q}^{p-1} \left\|\partial_x v(t)\right\|_{L_x^q}\right)^{r'} dt\right)^{\frac{1}{r'}} \\
&\le C\left\|v\right\|_{L^\infty(I;L^q)}^{p-1} \, T^\alpha \left\|\partial_x v\right\|_{L^r(I;L_x^q)} \\
&\le CT^\alpha \left\|v\right\|_Y^{p-1}\left\|\partial v\right\|_X .
\end{aligned}
$$

□

Proposition Let $\psi_0 \in H^1$. Then there exists $T = T\left(\|\psi_0\|_{H^1}\right) > 0$ so that there is a unique $u \in \bar{Y}\left([0,T]\right)$ with $u = G_0\psi_0 + iGF_u$.

Proof. Let $\Phi(u) = G_0\psi_0 + iGF(u)$.

Then $\Phi : B_R(Y) \circlearrowleft$ provided $R = C_1\|\psi_0\|_{H^1}, C_1 \gg 1$, absolute constant.

$$
\begin{aligned}
\left\|\Phi(u)\right\|_Y &\le C\left\|\psi_0\right\|_{H^1} + C\left(T + T^\alpha \left\|u\right\|_Y^{p-1}\right)\left\|u\right\|_Y \\
&\le C\left\|\psi_0\right\|_{H^1} + \underbrace{C\left(T + T^\alpha R^{p-1}\right)}_{<\frac{1}{2}} R \le R
\end{aligned}
$$

for $R = 2C\left\|\psi_0\right\|_{H^1}$ and $T = T(R) \ll 1$.

Let $E = \left\{B_R(Y), \left\|\cdot\right\|_X\right\}$.

$$
\left\|\Phi(u) - \Phi(v)\right\|_X \le C\left(T + T^\alpha R^{p-1}\right)\left\|u - v\right\|_X < \frac{1}{2}\left\|u - v\right\|_X .
$$

Suppose $\left\|u_n - u_m\right\|_X \underset{n,m\to\infty}{\longrightarrow}$ and $(u_n) \subset B_R(Y)$

$$
\begin{aligned}
&\Rightarrow u_n \to u \text{ in } \left\|\cdot\right\|_X \\
\langle \partial u_n(t), \phi(t)\rangle &= -\langle u_n(t), \partial\phi(t)\rangle \overset{n\to\infty}{\longrightarrow} -\langle u(t), \partial\phi(t)\rangle \\
&\Rightarrow u \ \in B_R(Y)
\end{aligned}
$$

Thus E is complete.

So by contraction argument we are done. □

Lemma. *Let $u = u_{\psi_0}$ be the solution from the previous Proposition. Then $H^1 \ni \psi_0 \mapsto u_{\psi_0} \in \bar{Y}$ is continuous in the following sense:*

Given $\psi_0 \in H^1$, there exists $T = T(\psi_0) > 0$ so that $\forall\varepsilon > 0, \exists\,\delta > 0$ such that if $\psi_1 \in H^1, \left\|\psi_0 - \psi_1\right\|_{H^1} \le \delta \Rightarrow u = u_{\psi_0}$ and $v = v_{\psi_1}$ exists in $\bar{Y}([0,T])$ and $\left\|u - v\right\|_Y \le \varepsilon$.

Proof. It is clear that we have $T = T\left(\|\psi_0\|\right)$.

$$
\begin{aligned}
\Phi(w) = \Phi_{\psi_0}(w) &= G_0\psi_0 + i\; GFw \\
\Rightarrow u &= \lim_{n\to\infty} \Phi_{\psi_0}^n\ (\psi_0)
\end{aligned}
$$

$$
\begin{aligned}
&\left\|u - \Phi_{\psi_0}^n(\psi_0)\right\|_X < \delta^n \\
&\left\|v - \Phi_{\psi_1}^n(\psi_1)\right\|_X < \delta^n \\
&\quad \Rightarrow \left\|u - v\right\|_X \le 2\delta^n + \left\|\Phi_{\psi_0}^n\ (\psi_0) - \Phi_{\psi_1}^n\ (\psi_1)\right\|_X
\end{aligned}
$$

Induction:

$$\begin{aligned}\left\|\Phi_{\psi_0}(\psi_0)-\Phi_{\psi_1}\ (\psi_1)\right\|_X &\le \left\|GF(\psi_0)-GF(\psi_1)\right\|_X+\left\|G_0\left(\psi_0-\Psi_1\right)\right\|_X\\ &\le C\left\|\psi_0-\psi_1\right\|_{L^2}+C\left(T+T^\alpha R^{p-1}\right)\left\|\psi_0-\psi_1\right\|_X\\ &\le C\left\|\psi_0-\psi_1\right\|_{L^2}+C\left\|\psi_0-\psi_1\right\|_{H^1}\end{aligned}$$

because $\Phi_\psi(\psi)\in B_R(Y)$

$$\begin{aligned}\left\|\Phi^2_{\psi_0}(\psi_0)-\Phi^2_{\psi_1}\ (\psi_1)\right\|_X &\le C\left\|\psi_0-\psi_1\right\|_2+C\left(T+T^\alpha R^{p-1}\right)\left\|\Phi_{\psi_0}(\psi_0)-\Phi_{\psi_1}\ (\psi_1)\right\|_X\\ &\le \dots\end{aligned}$$

$\Rightarrow$ (by induction)
$\left\|u-v\right\|_X < \varepsilon.$
(To be continued) □

Ma 142 3/13/2003

Last time: we showed that for every $\psi_0 \in H^1$

(DH) $$u = G_0\psi_0 + i\, GF(u)$$

has a unique solution

$$u \in \bar{Y} = \{v \in \bar{Y} : \partial v \in \bar{X}\} \subset C(I, H^1)$$
$$I = [0,T], \qquad T = T(\|\psi_0\|_{H^1})$$

• continuity in the following sense:
if $\psi_n \overset{H^1}{\longrightarrow} \psi_0$ then for some $T = T(\|\psi_0\|_{H^1}) > 0$ one has $\|u_n - u\|_X \longrightarrow 0$ where u_n, u are solutions of (DH) with ψ_0, ψ respectively.

Further observation: can take X_0 instead of X

$$\|u_n(t) - u(t)\|_{L^q(R^d)} \;\le\; C\, \|u_n(t) - u(t)\|_{L^2}^{\theta} \;\; \|\partial u_n(t) - \partial u(t)\|_{L^2}^{1-\theta}$$

for some $0 < \theta < 1$ provided $q < \frac{2d}{d-2}$

$$\text{Also :} \qquad \begin{cases} \|u_n(t) - u(t)\|_{L^2}^{\theta} \longrightarrow 0 \\ \|\partial u_n(t) - \partial u(t)\|_{L^2}^{1-\theta} \le \|\partial u\|_{L^\infty(I,L^2)}^{1-\theta} + \|\partial u_n\|_{L^\infty(I,L^2)}^{1-\theta} \end{cases}$$

Thus $\|u_n - u\|_{L^\infty(I;L^q)} \mapsto 0$
For future reference note that $\|u_n - u\|_{L^\infty(I,L^{q+\varepsilon})} \longrightarrow 0$ for small $\varepsilon > 0$.
Claim: $\|u_n - u\|_Y \longrightarrow 0$ as $n \longrightarrow \infty$

$$\begin{aligned} \partial u - \partial u_n &= G_0(\partial\psi_0 - \partial\psi_n) + iG\left(F'(u)\partial u - F'(u_n)\partial u_n\right) \\ &= G_0(\partial\psi_0 - \partial\psi_n) + iG\Big[F'(u_n)(\partial u - \partial u_n)\Big] + iG\Big[\left(F'(u) - F'(u_n)\right)\partial u\Big] \end{aligned}$$

$$\begin{aligned} \|G_0(\partial\psi_0 - \partial\psi_n)\|_X &\le C\|\partial\psi_0 - \partial\psi_n\|_2 \longrightarrow 0 \\ \|F'(u_n)(\partial u - \partial u_n)\|_{X^1} &\le \|F_1'(u_n)(\partial u - \partial u_n)\|_{L^1(I;L^2)} \\ &\quad + \|F_2'(u_n)(\partial u - \partial u_n)\|_{L^{r'}(I;L_x^{q'})} = A + B \end{aligned}$$

Repeat the contraction argument:

$$\left.\begin{aligned} A &\le CT\|\partial u - \partial u_n\|_{L^\infty(I,L^2)} \\ \text{Also, for } B : B &\le \; CT^\infty\|u_n\|_{X_0}^{p-1}\|\partial u - \partial u_n\|_{L^r\left(I,L_x^q\right)} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \; A + B \le \delta\, \|\partial u - \partial u_n\|_X$$

$$\circledast \qquad \left\|\left(F_1'(u) - F_1'(u_n)\right)\partial u\right\|_{L^1(I,L^q)} \overset{\text{claim}}{\mapsto} 0$$
$$\circledast\circledast \qquad \left\|\left(F_2'(u) - F_2'(u_n)\right)\partial u\right\|_{L^{r'}(I;L^{q'})} \mapsto 0$$

For $\circledast$,

$$\left|\left(F_1'(u)-F_1'(u_n)\right)(t,x)\cdot\partial u(t,x)\right|\le C\left|\partial u(t,x)\right|\in L^\infty(I;L^2)$$

It suffices to show that:

$$\left\|\left(F_1'(u)-F_1'(u_n)\right)\partial u(t)\right\|_{L^2_x}\to 0$$

for a.e. $t\in I$.

Assume that

$$u_n(t,\cdot)\longrightarrow u(t,\cdot)\text{ a.e. }\Rightarrow \underbrace{(F_1'(u_n))-F_1'(u)(t)\cdot\partial u(t)}_{\substack{\downarrow\\ \text{a.e.}\\ 0}}$$

And it has a positive L^2-majorant.

Fix "a.e." t. Then $u_{n_j}(t,\cdot)\to u(t,\cdot)$ a.e. Therefore:

$$\left\|\left(F(u_{n_j})-F(u)\right)(t)(\partial u)(t)\right\|_2\longrightarrow 0.$$

$$u_n(t)\overset{L^{q+\varepsilon},L^q}{\longrightarrow}u(t)$$

Given ε, there exists $\rho(t)>0$ s.t.

$$\sup_n\left[\left\|u_n(t)\right\|_{L^q\left({}^cB_{\rho(t)}\right)}+\left\|u(t)\right\|_{L^q\left({}^cB_{\rho(t)}\right)}\right]<\varepsilon$$

$$\begin{aligned}\left\|\left(F_2'(u_n)-F_2'(u)\right)(t)\partial u(t)\right\|_{L^{q'}\left({}^cB_{\rho(t)}\right)}\le\ & C\Big(\left\|u_n(t)\right\|^{p-1}_{L^q({}^cB_{\rho(t)})}\\ &+\left\|u(t)\right\|^{p-1}_{L^q({}^cB_{\rho(t)})}\Big)\left\|\partial u(t)\right\|_{L^q}<c\varepsilon^{p-1}\left\|\partial u(t)\right\|_{L^q}\qquad\circledast\circledast\circledast\end{aligned}$$

It suffices to show that

$$\left\|\left(F_2'(u_n)-F_2'(u)\right)(t)\partial u(t)\right\|_{L^{q'}_x}\overset{a.e.t}{\longrightarrow}0$$

$$\begin{aligned}\text{Since}\quad lhs&\le C\left(\left\|u_n(t)\right\|_q^{p-1}+\left\|u(t)\right\|_q^{p-1}\right)\ \left\|\partial u(t)\right\|_{L^q}\\ &\le C\ R^{p-1}\left\|\partial u(t)\right\|_{L^q}\in L^r(I)\hookrightarrow L^{r'}(I)\ .\end{aligned}$$

By $\circledast\circledast\circledast$ it suffices to show that

$$\int\limits_{|x|<\rho(t)}\left|\Big(F_2'\big(u_n(t,x)\big)-F_2'\big(u(t,x)\big)\Big)\partial u(t,x)\right|^{q'}dx\mapsto 0$$

$$S_{\delta_1,\delta_2}=\left\{|x|<\rho(t)\ :\ |u_n(t,x)-u(t,x)|<\delta_1\ ,\ |u_n(t,x)|<\frac{1}{\delta_2},|u(t,x)|<\frac{1}{\delta_2}\right\}$$

$$\left|B_{\rho(t)}\setminus S_\delta\right|\le\frac{1}{\delta_1^q}\ \left\|u_n(t)-u(t)\right\|_q^q+\delta_2^q\left(\left\|u_n(t)\right\|_q^q+\left\|u_n(t)\right\|_q^q\right)$$

So

$$\int\limits_{|X|< \ \rho(t)} \left|\left[F_2'\big(u_n(t,x)\big) - F_2'\big(u_n(t,x)\big)\right]\partial u(t,x)\right|^{q'} \ dx$$

$$\le \int\limits_{S_{\delta_1,\delta_2}} \varepsilon^{q'}\left|\partial u(t,x)\right|^{q'} \ dx + \underbrace{\Bigg(\int\limits_{B_{\rho(t)}\setminus S_{\delta_1,\delta_2}} \left|\partial u(t,x)\right|^q \ dx\Bigg)}_{<\varepsilon} \underbrace{\left(\left\|u_n(t)\right\|_q^{q(p-1)} + \left\|u(t)\right\|_q^{q(p-1)}\right)}_{<C \text{ unif. in } n} .$$

Therefore

$$\left\|\partial(u_n-u)\right\|_X \le C\left\|\psi_n-\psi_0\right\|_{H^1} + \delta\left\|\partial(u_n-u)\right\|_X + 0(1) \ \text{ as } n\to\infty \ .$$

Conclusion: $\left\|u_n-u\right\|_y \mapsto 0$.

Lemma. $\psi\in C(I,H^1)\Rightarrow F(\psi)\in C(I,H^1)$

Proof.

$$\left\|F_1\big(\psi(t)\big)-F_1\big(\psi(s)\big)\right\|_2 \le \ C\left\|\psi(t)-\psi(s)\right\|_2 \longrightarrow 0 \text{ as } s\longrightarrow t$$

$$\left\|F_2\big(\psi(t)\big)-F_2\big(\psi(s)\big)\right\|_{L^{q'}} \le \ C\Big(\left\|\psi(t)\right\|_q^{p-1} + \|\psi(s)\|_q^{p-1}\Big) \ \underbrace{\left\|\psi(t)-\psi(s)\right\|}_{\substack{\to 0 \text{ as } s\to t\\ \text{by Sobolev embedding}}}$$

We have shown $F(\psi)\in C\left(I;\ L^2+L^{q'}\right)\hookrightarrow L\left(I,H^{-1}\right)$ by duality (since $H^1\hookrightarrow L^2+L^q$, $L^2+L^{q'}\hookrightarrow H^{-1}$) ☐

Corollary. *Let ψ be the unique local solution of $\psi=G_0\psi_0+iGF(\psi)\phi_0\in H^1$ which is given by the previous construction.*

Then $\partial_t\psi\in C\left(I;H^{-1}\right)$ and $i\partial_t\psi+\frac{1}{2}\Delta\psi+F(\psi)=0$ pointwise in $t\in I$ as an identity in H^{-1}.

Proof.

$$\psi(t)=e^{i\frac{t}{2}\Delta}\psi_0+i\int\limits_0^t e^{i\,(t-s)\frac{\Delta}{2}}F\big(\psi(s)\big)ds \ .$$

$$\frac{1}{h}\big(\psi(t+h)-\psi(t)\big)=\frac{1}{h}(e^{i\frac{h}{2}\Delta}-I)e^{it\frac{\Delta}{2}}\psi_0+i\frac{1}{h}(e^{ih\frac{\Delta}{2}}-I)$$

$$\cdot\underbrace{\int\limits_0^t e^{i(t-s)\frac{\Delta}{2}}\,F(\psi(s))ds}_{GF(\psi)\in H^1}+i\,\frac{1}{h}\int\limits_t^{t+h} e^{i(t+h-s)\frac{\Delta}{2}}\,F\big(\psi(s)\big)\,ds+i\,\frac{1}{h}$$

$$\cdot\int\limits_t^{t+h} e^{i\,(t+h-s)\frac{\Delta}{2}}[F\big(\psi(s)\big)-F\big(\psi(t)\big)]\,ds$$

We prove

$$\left\|\frac{1}{h}\big(e^{i\,\frac{h}{2}\Delta}-I\big)f-\frac{i}{2}\,\Delta f\right\|_{H^1} \longrightarrow 0 \text{ as } h\to l \text{ for every } f\in H^1$$

because

$$\int \underbrace{\left|\left[\frac{1}{h}\left(e^{-i\frac{h}{2}\xi^2}-1\right)+\frac{i}{2}\xi^2\right]\right|^2}_{\leq C|\xi|^4 \text{ unif. in } h\ .} |\hat{f}(\xi)|^2 \ \left(1+\left|\xi\right|^2\right)\ d\xi \overset{\text{LDCT}}{\longrightarrow} 0$$

$$i\frac{1}{h}\int\limits_t^{t+h} e^{i(t+h-s)\frac{\Delta}{2}}\left[F\big(\psi(s)\big)-F\big(\psi(t)\big)\right]ds \longrightarrow 0 \text{ in } H^{-1}\ ,$$

because

$$\int \left|\frac{1}{h}\int\limits_0^h (e^{-ih\frac{\xi^2}{2}}\ du-1)\right|^2 (1+|\xi|^2)^{-1}\left|\hat{f}(\xi)\right|^2 d\xi \overset{\text{LDCT}}{\longrightarrow}\ 0$$

□

Corollary. *Suppose* $\psi_0 \in H^1$, $\phi \in C(I;H^1)\cap C^1\,(I;H^1)$

$$\begin{cases} i\partial_t\psi+\frac{1}{2}\,\Delta\,\psi+F(\psi)=0 \\ \psi\big|_{t=0}=\psi_0 \end{cases}$$

Then $\psi_0 = G_0\psi_0 + i\ GF(\psi)$.

Proof.

$$i\partial_t\big(e^{-it\frac{\Delta}{2}}\psi(t)\big) = ie^{-it\frac{\Delta}{2}}\ F\big(\psi(t)\big)$$

$$i\partial_t\big(e^{-it\frac{\Delta}{2}}\psi(t)\big) = i\left(-i\ \frac{\Delta}{2}\ e^{-it\frac{\Delta}{2}}\psi(t)+e^{-it\frac{\Delta}{2}}\partial_t\ \psi(t)\right)$$

$$= -e^{-it\frac{\Delta}{2}}\ F\big(\psi(t)\big)\ .$$

$$ie^{-it\frac{\Delta}{2}}\psi(t)-i\psi(0) = -\int\limits_0^t e^{-i\frac{s\Delta}{2}}F\big(\psi(s)\big)\ ds$$

$$\psi(t) = e^{it\frac{\Delta}{2}}\psi(0)+i\int\limits_0^t e^{i(t-s)\frac{\Delta}{2}}F\big(\psi(s)\big)\ ds$$

□

Salem-Salem '99

Ma 142 04/01

Theorem.

1) *Let* $0 < \sigma < \frac{2}{d-2}$. *For every* $\psi_0 \in H^1$
   $\exists\, T_* > 0$ *(max. time of existence),* $T_* = T_*(\|\psi_0\|_{H^1})$ *so that* $\exists!\psi \in C([0,T_*),H^1) \cap C^1([0,T_*),H^{-1})$,

$$i\partial_t\psi + \frac{1}{2}\Delta + \lambda\big|\psi\big|^{2\sigma}\psi = 0 \qquad \text{(pointwise in } H^{-1}) .$$

   Moreover, ψ *depends continuously on* ψ_0 *(w.r.t.* H^{-1}*).*

2) *If* $\psi_0 \in H^2$, *then* $\psi \in C([0,T_*),H^2) \cap C^1([0,T_*),L^2)$.

Remark

1) If $T_* < \infty$, then $\lim\limits_{t\to T_*} \big\|\psi(t)\big\|_{H^1} = \infty$.
   (because if $\lim\limits_{t\to T_*} \big\|\psi(t)\big\|_{H^1} < \infty$, then take $t_0 < T_*$ so close to T_* that you can continue beyond T_*).
2) You can include a potential term $V(t,x)\psi$ in part 1) of the theorem, provided

$$\sup_t \Big[\big\|V(t,x)\big\|_{L^\infty_x + L^{d/2}_x} + \big\|\nabla_x V(t,x)\big\|_{L^\infty_x + L^{d/2}}\Big] < \infty .$$

Back to contraction argument for Kato's theorem.

$$\tilde{F}(t,x,u) = V(t,x)u + \underbrace{F(u)}_{\lambda|u|^{2\sigma}u}$$

we needed

$$F : X_0 \longrightarrow X'$$
$$\tilde{F} : Y \;\longrightarrow Y'$$

$$X_0 = L^\infty(I;L^2 \cap L^g) \;\; q = 2\sigma + 2 < \tfrac{2d}{d-2}$$
$$X' = L^1(I;L^2) + L^{r'}(I,L^{q'})$$

Put $F = 0$ and $V(t,x) = \underset{\substack{\downarrow \\ L^\infty}}{\mathrm{V}_1}(t,x) + \underset{\substack{\downarrow \\ L^{d/2}}}{\mathrm{V}_2}(t,x)$

$$\big\|V_1(t,x)\psi\big\|_{L^1(I,L^2)} \le CT\big\|\psi\big\|_{L^\infty(I,L^2)}$$

$$\big\|V_2(t,x)\psi\big\|_{L^{r'}(I,L^2)} = \left(\int_0^T \big\|V_2(t,\cdot)\psi\big\|^{r'}_{L^{q'}}\,dt\right)^{1/r'}$$

$$\le \left(\int_0^T \big\|V_2(t,\cdot)\big\|^{r'}_{q'\left(\frac{q}{q'}\right)'}\big\|\psi(t)\big\|^{r'}_q\,dt\right)^{1/r'}$$

$$\le \big\|V_2\big\|_{L^\infty\left(\mathbb{R},L^{q'\left(\frac{q}{q'}\right)'}\right)} \underset{\substack{\downarrow \\ \text{gain} \\ \text{from} \\ r \text{ to } r'}}{\mathrm{T}^\alpha} \big\|\psi\big\|_{L^r(I,L^q)}$$

$$m = \frac{q}{q-1}\ \frac{q-1}{q-2} = \frac{q}{q-2} \quad \Rightarrow \quad \frac{1}{m} = 1 - \frac{2}{q}\ <\ 1 - \frac{d-2}{d} = \frac{2}{d}$$

$$\infty > m > \frac{d}{2} \qquad \text{so ok.}$$

Conserved quantities. Let ψ_0, ψ be as in the theorem.
Then

(i) $$\|\psi(t)\|_2 = \|\psi_0\|_2$$

(ii) $$\mathcal{H}[\psi(t)] = \int \left\{ \frac{1}{2}|\nabla\psi(t,x)|^2 - \frac{\lambda}{\sigma+1}(\psi)^{2\sigma+2}(t,x) \right\} d = \mathcal{H}[\psi_0]\ .$$

Proof.

(i) $$\langle i\partial_t\psi, \psi\rangle + \frac{1}{2}\underbrace{\langle\Delta\psi,\psi\rangle + \lambda\langle|\psi|^{2\sigma}\psi,\psi\rangle}_{\text{real}} = 0$$

$$\underbrace{i\langle\partial_t\psi,\psi\rangle + i\langle\psi,\partial_t\psi\rangle}_{= i\frac{d}{dt}\langle\psi,\psi\rangle(t)} = 0$$

(ii) $$\Big\langle i\partial_t\psi + \frac{1}{2}\Delta\psi + \lambda|\psi|^{2\sigma}\psi = 0\ ,\ \partial_t\psi\Big\rangle = 0$$

need $\psi \in H^2$ and then pass to the limit.

$$i\|\partial_t\psi\|_2^2 + \frac{1}{2}\underbrace{\langle\Delta\psi,\partial_t\psi\rangle}_{-\langle\nabla\psi,\partial_t\nabla\psi\rangle} + \lambda\langle|\psi|^{2\sigma}\psi,\partial_t\psi\rangle = 0$$

(add conjugate)

$$-\frac{1}{2}\langle\nabla\psi,\partial_t\nabla\psi\rangle - \frac{1}{2}\langle\partial_t\nabla\psi,\nabla\psi\rangle + \lambda\big(\langle|\psi|^{2\sigma}\psi,\partial_t\psi\rangle + \langle\partial_t\psi|\psi|^{2\sigma}\psi\rangle\big) = 0$$

$$0 = -\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|_2^2 + \frac{\lambda}{\sigma+1}\frac{d}{dt}\int |\psi|^{2\sigma+2}\,dx$$

$$\text{If } \psi_0^{(n)} \xrightarrow{H^1} \psi_0 \Rightarrow \psi^{(n)} \longrightarrow \psi \text{ in } C\left([0,T_*),H^1\right) \cap C^1\left([0,T_n),H^{-1}\right)$$

then

$$\mathcal{H}\left[\psi^{(n)}(t)\right] = \mathcal{H}\left[\psi^{(n)}\right].$$

$$\downarrow \text{(Sobolev embedding)}$$

$$\mathcal{H}[\psi(t)] = \mathcal{H}[\psi_0]$$

□

Theorem. *(Global existence) Let $\lambda \le 0$ (good) and $\sigma < \frac{2}{d-2}$, or $\lambda > 0$ and $\sigma < \frac{2}{d}$. Then $T_* = \infty$, $\forall\ \psi_0 \in H^1$*

Proof.

1) $\sup_{t<T^*} \frac{1}{2}\|\nabla\psi(t)\|_2^2 \le \mathcal{H}[\psi_0] \;\Rightarrow\; T_* = \infty.$
2) For $\lambda > 0$, we need Gagliardo-Nirenberg-Sobolev:

$$\|u\|_q \le C\;\|\nabla u\|_2^\theta \|u\|_2^{1-\theta}$$

$$\text{need}\;\; 2 \le q \le 2^* = \frac{2d}{d-2}$$

$$\left|\begin{array}{l} u(x) \longrightarrow u(\rho x) \\ \rho^{-d/q} = \rho^\alpha \rho^{-d\theta/2}\rho^{-d(1-\theta)/2} \\ \theta = \frac{d}{2} - \frac{d}{q} = d\left(\frac{1}{2} - \frac{1}{q}\right) \end{array}\right.$$

□

Proof of GNS. Show $\|u\|_{2^*} \le C\|\nabla u\|_2$, $\quad \forall u \in \mathcal{S}$
(endpoint is sharp Sobolev emb.)
(other endpoint is $\|u\|_2 \le \|u\|_2$), then interpolation

$$\hat{u}(\xi) = \frac{\xi}{|\xi|^2}\widehat{\nabla u}(\xi)$$

$$\|u\|_{2^*} = \left\|\left(\frac{\xi}{|\xi|^2}\widehat{\nabla u}(\xi)\right)^\vee\right\|_{2^*} \overset{?}{\le} C\|\nabla u\|_2$$

$$\rightarrow\; \text{need the multiplier } \frac{\xi}{|\xi|^2} \text{ to take } L^2 \to L^{2^*}.$$

compare to $m_0(\xi) = \frac{1}{|\xi|}$, $\check{m}_0(x) = \frac{C}{|x|^{d-1}}$,
fractional integration, takes $L^p \to L^r$, $1 < r < \infty$

$$\text{with} \qquad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{\frac{d}{d-1}} \quad \Rightarrow \quad \frac{1}{d} = \frac{1}{p} - \frac{1}{r}$$

$$\text{and} \qquad \frac{1}{d} = \frac{1}{2} - \frac{1}{2^*} \qquad \text{works}$$

To go from m_0 to

$$\vec{m} = \left(\frac{\xi_j}{|\xi|^2}\right)_{j=1}^d = \left(\hat{R}_j \frac{1}{|\xi|}\right)_{j=1}^d$$

and $\hat{R}_j$ is bounded from L^2 to L^2 and L^{2^*} to L^{2^*}

$$\text{then} \qquad \int \frac{1}{2}\left|\nabla\psi(t,x)\right|^2 dx \le \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1}\int |\psi|^{2\sigma+2}\,dx$$

$$\le \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1}\|\nabla\psi(t)\|_2^{\theta(2\sigma+2)} \underbrace{\|\psi(t)\|_2^{(1-\theta)(2\sigma+2)}}_{\|\psi_0\|_2^{(1-\theta)(2\sigma+2)}}$$

$$\text{need} \qquad \underset{d\left(\frac{1}{2}-\frac{1}{q}\right)}{\overset{\theta}{\downarrow}}\;(2\sigma+2) < 2$$

$$d\left(\frac{1}{2}-\frac{1}{2\sigma+2}\right)(2\sigma+2)<2\ ,\ \ \sigma<\frac{2}{d}\ \ \text{OK.}$$

□

When $\sigma = \frac{2}{d}$, Ok when small L^2 data.

Remark (small data global existence)

1) $\sigma = \frac{2}{d}$ and $\|\psi_0\|_2 < \varepsilon = \varepsilon(\lambda, \sigma, d) \Rightarrow T_* = \infty$.
2) $\frac{2}{d} < \sigma < \frac{2}{d-2}$ and $\|\psi_0\|_{H^1} < \varepsilon \Rightarrow T_* = \infty$.

Proof.

1)
$$\frac{1}{2}\|\nabla\psi(t)\|_2^2 \le \mathcal{H}\ [\psi_0] + \underbrace{\frac{\lambda}{\sigma+1}\varepsilon^{2\sigma}}_{<\frac{1}{4}\ \text{ok.}}\|\nabla\psi(t)\|_2^2$$

2)
$$\frac{1}{2}\|\nabla\psi(t)\|_2^2 \le \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1}\|\nabla\psi(t)\|_2^{d\sigma} \qquad \underbrace{\|\psi_0\|_2^{2\sigma+2-d\sigma}}_{=\frac{\lambda}{\sigma+1}\|\nabla\psi(t)\|^{d\sigma-2}\|\psi_0\|_2^{2\sigma+2-d\sigma}}$$

$$\stackrel{?}{\le} \frac{1}{4}\ \|\nabla\psi\|_2^2$$

$$\|\nabla\psi(t)\|^{d\sigma-2}\|\psi_0\|_2^{2\sigma+2-d\sigma}$$

$$\le \left(4\mathcal{H}[\psi_0]\right)^{\frac{d\sigma-2}{2}}\|\psi_0\|^{2\sigma+2-d\sigma}$$

need $< \frac{1}{4}\cdot\frac{\sigma+1}{\lambda}$ then proceed "inductively." □

NB: $\lambda \le 0 =$ defocussing
$\lambda > 0 =$ focussing

Theorem. *(Glassey) Let $\lambda > 0$ and $\frac{2}{d} \le \sigma < \frac{2}{d-2}$*
If $\psi_0 \in H^1$, $x\psi_0 \in L^2$ and $\mathcal{H}[\psi_0] < 0$.
$\Rightarrow T_* < \infty$.

Proof. We need to show that

$$\frac{d^2}{dt^2} \int |x|^2 |\psi(t,x)|^2 dx = 4\mathcal{H}[\psi_0] - \frac{2\lambda}{\sigma+1}(d\sigma - 2) \int_{R^d} |\psi|^{2\sigma+2}(t,x)\, dx$$

(nonlinear Viriel identity)
then

$$\frac{d^2}{dt^2} \int |x|^2 |\psi(t,x)|^2 dx < 0$$

$\int$ must become < 0 at some point, blowup (focusing at $x = 0$, L^∞ also blows up) □

Prop Let $\psi \in C\left([0,T_*), H^1\right) \cap C^1\left([0,T_*), H^1\right)$ be a solution of

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta + \lambda \left|\psi\right|^{2\sigma} = 0 \\ \psi\Big|_{t=0} = \psi_0 \in H^1. \end{cases}$$

Suppose $x\psi_0 \in L^2 \Rightarrow x\psi(t,x) \in L^2$ for all $0 < t < T_*$ and $t \to \int |x|^2 |\psi(t,x)|^2\, dx$ is C^2 and Viriel holds. Moreover, one has the pseudoconformal identity

$$\left\|(x - t_p)\psi(t)\right\|_2^2 - 2t^2 \frac{\lambda}{\sigma+1} \int_{\mathbb{R}} |\psi(t,x)|^{2\sigma+2}\, dx$$

$$= \left\|x\psi_0\right\|_2^2 + 2\frac{xd\sigma - 2}{\sigma+1} \int_0^t \tau \int_{\mathbb{R}^d} \left|\psi(\tau,x)\right|^{2\sigma+2}\, dx\, d\tau.$$

□

Proof. $\varepsilon > 0$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \|e^{-\varepsilon|x|^2} |x| \psi(t,x)\|_2^2 \\
&= \mathrm{Re}\, \langle e^{-2\varepsilon|x|^2} |x|^2 \psi_t, \psi\rangle \\
&= \mathrm{Re}\, i\langle e^{-2\varepsilon|x|^2} |x|^2 \psi, i\,\psi_t\rangle \\
&= -\mathrm{Re}\, i\langle e^{-2\varepsilon|x|^2} |x|^2 \psi, \frac{1}{2}\Delta\psi + \lambda|\psi|^{2\sigma}\,\psi\rangle \\
&= \frac{1}{2}\mathrm{Re}\, i\langle \nabla(e^{-2\varepsilon|x|^2}|x|^2\psi), \nabla\psi\rangle \\
&= \mathrm{Re}\, i\langle e^{-2\varepsilon|x|^2}|x| \cdot \psi, \nabla\psi\rangle \\
&\quad - 2\mathrm{Re}\, i\langle e^{2\varepsilon|x|^2} \varepsilon x|x|^2\psi,\ \nabla\psi\rangle
\end{aligned}$$

Gronwall uniformly in ε.

$$\frac{1}{2}\big\|e^{-\varepsilon|x|^2}|x|\psi(t,x)\big\|_2^2 \le \big\||x|\psi_0\big\|_2^2 + \int_0^t \|e^{-\varepsilon|x|^2}|x|\psi(\tau)\|_2\|\nabla\psi(\tau)\|_2\,d\tau + 2\int_0^t \underbrace{\|e^{-\varepsilon|x|^2}\varepsilon|x|^2\|_\infty}_{\substack{\text{bdd OK}\\ \text{unif. in }\varepsilon}} \|e^{-\varepsilon|t|^2}x\psi(\tau)\|_2 \underbrace{\big\|\nabla\psi(\tau)\big\|_2}_{\substack{\text{OK because}\\ H' \text{ solutions}\\ \text{away from } T_*}}\,d\tau$$

$$\le \||x|\psi_0\|_2^2 + \sup_{0\le\tau\le T<T_*}\|\nabla\psi(\tau)\|_2(1+2C_0)\int_0^t \|e^{-\varepsilon|x|^2}x\psi(\tau)\|_2\,d\tau$$

$$\le \||x|\psi_0\|_2^2 + K(T)^2 + T\int_0^t \|e^{-\varepsilon|x|^2}x\psi(\tau)\|_2^2\,d\tau$$

$$\longrightarrow \text{expo growth, } \sup_{\varepsilon>0}\|e^{-\varepsilon|x|^2}\,x\psi(\tau)\|_2<\infty\,. \quad \forall 0<\tau<T_*$$

□

4/1/2004

Last term we proved most of the following:

Theorem.

1) $0 < \sigma < \frac{2}{d-2}$. *Then for every* $\psi_0 \in H^1 \quad \exists 0 < T_* \le \infty$ *(max time of existence).* $T_* = T_*(\|\psi_0\|_{H^1})$, *so that* $\exists ! \psi \in C([0,T_*];H^1) \cap C^1([0,T_*],H^1)$

$$\text{with} \qquad \left[\begin{array}{l} i\partial_t \psi + \frac{1}{2}\Delta\psi + \lambda|\psi|^{2\sigma}\psi = 0 \\ \psi\big|_{t=0} = \psi_0 \end{array} \right.$$

Moreover, ψ *depends continuously on* $\psi_0 \in H^1$.

2) *If* $\psi_0 \in H^2$, *then* $\psi \in C([0,T];H^2) \cap C^1([0,T];L^2)$.

Remarks:

1) If $T_* < \infty$, then $\lim_{t\to T_*} \|\psi(t)\|_{H^1} = \infty$ (obvious from) $T_* = T_*(\|\psi_0\|_{H^1})$.
2) We can include a potential $V(t,x)$ in part (1) of theorem to treat $i\partial_t\psi + \frac{1}{2}\Delta\psi + V(t,x)\psi + \lambda|\psi|^{2\sigma}\psi = 0$
   Assumption: $\sup_t[\|V(t,x)\|_{L^\infty_x + L^{d/2}} + \|\nabla_x V(t,x)\|_{L^\infty_x+L^{d/2}}] < \infty$.
   We need to go back to contraction argument of Kato's Theorem.

   $\tilde{F}(t,x;u) = V(t,x)u + F(u) = V(t,x)u + \lambda|u|^{2\sigma}\psi$
   $\tilde{F}: \; X_0 \to X^1 \;,\; \tilde{F}: \; Y \to Y^1$
   $X_0 = L^\infty(I; L^2 \cap L^q) \;,\; X^1 = L^1(I;L^q) + L^{r^1}(I;L^{q'}) \quad , \; q = 2\sigma + 2 < \frac{2d}{d-2}$
   Set $F = 0$ and $V(t,x) = V_1(t,x) + V_2(t,x)$
   $\big\|V_1(t,x)\psi\big\|_{L^1(I;L^2)} \le CT\big\|\psi\big\|_{L^\infty(I;L^2)}$

$$\|V_2(t,x)\psi\|_{L^{r'}(I;L^{q'})} = \left(\int_0^T \|V_2(t,\cdot)\psi\|^{r'}_{L^{q'}}\,dt\right)^{\frac{1}{r'}} \le \left(\int_0^T \|V_2(t,\cdot)\|^{r'}_{q'\left(\frac{q}{q'}\right)}\|\psi\|^{r'}_q\,dt\right)^{\frac{1}{r'}}$$
$$\le \|V_2\|_{L^\infty\left(\mathbb{R},L^{q'\left(\frac{q}{q'}\right)'}\right)} T^\alpha \|\psi\|_{L^r(I;L^q)}$$

$$\text{with } m = q'\left(\frac{q}{q'}\right)' = \frac{q}{q-2} > \frac{d}{2} \text{ and } \alpha = \frac{1}{r'} - \frac{1}{r}\,.$$

Conserved quantities: Let ψ_0, ψ be as in Theorem. Then:

$(i) \quad \|\psi(t)\|_2 = \|\psi_0\|_2$

$$(ii) \quad \mathcal{H}[\psi(t)] = \int_{\mathbb{R}^d} \left\{\frac{1}{2}|\nabla\psi(t,x)|^2 - \frac{\lambda}{\sigma+1}|\psi(t,x)|^{2\sigma+2}\right\}\,dx = \mathcal{H}[\psi_0]\,.$$

Proof. (i) We have

$$i\langle\partial_t\psi,\psi\rangle + \frac{1}{2}\underbrace{\langle\Delta\psi,\psi\rangle + \lambda\langle\big|\psi\big|^{2\sigma}\psi,\psi\rangle}_{\text{real}} = 0$$

$$i\langle\partial_t\psi,\psi\rangle + i\;\langle\psi,\partial_t\psi\rangle = 0$$
$$i\tfrac{d}{dt}\langle\psi,\psi\rangle = 0$$

(ii) $\langle Eq, \partial_t\psi\rangle = 0$ so for $\psi_0 \varepsilon H^2$:

$$i\|\partial_t\psi\|_2^2 + \frac{1}{2}\underbrace{\langle\Delta\psi, \partial_t\psi\rangle}_{-\langle\nabla\psi, \partial_t\nabla\psi\rangle} + \lambda\langle|\psi|^{2\sigma}\psi, \partial_t\psi\rangle = 0$$

$$0 = -\frac{1}{2}\langle\nabla\psi, \partial_t\nabla\psi\rangle - \frac{1}{2}\langle\partial_t\nabla\psi, \nabla\psi\rangle + \lambda\big(\langle|\psi|^{2\sigma}\psi, \partial_t\psi\rangle + \langle\partial_t\psi, |\psi|^{2\sigma}\psi\rangle\big)$$

$$0 = -\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|_2^2 + \frac{\lambda}{\sigma+1}\frac{d}{dt}\int |\psi|^{2\sigma+2}\,dx$$

Now if $\psi_0 \in H^1$, take $\psi^{(n)} \to \psi_0$ in $H^1 \Rightarrow \psi^{(n)} \to \psi$ in $C([0,T_*];H^1) \cap C^1([0,T_*];H^{-1})$

Hence

$$\begin{array}{cc} \mathcal{H}[\psi^{(n)}(t)] & = \mathcal{H}[\psi^{(n)}] \\ \downarrow & \downarrow \\ \mathcal{H}[\psi(t)] & \mathcal{H}[\psi_0] \end{array}$$

□

Theorem. *(Global existence) Let $\lambda \le 0$ and $\sigma < \frac{2}{d-2}$, or $\lambda > 0$ and $\sigma < \frac{2}{d}$.*
Then $T_* = \infty \quad \forall\psi_0 \in H^1$

Proof.

$$\lambda \le 0 \quad \Rightarrow \sup_{t\le T_*}\frac{1}{2}\|\nabla\psi(t)\|_2^2 \le \mathcal{H}[\psi_0] \Rightarrow\ T_* = \infty$$

$\lambda > 0$— we need Gagliardo-Nirenberg-Sobolev:

$$\|u\|_q \le C\|\nabla u\|_2^\theta\|u\|_2^{1-\theta}$$

$$2 \le q \le 2^* = \frac{2d}{d-2}\ ,\ \frac{1}{2} - \frac{1}{2^*} = \frac{1}{d}$$

$$\theta = d\left(\frac{1}{2} - \frac{1}{q}\right) \text{ by scaling}$$

Proof of G-N-S: $\|u\|_{2^*} \le C\|\nabla u\|_2 \forall u \in \mathcal{S}$ (then extend to H^1)(plus Hölder for $2 < q < 2^*$)

$$\hat{u}(\xi) = \frac{\xi}{|\xi|^2}\widehat{\nabla u}(\xi)\ ,\ \text{so want}\ \left\|\left(\frac{\xi}{|\xi|^2}\widehat{\nabla u}(\xi)\right)^\vee\right\|_{2^*} \le C\|\nabla u\|_2$$

So want to bound the multiplier $\vec{m}(\xi) = \frac{\xi}{|\xi|^2}$ on $L^2 \to L^{2^*}$
Let $m_0(\xi) = \frac{1}{|\xi|}$; $\check{m}_0(x) = \frac{C}{|x|^{d-1}}$ is kernel $L^p \to L^r$ with $1 < r < \infty$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{d-1}{d} = 1 + \frac{1}{p} - \frac{1}{d}$, so $\frac{1}{d} = \frac{1}{p} - \frac{1}{r}$ $\left(= \frac{1}{2} - \frac{1}{2^*}\right)$

$$\text{Now } \vec{m} = \left(\frac{\xi_j}{|\xi|^2}\right)_{j=1}^d = \left(\hat{R}_j\frac{1}{|\xi|}\right)_{j=1}^d$$

with R_j - Riesz transf. (in Fourier space $\hat{R}_j = \frac{\xi_j}{|\zeta|}$)

$$\text{since } \frac{1}{|\xi|} : L^2 \to L^{2^*} \text{ and } R_j : L^p \to L^p\ (1 < p < \infty),\ \text{done}$$

$$\int \frac{1}{2}|\nabla\psi(t,x)|^2\,dx \le \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1}\int |\psi_0(t,x)|^{2\sigma+2}\,dx$$
$$\le \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1}\|\nabla\psi(t)\|_2^{\theta(2\sigma+2)} \underbrace{\|\psi(t)\|_2^{(1-\theta)}}_{\|\psi_0\|_2^{(1-\theta)(2\sigma+2)}}(2\sigma+2)$$

$$\text{Need } \theta(2\sigma+2) < 2\ ,\ \text{so } d\left(\frac{1}{2} - \frac{1}{2\sigma+1}\right)(2\sigma+2) < 2,\ \text{i.e., } \sigma < \frac{2}{d}$$

□

Remark: (small data global existence for $\lambda > 0$)

1) $\sigma = \frac{2}{d}$ and $\|\psi_0\|_2 < \varepsilon = \varepsilon(\lambda,\sigma,d) \quad \Rightarrow \quad T_* = \infty$
2) $\frac{2}{d} < \sigma < \frac{2}{d-2}$ and $\|\psi_0\|_{H^1} < \varepsilon \Rightarrow T_* = \infty$

Proof. 1) $\frac{1}{2}\|\nabla\psi(t)\|_2^2 \le \mathcal{H}[\psi_0] + \underbrace{\frac{\lambda}{\sigma+1}\varepsilon^{2\sigma}}_{\le\frac{1}{4}\Rightarrow\text{done}}\|\nabla\psi(t)\|_2^2$ (see previous proof)

2) $\frac{1}{2}\|\nabla\psi(t)\|_2^2 \le \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1}\|\nabla\psi(t)\|_2^{d\sigma}\|\psi_0\|_2^{2\sigma+2-d\sigma}\big(d\sigma = \theta(2\sigma+2)\big)$
Write $\|\nabla\psi\|_2^{d\sigma} = \|\nabla\psi\|_2^{d\sigma-2}\|\nabla\psi\|_2^2$ and need $\frac{\lambda}{\sigma+1}\|\nabla\psi\|_2^{d\sigma-2}\|\psi_0\|_2^{2\sigma+2-d\sigma} \le \frac{1}{4}$
So assume $\|\nabla\psi\|_2^2 \le 8\mathcal{H}[\psi_0]$ and take ε s.t. $\frac{\lambda}{\sigma+1}\left(8\mathcal{H}[\psi_0]\right)^{\frac{d\sigma-2}{2}}\|\psi_0\|_2^{2\sigma+2-d\sigma} \le \frac{1}{4}$
Then get $\|\nabla\psi\|_2^2 \le 4\mathcal{H}[\psi_0]$ up to that T using $\nearrow \Rightarrow$ can extend past T

□

Glassey's Theorem: Let $\lambda > 0, \frac{2}{d} \le \sigma < \frac{2}{d-2}$. If $\psi_0 \in H^1, x\psi_0 \in L^2$, and $\mathcal{H}[\psi_0] < 0$, then $T_* < \infty$.

Proof.

$$\frac{d^2}{dt^2}\int |X|^2|\psi(x,t)|^2\,dx \overset{\text{Virial identity}}{\overset{\downarrow}{=}} 4\mathcal{H}[\psi_0] - \frac{2\lambda}{\sigma+1}(d\sigma-2)\int_{\mathbb{R}^d}|\psi(t,x)|^{2\sigma+2}\,dx \le 4\mathcal{H}[\psi_0] < 0$$

□

Proposition: Let $\psi \in C([0,T_*];H^1)\cap C^1([0,T_*];H^{-1})$ be a solution if

$$\left[\begin{array}{l} i\partial_t\psi + \frac{1}{2}\Delta\psi + \lambda|\psi|^{2\sigma}\psi = 0 \\ \psi\big|_{t\infty} = \psi_0 \in H^1 \end{array}\right.$$

Suppose $x\psi_0 \in L^2 \Rightarrow x\psi(t,x) \in L^2$ for all $0 < t < T_*$ and $t \mapsto \int |x|^2|\psi(t,x)|^2\,dx$ is C^2 and *Virial identity* holds. Moreover, one has the *pseudo-conformal identity.*

$$\|(x-tp)\psi(t)\|_2^2 - 2t^2\frac{\lambda}{\sigma+1}\int_{\mathbf{R}^d}|\psi(t,x)|^{2\sigma+2}\,dx = \|x\psi_0\|_2^2 + \frac{2\lambda(d\sigma-2)}{\sigma+1}\int_0^t \tau\int_{\mathbb{R}^d}|\psi(t,x)|^{2\sigma+2}\,dx\,d\tau$$

Proof.

$$\varepsilon > 0; \frac{1}{2}\frac{d}{dt}\|e^{-\varepsilon|x|^2}|x|\psi(t,x)\|_2^2 = \text{Re } i\langle e^{-2\varepsilon|x|^2}|x|^2\psi, i\psi_t\rangle$$

$$= -\mathrm{Re}\; i\langle e^{-2\varepsilon|x|^2}|x|^2\psi, \frac{1}{2}\Delta\psi + \lambda|\psi|^{2\sigma}\psi\rangle$$
$$= \frac{1}{2}\mathrm{Re}\; i\langle \nabla(e^{-2\varepsilon|x|^2}|x|^2)\psi, \nabla\psi\rangle$$
$$= \mathrm{Re}\; i\langle e^{-2\varepsilon|x|^2}x\psi, \nabla\psi\rangle - 2\mathrm{Re}\; i\langle e^{-2\varepsilon|x|^2}\varepsilon x|x|^2\psi, \nabla\psi\rangle$$

So undo derivative:

$$\frac{1}{2}\|e^{-2\varepsilon|x|^2}x\psi\|_2^2 \leq \||x|\psi_0\|_2^2 + \int_0^t \|e^{-\varepsilon|x|^2}x\psi(\tau,x)\|_2\|\nabla\psi(\tau,x)\|_2\,d\tau$$
$$+ 2\int_0^t \underbrace{\|e^{-\varepsilon|x|^2}\|_\infty}_{\leq C \text{ unif. in } \varepsilon} \|e^{-\varepsilon|x|^2}x\psi(\tau)\|_2\|\nabla\psi(\tau)\|_2\,d\tau$$
$$\leq \||x|\psi_0\|_2^2 + \underbrace{\sup_{0\leq\tau\leq T<T_*}\|\nabla\psi(\tau)\|_2(1+2C_0)}_{\leq K(T)}\int_0^t \|e^{-\varepsilon|x|^2}x\psi(\tau)\|_2\,d\tau$$
$$\leq \||x|\psi_0\|_2^2 + K(T)\int_0^t \|e^{-\varepsilon|x|^2}x\psi(\tau)\|_2^2\,d\tau$$

So $\sup_{\varepsilon>0}\|e^{-\varepsilon|x|^2}x\psi(\tau)\|_2 < \infty \qquad \forall 0 < t < T_*$ (by Gronwall) $\Rightarrow x\psi(\tau,x) \in L^2$. □

Ma 142
4/8/2003 **Version 2**

H^1 critical equation $i\partial_i\psi + \frac{1}{2}\Delta\psi - |\psi|^4\psi = 0$ in $\mathbb{R}^3$ ($\sigma = \frac{2}{d-2s}$, $s = 1$, $d = 3$)

Major open problem to prove global existence —easy to show local existence for $\sigma = \frac{2}{d-2}$, $d \geq 3$.

Main difference: $T_* = T_*(\psi_0)$ but do not know if $T_* = T_*(\| \psi_0 \|_{H^1})$

Ideas: Strichartz and Sobolev.

Theorem. *(Cazenave-Weissler)*

Let $d \geq 3$, $\sigma = \frac{2}{d-2}$. For every $\psi_0 \in H^1$, there exists

$$\psi \in C\big([0,T^*], H^1\big) \cap C^1\big([0,T_*], H^{-1}\big)\ ,$$
$$T_* = T_*(\psi_0) = 0\ ,$$

of
$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi + \lambda|\psi|^{2\sigma}\psi = 0\ . \\ \psi|_{t=0} = \psi_0\ . \end{cases}$$

This solution belongs to $L^\gamma_{loc}\big([0,T_*), W^{1,\rho}\big)$, where $\gamma = \gamma(d)$, $\rho = \rho(d)$ are a particular pair of Strichartz exponents

$$\Big(\frac{2}{\gamma} + \frac{d}{p} = \frac{d}{2}\Big) \qquad \textit{and}$$

ψ is unique in $L^\infty_{loc}\big([0,T_*),H^1\big)\cap L^\gamma_{loc}\big([0,T_*),W^{1,\rho}\big)$. Finally, there is L^2 and energy cons.

Idea:

$$\text{set } f_n(u)=\lambda\begin{cases} u^\sigma\ , & 0\le u\le n\\ \text{smooth} & n<u<4n\\ (2n)^\sigma & u\ge 4n\ .\end{cases}$$

$$\text{s.t.}|f_n'(u)|\le C\cdot|u|^{\sigma-1}$$

Solve
$$\begin{cases} i\partial_t\psi^{(n)}+\frac{1}{2}\Delta\psi^{(n)}+f_n(|\psi^{(n)}|^2\psi^{(n)}=0\\ \psi^{(n)}|_{t=0}=\psi_0\end{cases}$$

$\to$ globally in H^1 because nonlinearity bounded. Thus find *a priori* bounds on $L^\gamma\big([0,T_*),W^{1,\rho}\big)$ and (determine T_*).

Call (r,q) admissible if $\frac{2}{r}+\frac{d}{q}=\frac{d}{2}$, $2<r\le\infty$

Lemma. *Let (r,q) be admissible and $\gamma=\frac{2d}{d-2}$ and $p=\frac{2d^2}{d^2-2d+4}$. Let $r\ge\frac{d}{d-2}$. Then*

$$\begin{aligned}\sup_n\parallel\nabla\big[f_n(|\psi|^2)\psi\big]\parallel_{L^{r'}I,L^{q'}}&\le C\cdot\parallel\psi\parallel^{2\sigma}_{L^\gamma(I,L^{\rho^*})}\cdot\parallel\nabla\psi\parallel_{L^a(I,L^b)}\\ &\le C\cdot\parallel\nabla\psi\parallel^{2\sigma}_{L^\gamma(I,L^\rho)}\cdot\parallel\nabla\psi\parallel_{L^a(I,L^b)}\end{aligned}$$

where
$$\frac{2\sigma}{p^*}+\frac{1}{b}=\frac{1}{q'}$$
$$\frac{2\sigma}{\gamma}+\frac{1}{a}==\frac{1}{r'}\ \textit{and } (a,b)\ \textit{satisfies}$$
$$\frac{2}{a}+\frac{d}{b}=\frac{d}{2}$$

Note: (γ,ρ) det as follows: take $b=q$, $a=r$, then actually $b=q=\rho$, $a=r=\gamma$.

Proof.

$$\begin{aligned}\Big|\nabla\big[f_n(|\psi|^2)\psi\big]\Big|&\le|f_n'(|\psi|^2)|\ |\psi|^2|\nabla\psi|\\ +|f_n(|\psi|^2)|\ |\nabla\psi|&\le C|\psi|^{2\sigma}|\nabla\psi|\\ \parallel\ |\psi|^{2\sigma}|\nabla\psi|\ \parallel_{L^{q'}_x}&\le\parallel\psi\parallel^{2\sigma}_{\rho^*}\parallel\nabla\psi\parallel_b\end{aligned}$$

$$1\le m\le\infty\text{ with }\quad\left.\begin{matrix}mq'=b\\ 2\sigma q'm'=\rho^*\end{matrix}\right\}\Rightarrow\frac{q'}{b}+\frac{2\sigma q'}{\rho^*}=1$$

$$\parallel\ |\psi|^{2\sigma}|\nabla\psi|\ \parallel_{L^{r'}(I,L^{q'})}\le\parallel\psi\parallel^{2\sigma}_{L^\gamma(I,L^{\rho^*})}\parallel\nabla\psi\parallel_{L^a(I,L^b)}$$

$$1\le\nu\le\infty\text{ s.t.}\quad\left.\begin{matrix}2\sigma r'\nu'=\gamma\\ r'v=a\end{matrix}\right\}\Rightarrow\frac{2\sigma}{\gamma}+\frac{1}{q}=\frac{1}{r'}$$

Check (a,b) is admissible: (still check $a\ge2$ afterwards).

$$\begin{aligned}\frac{2}{a}+\frac{d}{b}&=2\Big(\frac{1}{r'}-\frac{2\sigma}{\gamma}\Big)+d\Big(\frac{1}{q'}-\frac{2\sigma}{\rho^*}\Big)\\ &=2+d-\Big(\frac{2}{r}+\frac{d}{q}\Big)-2\sigma\Big(\frac{2}{\gamma}+\frac{d}{\rho^*}\Big)\end{aligned}$$

$$= 2 + d - \frac{d}{2} - \frac{4}{d-2}\underbrace{\Big(\frac{2}{\gamma} + \frac{d}{\rho}\Big)}_{\frac{d}{2}} + 2\gamma$$

$$= 2 + d - \frac{d}{2} - \frac{4}{d-2}\frac{d}{2} + \frac{4}{d-2}$$

$$= 2 + \frac{d}{2} - \frac{4}{d-2}\Big(\frac{d}{2} - 1\Big) = \frac{d}{2}\ .$$

$$\left(\begin{array}{lll} r \geq \frac{d}{d-2} & \Rightarrow & \text{bounds on } m, \nu \\ & \Rightarrow & a \geq 2 \end{array}\right)$$

□

Recall

$$(Gf)(t) = \int\limits_0^t e^{i(t-s)\Delta/2} f(s)\, ds$$

Lemma.

$$\| \, Gf \, \|_{L^r(I,L^q)} \leq C \cdot \| \, f \, \|_{L^{\gamma'}(I,L^{\rho'})}$$

for any admissible $(r, q), (\gamma, q)$

Proof.

$$\begin{cases} G : L^1(I, L^2) \to L^r(I, L^q) & (a) \\ G : L^{r'}(I, L^{q'}) \to L^r(I, L^q) & (b) \\ G : L^{r'}(I, L^{q'}) \to L^\infty(I, L^2) & (c) \end{cases}$$

NB:

$$\frac{2}{r} + \frac{d}{q} = \frac{d}{2}\begin{cases} 2 < r \leq \infty\ . \\ 2 \leq q \leq \frac{2d}{d-2} \end{cases}$$

Interp.

$$\begin{array}{ll} (a), (b) & q' \leq \rho' \leq 2\ , \quad 2 \leq \rho \leq q\ . \\ (b), (c) & 2 \leq q \leq \rho\ . \end{array}$$

□

Corollary. $\forall$ *admissible* (r, q)*, take* (γ, ρ) *as before*

(1) $$\| \, \nabla G\big(f_n(|\psi|^2\psi\big) \, \|_{L^r(I,L^q)} \leq C \, \| \, \nabla\psi \, \|^{2\sigma+1}_{L^\gamma(I,L^\rho)}$$

(2) $$\| \, G\big(f_n(|\psi|^2\psi\big) - G\big(f_n(|\phi|^2\phi\big)\|_{L^r(I,L^q)}$$
$$\leq C\big(\| \, \nabla\psi \, \|^{2\sigma}_{L^\gamma(I,L^\rho)} + \| \, \nabla\phi \, \|^{2\sigma}_{L^\gamma(I,L^\rho)}\big)$$
$$\| \, \psi - \rho \, \|_{L^\gamma(I,L^\rho)}$$

(will be applied with $(r, p) = (\gamma, \rho)$*).*

Proof. Combine lemmas 1 and 2. □

Proof of Theorem. Switch to Duhamel.

$$\psi^{(n)} = G_0\psi_0 + iGF_n(\psi^{(n)})$$
$$F_n(\psi) = f_n(|\psi|^2)\psi$$

(equivalent to strong version for H^1 solution).

$$\| \nabla\psi^{(n)} \|_{L^\gamma(I,L^\rho)} \leq \| G_0\nabla\psi_0 \|_{L^\gamma(I,L^\rho)} + C_0 \cdot \| \nabla\psi^{(n)} \|^{2\sigma+1}_{L^\gamma(I,L^\rho)}$$

We need $\sup_n \| \nabla\psi^{(n)} \|_{L^\gamma(I,L^\rho)} < \infty$

$$\rightarrow \text{ take } \| G_0\nabla\psi_0 \|_{L^\gamma(I,L^\rho)} < \delta \leq \| \nabla\psi_0 \|_2$$

then $$\sup_n \| \nabla\psi^{(n)} \|_{L^\gamma(I,L^\rho)} \leq 2\delta \text{ if } 2C_0(\delta)^{2\sigma} < 1$$

Det $\cdot T_*$ so that $< \delta \leq \| \nabla\psi_0 \|_2$ (or small initial data)

Use H^1 existence $\Rightarrow$ $\nabla\psi_0 \in L^2$ but what guarantees $\nabla\psi_0 \in L^\rho$? need to show that $\| \nabla\psi^{(n)} \|_{L^\gamma(I,L^\rho)} < \infty$.

$$\text{Kato } \Rightarrow \nabla\psi \in L^{\tilde{r}}(I, L^{\tilde{q}}) \ , \ \tilde{q} = 2\tilde{\sigma} + 2 \ .$$

where $\tilde{\sigma}$ is artibrary subcritical

$$\tilde{\sigma} < \frac{2}{d-2}$$
$$\rho = 2\tilde{\sigma} + 2 \ ,$$

$d = 3$. $$\frac{18}{7} = 2\tilde{\sigma} + 2 < 6 \ .$$

Conclusion: Uniform boundedness in n.

NB: we also have $\| \nabla\psi^{(n)} \|_{L^\infty(I,L^2)}$

$$\leq \| G_0\nabla\psi_0 \|_2 + C \| \nabla\psi^{(n)} \|^{2\sigma+1}_{L^\gamma(I,L^\rho)}$$

(by the corollary before).

$$\leq C \cdot \| \nabla\psi_0 \|_2 + C \cdot (2\delta)^{2\sigma} \| \nabla\psi_0 \|_2$$
$$\leq C \cdot \| \nabla\psi_0 \|_2$$

and also $\boxed{\sup_n \| \nabla\psi^{(n)} \|_{L^r(I,L^q)} < \infty \ .}$

Estimates on

$$\| \psi^{(n)} \|_{L^\gamma(I,L^\rho)} \leq \| G_0\psi_0 \|_{L^\gamma(I,L_\rho)} + \overbrace{C \cdot (2\delta)^{2\sigma}}^{<\frac{1}{2}} \cdot \| \psi^{(n)} \|_{L^\gamma(I,L^\rho)}$$

$$\boxed{\sup_n \| \psi^{(n)} \|_{L^\gamma(I,L^\rho)} < \infty \ .}$$

Pass to a limit

$$\| \psi^{(n)} - \psi^{(m)} \|_{L^\gamma(I,L^\rho)} = \| GF_n(\psi^{(n)}) - GF_m(\psi^{(m)}) \|_{L^\gamma(I,L^\rho)}$$

$$\leq \| G[F_n(\psi^{(n)}) - F_n(\psi^{(m)})] \|_{L^\gamma(I,L^\rho)}$$

(assume $n > m$)

$$\begin{aligned}
&+ \| G[F_n(\psi^{(m)}) - F(\psi^{(m)}) \|_{L^\gamma(I,L^\rho)} \\
&+ \|G[F_m(\psi^{(m)}) - F(\psi^{(m)}) \| \\
&\lesssim \big(\underbrace{\| \nabla\psi^{(n)} \|^{2\sigma}_{L^\gamma(I,L^\rho)} + \| \nabla\psi^{(m)} \|^{2\sigma}_{L^\gamma(I,L^\rho)}}_{\leq (2\delta)^{2\sigma}}\big) \| \psi^{(n)} - \psi^{(m)} \|_{L^\gamma(I,L^\rho)} \\
&+ \underbrace{2 \| \nabla\psi^{(m)}}_{\leq(2\delta)^{2\sigma}} \|^{2\sigma} \| \underbrace{\chi_{[|\psi^{(m)}|>m]}\psi^{(m)}}_{\leq m^{-\frac{(2-\rho)}{\rho}}\left(\int_I \|\nabla\psi^{(m)}(t)\|_{L^q}^{\frac{2^*}{\rho}\cdot\gamma}\, dt\right)^{\frac{1}{\gamma}}} \|_{L^\gamma(I,L^\rho)}
\end{aligned}$$

because $\leq \| F_n(\psi^{(m)}) - F(\psi^{(m)}) \|_{L^{\gamma'}(I,L^{\rho'})}$, Hölder and apply Sobolev emb. to get ∇.

$$\begin{aligned}
\text{(because } \| \chi_{[|\psi^{(m)}|>m]}\psi^{(m)} \|^\rho_{L^\rho} &\leq m^{-(2^*-\rho)} \int |\psi^{(m)}|^{2^*}\, dx \\
&\leq m^{-(2^*-\rho)} \| \nabla\psi^{(m)} \|^{2^*}_{L^2})
\end{aligned}$$

Cauchy seq. $\to$ strong limit that still satisfies the bounds.

Ma 142
4/10/2003 **Version 2**

Conclusion: Provided $\| G_0\nabla\psi_0 \|_{L^\gamma(I,L^\rho)} < \delta$ small then

(a) $\sup_n \| \nabla\psi^{(n)} \|_{L^\gamma(I,L^\rho)} \leq 2\delta$

(b) $\sup_n \| \nabla\psi^{(n)} \|_{L^r(I,L^q)} < \infty$ $\quad$ (r, q) admissible

(c) $\sup_n \| \psi^{(n)} \|_{L^r(I,L^q)} < \infty$

and then $\quad \| \psi^{(n)} - \psi^{(m)} \|_{L^\gamma(I,L^\rho)} \to 0$ as $n, m \to \infty$.

$$\psi^{(n)} \to \psi \text{ in } L^\gamma(I, L^\rho)$$

and ψ still satisfies (a), (b), (c)

(because e.g., $\nabla\psi \in L^\infty(I, L^2)$,

$$\begin{aligned}
|\langle\psi(t), \nabla\chi(t)\rangle| &= \lim |\langle\psi^{(n)}(t), \nabla\chi(t)\rangle| \\
&= \lim |\langle\nabla\psi^{(n)}(t), \chi(t)\rangle| \\
&\leq \sup_n \| \nabla\psi^{(n)}(t) \|_2 \cdot \| \chi(t) \|_2
\end{aligned}$$

$$\int |\langle\psi(t), \nabla\chi(t)\rangle|\, dt \leq \sup_n \| \nabla\psi^{(n)} \|_{L^\infty(I,L^2)} \cdot \| \chi \|_{L^1(I,L^2)}$$

Then check eq. satisfied

$$\psi^{(n)} = G_0\psi_0 + \underbrace{iGF_n(\psi^{(n)})} \text{ in } L^\gamma(I, L^\rho)$$

$$\downarrow$$

$$\psi$$

$$\text{in } L^\gamma(I,L^\rho)\ \underbrace{G[F_n(\psi^{(n)}) - F_n(\psi)]}_{\to 0} + G[F_n(\psi) - F(\psi)] + GF(\psi)$$

$$\text{because} \quad \leq C(\| \nabla\psi^{(n)} \|^{2\sigma} + \| \nabla\psi \|^{2\sigma}) \| \psi^{(n)} - \psi \|$$

and

$$G[F_n(\psi) - F(\psi)] \to 0 \qquad \text{because}$$

$$\begin{aligned} &\| GF_n(\psi) - GF(\psi) \|_{L^\gamma(I,L^\rho)} \\ \leq &\| F_n(\psi) - F(\psi) \|_{L^{\gamma'}(I,L^{\rho'})} \\ \leq &\| f_n(|\psi|^2)\psi - f(|\psi|^2)\psi \|_{L^{\gamma'}(I,L^{\rho'})} \\ \lesssim &\| |\psi|^{2\sigma}\psi\chi_{[|\psi|^2>n]} \|_{L^{\gamma'}(I,L^{\rho'})} \\ \lesssim &\| \nabla\psi \|^{2\sigma}_{L^\gamma(I,L^\rho)} \| \psi \underbrace{\chi_{[|\psi|^2>n]} \|_{L^{\gamma'}(I,L^{\rho'})}}_{\to 0} \end{aligned}$$

Also, $\psi \in C(I, H^1)$ is easy.

Uniqueness in $L^\infty(I, H^1) \cap L^\gamma(I, W^{1,\rho})$.

$$\psi = G_0\psi_0 + iGF(\psi)$$

$$\begin{aligned} \| \nabla\psi \|_{L^\gamma(I,L^\rho)} \leq &\overbrace{\| G_0\nabla\psi_0 \|}^{<\delta}{}_{L^\gamma(I,L^\rho)} \\ &+ C \| \nabla\psi \|^{2\sigma+1}_{L^\gamma(I,L^\rho)} \end{aligned}$$

take $2C_0(2\delta)^{2\sigma} < 1$

(these estimates derived directly from eq.)

$$\begin{aligned} &\Rightarrow \| \nabla\psi \|_{L^\gamma(I,L^\rho)} < 2\delta \, . \\ &\| \psi - \phi \|_{L^\gamma(I,L^\rho)} \leq \| GF(\psi) - GF(\phi) \|_{L^\gamma(I,L^\rho)} \\ &\qquad\qquad \leq C(\| \nabla\psi \|^{2\sigma}_{\gamma,\rho} + \| \nabla\phi \|^{2\sigma}_{\gamma,\rho}) \| \psi - \phi \|_{\gamma,\rho} \\ &\Rightarrow \psi = \phi \end{aligned}$$

Conservation laws

(i)

$$\| \psi^n(t) \|_2 = \| \psi_0 \|_2$$

$$\begin{aligned} \| \psi^{(n)} - \psi^{(m)} \|_{L^\infty(I,L^2)} &\leq \| GF_n(\psi^{(n)}) - GF_m(\psi^{(m)}) \|_{\infty,2} \\ &\leq \cdots \leq \| \psi^{(n)} - \psi^{(m)} \|_{\gamma,\rho} \to 0 \, . \end{aligned}$$

for almost every t, $\exists$ subseq s.t.

$$\psi^{(n)} \to \psi \text{ in } L^2_x$$

for a.e. t, $\| \psi(t) \|_2 = \| \psi_0 \|_2$ since $\psi(t)$ is co. in time, OK.

(ii)
$$\begin{aligned} \mathcal{H}_n[\psi^n_{(t)}] &= \mathcal{H}_n[\psi_0] \\ &= \int \Big[\frac{1}{2}|\nabla\psi_0|^2 - G_n(\psi_0)\Big]\, dx \end{aligned}$$

where $G_n(u) = \int_0^u F_n(v)\, dv$

$$\mathcal{H}_n[\psi^{(n)}(t)] = \int \Big[\underbrace{\frac{1}{2}|\nabla\psi^{(n)}(t,x)|^2}\, - G_n\big(\psi^n(t,x)\big)\Big]\, dx$$

weak convergence only $\to$ one-sided control by semi-continuity.

$$\sup_n |G_n(u)| \le C|u|^{2\sigma+2}\,, \qquad 2\sigma+2 = \frac{4}{d-2} + 2 = \frac{2d}{d-2} = 2^*$$

$$\| \psi^{(n)} - \psi^{(m)} \|_{2^*} \overset{?}{\to} 0$$

(not from $\| \psi^{(n)} - \psi^{(m)} \|_{L^r(I,L^q)} \le \cdots \to 0$ because $r = 2$, $q = 2^*$ is a forbidden endpoint.)

$$\begin{aligned} \text{Yes,} \qquad & \| \psi^{(n)}(t) - \psi^{(m)}(t) \|_{2^*} \\ & \le C \| \nabla\psi^{(n)}(t) - \nabla\psi^{(m)}(t) \|_q^{\theta} \\ & \quad \| \psi^{(n)}(t) - \psi^{(m)}(t) \|_2^{1-\theta} \end{aligned}$$

$$\Big(q > 2, \frac{1}{q} - \frac{1}{q^*} = \frac{1}{d} \text{ with } \frac{1}{q^*} = \frac{1}{q} - \frac{1}{d} < \frac{1}{2} - \frac{1}{d} = \frac{1}{s^*} \Rightarrow q^* > 2^*\Big)$$

$$\begin{aligned} \exists \text{ subseq, for a.e. } t,\ & \| \nabla - \nabla \| \text{ bdd}, \\ & \| \psi^{(n)}(t) - \psi^{(m)}(t) \| \to 0\,. \end{aligned}$$

$$\Rightarrow \int G_n\big(\psi^{(n)}(t,x)\big)\, dx \to \int G\big(\psi(t,x)\big)\, dx$$

$$\begin{aligned} \lim_{n\to\infty} \mathcal{H}_n[\psi_0] &= \mathcal{H}[\psi_0] \\ &= \lim_{n\to\infty} \mathcal{H}_n[\psi^{(n)}] \\ &= \lim_{n\to\infty} \mathcal{H}_n[\psi^{(n)}] \\ &\ge \mathcal{H}[\psi(t)] \end{aligned}$$

Concl.
$$\mathcal{H}[\psi_0] \ge \mathcal{H}[\psi(t)]$$

Time-reversal, previous result do not depend on it (though equation changes)

$$\begin{aligned} t_0 = 0 \le t \le t_1 \\ \phi(t) = \psi(t_1 - t) \end{aligned}$$

$$\psi(t) = e^{it\frac{\Delta}{2}}\psi_0 + i\int_0^t e^{i(t-s)\frac{\Delta}{2}} F\big(\psi(s)\big)\, ds\ .$$

Claim: ϕ satisfies

$$\phi(t) \overset{?}{=} e^{-it\frac{\Delta}{2}}\psi(t_1) - i\int_0^t e^{-i(t-s)\frac{\Delta}{2}} F\big(\phi(s)\big)\, ds$$

$$(-i\partial_t\psi + \frac{1}{2}\Delta\psi + F(\phi) = 0)$$

$$\phi(t) = e^{-it\frac{\Delta}{2}}\Big\{e^{it_1\frac{\Delta}{2}}\psi_0 + i\int_0^{t_1} e^{i(t_1-\tau)\frac{\Delta}{2}} F\big(\psi(\tau)\big) d\tau\Big\}$$

$$-i\int_{t_1-t}^{t_1} e^{-i(t-t_1+s)\frac{\Delta}{2}} F\big(\psi(s)\big)\, ds = e^{i(t_1-t)\frac{\Delta}{2}}\psi_0$$

$$+ie^{-it\frac{\Delta}{2}}\int_0^{t_1-t} e^{i(t_1-t)}\frac{\Delta}{2} F\big(\psi(\tau)\big)\, d\tau$$

(so Backwards Duhamel is satisfied)

Check $C^1(I, H^{-1})$

$$i\partial_t\psi + \frac{1}{2}\Delta\psi + \lambda \underbrace{|\psi|^{2\sigma}\psi}_{\text{co. in } H^{-1}} = 0\ .$$

d=3
$$\sigma = 2\ , \quad |\psi|^4\psi\ .$$
$$t \to \psi \in C(I, H^1)\ .$$
$$H^1 \hookrightarrow L^6(\mathbb{R}^3)\ , \quad L^{\frac{6}{5}} \hookrightarrow H^{-1}$$
$$t \to |\psi|^4\psi \in C(I, L^{\frac{6}{5}})$$
$$\text{and} \quad \in C(I, H^{-1})$$

the rest is okay.

Look at $i\partial_t\psi + \frac{1}{2}\Delta\psi - |\psi|^2\psi = 0$.

$$\begin{array}{ll} s = 1 & \text{subcritical for } s > \frac{1}{2} \\ s = \frac{1}{2} & \text{critical for } s = \frac{1}{2}. \end{array}$$

$$\sigma_{\text{critical}} = 1 = \frac{2}{d-2s} = \frac{2}{3-2s}\ , \quad s = \frac{1}{2}\ .$$

Claim: We have all the methods to prove local existence

$$\exists ce\ , \quad \text{both } s > \frac{1}{2}\ , \quad s = \frac{1}{2}$$

$$\psi = G_0\psi_0 + \lambda i G(|\psi|^2\psi)$$

$$\begin{aligned}
\| \langle\nabla\rangle^s \psi \|_{L^\gamma(I,L^q)} &= \| \big((1+|\xi|^2)^{\frac{s}{2}}\hat{\psi}\big)^\vee \|_{\gamma,q} \\
\downarrow \qquad\qquad &\leq \| G_0 \langle\nabla\rangle^s \psi_0 \|_{\gamma,q} \\
(1-\Delta)^{\frac{s}{2}} \qquad &\quad + C\cdot \| \langle\nabla\rangle^s |\psi^2\psi \|_{a',b'} \\
&\leq \| \langle\nabla\rangle^s \psi_0 \|_2 \\
&\quad + CT^\alpha \| \psi \|^2_{L^\gamma(I,L^\rho)} \cdot \| \langle\nabla\rangle^s \psi \|_{\infty,2}
\end{aligned}$$

$\alpha > 0$ because subcritical

Ma 142
4/15/2003
Last time:

$$i\partial_t\psi + \frac{1}{2}\Delta\psi - |\psi|^2\psi = 0$$
$$\psi|_{t=0} = \psi_0 \in H^1$$

We proved global well-posedness, energy conservation, L^2 conservation. Moreover, we have regularity in H^2 (Kato's paper.)

We proved pseudo-conformal identity:

$$\| (x-tp)\psi(t) \|_2^2 + t^2 \| \psi(t) \|_4^4 = \| x\psi_0 \|_2^2 - \int_0^t \tau \| \psi(t) \|_4^4 \; d\tau$$

which implies

$$\| \psi(t) \|_4^4 \leq \frac{\| x\psi_0 \|_2^2}{t^2} ,$$

which implies that there are no solitons.

Where can we expect well-posedness (local)?

$$\sigma_{\text{crit}} = \frac{2}{d-2s} \, , d=3 \, , s = \frac{1}{2} \, .$$

If $s > \frac{1}{2}$, we can prove local well-posedness:

We have real formulation $\psi = G_0\psi_0 + iG|\psi|^2\psi$ and

$$\| \langle\nabla\rangle^s\psi \|_{L^r(I;L^g)} \leq C \| \langle\nabla\rangle^s\psi_0 \|_2 + C \| \langle\nabla\rangle^s |\psi|^2\psi \|_{L^{a'}(I;L^{b'})}$$

If $\langle\nabla\rangle$ would be an ordinary ∇, we would have, by Leibnitz,

$$\| \langle\nabla\rangle^s\big(|\psi|^2\psi\big) \|_{b'} \lesssim \| \langle\nabla\rangle^s\psi \|_b \| \; |\psi|^2 \; \|_g \; .$$

We need to control the nonlinear term by means of $\| \langle\nabla\rangle^s\psi \|_b$.

$b < 6 = \frac{2d}{d-2}$ (the endpoint for Strichartz) and by Sobolev embedding $1 - \frac{1}{\frac{2b}{b-2}} \leq \frac{s}{3}$, i.e., $s \geq \frac{3}{b}$. So we have to have $s > \frac{1}{2}$.

$$\begin{aligned}
\| \langle\nabla\rangle^s\psi \|_{r,q} &\lesssim \| \psi_0 \|_{H^s} + \Big(\int_0^T \| \langle\nabla\rangle^s\psi(t) \|_2^{2a'} \| \langle\nabla\rangle^s\psi \|^{a'}_{L^b_x} \; dt\Big)^{\frac{1}{a'}} \\
&\lesssim \| \psi_0 \|_{H^s} + \| \langle\nabla\rangle^s\psi \|^2_{L^\infty(I;L^2)} \, T^{1-\frac{2}{a}} \, \| \langle\nabla\rangle^s\psi \|_{L^a(I;L^b)} \qquad \big(I=[0,T]\big) \, .
\end{aligned}$$

By admissibility, $\frac{2}{a}+\frac{3}{b}=\frac{3}{2}$, so $b<6 \Leftrightarrow a>2$, so $1-\frac{2}{a}>0$.

Set $r=\infty$, $g=2$, and let $\mu :=\| \langle\nabla\rangle^s\psi \|_{\infty,2} + \| \langle\nabla\rangle^s\psi \|_{a,b}$

So, $\mu \le C_0 \| \psi_0 \|_{H^s} + C_1 T^{1-\frac{2}{a}}\mu^3$ and, for T small enough, $\mu \le 2c_0 \| \psi_0 \|_{H^s}$

For $s=\frac{1}{2}$, we can still prove the well-posedness.

EXERCISE: Prove it.

As for global well-posedness, we don't know about it for every $s>\frac{1}{2}$ (at the moment we know for $s>\frac{4}{5}$).

Prove local and global well-posedness of $i\partial_t\psi+\frac{1}{2}\Delta\psi+\lambda|\psi|^{2\sigma}\psi=0$ when $d\sigma<2$.

Now we want to prove Leibnitz for $\langle\nabla\rangle^s$, $s>0$, i.e.,

$$\| \langle\nabla\rangle^s(fg) \|_p \lesssim \| \langle\nabla\rangle^s f \|_{p_1} \| g \|_{p_2} + \| \langle\nabla\rangle^s g \|_{p_3} \| f \|_{p_4} \ ,$$

where $1<p<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$, $p_1,p_2,p_3,p_4\in\langle 1,\infty\rangle$.

(Note that the estimate $\| \langle\nabla\rangle^s\big(|\psi|^2\psi\big) \|_{b'} \lesssim \| \langle\nabla\rangle^s\psi \|_b \| \ |\psi|^2 \|_q$ follows from Leibnitz for $\langle\nabla\rangle^s$ applied to the product $\psi\psi\bar{\psi}$.)

The question is: is $\frac{\nabla}{\langle\nabla\rangle}=\frac{\xi}{(1+|\xi|^2)^{\frac{1}{2}}}$ a bounded multiplier?

The idea of the proof of Leibnitz for $\langle\nabla\rangle^s$ is as follows:

$$f=\sum_{j\ge0}\Delta_j f \ , \quad g=\sum_{k\ge0}\Delta_k g \ .$$

$$\begin{aligned}
\| \langle\nabla\rangle^s(fg) \|_p &= \| \sum_{j,k\ge0}\langle\nabla\rangle^s\Delta_j f\Delta_k g \|_p \\
&= \| \sum_j \sum_{k\le j-10}\langle\nabla\rangle^s\Delta_j f\Delta_k g \|_p \quad \leftarrow A \ \text{Fourier support here is of size} \approx 2^j \\
&\quad + \| \sum_k \sum_{j\le k-10}\langle\nabla\rangle^s\Delta_j f\Delta_k g \|_p \quad \leftarrow B \\
&\quad + \| \sum_{|j-k|<10}\langle\nabla\rangle^s\Delta_j f\Delta_k g \|_p \quad \leftarrow C
\end{aligned}$$

A should behave like this:

$$\begin{aligned}
A &\lesssim \| \Big(\sum_{j\ge0} 2^{2js}|\Delta_j f|^2 |\sum_{k\le j-10}\Delta_k g|^2\Big)^{\frac{1}{2}} \|_p \quad \swarrow \text{this is called the paraproduct} \\
&\lesssim \| \Big(\sum_{j\le0} 2^{2js}|\Delta_j f|^2\Big)^{\frac{1}{2}} Mg \|_p \\
&\lesssim \| \Big(\sum_{j>0} 2^{2js}|\Delta_j f|^2\Big)^{\frac{1}{2}} \|_{p_1} \| Mg \|_{p_2} \\
&\lesssim \| f \|_{W^{s,p_1}} \| g \| \ p_2
\end{aligned}$$

the same for B

We expect

$$C \lesssim \| \sum_{|j-k|<10} 2^{js}|\Delta_j f|\,|\Delta_k g| \ \|_p$$
$$\lesssim \Big(\sum_j 2^{2js}|\Delta_j f|^2\Big)^{\frac{1}{2}} \Big(\sum_k |\Delta_k g|^2\Big)^{\frac{1}{2}} \ \|_p$$
$$\lesssim \| \ \Big(\sum_j 2^{2js}|\Delta_j f|^2\Big)^{\frac{1}{2}} \ \|_{p_1} \| \ \Big(\sum_k |\Delta_k g|^2\Big)^{\frac{1}{2}} \ \|_{p_2} \lesssim \| \ f \ \|_{W^{s,p_1}} \| \ g \ \|_{p_2}$$

Lemma.

(*i*) *Let* $\{f_k\}_{k\geq 0}$ *be some functions with* $\text{supp}\hat{f}_k \subseteq \{c_0 2^k \leq |\xi| < C_0 2^k\}$, *for* $k \geq 1$, *and* $supp\hat{f}_0 \subseteq \{|\xi| \leq C_0\}$.

Then $\| \ \langle\nabla\rangle^s \sum_{k=0}^{\infty} f_k \ \|_p \lesssim \| \ \Big(\sum_{k=0}^{\infty} 2^{ks}|f_k|^2\Big)^{\frac{1}{2}} \ \|_p$.

The reverse holds if $f_k = \Delta_k f$.

(*ii*) *Let* $\{f_k\}_{k\geq 0}$ *be such that* $\text{supp}\hat{f}_k \subset B(0, C_0 2^k)$.

Then $\| \ \langle\nabla\rangle^s \sum_{k=0}^{\infty} f_k \ \|_p \lesssim \| \ \Big(\sum_{k=0}^{\infty} 2^{2ks}|f_k|^2\Big)^{\frac{1}{2}} \ \|_p$.

Exercise:

Clean up A,B,C by the lemma.
(In part C, don't do C − S.
Comment:

$$\| \sum_{|j-k|\leq 10} \langle\nabla\rangle^s \Delta_j f \Delta_k g \ \|_p \lesssim \| \ \Big(\sum 2^{2sj}|\Delta_j f|^2 |\Delta_k g|^2\Big)^{\frac{1}{2}} \ \|_p$$
$$\lesssim \| \ \Big(\sum_{j\geq 0} 2^{2sj}|\Delta_j f|^2\Big)^{\frac{1}{2}} Mg \ \|_p \lesssim \| \ f \ \|_{W^{s,p_1}} \| \ g \ \|_{p_2} \quad .)$$

Proof of the lemma. Pick ν is large that dist (supp $\hat{f}_k$, supp $\hat{f}_{k+\nu}$) $\sim 2^k$, $\forall k$ (So, $c_0 2^\nu > C_0$)

$$\| \ \langle\nabla\rangle^s \sum_{k\geq 0} f_k \ \|_p \leq \sum_{j=0}^{\nu-1} \| \ \langle\nabla\rangle^s \sum_{k\equiv j (\text{mod } \nu)} f_k \ \|_p$$
$$= \sum_{j=0}^{\nu-1} \| \ \sum_{k\equiv j} \langle\nabla\rangle^s 2^{-ks}\psi_k * 2^{2k} f_k \ \|_p$$
$$= \sum_{j=0}^{\nu-1} \| \ (m_{s,j}\hat{F}_j)^\vee \ \|_p \ ,$$

where $F_j = \sum_{k\equiv j} 2^{ks} f_k$ and

$$m_{s,j}(\xi) = \sum (1+|\xi|^2)^{\frac{s}{2}} 2^{-ks}\hat{\psi}_k(\xi)$$

If ψ_k is obtained by scaling, $m_{s,j}$ is a Mikhlin multiplier and we have

$$\sum_{j=0}^{\nu-1} \| (m_{sj}\hat{F}_j)^{\vee} \|_p \lesssim \sum_{j=0}^{\nu-1} \| F_j \|_p \lesssim \sum_{j=0}^{\nu-1} \| (\sum_{k\equiv j} 2^{2ks}|f_k|^2)^{\frac{1}{2}} \|_p$$

$$\lesssim \| (\sum_{k\geq 0} 2^{2ks}|f_k|^2)^{\frac{1}{2}} \|_p \quad .$$

□

Ma 142
4/17/2003

Lemma from last time

(*i*) $\{f_k\}_{k\geq 0}$ so that supp $\hat{f}_k \subset \{C_0 2^k \leq |\xi| \leq C_0 2^k\}$ if $k \geq 1$ and supp $\hat{f}_0 \subset \{|\xi| \leq C_0\}$. Then

$$\| \langle \nabla \rangle^s \sum_{k\geq 0} f_k \|_{L^p} \lesssim \| \left(\sum_{k\geq 0} 2^{2sk}|f_k|^2 \right)^{\frac{1}{2}} \|_{L^p}$$

if $1 < p < \infty$ and $s \in \mathbb{R}$. Conversely, if $f_k = \Delta_k(f)$ (= Littlewood-Paley projection), then the converse holds.

(*ii*) If $\{f_k\}_{k\geq 0}$ are such that supp $\hat{f}_k \subset \{|\xi| \leq C_0 2^k\}$ for $k \geq 0$, then

$$\| \langle \nabla \rangle^s \sum_{k\geq 0} f_k \|_p \lesssim \| \left(\sum_{k\geq 0} 2^{2sk}|f_k|^2 \right)^{\frac{1}{2}} \|_p$$

if $s > 0$ and $1 < p < \infty$.

Proof (continuation): The converse

$$\| \left(\sum_{k\geq 0} 2^{2sk}|\Delta_k(f)|^2 \right)^{\frac{1}{2}} \|_p = \| \left(\sum_{k\geq 0} 2^{2sk}|\Delta_k(\tilde{\psi} \star f)|^2 \right)^{\frac{1}{2}} \|_p$$

where $\widehat{\Delta_k(f)}(\xi) = \hat{\psi}(2^{-k}\xi)\hat{f}(\xi)$.

($\hat{\tilde{\psi}} = 1$ on supp $\hat{\psi}$ and $\hat{\tilde{\psi}}$ does not pile up: $\sum_k |\hat{\tilde{\psi}}(\xi)|^2 \leq 3$.

$$\hat{\tilde{\psi}}_k(\xi) = \hat{\tilde{\psi}}(2^{-k}\xi)$$

$$\lesssim \| \sum_{k\geq 0} 2^{sk}\tilde{\psi}_k * f \|_\rho$$

↑
by LP

$$= \| \Big(\underbrace{\sum_{k\geq 0} 2^{sk}(1+|\xi|^2)^{-\frac{s}{2}}\hat{\tilde{\psi}}(2^{-k}\xi)}_{m_s(\xi)}(1+|\xi|^2)^{+\frac{s}{2}}\hat{f}(\xi) \Big)^{\vee} \|_p$$

with $\| m_s \|_\infty$ and $|\partial^\alpha m_s(\xi)| \leq C_\alpha(1+|\xi|)^{-|\alpha|}$ because, for example,

$$|\nabla m_s(\xi)| \lesssim \sum (1+|\xi|^2)^{-\frac{s}{2}} \{ (1+|\xi|)^{-1} 2^{sk} |\hat{\tilde{\psi}}(2^{-k}\xi)|$$

$$+\,2^{sk}\underbrace{2^{-k}|(\nabla\hat{\hat{\psi}})(2^{-k}\xi)}_{\sim(1+|\xi|)^{-1}|(\nabla\hat{\hat{\psi}})(2^{-k}\xi)|}\ \}$$

So m_s is a Mikhlin multiplier.

$$\lesssim\|\ \left((1+|\xi|^2)^{\frac{s}{2}}\hat{f}(\xi)\right)^{\vee}\ \|_p=\|\ f\ \|_{W^{s,p}}$$

Remark for (ii), what about $s=0$?

$$p=2,s=0\ ,\qquad \|\ \sum_k f_k\ \|_2^2\overset{?}{\lesssim}\sum_k\|\ f_k\ \|_2$$

would be true only if there was some orthogonality, not in general. (e.g., take $\hat{f}_k=\chi_{[0,1]}$, $N^2\overset{?}{\lesssim}N$, no.)
Proof of (ii). Use fact that weight $(1+|\xi|^2)^{\frac{s}{2}}$ is bigger at the endpoints of the interval $[-C_0 2^k, C_0 2^k]$ in 1-D.
From (i),

$$\begin{aligned}\|\ \sum_{k\ge 0}f_k\ \|_{W^{s,p}}&\lesssim\|\ \Big(\sum_{l\ge 0}2^{2sl}\Big|\Delta_l\sum_{k\ge l}f_k\Big|^2\Big)^{\frac12}\ \|_p\\&=\|\ \sum_{l\ge 0}\Big|\sum_{k\ge l}2^{s(l-k)}\psi_l*\underbrace{2^{sk}f_k}_{w_k}\Big|^2\Big)^{\frac12}\ \|_p\\&\overset{?}{\lesssim}\|\ \Big(\sum_{k\ge 0}\underbrace{2^{2sk}|f_k|^2}_{|w_k|^2}\Big)^{\frac12}\ \|_p\end{aligned}$$

is the statement

$$\begin{gathered}\|\ T\big(\{w_k\}_{k\ge 0}\big)\ \|_{L^p(l^2)}\overset{?}{\lesssim}\|\ \{w_k\}_{k\ge 0}\ \|_{L^p(l^2)}\\T\big(\{w_k\}_{k\ge 0}\big)=\{\sum_{k\ge l\ge 0}2^{s(l-k)}\psi_l*w_k\}_{l\ge 0}\end{gathered}$$

$\to$ vector-valued singular integral.
By Hilbert space-valued Calderon-Zygmund theory,

$\alpha)$ $T:L^2(l^2)\to L^2(l^2)$ bounded
$\beta)$ kernel matrix element $K_{kl}=2^{s(l-k)}\psi_l$,
$\|\ K(x)\ \|_{l^2\to l^2}\le B.|x|^{-d}$
$\gamma)$ $\|\ \nabla K(x)\ \|_{l^2\to l^2}\le B.|x|^{-d-1}$

why $\gamma)$ is needed? $1/|x|*f$ does not make sense.
CZ-theorem: $\alpha)$, $\beta)$, $\gamma)$ $\Rightarrow\|\ T\ \|_{L^p(l^2)\to L^p(l^2)}\le C_{d,p}B$,

$$1<p<\infty$$

Show α)

$$\begin{aligned}
\| T\{w_k\}_{k\geq 0} \|^2_{L^2(l^2)} &= \int \Big(\sum_{l\geq 0} \Big| \sum_{k\geq l\geq 0} K_{kl} * w_k \Big|^2 \Big)\, dx \\
&= \sum_{l\geq 0} \int \Big| \sum_{k\geq l} 2^{s(l-k)} \widehat{\psi(2^{-l}\xi)} \hat{w}_k(\xi) \Big|^2 \, d\xi \\
&\lesssim \int \underbrace{\sum_{l\geq 0} |\hat{\psi}(2^{-l}\xi)|^2}_{\leq C} \underbrace{\sum_{k\geq l} 2^{2s(l-k)}}_{\leq C(s)} \sum_{k\geq l} |\hat{w}_k(\xi)|^2 \, d\xi \\
&\leq C(s) \, \| \{w_k\}_{k\geq 0} \|^2_{L^2(l^2)} \quad .
\end{aligned}$$

Show β): Schur's test

$$\begin{aligned}
&(1) \quad \sup_l \sum_{k\geq l} |K_{kl}(x)| \overset{?}{\leq} B|x|^{-d} \\
&(2) \quad \sup_k \sum_{k\geq l\geq 0} |K_{kl}(x)| \overset{?}{\leq} B|x|^{-d}
\end{aligned}$$

$$\begin{aligned}
(1) \quad & \sum_{k\geq l} 2^{s(l-k)} |\psi_l(x)| \\
& \leq C(s)|\psi_l(x)| = C(s) 2^{ld} |\psi(2^l x)| \\
& \leq C_N(s) 2^{ld} (1 + 2^l|x|)^{-N} \qquad N \text{ big.} \\
& \lesssim \begin{cases} |x|^{-d} & \text{for } 2^l|x| < 1 \\ 2^{l(d-N)}|x|^{-N} < |x|^{-d} & \text{for } 2^l|x| > 1 \, . \end{cases} \\
(2) \quad & \sum_{k\geq l\geq 0} 2^{s(l-k)} |\psi_l(x)| \leq C_N \sum_{k\geq l\geq 0} 2^{s(l-k)} (1 + 2^l|x|)^{-N} \\
& \lesssim \underbrace{\sum_{\substack{k\geq l\geq 0 \\ 2^l|x|<1}} \underbrace{2^{s(l-k)}}_{\leq 1} 2^{ld}}_{\lesssim |x|^{-d}} + \underbrace{\sum_{\substack{k\geq l\geq 0 \\ 2^l|x|>1}} \underbrace{2^{s(l-k)}}_{\leq 1} 2^{ld} (2^l|x|)^{-N}}_{\lesssim |x|^{-d-1} \left(|x|^{-1} \right)^{-1} = |x|^{-d}} \\
& \qquad \text{take } N = d+1,
\end{aligned}$$

Show γ) idem, 2^{ld} becomes $2^{l(d+1)}$, etc.

Note: Vector-valued CZ theory trivially follows from scalar $C2$ theory.

Standing Waves and Calculus of Variations

$$i\partial_t\psi + \frac{1}{2}\Delta\psi + f(|\psi|^2)\psi = 0$$

Try solutions of the form $\psi(t,x) = e^{\frac{ix^2}{2}t}\phi(x)$

$$0 = -\frac{\alpha^2}{2}\phi e^{i\frac{\alpha^2}{2}t} + e^{i\frac{\alpha^2}{2}t}\frac{1}{2}\Delta\phi + f(|\phi|^2)\phi e^{i\frac{x^2}{2}t}$$
$$0 = -\frac{\alpha^2}{2}\phi + \frac{1}{2}\Delta\phi + f(|\phi|^2)\phi \qquad \text{(time indep.)}$$

Comments: $f(u) = \lambda u^\sigma$, $\sigma > 0$.

If $\lambda = -1$ (defocussing), it follows from the pseudo-conformal identity that

$$\| \psi(t) \|_{2\sigma+2}^{2\sigma+2} \lesssim t^{-2} \, .$$

$\Rightarrow$ You can't have standing waves for $\lambda < 0$. Furthermore, you want ϕ to decay nicely at ∞. The nonlinear term is higher-order $\Rightarrow$ negligible at ∞. Look at the linear part, solutions look like $\phi(x) \sim e^{-\alpha|x|}$, okay. ($\alpha^2 > 0$ ensures we are above/below the spectrum of $-\Delta/\Delta$).

Moreover

$$0 = \int -\frac{\alpha^2}{2}|\phi|^2 - \frac{1}{2}|\nabla\phi|^2 + \lambda|\phi|^{2\sigma+2}\, dx \Rightarrow \lambda > 0$$

Side note: $i\partial_t\psi + \frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi = 0$

$$\mathcal{H}\big(\psi(t)\big) = \int \Big\{ \frac{1}{2}|\nabla\psi|^2 - V|\psi|^2 - F(\psi) \Big\}\, dx$$
$$= \mathcal{H}(\psi_0)$$

Next time: non-uniqueness of solutions to

$$\begin{cases} -\Delta u = mu - |u|^{2\sigma}u \\ u = 0 \qquad \text{on } \partial\Omega \end{cases}$$

$$\mathcal{L}[u] = \int_\Omega \Big\{ \frac{1}{2}|\nabla u|^2 - F(u) \Big\}\, dx$$

i.e., non-uniqueness of the minimizers (local minima).

$$\inf_\gamma \max_{0\le t\le 1} \mathcal{L}[\gamma(t)] \, .$$

is here a saddle point.

Ma 142 4/22/2003

Please look through Exercises 4)–11) within this coming week (most important 4)–9)).

Last time we started with *standing wave*. Solutions to

$$i\partial_t\psi + \frac{1}{2}\Delta\psi + f(|\psi|^2)\psi = 0$$
$$\psi(t,x) = e^{i\frac{\alpha^2}{2}t}\phi(x)$$
$$\Rightarrow -\frac{1}{2}\alpha^2\phi + \frac{1}{2}\Delta\phi + f(\phi^2)\phi = 0$$

Continuing formally

$$\mathcal{G}_{\vec{v},y}(t) := e^{-it\frac{|v|^2}{2}}e^{-i\vec{v}\cdot x}e^{i(y+t\vec{v})\cdot\vec{p}}$$

classically:

$$x \mapsto x - y - t\vec{v}$$
$$\xi \mapsto \xi - \vec{v}$$
$$\mathcal{G}_{\vec{v},y}(t)e^{i\frac{t}{2}\Delta} = e^{i\frac{t}{2}\Delta}\mathcal{G}_{\vec{v},y}(0)$$

Define $\phi(t) = \mathcal{G}_{\vec{v},y}(t)\psi(t)$ for ψ a solution.

Then

$$\phi(t) = \mathcal{G}_{\vec{v},y}(t)\{e^{i\frac{t}{2}\Delta}\psi(0) + i\int_0^t e^{i(t-s)\frac{\Delta}{2}}f(|\psi(s)|^2)\psi(s)\, ds$$

$$= e^{i\frac{t}{2}\Delta}\phi(0) + i\int_0^t e^{i(t-s)\frac{\Delta}{2}}\underbrace{\mathcal{G}_{\vec{v},y}(s)f(|\psi(s)|^2)\psi(s)\, ds}_{f(|\phi(s)|^2)\phi(s)}$$

$$\Rightarrow \begin{cases} i\partial_t\phi + \frac{1}{2}\Delta\phi + f(|\phi|^2)\phi = 0 \\ \phi|_{t=0} = \mathcal{G}_{\vec{v},y}(0)\psi_0 \end{cases}$$

So, again for standing waves

$$e^{i\gamma}\mathcal{G}_{\vec{v},y}(t)[e^{i\frac{\alpha^2}{2}t}\phi] = e^{\frac{i}{2}(\alpha^2-|v|^2)t}e^{i\gamma}e^{ix\cdot\vec{v}}\phi(x-y-t\vec{v})$$

depends on $(\alpha,\gamma,y,\vec{v}) \in \mathbb{R}^{2d+2}$.

Solve: Let $m \geq 0$

$$\left[\begin{array}{ll} -\Delta u + mu = f(u) & \text{in } \Omega \subset \mathbb{R}^d \text{ bdd.} \\ u = 0 & \text{on } \partial\Omega \end{array}\right.$$

$$\mathcal{L}[u] = \int_\Omega \Big\{\frac{1}{2}|\nabla u|^2 + \frac{m}{2}u^2 - F(u)\Big\}(x)\, dx$$
$$u \in C_0^\infty(\Omega)$$
$$u_\varepsilon(x) = u(x) + \varepsilon\phi(x)$$

$$0 = \frac{d}{d\varepsilon}|_{\varepsilon=0}\mathcal{L}[u_\varepsilon] = \int_\Omega (\nabla u \cdot \nabla\phi + mu\phi - f(u)\phi)(x)\, dx$$

$$\uparrow$$
$$\forall\phi$$

$$= \int_\Omega \big(-\Delta u \cdot \phi + mu\phi - f(u)\phi\big)(x)\, dx$$

$$\Rightarrow \Delta u - mu + f(u) = 0\ .$$

Abstract machinery for finding critical points
$\mathcal{H}$ = Hilbert space
$I \in C^1(\mathcal{H}_i\mathbb{R})$
$I[u+w] = I[u] + \langle I'[u], w\rangle + 0(\| w \|^2)$
$I' : \mathcal{H} \to \mathcal{H}^*$ locally-Lipschitz
$\mathcal{A}_c\{u \in \mathcal{H} | I[u] \le c\}$
$\mathcal{K}_c\{u \in \mathcal{H} | I[u] = c, I'[u] = 0\}$
Palais-Smale condition (PS):
if $\{u_K\}_{K=1}^\infty$ in $\mathcal{H}$ s.t. $\{I[u_k]\}_k$ is bounded and $I'[u_k] \to 0$, then $\{u_k\}_k$ precompact in $\mathcal{H}$.

Remark:
For $F(u) = |u|^4$, $\mathcal{H} = H_0^1$

MP Theorem Let I be as above and (PS). Assume $I[0] = 0$, $\exists r, a > 0$ with $I[u] > a$ if $\| u \| = r$ and $\exists v \in \mathcal{H}$, $\| v \| > r$, $I(v) \le 0$. Let

$$C = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I[\gamma(t)]\ ,\ \Gamma = \{\gamma : (0,1) \to \mathcal{H} : \gamma(0) = 0\ ;\ \gamma(1) = v\}$$

Then C is a critical value.

Deformation Theorem I satisfies (PS) and $\mathcal{K}_c = \emptyset$. Then $\exists\varepsilon > \delta > 0$ and

$$\eta : [0,1] \times \mathcal{H} \to \mathcal{H} \text{ with}$$

(i) $\eta_0(u) = u$,
(ii) $\eta(t,u) = u$, $\forall t \in [0,1]$, $\forall u$ with $I[u]$ and $(c-\varepsilon, c+\varepsilon)$
(iii) $\eta_1(\mathcal{A}_{c+\delta}) \subset \mathcal{A}_{c-\delta}$;
(iv) $I[\eta_t(u)] \le I[u]$, $\forall u \in \mathcal{H}$, $\forall t \in [0,1]$

Proof. Idea is to solve:

$$\begin{cases} \dot\eta(t) = -I'[\eta] \\ \eta(0) = u \end{cases}$$

$$\frac{d}{dt} I[\eta] = \langle I'[\eta], \dot\eta\rangle = -\| I'[\eta] \|^2$$

modify this flow to move *only* those u with $I[u]$ close to c!

Claim: $\exists\sigma, \varepsilon > 0$ so that

$$\inf_{u:|I[u]-c|\le\varepsilon} \| I'[u] \| > \sigma > 0\ .$$

If not:

$$\left.\begin{array}{l} I'[u_k] \to 0 \\ I[u_k] \to C \end{array}\right\} \overset{PS}{\Rightarrow} \quad \begin{array}{c} u_k \overset{\mathcal{H}}{\to} u_\infty \ , \\ I'[u_\infty] = 0 \ ; \\ I[u_\infty] = c \end{array}$$

$$\Rightarrow u_\infty \in \mathcal{K}_c = \emptyset : \text{contradiction}$$

So solve:

$$\begin{cases} \dot\eta(t) = -I'[\eta]g(\eta) \\ \eta(0) = u \end{cases} \quad \left[\Rightarrow \frac{d}{dt} I[\eta] = - \parallel I'[\eta] \parallel^2 g(\eta)\right]$$

Define

$$A = \{u \in \mathcal{H} : I[u] \in (c-\delta, c+\delta)\}$$

$$B = \{u \in \mathcal{H} : I[u] \notin (c-\varepsilon, c+\varepsilon)\}$$

$$h(u) = \frac{\operatorname{dist}(u,B)}{\operatorname{dist}(u,B) + \operatorname{dist}(u,A)} \Rightarrow \begin{array}{l} h = 0 \text{ on } B \\ h = 1 \text{ on } A \end{array}$$

$$g(\eta) = h(\eta)$$

So solve on $t \in [0,1]$;

If $I[\eta_t(u)] \le c - \delta$ for some t and u $\Rightarrow$ okay.

WLOG: $\quad c + \delta > I[\eta_t(u)] > c - \delta$ for every $0 \le t \le 1$

$$\Rightarrow \quad I[\eta_1(u)] < c + \delta - \sigma^2 < c - \delta \text{ (contradiction for } \delta \ll 1).$$

□

Proof of MP. Suppose $\mathcal{K}_c = \emptyset$. Pick $\varepsilon, \delta > 0$ like in Deformation Theorem and

$$\max_{0 \le t \le 1} I[\gamma(t)] < c + \delta$$

Define $\tilde\gamma(t) = \eta_1[\gamma(t)]$

$$\tilde\gamma \in \Gamma \ , \ \eta_1(v) = 0 \ , \ n_1(v) = v$$

$$\text{as} \quad 0 = I(0) \notin (c-\varepsilon, c+\varepsilon)$$

$$0 \ge I(v) \notin (c-\varepsilon, c+\varepsilon)$$

$$\max_{0 \le t \le 1} I[\tilde\gamma(t)] \ < c - \delta : \text{contradiction}$$

□

Remark Existence of solution to $(*)$ on $[0,1]$):

Define $T = \sup\{t : \exists \eta\}$

$$T < \infty \Rightarrow \int_0^T \parallel I'[\eta(t)] \parallel g\big(\eta(t)\big)\, dt = \infty$$

$$\Rightarrow \exists T_1 < T \text{s.t.} I[\eta(T_1)] < c - 10^6$$

$$\text{as} \int_0^T \parallel I'[\eta] \parallel^2 g(\eta)\, dt = \infty$$

$$\Rightarrow \forall t < T_1, \| I'(\eta) \| \, g(\eta) = 0 \ : \ \text{contradiction}$$

$$I[u] = \int\limits_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + \frac{m}{2}u^2 - F(u) \right\} dx \ , \ m \geq 0 \ .$$

Theorem. *Suppose f is such that*

$$|f'(z)| \leq C(1 + |z|^{p-1})$$
$$|f(z)| \leq C(1 + |z|^p) \ , \ p < \frac{d+2}{d-2} \ .$$

$$F(z) = \int\limits_0^z f(u) \, du \ \textit{with} \ 0 \leq F(z) \leq \gamma z f(z) \ , \ \gamma < \frac{1}{2}$$
$$\forall z \in \mathbb{R} \ .$$

and $F(z) > \beta > 0$ if $|z| > R$

Then $\exists$ a non-zero weak solution $u \in H_0^1(\Omega)$ of $-\Delta u + mu = f(u)$., i.e.,

$$\int\limits_\Omega \nabla u \cdot \nabla v + muv = \int\limits_\Omega f(u) v$$
$$\forall v \in H_0^1(\Omega)$$
$$(H_0^1(\Omega) = \overline{C_0^\infty}^{\|\cdot\|_{H^1}})$$

Let $\mathcal{H} = H_0^1$

Lemma. *$I \in C^1(\mathcal{H};\mathbb{R})$, $I' \in Lip_{loc}$ and*

$$\langle I'[u], w \rangle = \int\limits_\Omega \{ \nabla u \cdot \nabla w + muw - f(u)w \} \, dx$$

Proof.

$$I[u+w] = I[u] + \overbrace{\int\limits_\Omega \{ \nabla u \cdot \nabla w + muw - f(u)w \} \, dx}^{\stackrel{def}{=} \langle I'[u], w \rangle}$$
$$\underbrace{+ \int\limits_\Omega \left(\frac{1}{2}|\nabla w|^2 + \frac{m}{2}w^2 \right) dx}_{= O(\|w\|_{\mathcal{H}}^2)} + \underbrace{\int\limits_\Omega \Big(-F(u+w) + f(u)w + F(u) \Big) \, dx}_{R}$$

Now $\| u \|_{\mathcal{H}} = \int_\Omega |\nabla u|^2$

$$|-R| = \Big| \int\limits_\Omega \Big(F(u+w) - F(u) - f(u)w \Big) \, dx \Big|$$

$$
\begin{aligned}
&\leq \int_\Omega \Big(\int_0^1 (1-s)|f'(u+sw)||w|^2 \, ds \Big) \, dx \\
&\lesssim \int_\Omega (1+|u|^{p-1}+|w|^{p-1})|w|^2 \, dx \\
&\lesssim \Big(\int (1+|u|+|w|\Big)^{p+1}\Big)^{\frac{p-1}{p+1}} \underbrace{\Big(\int_\Omega |w|^{p+1} \Big)^{\frac{2}{p+1}}}_{0(\|w\|_{\mathcal{H}}^2)} \text{ (by Sobolev Embedding)}
\end{aligned}
$$

$$
\begin{aligned}
&\| I'[u_1] - I'[u_2] \|_{\mathcal{H}^*} = \sup_{\|v\|_{\mathcal{H}} \leq 1} |\langle I'[u_1] - I'[u_2], v \rangle| \\
&\lesssim \sup_{\|v\|_{\mathcal{H}} \leq 1} \Big\{ \int_\Omega \big(|\nabla(u_1-u_2)||\nabla v| + m|u_1-u_2||v| \big) \, dx + \int_\Omega |f(u_1)-f(u_2)||v| \, dx \Big\} \\
&\lesssim \| u_1 - u_2 \|_{\mathcal{H}} + \sup_{\|v\|_{\mathcal{H}} \leq 1} \int (1+|u_1|^{p-1}+|u_2|^{p-1})|u_1-u_2||v| \\
&\lesssim \| u_1 - u_2 \|_{\mathcal{H}} + \sup_{\|v\|_{\mathcal{H}} \leq 1} \underbrace{\Big(\int 1+|u_1|^{p+1}+|u_2|^{p+1} \Big)^{\frac{p-1}{p+1}}}_{C_R} \underbrace{\| u_1 - u_2 \|_{p+1}}_{\|u_1-u_2\|_{\mathcal{H}}} \| v \|_{p-1}
\end{aligned}
$$

□

Ma 142 4/24/2003

We are solving

$$
\left[\begin{array}{rl} -\Delta u + mu = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \right.
$$

conditions:

$$
m \geq 0 \ , \ \begin{array}{l} |f(z)| \leq C(1+|z|^p) \\ |f'(z)| \leq C(1+|z|^{p-1}) \end{array}
$$

$$
1 < p < \frac{d+2}{d-2}
$$

$$
\Big(\text{point: } \ p+1 < \frac{2d}{d-2} = 2^* \Big)
$$

$$
F(z) := \int_0^z f(w) \, dw \ , \ 0 \leq F(z) \leq \gamma z f(z)
$$

$$
0 < \gamma < \frac{1}{2}
$$

$$\mathcal{H} := H_0^1(\Omega)\ ,\ \| u \|_{\mathcal{H}}^2 := \int_\Omega |\nabla u|^2$$

$\Omega \subset \mathbb{R}^d$ bounded, nice boundary.

Lemma. *Let*

$$I[u] = \int_\Omega dx\Big(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}mu^2 - F(u)\Big)$$

Then $I \in C^1(\mathcal{H}, \mathbb{R})$, $I' \in Lip_{loc}$

$$\langle I'[u], v\rangle = \int_\Omega \big(\nabla u \nabla v + muv - f(u)v\big)\ dx$$

Lemma. *I satisfies Palais-Smale condition*

Proof. $\{u_k\} \in \mathcal{H}$, $I[u_k]$ bounded and

$$I'[u_k] \underset{\|\cdot\|_{\mathcal{H}^*}}{\longrightarrow} 0$$

$$|\langle I'[u_k], u_k\rangle| < \varepsilon \| u_k \|_{\mathcal{H}}$$

$$-C < \int_\Omega \frac{1}{2}|\nabla u_k|^2 + \frac{m}{2}u_k^2 - F(u_k) < C$$

$$-\varepsilon \| u_k \|_{\mathcal{H}} < \int_\Omega \frac{1}{2}|\nabla u_k|^2 + \frac{m}{2}u_k^2 - \frac{1}{2}f(u_k)u_k < \varepsilon \| u_k \|_{\mathcal{H}}$$

$$\Big(\frac{1}{2} - \gamma\Big)\int u_k f(u_k) \le \int \frac{1}{2}f(u_k)u_k - F(u_k) < C + \varepsilon \| u_k \|_{\mathcal{H}}$$

$$\frac{\frac{1}{2}-\gamma}{\gamma}\int_\Omega F(u_k)$$

$$\Rightarrow \frac{1}{2}\| u_k \|_{\mathcal{H}}^2 = \int \frac{1}{2}|\nabla u_k|^2 < C + \frac{\gamma}{\frac{1}{2}-\gamma}\Big(C + \varepsilon \| u_k \|_{\mathcal{H}}\Big)$$

$$\sup_k \| u_k \|_{\mathcal{H}} < \infty$$

$$\Rightarrow u_{kj} \to u_\infty \text{ in } H_0^1(\Omega) \hookrightarrow L^{P+1}(\Omega)\ , H_0^1(\Omega) \hookrightarrow W^{p+1,\varepsilon}$$

$$\| -\Delta u_k + mu_k - f(u_k) \|_{\mathcal{H}^*} \to 0$$

Exp of non-compact embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}$

$$\| \phi\Big(\frac{x}{\delta}\Big)\delta^{-\frac{d}{2}} \|_2$$

$$u_{k_n} \to u_\infty \text{ in } L^{p+1}(\Omega)$$

$$\| u_{kj} - u_{kl} \|_2 \to 0 \Rightarrow \| u_{kj} - u_{kl} \|_{\mathcal{H}^*}$$

$$= \sup |\langle u_{kj} - u_{kl}, v\rangle_{L^2}|$$

$$0 \overset{?}{\longleftarrow} \| f(u_{kj}) - f(u_{kl}) \|_{\mathcal{H}^*}\ 1 \ge \| v \|_{H_0^1}(\Omega)$$

$$= \sup_{\|v\|_{H_0^1(\Omega)} \leq 1} |\langle f(u_{kj}) - f(u_{kj}), v\rangle| \qquad \leq \| u_{kj} - u_{kl} \|_2 \| v \|_2$$

Thus

$$\| u_{kj} - u_{kl} \|_{\mathcal{H}^*} \to 0$$

$$\begin{aligned} &\lesssim \| f(u_{kj}) - f(u_{kl}) \|_{L^{\frac{p+1}{p}}} \\ &\lesssim \| (1 + |u_{kj}|^{p-1} + |u_{kl}|^{p-1})|u_{kj} - u_{kl}| \|_{L^{\frac{p+1}{p}}} \\ &\lesssim \| 1 + |u_{kj}| + |u_{kl}| \|_{p+1}^{p-1} \| u_{kj} - u_{kl} \|_{p+1} \to 0 \end{aligned}$$

$$0 \leftarrow \| -\Delta u_{kj} + \Delta u_{kl} \|_{\mathcal{H}^*}$$

$$\int |\nabla(u_{kj} - u_{kl}|^2 \leq \varepsilon \| u_{kj} - u_{kl} \|_{\mathcal{H}} = \varepsilon\Big(\int |\nabla(u_{kj} - u_{kl})|^2\Big)^{\frac{1}{2}}$$

□

Comment on last lecture:
Use/Solve:

$$\dot{\eta}_t = -\frac{I'[\eta_t]}{1 + I'[\eta_t]} g(\eta_t)$$

Theorem. *Assume in addition that*

$$F(z) = \beta > 0 \ \textit{if} \ |z| > R \, .$$

Then $\exists$ *a weak solution* $u \neq 0$ *of*

$$\left[\begin{array}{rcll} -\Delta u + mu & = & f(u) & \textit{in} \ \ \Omega \\ u & = & 0 & \textit{on} \ \partial\Omega \\ u \in H_0^1(\Omega) & & & \end{array}\right.$$

Proof.

$$\text{check these 3} \begin{cases} I[0] = 0 & \\ I[u] > a > 0 & \text{if} \quad \| u \|_{\mathcal{H}} = \tau > 0 \\ \exists v \in \mathcal{H}, \| v \|_{\mathcal{H}} > \tau \, , & I[v] \leq 0 \end{cases}$$

(1)
$$F(0) = 0$$

(2)
$$\begin{aligned} I[u] &\geq \frac{1}{2} \| u \|_{\mathcal{H}}^2 - \int\limits_{\{\Omega, |u| > \delta\}} |F(u)| dx - \int\limits_{\{\Omega, |u| < \delta\}} |F(u)| dx \\ &> \frac{1}{2} \| u \|_{\mathcal{H}}^2 - C_\delta \int\limits_{\Omega} |u|^{p+1} - \varepsilon \int\limits_{\Omega} |u|^2 \, dx \\ &> \Big(\frac{1}{2} - C^{\varepsilon}_{\text{Poincare}}\Big) \| u \|_{\mathcal{H}}^2 - C \| u \|_{\mathcal{H}}^{p+1} \qquad \text{Sobolev embed.} \\ &> \frac{1}{4} r^2 - C r^{p+1} \end{aligned}$$

$$\int |u|^2 \leq C_{\text{Poincare}} \int |\nabla_u|^2$$

(3)
$$\begin{aligned}
\big(z^{-\mu}F(z)\big)' &= z^{-\mu-1}\big(zF' - \mu F(z)\big)\\
&\geq z^{-\mu-1}\Big(\frac{1}{\gamma} - \mu\Big)F(z) > \beta\Big(\frac{1}{\gamma} - \mu\Big)z^{-\mu-1}\\
z^{-\mu}F(z) - R^{-\mu}F(R) &> \beta\Big(\frac{1}{2} - \mu\Big)\Big(\frac{R^{-\mu}}{\mu} - \frac{z^{-\mu}}{\mu}\Big) \quad \text{if } z \geq R\\
&\Rightarrow z > 200R \text{ implies}\\
F(z) &> z^{\mu}\frac{\beta\Big(\frac{1}{\gamma} - \mu\Big)}{2\mu}R^{-\mu}\\
F(z) &> a_1 z^{\mu} - A_1, \forall z
\end{aligned}$$

(Mountain pass in Rabinowitz)

$$\begin{aligned}
I[tv] &\leq C_v t^2 + A_1|\Omega| - a_1 \int_\Omega t^{\mu}|v|^{\mu}\, dx\\
&\longrightarrow -\infty \text{ for } \ t \to \infty
\end{aligned}$$

By MP $\exists u \neq 0$ with $\langle I'[u], v\rangle = 0 \ \forall v \in \mathcal{H}$

$$\Rightarrow \int \nabla u \nabla v + muv - f(u)v = 0 \qquad \forall v \in H_0^1(\Omega)$$

this is a weak solution ∎

— Regularity;
 — Is the weak sol. a classical solution?
 — Positivity of solutions?
 — Is it true that the solution $\in C^{\infty}(\bar{\Omega})$?

Regularity:

$$-\Delta u = f \in L^2_{loc}(\Omega) \Rightarrow u \in W^{2,2}_{loc}(\Omega)$$

$$\chi \text{ comp. supp } -\Delta(\chi u) = -\chi f - 2\nabla\chi\nabla u - u\nabla\chi$$

$$-\Delta u = f(u) \overset{?}{\in} L^2$$

$$\int |u|^{2p}, \infty \ ?$$

$$\frac{1}{2} - \frac{1}{2p} < \frac{1}{d}\,,\ \frac{1}{p'} < \frac{2}{d}\,,\ p' > \frac{d}{2}$$

$$p < \frac{d}{d-2}\,,\ p < \frac{d+2}{d-2}$$

Theorem. *Let f be as at the start at this lecture*

$$-\Delta u = f(u) \quad \textit{weak solution in } \Omega$$

$$\Rightarrow u \in W^{2,2}_{loc}(\Omega)$$

$$\Omega' + h\vec{e_i} \Subset \Omega$$
$$v_h(x) = \frac{1}{h}\big(u(x+h\vec{e_i}) - u(x)\big)$$
$$-\Delta v_h = \frac{1}{h}\Big(f\big(u(x+h\vec{e_i})\big) - f\big(u(x)\big)\Big)$$
$$\sup_{|h|<\varepsilon} \parallel v_h \parallel_{H^1(\Omega')} < \infty$$

$$\int \nabla v_h \nabla(\phi^2 v_h) = \int \frac{1}{h}\Big(f\big(u(x+h\vec{e_i})\big) - f\big(u(x)\big)\Big)\phi^2 v_h \, dx$$

$$\begin{aligned}
\int_\Omega \phi|\nabla v_h|^2 &\le 2\int_\Omega \phi|\nabla\phi|\,|v_h|\,|\nabla v_h| \\
&+ C\int_\Omega \big(1 + |u(x+h\vec{e_i})|^{p-1} + |u(x)|^{p-1}\big)|v_h(x)|^2\phi^2\,dx \\
&\le \frac{1}{2}\int \phi^2|\nabla v_h|^2 + 2\int|\nabla\phi|^2|v_h|^2 + \\
&+ \Big(\int\big(1+|u(x+h\vec{e_i})| + |u(x)|\big)^{(p-1)m'}\Big)^{\frac{1}{m'}} \times \Big(\int|\phi v_h|^{2m}\Big)^{\frac{1}{m}} = X
\end{aligned}$$

$$\left[\begin{array}{c} 2m < 2^* \\ (p-1)m' < 2^* \end{array}\right.$$

$$\begin{aligned}
2m = 2^* \Rightarrow m &= \frac{2^*}{2} = \frac{d}{d-2} \\
m' &= \frac{d}{2} \\
(p-1)\frac{d}{2} &\overset{?}{<} 2^* = \frac{2d}{d-2} \\
p-1 &< \frac{4}{d-2} \\
p &< \frac{d+2}{d-2} \qquad \checkmark
\end{aligned}$$

$\circledast \qquad 0 < \theta < 1$

$$\Big(\int|\phi v_h|^{2m}\Big)^{\frac{1}{m}} \le \Big(\int|\nabla(\phi v_k)|^2\Big)^{\theta} \times \Big(\int|\phi v_h|^2\Big)^{(1-\theta)}$$

$$\int_{\Omega'}|v_h|^2\,dx < C\int_\Omega|\nabla u|^2\,dx$$

$$\sup_{|h|<\varepsilon}\int_{\Omega'}|\nabla v_h|^2\,dx < \infty$$

Ma 142 5/1/2003

Sogge's Estimate:

$$\| H_k f \|_{L^{p'}(S^{d-1})} \leq C(1+k)^{1-\frac{2}{d}} \| f \|_{L^p(S^{d-1})} \qquad p = \frac{2d}{d+2}\ ,\ p' = \frac{2d}{d-2}$$

(H_k = orthogonal projection onto k^{th} spherical harmonics, $k \geq 0$)

$$(-\Delta_{S^{d-1}} Y_k = k(k+d-2)Y_k)$$

Carleman:

$$\| \ |x|^{-\tau} u \|_{L^{q'}(\mathbb{R}^d)} \leq C_\delta \| \ |x|^{-\tau} \Delta u \|_{L^q(\mathbb{R}^d)}\ ,\ q = \frac{2d}{d+2}\ ,\ q' = \frac{2d}{d-2}$$

$$\tau \geq 0\ ,\ \text{dist}\Big(\tau, \mathbb{Z} + \frac{\mathrm{d}-2}{2}\Big) > \delta > 0$$

$$u \in C^\infty_{\text{comp}}(\mathbb{R}^d \backslash \{0\})$$

Proof. (Using Sogge)

$$r = e^t\ ,\ -\infty < t < \infty\ ,\ w \in S^{d-1}$$

$$\Delta = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{d-1}}\ ,\ \partial_r = \frac{1}{r}\partial_t\ ,\ \partial_r^2 = \frac{1}{r^2}\partial_t^2 - \frac{1}{r^2}\partial_t$$

$$\Delta = e^{-2t}[\partial_t^2 + (d-2)\partial_t + \Delta_{S^{d-1}}]\ ,\ dx = e^{dt}\ dt\ dw$$

$$\| \ |x|^{-\tau} u \|_{L^{q'}(dx)} = \| e^{-t\tau} d^{\frac{dt}{q'}} u \|_{L^{q'}(dt\ dw)} = \| e^{-t\tau} e^{\frac{d-2}{2}t} u \|_{L^{q'}(dt\ dw)}$$

$$\| \ |x|^{-\tau} \Delta u \|_{L^q(dx)} = \| e^{-t\tau} e^{\frac{d+2}{2}t} e^{-2t} Qu \|_{L^q(dt\ dw)} \qquad Q = \partial_t^2 + (d-2)\partial_t + \Delta_{S^d}$$

$$\| e^{-t(\tau - \frac{d-2}{2})} u \|_{L^{q'}_{t,w}} \overset{?}{\leq} C \| e^{-t(\tau-\frac{d-2}{2})} Qu \|_{L^q_{t,w}}$$

$$\| u \|_{L^{q'}_{t,w}} \overset{?}{\leq} C \| Q_\tau u \|_{L^q_{t,w}} \quad \text{where } Q_\tau = e^{-t(\tau-\frac{d-2}{2})} Q e^{t(\tau - \frac{d-2}{2})}$$

supp $u(\cdot, w)$ is compact, so (there u suddenly becomes f):

in L^2_w $\downarrow$ Fourier and inv. Fourier (basically $\hat{\delta}_{t-s} = e^{i\eta(t-s)}$) $\downarrow$

$$f(t,w) = \sum_{k\geq 0} \big(H_k f(t,\cdot)\big)(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k \geq 0} \big(H_k f(s,\cdot)\big)(w) e^{i\eta(t-2)} ds\, d\eta$$

$$(Q_\tau f)(t,w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k\geq 0} e^{-t[\tau - \frac{d-2}{2}]} Q\big\{\big(H_k f(s,\cdot)\big)(w) e^{t[i\eta + \tau - \frac{d-2}{2}]}\big\} e^{-i\eta s} ds\ d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k \geq 0} e^{i\eta t} \Big(-k(k+d-2) + \Big(i\eta + \tau - \frac{d-2}{2}\Big)^2$$

$$+ (d-2)\Big(i\eta + \tau - \frac{d-2}{2}\Big)\Big) H_k f(s,w) e^{-i\eta s} ds\, d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k\geq 0} \Big((i\eta+\tau)^2 - \Big(k + \frac{d-2}{2}\Big)^2\Big) H_k f(s,w) e^{i\eta(t-s)} ds\, d\eta$$

$(Q_t^{-1} f)(t,w) = (T_\tau f)(t,w)$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k\geq 0} \Big(i\eta + \tau + k + \frac{d-2}{2}\Big)^{-1} \Big(i\eta\tau - k - \frac{d-2}{2}\Big)^{-1} H_k f(s,w) e^{i\eta(t-s)} ds\, d\eta$$

Need to show $\| T_\tau f \|_{L^{q'}_{t,w}} \leq C \| f \|_{L^q_{t,w}}$

$$\| T_\tau f(t,\cdot) \|_{L^{q'}_w} \leq \int_{-\infty}^{\infty} \sum_{k\geq 0} |m_k(\tau,t,s)| \; \| H_k f(\delta,\cdot) \|_{L^{q'}_w} \; ds$$

$$\overset{\text{Sogge}}{\leq} C \int_{-\infty}^{\infty} \sum_{k\geq 0} |m_k(\tau,t,s)| (1+k)^{1-\frac{2}{d}} \| f(s,\cdot) \|_{L^q_w} \; ds \qquad \text{where}$$

$$m_k(\tau,t,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\eta(t-s)}\, d\eta}{(i\eta + \tau + k + \frac{d-2}{2})(i\eta + \tau - k - \frac{d-2}{2})} \quad \Big(\text{note: } \Big|\tau \pm \Big(k + \frac{d-2}{2}\Big)\Big| > \delta\Big)$$

so $|m_k(\tau,t,s)| \lesssim (1+k)^{-1} \exp(-|t-s| - |\tau \pm k \pm \frac{d-2}{2})$ by a residue calculation. So

$$\| T_\tau f(t,\cdot) \|_{L^{q'}_w} \leq C \int_{-\infty}^{\infty} \underbrace{\sum_{k\geq 0} (1+k)^{-\frac{2}{d}} \exp(-|t-s| \Big|\tau \pm k \pm \frac{d-2}{2}\Big|)}\; \| f(s,\cdot) \|_{L^q_w} \; ds$$

$$\leq C \int_{-\infty}^{\infty} C_\delta |t-s|^{-1+\frac{2}{d}} \; \| f(s,\cdot) \|_{L^q_s} \; ds$$

$$\leq C_\delta \| f \|_{L^q_{t,w}} \qquad \text{by fract. integration}$$

□

Derrick-Pohozaev Identity:

We have

$$-\Delta u + mu = f(u) = |u|^{p-1} u \;,\; m \geq 0 \;, 1 < p < \frac{d+2}{d-2} \qquad (*)$$

Proceed formally in $\mathbb{R}^d$

$$I[u] = \int \Big\{ \frac{1}{2} |\nabla u|^2 + \frac{m}{2} |u|^2 - F(u) \Big\}\, dx$$

has crit. point at u

$$u_r(x) = u(rx)\ ,\ \text{so}\ \frac{d}{dr} I[u_r]\Big|_{r=1} = 0$$

$$I[u_r] = r^{-d} \int \Big\{ \frac{1}{2} r^2 |\nabla u|^2 + \frac{m}{2}|u|^2 - F(u) \Big\}\, dx$$

$$\frac{d}{dr}\Big|_{r=1} I[u_r] = -\frac{d-2}{2} \int |\nabla u|^2 - \frac{d}{2} \int m u^2 + d \int F(u) = 0 \qquad (\text{ = Derrick Pohozaev})$$

Also, we have $\int |\nabla u|^2 + \int m u^2 - \int |u|^{p+1} = 0$ (integrate $(*)$ by parts). So

$$0 = \Big(\frac{d}{p+1} - \frac{d-2}{2} \Big) \int |\nabla u|^2 + \Big(\frac{d}{p+1} - \frac{d}{2} \Big) m \int u^2 \qquad (< 0 \text{ if } p > \frac{d+2}{d-2} \text{ or } m = 0)$$

So if

$$\left. \begin{array}{l} m > 0 \quad \& \quad p \geq \frac{d+2}{d-2} \\ m \geq 0 \quad \& \quad p > \frac{d+2}{d-2} \end{array} \right\} \Rightarrow u \equiv 0$$

Now, what about in a bdd domain Ω?

Proposition: Let Ω be bdd domain in $\mathbb{R}^d$, star-shaped w.r.t. some interior point. Let $u \in C^2(\bar{\Omega})$. Solve

$$\begin{cases} -\Delta u + m u = |u|^{p-1} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

If $p > \frac{d+2}{d-2}$, $m \geq 0$, or $p \geq \frac{d+2}{d-2}$, $m > 0$, then $u \equiv 0$.

Proof.

$$\int (-\Delta u + m u) x_j \partial_j u = \int |u|^{p-1} u x_j \partial_j u \qquad \circledast$$

$\downarrow$ summed over l, j

$$\int_\Omega -\Delta u x_j \partial_j u = \int_{\partial\Omega} -\frac{\partial u}{\partial n} x_j \partial_j u + \int_\Omega (\partial_j u)^2 + \partial_l u x_j \partial_{jl} u\ . \qquad \frac{\partial u}{\partial n} = \vec{\nu} \cdot \nabla u$$

$\vec{\nu}$ = outer norm

$$\int_\Omega \partial_l u x_j \partial_{jl} u = \int_\Omega x_j \partial_j \frac{1}{2} |\nabla u|^2 = \frac{1}{2} \int_{\partial\Omega} \vec{\nu} \cdot \vec{x} |\nabla u|^2 - \frac{d}{2} \int_\Omega |\nabla u|^2$$

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \vec{x} \cdot \vec{\nabla} u = \int_{\partial\Omega} \vec{x} \cdot \vec{\nu} |\nabla u|^2$$

So

$$\int_\Omega -\Delta u x_j \partial_j u = -\frac{1}{2} \int_{\partial\Omega} \vec{x} \cdot \vec{\nu} |\nabla u|^2 + \int_\Omega |\nabla u|^2 \Big(\frac{2-d}{2} \Big)$$

$$\int_\Omega u x_j \partial_j u = \int_\Omega x_j \partial_j \frac{1}{2} u^2 = \int_{\partial\Omega} \vec{x} \cdot \vec{\nu} \frac{1}{2} u^2 - \frac{d}{2} \int_\Omega u^2 = -\frac{d}{2} \int_\Omega u^2$$

$$\int_\Omega |u|^{p-1} u x_j \partial_j u = \int_\Omega x_j \partial_j \Big(\frac{|u|^{p+1}}{p+1} \Big) = -\frac{d}{p+1} \int_\Omega |u|^{p+1}$$

So ⊛ is

$$\frac{d-2}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_{\partial,\Omega} \vec{x} \cdot \vec{\nu} |\nabla u|^2 + \frac{md}{2} \int_\Omega u^2 = \frac{d}{p+1} \int_\Omega |u|^{p+1}$$

and

$$\int_\Omega |\nabla u|^2 + m \int u^2 = \int |u|^{p+1}$$

$$\Big(\frac{d-2}{2} - \frac{d}{p+1} \Big) \int_\Omega |\nabla u|^2 + m \Big(\frac{d}{2} - \frac{d}{p+1} \Big) \int_\Omega u^2 \leq 0$$

because $\vec{x} \cdot \vec{\nu} \geq 0$ if 0 is the "star" point □

Variational approach (for $p < \frac{d+2}{d-2}$).

$$\begin{array}{rcll} -\Delta u + mu &=& |u|^{p-1} u & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array} \qquad p < \frac{d+2}{d-2}$$

$$E[u] = \int \frac{1}{2} |\nabla u|^2 + \frac{m}{2} u^2$$

$$G[u] = \frac{1}{p+1} \int |u|^{p+1}$$

Step 1: $\inf\{E[u] | u \in H_0^1(\Omega) \ , \ G[u] = 1\}$ is attained. Let $\{u_j\}$ be a minimizing sequence in $H_0^1(\Omega) \Rightarrow u_j \rightharpoonup u_\infty$ in $H_0^1(\Omega)$. So $u_j \to u_\infty$ in $L^{p+1}(\Omega)$ (since $p+1 < 2^*$) $\Rightarrow G[u_\infty] = 1$, $E[u_\infty] \leq \inf E[u_j]$. So u_∞ is a minimizer.

Step 2: $u_\varepsilon = (u_\infty + \varepsilon v) \big(h(\varepsilon) \big)^{-1}$ where $\big(h(\varepsilon) \big)^{p+1} = G[u_\infty + \varepsilon v]$. Then $G[u_\varepsilon] = 1$ and

$$\begin{aligned} 0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E[u_\varepsilon] &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} h(\varepsilon)^{-2} E[u_\infty + \varepsilon v] \\ &= -2h'(0)h(0)E[u_\infty] + \langle E'[u_\infty], v \rangle \\ &= -2 \frac{\langle G'[u_\infty], v \rangle}{p+1} E[u_\infty] + \langle E'[u_\infty], v \rangle \end{aligned}$$

So

$$\int \nabla u_\infty \nabla v + m u_\infty v = \underbrace{\frac{2}{p+1} E[u_\infty]}_{\lambda > 0} \int |u_\infty|^{p-1} u_\infty v$$

So u_∞ is a weak solution of

$$\begin{array}{rcll} -\Delta u + mu &=& \lambda |u|^{p-1} u & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega \end{array}$$

Step 3:

$$u=\lambda^{-\beta}u_\infty \text{ for } \beta=-\frac{1}{p-1} \Rightarrow \begin{array}{rcll} -\Delta u+mu & = & |u|^{p-1}u & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array}$$

Suppose $u_\infty = u_\infty^+ - u_\infty^-$ where $u_\infty^+ \not\equiv 0$, $u_\infty^- \not\equiv 0$.

$$1=G[u_\infty]=\alpha_+^{p+1}G\Big[\frac{u_\infty^+}{\alpha_+}\Big]+\alpha_-^{p+1}G\Big[\frac{u_\infty^-}{\alpha_-}\Big] \qquad (\alpha\text{'s s.t. the } G\text{'s are } 1)$$

$$E[u_\infty]=\alpha_+^2E\Big[\frac{u_\infty^+}{\alpha_+}\Big]+\alpha_-^2E\Big[\frac{u_\infty^-}{\alpha_-}\Big]\geq(\alpha_+^2+\alpha_-^2)E[u_\infty] \qquad (u_\infty \text{ is minimizer})$$

Here $1=\alpha_+^{p+1}+\alpha_-^{p+1}$ and $0<\alpha_\pm<1 \Rightarrow \alpha_+^2+\alpha_-^2<1 \Rightarrow$ contradiction.

Now $u_\infty^+\equiv 0$ or $u_\infty^-\equiv 0$ (assume $u_\infty\geq 0$)

Assume $p>2 \Rightarrow u_\infty \in C^2(\bar{\Omega})$

$$-\Delta u_\infty + mu_\infty \geq 0$$

By strong max principle: $u_\infty > 0$ in Ω

Evans — Ch. 9 $\Rightarrow u_\infty$ radial

So we have a solution $u_\infty > 0$, radial (same for u).

Ma 142 5/13/2003

Solve $-\Delta u + mu = g(u)$ in $\mathbb{R}^d$.

$m>0 \quad g$ nondegenerate ($g\neq 0$) subcritical (Derrick-Pohozaev)

$$|g'(u)|\lesssim 1+|u|^{p-1} \qquad 1<p<\frac{d+2}{d-2}$$

$$G(u)=\int_0^u g(v)dv$$

$$G(u)>\frac{1}{2}mu^2 \text{ for some } u>0$$

$$G(u)=0(u^2) \text{ as } u\mapsto 0$$

$$I[u]=\frac{1}{2}\int|\nabla u|^2$$

$$V[u]=\int\Big(G(u)-\frac{1}{2}mu^2\Big)\,dx \qquad \text{functionals on } H^1.$$

$$|G(u)|\leq\varepsilon u^2+C_\varepsilon|u|^{p+1} \qquad \text{by Gagliardo } -N-S$$

We know that $\| u \|_{p+1}\leq C \| \nabla u \|_2^{1-\theta} \| u \|_2^{\theta}$ for some $0<\theta<1$.

Theorem. *$\mathcal{A} := \{u \in H^1(\mathbb{R}^d) : V(u) = 1\} \neq \emptyset$ and $I_\infty = \inf_{u\in\mathcal{A}} I[u]$ is attained at some $u_0 \in \mathcal{A}$ which is radial, non-increasing and non-negative. Moreover, $\exists \lambda > 0$, s.t. $-\Delta u = \lambda\big[g(u_0) - mu_0\big]$ in the weak sense.*

Corollary. *$d = 3$ ⊛ has a classical solution ($\in C^2(\mathbb{R}^d)$) which is positive, radial, and non-increasing.*

Proof. $u(x) = u_0(\lambda^{-\frac{1}{2}}x)$

$$\begin{aligned}-\Delta u = -\lambda^{-1}(\Delta u_0)(\lambda^{-\frac{1}{2}}x) &= \big(g(u_0) - mu_0\big)(\lambda^{-\frac{1}{2}}x)\\ &= g(u) - mu = f(u)\end{aligned}$$

Recall interior regularity theory $\Rightarrow u \in H^2_{\text{loc}}(\mathbb{R}^d)$.

Idea: $-\Delta\partial_j u = f'(u)\partial_j u$

$$\int \nabla\partial_j u \cdot \nabla(x^2\partial_j \cdot u) = \int f'(u)\partial_j u x^2 \partial_j u$$

$$\begin{aligned}\int \chi^2|\nabla\partial_j u|^2 &\le 2\int \chi|\nabla\chi||\nabla\partial_j u||\partial_j u| + \int \underbrace{|f'(u)|}_{1+|u|^{p-1}}|\partial_j u|^2\chi^2\\ &\le \frac{1}{4}\int \chi^2|\nabla\partial_j u|^2 + 4\int|\nabla x|^2|\partial_j u|^2 + C\underbrace{\left(\int_\Omega (1+|u|^{p-1})^{\frac{p+1}{p-1}}\right)^{\frac{p-1}{p+1}}}_{\lesssim \|u\|_{H^1}}\end{aligned}$$

$$\left(\int |\chi\partial_j u|^{p+1}\right)^{\frac{2}{p+1}}$$

For $d = 3$ $\quad \frac{1}{2} - \frac{1}{p} = \frac{2}{3} \Rightarrow H^2_{\text{loc}}(\mathbb{R}^3) \hookrightarrow C^{0,\alpha}_{\text{loc}}(\mathbb{R}^3)$

$$-\Delta u = f(u) \in C^\alpha_{\text{loc}}$$

Schauder estimates $\Rightarrow u \in C^{2,\alpha}_{\text{loc}}$

$$[\Delta^2\phi]_\alpha \le C[\Delta\phi]_\alpha \text{ if } \phi \in C^2_0$$

↑ In harmonic analysis last year

↖ compactly supported

In order to get rid of the compact support assumption → extra technical complications.

We know u is a radial, non-increasing & $u \ge 0$.

Suppose $u(x_0) = 0$ for some $x_0 \Rightarrow u(x)$ if $|x| \ge x_0$.

$$-\Delta u + mu = q(x) \underbrace{u}_{\geq 0.} \qquad\qquad q(x) = \frac{g\big(u(x)\big)}{u(x)}$$

$$-\Delta u \geq 0 \qquad \shortparallel$$

$$(q^+ - q^-)u$$

$$-\Delta u + (\underbrace{m + q^-}_{\geq 0})u = q^+ u \geq 0$$

If $u = 0$ anywhere $\Rightarrow u \equiv 0$ □

Proof of theorem

1) $\mathcal{A} \neq \phi$ we know $\mu_0 = G(w) - \frac{1}{2}mw^2 > 0$ for some fixed $w > 0$.

$$u(x) = \begin{cases} w \text{ if } |x| < R \\ w(R + 1 - |x|)_+ \text{ of } |x| > R \end{cases}$$

$$V[u] \geq C_d \mu_s R^d - CR^{d-1} > 0 \text{ if } R \text{ large}$$

$$V[u(\sigma\cdot)] = \sigma^{-d} V(u) = 1$$

2) Pick a minimizing sequence $\{u_n\}$

$$I[u_n] \searrow I_\infty \; V[u_n] = 1$$

make u_n radial and decreasing. More precisely, let $v_n = |u_n|^*$

[Digression: decreasing rearrangements, wedding cake principle . . .

$$V[u_n] = V[\,|u_n|\,] = V[\,|u_n|^*]$$

$$\int |\nabla u_n|^2 = \int \big|\nabla |u_n|\big|^2 \overset{\boxed{\text{Lemma}}}{\geq} \int \big|\nabla |u_n|^*\big|^2 .$$

Lemma. *If* $f \in H^1(\mathbb{R}^d) \Rightarrow |f|^* \in H^1$ *and* $\int \big|\nabla |f|^*\big|^2 \leq \int |\nabla f|^2$

Proof. (Lieb)

$$f_1 g_1 h \geq 0 \text{ measureable } \Rightarrow \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f(x)g(x-y)h(y)dx\,dy$$

$$\leq \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} f^*(x)g^*(x-y)h^*(y)dx\,dy$$

$$g_t(x) = \frac{C}{t^{\frac{d}{2}}} e^{-\frac{\pi x^2}{t}} \qquad \hat{g}_t(\xi) = e^{-\pi t \xi^2}$$

$$f\phi \in L^2 I_t[\phi] = \frac{1}{t}\Big\{\int\limits_{\mathbb{R}^d} |\phi|^2 - \int \Phi(x)\Phi(y)g_t(x-y)dx\,dy\Big\}$$

$$= \frac{1}{t}\int |\hat{\phi}|^2(\xi)(\underbrace{1-e^{\pi t\xi^2}}_{\leq \pi t\xi^2})\,d\xi$$

If $f \in H^1$

$f^* \in L^2$

$$\begin{array}{ccc} I_t[f] & \geq & I_t[f^*] \\ \text{DCT}\downarrow & & \downarrow\text{Fatou} \\ C_0\int|\nabla f|^2 & & C_0\int|\nabla f^*|^2 \end{array}$$

□

WLOG $$u_n = v_n \qquad \sup_n \| u_n \|_{H^1} < \infty$$

$$L^2 \text{ part} \qquad 1 + \frac{1}{2}\int m u_n^2 \leq \int G(u_n)$$

$$\leq \varepsilon \int u_n^2 + C_\varepsilon \int |u_n|^{2*} \leq \varepsilon \int u_n^2 + C_\varepsilon \int |\nabla u_n|^2$$

WLOG
$$u_n \longrightarrow u_\infty \text{ in } H^1$$
$$u_n \longrightarrow u_\infty \text{ in } L_{\text{loc}} \qquad \text{for any } 1 \leq q \leq p+1$$
$$u_n \longrightarrow u_\infty \text{ a.e.}$$

$$I[u_\infty] \leq \lim I[u_n] = I_\infty$$

$$\int\limits_{\mathbb{R}^d} G(u_n) \mapsto \int\limits_{\mathbb{R}^d} G(u_\infty)$$

Need to prevent the bulk of u_n to dmsh off to ∞.

Lemma. *Let $f \in H^1(\mathbb{R}^d) d \geq 3$ and radial. Then $\exists R = R(d) > 0$ and a continuous function $F(x)$ s.t.*

$F(x) = f(x)$ for a.e. $|x| > R$ and $\big|F|x|\big| \leq C_d|x|^{\frac{1-d}{\alpha}} \| f \|_{H^1}$

$$\underline{\lim}\Big|\int\limits_{\mathbb{R}^d}\big(G(u_n) - G(u_\infty)\big)\Big| \leq \overbrace{\underline{\lim}\int\limits_{|*|<R} |G(u_n) - G(u_\infty)|}^{=0}$$

$$+ 2\,\underline{\lim}\int\limits_{|x|>R} |G(u_n)| + \int\limits_{|x|>R} |G(u_\infty)|$$

$$\leq \varepsilon \sup_n \| u_n \|_2^2 \leq \varepsilon C$$

$$1 = \overline{\lim}\, V[u_n] = \overline{\lim} \int \big(G(u_n) - \frac{1}{2} m u_n^2\big) = \int G(u_\infty) - \frac{1}{2} m \underline{\lim} \int u_n^2$$
$$\le V[u_\infty]$$

If
$$V[u_\infty] > 1 \qquad V[u_\infty(\sigma \cdot)] = \sigma^{-d} V[u_\infty] = 1$$
$$I[u_\infty(\sigma\cdot)] = \sigma^{(2-d)} \cdot I[u_\infty] < I[u_\infty]$$
$$V[u_\infty] = 1\ ,\ u_\infty \in \mathcal{A} \qquad I[u_\infty] = I_\infty$$

Ma 142 5/15/2003

$$-\Delta u + mu = g(u) \qquad \text{in } \mathbb{R}^d,\ d \ge 3\ . \qquad \circledast$$

$m > 0$, g odd, $g \in C^1(\mathbb{R})$.

$$|g'(u)| \le C(1 + |u|)^{p-1}\ ,\ 1 < p < \frac{d+2}{d-2}$$

$G(u) = \int_0^u g(v)\, dv$ satisfies

$$G(w) > \frac{1}{2} m w^2 \ \text{ for some } w > 0.$$
$$G(u) = \sigma(u^2) \ \text{ as } u \to 0.$$
$$I[u] = \int \frac{1}{2} |\nabla u|^2\ ,\ V[u] = \int G(u) - \frac{1}{2} m u^2\ .$$

Theorem. *(Bereslycki-Lions, ~ 80)*

Let $\mathcal{A} = \{u \in H^1(\mathbb{R}^d)\ ;\ V[u] = 1\}$. Then $\inf_{u \in \mathcal{A}} I[u]$ is attained at some $u_\infty \in \mathcal{A}$ where u_∞ is radial, decreasing, nonnegative, and $\exists \lambda > 0$ s.t. u_∞ is a weak solution of $-\Delta u = \lambda\big(g(u) - mu\big)$.

Proof. We found the minimizer u_∞ as a weak limit in H^1 of the symmetric, nonincreasing, rearrangement of some other sequence of minimizers. $I[u_\infty] = I_\infty$, $u_\infty \ge 0$.

$$V\big[(u_\infty + \varepsilon\theta)(\sigma(\varepsilon)\cdot)\big] = 1 = \sigma(\varepsilon)^{-d} V[u_\infty + \varepsilon\theta]$$
$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I\big[(u_\infty + \varepsilon\phi)(\sigma(\varepsilon)\cdot)\big] = 0$$

$$\underbrace{V[w + \varepsilon v]} = V[w] + \varepsilon \langle V'[w], v\rangle + \sigma(\varepsilon)\ .$$
$$\int G(w + \varepsilon v) - \frac{1}{2} m (w + \varepsilon v)^2$$
$$= V[w] + \int \big(G(w + \varepsilon v) - G(w)\big) - \varepsilon m w v + O(\varepsilon^2)\ .$$

$$= V[w] + \underbrace{\varepsilon\langle V'[w], v\rangle}_{\substack{\text{by definition,}\\ \int (g(w)v - mwv)}} + \int \big(G(w+\varepsilon v) - G(w)\big) - \varepsilon g(w)v\big) + O(\varepsilon^2)$$

with

$$\begin{aligned}
&\Big|\int G(w+\varepsilon v) - G(w) - \varepsilon g(w)v\Big| \\
&\quad = \Big|\int \int_0^1 \frac{d}{dt} G(w+\varepsilon tv)dt - \varepsilon g(w)v\Big| \\
&\quad \le \int_0^1 \int_{\mathbb{R}^d} |g(w+\varepsilon tv) - g(w)||\varepsilon|v|dx\ dt \\
&\quad \le \int_0^1 \int_0^1 \int_{\mathbb{R}^d} |g'(w+\varepsilon tsv)|\varepsilon^2 t|v|^2 dx\ dt\ ds \\
&\quad \le C\varepsilon^2 \int_{\mathbb{R}^d} (1 + |w|^{p-1} + |v|^{p-1})|v|^2 \le C\varepsilon^2
\end{aligned}$$

$$\begin{aligned}
v, w \in H^1 &\hookrightarrow L^{p+1}(\mathbb{R}^d) \\
&\hookrightarrow L^{2*}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)
\end{aligned}$$

For $|w|^{p-1}|v|^2$, interpolate by Gagliardo - N-S.

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} I\big[(u_\infty + \varepsilon\phi)\big(\sigma(\varepsilon)\cdot\big)\big] \\
&= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sigma(\varepsilon)^{2-d} I[u_\infty + \varepsilon\phi] \\
&\text{N.B.} \begin{cases} \sigma'(0) \text{ exists because } -d\sigma'(0) + \langle V'[u_\infty], \phi\rangle = 0 \\ \sigma(0) = 1 \end{cases} \\
&= (2-d)\sigma'(0) I[u_\infty] + \langle I'[u_\infty], \phi\rangle \\
&\Rightarrow \langle I'[u_\infty], \phi\rangle = \underset{\frac{d-2}{d} I[u_\infty] > 0}{\underset{\shortparallel}{\lambda}} \langle V'[u_\infty], \phi\rangle .
\end{aligned}$$

And then rescale

$$-\Delta u_\infty = \lambda\big(g(u_\infty) - mu_\infty\big)$$

to get

$$-\Delta u + mu = g(u)$$

□

Decay lemma for radial H^1 functions.

Let $f \in H^1(\mathbb{R}^d), d \geq 3$, be radial. Then $f(x) = F(x)$ a.e., where F is continuous away from 0 (Jacobian factor, f is actually Hölder sth along rays by Sobolev embedding), and $|F(x)| \leq C|x|^{\frac{1-d}{2}} \parallel f \parallel_{H^1}$ for $|x| > R(d)$.

Proof. Assume first that $f \in C_0^\infty$. $m = \frac{d-1}{2}$

$$\begin{aligned}\frac{d}{dr}\big(r^m f(r)\big)^2 &= 2\big(r^m f(r)\big)' r^m f(r) \\ &\leq \Big(\big(r^m f(r)\big)'\Big)^2 + \big(r^m f(r)\big)\end{aligned}$$

Integrate $\int\limits_0^r$:

$$\begin{aligned}\big(r^m f(r)\big)^2 &\leq \int\limits_0^r \Big[\big(\rho^m f(\rho)\big)'\Big]^2 d\rho \\ &\quad + \int\limits_0^r \rho^{2m} f(\rho)^2 d\rho \\ &= \int\limits_0^r \rho^{2m} f'(\rho) d\rho + 2m \int\limits_0^r \rho^{2m-1} f'(\rho) \overbrace{f(\rho) d\rho}^{A} \\ &\quad + m^2 \int\limits_0^r \rho^{2m-2} \underbrace{f(\rho)^2}_{B} d\rho + \underbrace{\int\limits_0^r \rho^{2m} f(\rho)^2 d\rho}_{\leq \|f\|_{H^1}^2}\end{aligned}$$

$$A = \int\limits_0^r \rho^{2m-1} \frac{1}{2}\Big(\big(f(\rho)\big)^2\Big)' = -\frac{2m-1}{2} \int\limits_0^r \rho^{2m-2} f(\rho)^2 d\rho + \frac{r^{2m-1}}{2} f(\rho)^2$$

(no contrib. at 0 because $2m - 1 = d - 2 \geq 1$).

$$\big(r^m f(r)\big)^2 \leq C \parallel f \parallel_{H^1}^2 + \underbrace{\big[m^2 - (2m-1)m\big]}_{=m-m^2 \leq 0} \int\limits_0^r \rho^{2m-2} f^2 \, d\rho + m r^{2m-1} f(r)^2$$

$$r^{2m}\Big(1 - \frac{m}{r}\Big) f(r)^2 \leq C \parallel f \parallel_{H^1}^2$$

Approximate H^1 radial by C^∞ radial. □

Recall the main corollary: ⊛ has a C^2 solution u which is radial, nonincreasing and positive. Moreover, u decays exponentially, $\forall \varepsilon > \exists C_\varepsilon : |u(x)| \leq C_\varepsilon e^{(-\sqrt{m}-\varepsilon)|x|}$.

Proof.

$$u(x) = u_\infty(\lambda^{-\frac{1}{2}}x)\ , \qquad \text{check that this is a weak solution of } \circledast$$
$$-\Delta u = f(u) = g(u) - mu$$
$$u \in H^2_{loc}(\mathbb{R}^d) \overset{d=3}{\hookrightarrow} C^{0,\alpha}_{\text{loc}} \Rightarrow f(u) \in C^\alpha_{loc} \text{ (because } g \in C^1)$$

Recall Schauder estimates, $0 < \alpha < 1$

$$[D^2 g]_\alpha \leq C_{\alpha,d}[\Delta g]_\alpha \qquad \forall g \in C_0^\infty$$
$$\Rightarrow [D^2 u]_\alpha < \infty$$
$$-\Delta u = f(u) \text{ a.e.}$$

Observe that $\partial_j u$ is a weak solution of

$$-\Delta(\partial_j u) = \underbrace{f'(u)}_{\in L^\infty} \partial u$$

$$\text{because } u \in \begin{cases} H^1(\mathbb{R}^d) \\ C^{0,\alpha}_{loc} \end{cases} \underbrace{\text{and decays by}}_{\in L^2(\mathbb{R}^d)} \text{the lemma}$$

$$\Rightarrow \partial_j u \in H^2(\mathbb{R}^d) \Rightarrow u \in H^3(\mathbb{R}^d) \overset{d=3}{\hookrightarrow} C^{1,\alpha}$$

In order to apply Schauder, mollify,

$$u_\varepsilon = u * \eta_\varepsilon$$
$$-\Delta u_\varepsilon = f(u) * \eta_\varepsilon$$
$$-\Delta(\chi u_\varepsilon) = \chi\big(f(u) * \eta_\varepsilon\big) - \Delta\chi u_\varepsilon - 2\nabla\chi\nabla u_\varepsilon$$
$$[D^\alpha(\chi u_\varepsilon)]_\alpha \leq C\big[\chi \cdot \big(f(u) * \eta_\varepsilon\big)\big]_\alpha + C \cdot [\Delta\chi u_\varepsilon]_\alpha + C[\nabla\chi\nabla u_\varepsilon]_\alpha\ .$$

Note that

$$\sup_{1>\varepsilon>0} \big[\chi \cdot f(u) * \eta_\varepsilon\big]_\alpha < \infty$$
$$\sup_{0<\varepsilon<1} [\Delta\chi \cdot u_\varepsilon]_\alpha < \infty$$
$$\sup_{0<\varepsilon<1} \big[\nabla\chi \cdot \nabla u_\varepsilon\big]_\alpha < \infty$$
$$\Rightarrow \sup_{0<\varepsilon<1} \big[D^2(\chi u_\varepsilon)\big]_\alpha < \infty$$

$\chi u \in H^2\ ,$ $\qquad |D^2(\chi u_\varepsilon)(x) - D^2(\chi u_\varepsilon)(y)| \leq C|x-y|^\alpha$

$$|D^2(\chi u_\varepsilon)(x) - \underbrace{\fint_{B(0,1)} D^2(\chi u_\varepsilon)(y)\,dy}_{\downarrow \varepsilon\to 0} \le C(1+|x|)^\alpha$$

$$\fint_{B(0,1)} D^2(\chi_u)(y)\,dy \text{by LDCT.}$$

$$\Rightarrow \begin{cases} \{D^2(\chi u_\varepsilon)\}_{\varepsilon>0} \text{ is locally uniformly bdd} \\ \text{and equicontinuous} \end{cases}$$

$$D^2(\chi u_{\varepsilon_j}) \xrightarrow{L^\infty} D^2(\chi_u) \text{ a.e.}$$

$$\Rightarrow u \text{ is } C^2$$

$$\text{Actually } D^2u \in C^{0,\alpha}$$

Decay:

$$-\Delta u - \frac{g(u)}{u}u = -mu$$

$$V(x) = \begin{cases} -\dfrac{g\big(u(x)\big)}{u(x)} & \text{if } u(x) \ne 0 \\ 0 & \text{else.} \end{cases}$$

$$Hu = -\Delta u - \frac{g(u)}{u}u$$

Apply Agmon. V is continuous and $V(x) \to 0$ as $|x| \to \infty$, and

$$\lim_{R\to\infty} \sup_{|x|>R} |V(x)| = 0\ .$$

(use u radial and $g(u) = \sigma(u)$ at the origin).

$$\int_{\mathbb{R}^d} e^{2(\sqrt{m}-\varepsilon)|x|}|u(x)|^2 \le C_\varepsilon$$

□

Back to NLS:

$$i\partial_t\psi + \frac{1}{2}\Delta\psi + f(|\psi|^2)\psi = 0$$

$$\psi = 0 \text{ on } \partial\Omega \times [0,\infty)$$

$$\psi = u\ ,\ t = 0\ .$$

$$\psi(t) = e^{\frac{i\alpha^2}{2}t}u(x)$$

$$\Rightarrow \begin{cases} \frac{-\alpha^2}{2}u + \frac{1}{2}\Delta u + f(u^2)u = 0 \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

e.g., $f(u^2) = |u|^{2\sigma}$, focusing case,

$$0 < \sigma < \frac{4}{d-2}$$

Stability of solitons.

$$\tilde{\psi}(t,x) = u(x)\big(1 + \varepsilon r(t,x) + O(\varepsilon^2)\big)e^{i(\frac{\alpha^2}{2}t + \varepsilon s(t,x) + 0(\varepsilon^2))}$$

(Heuristics here)

$$= \big(u + \varepsilon \underbrace{ru}_{w} + \varepsilon i \underbrace{su}_{v} + O(\varepsilon^2)\big)e^{\frac{i}{2}\alpha^2 t}$$

→ Find estimates of boundedness on v, w when initially of size $O(1)$.

Answer: stable for $\sigma d < 2$, unstable for $\sigma d > 2$.

Ma 142 5/20/2003

$$\begin{aligned}
&i\partial_t\psi + \frac{1}{2}\Delta\psi + |\psi|^{2\sigma}\psi = 0 \qquad (\text{or with } f(|\psi|^2)\psi) \qquad \sigma < \frac{2}{d-2}\\
&\psi(t,x) = e^{i\frac{\alpha^2}{2}t}u(x;\alpha)\\
&-\frac{\alpha^2}{2}u + \frac{1}{2}\Delta u + u^{2\sigma+1} = 0 \qquad (\text{or with } f(u^2)u)\\
&u < 0\ ,\ \text{radial, etc. (in } d = 3)
\end{aligned}$$

$$d = 1 \qquad \& f(u) = u^\sigma \Rightarrow \phi(x) = \frac{(\sigma+1)^{\frac{1}{2\sigma}}}{\cosh^{\frac{1}{\sigma}}(\sigma x)} \qquad (0 < \sigma < \infty) \qquad \text{solves on } \mathbb{R}:$$

$$\phi'' - \phi + |\phi|^{2\sigma}\phi = 0$$

Heuristic discussion of stability:

$$\begin{aligned}
\psi(t,x) &= u(x,\alpha)\big(1 + \varepsilon r(t,x) + O(\varepsilon^2)\big)e^{i(\frac{\alpha^2}{2}t + \varepsilon\beta(t,x) + O(\varepsilon^2))}\\
&\approx (u + \varepsilon \underbrace{ur}_{w} + i\varepsilon \underbrace{u\beta}_{v})e^{i\frac{\alpha^2}{2}t}\\
0 &= i(\varepsilon\dot{w} + i\varepsilon\dot{v}) + \frac{1}{2}(\Delta u + \varepsilon\Delta w + i\varepsilon\Delta v) + f(|u + \varepsilon w + i\varepsilon v|^2)(u + \varepsilon w + i\varepsilon v) - \frac{\alpha^2}{2}(u + \varepsilon w + i\varepsilon v)\\
&\qquad\qquad\approx [f(u^2) + f'(u^2)2\varepsilon uw](u + \varepsilon w + i\varepsilon v)\\
0 &= -\dot{v} + \frac{1}{2}\Delta w + f(u^2)w + 2f'(u^2)u^2 w - \frac{\alpha^2}{2}w\\
0 &= \dot{w} + \frac{1}{2}\Delta v + f(u^2)v - \frac{\alpha^2}{2}v
\end{aligned}$$

So:

$$\begin{pmatrix} \dot w \\ \dot v \end{pmatrix} = \begin{pmatrix} 0 & L_0 \\ -L_1 & 0 \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} \quad \text{with} \quad \begin{aligned} L_0 &= \left(-\tfrac{1}{2}\Delta + \tfrac{\alpha^2}{2}\right) - f(u^2) \\ L_1 &= \left(-\tfrac{1}{2}\Delta + \tfrac{\alpha^2}{2}\right) - f(u^2) - 2f`(u^2)u^2 \end{aligned}$$

or

$$\ddot w = -L_0 L_1 w$$

$$\dot v = -L_1 L_0 v$$

$L_0 u = 0 \Rightarrow u$ is e-func. of L_0.

$\sigma(L_0) = [\frac{\alpha^2}{2}, \infty) \cup \{\text{e-vals below} \frac{\alpha^2}{2}\}$ (since $f(u^2)$ is rapidly decaying).

Ground state of L_0 is simple and positive (and orthogonal to u if different from u). Since $u > 0 \Rightarrow \inf \sigma(L_0) = 0$ and u is the ground state.

Theorem. *$H = -\Delta + V$ and $V(x) \to 0$ as $|x| \to \infty$. Assume $\langle Hf, f\rangle < 0$ for some $f \in Dom(H) = H^2$. Then $\inf_{\substack{\|f\|_2=1 \\ f\in H^2}} \langle Hf, f\rangle$ is attained at some $f_0 > 0$, $\| f_0 \|_2 = 1$ and $\langle Hf_0, f_0\rangle = \lambda_0 < 0$ is a simple e-val.*

Proof. (Sketch):

$$\left.\begin{aligned} &\langle Hf_n, f_n\rangle \searrow \lambda_0 \qquad \& \| f_n \|_2 = 1 \\ &\int |\nabla f_n|^2 + V f_n^2 = I(f_n) \end{aligned}\right\} \overset{(V\in L^\infty)}{\Longrightarrow} \sup_n \| f_n \|_{H^1} < \infty$$

So

$$\left. f_n \xrightarrow{H^1} f_\infty \ \& \ \begin{aligned} f_n &\xrightarrow{L^2_{\text{loc}}} f_\infty \\ f_n &\xrightarrow{a.e.} f_\infty \end{aligned}\right\} \overset{(\text{Fatou})}{\Longrightarrow} \int f_\infty^2 \le 1 \ \& \int\limits_{|x|>R} V f_n^2 \to \int\limits_{|x|>R} V f_\infty^2$$

Unit in n. ∎

Claim:

$$\lim_{R\to\infty} \sup_n \int\limits_{|x|>R} f_n^2 \to 0$$

Proof. Let

$$\tilde f_n(x) = \frac{f_n(x)\chi(x/R_n)}{\| \text{—"—} \|_2} \text{ if } \int\limits_{|x|>R_n} f_n^2 > \delta > 0 \text{ for some } n \to \infty$$

$$I(\tilde f_n) \ge \| \cdot \|_2^{-2} \int \left\{ |\nabla f_n|^2 \chi^2\left(\frac{x}{R_n}\right) - C R_n^{-2} |f_n|^2 - \| V \|_{L^\infty(|x|>R_n)} |f_n|^2 \right\} \text{ etc.}$$

Actually

$$\left.\begin{aligned} &\| f_\infty \|_{H^1} \le \underline{\lim} \| f_n \|_{H^1} \\ &\int V f_n^2 \to \int V f_\infty^2 \ (\text{since } V \to 0) \end{aligned}\right\} \Rightarrow I(f_\infty) \le \underline{\lim}\, I(f_n) = \lambda_0$$

Since $\lambda_0 < 0 \Rightarrow \| f_\infty \|_2 = 1$ and f_∞ is the minimizer. So

$$L_0 \geq 0 \ \& \ L_1 \partial_\alpha u = -\alpha u \text{ (from } \partial_\alpha(L_0 u(\cdot;\alpha)) = 0)$$

$$L_0^{\frac{1}{2}} \ddot{v} = -L_0^{\frac{1}{2}} L_1 L_0^{\frac{1}{2}} L_0^{\frac{1}{2}} v$$
$$L_0^{-\frac{1}{2}} \ddot{w} = -\underbrace{L_0^{\frac{1}{2}} L_1 L_0^{\frac{1}{2}}}_{A} L_0^{-\frac{1}{2}} w$$

□

Exercise: Show that A is self-adjoint with $\mathrm{Dom}(A) = H^4$

$$L_0^{\frac{1}{2}} v = C_0 \cos(\sqrt{A}\, t)\phi_0 + C_1 \sin(\sqrt{A}\, t)\phi_1$$

If $\sigma(A) \subset [0, \infty)$, then "stability." If not, then "instability."

$$\inf_{g \in H^4} \frac{\langle g|A|g\rangle}{\langle g|g\rangle} = \inf_{g \in H^4} \frac{\langle \overbrace{L_0^{\frac{1}{2}} g}^{h} | L_1 | L_0^{\frac{1}{2}} g\rangle}{\langle g|g\rangle} = \inf_{\substack{h \in H^2 \\ h \perp u}} \frac{\langle h|L_1|h\rangle}{\langle h|L_0^{-1}|h\rangle}$$

Since $L_0 \geq 0$, the question is $\langle h|L_1|h\rangle \overset{?}{\geq} 0$ for all $h \in H^2$, $h \perp u$.

$L_1(\partial_j u) = \partial_j(L_0 u) = 0 \Rightarrow \partial_j u$ is e-func of L_1 with e-val 0.

In $d = 1$, we have ϕ' having 1 node $\Rightarrow$ there exists unique negative e-val.

In all d expect $\mathrm{Ker} L_1 = [\partial, u, \ \ldots \ , \partial_d u]$ and there exists unique and negative e-val.

Solve variational problem $\inf\limits_{\substack{h \perp u \\ \|h\|_2 = 1}} \langle h|L_1|h\rangle$ (minimizer h_0 and $\langle h_0|L_1|h_0\rangle = \beta$.

So $L_1 h_0 = \alpha u + \beta h_0$ or $(L_1 - \beta)h_0 \equiv \alpha u$

We know $\beta \geq \mu_0$ (= negative e-val of L_1). Assume $\beta < 0$.

First: $\beta \neq \mu_0$ (otherwise h is unique "positive" ground state of $L_1 \Rightarrow u \not\perp h_0 \Rightarrow\Rightarrow\Leftarrow$)

So: $0 > \beta > \mu_0 \Rightarrow \alpha \neq 0$ (because we claim $\sigma(L_1) \cap (\mu_0, 0) = \emptyset$).

$$h_0 = \mathcal{L}(L_1 - \beta)^{-1} u$$
$$0 = \langle u|(L_1 - \beta)^{-1}|u\rangle$$

Let $w(\lambda) = \langle u|(L_1 - \lambda)^{-1}|u\rangle \Rightarrow$

$w(\lambda) \to -\infty$ as $\lambda \downarrow \mu_0$ (because $u > 0$ and ground state of L_1 is > 0).

$$w'(\lambda) = \langle u|(L_1 - \lambda)^{-2}|u\rangle > 0$$

$$\downarrow L_1 \partial_\alpha u = -\alpha u$$

$$w(0)=\langle u|L_1^{-1}|u\rangle=-\frac{1}{\alpha}(u,\partial_\alpha u)=-\frac{1}{2\alpha}\frac{\partial}{\partial_\alpha}\parallel u\parallel_2^2$$

$$w(0)>0\Rightarrow \text{instability}$$

$$w(0)<0\Rightarrow \text{stability}$$

$$u(x;1)=g\text{ solves }-\frac{g}{2}+\frac{1}{2}\Delta g+g^{2\sigma+1}=0$$

$$\frac{1}{2}\Delta\big(\alpha^m g(\alpha x)\big)=\alpha^{2+m}\Big(\frac{g}{2}-g^{2\sigma+1}\Big)=\alpha^2\frac{(\alpha^m g)}{2}-(\alpha^m g)^{2\sigma+1}\text{ if }m=\frac{1}{\sigma}$$

So

$$u(x,\alpha)=\alpha^{\frac{1}{\sigma}}g(\alpha x)$$

$$\frac{\partial}{\partial\alpha}\parallel u(\cdot,\alpha)\parallel_2^2=\frac{\partial}{\partial\alpha}\alpha^{\frac{2}{\sigma}-d}\parallel g\parallel_2^2\begin{cases}>0 & \text{if } 2>d\sigma \quad (\Rightarrow\beta<0)\\ <0 & \text{if } 2<d\sigma \quad (\Rightarrow\beta>0)\end{cases}$$

Ma 142 5/22/2003

Last time $H=-\Delta+V$ $\langle Hf,f\rangle<0$
$\{f_n\}$ $\parallel f_n\parallel_2=1$ $\quad\langle Hf_n,f_n\rangle\searrow\mathcal{A}_0<0$
$f_n\rightarrow f_\infty$ in H^1 $f_n\rightarrow f_\infty$ L^2_{loc} $\parallel f_\infty\parallel_2\leq 1$
Claim $\lim_{R\to\infty}\sup_n\int_{|x|>\mathbb{R}}|f_n|^2=0$ otherwise then

$$\exists n\nearrow\ R_n\nearrow\infty\ \int_{|x|>R_n}|f_n|^2>\delta>0$$

$$\tilde{f}_n(x)=\frac{x\Big(\frac{x}{R_n}\Big)f_n}{\parallel\chi_{(\dot{\overline{R_1}})}f_n\sqrt{1+\delta}\parallel_{L^2}}\text{sup}\chi\subset B_{(0,2)}$$

$$\begin{aligned}I[\tilde{f}_n]&=\int\frac{1}{2}|\nabla\tilde{f}_n|^2+\frac{1}{2}V\tilde{f}_n^2\\&\leq\frac{1}{1-\delta}\int\frac{1}{2}|\nabla f_n|^2+\frac{1}{2}Vf_n^2+O(1)=\frac{1}{1-\delta}I(f_n)\\&\Rightarrow\frac{\lambda_0}{1-\delta}<\lambda_0<0\end{aligned}$$

Theorem. *(Weinstein '86)* $d\sigma<2$
$g(\cdot\alpha)$ the ground state, i.e., $-\frac{1}{2}\Delta g+\frac{\alpha^2}{2}g-g^{2\sigma+1}=0$
$g>0$ *radial*
$\varepsilon>0\quad\exists\delta>0$ *s.t.*
$\forall\psi\in H^1$ *s.t.* $\parallel\phi-e^{i\gamma g(\cdot\ +y)}\parallel_{H^1}<\delta$
then $\sup_t\inf_{\gamma\cdot y}\parallel\psi(t,\cdot)-e^{i\gamma g(\cdot\ +y)}\parallel_{H^1}<\varepsilon$

where

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi + |\psi|^{2\sigma}\psi = 0 \\ \psi\big|_{t=0} = \phi \end{cases}$$

Proof.

$$\theta_f = \{e^{i\gamma}f(\cdot + y) : \gamma \in R \quad y \in R^d\}$$
$$\rho(\psi(t), \theta_g) = \inf_{\gamma,y}\{\parallel \nabla\psi - e^{i\gamma}\nabla g(\cdot + y) \parallel_2^2 + \alpha^2 \parallel \psi - e^{i\gamma}g(\cdot + y) \parallel_2^2\}$$

□

Exercise: Check that $\exists\gamma = \gamma(t)$ $y = y(t)$ continuous such that at those values. inf is attained.

$$e^{-i\gamma(t)}\psi\big(t; -y(t)\big) = g + \overbrace{u(t,\cdot) + iv(t,\cdot)}^{w(t,\cdot)}$$
$$\min(\alpha^2, 1) \parallel w(t) \parallel_{H^1}^2 \leq \rho(\psi(t), \theta g) \leq \max(\alpha^2, 1) \parallel w(t) \parallel_{H^1}^2$$

action $$\zeta[f] = \int \frac{1}{2}|\nabla f|^2 + \frac{\alpha^2}{2}f^2 - \frac{1}{\sigma+1}|f|^{2\sigma+2}\, dx$$

Fact 1) $$\zeta'[g] = 0$$

2) $$\zeta[\psi(t)] = \zeta[\phi] = \mathcal{H}[\psi(t)] + \frac{\alpha^2}{2}\int |\psi(t)^2|$$

$$\langle\zeta'[g], h\rangle = \int \frac{1}{2}2\nabla g \cdot \nabla h + \frac{\alpha^2}{2}2gh - 2g^{2\sigma+1}h = 0$$

3) $$\delta\zeta[\phi] = \zeta[\phi] - \zeta[g] = \zeta\big[e^{-i\gamma(0)}\phi\big(\cdot - y(0)\big)\big] - \zeta[g] < \varepsilon$$

$$\delta\zeta[\psi] = \zeta[g + w(t)] - \zeta[g] \underset{goal}{>} C_0 \parallel \underbrace{W(t) \parallel_{H^1}^2 - C(\parallel W(t) \parallel_{H^1}^{2+\mu}}_{h(\|W(t)\|_{H^1})} \parallel W \parallel_{H^1}^{\sigma})$$

$$\Rightarrow \parallel W(t) \parallel_{H^1} < \varepsilon$$

$$\zeta[g + w] - \zeta[g]$$
$$= \int \frac{1}{2}\big|\nabla(g + w)\big|^2 + \frac{\alpha^2}{2}|g + w|^2 - \frac{1}{\sigma+1}|g + w|^{2\sigma+2} - \int \frac{1}{2}|\nabla g|^2 + \frac{\alpha^2}{2}|g|^2 - \frac{|g|^{2\sigma+2}}{\sigma+1}$$

g real $w = u + iv$

$$= \int \frac{1}{2}2\nabla g\nabla u + \frac{\alpha^2}{2}2gu - 2g^{2\sigma+}u\, dx +$$
$$\int \frac{1}{2}|\nabla u|^2 + \frac{\alpha^2}{2}|u|^2 + \int \frac{1}{2}|\nabla v|^2 + \frac{\alpha^2}{2}v^2$$
$$- \int \frac{1}{\sigma+1}\big(|g + w|^{2\sigma+2} - |g|^{2\sigma+2} - (2\sigma + 2)g^{2\sigma+1}u\big)$$

$\sigma = 1$ non-linear term

$$|g+w|^4 - |g|^4 - 4g^3u$$
$$= |g+u|^4 + v^4 + 2(g+u)^2v^2 - g^4 - 4g^3u$$
$$= 6g^2u^2 + 4gu^3 + u^4 + 2g^2v^2 + 4guv^2 + 2u^2v^2 + v^4$$
$$= \int \underbrace{\Big(\frac{1}{2}|\nabla u|^2 + \frac{\alpha^2}{2}u^2 - 3g^2\,u^2\Big)}_{(L_1u,u)} + \int \underbrace{\Big(\frac{1}{2}|\nabla u|^2 + \frac{\alpha^2}{2}v^2 - g^2v^2\Big)}_{(L_0v,v)} + O(\| w \|_3^3 + \|w\|_4^4)$$

$$\langle L_1u,u\rangle + \langle L_0v,v\rangle - C\big(\| w \|_{H^1}^3 + \| w \|_{H^1}^4\big)$$
$$L_0 = -\frac{\Delta}{2} + \frac{\alpha^2}{2} - g^{2\sigma} \qquad L_1 = -\frac{\Delta}{2} + \frac{\alpha^2}{2} - (1+2\sigma)g^{2\sigma}$$

Find the constraint on $\gamma(t)y(t)$

$$\| \nabla\psi - e^{i\gamma}\nabla g(\cdot+y) \|_2^2 + \alpha^2 \| \psi - e^{-i\gamma}g(\cdot+y) \|_2^2$$
$$= \| \nabla\psi \|_2^2 + \| \psi \|_2^2 + \| \nabla g \|_2^2 + \alpha^2 \| g \|_2^2$$
$$- 2\mathrm{Re}\big[\langle \nabla\psi, e^{i\gamma}\nabla g(\cdot+y)\rangle + \alpha^2\langle \psi, e^{i\gamma}g(\cdot+y)\rangle\big]$$
$$\frac{\partial}{\delta\gamma}\mathrm{Re}[\quad] = 0 = \mathrm{Im}\big[\langle \nabla\psi, e^{i\gamma}\nabla g(\cdot+y)\rangle + \alpha^2\langle \psi, e^{i\gamma}g(\cdot+y)\rangle\big]$$
$$= \mathrm{Im}\big[\langle \nabla g + \nabla w\ ,\ \nabla g\rangle + \alpha^2\langle g+w, g\rangle\big]$$
$$= \langle \nabla v, \nabla g\rangle + \alpha^2\langle v, g\rangle \ = \langle v, -vg + \alpha^2 g\rangle$$
$$= 2\langle v, g^{2\sigma+1}\rangle$$

$$\frac{\partial}{\partial y_i}\mathrm{Re}[\cdots] = 0 \Rightarrow \int g^{2\sigma}\frac{\partial g}{\partial \chi_i}\cdot u = 0$$

Lemma. $\forall v \perp g^{2\sigma+1} \Rightarrow (L_0v,v) \geq C_0 \| v \|_{H^1}^2 \qquad C_0 > 0$

Proof.

$$(L_0v,v) \geq 0 \quad \forall v \in H^2$$
$$\inf_{f\in H^1}(L_0f,f) = \beta_0 \geq 0$$
$$\| f \|_2 = 1 \ f \perp g^{2\sigma+1} \qquad f_n \text{ minimizer sequence}(L_0f_n,f_n) \searrow \beta_0$$
$$\text{if } \beta_0 = 0 \qquad \forall\eta > 0 \text{ small}$$
$$\int \frac{1}{2}|\nabla f_n|^2 + \frac{\alpha^2}{2}|f_n|^2 < \int g^{2\sigma}|f_n|^2 + \eta$$
$$f_n \overset{H^1}{\rightharpoonup} f_\infty \qquad f_n \to f_\infty\ L^2_{loc}\ a.c. \int |f_\infty|^2 \leq \infty$$
$$\Rightarrow \frac{\alpha^2}{2} < \int g^{2\sigma}|f_\infty|^2 + \eta \Rightarrow f_\infty \neq 0$$

$$\tilde{f}_a = \frac{f_\infty}{\| f_\infty \|_2} \perp g^{2\sigma+1}$$

$$\langle L_0 \tilde{f}_\infty, \tilde{f}_\infty \leq 0 \qquad \Rightarrow \langle L_0 \tilde{f}_\infty, \tilde{f}_\infty \rangle = 0$$

$$\tilde{f}_\infty \in \ker L_0 = \text{span}\{g\} \Rightarrow\Leftarrow \Rightarrow \tilde{f}_\infty \perp g^{2\sigma+1} \text{positive function}$$

$$\Rightarrow \langle L_0 v, v \rangle \geq C_0 \| v \|_2^2$$

$$(L_0 v, v) = (1-\delta)(L_0 v, v) + \delta \int |\nabla v|^2 + \frac{\delta \alpha^2}{2} \int |v|^2 - \delta \int g^{2\sigma} |v|^2$$

$$\delta \quad \text{small} \qquad |g^{2\sigma} \cdot |_\infty \cdot \delta < \frac{\alpha_0}{2}(1-\delta)$$

$$\geq C\delta \int |\nabla v|^2 + \delta \frac{\alpha^2}{2} \int |v|^2$$

□

Ma 142 5/27/2003

Weinstein's orbital stability theorem (~ 86)

Let $g = g(x;\alpha)$ be the ground state of

$$-\frac{1}{2}\Delta g + \frac{\alpha^2}{2} g - g^{2\sigma+1} = 0$$

Suppose $d\sigma < 2$. Then the soliton $\Psi_0(t,x) = e^{i\frac{\alpha^2}{2}t} g(x;\alpha)$ is *orbitally* stable: i.e., $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\text{if } \circledast : \inf_{\substack{\theta \in \mathbb{R}, \\ y \in \mathbb{R}^d}} \| e^{i\theta} \phi(\cdot + y) - g \|_{H^1} < \delta \text{ , then}$$

$$\sup_t \int_{\substack{\theta \in \mathbb{R} \\ y \in \mathbb{R}^d}} \| e^{i\theta} \Psi(t, \cdot + y) - g \|_{H^1} < \varepsilon$$

where

$$\left[\begin{array}{l} i\partial_t \Psi + \frac{1}{2}\Delta \Psi + |\Psi|^{2\sigma} \Psi = 0 \\ \Psi\big|_{t=0} = \phi \end{array} \right.$$

Proof.

$$\theta_F = \{ e^{i\gamma} f(\cdot + y) \big| \gamma \in \mathbb{R}, y \in \mathbb{R}^d \}$$

$$\rho(\Psi(t), g) := \inf_{\theta, y} \{ \| e^{i\theta} \nabla \Psi(t, \cdot + y) - \nabla g \|_2^2 + \alpha^2 \| e^{i\theta} \Psi(t, \cdot + y) - g \|_2^2 \} \text{ .}$$

□

Check: Since $t \mapsto \Psi(t)$ is H^1-continuous, $\exists y(t)$, $\theta(t)$ cont. so that $\rho\big(\Psi(t), g\big)$ is attained at $y(t)$ and $\theta(t)$.

Let

$$e^{i\theta(t)}\Psi\big(t, \cdot + y(t)\big) = g + w(t) = g + u(t) + iv(t)$$
$$\Rightarrow \int g^{2\sigma+1} v\, dx = 0 \;, \int g^{2\sigma}\frac{\partial g}{\partial x_e} u\, dx = 0 \;, \forall 1 \le l \le d \,.$$

Action

$$\zeta[f] = \int \Big\{\frac{1}{2}|\nabla f|^2 + \frac{\alpha^2}{2}|f|^2 - \frac{1}{\sigma+1}|f|^{2\sigma+2}\Big\} dx$$
$$= \mathcal{H}[f] + \frac{\alpha^2}{2}\int |f|^2$$

$$\Rightarrow \zeta[\Psi(t)] = \zeta[\Psi(0)] \;\; = \zeta[\phi]$$
$$\zeta[\Psi(t)] = \zeta\big[e^{i\theta(t)}\Psi\big(t, \cdot + y(t)\big)\big]$$
$$= \zeta[g + w(t)]$$
$$\underset{\substack{\text{small}}}{\underset{\delta\zeta[\Psi(0)]}{\overset{\delta\zeta[\Psi(t)]}{\shortparallel}}} \; = \zeta[\Psi(t)] - \zeta[g] = \zeta[g + w(t)] - \zeta[g]$$

$$\underset{\text{Claim}}{>} C_1 \parallel w(t) \parallel^2_{H^1} - C(\parallel w(t) \parallel^{2+\mu}_{H^1} + \parallel w(t) \parallel^6_{H^1}) \qquad \mu > 0$$

$$\sigma = 1\;,\; d = 1 \qquad \zeta[f] = \int \frac{1}{2}|\nabla f|^2 + \frac{\alpha^2}{2}|f|^2 - \frac{1}{2}|f|^4$$
$$\zeta[g + w] - \zeta[g] \ge (L_0 v, v) + (L_1 u, u) - C(\underset{\substack{\uparrow \\ H^1}}{\parallel w \parallel^3_3} + \underset{\substack{\uparrow \\ H^1}}{\parallel w \parallel^4_4})$$

Sob. emb.

$$L_0 = -\frac{1}{2}\Delta + \frac{\alpha^2}{2} - g^{2\sigma}$$
$$L_0 g = 0$$
$$L_1 = -\frac{1}{2}\Delta + \frac{\alpha^2}{2} - (2\sigma+1)g^{2\sigma}$$
$$L_1(\vec{\nabla} g) = 0\;,\; L_1(\partial_\alpha g) = -\alpha g$$

$\Rightarrow \cdot g$ is the unique positive ground state of L_0

- **Fact**: $\mathrm{Ker} L_1 = \mathrm{span}\{\partial_j g | 1 \le j \le d\}$ and L_1 has unique negative eigenvalue $\mu_0 < 0$.

Lemma. $(L_0 v, v) \ge C_0 \parallel v \parallel^2_{H^1}$, $\forall v \perp g^{2\sigma+1}$ *(done last time)*

Lemma. $(L_1 f, f) \geq 0$, $\forall f \perp g$

Proof. Suppose

$$\inf_{\substack{\|f\|_2^2=1 \\ f\perp g}} (L_1 f, f) = \beta_1 < 0$$

Let f_∞ be the minimizer:

$$\Rightarrow L_1 f_\infty = \beta_1 f_\infty + \beta_2 g \ , \ \mu_0 < \beta_1 < 0 \Rightarrow \beta_2 \neq 0 \ .$$
$$(L_1 - \beta_1) f_\infty = \beta_2 g$$

Remark: If $\beta_1 = \mu_0$, then f_∞ is the ground state. If $L_1 \Rightarrow f_\infty > 0$: contradicts $\int f_\infty g = 0$ as $g > 0$.

So

$$f_\infty = \beta_2 (L_1 - \beta_1)^{-1} g$$
$$\Rightarrow 0 = \omega(\beta_1) = \langle g | (L_1 - \beta_1)^{-1} | g \rangle$$
$$\text{where } \omega(\lambda) = \langle g | (L_1 - \lambda)^{-1} | g \rangle \text{ for } \mu_0 < \lambda < 0.$$

So

$$\omega'(\lambda) > 0$$
$$\Rightarrow \omega(0) > 0$$

$$\begin{aligned} \omega(0) &\underset{\uparrow}{=} \langle g | L_1^{-1} | g \rangle \\ &\text{as } g \perp \text{Ker} L_1 = \text{span}\{\vec{\nabla} g\} \\ &= -\frac{1}{\alpha} \langle g | \partial_\alpha g \rangle \\ &= -\frac{1}{2\alpha} \underbrace{\partial_\alpha \| g \|_2^2}_{>0 \text{ as } d\sigma < 2} < 0 \\ &\text{contradiction} \end{aligned}$$

□

Lemma.

$$\exists C_1 > 0 \ , \ (L_1 f, f) \geq C_1 \| f \|_2^2 \ , \forall f \perp g \ \textit{and}$$
$$f \perp g^{2\sigma} \frac{\partial g}{\partial x^l} \ , 1 \leq l \leq d \ .$$

Proof.

$$\inf_{\substack{f\in H^1\\ f\perp g, f\perp\vec{\nabla}g\cdot g^{2\sigma}\\ \|f\|_2^2=1}} (L_1 f, f) = \beta_1 \geq 0 \quad \text{By previous lemma.}$$

Assume $\beta_1 = 0$, $\Rightarrow \exists f_\infty$ minimizer satisfying

$$\begin{aligned} L_1 f_\infty &= \overbrace{\beta_1 f_\infty}^{=0} + \beta_2 g + \vec{\beta}_3 \cdot \vec{\nabla} g g^{2\sigma} \\ (L_1 f_\infty, g) &= \beta_2 (g, g) \\ (\underset{\substack{\shortparallel\\0}}{f_\infty \; L_1 \partial_j g})(L_1 f_\infty, \partial_j g) &= \beta_{3,l} \underbrace{\int \partial_l g g^{2\sigma} \partial_j g}_{A_j l} \end{aligned}$$

$$(A\xi, \xi) = \int g^{2\sigma} |\vec{\nabla} g \cdot \xi|^2 dx > 0 \text{ if } \xi \neq 0$$

So $\vec{\beta}_3 = 0$

$$\begin{aligned} L_1 f_\infty &= \beta_2 g \\ f_\infty &= -\frac{\beta_2}{\alpha} \partial_\alpha g + \vec{c} \cdot \vec{\nabla} g \\ (f_\infty, g) = 0 &= -\frac{\beta_2}{\alpha} \underbrace{(g, \partial_\alpha g)}_{<0} \Rightarrow \beta_2 = 0 \end{aligned}$$

So $f_\infty = \vec{c} \cdot \vec{\nabla} g$

$$\begin{aligned} 0 \underset{\substack{\uparrow\\ f_\infty \perp g^{2\sigma}\vec{\nabla} g}}{=} \left(f_\infty, g^{2\sigma} \frac{\partial g}{\partial x^l} \right) &= c_j \underbrace{\int \frac{\partial g}{\partial x^j} g^{2\sigma} \frac{\partial g}{\partial x^l}}_{\forall j,l} \Rightarrow \vec{c} = 0 \\ \Rightarrow f_\infty = 0: \text{ contradicts } \| f_\infty \|_2 = 1 \end{aligned}$$

$\square$

So we get

$$\begin{aligned} (L_1 f, f) &\leq C_1 \| f \|_{H^1}^2 \\ \forall f \perp g \ &, \ f \perp \vec{\nabla} g \cdot g^{2\sigma} \end{aligned}$$

$$(L_1 f, f) = \int \frac{1}{2} |\nabla f|^2 + \frac{\alpha^2}{2} |f|^2 - (2\sigma + 1) g^{2\sigma} |f|^2$$

If $\| \nabla f \|_2 < \gamma^{-1} \| f \|_2$, you're okay.

$$(L_1 f, f) \geq \frac{C_1}{2} \| f \|_2^2 + \frac{C_1}{2}\gamma^2 \| \nabla f \|_2^2$$

If $\| \nabla f \|_2 \geq \gamma^{-1} \| f \|_2$

Then

$$\begin{aligned}(L_1 f, f) &\geq \frac{1}{2}(L_1 f, f) + \frac{C_1}{2} \| f \|_2^2 \\ &\geq \frac{1}{4}\int |\nabla f|^2 \, dx + \frac{1}{4}\gamma^{-2}\int |f|^2 - \Big(\frac{\alpha^2}{2} + (2\sigma+1) \| g \|_\infty^{2\sigma}\Big)\int |f|^2 + \frac{C_1}{2} \| f \|_2^2\end{aligned}$$

with γ small enough so that

$$\frac{1}{4}\gamma^{-2} > \frac{\alpha^2}{2} + (2\sigma+1) \| g \|_\infty^{2\sigma}$$

Observation: It's enough to consider ϕ so that

$$\| \phi \|_2 = \| g \|_2 \ .$$

$$\circledast \Rightarrow \Big| \| \phi \|_2 - \| g \|_2 \Big| < \delta$$

Take $g_\lambda(x, \alpha) := \lambda^{\frac{1}{\sigma}} g(\lambda x, \alpha)$

$$-\frac{1}{2}\Delta g_\lambda - g_\lambda^{2\sigma+1} = \lambda^{\frac{1}{\sigma}}\lambda^2\big(-\frac{1}{2}(\Delta g)(\lambda x) - g^{2\sigma+1}(\lambda x)\big) =$$

Nonlinearity:

$$\frac{2\sigma+1}{\sigma} = 2 + \frac{1}{\sigma}$$

$$= -\lambda^{\frac{1}{\sigma}}\lambda^2\frac{\alpha^2}{2}g(\lambda x, \alpha) = -\frac{(\lambda\alpha)^2}{2}g_\lambda(x, \alpha)$$

So $g_\lambda(x, \alpha) = g(x, \lambda\alpha)$

$$\| g_\lambda \|_2 = \lambda^{\frac{1}{\sigma}}\lambda^{-\frac{2}{\alpha}} \| g \|_2$$

So $\exists\lambda$ close to 1 s.t. $\| \phi \|_2 = \| \lambda^{\frac{1}{\sigma}} g(\lambda\cdot) \|_2$

Assume we've proved theorem for $\| \phi \|_2 = \| g \|_2$. That is we've changed g to g_λ for λ small. But $\| g - g_\lambda \|_{H^1}$ small $\Rightarrow$ OK.

Lemma.

$$(L_1 u, u) \geq C_1 \| u \|_{H^1}^2 - C(\| w \|_{H^1}^4 + \| w(t) \|_{H^1}^3)$$

with $u(t)$ as above (plus L^2- preservation)

Proof.

$$\begin{aligned}\| g \|_2^2 = \| \Psi(t) \|_2^2 &= \| g + w(t) \|_2^2 = \| g + u(t) \|_2^2 + \| v(t) \|_2^2 \\ &= \| g \|^2 + 2(g, u) + \| w \|_2^2 \\ \Rightarrow \big(g, u(t)\big) &= -\frac{1}{2} \| w(t) \|_2^2\end{aligned}$$

$$u(t) = u_{\|}(t) + u_{\perp}(t)$$

$$(\text{wlog} \parallel g \parallel_2 = 1) \qquad u_{\|} = (u, g,)g\ .$$

$$(L_1 u, u) = \underbrace{(L_1 u_\perp, u_\perp)}_{\substack{\geq C_1 \|u_\perp\|^2_{H^1} \\ \uparrow \\ (\text{as } g \perp g^{2\sigma}\vec{\nabla} g \Rightarrow u_\perp \perp g^{2\sigma}\vec{\nabla} g)}} + 2(L_1 u_\perp, u_\|) + (L_1 u_\|, u_\|)\ .$$

$$(L_1 u_\|, u_\|) = (u, g)^2 (L_1 g, g) = \underset{\substack{\uparrow \\ \text{possibly negative}}}{\mathrm{c}} \parallel w(t) \parallel_2^4$$

$$\begin{aligned} (L_1 u_\|, u_\perp) &= (u, g)(L_1 g, u) - (u, g)^2 (L_1 g, g) \\ &\geq -\frac{1}{2} \parallel w(t) \parallel_2^2 \parallel w(t) \parallel_2 \parallel L_1 g \parallel_2 - c \parallel w(t) \parallel_2^4 \end{aligned}$$

$$\parallel u_\perp \parallel^2_{H^1} = \parallel u - (u, g) g \parallel^2_{H^1} = \parallel u \parallel^2_{H^1} - \overbrace{2(u, g)_{H^1}(u, g)_{L^2}}^{\substack{\sim \|w\|^3 \\ \Uparrow}} + (u, g)^2 \parallel g \parallel^2_{H^1}$$

□