UNSTABLE QUASI-GEODESICS IN TEICHMÜLLER SPACE

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In this note we will construct an example of a quasi-geodesic in the Teichmüller space of a surface with the following instability property:

Theorem *For any hyperbolic surface S whose Teichmüller space $\mathcal{T}(S)$ has dimension at least 4, there is a bi-infinite quasi-geodesic L in $\mathcal{T}(S)$ such that*

1. *L projects to a compact part of the moduli space $\mathcal{M}(S)$, but*
2. *L does not lie in a bounded neighborhood of any geodesic.*

(Recall that a quasi-geodesic, or (K,ϵ)-quasi-geodesic for $K>1, \epsilon>0$, is a path for which the length ℓ and distance d between endpoints of any subsegment satisfy $\ell \leq Kd+\epsilon$.)

This theorem answers in the negative a question asked us by M. Mitra (see [7] for a related sharper question), and should be contrasted with Theorem 4.2 from [6], which gives the following stability property: Given a fixed compact set $\mathcal{C} \subset \mathcal{M}(S)$ and $K>1, \epsilon>0$, there is a constant R so that, if G is any geodesic whose projection to $\mathcal{M}(S)$ lies in $\mathcal{C}$, and if L is any (K,ϵ)-quasi-geodesic joining two points of G, then L lies in an R-neighborhood of G. This stability property holds, by comparison, in hyperbolic spaces for *all* geodesics G, but in Teichmüller space it fails if G does not project to a compact set.

This suggests that "in the thick part of Teichmüller space things are essentially hyperbolic". Our theorem, on the other hand, shows that the roles of the geodesic and the quasi-geodesic cannot be reversed, even in the thick part, and hence that this stability is somewhat delicate.

Note of course that when $dim(\mathcal{T}(S))=2$ the theorem is false: $\mathcal{T}(S)$ is then the hyperbolic plane, in which all quasi-geodesics are uniformly close to geodesics.

We refer the reader to [4] for basic notions of Teichmüller theory; to [2] and [3] for material on Thurston's measured lamination (or foliation) spaces and on pseudo-Anosov homeomorphisms; and to [1] for more on quasi-geodesics and their stability properties in hyperbolic spaces.

Proof

Fix any essential proper subdomain Y of S, other than an annulus or a thrice-punctured sphere. By "essential" we mean that $\pi_1(Y)$ injects into $\pi_1(S)$. Let $h: S \to S$ be a homeomorphism which preserves Y, is the identity outside Y, and whose restriction to Y is in a pseudo-Anosov mapping class.

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The action of h on $\mathcal{T}(S)$ takes a conformal structure x to its pullback by h^{-1}.

Fix $x_0 \in \mathcal{T}(S)$ and let $x_n = h^n(x_0)$. Join x_0 to x_1 by a fixed path (say geodesic) p_0, let $p_n = h^n(p_0)$, and let L denote the infinite concatenation $\cdots * p_{-2} * p_{-1} * p_0 * p_1 * p_2 * \cdots$.

Clearly L projects to a compact subset of $\mathcal{M}(S)$, namely the image of p_0.

We first show that L is a (K, ϵ)-quasi-geodesic, for some fixed $K > 1$ and $\epsilon > 0$. Since each segment p_n has fixed length $|p_0|$, and since L is invariant under h, it suffices to show for some $C > 0$ that $d(x_0, x_n) \geq Cn$ for all $n > 0$. Let μ be the unstable lamination of h in its action on Y. Then there exists $\lambda > 1$ such that $h^{-n}(\mu) = \lambda^n \mu$. Let $\mathcal{ML}(S)$ denote Thurston's space of measured laminations on S. For any measured lamination ν let $E_x(\nu)$ denote the extremal length of ν on $x \in \mathcal{T}(S)$ – this was defined for all measured laminations by Kerckhoff [5] and generalizes the extremal length of a curve. In particular E_x is homogeneous of degree 2 in lamination space. Then

$$E_{x_n}(\mu) = E_{x_0}(h^{-n}(\mu)) = \lambda^{2n} E_{x_0}(\mu).$$

Kerckhoff also established in [5] that Teichmüller distance can be given as:

$$d(x,y) = \frac{1}{2} \log \sup_{\nu \in \mathcal{ML}(S)} \frac{E_y(\nu)}{E_x(\nu)}.$$

It follows immediately that

$$d(x_0, x_n) \geq n \log \lambda.$$

This establishes that L is a quasi-geodesic.

Now we show that L cannot stay in a bounded neighborhood of any geodesic. Fix $M > 0$ and let y_{-n} and y_n be any two points within M of x_{-n} and x_n respectively. Let G_n be the Teichmüller geodesic joining y_{-n} to y_n. Let Φ_n be the unit-area quadratic differential on y_{-n} associated to G_n. Let α be a boundary curve of Y.

We first claim that, for any unit-area quadratic differential Φ on any y_i within distance M of x_i, $\ell_\Phi(\alpha)$ is bounded uniformly above and below by positive constants, depending on M, where ℓ_Φ denotes minimal length of a homotopy class in the singular flat metric $|\Phi|$. Indeed, if Φ is any unit-area quadratic differential on a surface y_0 within distance M of x_0 then $\ell_\Phi(\alpha)$ is bounded above and below by positive constants, just by compactness of the set of unit-area quadratic differentials on a compact set of surfaces in Teichmüller space. Transporting y_0 to y_i by h^i and noting that $h(\alpha) = \alpha$, we obtain the desired estimate.

If $\ell_{\Phi,h}$ and $\ell_{\Phi,v}$ denote lengths in the horizontal and vertical directions of Φ respectively, then both of these are also bounded above for α by the previous estimate. Apply this to Φ_n on y_{-n}, and then to its image on y_n by the Teichmüller flow along G_n, which we call Φ'_n. On both of these we then get uniform upper bounds on the horizontal and vertical lengths of α. However, $\ell_{\Phi_n,h}$ increases exponentially along G_n, and $\ell_{\Phi_n,v}$ decreases exponentially. It follows that on the midpoint m_n of G_n, both are bounded above by ce^{-n} for a fixed c.

Thus the length of α on m_n with respect to the unit-area differential is going to zero as $n \to \infty$. This means m_n has to leave every compact set of the moduli space. For in any compact set of the moduli space there is a lower bound on ℓ_Φ over all unit-area quadratic differentials and simple closed curves.

It follows that G_n cannot remain in a bounded neighborhood of L for all n. Now, if L were in a bounded neighborhood of some geodesic G then one easily sees that G would be in a bounded neighborhood of L. An exhaustion of G by subsegments G_n would then lead to a contradiction. This establishes the theorem.

References

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