On continuous time bubbling for the harmonic map heat flow in two dimensions

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Harmonic map heat flow

Gradient flow of the Dirichlet energy

\[
E(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx,
\]

\[u : \mathbb{R}^2 \to S^2\]

solves the heat equation (Eells, Sampson '64):

\[
u_t = \Delta u + |\nabla u|^2 u = T(u)
\]

\[u(0, \cdot) = u_0(\cdot)\]

Tension: \(T(u) = \Pi_{T_u} \Delta u\) projection onto the tangent plane \(T_u\)

Energy monotone:

\[
E(u(0)) - E(u(t)) = \int_0^t \|\partial_s(s, \cdot)\|^2_2 \, ds
\]

Existence, regularity, energy concentration and singularities in finite time: (Struwe '85). Harmonic maps are stationary solutions to HMHF.
Let $\mathcal{M}, \mathcal{N}$ be general Riemannian manifolds, $\dim \mathcal{M} = 2$.

**Theorem (Struwe ’85)**

Initial data $u_0 \in \dot{H}^1(\mathcal{M}; \mathcal{N})$, there exists unique global HMHF energy evolution on $[0, \infty) \times S^2$ which is smooth up to finitely many points $(x_\ell, T_\ell)$ characterized by the condition

$$
\limsup_{t \to T_\ell} E_R(u(t, \cdot), x_\ell) > \varepsilon_0 > 0
$$

for all $0 < R \leq R_0$.

Local compactness in $\dot{H}^2(\mathcal{M}; \mathcal{N})$ if energy does not concentrate, and $\int_P |\nabla u|^4 \, dt \, dx < \infty$ where $P$ is a parabolic cylinder.

Energy concentration the only obstruction to local $\dot{H}^2$ compactness of a Palais-Smale sequence relative to energy and its $L^2$-gradient. Harmonic sphere bubbles off at any singular time.

Chang, Ding, Ye ’92: Finite time blowup.
Qing’s bubbling theorem

Jie Qing ’95 characterized singularity formation in Struwe’s HMHF \( \mathbb{R}^2 \to \mathbb{S}^2 \) via a bubble decomposition along a carefully chosen sequence of times approaching one of the singular times \( T_\ell \).

**Theorem (Qing ’95)**

Let \( (x_0, T_0) \) be a singularity of \( u : [0, \infty) \times \mathbb{R}^2 \to \mathbb{S}^2 \), HMHF solution. There exist \( t_n \to T_0^- \), harmonic spheres \( \omega_k : \mathbb{R}^2 \to \mathbb{S}^2 \)

\[
\lim_{t \to T_0^-} E_R(u(t, \cdot), x_0) = E_R(u(T_0, \cdot), x_0) + \sum_{k=1}^{p} E(\omega_k)
\]

\[
u(t_n, \cdot) = u(T_0, \cdot) + \sum_{k=1}^{p} \left( \omega_k \left( \frac{\cdot - a_k^n}{\lambda_k^n} \right) - \omega_k(\infty) \right) + o_{W^{1,2}(B_R)}(1)
\]

\( R > 0 \) small, \( \lambda_k^n \to 0 \), \( a_k^n \to x_0 \). Bubbles asymptotically orthogonal.

Proved via bubbling for a Palais-Smale sequence.
Asymptotic orthogonality of the bubbles

For all \( k \neq \ell \), \( n \to \infty \)

\[
\frac{\lambda_n^k}{\lambda_n^\ell} + \frac{\lambda_n^\ell}{\lambda_n^k} + \frac{|a_n^k - a_n^\ell|^2}{\lambda_n^k \lambda_n^\ell} \to \infty
\]  

(1)
Theorem

\[ u : \mathbb{R}^2 \to S^2 \] weak non-constant solution of \( \Delta u + u|\nabla u|^2 = 0 \) of finite energy. Then \( u : S^2 \to S^2 \) smooth harmonic map (Hélein, Sacks-Uhlenbeck), nonzero degree. Conformal modulo orientation (Eells-Wood). Cauchy-Riemann system

\[ \partial_1 u \mp u \times \partial_2 u = 0 \iff \partial_2 u \pm u \times \partial_1 u = 0 \]

holds, \( u \) unique minimizer of energy in its homotopy class, \( E(u) = 4\pi |\deg(u)| \). There exist \( P, Q \in \mathbb{C}[z] \) without common linear factor satisfying

\[ \max(\deg(P), \deg(Q)) = |\deg(u)| \geq 1 \]

and such that \( u = \frac{P}{Q} \) for \( \deg(u) > 0 \), or \( \bar{u} = \frac{P}{Q} \) for \( \deg(u) < 0 \).
Key steps in the proof

- Hélein's regularity theorem (false in $\mathbb{R}^d, d \geq 3$). Div, curl structure, Hardy space compensated compactness (Coifman, Lions, Meyer, Semmes '92): continuity of weak solution. Then by elliptic regularity $\nabla u \in L^p$, $u \in C^\infty(\mathbb{R}^2)$

- Hopf quadratic differential

$$\varphi \, dz^2 = \langle \partial_z u, \partial_z u \rangle \, dz^2 = (|\partial_x u|^2 - |\partial_y u|^2 - 2i \langle u_x, u_y \rangle) \, dz^2$$

Harmonic map: $\partial_{\bar{z}} \varphi = 0$ holomorphic on $S^2$, constant. Vanishes at $z = \infty$ so conformality follows:

$$|\partial_x u|^2 - |\partial_y u|^2 - 2i \langle u_x, u_y \rangle = 0$$

- Bogomolnyi identity:

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\partial_1 u - u \times \partial_2 u|^2 + \int_{\mathbb{R}^2} \partial_1 u \cdot u \times \partial_2 u$$
Elliptic compactness lemma: bubbling in energy and $L^\infty$

$u_n : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ with $\limsup_{n \to \infty} E(u_n) < \infty$, and

$$\lim_{n \to \infty} \rho_n \|T(u_n)\|_{L^2} = 0$$

for some $\rho_n \in (0, \infty)$. For arbitrary $y_n \in \mathbb{R}^2$, $\exists R_n \to \infty$ with

$$u_n - \omega_0(\frac{\cdot - y_n}{\rho_n}) - \sum_{j=1}^{M} (\omega_j(\frac{\cdot - b_{j,n}}{\mu_{j,n}}) - \omega_j(\infty)) \to 0$$

in energy and uniformly on $D(y_n, R_n \rho_n) \supset D(b_{j,n}, \mu_{j,n})$

- harmonic maps $\omega_j$, nonconstant if $j \geq 1$
- orthogonality of scales as in (1)
- separation of $D(b_{j,n}, \mu_{j,n})$ from $\partial D(y_n, R_n \rho_n)$
- quantization of energy: $E(u_n; D(y_n, R_n \rho_n)) = 4\pi K + o(1)$

Qing '95, Ding-Tian '95, Wang '96, Qing-Tian '97, Lin-Wang '98
Disks in the bubble tree

$D(b_{j,n}, \mu_{j,n})$

$D(y_n, R_n \rho_n)$
Local Palais-Smale sequences for the heat flow

Smooth HMHF \( u : [0, T) \times S^2 \to S^2 \), singularity at \( t = T \). Energy dissipation

\[
\int_0^T \| T(u)(t) \|_2^2 \, dt < \infty \tag{2}
\]

- If \( T = \infty \), then \( \exists t_n \to \infty \) with \( \sqrt{t_n} \| T(u(t_n)) \|_2 \to 0 \)
- If \( T < \infty \), then \( \exists t_n \to T^- \) with \( \sqrt{T - t_n} \| T(u(t_n)) \|_2 \to 0 \)

Elliptic compactness applies at these parabolic scales. Rescale

- If \( T = \infty \), then \( u_n(y) := u(t_n, y_n + \sqrt{t_n} y) \) is Palais-Smale
- If \( T < \infty \), then \( u_n(y) := u(t_n, y_n + \sqrt{T - t_n} y) \) is Palais-Smale

Bubbling for HMHF locally at parabolic scales along a time sequence \( t_n \) determined by \( L^2 \) integrability (2).
Open problems

- If $T < \infty$, is the body map $u(T, \cdot)$ continuous?
- If $T = \infty$, are the points of energy concentration unique?
- **Uniqueness** of harmonic bubbles? Counterexamples by Topping if target manifold not $\mathbb{S}^2$ (nonanalytic)
- Continuous in time bubbling (soliton resolution)?

Progress by Topping, ‘97, ‘04 for maps $\mathbb{S}^2 \to \mathbb{S}^2$.

**Theorem (Topping, ‘97, ‘04)**

*If $T = \infty$ and if all the *concentrating* bubbles in the sequential decomposition have the *same orientation*, then the points of energy concentration $\{x_\ell\} \subset \mathbb{S}^2$ are unique. Moreover, the body map is unique, i.e., there exists a harmonic map $\omega_\infty: \mathbb{S}^2 \to \mathbb{S}^2$ such that $u(t) \rightharpoonup \omega_\infty$ as $t \to \infty$, weakly in $\dot{H}^1$ and strongly in $C^k_{loc}(\mathbb{S}^2 \setminus \{x_\ell\})$.*
Consider \( k \)-equivariant maps \( u : \mathbb{R}^2 \to \mathbb{S}^2 \), i.e.,

\[
u(t, re^{i\theta}) = (\sin \psi(t, r) \cos k\theta, \sin \psi(t, r) \sin k\theta, \cos \psi(t, r))\]

Harmonic maps given by \( \psi(t, r) = m\pi \pm Q(r/\lambda) \) for \( m \in \mathbb{Z}, \lambda > 0 \), and \( Q(r) = 2 \arctan(r^k) \).

**Theorem (Jendrej-Lawrie '22)**

Let \( \psi(t, r) \) solve the HMHF. Suppose \( T = \infty \). Then, there exist \( m \in \mathbb{Z}, N \in \mathbb{N} \) and \( C^1 \) functions \( 0 < \lambda_1(t) < \cdots < \lambda_N(t) \) such that,

\[
\lim_{t \to T} \| \psi(t, \cdot) - m\pi - \sum_{j=1}^{N} \pm (Q(\cdot/\lambda_j(t)) - \pi) \|_\mathcal{E} = 0
\]

and \( \lim_{t \to T} \sum \lambda_j(t)/\lambda_{j+1}(t) = 0 \). Similar when \( T < \infty \).

Note: \( \lambda_{N+1}(t) := \sqrt{t} \), and subsequent equivariant bubbles always have opposite orientations as maps \( \mathbb{R}^2 \to \mathbb{S}^2 \).
Van der Hout ('03): same result in the case $T < \infty$ by showing there are no non-trivial equivariant bubble towers in finite time. In the case $T = \infty$, non-trivial bubble towers can occur; see for example Del Pino, Musso, Wei ('21) for a construction for the closely related energy critical heat equation.

Finite time blow up solutions with one bubble (including a stable regime) were discovered by Raphaël-Schweyer ('13, '14) for $k = 1$. See also Guan, Gustafson, Tsai ('09) and Gustafson, Nakanishi, Tsai ('10) who proved asymptotic stability of $Q$ for $k \geq 3$, and Davila, Del Pino, Wei ('20) for blow up outside of equivariant symmetry.

Remainder of the talk: discuss a continuous in time bubble decomposition in the general case, i.e., for maps $\mathbb{R}^2 \rightarrow S^2$ without symmetry assumptions (as in Jendrej-Lawrie '22), and without assumptions on the orientations of the bubbles (as in Topping '97, '04).
Centers and scales of harmonic maps: \( \omega : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3 \) positive energy, \( \gamma_0 \in (0, 2\pi) \), scale of \( \omega \)

\[
\lambda(\omega; \gamma_0) := \inf\{\lambda \in (0, \infty) \mid \exists \, a \in \mathbb{R}^2 \text{ s.t. } E(\omega; D(a, \lambda)) \geq E(\omega) - \gamma_0\}.
\]

Center of \( \omega \): fix \( a = a(\omega; \gamma_0) \in \mathbb{R}^2 \) with

\[
E(\omega; D(a(\omega; \gamma_0), \lambda(\omega; \gamma_0))) \geq E(\omega) - \gamma_0.
\]

\( M \)-bubble configuration \( \Omega = (\omega_0, \omega_1, \ldots, \omega_M) \)

\[
\mathcal{Q}(\Omega; x) = \omega_0 + \sum_{j=1}^{M} (\omega_j(x) - \omega_j(\infty))
\]

where \( \omega_0 = \text{const} \in S^2 \), \( \omega_j : \mathbb{R}^2 \to S^2 \), \( j \geq 1 \) non-constant harmonic maps, \( \omega_j(\infty) := \lim_{|x| \to \infty} \omega_j(x) \). Constant maps: \( M = 0 \).
Distance to a multi-bubble configuration

Smooth map $u : \mathbb{R}^2 \to S^2$, multi-bubble $Q(\Omega)$, disk $D(y; \rho) \subset \mathbb{R}^2$, auxiliary scales $\vec{\nu} = (\nu, \nu_1, \ldots, \nu_M)$, $\vec{\xi} = (\xi, \xi_1, \ldots, \xi_M)$.

Distance $d(u, Q(\Omega); D(y, \rho); \vec{\nu}, \vec{\xi}) \ll 1$ means

- closeness in energy to multi-bubble on the large disk:
  $$E(u - Q(\Omega); D(y, \rho)) \ll 1$$

- near constancy on the exterior neck region:
  $$E(u; D(y, \nu) \setminus D(y, \xi)) + \|u - \omega_0\|_{L^\infty(D(y, \nu) \setminus D(y, \xi))} \ll 1$$

- large exterior neck: $\xi \ll \rho \ll \nu$

- orthogonality of bubbles scales/centers: $\lambda(\omega_j) \ll \lambda(\omega_k)$ or $\lambda(\omega_j) \gg \lambda(\omega_k)$ or $|a(\omega_j) - a(\omega_k)| \gg \lambda(\omega_j)$

- separation from exterior neck:
  $$\xi_j \ll \lambda(\omega_j) \ll \text{dist}(a(\omega_j), \partial D(y, \xi))$$
$L^\infty$ control on the bubbles, removal of sub-bubbles

- uniform closeness of $u, \omega_j$ after removal of interior bubbles:
  $$\|u - \omega_j\|_{L^\infty(D^*_j)} \ll 1$$

- Swiss cheese (holes are of the same size):
  $$D^*_j := D(a(\omega_j), \nu_j) \setminus \bigcup_k D(a(\omega_k), \xi_j).$$

- separation from boundaries:
  $$\xi_j \ll \text{dist}(a(\omega_k), \partial D(a(\omega_j), \nu_j)), \quad \lambda(\omega_j) \ll \nu_j$$

Local multi-bubble proximity function:

$$\delta(u; D(y, \rho)) := \inf_{\Omega, \vec{v}, \vec{\xi}} d(u, Q(\Omega); D(y, \rho); \vec{v}, \vec{\xi})$$

Infimum taken over all multi-bubble configurations, and scales $\vec{v}, \vec{\xi}$
Exterior neck region

\[ D(y, \nu) \quad D(y, \rho) \quad D(y, \xi) \]
Swiss cheese structure
Theorem (Jendrej, Lawrie, S. ’23)

\( u(t) : [0, T_+) \times \mathbb{R}^2 \rightarrow S^2 \) smooth HMHF solution, maximal

\( T_+ = T_+(u_0) \in (0, \infty] \). If \( T_+ < \infty \), then \( \forall y \in \mathbb{R}^2 \),

\[
\lim_{t \to T_+} \delta(u(t); D(y, \sqrt{T_+ - t})) = 0.
\]

Arbitrary \( t_n \to T_+ \) and \( D(y_n, R_n\rho_n) \subset D(y, \sqrt{T_+ - t}) \), \( R_n \to \infty \), assume energy evacuates from necks of disks. Then,

\[
\lim_{n \to \infty} \delta(u(t_n); D(y_n, \rho_n)) = 0.
\]

Analogous statement on \( D(y, \sqrt{t}) \) if \( T_+ = \infty \).

Solution remains close to multi-bubble configurations at parabolic scales, and on all smaller disks whose boundaries do not intersect bubbles, for all times up to \( T_+ \).
Comments on the theorem

- Analogous result when $T_+ = \infty$
- Does not give the uniqueness of bubbles.
- How to think about the theorem: non-existence of bubble collisions that destroy multi-bubble structure.
- As a corollary, we obtain a sequential bubble decomposition as in Qing along every time sequence $t_n \to T_+$ after passing to a subsequence (not just along Palais-Smale sequences)
Comments on the proof

- Proof by contradiction: $u(t)$ cannot come close to, and then move away from multi-bubble configurations (MBCs) infinitely many times. Reminiscent of invariant manifold theory in dynamical systems, theory of $\omega$-limit sets.

- However: linearized operator here has no spectral gap, no stable/unstable manifolds

- By *sequential* soliton resolution (bubbling along sequence of times) we know that we approach MBCs infinitely many times.

- If theorem fails, $\delta(u(t_n); D(y_n, \rho_n)) > \eta > 0$ for $t_n \to T_{+-}$. By energy dissipation and compactness lemma exist $\sigma_n$ with $\delta(u(\sigma_n); D(y_n, \rho_n)) \to 0$ where $0 < t_n - \sigma_n \ll \rho_n^2$

- Notions of collision intervals and minimal collision energy needed to lead this to a contradiction. This was essential for soliton resolution for wave maps by Jendrej, Lawrie '21.
Local energy propagation (Struwe '85): $0 < t_1 < t_2 < T_+$,

$$\int_{\mathbb{R}^2} |\nabla u(t_2, x)|^2 \phi(x)^2 \, dx \leq \int_{\mathbb{R}^2} |\nabla u(t_1, x)|^2 \phi(x)^2 \, dx + CE(u_0) \frac{t_2 - t_1}{R^2}$$

$$\int_{\mathbb{R}^2} |\nabla u(t_2, x)|^2 \phi(x)^2 \, dx \geq \int_{\mathbb{R}^2} |\nabla u(t_1, x)|^2 \phi(x)^2 \, dx$$

$$- C \left( E(u_0) \frac{(t_2 - t_1)}{R^2} + |E(u(t_1)) - E(u(t_2))| \right)$$

$\phi$ cut-off adapted to $D(x_0, R)$.

- Integrate HMHF by parts against $u_t \phi^2$. Nonlinear term drops out, normal vector field.
- Controls energy flow on parabolic regions.
- Energy evacuates from boundaries of parabolic regions. No self-similar energy concentration both in finite (Topping) and infinite times.
Lemma: Solution of $\partial_t v - \Delta v = F$, $v(0) = v_0$ satisfies

$$
\| v \|_{L^2(I; L^\infty(\mathbb{R}^2))} \leq C_0 \left( \| v_0 \|_{L^2(\mathbb{R}^2)} + \| F \|_{L^1(I; L^2(\mathbb{R}^2))} \right)
$$

With $(Tf)(t) := e^{t\Delta} f$ one has $T^* F = \int_0^\infty e^{s\Delta} F(s) \, ds$. From

$$(TT^* F)(t) = \int_0^\infty e^{(t+s)\Delta} F(s) \, ds$$

conclude

$$
\| (TT^* F)(t) \|_\infty \lesssim \int_0^\infty (t + s)^{-1} \| F(s) \|_1 \, ds
$$

$$
\| TT^* F \|_{L^2((0,\infty), L^\infty(\mathbb{R}^2))} \lesssim \| F \|_{L^2((0,\infty), L^1(\mathbb{R}^2))}
$$

$$
\langle TT^* F, F \rangle = \| T^* F \|_2^2 \lesssim \| F \|_{L^2((0,\infty), L^1(\mathbb{R}^2))}^2
$$
**Lemma:** On a Swiss cheese region with $L \geq 0$ congruent, well-separated holes, assume

$$\|u_{n,0} - \omega\|_{L^\infty(D(0,4R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell, 4^{-1}\varepsilon_n))}$$

$$+ E\left(u_{n,0} - \omega; D(0,4R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell, 4^{-1}\varepsilon_n)\right) \to 0.$$ 

Then, if $\tau_n \ll \varepsilon_n^2$ (or $\tau_n \ll R_n^2$ if $L = 0$),

$$\|u_n(\tau_n) - \omega\|_{L^\infty(D(0,R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell,\varepsilon_n))} \to 0.$$ 

- Contraction of heat flow on $L^\infty$
- Tao’s parabolic Strichartz estimate
- Struwe’s small energy local $\int (|\nabla u|^4 + |\Delta u|^2) \, dt \, dx$ bound
Definition: $K \geq 1$ minimal with the following properties.

$\exists \ y_n \in \mathbb{R}^2, \rho_n, \varepsilon_n \in (0, \infty), \sigma_n, \tau_n \in (0, T_+) \text{ and } \eta > 0$, with
$\varepsilon_n \to 0, \ 0 < \sigma_n < \tau_n < T_+, \sigma_n, \tau_n \to T_+$, so that

1. $\delta(u(\sigma_n); D(y_n, \rho_n)) \leq \varepsilon_n$;
2. $\delta(u(\tau_n); D(y_n, \rho_n)) \geq \eta$;
3. the interval $I_n := [\sigma_n, \tau_n]$ satisfies $|I_n| \leq \varepsilon_n \rho_n^2$;
4. $E(u(\sigma_n); D(y_n, \rho_n)) \to 4K\pi \text{ as } n \to \infty$;

We call $\sigma_n$ bubbling times, and $\tau_n$ ejection times.

Lemma: If theorem fails, then $K \geq 1$ well-defined with collision intervals $[\sigma_n, t_n]$.

Based on energy dissipation and localized sequential bubbling. For $K > 0$ need propagation estimates, both in energy and $L^\infty$. 

Jendrej, Lawrie, S.  Continuous time bubbling for HMHF
Key Lemma: Let $K \geq 1$ minimal collision energy, $I_n := [\sigma_n, \tau_n]$ associated collision intervals. $\exists \varepsilon > 0$ such that if $s_n \in I_n$ satisfies

$$\delta(u(s_n); D(y_n, \rho_n)) \leq \varepsilon$$

Then,

$$\tau_n - s_n \geq \varepsilon \max_{j \in \{1, \ldots, M\}} \lambda(\omega_j)^2 =: \varepsilon \lambda_{\text{max}, n}^2.$$  \hspace{1cm} (3)

where scales $\lambda(\omega_j)$ correspond to any MBC $Q(\omega)$ for which

$$\varepsilon \leq d(u(s_n), Q(\omega); D(y_n, \rho_n); \vec{v}, \vec{\xi}) \leq 2\varepsilon.$$  \hspace{1cm} (4)

Proof Sketch: If lemma fails, $\exists \tilde{\sigma}_n \in I_n$ with $\tau_n - \tilde{\sigma}_n \ll \lambda_{\text{max}, n}^2$ and for which $\delta(u(\tilde{\sigma}_n); D(y_n, \rho_n)) \to 0$ and $\delta(u(\tau_n); D(y_n, \rho_n)) \geq \eta$. 
Key lemma: proof sketch

\[ E \leq 4\pi (K - 1) + o(1) \]

multi-bubble configuration

\[ E \leq 4\pi (K - 1) + o(1) \]

NOT multi-bubble configuration

\[ t = \tilde{\sigma}_n \]

\[ t = \tau_n \]

- By propagation estimates, multi-bubble structure is preserved at scale \( \lambda_{\text{max},n} \) on the interval \([\tilde{\sigma}_n, \tau_n]\).
- Hence, it is lost at a smaller scale (pink disks, radius \( \sqrt{\tau_n - s_n} \ll \tilde{\rho}_n \ll \lambda_{\text{max},n} \)), contradicting minimality of \( K \).
Main theorem: proof sketch

- Use key lemma: fix $\varepsilon > 0$ and $J_n := [s_n, \tau_n] \subset I_n$ so that

$$\tau_n - s_n \geq \varepsilon \lambda_{\text{max}, n}^2, \quad \delta(u(t); D(y_n, \rho_n)) \geq \varepsilon, \quad \forall t \in J_n$$

("no return property" on $J_n$).

- Then,

$$\lambda_{\text{max}, n} \| T(u(t)) \|_2 \geq c_0 > 0 \text{ for all } t \in J_n$$

Otherwise, bubbling at scale $\lambda_{\text{max}, n}$ at some $t_n \in J_n$ by elliptic compactness lemma, contradicting no-return property of $J_n$.

- Contradiction with the energy identity:

$$\infty = \sum_n \int_{s_n}^{\tau_n} c_0 \lambda_{\text{max}, n}^{-2} \, dt \leq \sum_n \int_{s_n}^{\tau_n} \| T(u(t)) \|_{L^2}^2 \, dt \leq \int_0^{T_+} \| T(u(t)) \|_{L^2}^2 \, dt < \infty$$
Happy Birthday, Frank! Many happy returns, much continued joy with mathematics.

Many thanks to


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