Decay of linear waves on curved backgrounds

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An overview

- Pointwise decay for the free wave and Schrödinger evolutions
- Perturbations by a (magnetic) potential, local L^2 vs. global L^∞ decay. Role of zero energy resonances. Laplace transform method. Global from local decay.
- A nonlinear application to center-stable manifold for NLW.
- Change of metric, trapping vs. nontrapping.
- Surfaces of revolution, decay of waves on them. Periodic geodesic, asymptotically conical.
- Theorems: Decay at fixed angular momentum ℓ, summation over ℓ; large $\ell \rightsquigarrow$ semiclassical formulation. Role of negative curvature. Elliptic vs. hyperbolic periodic geodesics.
- Reduction to a one-dimensional problem with a smooth, asymptotically inverse square potential on $\mathbb{R}$ ('critical decay'). WKB in the double asymptotic regime ($\hbar \to 0$, $E \to 0$).
- Mourre estimate at the top energy. Semiclassical Hunziker-Sigal-Soffer propagation estimates.
- Waves on a Schwarzschild black-hole background, Price's law.

The free case

Schrödinger evolution $\psi(t) = e^{it\Delta}\psi_0$ in $\mathbb{R}^{d+1}_{t,x}$ satisfies:

$$\|\psi(t)\|_{H^s} = \|\psi_0\|_{H^s}$$

$$\|\psi(t)\|_\infty \leq Ct^{-\frac{d}{2}}\|\psi_0\|_1$$

Follow from, respectively,

$$\begin{aligned}\psi(t,x) &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(t|\xi|^2 + x\cdot\xi)}\widehat{\psi_0}(\xi)\, d\xi \\ &= c(d)t^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}}\psi_0(y)\, dy\end{aligned}$$

Wave equation $\Box u = \partial_t^2 u - \Delta u = 0$ in $\mathbb{R}^{d+1}$ satisfies

$$\mathcal{E}(u) = \|\nabla u\|_2^2 + \|\partial_t u\|_2^2 = \mathrm{const}$$

and dispersive decay

$$\|u(t)\|_\infty \lesssim t^{-\frac{d-1}{2}}(\|u(0)\|_{\dot{B}^{\frac{d+1}{2}}_{1,1}} + \|\partial_t u(0)\|_{\dot{B}^{\frac{d-1}{2}}_{1,1}})$$

Besov norm $\|f\|_{\dot{B}^\alpha_{1,1}} = \sum_{j\in\mathbb{Z}} 2^{\alpha j}\|P_j f\|_1$.

Set $j = 0$. Apply stationary phase to

$$P_0 e^{\pm it|\nabla|} f(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i((x-y)\cdot\xi \pm t|\xi|)} \chi(\xi)\, d\xi\, f(y)\, dy$$

in polar coordinates. Note: $D^2_\xi |\xi|$ degenerate in radial direction.
In odd dimensions stronger bound

$$\|u(t)\|_\infty \lesssim t^{-\frac{d-1}{2}} (\|u(0)\|_{\dot{W}^{\frac{d+1}{2},1}} + \|\partial_t u(0)\|_{\dot{W}^{\frac{d-1}{2},1}})$$

$\dot{W}^{\alpha,p}$ is homogeneous Sobolev space.
In $\mathbb{R}^3$,

$$\|u(t)\|_\infty \lesssim t^{-1}(\|D^2 u(0)\|_{L^1(\mathbb{R}^3)} + \|D\partial_t u(0)\|_{L^1(\mathbb{R}^3)})$$

Follows from the Kirchhoff formula:

$$u(t,x) = (4\pi t)^{-1} \int_{tS^2} g(x+y)\, \sigma_{tS^2}(dy)$$

solves $\Box u = 0$, $(u(0), \partial_t u(0)) = (0, g)$. Apply Gauss-Green divergence theorem, Sobolev imbedding $\dot{W}^{1,1} \hookrightarrow L^{\frac{3}{2}}$.

Lower order perturbations

Consider $H = -\Delta + V$ or $H = (i\nabla + A)^2$ with Schrödinger and wave evolutions

$$e^{itH}, \qquad \cos(t\sqrt{H}), \ \frac{\sin(t\sqrt{H})}{\sqrt{H}}$$

V, A real-valued, sufficiently regular, decaying at infinity. H self-adjoint.

Question: Decay estimates as in free case?
Obvious problem: bound states $H\psi = E\psi$, $E \leq 0$. So restrict attention to $HP_c = H\chi_{(0,\infty)}(H)$.

Jensen-Kato **local decay** theorem, late 1970's:

$$\|\langle x\rangle^{-\sigma} e^{itH} P_c f\|_{L^2(\mathbb{R}^3)} \lesssim \langle t\rangle^{-\frac{3}{2}} \|\langle y\rangle^{\sigma} f\|_{L^2(\mathbb{R}^3)} =: \langle t\rangle^{-\frac{3}{2}} \|f\|_{L^{2,\sigma}(\mathbb{R}^3)}$$

for some $\sigma > 0$, V polynomially decaying.

Essential condition: **zero energy is neither an eigenvalue nor a resonance** of H (zero is regular)

Lower order perturbations, local decay

This means:

- $\sup_{\mathrm{Im}\, z>0} \|\langle x\rangle^{-\sigma}(-\Delta + V - z)^{-1}\langle x\rangle^{-\sigma}\|_{2\to 2} < \infty$
- Nonexistence of $f \not\equiv 0$ with

$$Hf = 0, \quad f \in \bigcap_{\varepsilon>0} L^{2,-\frac{1}{2}-\varepsilon}(\mathbb{R}^3)$$

Laurent expansion of resolvent: as $z \to 0$ in $\mathrm{Im}\, z > 0$,

$$R(z) := (-\Delta + V - z)^{-1} = z^{-1}B_{-1} + z^{-\frac{1}{2}}B_{-\frac{1}{2}} + B_0 + z^{\frac{1}{2}}B_{\frac{1}{2}} + \rho(z)$$

- $B_{-1}, \ldots, B_{\frac{1}{2}}$ bounded in $L^{2,\sigma}$
- $\|\langle x\rangle^{-\sigma}\rho(z)f\|_2 \lesssim |z|\|\langle x\rangle^{\sigma} f\|_2$ for z small.
- B_{-1} is the orthogonal projection onto the zero eigenspace
- zero energy is regular iff $B_{-1} = B_{-\frac{1}{2}} = 0$
- $B_{-1}, B_{-\frac{1}{2}}$ are of finite rank
- Jensen-Kato theorem: $\int_0^\infty e^{it\lambda}[R(\lambda) - R(\lambda)^*]\, d\lambda$

Lower order perturbations, local decay

Examples:

- $V = 0$ in three dimensions, $z = \zeta^2$:

$$(-\Delta - \zeta^2)^{-1}(x, y) = \frac{e^{i\zeta|x-y|}}{4\pi|x-y|}, \quad \operatorname{Im}\zeta > 0$$

  Taylor expand exponential. Zero energy regular.

- $V = 0$ in one dimension:

$$(-\Delta - \zeta^2)^{-1}(x, y) = \frac{e^{i\zeta|x-y|}}{2i\zeta}, \quad \operatorname{Im}\zeta > 0$$

  Zero energy is a *resonance*.

- In $\mathbb{R}^d$:

$$(-\Delta - \zeta^2)^{-1}(x, y) = c_d\, \zeta^{\frac{d-2}{2}} |x-y|^{-\frac{d-2}{2}} H^{+}_{\frac{d-2}{2}}(\zeta|x-y|)$$

  with Hankel function. If d even, logarithmic branch point at $\zeta = 0$.

Laplace transform method, Hille-Yoshida theorem

$$e^{itH} P_c = \frac{1}{2\pi i} \int_{p_0 - \infty}^{p_0 + \infty} e^{tp} (H + ip)^{-1} P_c \, dp \qquad p_0 > 0$$

- Meromorphic continuation of $(H + ip)^{-1}(x, y)$ to $\mathrm{Re}\,(p) \leq 0$ (for example, $H = -\Delta + V$, V compactly supported), poles equal complex resonances.
- Deform contour into "thermometer" around $(-\infty, 0]$. Residues contribute $\sum_j e^{\zeta_j t} \phi_j$, $\mathrm{Re}\,(\zeta_j) < 0$.
- As $t \to \infty$, dominant tail comes from expansion around $p = 0$:

$$\int_0^\infty e^{-tp} p^\alpha \, dp = t^{-\alpha - 1} \Gamma(\alpha + 1)$$

- So $t^{-\frac{1}{2}}$ if $\alpha = -\frac{1}{2}$ as in the resonant case for $d = 3$, and $t^{-\frac{3}{2}}$ if zero is regular $(\alpha = \frac{1}{2})$.
- In odd dimension $d > 3$ branching starts at $p^{\frac{d-2}{2}} \rightsquigarrow t^{-\frac{d}{2}}$.

$$u(t) = \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c g = \frac{1}{2\pi i} \int_{p_0 - i\infty}^{p_0 + i\infty} e^{tp} (H + p^2)^{-1} P_c \, g dp, \qquad p_0 > 0$$

- In odd dimensions, $R(p^2)$ is analytic at $p = 0 \rightsquigarrow$ **exponential local decay**. *Sharp Huygens principle* (SHP)
- In even dimensions, $R(p^2)$ exhibits logarithmic branching at $p = 0 \rightsquigarrow$ specific power law for the local decay (failure of SHP).

Summary: *Local decay for Schrödinger and wave evolutions determined by smallest non-analytic contribution to the resolvent as* $p \to 0$.

Some history

- Vainberg, Rauch 70's: local decay for wave and Schrödinger for exponentially decreasing potentials, role of resonance for $d = 3$
- Jensen, Kato late 70's: expansion of the local evolution in powers of time for polynomially decaying V
- Murata, early 80's: most complete analysis of the local decay for Schrödinger, asymptotic expansion in time, also for the case of zero energy being singular
- **Global** $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ decay for $e^{it(-\Delta+V)}$, $d \geq 3$ under decay and regularity assumptions on V, zero energy regular, by Journé, Soffer, Sogge 1991 (JSS).
- Beals, Strauss 93,94: global pointwise decay for wave equation, $V \geq 0$ or V small.
- Yajima 1995-2005: boundedness of the wave-operators $W_\pm := \lim_{t\to\pm\infty} e^{-itH}e^{itH_0}$ on L^p and $W^{k,p}$, $1 \leq p \leq \infty$. W intertwines evolutions: $f(H)P_c(H) = Wf(H_0)W^*$. Improves previous global decay results.

Some history

- 2000 - present: Rodnianski, S., Krieger, Goldberg, Erdogan, Beceanu, Vodev, Moulin, Cuccagna, d'Ancona, Georgiev obtained various results weakening assumptions on V
- time-dependent potentials: present major difficulties, no general theory. Partial results by Rodnianski-S., Goldberg, Beceanu. For time-periodic case (ionization problem) major advance by Costin, Lebowitz, Tanveer, as well as Yajima et al.
- Magnetic case: **No pointwise global decay results known**. Strichartz estimates by Erdogan, Goldberg, S., and Metcalfe, Tataru, Marzuola, 2006, 2007.
- Applications to asymptotic stability problems for nonlinear Schrödinger and wave equations: Soffer-Weinstein, Buslaev-Perelman, Rodnianski-S.-Soffer, Krieger-S., Cuccagna, Mizumachi.

Global decay for Schrödinger

Ginibre's argument: $H = H_0 + V$, $|V(x)| \lesssim \langle x \rangle^{-2\sigma}$, assume

$$\|e^{itH_0} f\|_{L^2 + L^\infty(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\alpha} \|f\|_{L^1 \cap L^2(\mathbb{R}^d)}$$
$$\|\langle x \rangle^{-\sigma} e^{itH} P_c f\|_{L^2(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\alpha} \|\langle y \rangle^{\sigma} f\|_{L^2(\mathbb{R}^d)}$$

Applying Duhamel twice yields

$$\begin{aligned} e^{itH} P_c &= e^{itH_0} P_c + i \int_0^t e^{-i(t-s)H_0} V e^{isH} P_c \, ds \\ &= e^{itH_0} P_c + i \int_0^t e^{i(t-s)H_0} V P_c e^{isH_0} \, ds \\ &+ \int_0^t \int_0^s e^{i(t-s)H_0} V e^{i(s-s')H} P_c V e^{is' H_0} \, ds' \, ds \end{aligned}$$

Important feature: evolution of H sandwiched between two weights (namely V) and P_c placed correctly. So can use *local decay* for H.

Global decay for Schrödinger

If $\alpha > 1$, then for $\|f\|_{L^1 \cap L^2(\mathbb{R}^d)} = 1$ one has

$$\|e^{itH} P_c f\|_{L^\infty + L^2(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\alpha} + \int_0^t \langle t-s \rangle^{-\alpha} \langle s \rangle^{-\alpha} \, ds$$
$$+ \int_0^t \int_0^s \langle t-s \rangle^{-\alpha} \langle s-s' \rangle^{-\alpha} \langle s' \rangle^{-\alpha} \, ds' \, ds \lesssim \langle t \rangle^{-\alpha}$$

For $H_0 = -\Delta$, works for $d \geq 3$: $\alpha = \frac{d}{2}$. Remove L^2: difficulty of $(t-s)^{-\alpha}$, nonintegrable at $s = t$. Use

$$\sup_{1 \leq p \leq \infty} \|e^{-it\Delta} V e^{it\Delta}\|_{p \to p} \leq \|\hat{V}\|_1$$

Some applications

- Energy critical wave equation $\Box u - u^5 = 0$ in $\mathbb{R}^{1+3}$.
- locally well-posed in $\dot{H}^1 \times L^2$, global existence for small data, large data can blow up in finite time.
- Stationary solutions $W_\lambda(x) := \lambda(1+\lambda^2|x|^2/3)^{-\frac{1}{2}}$ for $\lambda > 0$ (extremizers of $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$)
- Linearizing around W_λ leads to $H = -\Delta - 5W_\lambda^4$
- Negative eigenvalue, $\partial_\lambda W_\lambda$ is a resonant mode of zero energy.
- W_λ is linearly exponentially unstable.
- There exist data arbitrarily close to W_λ in energy which blow up in finite time (Krieger-S.-Tataru, 07). Duykaerts, Kenig, Merle 09: all radial type II blowup near W of this nature.
- There exists a codimension one Lipschitz manifold near W_λ in the space of radial data with enough regularity and decay such that data on it obey asymptotic stability. $\{W_\lambda\}$ acts as an attractor. **Exists in energy space, center stable mf?**

Theorem

$V \in \mathbb{R}$, $|V(x)| \lesssim \langle x\rangle^{-\kappa}$ with $\kappa > 3$. If zero energy is regular for $H = -\Delta + V$, then

$$\left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f \right\|_\infty \lesssim t^{-1} \|f\|_{W^{1,1}(\mathbb{R}^3)}$$

for all $t > 0$. If zero is a resonance but not an eigenvalue of $H = -\Delta + V$, let ψ be the unique resonance function normalized so that $\int V\psi(x)\, dx = 1$. Then $\exists\, c_0 \neq 0$ s.t.

$$\left\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f - c_0(\psi \otimes \psi) f \right\|_\infty \lesssim t^{-1} \|f\|_{W^{1,1}(\mathbb{R}^3)}$$

for all $t > 0$.

Metric perturbations

- $-\Delta \rightsquigarrow H := -\sum_{j,k=1}^{d} \partial_j(a_{jk}(x)\partial_k)$
- New obstruction: trapping. Classical Hamiltonian flow

$$\dot{x}_k := 2\sum_{j=1}^{d} a_{jk}(x)\xi^j, \qquad \dot{\xi}_\ell = \sum_{j,k=1}^{d} \partial_\ell a_{jk}(x)\xi^k\xi^j$$

  exhibits time-periodic trajectories.
- Murata 1984: if $I \subset (0,\infty)$ has no trapped energies, then

$$\sup_{\mathrm{Im}\, z>0,\, \mathrm{Re}\, z\in I} \|\langle\cdot\rangle^{-\sigma}(H-z)^{-1}\chi_I(H)\langle\cdot\rangle^{-\sigma}\|_{2\to 2} < \infty$$

- Tsutsumi 1984: local decay for Schrödinger outside a nontrapping obstacle, Dirichlet BC.
- Ikawa 1988: wave equation outside of several convex bodies, trapped rays, local energy decay, complex resonances.
- Doi 1996: trapped trajectories destroy the $\frac{1}{2}$-Kato smoothing effect of the Schrödinger flow
- witout trapping: Craig, Kappeler, Strauss; Staffilani, Tataru; Rodnianski, Tao; Hassel, Tao, Wunsch; Tataru; Nakamura

Elliptic versus hyperbolic geodesics, continued

- quantify "destroy": no global in time Strichartz estimates possible; local in time: Burq, Gerard, Tzvetkov obtained Strichartz estimates for compact $\mathcal{M}$ with losses of derivatives, some losses necessary.
- **Flat** 2-**dim torus**: Bourgain early 90's obtained L^4_{tx} Strichartz *without loss*, Tatakoa-Tvetkov same for $S^1 \times \mathbb{R}$.
- **hyperbolic case**: pioneered by Ikawa (starting 80's), remove convex obstacles from $\mathbb{R}^3$, distribution of resonances, local energy decay. Some loss in terms of data, but same exponential decay for local energy as in non-trapping case.
- Beginning systematic developments (2000-present): Anantharaman, Nonnenmacher, Zworski, Christianson, Burq, Guillarmou, Hassell. Find ε-loss or no loss in Strichartz, study semi-classical resonances
- Doi: some loss must occur in smoothing estimate for the Schrödinger if there is a trapped trajectory.
- No general theory at this point.

Surfaces of revolution, conic ends

- $\Omega \subset \mathbb{R}^N$ embedded compact d-dimensional Riemannian mfld
- Define the $(d+1)$-dimensional manifold

$$\mathcal{M} := \{(x, r(x)\omega) \mid x \in \mathbb{R},\ \omega \in \Omega\}$$
$$ds^2 = r^2(x) ds_\Omega^2 + (1 + r'(x)^2) dx^2$$

$r \in C^\infty(\mathbb{R})$ and $\inf_{x\in\mathbb{R}} r(x) > 0$.

- *conical ends*:

$$r(x) = |x| \, (1 + h(x)), \quad h^{(k)}(x) = O(x^{-2-k}) \quad \forall\, k \geq 0$$

as $x \to \pm\infty$.

- Example: one-sheeted hyperboloid, $r(x) = \sqrt{1+x^2}$.
- Geodesic flow trapped on $(x_0, r(x_0)\Omega)$ provided $r'(x_0) = 0$
- For simplicity: $\Omega = S^1$.

Decay of waves on $\mathcal{M}$

Consider $e^{it\Delta_{\mathcal{M}}}$ and $\frac{\sin(t\sqrt{-\Delta_{\mathcal{M}}})}{\sqrt{-\Delta_{\mathcal{M}}}}$, $\cos(t\sqrt{-\Delta_{\mathcal{M}}})$.

- What type of local/global decay does one have?
- Does the trapped geodesic destroy the Euclidean decay rates?
- What is the difference between a one-sheeted hyperboloid and $\mathcal{M}$ that has an equatorial sector of S^2 in the middle?

Some answers:

- For **fixed angular momentum** ℓ the same global decay holds as for $\mathbb{R}^2$.
- In fact, one has **faster local decay** for $\ell > 0$. These rates are *universal*, i.e., independent of the local geometry. *Non-Euclidean behavior*.
- The local geometry determines the constants $C(\ell)$ involved in the decay bounds. Summation over ℓ possible only if $\mathcal{M}$ has negative curvature (can be relaxed somewhat).

Decay of waves on $\mathcal{M}$

Theorem (S.-Soffer-Staubach, Donninger-S.-Soffer)

$\mathcal{M}$ a surface of revolution as above. Define weights $w_\sigma(x) := \langle x \rangle^{-\sigma}$ on $\mathcal{M}$. $\forall\, \ell \geq 0$, $\forall\, 0 \leq \sigma \leq \sqrt{2}\ell$, $\exists\, C(\ell, \mathcal{M}, \sigma)$, $C_1(\ell, \mathcal{M}, \sigma)$ s.t. $\forall\, t > 0$

$$\| w_\sigma \, e^{it\Delta_{\mathcal{M}}} f \|_{L^\infty(\mathcal{M})} \leq \frac{C(\ell, \mathcal{M}, \sigma)}{t^{1+\sigma}} \Big\| \frac{f}{w_\sigma} \Big\|_{L^1(\mathcal{M})}$$

$$\| w_\sigma \, e^{it\sqrt{-\Delta_{\mathcal{M}}}} f \|_{L^\infty(\mathcal{M})} \leq \frac{C_1(\ell, \mathcal{M}, \sigma)}{t^{\frac{1}{2}+\sigma}} \Big\| \frac{(\partial_x f, \ f)}{w_\sigma} \Big\|_{L^1(\mathcal{M})}$$

provided $f = f(x, \theta) = e^{i\ell\theta} \tilde{f}(x)$.

- Note the non-Euclidean decay for $\sigma > 0$!
- Rapid growth: $C(\ell) \sim e^{\ell^{2+}}$
- For fixed ℓ: change of variables reduces to 1-dim evolution $e^{it(-\partial_{xx}+V)}$, $\ell = 0$ zero resonance, $\ell > 0$ non-resonant.

Underlying one-dimensional problem

- Separation of variables for fixed $\ell \geq 0$. Reduction to operator in $\xi =$ arclength along a generator of $\mathcal{M}$.
- $\mathcal{H}_\ell = -\partial_\xi^2 + V_\ell(\xi)$

$$V_\ell(\xi) = \frac{\nu^2 - \frac{1}{4}}{\xi^2} + O(\xi^{-3}), \quad |\xi| \to \infty$$

  with $\nu := \sqrt{2}\,\ell$. Inverse square decay "critical".
- Determine local/global decay of

$$e^{it\mathcal{H}_\ell}, \qquad \frac{\sin(t\sqrt{\mathcal{H}_\ell})}{\sqrt{\mathcal{H}_\ell}}, \ \cos(t\sqrt{\mathcal{H}_\ell})$$

  on the line.
- Essential issue as before: *Zero energy resonance or not?*
- In the surface of revolution case $\ell = 0$ leads to zero energy resonance (as in $\mathbb{R}^2$), but $\ell > 0$ does not.
- No accelerated local decay possible for $\nu = 0$, for $\nu > 0$ one has faster local decay.
- Open problem: understand $\nu > 0$ in the resonant case.

Decay of waves on $\mathcal{M}$, summation in ℓ

Theorem (Donninger-S.-Soffer, fall 2009)

$\mathcal{M}$ as before, $K < 0$. Then for all $t > 0$, and any $\varepsilon > 0$,

$$\big\| w_{1+\varepsilon} e^{it\Delta_{\mathcal{M}}} w_{1+\varepsilon} f \big\|_{L^2(\mathcal{M})} \le \frac{C(\mathcal{M},\varepsilon)}{\langle t \rangle} \big\| (1-\partial_\theta^2)\, f \big\|_{L^2(\mathcal{M})}$$

$$\big\| w_1 e^{it\Delta_{\mathcal{M}}} w_1 f \big\|_{L^\infty(\mathcal{M})} \le \frac{C(\mathcal{M},\varepsilon)}{t} \big\| (1-\partial_\theta^2)^{2+\varepsilon} f \big\|_{L^1(\mathcal{M})}$$

For the wave equation one has, with $\mathcal{L} := 1 - \partial_\theta^2$,

$$\big\| w_{1+\varepsilon} e^{\pm it\sqrt{-\Delta_{\mathcal{M}}}} w_{1+\varepsilon} f \big\|_{L^2(\mathcal{M})} \le \frac{C_1(\mathcal{M},\varepsilon)}{\langle t \rangle^{\frac{1}{2}}} \big\| \mathcal{L}^{\frac{5}{4}} (\partial_x f, f) \big\|_{L^2(\mathcal{M})}$$

$$\big\| w_{\frac{1}{2}} e^{\pm it\sqrt{-\Delta_{\mathcal{M}}}} w_{\frac{1}{2}} f \big\|_{L^\infty(\mathcal{M})} \le \frac{C_1(\mathcal{M},\varepsilon)}{t^{\frac{1}{2}}} \big\| \mathcal{L}^{\frac{9}{4}+\varepsilon} (\partial_x f, f) \big\|_{L^1(\mathcal{M})}$$

Also admissible: $K < 0$ away from unique geodesic, $K = 0$ on it, but finitely degenerate. Lose higher powers of ∂_θ depending on order of degeneracy.

Elliptic versus hyperbolic geodesics, some history

- $\mathcal{M}$ 2-dim manifold, Γ periodic geodesic. Poincaré map on $T_0(T_p^*\mathcal{M})$ has eigenvalues $1, 1, \lambda_1, \lambda_2$.
- **elliptic:** $\lambda_2 = \overline{\lambda_1} \in S^1$, **hyperbolic:** $|\lambda_1| < 1 < |\lambda_2|$
- *elliptic case*: under further *stability conditions* on Γ, there exist **quasimodes**: u_ε concentrated near Γ in $\varepsilon^{\frac{1}{2}}$ neighborhood,

$$\|(\Delta_{\mathcal{M}} - E(\varepsilon))u_\varepsilon\|_2 \leq C_M\, \varepsilon^M \|u_\varepsilon\|_2$$

$$u_\varepsilon(x) = e^{i\psi(x)/\varepsilon}(a_0(x) + \varepsilon a_1(x) + \ldots)$$

  Gaussian beam, "geometric optics": $Im\psi \geq 0$ satisfies eikonal equation, a_j transport equations.
  Studied from 1960's-present by Keller, Babich, Lazutkin, Ralston, Colin de Verdiere, Stefanov, Zelditch, Zworski etc.; important relation to spectrum of $\Delta_{\mathcal{M}}$, resonances
- quasimodes **destroy** local energy decay, Stichartz estimates

Proof ideas

- Define arclength $d\xi^2 = (1 + r'(x)^2)\, dx^2$. For ℓ fixed
$$e^{-i\ell\theta} r^{\frac{1}{2}}(\xi) \Delta_{\mathcal{M}}(r^{-\frac{1}{2}}(\xi) e^{i\ell\theta} f(\xi)) = \mathcal{H}_\ell f$$
with
$$\mathcal{H}_\ell = -\partial_\xi^2 + V_\ell, \qquad V_\ell(\xi) = \frac{2\ell^2 - \frac{1}{4}}{\langle \xi \rangle^2} + O(\langle \xi \rangle^{-3})$$
- Inverse square decay "critical"; Jost solutions $\mathcal{H}_\ell f_\pm(\cdot, \lambda) = \lambda^2 f_\pm(\cdot, \lambda)$ not continuous as $\lambda \to 0$:
$$f_+(\xi, \lambda) = e^{i\xi\lambda} + \int_\xi^\infty \frac{\sin(\lambda(\eta - \xi))}{\lambda} V_\ell(\eta)\, f_+(\eta, \lambda) d\eta$$
- Behavior of these functions near $\lambda = 0$ crucial; for $\xi > \xi'$
$$e^{itH_\ell}(\xi, \xi') = \int_0^\infty e^{it\lambda} E_\ell(d\lambda)(\xi, \xi')$$
$$= \frac{2}{\pi} \int_0^\infty e^{it\lambda^2} \operatorname{Im} \Big[\frac{f_+(\xi, \lambda) f_-(\xi', \lambda)}{W(\lambda)} \Big] \lambda \, d\lambda$$

Proof Ideas 2

- At $\xi = \xi' = 0$ and for small λ reduces to

$$\left| \int_0^\infty e^{it\lambda^2} \lambda^{1+2\nu} \chi(\lambda)\, d\lambda \right| \leq Ct^{-1-\nu}$$

where $\nu := \sqrt{2}\,\ell$.

- Why is $E(d\lambda^2)$ so flat near $\lambda = 0$?
- To motivate, we demonstrate for $\ell > 0$

$$W_\ell(\lambda) = c\lambda^{1-2\nu}(1 + o(1)) \qquad \lambda \to 0$$

- WKB heuristics:

$$W(\lambda) = \frac{-2i\lambda}{T(\lambda)} = -2i\lambda e^{S(\lambda)}$$

where the action $S(\lambda)$ is

$$S(\lambda) = \int_{x_0}^{x_1} \sqrt{\nu^2 \langle y \rangle^{-2} - \lambda^2}\, dy$$

with $x_0 < 0 < x_1$ being the turning points

Proof Ideas 3

- Therefore

$$S(\lambda) = 2\nu|\log\lambda|(1+o(1)) \qquad \lambda \to 0$$

which gives the claim on $W_\ell(\lambda)$ above.

- Rigorous proof constructs $f_+(\cdot,\lambda)$ perturbatively. For $|\xi|\lambda \ll 1$ perturb in λ around zero energy solutions. For $|\xi|\lambda > \lambda^\epsilon$ perturb around Jost solutions of the Bessel equation

$$\mathcal{H}_{0,\nu} := -\partial_\xi^2 + (\nu^2 - \frac{1}{4})\xi^{-2}$$

- Then conclude by *matching* these representations in the overlap.
- This method applies to *all* $\ell \geq 0$ (special care required for $\ell = 0$ in the surface case: zero energy resonance!), but gives super-exponential growth in ℓ. *Unsuitable* for summing in ℓ.
- For summation, convert to a **semiclassical** representation, $\hbar := \ell^{-1}$, control constants in terms of powers of $\hbar^{-1}$.

WKB

- Given $\hbar^2 f''(x) = Q(x)f(x)$ with $Q \neq 0$ on interval I.
- Seek f in the form

$$Q(x)^{-\frac{1}{4}} e^{\frac{1}{\hbar}\int_{x_0}^{x} \sqrt{Q}(y)\,dy}(1 + \hbar a(x,\hbar))$$

- $a(x,\hbar)$ satisfies a Volterra integral equation, can be controlled away from points where $Q = 0$.
- Special case of **Liouville-Green transform**: define $g(w) := (w'(x))^{\frac{1}{2}} f(x)$ where $w = w(x)$ diffeomorphism. Then $f'' = Qf$ equivalent to $g''(w) = \tilde{Q}(w)g(w)$ where

$$\begin{aligned}\tilde{Q}(w) &:= \frac{Q(x)}{(w'(x))^2} - (w'(x))^{-\frac{3}{2}} \partial_x^2 (w'(x))^{-\frac{1}{2}} \\ &= \frac{Q(x)}{(w'(x))^2} - \frac{3}{4}\frac{(w''(x))^2}{(w'(x))^4} + \frac{1}{2}\frac{w'''(x)}{(w'(x))^2}\end{aligned}$$

- Choose $\frac{Q(x)}{(w'(x))^2}$ to be simple, such as constant or linear

Proof Ideas 4

- Challenge for WKB: need precise error bounds in the whole range $0 < \hbar < \hbar_0$, $0 < \lambda < \epsilon$. We restrict to positive (asymptotically) inverse square potentials.
- It turns out that one needs to modify the positive inverse square potential before applying WKB by adding $\frac{1}{4}\hbar^2\langle x\rangle^{-2}$ to it.
- Losses in $\hbar^{-1}$ come from the *top of the potential*, nowhere else. Top is a nondegenerate maximum by assumption.
- Could use suitable WKB near the top as well. Instead we rely on **Mourre** estimate, followed by Hunziker-Sigal-Soffer type propagation estimates (elegant time dependent approach to Mourre theory).
- Mourre despite trapping: use semiclassical harmonic oscillator (or HUP) as comparison, Briet-Combes-Duclos, Shu Nakamura (mid 80's).

Proof Ideas 5

Theorem (Hunziker-Sigal-Soffer)

A, H s-a on H-space; Mourre estimate, $\theta > 0$, $I \subset \mathbb{R}$ compact:

$$E_I i[H, A] E_I \geq \theta E_I \tag{1}$$

$[A, f(H)]$, $[A, [A, f(H)]]$ etc bounded, $f \in C_0^\infty(\mathbb{R})$. Then $\forall\ m \geq 1$,

$$\|\chi^-(A - a - \theta' t) e^{iHt} g(H) \chi^+(A - a)\| \leq C(m, \theta, \theta')\, t^{-m}$$

$\forall\ g \in C_0^\infty(I)$, any $0 < \theta' < \theta$, uniformly in $a \in \mathbb{R}$. $\forall\ \alpha > 0$

$$\|\langle A\rangle^{-\alpha} e^{iHt} g(H) \langle A\rangle^{-\alpha}\| \leq C(\alpha)\, \langle t\rangle^{-\alpha}$$

$V(x) = 1 - \frac{1}{2}\langle Qx, x\rangle + O(|x|^3)$, $Q > 0$, $h(x,\xi) = \frac{1}{2}\xi^2 + V(x)$, $a(x,\xi) = x\xi$,

$$\{h, a\} = \xi^2 - x \cdot \nabla V = \xi^2 + \langle Qx, x\rangle + O(|x|^3) \geq \theta(\xi^2 + x^2)$$

Use harmonic oscillator, or HUP

Proof Ideas 6

Theorem (Costin-S.-Staubach-Tanveer)

$0 < V \in C^\infty(\mathbb{R})$, $V(x) = \mu_\pm^2 x^{-2} + O(x^{-3})$. Let

$$V_0(x;\hbar) := V(x) + \frac{\hbar^2}{4}\langle x\rangle^{-2} \tag{2}$$

turning points, $E > 0$ small, $x_2(E;\hbar) < 0 < x_1(E;\hbar)$. Define

$$S(E;\hbar) := \int_{x_2(E;\hbar)}^{x_1(E;\hbar)} \sqrt{V_0(y;\hbar) - E}\, dy$$

$$T_+(E;\hbar) := x_1(E;\hbar)\sqrt{E} - \int_{x_1(E;\hbar)}^{\infty} \left(\sqrt{E - V_0(y;\hbar)} - \sqrt{E}\right) dy$$

$$T_-(E;\hbar) := -x_2(E;\hbar)\sqrt{E} - \int_{-\infty}^{x_2(E;\hbar)} \left(\sqrt{E - V_0(y;\hbar)} - \sqrt{E}\right) dy$$

$T(E;\hbar) := T_+(E;\hbar) + T_-(E;\hbar)$.

Proof Ideas 6, continued

Theorem (continued)

$0 < \forall\, \hbar < \hbar_0$, $\hbar_0 = \hbar_0(V) > 0$ small, $0 < E < E_0$

$$\Sigma_{11}(E;\hbar) = e^{-\frac{1}{\hbar}(S(E;\hbar)+iT(E;\hbar))}(1 + \hbar\,\sigma_{11}(E;\hbar))$$
$$\Sigma_{12}(E;\hbar) = -ie^{-\frac{2i}{\hbar}T_+(E;\hbar)}(1 + \hbar\,\sigma_{12}(E;\hbar))$$

correction terms satisfy

$$|\partial_E^k\,\sigma_{11}(E;\hbar)| + |\partial_E^k\,\sigma_{12}(E;\hbar)| \le C_k\,E^{-k} \quad \forall\, k \ge 0,$$

C_k only depends on k and V.

- coordinates $(t, r, (\theta, \phi)) \in \mathbb{R} \times (2M, \infty) \times S^2$, metric

$$g = -F(r)dt^2 + F(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

with $F(r) = 1 - \frac{2M}{r}$, mass $M > 0$

- *Regge–Wheeler tortoise coordinate* r_* defined by $F = \frac{dr}{dr_*}$.
Metric

$$g = -F(r)dt^2 + F(r)dr_*^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Explicitly, $r_* = r + 2M \log\left(\frac{r}{2M} - 1\right)$.

- With $\psi(t, r_*, \theta, \phi) = r(r_*)\tilde{\psi}(t, r_*, \theta, \phi)$ wave equation $\Box_g \tilde{\psi} = 0$ becomes

$$-\partial_t^2\psi + \partial_x^2\psi - \frac{F}{r}\frac{dF}{dr}\psi + \frac{F}{r^2}\Delta_{S^2}\psi = 0$$

where $x = r_*$.

Analogies with surface of revolution case:

- Separate variables, i.e., project onto spherical harmonic $Y_{m,\ell}$
- Reduces to one-dimensional wave equation with a potential

$$V_{\ell,\sigma}(x) = \left(1 - \frac{2M}{r(x)}\right)\left(\frac{\ell(\ell+1)}{r^2(x)} + \frac{2M\sigma}{r^3(x)}\right)$$

  where $\sigma = -3, 0, 1$.
- Basic question (from physics): local decay of solutions to this wave equation. "Price's law": $t^{-2\ell-3}$.
- $V_{\ell,\sigma}$ has unique nondegenerate maximum: **photon sphere**, trapped geodesics. Decays exponentially to the left, inverse square to the right. Harder to deal with than in the surface of revolution case; x^{-3} decay $\rightsquigarrow$ t^{-3} for $\ell = 0$.
- Exclude gauge modes $(\sigma, \ell) \in \{(-3,1), (-3,0), (0,0)\}$. These are precisely the values which lead to zero energy resonance!

Result for Schwarzschild

Theorem (Donninger-S.-Soffer, 09)

$\Box_g \psi = 0$, data $\psi[0] = (\psi_0, \psi_1)$, satisfies the following local decay:

$$\|\langle x\rangle^{-\frac{9}{2}-}\psi(t)\|_{L^2} \lesssim \langle t\rangle^{-3}\|\langle x\rangle^{\frac{9}{2}+}(\not\nabla^5\partial_x\psi_0, \not\nabla^5\psi_0, \not\nabla^4\psi_1)\|_{L^2}$$
$$\|\langle x\rangle^{-4}\psi(t)\|_{L^\infty} \lesssim \langle t\rangle^{-3}\|\langle x\rangle^{4}(\not\nabla^{10}\partial_x\psi_0, \not\nabla^{10}\psi_0, \not\nabla^9\psi_1)\|_{L^1}$$

$\not\nabla$ = angular derivative, $L^2 := L^2_x(\mathbb{R}; L^2(S^2))$, $L^1 := L^1_x(\mathbb{R}; L^1(S^2))$, and $L^\infty := L^\infty_x(\mathbb{R}; L^\infty(S^2))$.

- Similar and simultaneous result by Tataru.
- Previous work by Blue-Soffer, Finster-Smoller-Yau, Dafermos-Rodnianski, Blue-Sterbenz.
- Many more questions remain (fundamental solution, optimal estimates, pointwise bounds for Ikawa's model, etc.)!
- Vielen Dank für Ihre Aufmerksamkeit!