Subharmonic techniques in multiscale analysis: Lecture 4

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Maiori, September 2013

AL theorem for Z^2

We now turn to the higher-dimensional case. For any $\underline{\omega}, \underline{\theta} \in \mathbb{T}^2$ consider the Schrödinger operator on $\ell^2(\mathbb{Z}^2)$ defined as

$$H_{\underline{\omega}}(\underline{\theta}) = -\Delta + \lambda V(\underline{\theta}).$$

The potential $V(\underline{\theta})$ at site $(n_1, n_2) \in \mathbb{Z}^2$ is given by

$$V(\underline{\theta})(n_1, n_2) = v(\theta_1 + n_1\omega_1, \theta_2 + n_2\omega_2)$$

with a real-analytic function v on $\mathbb{T}^2$ which is nonconstant on every horizontal and vertical line.

Theorem (Bourgain-Goldstein-S., 2002)

For any $\varepsilon > 0$, $\underline{\theta} \in \mathbb{T}^2$ there is a set $\mathcal{F}_\varepsilon = \mathcal{F}_\varepsilon(\underline{\theta}) \subset \mathbb{T}^2$ so that $\mathrm{mes}(\mathbb{T}^2 \setminus \mathcal{F}_\varepsilon) < \varepsilon$ and such that for any $\underline{\omega} \in \mathcal{F}_\varepsilon$ and any $\lambda \geq \lambda_0(\varepsilon, v)$ the operator $H_{\underline{\omega}}(\underline{\theta})$ displays AL.

Applies equally well to long-range operators in place of the Laplacian.

We define Green functions on cubes $\Lambda \subset \mathbb{Z}^2$ as

$$G^{\Lambda}_{\underline{\omega}}(\underline{\theta}, E)(n, m) := \left[(H_{\underline{\omega}}(\underline{\theta}) - E) \restriction \Lambda\right]^{-1}(n, m)$$

As before, we call a Green function **good** if

- $\|G^{\Lambda}_{\underline{\omega}}(\underline{\theta}, E)\| < e^{N^b}, \quad b < 1$
- $|G^{\Lambda}_{\underline{\omega}}(\underline{\theta}, E)(n, m)| \lesssim \exp(-\gamma|n - m|), \quad |n - m| > N/10$

where $N = \mathrm{diam}\Lambda$.

Two main ingredients in the proof:

(A) Large deviation estimates for fixed energies

$$\mathrm{mes}\left[\underline{\theta} \in \mathbb{T}^2 \ : \ G^{\Lambda}_{\underline{\omega}}(\underline{\theta}, E) \text{ is bad}\right] < e^{-(\mathrm{diam}\,\Lambda)^{\sigma}} \qquad (1)$$

We need this for *all slices*.

(B) Elimination of the energy.

We prove (1) by induction on scales, starting from large N_0. Then (1) holds for $\lambda \geq \lambda_0(N_0)$. This is the only reason for the largeness assumption.

To pass to the next scale $N_1 = N_0^C$ we partition a cube of that size into smaller N_0 cubes. The number of bad cubes on the small scale is

$$\#\Big\{(n_1, n_2) \in [-N_1, N_1]^2 \ : \ (n_1\omega_1, n_2\omega_2) \in \mathcal{B}_{N_0,\underline{\omega}}(E)\Big\} \qquad (2)$$

where $\mathcal{B}_{N_0,\underline{\omega}}(E) := \{\underline{\theta} \in \mathbb{T}^2 \ : \ G_{\underline{\omega}}^{\Lambda_0}(\underline{\theta}, E) \text{ is bad}\}$, Λ_0 being a square centered at the origin of side length N_0.

In order to apply the resolvent identity and the matrix-valued Cartan theorem from above we need to show that (2) is bounded by N_1^b, $b < 1$.

Indeed: We cannot apply the res-id. if there is a **linear chain of bad cubes**.

The chains in the resolvent identity

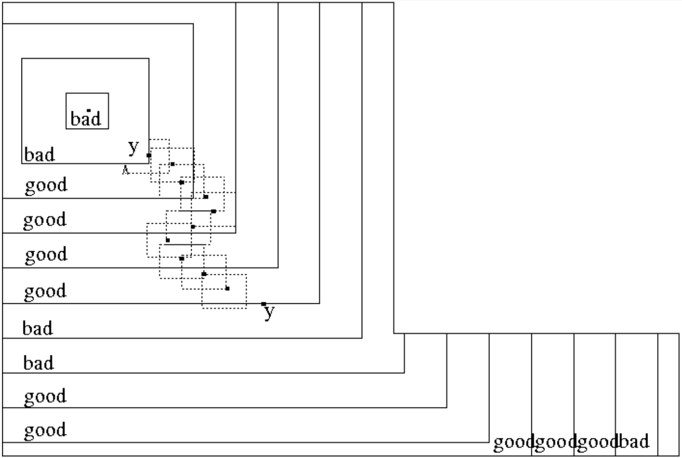


Figure: Applying the resolvent identity with good and bad cubes

Notice that we need more general regions than cubes: the difference of two cubes may have a corner. This is only technical.

How to ensure the **sublinear estimate?**

Proposition

$\mathcal{A} \subset [0,1]^2$ semi-algebraic of degree $\leq B$. Sections satisfy

$$\operatorname{mes}(\mathcal{A}_{\theta_1}) < \eta, \ \operatorname{mes}(\mathcal{A}_{\theta_2}) < \eta \qquad \forall\, (\theta_1, \theta_2) \in \mathbb{T}^2 \tag{3}$$

Let

$$\log B \ll \log N \ll \log \frac{1}{\eta}. \tag{4}$$

There exists Ω_N with

$$\operatorname{mes}([0,1]^2 \setminus \Omega_N) < N^{-\varepsilon},$$

and for all $\underline{\omega} \in \Omega_N$, $\delta_0 > 0$ some absolute constant,

$$\#\left\{(n_1, n_2) \in \mathbb{Z}^2 \;\middle|\; |n_1| \vee |n_2| < N, (n_1\omega_1, n_2\omega_2) \in \mathcal{A} \pmod{\mathbb{Z}^2}\right\}$$
$$< N^{1-\delta_0}.$$

An arithmetic condition

Several comments:

- The set Ω_N determined by purely arithmetic considerations that do not depend on the curve Γ. In particular, elimination of "bad" ω does not involve the potential at this stage.
- The idea behind the lemma: if too many points $(n_1\omega_1, n_2\omega_2)$ fall very close to an algebraic curve Γ, then there would have to be many small triangles with vertices close to Γ. Here "small" means both small sides and small area. So we need to precisely exclude such ω.
- Now we can pass to the next scale measure estimate along slices by means of the matrix Cartan theorem of Lecture 3. This gives the LDT at the large scale N_1 on each horizontal and vertical slice.
- For AL then use semi-algebraic sets to eliminate double resonances.

The arithmetic lemma

Lemma

$0 < N \in \mathbb{Z}$ be a positive integer. $\exists\, \Omega_N \subset [0,1]^2$ so that $\mathrm{mes}([0,1]^2 \setminus \Omega_N) < N^{-\varepsilon}$ s.t. $\underline{\omega} = (\omega_1, \omega_2) \in \Omega_N$ satisfies: let $q_1, q_1', q_2, q_2' \in \mathbb{Z} \setminus \{0\}$ bounded in abs.val. by N and suppose

$$\begin{cases} \theta_1 \equiv q_1\omega_1, & \theta_1' \equiv q_1'\omega_1 \\ \theta_2 \equiv q_2\omega_2, & \theta_2' \equiv q_2'\omega_2 \end{cases} \mod \mathbb{Z}$$

satisfy

$$|\theta_i|,\ |\theta_i'| < N^{-1+\delta} \qquad (i = 1,2) \tag{5}$$

and

$$-N^{-3+\delta} < \begin{vmatrix} \theta_1 & \theta_1' \\ \theta_2 & \theta_2' \end{vmatrix} < N^{-3+\delta}$$

with $\delta > 0$ sufficiently small. Then with $\delta' \to 0$ with δ

$$\gcd(q_1, q_1') > N^{1-\delta'},\ \gcd(q_2, q_2') > N^{1-\delta'}$$

A simple observation

What is the relevance of the *greatest common divisor?*

Let $0 < n, m \in \mathbb{Z}$, $1 > \rho > 0$. Then

$$\mathrm{mes}[\theta \in \mathbb{T} \mid \|\theta m\| < \rho, \ \|\theta n\| < \rho] \simeq \rho^2 + \frac{\rho \gcd(m,n)}{m+n}, \qquad (6)$$

where $\|\cdot\|$ denotes the distance to the nearest integer. This implies that the fractional parts of $\theta m, \theta n$, considered as random variables, are strongly dependent if and only if $\gcd(m,n)$ is large relative to $m+n$.

(5)+DC gives $q_1, q_1' > N^{-1-2\delta}$. If $\gcd(q_1, q_1') < N^{1-\delta'}$, $\delta' = 2\delta$, then by (6) measure of those ω_1 is

$$\lesssim N^{-1+\delta-\delta'} = N^{-1-\delta}$$

But this cannot be summed over q_1, q_1'.

$$|\theta_i| = \|q_i\omega_i\| = |q_i\omega_i - m_i| < N^{-1+}$$
$$|\theta'_i| = \|q'_i\omega_i\| = |q'_i\omega_i - m'_i| < N^{-1+}$$

So by Diophantine condition, $|q_i|, |q'_i| > N^{1-}$. Partition $\mathbb{T}$ into intervals of size N^{-2}. Number of admissible (q, m) with

$$|q\omega_i - m| < N^{-1+}$$

is at most N^{0+}. Area condition with $\omega_i = \omega_{i,0} + \kappa_i$, $|\kappa_i| < N^{-2}$ and fixed $\omega_{i,0}$. Write

$$-N^{-3+} < \begin{vmatrix} q_1\omega_1 - m_1 & q'_1\omega_1 - m'_1 \\ q_2\omega_2 - m_2 & q'_2\omega_2 - m'_2 \end{vmatrix} < N^{-3+}$$

in the form

$$|(q_1q'_2 - q'_1q_2)\kappa_1\kappa_2 + \alpha_1\kappa_1 + \alpha_2\kappa_2 + \beta| < N^{-3+}$$

Proof of lemma

This determines an area, assuming $|q_1 q_2' - q_1' q_2| \geq 1$ of size

$$\lesssim \frac{N^{-3+}}{|q_1 q_2' - q_1' q_2|} \log N$$

If $|q_1 q_2' - q_1' q_2| \geq N^{1+\varepsilon}$, then this is $\lesssim N^{-4-\varepsilon+}$. Summing over the N^4 choices of little squares gives a *total measure* $\lesssim N^{-\varepsilon+}$ in that case. If $|q_1 q_2' - q_1' q_2| < N^{1+\varepsilon}$, simplify: $\gcd(q_1, q_1') = 1$, $\gcd(q_2, q_2') = 1$. By definition, have

$$|m_i/q_1 - m_i'/q_i'| < N^{-2+}, \quad |m_i q_i' - m_i' q_i| < N^{0+}$$

Number of admissible (q_1, q_1', q_2, q_2') is $\lesssim N^{3+}$, and of (m_1, m_1', m_2, m_2') is $\lesssim N^{0+}$. So measure bound in (ω_1, ω_2) space is $\lesssim N^{-4+} \times N^{3+} = N^{-1+}$. We have N^{1-} room to move here, which allows summing up over all cases $\min_{i=1,2} \gcd(q_i, q_i') < N^{1-}$.

The geometry behind the arithmetic proposition

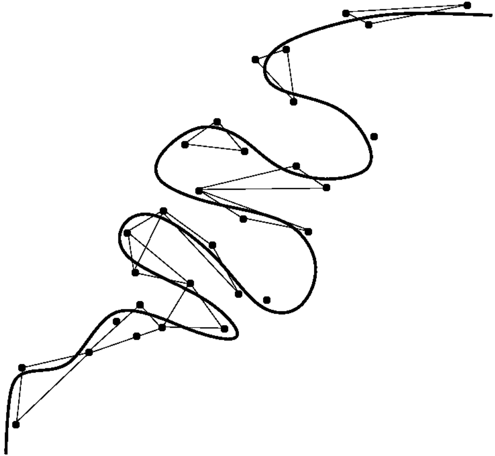


Figure: Curve, points, triangles

If an algebraic curve comes very close to N^{1-}-many points in $(n_1\omega_1, n_2\omega_2) \mod \mathbb{Z}^2$ with good (ω_1, ω_2), then this increases the degree of the curve due to many oscillations.

More semi-algebraic machinery

Gromov-Yomdin triangulation/uniformization theorem.

Theorem

For any positive integers r, d *there exists* $C = C(d, r)$ *with the following property: any semi-algebraic set* $\mathcal{S} \subset [0,1]^d$ *can be triangulated into* $N \lesssim (1 + \deg \mathcal{S})^C$ *simplices, where for every closed* k*-simplex* $\Delta \subset \mathcal{S}$ *there exists a homeomorphism* h_Δ *of the standard simplex* $\Delta_{k,0} \subset \mathbb{R}^k$ *with unit edge-length onto* Δ *such that* h_Δ *is real analytic on the interior of each face* Δ*. Furthermore,* $\sup_{x\in\Delta} \sup_{|\alpha|\leq r} |\partial^\alpha h_\Delta(x)| \leq 1$*.*

Gromov, M. *Entropy, homology, and semialgebraic geometry*, Séminaire Bourbaki, Vol. 1985/86. Astérisque No. 145–146 (1987), 5, 225–240.

Connecting curves

We may connect any two points along a curve passing through a chain of simplices given by the previous triangulation theorem.

Corollary

Let $\mathcal{S} \subset [0,1]^d$ be semi-algebraic of degree B.

(A) *If $p, q \in \mathcal{S}$, then there exists a path $\gamma : [0,1] \to \mathcal{S}$ s.t. $\gamma(0) = p, \gamma(1) = q$ and $|\dot{\gamma}| < B^C$.*

(B) *Let $\mathcal{S} \subset \prod_{j=1}^{d}[0, \rho_j]$ be semi-algebraic of degree B. If $p, q \in \mathcal{S}$, then there exists a path $\gamma : [0,1] \to \mathcal{S}$ s.t. $\gamma(0) = p, \gamma(1) = q$ and*

$$\sum_{j=1}^{d} \rho_j^{-1} |\dot{\gamma}_j| < B^C$$

(C) *If $\mathrm{mes}(\mathcal{S}) < \varepsilon^n$, then $\mathcal{S}$ can be covered by at most $B^C \varepsilon^{1-n}$ balls of radius ε.*

Proof of proposition

If it fails then

$$\#\{(n_1, n_2) \in \mathbb{Z}^2 \ : \ |n_i| \le N, \ (n_1\omega_1, n_2\omega_2) \in \mathcal{A} \mod \mathbb{Z}\} > N^{1-}$$

We have $\mathrm{mes}(\mathcal{A}) < \eta$, $\mathrm{dist}(x, \partial\mathcal{A}) < \sqrt{\eta}$ for all $x \in \mathcal{A}$.
Yomdin-Gromov:

$$\#\{(n_1, n_2) \in \mathbb{Z}^2 : |n_i| \le N, \ \mathrm{dist}((n_1\omega_1, n_2\omega_2), \Gamma) < \sqrt{\eta} \mod \mathbb{Z}\} > N^{1-}$$

where Γ parametrized by

$$\gamma : [0, 1] \to \Gamma, \ |\gamma'| < 1, |\gamma''| < 1$$

Cover Γ by $N^{1-\delta}$ disks D_α of radius $N^{-1+\delta}$. Each corresponds to $\lesssim N^{\delta+}$ pairs $(n_1, n_2) \in K_1$. There is α so that $\#K_\alpha > N^{\delta-}$ and if $P_0, P_1, P_2 \in D_\alpha \cap \Gamma$ then

$$\angle(P_0P_1, P_0P_2) < N^{-1+\delta}$$

From DC and measure conditions can ensure that there are $\le N^{0+}$ points $n_i\omega_i$ on any slice.

Turning of the curve

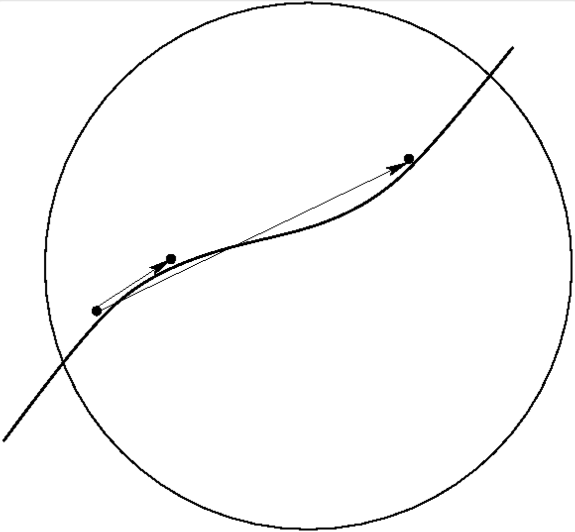


Figure: The geometry in the proposition

Proof of proposition

Cover D_α by disks of radius $N^{\delta-\delta^2}$. Find $\bar{n}, n \in K_\alpha$ with distinct coordinates such that

$$\|(n_1 - \bar{n}_1)\omega_1\| + \|(n_2 - \bar{n}_2)\omega_2\| < N^{-1+\delta^2}$$

Set $q_i = n_i - \bar{n}_i$, $\theta_i \equiv q_i\omega_i$ same with prime. Then $|\theta_i| < N^{-1+\delta^2}$, $|\theta_i'| < N^{-1+\delta}$. Area of triangle $\lesssim N^{-3+3\delta}$.

Hence by the Lemma: $r_i = \gcd(q_i, q_i') > N^{1-\delta'}$, $i = 1, 2$.
Write $q_i = r_i Q_i$, $q_i' = r_i Q_i'$ with $|Q_i|, |Q_i'| < N^{\delta'}$ relatively prime.
Euclidean algorithm gives

$$k_i Q_i + k_i' Q_i' = 1, \quad |k_i|, |k_i'| < N^{\delta'}$$

One has

$$\|r_i\omega_i\| \le |k_i|\|q_i\omega_i\| + |k_i'|\|q_i'\omega_i\| < N^{\delta'} N^{-1+\delta} < N^{-\frac{1}{2}}$$

Proof of proposition

Therefore

$$|\theta_i| = \|q_i\omega_i\| = \|Q_i r_i \omega_i\| = |Q_i|\|r_i\omega_i\|, \quad |\theta_i'| = |Q_i'|\|r_i\omega_i\|$$

Hence

$$\frac{|\theta_i|}{|\theta_i'|} = \frac{|q_i|}{|q_i'|}, \quad |\theta_i'| = \frac{|\theta_i|}{|q_i|}|q_i'| \leq N \cdot N^{-2+2\delta^2+}$$

since $|\theta_i| < N^{-1+\delta^2}, |q_i| > N^{1-\delta^2-}$. This means that

$$\|q_i'\omega_i\| = |\theta_i'| < N^{-1+2\delta^2+}$$

But by DC the number of such $q_i' \in [0, N]$ is at most $N^{2\delta^2+}$. So the total number of points in the smaller disk is $\lesssim N^{4\delta^2+}$, contradicting the lower bound from above. $\square$

Some remarks

- Together with matrix-valued Cartan theorem this allows one to pass to the next scale in the exponential measure estimate on the event of having a bad Green function at fixed energy. In fact, along all coordinate slices.
- For AL then need elimination of energy, done by combining exponential estimate with semi-algebraic machinery. We'll discuss this later.
- Easier case $v(x_1, x_2) = F(x_1 + x_2)$ where F analytic on $\mathbb{T}$. Can allow for any number of variables. No need for arithmetic lemma, reduction to one dimension.
- Previous example shows that non-perturbative version of the theorem is FALSE. See Bourgain's book.
- Obstacle in passing to higher dimensions $\mathbb{Z}^3$ etc: **sub-linear bound**. We will next describe a non-arithmetic way of overcoming this issue.

Bourgain's method for $\mathbb{Z}^d$, $d \geq 3$

We now present Bourgain's 2007 GAFA paper, extending AL theorem to higher-dimensional lattices.

- No arithmetic analysis, carry out inductive scheme based on resolvent identity for exponential measure estimates by removing sets of $\underline{\omega}$ depending on the potential.
- Use a multi-dim generalization of the lemma on steep lines to exclude these sets. Apply Yomdin parametrization and other semi-algebraic techniques.
- This implies no chains of bad boxes satisfying certain lacunarity condition between boxes. These chains are long but of finite length (16 in $\mathbb{Z}^3$).
- Ultimately leads to bound of N^ε for the number of bad sites in N-box, thus much better than sublinear.
- Apply matrix-valued Cartan theorem along all slices with Riesz mass of this size. This gives the exponential estimate at the larger scale.

The localization theorem

Let with $v : \mathbb{T}^d \to \mathbb{R}$ analytic

$$\mathcal{H} = \Delta + \lambda v(x + n\omega)\delta_{nn'}$$

and $n\omega = (n_1\omega_1, \ldots, n_d\omega_d)$. Assume that

$$\theta_i \mapsto v(\theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_d)$$

nonconstant.

Theorem

Fix $x \in \mathbb{T}^d$, say $x = 0$ and $\delta > 0$. There exists $\lambda(v, \delta)$ so that for $\lambda > \lambda(v, \delta)$ there is $\Omega = \Omega_{\lambda v} \subset \mathbb{T}^d$ with $\mathrm{mes}(\mathbb{T}^d \setminus \Omega) < \delta$ and for $\omega \in \Omega$ the operator $\mathcal{H}$ exhibits AL.

Applies equally well to long-range operators in place of the Laplacian.

Semialgebraic elimination of variables

Lemma

Let $\mathcal{S} \subset [0,1]^{d+1}$ *be semi-algebraic of degree* B *and assume*

$$\forall\ t \in [0,1], \quad \mathrm{mes}_d(\mathcal{S}(t)) < \eta$$

Then

$$\{(x_1, x_2) \in [0,1]^{2d} \mid (x_1, t), (x_2, t) \in \mathcal{S}\}$$

is semi-algebraic of degree B^C *and measure*

$$\mathrm{mes}_{2d}(\mathcal{S}) < B^C \eta^{1/d}$$

Analogous statement obtained by iteration: $t \in \mathbb{R}^r$ *intersect* 2^r *slices.*

Main ideas: For s-a property apply projection theorem to

$$\{(x_1, x_2, t) \in [0,1]^{2d+1} \mid (x_1, t), (x_2, t) \in \mathcal{S}\}$$

Semialgebraic elimination of variables

By measure condition on slices:

$$\mathrm{dist}(x, \partial(A(t))) < \eta^{1/d} = \eta_1$$

Suffices to show the covering number bound

$$\mathcal{N}(\{(x_1, x_2) \in [0,1]^{2d} | \exists\ t \in [0,1] \text{ s.t. } x_1, x_2 \in \partial(A(t))\}, \eta_1) < B^C \eta_1^{1-2d}$$

This follows from

$$\{(x_1, x_2) \in [0,1]^{2d} \mid \exists\ t \in [0,1] \ \text{ with } \ x_1, x_2 \in \partial(A(t))\}$$

is the union of at most B^C s-a sets of dimension at most $2d-1$ and degrees at most B^C. Use implicit function theorem, with singularities confined to lower dimensions.

Illustration of lemma

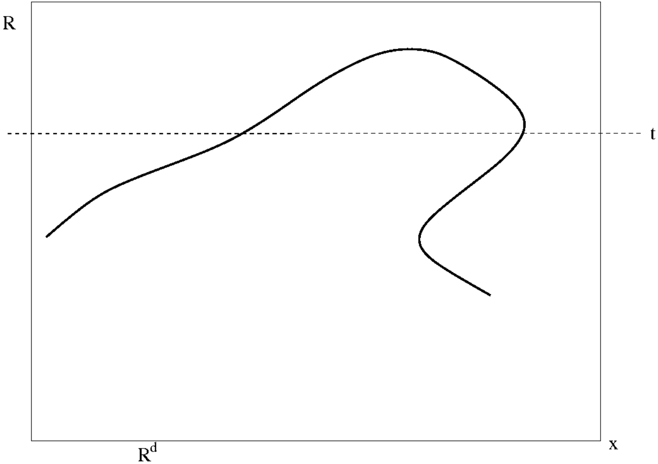


Figure: The (x, t) variables in the lemma

Lemma

Let $\mathcal{A} \subset [0,1]^{rd}$ s-a of degree B and $\mathrm{mes}_{rd}(\mathcal{A}) < \eta$. Let $\mathcal{N}_1, \dots, \mathcal{N}_{d-1} \subset \mathbb{Z}^r$ be finite s.t. with $C = C(d,r)$

$$\min_{1\le s\le r} |n_s| > (B \max_{1\le s\le r} |m_s|)^C \qquad n \in \mathcal{N}_i,\ m \in \mathcal{N}_{i-1} \tag{7}$$

Assume also that

$$\eta^{-1} > \max_{n\in\mathcal{N}_{d-1}} |n|^C$$

Then with $\delta^{-1} = \min_{n\in\mathcal{N}_1} \min_{1\le s\le r} |n_s|$ one has

$$\mathrm{mes}(\{\omega \in [0,1]^r | (\omega, n^{(1)}\omega, \dots, n^{(d-1)}\omega) \in \mathcal{A},\ n^{(i)} \in \mathcal{N}_i\}) < B^C \delta$$

Decomposition of semi-algebraic sets

Lemma

Let $\mathcal{S} \subset [0,1]^{d_1+d_2=d}$ be semi-algebraic of degree B and $\mathrm{mes}_d(\mathcal{S}) < \eta$ where $\eta < B^{-N}$ with large N. Denote by $(x, y) \in \mathbb{R}^{d_1+d_2}$ and fix $\varepsilon > \eta^{1/d}$. Then $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ where

$$\mathrm{mes}_{d_1}(\mathrm{Proj}_x \mathcal{S}_1) < B^C \varepsilon$$

$$\mathrm{mes}_{d_2}(\mathcal{S}_2 \cap L) < B^C \varepsilon^{-1} \eta^{1/d}$$

for all L, d_2-dimensional planes in $\mathbb{R}^d$ so that

$$\max_{1 \le j \le d_1} |\mathrm{Proj}_L(e_j)| < \varepsilon$$

where e_j is a coordinate unit vector.

Proof immediate from Yomdin: consider a single simplex, distinguish the two cases based on size of first order derivatives. See following figure.

Semi-algebraic decomposition

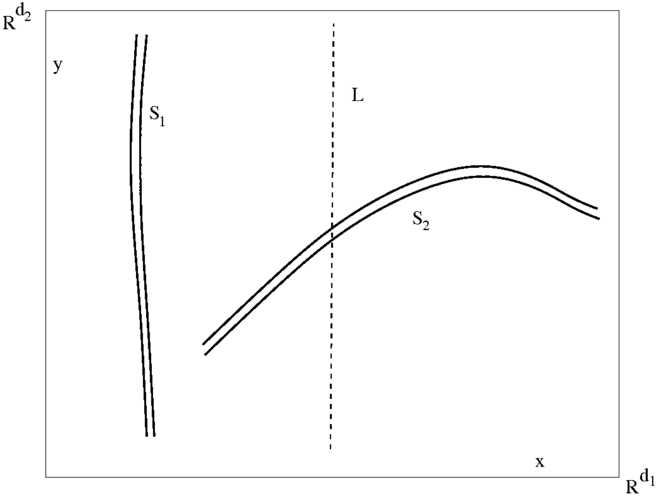


Figure: The sets S_1 and S_2

Proof of the steep planes lemma

Main ideas: partition $[0,1]^r$ into cubes I_α of size N_{d-1}^{-1} where

$$\begin{aligned} N_i &= \max_{n\in\mathcal{N}_i} \max_{1\le s\le r} |n_s| \\ M_i &= \min_{n\in\mathcal{N}_i} \min_{1\le s\le r} |n_s| \end{aligned} \tag{8}$$

For each α consider r-dim segment

$$L_\alpha = \{(\omega, n^{(1)}\omega, \ldots, n^{(d-1)}\omega) \mid \omega \in I_\alpha\}$$

with $n^{(i)} \in \mathcal{N}_i$ fixed. Main estimate:

$$|\mathrm{Proj}_{L_\alpha}(e_j)| < \frac{N_{d-2}}{M_{d-1}} \quad \forall\, j \le r(d-1) \tag{9}$$

Now apply decomposition for s-a sets as above with this ε, and then sum over the $n^{(i)}$ one after the other.

Measure estimate at fixed energy E

The exponential Green function measure estimate reads as follows:

Proposition

We can take λ *large and* $\underline{\omega}$ *outside of a small set such that the following holds:* $\exists\ \rho, c_1 \in (0,1)$ *s.t.* $\forall\ N$ *there is a subset* $X = X_N \subset [0,1]^d$

$$\mathrm{mes}(\{\theta \mid (x_1, \ldots, x_{i-1}, \theta, x_{i+1}, \ldots) \in X\}) < e^{-N^{c_1}}$$

for all $1 \le i \le d$ *and* x_i*, and if* $x \notin X$ *one has*

$$\|G_N(\underline{\omega}; E, x)\| < e^{N^\rho}$$
$$|G_N(\underline{\omega}; E, x)(n, n')| < e^{-\sigma N} \quad \forall\ |n - n'| > N/10$$

σ here is actually large, on the order of $\log \lambda$. Plays the role of "Lyapunov exponent".

Main ideas for the induction on scales

Fix some large initial scale N_0 and take λ large so that the proposition holds. This does not require elimination of frequencies. Let $N < M < \bar{N} = e^{N^c}$. The two lemmas imply the following:

there exists $\Omega_N \subset \mathbb{T}^r$ *with* $\mathrm{mes}(\Omega_N^c) < N^{-\sigma}$, $\sigma > 0$ *small s.t. for all* $\underline{\omega} \in \Omega_N$ $\nexists$ *sequence* $n^{(j)} \in \mathbb{Z} \cap [-\bar{N}, \bar{N}]^d$, $1 \le j \le C(d) = 2^{d+1}$ *with*

$$\min_s |n_s^{(1)}| > M^C$$

$$\min_s |n_s^{(j+1)}| > \left(M \max_s |n_s^{(j)}| \right)^C$$

and such that Green functions $G_\Lambda(\underline{\omega}; E, x)$ *for arbitrary but fixed* (E, x) *are all bad with* $\Lambda = [-M, M]^d + n^{(\alpha)}$.

The chains of cubes illustrated in $\mathbb{Z}^2$

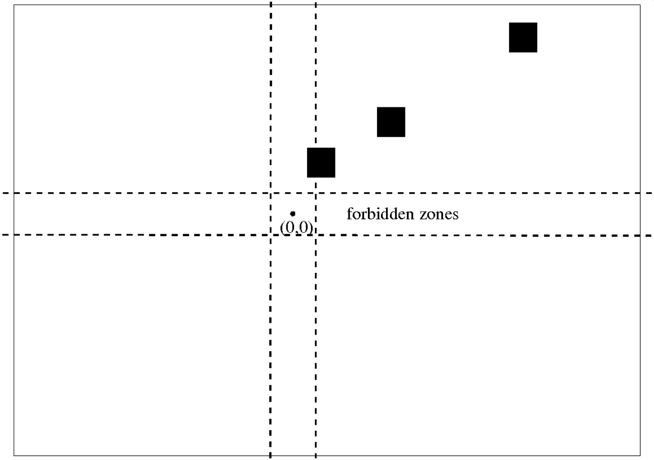


Figure: The type of chain arising in the elimination of (E, x) and $\underline{\omega}$

Note that $\min_s |n_s|$ of one cube needs to by far exceed the $\max_s |n_s|$ of the previous cube.

This of course requires **elimination of the variables** (E, x). Use the semi-algebraic elimination lemma from above based on *intersection of slices* applied to the following set:

$$\{(E, \underline{\omega}, x, y) \in [-C, C] \times \Omega_M \times [0,1]^{2d} \mid G_M(\underline{\omega}; E, x+y) \text{ is bad}\}$$

This is exactly what is needed to obtain the translation structure relative to the y-variable, which then becomes $n\omega$ via the lemma on steep planes.

What have we accomplished? The resonant sites are now in the forbidden zones, which are still super-linear in the large scale - and thus far too numerous.

Ergo: Need to repeat the construction inside of the forbidden zones, which means we only translate parallel to it: $I \subset \{1, 2, \ldots, d\}$ and now condition on partially translated Green function $G_M(\underline{\omega}; E, \{x_i + y_i\}_{i \in I}, \{x_j\}_{j \notin I})$ being resonant.

How to use the long chains

We need to go from the nonexistence of long resonant chains to a statement involving manageable regions in $\mathbb{Z}^d$ which are free of any resonant cubes.
After all: We wish to confine the resonant cubes within a limited volume (**sub-linear bound**).

Main claim: *For any $x \in \mathbb{T}^d, E \in \mathbb{R}, \underline{\omega} \in \Omega_N$ the following holds. Given $m \in \mathbb{Z}^d$ there exists $N < M < N^C$ so that for the region*

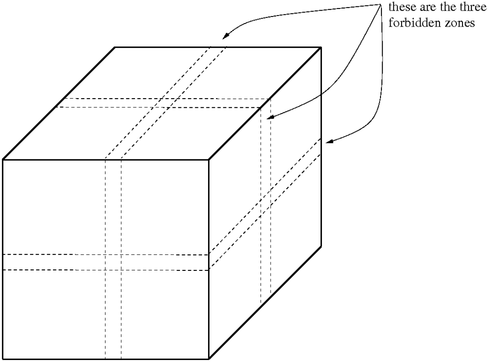
$$\Sigma := \prod_{i=1}^{d}[m_i - M, m_i + M] \setminus \prod_{i=1}^{d}[m_i - M^{\varepsilon}, m_i + M^{\varepsilon}]$$

the Green function $G_{\Sigma}(\underline{\omega}; E, x)$ is good.

We can take $\varepsilon = \frac{1}{2d}$ so that the total volume that we remove is sub linear in the diameter of the large box.
Note: We cannot expect to make such a statement for a FIXED box. The choice of the scale M depends on $x, E, m, \underline{\omega}, N$.

How to use the long chains

We begin with a simpler version of the claim, in which we squeeze only a single coordinate. If this were false, then we could get a nested family of the following basic building block, so that in the cubes outside the forbidden zones there is always a resonant site.



But then we have a chain which we excluded. Forbidden zones dictated by the min-max condition in **lemma on steep planes**. Now squeeze each coordinate one after the other.

Boxing in the resonant sites

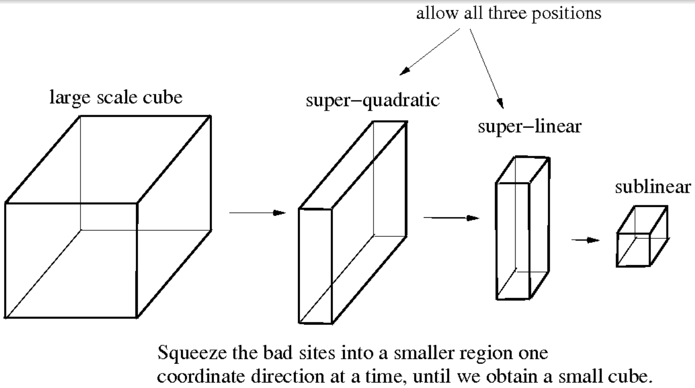


Figure: The stepwise confinement of resonant sites

This is achieved by applying lemma on steep planes with different configurations of coordinates: first all of them, then reduce by one etc., until down to a single coordinate. Now apply matrix Cartan to obtain the exponential measure estimate at the large scale.

This is now fairly straightforward. Requires further elimination in $\underline{\omega}$ to remove double resonance.
Begin with solution $\mathcal{H}_{\underline{\omega}}(0)\xi = E\xi$ where ξ grows polynomially.
Step 1: Given N large find box of size in $[N, N^C]$ for which the annulus Σ from above centered at the origin is good. This gives a scale $\bar{N}$ such that

$$|\xi_n| \leq \exp(-\sigma \bar{N}) \quad \forall\ |n| = \bar{N}$$

which implies with $Q_{\bar{N}} = [-\bar{N}, \bar{N}]^d$ and with R the restriction operator to this box

$$\mathrm{dist}(E, \mathrm{spec}(RH_{\underline{\omega}}(0)R)) < \exp(-\sigma \bar{N}) \tag{10}$$

This means that we can replace E with one of the eigenvalues of the finite volume operator.

Step 2: With E as in (10) above, we need to throw out those $\underline{\omega}$ for which we *do not have the following* for $x = 0$: for all $N^C < |k| < N^{C'}$

$$\begin{aligned} \|G_N(E; k\underline{\omega})\| &< e^{N^\rho} \\ |G_N(E; k\underline{\omega})(n, n')| &< e^{-N} \quad \forall |n - n'| > N/10 \end{aligned} \tag{11}$$

This is precisely the double resonance condition. Consider set

$$\mathcal{S} = \Big\{ (\underline{\omega}, E, x) \in \Omega_N \times \mathbb{R} \times \mathbb{T}^d \,\big|\, \det(H_{\underline{\omega}}(0), E) = 0, \text{ and } \sum_{n,n'} |G_N(\underline{\omega}, E, x)(n, n')|^2 > e^{2N^\rho} \text{ or } \sum_{|n-n'|>N/10} |G_N(\underline{\omega}, E, x)(n, n')|^2 > e^{-2N} \Big\}$$

This satisfies two main conditions on complexity and measure.

- Semi-algebraic of degree N^C. Can project away energy E and this stays the same.
- Measure along each one-dimensional slice is small, on the order of $e^{-N^{\varepsilon}}$. In particular, entire measure is small, as well as that of any other higher-dimensional slices.

Now use the semi-algebraic decomposition from above, as well as the steep lines to eliminate the problematic frequencies:

$$(\underline{\omega}, k\underline{\omega}) \in \mathrm{Proj}_{\underline{\omega},x}(\mathcal{S})$$

No new ideas involved at this stage.

Summary of Lecture 4

- On lattices $\mathbb{Z}^d$ rely on (i) resolvent identity (ii) matrix-valued Cartan to get exponential estimate on the probability of having a bad Green function at fixed energy. Induction on scales, method is perturbative.
- Challenge: control the Riesz mass in Cartan estimate, i.e., number of resonant sites at the lower scale. Need sub-linear bound in diameter at large scale.
- In $\mathbb{Z}^2$ this was done by an arithmetic elimination process - higher dimensions? Does not depend on the potential.
- In higher dimensions we can use (i) semi-algebraic elimination of variables (ii) method of steep lines or planes to insure that no long chains (but of fixed length depending only on the dimension) of bad sites can form.
- Ultimately this gives the sub-linear bound on the number of bad sites.
- Elimination of energy then fairly standard, leading to Anderson Localization.

References

See Lecture 1