Subharmonic techniques in multiscale analysis: Lecture 1

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Introduction

Consider an operator $H(z)$ depending on a complex parameter $z \in \Omega \subset \mathbb{C}$. For example,

$$(H(z)\psi)_n = \psi_{n+1} + \psi_{n-1} - v_n(z)\psi_n, \quad n \in \mathbb{Z}$$

where $\{v_n(z)\}_{n\in\mathbb{Z}}$ collection of analytic functions such as

$$v_n(z) := F(z \cdot \omega^n), \quad |\omega| = 1,$$

F analytic near $S^1 := \{|z| = 1\}$. Restricting to finite volume $[-N, N]$ with Dirichlet conditions, we obtain a matrix $H_N(z)$. Assume $H_N(z)$ self-adjoint on S^1.
It is often important to study the resolvent

$$G_N(E; z) := (H_N(z) - E)^{-1}$$

on S^1.

Introduction

By Cramer's rule, we are lead to consider the vanishing of the analytic function

$$f(z) := \det(H_N(z) - E)$$

Basic question: What is the measure of $z \in S^1$ for which $|f(z)|$ is **close to zero**?

We will need some type of *transversality condition* to ensure *nondegeneracy* of $H_N(z) - E$. This is the same as asking when $u(z) = \log |f(z)|$ is *very large and negative*.

The function $u(z)$ is an example of a **subharmonic function**, and classical potential theory provides tools for that purpose.

A subharmonic function is one for which the Laplacian $\Delta u = \mu$ is a non-negative measure. Two important (and elementary) tools:

- Riesz representation of subharmonic functions
- Cartan's estimate on the size of level sets of subharmonic functions near $-\infty$.

Subharmonic functions 1

Definition

We say $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is subharmonic provided

- upper semi-continuity:

$$u(z) \geq \limsup_{\zeta \to z} u(\zeta), \quad \zeta \neq z$$

- sub mean-value property: for all $z \in \Omega$ there exists $r_0(z) > 0$

$$u(z) \leq \int_0^1 u(z + re(\theta))\, d\theta, \quad \forall r < r_0(z)$$

and $e(\theta) := e^{2\pi i \theta}$.

Examples: $\log|z - z_0|$, $\log|f(z)|$ for any analytic $f(z)$, logarithmic potential $\int \log|z - \zeta|\, \mu(d\zeta)$ for measure $\mu \geq 0$; Fatou gives upper s-c, e.g. for $\mu = \sum 2^{-n} \delta_{2^{-n}}$ the potential at $z = 0$ not continuous, only upper s-c. However: in applications typically have continuity.

Subharmonic functions 2

About upper-semicontinuous functions: sub-level sets $\{u < \lambda\}$ are open, they attain supremum on compact sets.

Some properties of subharmonic functions (s-h) $u(z)$:

- φ increasing, convex. Then $\varphi(u(z))$ is s-h. So $e^u, |f|^\lambda$ are s-h where f analytic, $\lambda > 0$.
- u_α s-h collection, then $\sup_\alpha u_\alpha$ is s-h provided it is upper s-c.
- limit of decreasing or uniformly convergent sequence of s-h functions is s-h
- finite sums of s-h functions are s-h
- **maximum principle**: if u has a local maximum, then constant. Indeed, if $M = u(z_0)$ local max, then we see that $u = M$ locally near z_0 by sub-mv and upper s-c property. So $\{u = M\}$ open. Closed by upper s-c.
- Averages

$$N(r, z, u) := \int_0^1 u(z + re(\theta))\, d\theta, \quad r < \operatorname{dist}(z, \partial\Omega)$$

increasing in r, $\lim_{r\to 0} N(r, z, u) = u(z)$. Convex in $\log r$.

Every s-h function is decreasing limit of smooth s-h functions:
$\varphi \geq 0$ radial smooth compactly supported bump-function

$$f_\varepsilon(z) := \int u_\varepsilon(z-w)\varphi_\varepsilon(w)\,dw,\ \ u_\varepsilon := \max(u, -\varepsilon^{-2})$$

$$\varphi_\varepsilon(w) := \varepsilon^{-2}\varphi(w/\varepsilon), \quad 0 < \varepsilon < 1$$

with $2\pi \int_0^\infty s\varphi(s)\,ds = 1$. Then $u_\varepsilon, f_\varepsilon$ are s-h, f_ε smooth. Further, we have

$$f_\varepsilon(z) = 2\pi \int_0^\infty N(\varepsilon s, z, u_\varepsilon) s\varphi(s)\,ds \searrow u(z)$$

since

$$N(\varepsilon s, z, u_\varepsilon) \leq N(\varepsilon s, z, u_{\varepsilon'}) \leq N(\varepsilon' s, z, u_{\varepsilon'})$$

for $\varepsilon < \varepsilon'$. And $N(\varepsilon s, z, u_\varepsilon) \leq N(\varepsilon' s, z, u_\varepsilon) \searrow N(\varepsilon' s, z, u)$ as $\varepsilon \to 0$ by monotone convergence; then $N(\varepsilon' s, z, u) \searrow u(z)$ as $\varepsilon' \to 0$. Apply monotone convergence again.

Subharmonic functions 4

$u \in C^2(\Omega)$ s-h if and only if $\Delta u \geq 0$ in Ω

Green's formula:

$$\int_D (v\Delta u - u\Delta v)\, dx = \int_{\partial D} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\sigma$$

Apply to u above and $v(w) = \frac{1}{2\pi}\log\frac{R}{|z-w|}$, $D = D(z,R)$. Then $\Delta v = -\delta_w$ and

$$u(z) + \frac{1}{2\pi}\int_{|z-w|<R} \log\frac{R}{|z-w|}\Delta u(w)\, m(dw) = N(z,R,u)$$

If $\Delta u(z) < 0$, then get a contradiction for $R > 0$ small.

S-h functions remain s-h under conformal change of variables: $\Delta(u \circ F) = |F'|^2\,(\Delta u) \circ F$

The distributional Laplacian of any s-h function is a positive Borel measure

First, if u is s-h, then it is in fact a distribution. We need to check that $\int u\varphi$ is finite (thus, not $-\infty$) for any test function φ. By sub-mv property, if $u = -\infty$ on an open set, then it is in fact constant $= -\infty$ in Ω. So we can assume that $u > -\infty$ on a dense set.
But then $\int_D u > -\infty$ for any (small) disk $D \subset \Omega$: consider $N(z, r, u) \geq u(z) > -\infty$ for a suitably chosen point $z \in \Omega$, and integrate in $r_1 \leq r \leq r_2$ so that this annulus contains the disk. So u is a distribution.
Let $u_\varepsilon \searrow u$ with smooth s-h functions u_ε. Then by monotone conv

$$0 \leq \int \varphi \Delta u_\varepsilon = \int u_\varepsilon \Delta\varphi \to \int u \Delta\varphi = \langle \Delta u | \varphi \rangle$$

for all test functions $\varphi \geq 0$. Riesz representation (Rudin RCA) says that $\frac{1}{2\pi}\Delta u = \mu$ is a positive Borel measure, called *Riesz mass of u*. We have $\mu(K) < \infty$ for all compact $K \subset \Omega$.

Disk and annulus

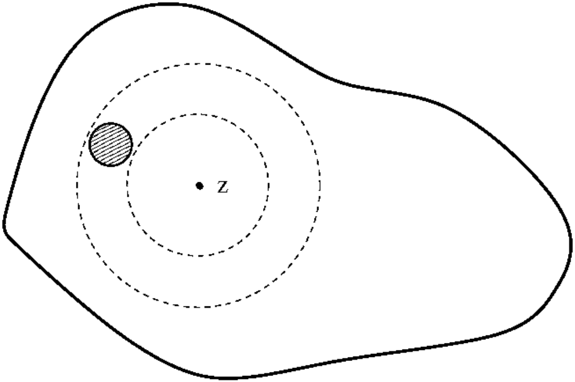

Figure : The geometry in the previous proof

Riesz representation theorem for subharmonic functions

Theorem

Let u be s-h in Ω. Then for any open $G \Subset \Omega$

$$u(z) = \int_G \log|z-\zeta|\,\mu(d\zeta) + h_G(z) \quad \forall\, z \in G$$

and h_G harmonic in G.

Let $v_\varepsilon(\zeta) = \frac{1}{2}\log(|z-\zeta|^2 + \varepsilon^2)$. Then $\Delta v_0 = 2\pi\delta_z$, and

$$\langle \mu, \chi v_\varepsilon \rangle = \frac{1}{2\pi}\langle u, v_\varepsilon \Delta\chi + 2\nabla v_\varepsilon \cdot \nabla\chi + \chi\Delta v_\varepsilon \rangle$$

where χ compactly supported in Ω, $\chi = 1$ on G. Limit $\varepsilon \to 0$

$$u(z) = \int_G \log|z-\zeta|\,\mu(d\zeta) - \frac{1}{2\pi}\langle u, v\Delta\chi + 2\nabla v \cdot \nabla\chi \rangle + \int_{\Omega\setminus G} \chi(\zeta)\log|z-\zeta|\,\mu(d\zeta)$$

as desired.

Examples of Riesz representation

Let $f(z) = P(z)g(z)$ where f analytic in Ω, polynomial $P(z) = \prod_{j=1}^{N}(z - z_j)$ with z_j (multiple) zeros, and $g \neq 0$ analytic in Ω. Then

$$\log|f(z)| = \sum_{j=1}^{N} \log|z - z_j| + \log|g(z)|, \quad \mu = \sum_j \delta_{z_j}$$

Total Riesz mass $\mu(\mathbb{C}) = \deg(P)$. Recall Jensen's formula: if $f(z) \neq 0$ then

$$\int_0^1 \log|f(z + re(\theta))|\, d\theta - \log|f(z)| = \sum_{j:|z_j - z|<r} \log \frac{r}{|z_j - z|}$$

So we have a zero count:

$$\#\{j : |z - z_j| < r/2\} \leq C\Big(\sup_{w\in D(z,r)} \log|f(w)| - \sup_{\zeta\in D(z,r/2)} \log|f(\zeta)|\Big)$$

A more quantitative version of Riesz representation

Introducing estimates in the previous argument yields the following more useful quantitative version, see Lemma 2.2, Goldstein-S., GAFA 2008.

Theorem

Let $u : \Omega \to \mathbb{R}$ be s-h on $\Omega \subset \mathbb{C}$. There exists a positive measure μ on Ω such that for any $\Omega_1 \Subset \Omega$

$$u(z) = \int_{\Omega_1} \log|z - \zeta|\, \mu(d\zeta) + h(z)$$

where h is harmonic on Ω_1 and μ is unique with this property. Moreover, μ and h satisfy the bounds

$$\mu(\Omega_1) \le C(\Omega, \Omega_1) \left(\sup_{\Omega} u - \sup_{\Omega_1} u\right)$$

$$\|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_2)} \le C(\Omega, \Omega_1, \Omega_2) \left(\sup_{\Omega} u - \sup_{\Omega_1} u\right)$$

for any $\Omega_2 \Subset \Omega_1$.

A more quantitative version of Riesz representation

The control of the Riesz mass here follows from the analogue of Jensen's formula:

$$\int_0^1 u(z + re(\theta))\, d\theta - u(z) = \int_0^r \frac{\mu(D(z,t))}{t}\, dt$$

Controlling deviations from above allows to bound the Riesz mass.
In the following lecture we will address the question about the structure of the set where a subharmonic function can be very *large and negative*. This will be covered by *Cartan's estimate*.

An application to large deviation estimates

Let V be analytic, real-valued on $\mathbb{T}^d$, and $Tx := x + \omega$ ergodic shift. Consider the Schrödinger equation on $\mathbb{Z}$

$$(H_x\, \psi)(n) = -\psi(n+1) - \psi(n-1) + V(T^n x)\psi(n) = E\psi(n) \quad (1)$$

Rewrite as a system (linear cocycle):

$$\begin{bmatrix} \psi(n+1) \\ \psi(n) \end{bmatrix} = A(T^n x, E) \begin{bmatrix} \psi(n) \\ \psi(n-1) \end{bmatrix},$$

$$A(x, E) = \begin{bmatrix} V(x) - E & -1 \\ 1 & 0 \end{bmatrix}.$$

Propagator $M_n(x, E) := A(T^n x, E) \ldots A(Tx, E)$. Lyapunov exp.:

$$L_n(E) := \frac{1}{n} \int_{\mathbb{T}^d} \log \|M_n(x, E)\|\, dx$$

Subadditivity: $L_n(E) \to L(E)$ exists. Since $\det A = 1$, one has $L(E) \geq 0$. Pointwise: for a.e. x

$$L(E) = \lim_{n\to\infty} n^{-1} \log \|M_n(x, E)\|$$

LDT

We now establish a quantitative version of this convergence under a Diophantine condition: $\|n\omega\| > n^{-1}(\log n)^{-2}$ for all $n \geq n_0(\omega)$. A.e. $\omega \in \mathbb{T}$ satisfies such a condition.

Theorem

Let $\omega \in \mathbb{T}$ satisfy a Diophantine condition. Then there exists $\sigma > 0$ such that

$$\left|\left\{x \in \mathbb{T} : |n^{-1} \log \|M_n(x, E)\| - L_n(E)| > n^{-\frac{1}{4}}\right\}\right| < e^{-n^{\sigma}} \quad \text{(LDT)}$$

for all $n \geq n_0(E, V, \omega)$.

On some strip around the real line

$$u(z) := n^{-1} \log \|M_n(z, E)\|$$

subharmonic, ≥ 0, and 1-periodic, and of size $\lesssim 1$. By RRT,

$$u(x) = \int \log |e(x) - \zeta| \, \mu(d\zeta) + h(x) \quad \forall\, x \in \mathbb{R} \qquad (2)$$

Plot of log of norm for almost Mathieu

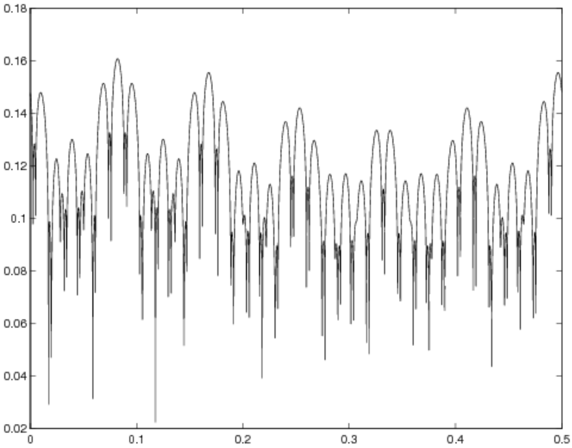

Figure : The graph of $\frac{1}{100}\log\|M_{100}(x)\|$, $\omega=\sqrt{2}$, $\lambda=2.2$

Plot of log of norm

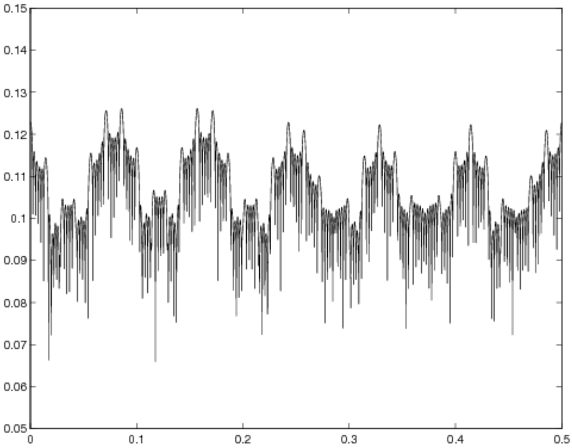

Figure : The graph of $\frac{1}{200} \log \|M_{200}(x)\|$

Plot of log of norm

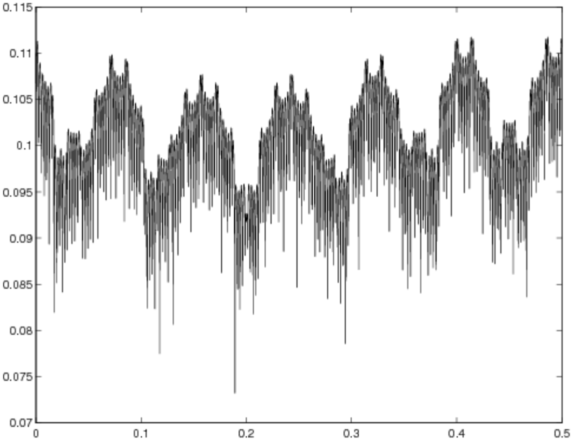

Figure : The graph of $\frac{1}{400} \log \|M_{400}(x)\|$

Proof of LDT

$\mu \geq 0$ and finite on some nbhd of $\{|z| = 1\}$ and h harmonic and 1-periodic on a strip around the real line, and $|h| \lesssim 1$ there. From this alone cannot derive (LDT).

Examples: $P_1(z) = (z-1)^n$, $P_2(z) = z^n - 1$, generate s-h functions

$$u_1(z) = n^{-1} \log |P_1(z)| = \log |z-1|, \quad u_2(z) := n^{-1} \log |z^n - 1|$$

of Riesz mass 1, average $= 0$ over S^1. Deviation estimates:

$$|\{x \in \mathbb{T} : |u_1(e(x))| > n^{-\frac{1}{2}}\}| \sim 1$$
$$|\{x \in \mathbb{T} : |u_2(e(x))| > n^{-\frac{1}{2}}\}| \sim e^{-\sqrt{n}}$$

We want to be closer to the second example, obviously. Note: u_2 is $1/n$ periodic, and u is almost invariant under shift by ω:

$$\sup_{x \in \mathbb{T}} |u(x + k\omega) - u(x)| \leq C\frac{k}{n} \tag{3}$$

from the matrix product structure of M_n.

Indeed:

$$u(x+k\omega)-u(x)=\frac{1}{n}\log\frac{\|M_n(x+k\omega,E)\|}{\|M_n(x,E)\|}$$

$$\begin{aligned}\|M_n(x+k\omega,E)\| &\le \|A(T^{n+k}x,E)\ldots A(T^{n+1}x,E)\|\cdot \\ &\quad \cdot\|M_n(x,E)\|\|A(Tx,E)^{-1}\ldots A(T^{k+1}x,E)^{-1}\| \\ &\le C^k\|M_n(x,E)\|\end{aligned}$$

In order to prove the LDT, we need to combine (2) and (3). Use Fourier series, with $K=K(n)\gg n$

$$u(x)=L_n(E)+\sum_{0<|\nu|\le K}\hat{u}(\nu)e(\nu x)+\sum_{|\nu|>K}\hat{u}(\nu)e(\nu x) \tag{4}$$

From (2)

$$\hat{u}(\nu)=\int\int_0^1\log|e(\theta)-\zeta|e(-\nu\theta)\,d\theta\,\mu(d\zeta)+\int_0^1 h(\theta)e(-\nu\theta)\,d\theta$$

Now $\hat{h}(\nu)=O(\nu^{-N})$, whereas the logarithmic integral satisfies

Proof of LDT

$$\int_0^1 \log|e(\theta) - \zeta|e(-\nu\theta)\, d\theta = O(\nu^{-1})$$

Since $\mu(\mathbb{C}) \lesssim 1$, we have $|\hat{u}(\nu)| \lesssim |\nu|^{-1}$. Thus,

$$\|u_K\|_2 \lesssim K^{-\frac{1}{2}}, \qquad u_K(x) := \sum_{|\nu|>K} \hat{u}(\nu)e(\nu x) \tag{5}$$

We need to use the properties of the shift by ω. Average (4) using almost invariance (3):

$$\begin{aligned} u(x) &= \frac{1}{m}\sum_{j=1}^{m} u(x+j\omega) + O(\frac{m}{n}) \\ &= L_n(E) + \sum_{0<|\nu|\le K} \hat{u}(\nu)\frac{1}{m}\sum_{j=1}^{m} e(\nu(x+j\omega)) \\ &\quad + \frac{1}{m}\sum_{j=1}^{m} u_K(x+j\omega) + O(\frac{m}{n}) \end{aligned} \tag{6}$$

Proof of LDT

Now

$$\sup_x \left| \frac{1}{m} \sum_{j=1}^{m} e(\nu(x + j\omega)) \right| \lesssim \min(1, m^{-1} \|\nu\omega\|^{-1}) \tag{7}$$

Inserting this into (6) yields by Diophantine condition

$$\begin{aligned} &\sup_x \left| \sum_{0<|\nu|\leq K} \hat{u}(\nu) \frac{1}{m} \sum_{j=1}^{m} e(\nu(x + j\omega)) \right| \\ &\lesssim \sum_{0<|\nu|\leq K} |\nu|^{-1} \min(1, m^{-1} \|\nu\omega\|^{-1}) \\ &\lesssim m^{-1} (\log K)^3 \lesssim n^{-\frac{1}{3}+3\sigma} \end{aligned} \tag{8}$$

where we set $m = n^{\frac{1}{3}}, K = \exp(2n^{\sigma})$. To pass to last line, divide into cases $\|\nu\omega\| \leq m^{-1}$, $2^k m^{-1} \leq \|\nu\omega\| \leq 2^{k+1} m^{-1}$.

Again from (6) we conclude that for σ small and n large,

$$\left|\left\{x \in \mathbb{T} : |u(x) - L_n(E)| > n^{-\frac{1}{4}}\right\}\right|$$
$$\leq \left|\left\{x \in \mathbb{T} : \left|\frac{1}{m}\sum_{j=1}^{m} u_K(x + j\omega)\right| > \frac{1}{2}n^{-\frac{1}{4}}\right\}\right| \lesssim n^{\frac{1}{2}} K^{-1} \lesssim e^{-n^{\sigma}}$$

which is the LDT. Under our strong Diophantine condition one can prove a sharper estimate, see Goldstein-S., Annals Math, 2001.

Theorem (Sharper LDT)

For any $\delta > 0$ and any positive integer n,

$$\left|\{x \in \mathbb{T} : |u_n(x) - L_n(E)| > \delta\}\right| \leq \exp\left(-c\delta^2 n + C(\log n)^A\right).$$

The constants c, C only depend on the size of E, the potential, and ω. A is absolute.

Notice the δ, δ^2 dependence. Can we do better? Yes if $L > 0$.

Further remarks on LDT

Under a weaker Diophantine condition $\|n\omega\| > n^{-a}, a > 1$, the same proof gives this weak LDT

$$\left|\{x \in \mathbb{T} : |u_n(x) - L_n(E)| > n^{-\sigma}\}\right| < e^{-n^\sigma}$$

where $\sigma > 0$.

- What is the LDT good for? When do we need the sharp LDT?
- How to prove the LDT on higher-dimensional tori with the shift dynamics? Or for other underlying dynamics, such as the skew shift $T(x, y) = (x + \omega, x + y)$? Note: this involves $n^2\omega$. Is there a proof that does not involve the Fourier transform?

Main applications are (i) Anderson Localization for (1) assuming $L > 0$ (LDT + elimination of energies via exclusion of DOUBLE RESONANCES) Bourgain-Goldstein, Annals, 2000. (ii) Regularity properties of the integrated density of states, distribution and separation of eigenvalues of (1) on finite volume, gaps in the spectrum in infinite volume Goldstein-S., Annals, 2001, GAFA 2008, Annals 2009.

We know from subadditivity that $L_n(E) \to L(E)$. What can we say about the rate? In other words, can we control $L_n(E) - L_{2n}(E)$? It turns out we can, assuming $L(E) > 0$. We need a tool that produces exponential growth in the norm of a long product of matrices. This need not be the case: $AA^{-1}AA^{-1}AA^{-1}AA^{-1}\dots$.

Proposition (Goldstein-S)

Let $A_1, \dots, A_n \in SL(2, \mathbb{R})$ *so that*

$$\min_{1 \le j \le n} \|A_j\| \ge \mu > n \tag{9}$$

$$\max_{1 \le j < n} [\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1}A_j\|] < \frac{1}{2} \log \mu \tag{10}$$

Then

$$\left| \log \|A_n \cdot \dots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\| \right| < C\frac{n}{\mu} \tag{11}$$

Proof of Avalanche Principle

Fix $K \in SL(2,\mathbb{R})$. Polar decomposition: $K = RP$, R rotation, $P > 0$. Eigenvectors of $P = \sqrt{K^*K}$ are $\mathbf{u}_K^+$, $\mathbf{u}_K^-$, respectively. One has $K\mathbf{u}_K^+ = \|K\|\mathbf{v}_K^+$, $K\mathbf{u}_K^- = \|K\|^{-1}\mathbf{v}_K^-$ with unit vectors $\mathbf{v}_K^+$, $\mathbf{v}_K^-$. Given $K, M \in SL(2,\mathbb{R})$, let $b^{(+,+)}(K,M) = \mathbf{v}_K^+ \cdot \mathbf{u}_M^+$, similar for $b^{(+,-)}$, $b^{(-,+)}$, $b^{(-,-)}$. We have

$$\begin{aligned}&\|K\|\|M\||b^{(+,+)}(K,M)| - \|K\|\|M\|^{-1} \le \|MK\| \\ &\le \|K\|\|M\||b^{(+,+)}(K,M)| + \|K\|^{-1}\|M\| + \|K\|\|M\|^{-1}.\end{aligned}$$

In particular,

$$\begin{aligned}&\frac{\|A_{j+1}A_j\|}{\|A_{j+1}\|\|A_j\|} - \frac{1}{\|A_j\|^2} \le |b^{(+,+)}(A_j, A_{j+1})| \\ &\le \frac{\|A_{j+1}A_j\|}{\|A_{j+1}\|\|A_j\|} + \frac{1}{\|A_j\|^2} + \frac{1}{\|A_{j+1}\|^2}.\end{aligned}$$

Binary structure in the proof of the Avalanche Principle

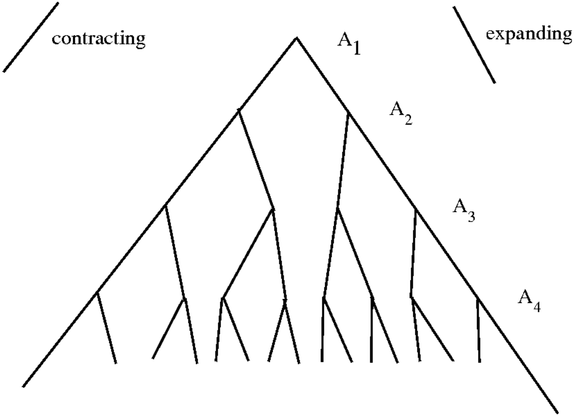

Figure : The expanding, contracting structure in the proof

Proof of Avalanche Principle

In view of our assumptions therefore

$$1 - \frac{\sqrt{\mu}}{\mu^2} \leq |b^{(+,+)}(A_j, A_{j+1})| \frac{\|A_{j+1}\| \|A_j\|}{\|A_{j+1} A_j\|} \leq 1 + \frac{2\sqrt{\mu}}{\mu^2} \qquad (12)$$

which implies $|b^{(+,+)}(A_j, A_{j+1})| \geq \frac{1}{\sqrt{\mu}}(1 - \mu^{-\frac{3}{2}}) \geq \frac{1}{2}\mu^{-\frac{1}{2}}$ if $n \geq 2$, say. One checks easily by induction that for any vector u

$$\begin{aligned} & A_n \cdot \ldots \cdot A_1 u \\ &= \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \|A_n\|^{\varepsilon_n} \prod_{j=1}^{n-1} \|A_j\|^{\varepsilon_j} b^{(\varepsilon_j, \varepsilon_{j+1})}(A_j, A_{j+1}) (u_{A_1}^{\varepsilon_1} \cdot u) \; v_{A_n}^{\varepsilon_n} \end{aligned}$$

Hence

$$\|A_n \cdot \ldots \cdot A_1 u\| = \|A_n\| \prod_{j=1}^{n-1} \|A_j\| |b^{(+,+)}(A_j, A_{j+1})| |u_{A_1}^{+} \cdot u| [1 + R_n(u)]$$

Proof of AP

$$|R_n(u)| \le \sum_{\substack{\varepsilon_1,\dots,\varepsilon_n=\pm1\\ \min_j \varepsilon_j=-1}} \prod_{j=1}^{n} \|A_j\|^{\varepsilon_j-1} \prod_{k=1}^{n-1} \left| \frac{b^{(\varepsilon_k,\varepsilon_{k+1})}(A_k,A_{k+1})}{b^{(+,+)}(A_k,A_{k+1})} \right|$$

$$\le \sum_{\ell=1}^{n} \binom{n}{\ell} \mu^{-2\ell}(2\sqrt{\mu})^{2\ell} = \sum_{\ell=1}^{n} \binom{n}{\ell} (4/\mu)^{\ell} = \left(1+\frac{4}{\mu}\right)^n - 1 < 4e^4\ \frac{n}{\mu}$$

So we have shown with $b^{(+,+)}(A_j,A_{j+1}) =: b_j^{(+,+)}$,

$$\left| \log\|A_n\cdot\ldots\cdot A_1\| - \sum_{j=1}^{n}\log\|A_j\| - \sum_{j=1}^{n-1}\log|b_j^{(+,+)}| \right| < C\frac{n}{\mu}$$

(11) follows from this and the sum of (12):

$$\left| \sum_{j=1}^{n-1} \left[\log|b_j^{(+,+)}| - \log\|A_{j+1}A_j\| + \log\|A_j\| + \log\|A_{j+1}\| \right] \right| \le C\mu^{-\frac{3}{2}} n$$

Rate of convergence for Lyapunov exponents

Assume $L(E) > \gamma > 0$. Let $N = nk$ and apply AP to

$$M_N(x,E) = \prod_{j=0}^{n-1} A_j, \quad A_j := M_k(T^{jk}x, E)$$

Let $k \sim C(\log N)^{\frac{1}{\sigma}}$, and apply (weak) LDT. Then on $\mathbb{T} \setminus \mathcal{B}$ with $|\mathcal{B}| < N^{-10}$, conditions (9), (10) hold (using that $L_k(E) - L_{2k}(E) \ll 1$ for k large):

$$\begin{aligned} |\log \|A_j\| - kL_k(E)| &< k^\sigma \\ |\log \|A_{j+1}A_j\| - 2kL_{2k}(E)| &< k^\sigma \end{aligned}$$

whence averaging over x yields

$$\begin{aligned} |NL_N(E) + (n-2)kL_k(E) - 2k(n-1)L_{2k}(E)| &< N^{-9} \\ |L_N(E) + L_k(E) - 2L_{2k}(E)| &< \frac{k}{N} \end{aligned} \tag{13}$$

Repeat the same analysis with twice as many matrices, i.e., with $2N$ instead of N.

Applying the AP to the propagators M_N

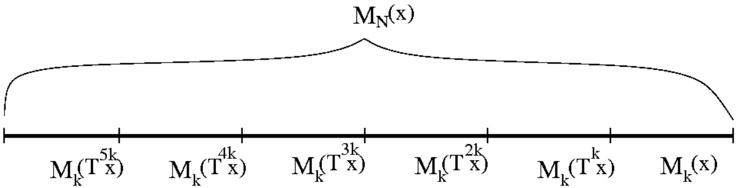

Figure : Writing M_N as product of shifted shorter M_k

Hence

$$|L_{2N}(E) + L_k(E) - 2L_{2k}(E)| < \frac{k}{N}$$

Subtracting from the previous estimate we obtain

$$0 \le L_N(E) - L_{2N}(E) < \frac{(\log N)^C}{N} \tag{14}$$

whence $L_N(E) - L(E) \lesssim (\log N)^C N^{-1}$ (works for any Diophantine condition). By a more careful rendition of the same argument we see that in fact

$$L_N(E) - L(E) \lesssim N^{-1}$$

and that this is optimal unless the convergence takes place exponentially fast due

$$|L(E) + L_n(E) - 2L_{2n}(E)| < e^{-n^\sigma}$$

Moreover, convergence is uniform on any compact interval on which $L > 0$.

Combine (14) with (13), this time using the sharper LDT:

$$|L(E) + L_k(E) - 2L_{2k}(E)| < \exp(-ck)$$

and so taking difference for E, E' and differentiating on small scale $L_k(E)$ one obtains:

$$\begin{aligned} |L(E) - L(E')| &\leq |L_k(E) - L_k(E')| + 2|L_{2k}(E) - L_{2k}(E')| + e^{-ck} \\ &\leq e^{Ck}|E - E'| + e^{-ck} \end{aligned}$$

Theorem

Assume sharp Diophantine condition, and $L(E) > \gamma > 0$ for $E \in I = [E_0, E_1]$. Then L is Hölder continuous on I.

The Hölder exponent depends on γ, but in fact in can be shown by a refinement of this argument from G-S 2001 that it does not. Under a weaker Diophantine condition we have $\exp(-|\log|E - E'||^b)$ continuity with $b < 1$. However: You, Zhang 2013 refined LDT and showed **Hölder for all Diophantine** ω.

$E_{\Lambda,j}(x)$, $j = 1, \dots, b - a + 1 = |\Lambda|$ are the eigenvalues of (1) restricted to the interval $\Lambda = [a, b]$ with Dirichlet BC $\varphi(a-1) = \varphi(b+1) = 0$. Consider

$$N_\Lambda(E, x) = \frac{1}{|\Lambda|} \sum_j \chi_{(-\infty, E)}(E_{\Lambda,j}(x)).$$

The limit (in the weak sense of measures)

$$\lim_{a \to -\infty, b \to +\infty} N_\Lambda(\cdot, x) = N(\cdot)$$

exists for a.e. x and is deterministic. $N(\cdot)$ is the IDS. Connection with Lyapunov exponent given by the Thouless formula

$$L(E) = \int \log |E - E'| \, dN(E'). \tag{15}$$

In other words, L is the Hilbert transform of N. Hence we have a corollary of Hölder regularity theorem: the IDS is also Hölder continuous.

Controlling the resolvent

The entries of the propagator matrix $M_n(x, E)$ are as follows:

$$M_n(x,E) = \begin{bmatrix} f_{[1,n]}(x,E) & -f_{[1,n-1]}(Tx,E) \\ f_{[1,n-1]}(x,E) & -f_{[1,n-2]}(Tx,E) \end{bmatrix} \tag{16}$$

Here $f_{[a,b]}(x, E)$ stands for the characteristic polynomial of the problem (1) on the interval $[a, b]$ with zero boundary conditions $\psi(a-1) = 0$, $\psi(b+1) = 0$, i.e.,

$$f_{[a,b]}(E) = \begin{vmatrix} v(a)-E & -1 & 0 & \cdots & 0 \\ -1 & v(a+1)-E & -1 & \cdots & 0 \\ & & & & \\ & & & & \\ 0 & \cdots & 0 & -1 & v(b)-E \end{vmatrix}$$

Is there a LDT for the entries rather than the whole matrix? Yes, by Goldstein-S. GAFA, 2008. Exploit that determinants are solutions of Schrödinger equation, and relation between entries given by $\det M_n = 1$.

Connection with the resolvent:

$$(H_N(x) - E)^{-1}(m,n) = \frac{f_{[1,m]}(x,E)\, f_{[n+1,N]}(x,E)}{f_{[1,N]}(x,E)}$$

We have a LDT for the determinants:

Proposition

For some small constant $\tau > 0$

$$\left|\{x \in \mathbb{T} \,:\, |N^{-1}\log|f_{[1,N]}(x,E)| - L_N(E)| > N^{-\tau}\}\right| \le e^{-N^{\tau}}.$$

By $|L_N(E) - L(E)| \lesssim N^{-1}$ we may replace $L_N(E)$ with $L(E)$.
Conclusion: if $L(E) > 0$ then

$$|(H_N(x) - E)^{-1}(m,n)| \lesssim \exp(-L(E)(m-n) + N^{1-\tau})$$

up to set of measure $\lesssim \exp(-N^{\tau})$.

Uniform upper bounds

We do not need LDT for the numerator: subharmonic functions can only fall significantly below the average but not rise much above it. Proof of LDT, work with a regularized form of (2):

$$u(x) \le \int \log(|e(x)-\zeta|+\delta)\,\mu(d\zeta) + h(x) \quad \forall\, x \in \mathbb{R}$$

with $\delta = N^{-1}$. Work with $\frac{1}{m}\sum_{j=1}^{m} u(T^j x)$ as in the proof of LDT. Harmonic part is harmless, and we have better decay of the Fourier transform

$$\Big| \int_0^1 \log(|e(x)-\zeta|+\delta)\, e(-\nu x)\, dx \Big| \lesssim \min(|\nu|^{-1}, \delta^{-1}|\nu|^{-2})$$

This gives

$$\sup_x N^{-1} \log \|M_N(x,E)\| \le L(E) + N^{-\frac{1}{2}+}$$

Assuming $L(E) > 0$ we may further improve this to (G-S 2008)

$$\sup_x N^{-1} \log \|M_N(x,E)\| \le L(E) + N^{-1}(\log N)^A$$

Higher-dimensional matrices

Let $A : \mathbb{T} \to GL(d, K)$ be continuous, co-cycle

$$\mathbb{T} \times K^d \in (x, v) \mapsto (x + \omega, A(x)v) \tag{17}$$

where ω is (strongly) Diophantine. $(\mathcal{E}, \mathfrak{d})$ compact metric space.

Theorem (S., 2012)

$A : \mathbb{T} \times \mathcal{E} \to GL(d, K)$ continuous ($K = \mathbb{R}, \mathbb{C}$), analytic $x \mapsto A(x, E)$ uniformly in $E \in \mathcal{E}$. Suppose $E \mapsto A(x, E)$ Hölder continuous, uniformly in $x \in \mathbb{T}$. Lyapunov exponents satisfy the gap condition

$$\lambda_j(E) - \lambda_{j+1}(E) > \kappa > 0 \quad \forall\, E \in \mathcal{E}, \forall\, 1 \leq j < d \tag{18}$$

Then all $\lambda_j(E)$ are Hölder continuous as a function of $E \in \mathcal{E}$. If (18) *holds at some point $E_0 \in \mathcal{E}$, then each $\lambda_j(E)$ is Hölder continuous locally around E_0. In other words, if all exponents are distinct at E_0, then they are all Hölder continuous locally around E_0, and therefore also remain distinct near E_0.*

Avalanche Principle for $d \times d$ matrices

Lemma

$\{A_j\}_{j=1}^n \subset GL(d,K)$ satisfy: for each $1 \le j \le n$ $\exists$ a 1-dimensional subspace $\mathcal{S}_j \subset K^d$ s.t.

$$\begin{gathered} |A_j v| = \|A_j\|\,|v| \quad \forall v \in \mathcal{S}_j \\ |A_j w| \le \alpha_j |w| \quad \forall w \in \mathcal{S}_j^\perp \\ \|A_j\| \ge \alpha_j \mu, \quad \mu \ge 16n^2 \end{gathered}$$

In addition, assume

$$\|A_{j+1}\|\,\|A_j\| < \mu^{\frac{1}{4}} \|A_{j+1}A_j\| \qquad \forall\, 1 \le j < n.$$

Then

$$\left| \log \|A_n \cdot \ldots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\| \right| < C \frac{n}{\sqrt{\mu}}$$

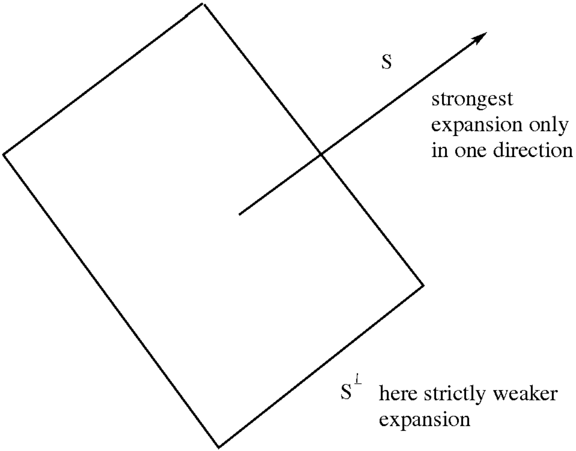

Figure : The unique expanding direction condition

Remarks on higher dimensions

The 2×2 AP is a special case, "line condition" holds automatically. Gap condition is the same as $L(E) > 0$, since the two Lyapunov exponents are $\{L(E), -L(E)\}$.

Oseledec multiplicative ergodic theorem guarantees a filtration of $\mathbb{R}^d$ associated with the sequence of Lyapunov exponents.

Higher-dimensional cocycles arise in the transfer-matrix formalism of higher-order difference equations. Duarte, Klein 2013: very general investigation of avalanche principle, LDT etc. for such cocycles. In particular, they also investigate the situation where the Lyapunov gaps collapse and obtain a version of the AP in that case. They also announce that they will come out with a paper in which they give sufficient conditions for different types of Lyapunov spectrum.

For the 1-dimensional Schrödinger operator we can change the underlying dynamics: instead of $\mathbb{T}$ we consider a potential on $\mathbb{T}^d$ with the multi-dim shift, or the skew shift.

Or we switch from a 1-dim operator to a higher-dimensional one, and we lose the entire transfer matrix formalism.

Summary of Lecture 1

- Subharmonic functions arise naturally in the context of Schrödinger operators with potentials determined by evaluating analytic functions along an orbit of an *ergodic deterministic transformation*. Examples: $\log \|M_n\|$, $\log |\det H_\Lambda|$. Also relevant to higher-dimensional lattices $\mathbb{Z}^d$ where we do not have a transfer matrix formalism; then s-h in each coordinate direction (pluri-s-h).
- Modulo harmonic functions, subharmonic ones are logarithmic potentials of positive Borel measures (Riesz representation). The latter are the key to analytic control, and the derivation of large deviation estimates.
- But size of Riesz measure alone too crude to imply a large deviation estimate. We require also some *structural information* on the measure, such as near invariance of the function itself under the dynamics. Easier to implement structure directly for the subhamonic function itself rather than for its Riesz measure.

- Avalanche principle in combination with LDT allows for *induction on scales* arguments. Essential for nonperturbative approach assuming only positivity of the Lyapunov exponent. Gives rates of convergence for these exponents. Can also be used to derive perturbatively (need big potentials for the initial step) large deviation estimates, such as for the skew shift. See Lecture 3.
- In the PDE setting (higher-dimensional lattices $\mathbb{Z}^d$) cannot use AP, instead rely on resolvent identity. This will again be perturbative. We control the Green functions directly in this way. A key analytical ingredient in this approach are the Matrix-valued Cartan estimates, see Lecture 3.
- All of the analysis in this lecture was for fixed energy. In the following lecture, we will encounter the so-called *elimination of the energy* which is needed for Anderson Localization.

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