Center Manifolds and Hamiltonian Evolution Equations

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An overview

- Equations: focusing nonlinear Klein-Gordon, Schrödinger, critical wave
- Review of local well-posedness theory, global existence vs. finite-time blowup. Forward scattering set $\mathcal{S}_+$
- Stationary solutions, ground states, variational analysis
- Some questions about $\mathcal{S}_+$, and some answer
- Payne-Sattinger theory: global dynamics below the ground state energy, functionals J and K.
- Raising the bar: energies above the ground state energy.
- Stable, Unstable, Center manifolds
- Hyperbolic dynamics, ejection lemma
- One-pass theorem, absence of almost homoclinic orbits
- Conclusion

Introduction

Energy subcritical equations:

$$\Box u + u = |u|^{p-1} u \;\; \text{in} \;\; \mathbb{R}^{1+1}_{t,x}(\text{even}), \mathbb{R}^{1+3}_{t,x}$$
$$i\partial_t u + \Delta u = |u|^2 u \;\; \text{in radial} \;\; \mathbb{R}^{1+3}_{t,x}$$

Energy critical case:

$$\Box u = |u|^{2^*-2} u \;\; \text{in radial} \;\; \mathbb{R}^{1+d}_{t,x} \tag{1}$$

$d = 3, 5.$

Goals: Describe transition between blowup/global existence and scattering, "Soliton resolution conjecture". Results apply only to the case where the energy is at most slightly larger than the energy of the "ground state soliton".

Basic well-posedness, focusing cubic NLKG in $\mathbb{R}^3$

$\forall\ u[0] \in \mathcal{H}$ there $\exists!$ strong solution $u \in C([0, T); H^1)$, $\dot{u} \in C^1([0, T); L^2)$ for some $T \geq T_0(\|u[0]\|_{\mathcal{H}}) > 0$. Properties: continuous dependence on data; persistence of regularity; energy conservation:

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4\Big)\, dx$$

If $\|u[0]\|_{\mathcal{H}} \ll 1$, then global existence; let $T^* > 0$ be maximal forward time of existence: $T^* < \infty \implies \|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3))} = \infty$. If $T^* = \infty$ and $\|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3))} < \infty$, then u scatters: $\exists\,(\tilde{u}_0, \tilde{u}_1) \in \mathcal{H}$ s.t. for $v(t) = S_0(t)(\tilde{u}_0, \tilde{u}_1)$ one has

$$(u(t), \dot{u}(t)) = (v(t), \dot{v}(t)) + o_{\mathcal{H}}(1) \quad t \to \infty$$

$S_0(t)$ free KG evol. If u scatters, then $\|u\|_{L^3([0,\infty),L^6(\mathbb{R}^3))} < \infty$. Finite prop.-speed: if $\vec{u} = 0$ on $\{|x - x_0| < R\}$, then $u(t, x) = 0$ on $\{|x - x_0| < R - t, 0 < t < \min(T^*, R)\}$.

Finite time blowup, forward scattering set

$T > 0$, exact solution to cubic NLKG

$$\varphi_T(t) \sim c(T-t)^{-\alpha} \qquad \text{as} \quad t \to T_+$$

$\alpha = 1$, $c = \sqrt{2}$.
Use finite prop-speed to cut off smoothly to neighborhood of cone $|x| < T - t$. Gives smooth solution to NLKG, blows up at $t = T$ or before.
Small data: global existence and scattering. **Large data:** can have finite time blowup.
Is there a criterion to decide finite time blowup/global existence?
Forward scattering set: $S(t)$ = nonlinear evolution

$$\mathcal{S}_+ := \Big\{(u_0, u_1) \in \mathcal{H} := H^1 \times L^2 \mid u(t) := S(t)(u_0, u_1) \,\exists\, \forall \text{ times}$$
$$\text{and scatters to zero, i.e.,} \quad \|u\|_{L^3([0,\infty);L^6)} < \infty\Big\}$$

$\mathcal{S}_+$ satisfies the following properties:

- $\mathcal{S}_+ \supset B_\delta(0)$, a small ball in $\mathcal{H}$,
- $\mathcal{S}_+ \neq \mathcal{H}$,
- $\mathcal{S}_+$ is an open set in $\mathcal{H}$,
- $\mathcal{S}_+$ is path-connected.

Some natural questions:

1. Is $\mathcal{S}_+$ bounded in $\mathcal{H}$?
2. Is $\partial\mathcal{S}_+$ a smooth manifold or rough?
3. If $\partial\mathcal{S}_+$ is a smooth mfld, does it separate regions of FTB/GE?
4. Dynamics starting from $\partial\mathcal{S}_+$? Any special solutions on $\partial\mathcal{S}_+$?

Stationary solutions, ground state

Stationary solution $u(t,x) = \varphi(x)$ of NLKG, weak solution of

$$-\Delta\varphi + \varphi = \varphi^3 \tag{2}$$

Minimization problem

$$\inf\left\{\|\varphi\|_{H^1}^2 \mid \varphi \in H^1,\ \|\varphi\|_4 = 1\right\}$$

has radial solution $\varphi_\infty > 0$, decays exponentially, $\varphi = \lambda\varphi_\infty$ satisfies (2) for some $\lambda > 0$.
Coffman: **unique ground state** Q.
Minimizes *the* *stationary energy (or action)*

$$J(\varphi) := \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 - \frac{1}{4}|\varphi|^4\right) dx$$

amongst all nonzero solutions of (2). Dilation functional:

$$K_0(\varphi) = \langle J'(\varphi)|\varphi\rangle = \int_{\mathbb{R}^3} (|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4)(x)\, dx$$

Some answers

Theorem

Let $E(u_0, u_1) < E(Q, 0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{\mathrm{rad}}$. In $t \geq 0$ for NLKG:

1. *finite time blowup*
2. *global existence and scattering to 0*
3. *global existence and scattering to Q:*
   $u(t) = Q + v(t) + O_{H^1}(1)$ *as* $t \to \infty$, *and*
   $\dot{u}(t) = \dot{v}(t) + O_{L^2}(1)$ *as* $t \to \infty$, $\Box v + v = 0$, $(v, \dot{v}) \in \mathcal{H}$.

All 9 combinations of this trichotomy allowed as $t \to \pm\infty$.

- Applies to dim $= 3$, cubic power, or dim $= 1$, all $p > 5$.
- Under *energy assumption* (EA) $\partial\mathcal{S}_+$ is connected, smooth mfld, which gives (3), separating regions (1) and (2). $\partial\mathcal{S}_+$ contains $(\pm Q, 0)$. $\partial\mathcal{S}_+$ forms the **center stable manifold** associated with $(\pm Q, 0)$.
- $\exists$ 1-dimensional stable, unstable mflds at $(\pm Q, 0)$. Stable mfld: Duyckaerts-Merle, Duyckaerts-Holmer-Roudenko

Hyperbolic dynamics

$\dot{x} = Ax + f(x)$, $f(0) = 0$, $Df(0) = 0$, $\mathbb{R}^n = X_s + X_u + X_c$, A-invariant spaces, $A \restriction X_s$ has evals in $\mathrm{Re}\, z < 0$, $A \restriction X_u$ has evals in $\mathrm{Re}\, z > 0$, $A \restriction X_c$ has evals in $i\mathbb{R}$.
If $X_c = \{0\}$, Hartmann-Grobman theorem: conjugation to e^{tA}.

If $X_c \neq \{0\}$, **Center Manifold Theorem:** $\exists$ local invariant mflds around $x = 0$, tangent to X_u, X_s, X_c.

$$X_s = \{|x_0| < \varepsilon \mid x(t) \to 0 \text{ exponentially fast as } t \to \infty\}$$
$$X_u = \{|x_0| < \varepsilon \mid x(t) \to 0 \text{ exponentially fast as } t \to -\infty\}$$

Example:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x + O(|x|^2)$$

$\mathrm{spec}(A) = \{1, -1, i, -i\}$

Hyperbolic dynamics near $\pm Q$

Linearized operator $L_+ = -\Delta + 1 - 3Q^2$.

- $\langle L_+ Q | Q \rangle = -2\|Q\|_4^4 < 0$
- $L_+\rho = -k^2\rho$ unique negative eigenvalue, no kernel over radial functions
- Gap property: L_+ has no eigenvalues in $(0,1]$, no threshold resonance (delicate!)

Plug $u = Q + v$ into cubic NLKG:

$$\ddot{v} + L_+ v = N(Q, v) = 3Qv^2 + v^3$$

Rewrite as a Hamiltonian system:

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(Q, v) \end{pmatrix}$$

Then $\mathrm{spec}(A) = \{k, -k\} \cup i[1, \infty) \cup i(-\infty, -1]$ with $\pm k$ simple evals. Formally: $X_s = P_1 L^2$, $X_u = P_{-1} L^2$. X_c is the rest.

The invariant manifolds

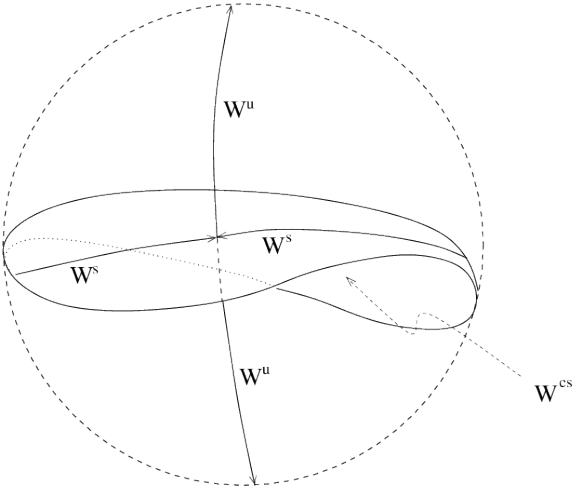

Figure: Stable, unstable, center-stable manifolds

Variational characterization

$$\begin{aligned} J(Q) &= \inf\{J(\varphi) \mid \varphi \in H^1 \setminus \{0\},\ K_0(\varphi) = 0\} \\ &= \inf\{J(\varphi) - \frac{1}{4}K_0(\varphi) \mid \varphi \in H^1 \setminus \{0\},\ K_0(\varphi) \le 0\} \end{aligned} \tag{3}$$

Note: if minimizer $\exists\, \varphi_\infty \ge 0$ (radial), then Euler-Lagrange: $J'(\varphi_\infty) = \lambda K_0'(\varphi_\infty)$, $K_0(\varphi_\infty) = 0$. So

$$0 = K_0(\varphi_\infty) = \langle J'(\varphi_\infty)|\varphi_\infty\rangle = \lambda\langle K_0'(\varphi_\infty)|\varphi_\infty\rangle = -2\lambda\|\varphi_\infty\|_4^4$$

$\lambda = 0 \Longrightarrow J'(\varphi_\infty) = 0 \Longrightarrow \varphi_\infty = Q$.

- Energy near $\pm Q$ a "saddle surface": $x^2 - y^2 \le 0$
- Better analogy $q(\xi) = -\xi_0^2 + \sum_{j=1}^\infty \xi_j^2$ in $\ell^2(\mathbb{Z}_0^+)$, "needle like"
- Similar picture for $E(u, \dot{u}) < J(Q)$. Solution trapped by $K \ge 0$, $K < 0$ in that set.

Schematic depiction of J, K_0

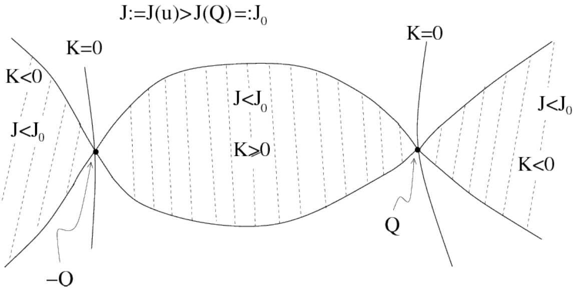

Figure: The splitting of $J(u) < J(Q)$ by the sign of $K = K_0$

- Energy near $\pm Q$ a "saddle surface": $x^2 - y^2 \leq 0$
- Better analogy $q(\xi) = -\xi_0^2 + \sum_{j=1}^{\infty} \xi_j^2$ in $\ell^2(\mathbb{Z}_0^+)$, "needle like"
- Similar picture for $E(u, \dot{u}) < J(Q)$. Solution trapped by $K \geq 0$, $K < 0$ in that set.

Payne-Sattinger theory I

$j_\varphi(\lambda) := J(e^\lambda \varphi)$, $\varphi \neq 0$ fixed.

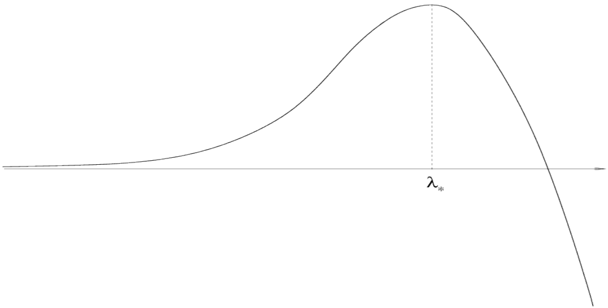

Figure: Payne-Sattinger well

Normalize so that $\lambda_* = 0$. Then $\partial_\lambda j_\varphi(\lambda)\big|_{\lambda=\lambda_*} = K_0(\varphi) = 0$.
"Trap" the solution in the well on the left-hand side: need
$E < \inf\{j_\varphi(0) \mid K_0(\varphi) = 0, \varphi \neq 0\} = J(Q)$ (lowest mountain pass).
Expect global existence in that case.

Invariant decomposition of $E < J(Q)$:

$$\mathcal{PS}_+ := \{(u_0, u_1) \in \mathcal{H} \mid E(u_0, u_1) < J(Q),\ K_0(u_0) \geq 0\}$$
$$\mathcal{PS}_- := \{(u_0, u_1) \in \mathcal{H} \mid E(u_0, u_1) < J(Q),\ K_0(u_0) < 0\}$$

In $\mathcal{PS}_+$ **global existence** in $\mathbb{R}$: $K_0(u(t)) \geq 0$ implies

$$\|u(t)\|_4^4 \leq \|u(t)\|_{H^1}^2 \Longrightarrow E \geq \frac{1}{4}\|u(t)\|_{H^1}^2 + \frac{1}{2}\|\dot{u}(t)\|_2^2 \simeq E$$

In $\mathcal{PS}_-$ finite time blowup in **both** positive and negative times.
Convexity argument: $y(t) := \|u(t)\|_{L^2}^2$ satisfies $K_0(u(t)) < -\delta$,

$$\begin{aligned}\ddot{y} &= 2[\|\dot{u}\|_2^2 - K_0(u(t))] \\ &= 6\|\dot{u}\|_2^2 - 8E(u, \dot{u}) + 2\|u\|_{H^1}^2\end{aligned}$$
$$\partial_{tt}(y^{-\frac{1}{2}}) = -\frac{1}{2}y^{-\frac{5}{2}}\left[y\ddot{y} - \frac{3}{2}\dot{y}^2\right] < 0$$

So finite time blowup.

Corollary: Q *unstable*.

$v_j = \lambda_j \rho + w_j$, $j = 0, 1$, $w_j \perp \rho$, $\omega = \sqrt{L_+ P_\rho^\perp}$

$$E(Q + v_0, v_1) = J(Q) + \frac{1}{2}(\langle L_+ v_0 | v_0 \rangle + \|v_1\|_2^2) + O(\|v_0\|_{H^1}^3)$$
$$= J(Q) + \frac{1}{2}(\lambda_1^2 - k^2 \lambda_0^2) + \frac{1}{2}(\|\omega w_0\|_2^2 + \|w_1\|_2^2) + O(\|v_0\|_{H^1}^3)$$
$$K_0(Q + v_0) = -2\langle Q^3 | v_0 \rangle + O(\|v_0\|_{H^1}^2)$$

Specialize: $v_0 = \varepsilon \rho$, $v_1 = 0$:

$$E(Q + v_0, 0) = J(Q) - \frac{k^2}{2}\varepsilon^2 + O(\varepsilon^3) < J(Q)$$
$$K_0(Q + v_0) = -2\varepsilon \langle Q^3 | \rho \rangle + O(\varepsilon^2)$$

So $\mathrm{sign}(K_0)$ determined by $\mathrm{sign}(\varepsilon)$.

Numerical 2-dim section through $\partial\mathcal{S}_+$ (with R. Donninger)

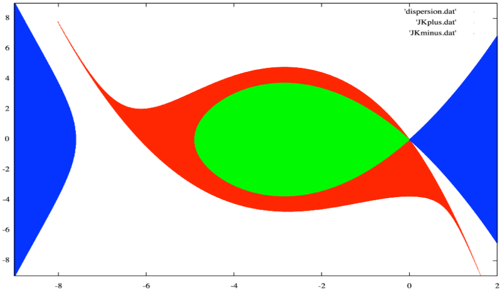

Figure: $(Q + Ae^{-r^2}, Be^{-r^2})$

- soliton at $(A, B) = (0, 0)$, (A, B) vary in $[-9, 2] \times [-9, 9]$
- RED: global existence, WHITE: finite time blowup, GREEN: $\mathcal{PS}_-$, BLUE: $\mathcal{PS}_+$
- Our results apply to a neighborhood of $(Q, 0)$, boundary of the red region looks smooth (caution!)

Solve NLKG with $u = \pm(Q+v) \to \ddot{v} + L_+ v = N(Q,v) \to$

$$\dot{\lambda}_+ - k\lambda_+ = \frac{1}{2k} N_\rho(Q,v) \tag{4}$$

$$\dot{\lambda}_- + k\lambda_- = -\frac{1}{2k} N_\rho(Q,v) \tag{5}$$

$$\ddot{\gamma} + L_+\gamma = P_\rho^\perp N(Q,v) \tag{6}$$

$P_\rho N(Q,v) = N_\rho(Q,v)\rho$, $v = \lambda\rho + \gamma$. ODE $\ddot{\lambda} - k^2\lambda = N_\rho(Q,v)$ is diagonalized by

$$\lambda_\pm = \frac{1}{2}(\lambda \pm k^{-1}\dot{\lambda})$$

(4) corresponds to eval k of $A = \begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix}$; (5) eval $-k$; (6) to essential spectrum $iR \setminus (-i,i)$ of A. **"Stabilize" exponential growth** in (4): if $N_\rho \equiv 0$, means $\lambda_+(0) = 0$. In general:

Solving the system (4)-(6)

Stability condition:

$$0 = \lambda_+(0) + \frac{1}{2k}\int_0^\infty e^{-sk} N_\rho(Q,v)(s)\,ds \tag{7}$$

yields (recall $v = \lambda\rho + \gamma$)

$$\lambda(t) = e^{-kt}\left[\lambda(0) + \frac{1}{2k}\int_0^\infty e^{-ks} N_\rho(s)\,ds\right] + \frac{1}{2k}\int_0^\infty e^{-k|t-s|} N_\rho(s)\,ds$$
$$\ddot\gamma + L_+\gamma = P_\rho^\perp N$$

Solve via Strichartz estimates for $\partial_{tt} + L_+$. **Conclusion:**
$\exists \mathcal{M} \ni (\pm Q, 0)$ small smooth, codim 1 mfld, $(u_0, u_1) \in \mathcal{M} \Rightarrow$
$u = Q + v + o_{H^1}(1)$ as $t \to \infty$, v free KG wave, $\mathcal{M}$ parametrized by $(\lambda(0), \gamma_\infty(0))$, where γ_∞ is the scattering solution of γ. *Energy partition:* $E(u,\dot u) = J(Q) + E_0(\gamma_\infty, \dot\gamma_\infty)$ $\quad \mathcal{M}$ *unique*: if u
$\exists \forall\ t \geq 0$, $\mathrm{dist}((u,\dot u),(\pm Q,0))$ small $\forall\ t \geq 0$, $\Rightarrow (u,\dot u) \in \mathcal{M}$.

If $(u, \dot{u}) \to (Q, 0)$ as $t \to \infty$, then $E(\vec{u}) = J(Q) \Rightarrow \gamma_\infty \equiv 0$. So $\vec{u}$ parametrized by $\lambda(0)$.

Three cases: $\lambda > 0$, $\lambda \equiv 0$, $\lambda < 0$.

Main (λ, γ)-system $\Rightarrow \lambda(t)$ decays exponentially as $t \to \infty$.

Duyckaerts-Merle type solutions: $W_\pm(t - t_0)$.

as $t \to -I$, W_+ blows up in finite time, W_- scatters to 0.

Remark: Construction more involved in the presence of symmetries (non-radial NLKG, radial or nonradial NLS). **Beceanu's linear estimates**: $\mathcal{H} = \mathcal{H}_0 + V$ matrix NLS Hamiltonian, $Z = P_c Z$,

$$\mathcal{H} = \begin{pmatrix} \Delta - \mu & 0 \\ 0 & -\Delta + \mu \end{pmatrix} + \begin{pmatrix} W_1 & W_2 \\ -W_2 & W_1 \end{pmatrix}$$

$$i\partial_t Z - iv(t)\nabla Z + A(t)\sigma_3 Z + \mathcal{H}Z = F, \; Z(0) \text{ given},$$

$\|A\|_\infty + \|v\|_\infty < \epsilon$, no eigenvalues or resonances of $\mathcal{H}$ in $(-\infty, -\mu] \cup [\mu, \infty)$. Then

$$\|Z\|_{L_t^\infty L_x^2 \cap L_t^2 L_x^{6,2}} \leq C\Big(\|Z(0)\|_2 + \|F\|_{L_t^1 L_x^2 + L_t^2 L_x^{6/5,2}}\Big)$$

Unstable dynamics off the center-stable mfld $\mathcal{M}$

$\mathcal{M}$ is **repulsive** (restatement of uniqueness of $\mathcal{M}$).

Goal: *Stabilize* $\mathrm{sign}(K_0(u(t))), \mathrm{sign}(K_2(u(t)))$. *Virial functional:*

$K_2(u) = \langle J'(u)|Au\rangle = \partial_\lambda|_{\lambda=0} J(e^{\frac{3\lambda}{2}}u(e^\lambda\cdot)),\ A = \frac{1}{2}(x\cdot\nabla + \nabla\cdot x)$,

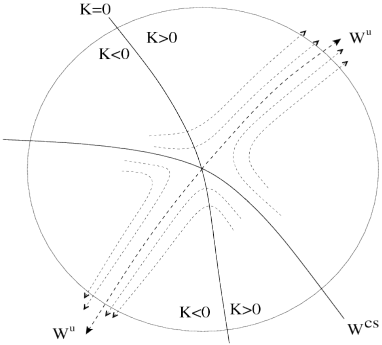

Figure: Sign of $K = K_0$ upon exit

"Stabilize": $u(t)$ defined on $[0, T_*)$, then $\mathrm{sign}(K(u(t)) \geq 0$ or < 0 on (T_{**}, T_*).

Ejection of trajectories along unstable mode

Lemma (Ejection Lemma)

$\exists\, 0 < \delta_X \ll 1$ *s.t.: $u(t)$ local solution of NLKG3 on* $[0, T]$ *with*

$$R := d_Q(\vec{u}(0)) \le \delta_X, \quad E(\vec{u}) < J(Q) + R^2/2$$

and for some $t_0 \in (0, T)$, one has the ejection condition:

$$d_Q(\vec{u}(t)) \ge R \quad (0 < \forall t < t_0). \tag{8}$$

Then $d_Q(\vec{u}(t)) \nearrow$ until it hits δ_X, and

$$\begin{aligned} & d_Q(\vec{u}(t)) \simeq -\mathfrak{s}\lambda(t) \simeq -\mathfrak{s}\lambda_+(t) \simeq e^{kt} R, \\ & |\lambda_-(t)| + \|\vec{\gamma}(t)\|_E \lesssim R + d_Q^2(\vec{u}(t)), \\ & \min_{s=0,2} \mathfrak{s} K_s(u(t)) \gtrsim d_Q(\vec{u}(t)) - C_* d_Q(\vec{u}(0)), \end{aligned}$$

for either $\mathfrak{s} = +1$ or $\mathfrak{s} = -1$.

Variational structure above $J(Q)$ (Noneffective!)

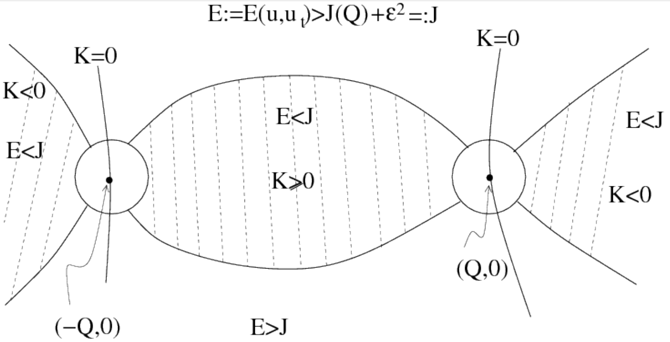

Figure: Signs of $K = K_0$ away from $(\pm Q, 0)$

$\forall\, \delta > 0 \;\; \exists\, \varepsilon_0(\delta), \kappa_0, \kappa_1(\delta) > 0$ s.t. $\forall \vec{u} \in \mathcal{H}$ with
$E(\vec{u}) < J(Q) + \varepsilon_0(\delta)^2, \quad d_Q(\vec{u}) \geq \delta$, one has following dichotomy:

$$K_0(u) \leq -\kappa_1(\delta) \;\text{ and }\; K_2(u) \leq -\kappa_1(\delta), \qquad \text{or}$$

$$K_0(u) \geq \min(\kappa_1(\delta), \kappa_0 \|u\|_{H^1}^2) \;\text{ and }\; K_2(u) \geq \min(\kappa_1(\delta), \kappa_0 \|\nabla u\|_{L^2}^2).$$

Crucial no-return property: Trajectory does **not return to balls around** $(\pm Q, 0)$. Suppose it did; Use *virial identity*

$$\partial_t \langle w\dot{u} | Au \rangle = -K_2(u(t)) + \text{error}, \quad A = \frac{1}{2}(x\nabla + \nabla x) \qquad (9)$$

where $w = w(t,x)$ is a space-time cutoff that lives on a rhombus, and the "error" is controlled by the external energy.

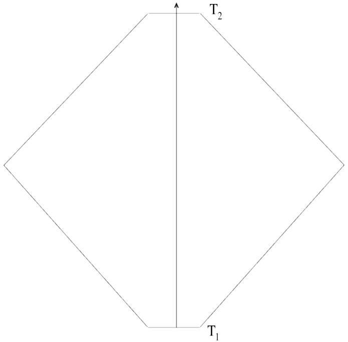

Figure: Space-time cutoff for the virial identity

One-pass theorem II

Finite propagation speed $\Rightarrow$ error controlled by free energy outside large balls at times T_1, T_2.

Integrating between T_1, T_2 gives contradiction; the **bulk** of the integral of $K_2(u(t))$ here comes from exponential ejection mechanism near $(\pm Q, 0)$.

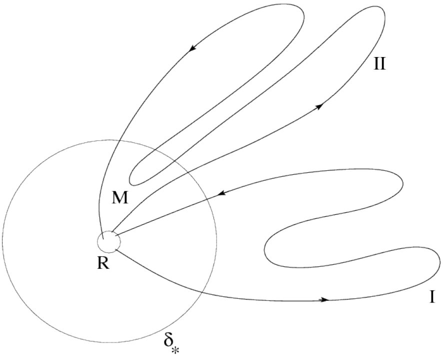

Figure: Possible returning trajectories

After integration of virial:

$$\langle w\dot{u}|Au\rangle\Big|_{T_1}^{T_2} = \int_{T_1}^{T_2} [-K_2(u(t)) + \mathrm{error}]\, dt$$

where T_1, T_2 are exit, and first re-entry times into R-ball.
Left-hand side: absolute value

$$\lesssim R + SR^2 \lesssim R \quad \text{inner radius}$$

were $S \simeq |\log R|$ size of base ($Q \ll R$ outside that ball).
Right-hand side: lower bound on $|K_2(u(t))|$ outside δ_*-ball by variational lemma.
Exponentially increasing dynamics gives

$$\int_{T_1}^{T_1^*} |K_2(u(t))|\, dt \gtrsim \delta_* \quad \text{outer radius}$$

where T_1^* exit-time from δ_*-ball

One-pass theorem IV

Some further issues:

- For trajectories of type I, this argument works; for type II, use ejection lemma at minimum point M.
- In the $K(u(t)) < 0$ region the above argument is sufficient, since error can be made small compared to $\kappa(\delta_*)$ by taking R small (and thus S large).
- In the $K(u(t)) \geq 0$ case, one has a possible complication due to $\int_{T_1}^{T_2} \|\nabla u(t)\|_2^2 \, dt$ being too small. In that case error becomes a problem (since we have no control over $T_2 - T_1$).
- Overcome that by showing $\exists\, \mu_0 > 0$ s.t.: if for some $\mu \in (0, \mu_0]$

$$\|\vec{u}\|_{L_t^\infty(0,2;\mathcal{H})} \leq M, \quad \int_0^2 \|\nabla u(t)\|_{L^2}^2 \, dt \leq \mu^2$$

then u exists globally and scatters to 0 as $t \to \pm\infty$, $\|u(t)\|_{L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^3)} \ll \mu^{1/6}$.

Further results I

- Nonradial NLKG3: use relativistic energy (Lorentz invariant)

$$E_m(\vec{u})^2 = E(\vec{u})^2 - |P(\vec{u})|^2$$

  where $P(\vec{u})$ is the conserved momentum. This works if $|E| > |P|$, the other case being reduced to Payne-Sattinger. For the orbital stability form of 9-set theorem restrict to normalized solutions, i.e., with $P(\vec{u}) = 0$. Center-stable mflds: Instead of Q, need to work with 6-parameter family of ground states (translated, "boosted"). Q gets squashed by Lorentz contraction. Need a variant of Beceanu's linear dispersive estimates.
- NLS equation: only radial; two modulation parameters for Q: phase, mass $e^{i\alpha^2 t+\gamma}\alpha Q(\alpha x)$. We "mod out" these symmetries (at least for the orbital stability part which does not involve the center-stable manifold); α is controlled by the mass of the solution, for the phase write $u = e^{i\theta}(Q + v)$.

Further results II

- NLS equation: Major difference in the one-pass theorem from NLKG: absence of finite propagation speed. So crucial virial argument is different; no time-dependent cutoffs. $K(u(t)) < 0$ case (for blowup and one-pass theorem) treated by a variant of the Ogawa-Tsutsumi argument. More difficult to treat $K(u(t)) \geq 0$. Use the following Morawetz identity due to Nakanishi, 1999:

$$\partial_t \langle u | \frac{t}{4\lambda} u + i \frac{r}{2\lambda} u_r \rangle$$
$$= \int_{\mathbb{R}^3} \left\{ \frac{t^2}{\lambda^3} |\nabla M u|^2 - \frac{|u|^4}{4} \left[\frac{2}{\lambda} + \frac{t^2}{\lambda^3} \right] + \frac{15 t^4}{4\lambda^7} |u|^2 \right\} dx,$$

where $\lambda := \sqrt{t^2 + r^2}$ and $M := e^{i|x|^2/(4t)}$. Right-hand side can be rewritten in terms of $K(u) = \|\nabla u\|_2^2 - \frac{1}{4}\|u\|_4^4$ and expressions which are integrable in time.

Critical wave equation I

$$\ddot{u} - \Delta u = |u|^{2^*-2}u, \quad u(t,x) : \mathbb{R}^{1+d} \to \mathbb{R}, \quad 2^* = \frac{2d}{d-2} \ (d = 3 \text{ or } 5),$$

Static Aubin, Talenti solutions

$$W_\lambda = T_\lambda W, \quad W(x) = \left[1 + \frac{|x|^2}{d(d-2)}\right]^{1-\frac{d}{2}},$$

T_λ is $\dot{H}^1$ preserving dilation

$$T_\lambda \varphi = \lambda^{d/2-1}\varphi(\lambda x)$$

Positive radial solutions of the static equation

$$-\Delta W - |W|^{2^*-2}W = 0$$

Variational structure:

$$J(\varphi) := \int_{\mathbb{R}^d} \left[\frac{1}{2}|\nabla\varphi|^2 - \frac{1}{2^*}|\varphi|^{2^*}\right] dx$$

$$K(\varphi) := \int_{\mathbb{R}^d} [|\nabla\varphi|^2 - |\varphi|^{2^*}]\, dx$$

Critical wave equation II

Radial $\dot{H}^1 \times L^2$, $E(\vec{\varphi}) < J(W) + \varepsilon^2$, outside soliton tube

$$\{\pm \vec{W}_\lambda \mid \lambda > 0\} + O(\varepsilon)$$

There exists four open disjoint sets which lead to all combinations of FTB/GE and scattering to 0 as $t \to \pm I$.

NOTE:

- We do not have a complete description of all solutions with energy $E(\vec{\varphi}) < J(W) + \varepsilon^2$.
- We do not know if the center-stable manifold exists in $\dot{H}^1 \times L^2$ (but in 05 Krieger-S. showed that there is such an object in a stronger non-invariant topology).
- Inside the soliton tube there exist blowup solutions, as found by Krieger-S.-Tataru. Duykaerts-Kenig-Merle showed that all type II blowup are of the KST form, as long as energy only slightly above $J(Q)$. So trapping by the soliton tube cannot mean scattering to $\{W_\lambda\}$ as it did in the subcritical case.