Introduction to Lectures 1-4.

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Consider the periodic second order ODE

$$-y''(x) + V(x)y(x) = Ey(x), \quad E \in \mathbb{R} \tag{1}$$

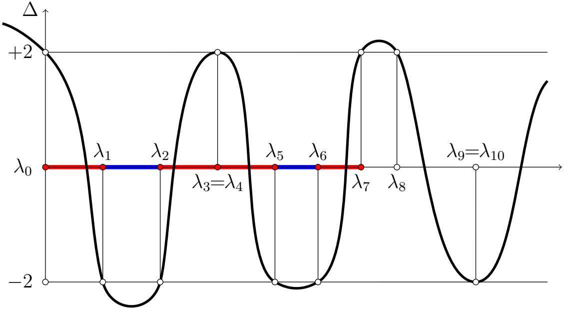
on the line. Assume $V(x+L) = V(x)$ real-valued (may take $L = 1$). Then by Floquet theory any solution of (1) is of the form $y(x) = e^{ik(E)x}a(x,E)$ where $a(x+L,E) = a(x,E)$. This comes from considering the propagator (fundamental matrix) $S(L)$ and its eigenvalues. Since $\det S(L) = 1$, either both eigenvalues lie on the unit circle (and complex conjugates), or they are real-valued and reciprocal.

What does $\mathrm{spec}(H)$ look like, where

$$(Hy)(x) = -y''(x) + V(x)y(x) \quad ?$$

We need to find those E for which $k(E)$ is real-valued.

The periodic problem: Hill's discriminant Δ



HILL'S DISCRIMINANT

- THE RED INTERVALS ARE THE BANDS OF THE SPECTRUM
- THE BLUE INTERVALS ARE THE GAPS
- THE DOUBLE EIGENVALUES, LIKE $\lambda_3 = \lambda_4$ FALL INTO THE SPECTRUM

Figure: The bands in the spectrum

Here $\Delta = \mathrm{trace}(S(L))$. The red intervals are precisely the ones where the eigenvalues are on the unit circle.