GEOMETRY OF THE COMPLEX OF CURVES II: HIERARCHICAL STRUCTURE

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Contents

1. Introduction

The complex of curves on a surface (Harvey [12]) is a finite dimensional and locally infinite complex $\mathcal{C}(S)$ associated to a surface S, which admits an action by the mapping class group $\mathrm{Mod}(S)$. The geometry and combinatorics of $\mathcal{C}(S)$ can be applied to study group-theoretic properties of $\mathrm{Mod}(S)$, and the geometry of Kleinian representations of $\pi_1(S)$.

Our goal in this paper is to develop tools for such applications, building on the results obtained in part I of this project [21]. We hasten to reassure the reader that this paper can be read independently of [21].

The vertices of $\mathcal{C}(S)$ are the homotopy classes of essential simple closed curves in S, and simplices correspond to k-tuples of disjoint curves (see Section 2 for more precise definitions). The resulting complex can be made into a metric space by letting each simplex be Euclidean of sidelength 1, and in [21] we showed that with this metric, $\mathcal{C}(S)$ is an infinite diameter δ-hyperbolic space in all but a small number of trivial cases. This result suggests that one try to apply the techniques of hyperbolic spaces and groups

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to study $\mathcal{C}(S)$ and its $\mathrm{Mod}(S)$-action, considering for example such questions as the word problem, conjugacy problem and quasi-isometric rigidity. The barrier to doing this is that the complex is locally infinite, and hence the distance bounds one obtains in a typical geometric argument give little a-priori information.

The structure that we will develop in the present paper is intended to help lift this barrier. The organizing philosophy is roughly this: Links of vertices in $\mathcal{C}(S)$ are themselves complexes associated to subsurfaces. The geometry of these links is tied to the geometry of $\mathcal{C}(S)$ by a family of *subsurface projection maps* (see also Ivanov [17, 16]), which are analogous to closest-point projections to horoballs in classical hyperbolic space. This gives a layered structure to the complex, with hyperbolicity at each level, and the main construction of our paper is a combinatorial device used to tie these levels together, which we call a *hierarchy of tight geodesics*.

Using these constructions, we derive a number of properties of $\mathcal{C}(S)$ which are similar to those of locally finite complexes, such as a finiteness result for geodesics with given endpoints (Theorem 6.14), and a convergence criterion for sequences of geodesics (Theorem 6.13). We then apply these ideas to study the conjugacy problem in $\mathrm{Mod}(S)$, deriving a linear bound on the shortest word conjugating two pseudo-Anosov mapping classes (Theorem 7.2). Along the way we describe a class of quasi-geodesic words in $\mathrm{Mod}(S)$ (Theorem 7.1), whose lengths can be estimated using the subsurface projection maps (Theorem 6.12).

The rest of Section 1 gives a more detailed outline of our results, and works through some explicit examples that motivate our constructions. Section 2 presents our definitions and notation, and proves some basic lemmas. Section 3 proves our fundamental result on subsurface projections, Sections 4 and 5 develop the machinery of hierarchies and their resolutions into sequences of markings, Section 6 proves our basic geometric control theorems, and Section 7 proves the conjugacy bound theorem for $\mathrm{Mod}(S)$. Section 8 is a brief remark on a variation of the hierarchy construction, suitable for considering elementary moves between pants decompositions rather than markings.

1.1. **Subsurface Projections.**

A basic analogy for thinking about $\mathcal{C}(S)$ is provided by the geometry of a family $\mathcal{F}$ of disjoint, uniformly spaced horoballs in $\mathbf{H}^n$, for example the uniform cusp horoballs of a Kleinian group. The non-proper metric space $X_{\mathcal{F}}$ obtained by collapsing each horoball to a point is itself δ-hyperbolic – see Farb [8] and Klarreich [18] – and the horoballs play a role similar to links of vertices in $\mathcal{C}(S)$.

If B is a horoball and L is a hyperbolic geodesic disjoint from B, then the closest-point projection of L to B has uniformly bounded diameter, independently of L or B. Interestingly, one can sensibly define a "projection" from the collapsed space $X_{\mathcal{F}}$ to B which similarly sends $X_{\mathcal{F}}$-geodesics avoiding B

to bounded sets. This turns out to be a crucial property in understanding the geometry of $X_{\mathcal{F}}$ and its relation to the geometry of $\mathbf{H}^n$.

In our context, vertices of $\mathcal{C}(S)$ are simple closed curves in S (see §2.1) and the link of a vertex v is closely related to the complexes $\mathcal{C}(Y)$ for the complementary subsurfaces Y of v. We will define projections π_Y from $\mathcal{C}(S)$ to $\mathcal{C}(Y)$ as follows: given a simple closed curve on S take its arcs of intersection with Y and perform a surgery on them to obtain closed curves in Y. (More precisely π_Y sends vertices in $\mathcal{C}(S)$ to finite sets in $\mathcal{C}(Y)$). We will prove the following analogue to the situation with horoballs:

Theorem 3.1 (Bounded Geodesic Image) *If Y is an essential subsurface of Z and g is a geodesic in $\mathcal{C}(Z)$ all of whose vertices intersect Y nontrivially, then the projected image of g in $\mathcal{C}(Y)$ has uniformly bounded diameter.*

The family $\mathcal{F}$ of horoballs also satisfies the closely related "bounded coset penetration property" of Farb [8], which roughly speaking is a stability property for paths in $\mathbf{H}^n$ whose images in $X_{\mathcal{F}}$ are quasi-geodesics: if two such paths begin and end near each other, then up to bounded error they penetrate through the same set of horoballs in the same way. This property does not hold in our case but a certain generalization of it does. This will be the content of Lemmas 6.2 (Large Link) and 6.6 (Common Links), which will be briefly discussed in §1.4 below.

1.2. **The conjugacy problem.** Fix a set of generators for $\mathrm{Mod}(S)$ and let $|\cdot|$ denote the word metric. As one application of our techniques, in Section 7 we establish the following bound:

Theorem 7.2 (Conjugacy Bound) *Fix a surface S of finite type and a generating set for $\mathrm{Mod}(S)$. If h_1, h_2 are words describing conjugate pseudo-Anosov elements, then the shortest conjugating element w has word length*

$$|w| \le C(|h_1| + |h_2|),$$

where the constant C depends only on S and the generating set.

This linear growth property for the shortest conjugating word is shared with word-hyperbolic groups (see Lysënok [20, Lemma 10]), although except in a few low-genus cases the mapping class group is not word hyperbolic since it contains abelian subgroups of rank at least 2 generated by Dehn twists about disjoint curves. Our proof is based on a proof which works in the word-hyperbolic case. The case of general elements of $\mathrm{Mod}(S)$ introduces complications similar to those that occur for torsion elements of word-hyperbolic groups. We hope to address the general case in a future paper.

This bound is related to the question of solubility of the conjugacy problem, since a computable bound on w provides a bounded search space for an algorithm seeking to establish or refute conjugacy. Hemion [15] proved that the conjugacy problem for $\mathrm{Mod}(S)$ is soluble, and Mosher [25] gave an explicit algorithm for determining conjugacy for pseudo-Anosovs. In both cases, no explicit bound on the complexity was given (although Mosher's algorithm is fast in practice). Theorem 7.2 is still short of a good complexity bound since we have not described an efficient way to search through the possible conjugating words. However, we are hopeful that the techniques of this paper can be extended to give a more complete algorithmic approach.

1.3. **Finiteness results.** In a locally finite graph, there are finitely many geodesics between any two points. In $\mathcal{C}(S)$ this is easily seen to be false even for geodesics of length 2. However we shall introduce a finer notion of *tight geodesic* (§4.1), for which we can establish:

Theorem 6.14 (Finite Geodesics) *Between any two vertices in $\mathcal{C}(S)$ there are finitely many tight geodesics.*

This is part of a collection of results showing that in several useful ways $\mathcal{C}(S)$ is like a locally finite complex. Another is Theorem 6.13 (Convergence of Hierarchies), which generalizes the property of a locally finite complex that any sequence of geodesics meeting a compact set has a convergent subsequence. In the locally finite setting this involves a simple diagonalization argument, and this is replaced here by an application of Theorem 3.1 and the hierarchy construction.

The following is an application of this result:

Proposition 7.6 (Axis) *Any pseudo-Anosov element $h \in \mathrm{Mod}(S)$ has a quasi-invariant axis in $\mathcal{C}(S)$: that is, a bi-infinite geodesic β such that $h^n(\beta)$ and β are 2δ-fellow travelers for all $n \in \mathbf{Z}$.*

That a *quasi-geodesic* exists which fellow-travels its h-translates is a consequence of work in [21]. The geodesic with this property is obtained by a limiting process using Theorem 6.13.

1.4. **Hierarchies of geodesics.** A geodesic in $\mathcal{C}(S)$ is a sequence of curves in S, but words in $\mathrm{Mod}(S)$ are more closely related to sequences of pants decompositions separated by elementary moves (replacement of one curve at a time). The hierarchy construction is based on the idea that a geodesic can be "thickened" in a natural way to give a family of pants decompositions. We will illustrate this in one of the simplest examples, that of the five-holed sphere $S_{0,5}$, below. We will then give a more general discussion of the construction and state some of our main results about it. Finally in §1.5 we will give a more extended, but still relatively simple, collection of examples.

A pants decomposition P in $S = S_{0,5}$ is a pair of disjoint curves α, β; i.e. an edge of $\mathcal{C}(S)$. An *elementary move* of pants $P \to P'$ fixes one of the curves, say α, and replaces β with a curve β' which intersects β minimally and is disjoint from α.

Now given $P = \{\alpha, \beta\}$, take some $\psi \in \mathrm{Mod}(S)$, and consider ways of connecting P to $\psi(P) = \{\alpha', \beta'\}$. Choose a geodesic in $\mathcal{C}(S)$ whose vertices are $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_N = \alpha'$. The subsurface $S \setminus \alpha_0$ has two components, a three-holed sphere, and a four-holed sphere which contains α_1 and β. Let

S_{α_0} denote the four-holed sphere. The complex of curves $\mathcal{C}(S_{\alpha_0})$ is isomorphic to the Farey graph (see §1.5), so let us join β to α_1 by a geodesic $\beta = \gamma_0, \gamma_1, \ldots, \gamma_m = \alpha_1$ in $\mathcal{C}(S_{\alpha_0})$. The transition (α_0, γ_i) to (α_0, γ_{i+1}) is an elementary move in pants. This path concludes with the pants decomposition $\{\alpha_0, \alpha_1\}$ (see Figure 1). Now working in S_{α_1} join α_0 to α_2 by a geodesic, giving a path of elementary moves ending with $\{\alpha_1, \alpha_2\}$. We repeat this procedure, eventually ending with $\psi(P_0)$.

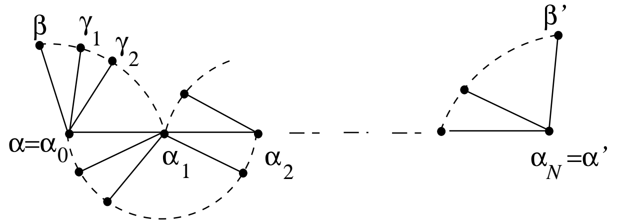

FIGURE 1. A sequence of pants decompositions in the five-holed sphere. Each edge represents a pants decomposition. Transitions between edges are elementary moves.

In this example, the same final pants decomposition $\psi(P_0)$ could have been written as $\psi(\theta(P_0))$ where θ is any product of Dehn twists around α and β. Thus in order to keep track of elements in the mapping class group, and not just pants decompositions, we will also need to keep track of twisting information around pants curves. In fact twisting data is implicit everywhere in this example: in the geodesic in $\mathcal{C}(S_{\alpha_0})$, for each $i \in (0, m)$, γ_{i-1} and γ_{i+1} both intersect γ_i minimally, and hence differ by a product of Dehn twists (or half-twists) about γ_i. Keeping track of this information will require the introduction of complexes associated to annular subsurfaces (see §2.4).

Ultimately we will be considering sequences of *complete markings*, which are pants decompositions together with twisting data, such that successive markings are separated by appropriately defined elementary moves. (More generally the markings need not be complete, but let us assume for the rest of this discussion that they are.) In considering such a sequence carefully, one finds in it segments where some sub-marking is fixed and all the elementary moves take place in a subsurface of S. Thus one obtains some interlocking structure of paths in subcomplexes of $\mathcal{C}(S)$.

Hierarchies of geodesics will be our method for constructing and manipulating such structures. Roughly, a hierarchy is a collection H of geodesics, each geodesic h contained in a complex $\mathcal{C}(Y)$ where Y is the *domain* of h. The geodesics will satisfy a technical condition called *tightness*, which makes them easier to control. There will be one "main geodesic" whose domain is all of S, and in general the geodesics will interlock via a relation called "subordinacy", which is related to the nesting of their domains. There will be an *initial* and a *terminal* marking, called $\mathbf{I}(H)$ and $\mathbf{T}(H)$, and a partial order on the geodesics which is related to the linear order of a sequence of elementary moves connecting $\mathbf{I}(H)$ to $\mathbf{T}(H)$ (the reason for a partial rather than linear order is that some elementary moves commute because they take place in disjoint subsurfaces).

Any hierarchy will admit a *resolution* of its partial order to a linearly ordered sequence of markings, separated by elementary moves, connecting $\mathbf{I}(H)$ to $\mathbf{T}(H)$. This resolution will be nonunique, but efficient in the following sense. Let $\widetilde{\mathcal{M}}$ be the graph whose vertices are complete markings and whose edges are elementary moves. We then have:

Theorem 6.10 (Efficiency of Hierarchies) *Any resolution of a hierarchy H into a sequence of complete markings is a quasi-geodesic in $\widetilde{\mathcal{M}}$, with uniform constants.*

In the case where $\mathbf{T}(H) = \psi(\mathbf{I}(H))$ for some $\psi \in \mathrm{Mod}(S)$, a resolution gives rise to a quasi-geodesic word in $\mathrm{Mod}(S)$ (Theorem 7.1).

Hierarchies will be constructed inductively, with the main geodesic chosen first and then further geodesics in subsurfaces determined by vertices of the previous ones. At every stage a geodesic is not uniquely determined, although hyperbolicity implies that all choices are fellow travelers. This is a priori a fairly loose constraint, but it has the following rigidity property, which is our generalization of Farb's bounded coset penetration property.

Lemma 6.6 (Common Links) *Suppose H and H' are hierarchies whose initial and terminal markings differ by at most K elementary moves. Then there is a number $M(K)$ such that, if a geodesic h appears in H and has length greater than M, then H' contains a geodesic h' with the same domain. Furthermore, h and h' are fellow-travelers with a uniform separation constant.*

This lemma is a consequence of the following lemma, which characterizes in terms of the subsurface projections π_Y when long geodesics appear in a hierarchy. For two markings μ and μ' we let $d_Y(\mu, \mu')$ denote the distance in $\mathcal{C}(Y)$ of their projections by π_Y (see §2.5).

Lemma 6.2 (Large Link) *There exists $M = M(S)$ such that, if H is any hierarchy in S and $d_Y(\mathbf{I}(H), \mathbf{T}(H)) \geq M$ for a subsurface Y in S, then Y is the domain of a geodesic h in H.*

Furthermore if h is in H with domain Y then its length $|h|$ and the projection distance $d_Y(\mathbf{I}(H), \mathbf{T}(H))$ are within a uniform additive constant of each other.

Both of these results follow from Theorem 3.1 (Bounded Geodesic Image) together with the structural properties of hierarchies, which are summarized by Theorem 4.7.

Applications of Lemmas 6.2 and 6.6 are based on the idea that, whenever geodesics in a hierarchy have short length, one can apply arguments that work for locally finite complexes. Whenever geodesics become long, one has this rigidity for all "nearby" hierarchies, and can work inductively in the shared domains of the long geodesics. Theorems 6.14 (Finite Geodesics), 6.13 (Convergence of Hierarchies) and 7.6 (Axis) are all consequences of this sort of argument.

1.5. **Motivating Examples.** To illustrate the above theorems, we will work through some more extended low-genus examples.

Let us first take a closer look at the case where S is a once-punctured torus or four-times punctured sphere. Then $\mathcal{C}(S)$ is the Farey graph (See figure 2 and §2.1), and in spite of the fact that the link of every vertex is infinite we have fairly explicit and rigid control of geodesics. In particular we note the following phenomenon. Let h be a geodesic and v a vertex in h, preceded by u and followed by w. The link of the vertex v can be identified with $\mathbf{Z}$, and we can measure the distance between u and w in this link, an integer $d_v(u, w)$. If h' is a geodesic with the same endpoints as h, then h' must pass through v *provided* $d_v(u, w)$ *is suffiently large* (5 will do). In fact the same holds if h' has endpoints, say, distance 1 from those of h. Furthermore, h' must enter the link of v at a point within 1 of u and exit within 1 of w. All of these claims are easy to show starting from the basic fact that any edge in the Farey graph separates it.

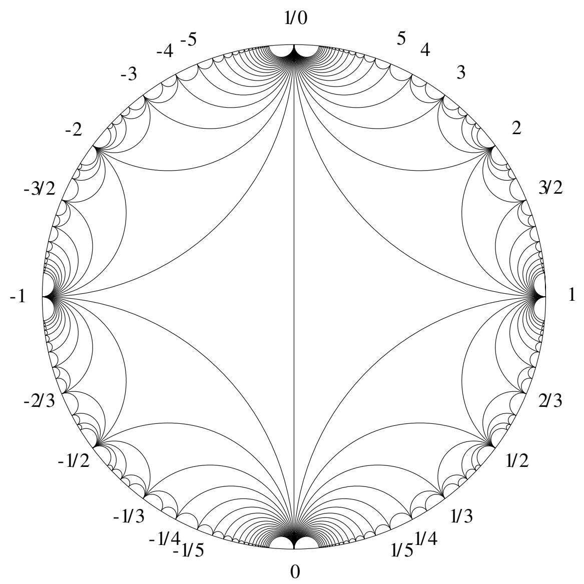

FIGURE 2. The complex of curves for a once-punctured torus or 4-times punctured sphere is the classical Farey graph. Vertices are labelled by slopes of the corresponding curves relative to some fixed homology basis.

This phenomenon, that a large link distance generates strong constraints on fellow traveling geodesics, persists in higher genus (even though the separation property of edges does not generalize), and gives rise to Lemmas

6.2 and 6.6. Let us now demonstrate this generalized phenomenon, together with the main features of our hierarchy construction, in the case where S is a closed genus 2 surface.

Let h be a geodesic in $\mathcal{C}(S)$ with a segment $..,u,v,w,...$ occuring somewhere in h. Let h' be a fellow traveler of h – for concreteness suppose the endpoints of h and h' are distance 1 or less apart, and occur at a distance at least $2\delta+2$ from u,v and w (where δ is the hyperbolicity constant of $\mathcal{C}(S)$). Hyperbolicity of $\mathcal{C}(S)$ implies that h and h' are $2\delta+1$-fellow travelers.

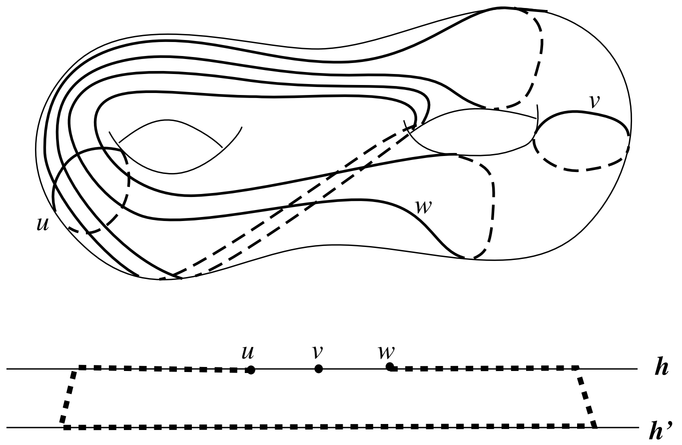

FIGURE 3. The short-cut argument for a genus 2 surface, where v is non-separating. If as shown h' does not meet v then the dotted rectangle bounds $d_Y(u,w)$.

Suppose first that the subsurface $Y = S \setminus v$ is connected – a 2-holed torus (figure 3). Then u and w give points in $\mathcal{C}(Y)$ and let us denote their distance in $\mathcal{C}(Y)$ by $d_Y(u,w)$. We can show the following statement:

If the "link distance" $d_Y(u,w)$ is sufficiently large then the fellow-traveler h' must also pass through v.

Suppose not – then every vertex of h' has nontrivial intersection with Y. Consider a path beginning at w, moving forward in h a distance $2\delta+2$, across to h' by a path of length at most $2\delta+1$, back along h' and over to h by another path of length at most $2\delta+1$, which lands at a point $2\delta+2$ behind u, and from there back up to u along h. By the triangle inequality, every point of this path not on h' has distance at least 2 from v (except the endpoints u and w which are in Y). Together with the assumption about h' we have that every point on the path represents a curve having nontrivial intersection with Y. The length of the segment on h' is bounded by $8\delta+8$ by the triangle inequality, so the total length of the path from w to u is at most $16\delta+14$. If we replace every curve with one arc of its intersection with Y, we obtain a sequence of properly embedded arcs or curves in Y, each disjoint from the previous. As we will see in Lemma 2.2, these can each be replaced with a simple closed curve, so that each one is distance at most 2 from its predecessor in $\mathcal{C}(Y)$. (This is the subsurface projection π_Y.) We therefore obtain a path in $\mathcal{C}(Y)$ connecting w to u, of length at most $32\delta + 28$. If we assumed $d_Y(u, w) > 32\delta + 28$ this would be a contradiction, and then h' would have to pass through the vertex v.

In that case, we can say more. Let u' be the predecessor and w' the successor of v along h'. The same kind of argument, applied to the segments of our path joining u to u' and w to w', gives an upper bound for $d_Y(u, u')$ and $d_Y(w, w')$. Joining u and w by a geodesic k in $\mathcal{C}(Y)$ and u' and w' by a geodesic k' in $\mathcal{C}(Y)$, we now know by hyperbolicity of $\mathcal{C}(Y)$ that k and k' are fellow-travelers.

Now suppose instead that v divides S into components Y_1 and Y_2, each necessarily a one-holed torus (see figure 4). Since u, v, w is a geodesic, u and w must intersect nontrivially and hence belong to the same component, say Y_1. The previous "short-cut" argument now implies that, if $d_{Y_1}(u, w) > 32\delta + 28$, some curve v' of h' must miss Y_1. We could again have $v' = v$, or now the additional possibility that v' lies (nonperipherally) in Y_2. Suppose this case happens. Set $Y' = S \setminus v'$, noting that it must be a single two-holed torus containing Y_1, and let u' and w' be the predecessor and successor of v' in h'. We again apply the short-cut argument to conclude that $d_{Y'}(u', w') \leq 32\delta + 28$; for otherwise v' would appear in h, but it is not u, v or w and is distance 1 from v, so this contradicts the fact that h is a geodesic. Let m' be a geodesic in $\mathcal{C}(Y')$ joining u' and w'. If every vertex of m' intersects Y_2, then using m', and thus bypassing v', we can find a path of some bounded length joining u and w, such that every point on it represents a curve that meets Y_1. Thus assuming $d_{Y_1}(u, w)$ is sufficiently large, m' must pass through a curve missing Y_1. Since it is an essential curve in Y', this curve in fact can only be v itself. Let y' and z' be the predecessor and successor of v along m'. They must lie in Y_1 and now in fact the same argument gives an upper bound for the distance in $\mathcal{C}(Y_1)$ between y' and u and between z' and w. Again by hyperbolicity any geodesic k' in $\mathcal{C}(Y_1)$ joining y' and z' fellow travels the geodesic k joining u and w. This is essentially the content of Theorem 6.6 (Common Links) in this case.

So far, we have constructed over h' a "hierarchy" of geodesics: m' is obtained as a geodesic in the link of v', joining its predecessor and its successor in h'. k' is obtained in the link of v, appearing in m', in the same way. We say that m' is *subordinate* to h', and k' to m'.

For the hierarchy over h we have something similar, with the geodesic k supported in one of the complementary domains Y_1 of v, and hence subordinate to h, but we have not constructed anything in the domain Y_2. A geodesic in Y_2 does arise naturally, in the following way. Let $U = S \setminus u$ and $W = S \setminus w$, noting that both of these are two-holed tori containing Y_2. There are geodesics p supported in U and q supported in W, so that p joins the predecessor of u to its successor v, and q joins the predecessor v of w to its successor (see figure 5 for schematic). Let s be the vertex of p preceding v, and let t be the vertex of q following v. Each is disjoint from

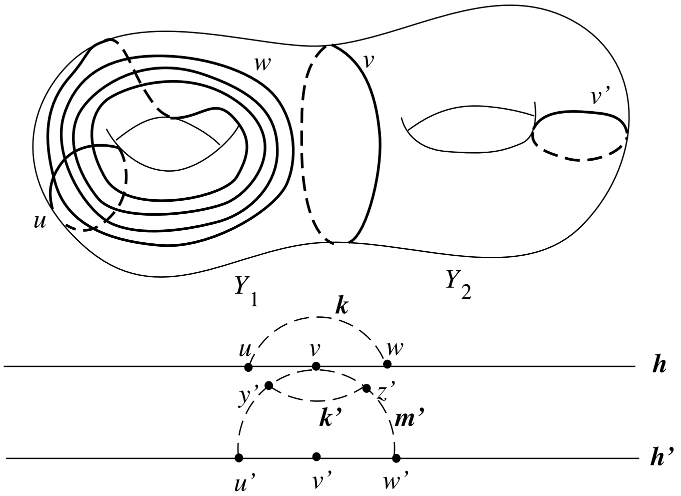

FIGURE 4. When v separates S into Y_1 and Y_2, h' can pass through v' in Y_2, But if $d_{Y_1}(u,w)$ is large then m', supported in $Y' = S \setminus v'$, must pass through v.

v, and therefore must lie in Y_2. We therefore may join s to t by a geodesic r in $\mathcal{C}(Y_2)$. In the notation we will later develop, r is *forward subordinate* to q, since it is supported in the domain of q minus the vertex v, and its last vertex is the successor of v. Similarly r is *backward subordinate* to p.

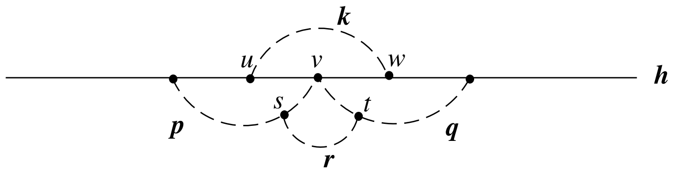

FIGURE 5. The geodesic r, supported in Y_2, arises naturally after the geodesics p and q are constructed in the links of u and w.

Let us see how pants decompositions arise in this structure. In the hierarchy over h', the vertices v' at bottom level (in h'), v on the next level (in m'), and any vertex x in the geodesic k', form a pants decomposition, which we also call a *slice* of the hierarchy. If x' is the successor of x in k' (so x and x' are neighbors in the Farey graph $\mathcal{C}(Y_1)$), the transition from (v', v, x) to (v', v, x') is an *elementary move.* In the hierarchy over h we can see a slice with different organization: starting with v at bottom level, we take any vertex a in k and b in r, and the triple (v, a, b) make a pants decomposition. We can move a and b independently in their respective geodesics, since their domains (Y_1 and Y_2) are disjoint. This kind of idea will give a way to "resolve" a hierarchy (non-uniquely) into a sequence of slices, or markings, which will then enable us to describe a useful class of words in the mapping class group.

In these examples we have only produced pants decompositions, but in our final construction there will be complete markings, which include twisting data around each pants curve. This will be done using "annulus complexes," which are analogous to the links of vertices in the Farey graph.

1.6. **Other applications and directions.** We hope that the tools developed here can be used to give an algorithmic approach to $\mathrm{Mod}(S)$ in which the complexity of problems such as the conjugacy problem can be computed. In particular, the conjugacy bound of Theorem 7.2, together with the quasi-geodesic words constructed from hierarchies, are a good start provided that one can give an effective algorithm to *construct* hierarchies with a Turing machine.

The word problem, by comparison, admits a quadratic-time solution because $\mathrm{Mod}(S)$ is known to have an *automatic structure* (see Mosher [26]). A stronger condition known as a *biautomatic structure* (see [7] for definitions of these terms) would give bounds on the conjugacy problem, but whether one exists remains open. Finding a biautomatic structure was an initial motivation for this paper, but significant problems remain. In particular the paths obtained from resolutions of hierarchies are *not* a bicombing of $\mathrm{Mod}(S)$, because of the presence of disjoint domains in S, whose order of traversal can differ in different paths. The standard "diagonalization" method of moving in both domains at once runs into some significant technical problems in our setting. However, we believe that the hierarchy structure should be powerful enough by itself to give algorithmic results.

A rather different application of our ideas is to questions of rigidity and classification for hyperbolic 3-manifolds. In [23], Kleinian representations of the fundamental group of the punctured torus were studied via the length functions they induce on its curve complex, the Farey graph. A connection between the combinatorics of this graph and the geometry of the corresponding 3-manifolds was established, which was a primary ingredient in the proof of Thurston's Ending Lamination Conjecture in that case. In general, given a representation $\rho : \pi_1(S) \to PSL(2, \mathbf{C})$ one can study the complex translation lengths of conjugacy classes of simple curves, viewed as a function on $\mathcal{C}(S)$. In [24, 22] , some quasi-convexity properties are established for these functions, analogous to results in [23]. A relationship is also established between a-priori geometric bounds on the representation and a combinatorial condition involving the subsurface projections π_Y. We hope these results will prove useful in studying the general classification problem for Kleinian groups.

2. Complexes and subcomplexes of curves

We review here the definitions of the various complexes of curves, paying particular attention to the way in which subsurfaces of a given surface give rise to sub-complexes. We will prove Lemma 2.2 relating arc complexes to curve complexes, define projections from a complex to its sub-complexes and prove Lemma 2.3 (Lipschitz Projection).

We will also treat the case of annuli, which are exceptional in various respects, and conclude with a discussion of markings and elementary moves.

2.1. **Basic definitions and notation.** Let $S = S_{\gamma,p}$ be an orientable surface of finite type, with genus $\gamma(S)$ and $p(S)$ punctures. It will be convenient to measure the complexity of S by $\xi(S) = 3\gamma(S) + p(S)$. Note that ξ is not equivalent to Euler characteristic, but has the property that if $T \subset S$ is an incompressible proper subsurface then $\xi(T)$ is strictly smaller than $\xi(S)$. We will only consider surfaces with $\xi > 1$, thus excluding the sphere and disk. We will also exclude the standard torus (which does not arise as a subsurface of a hyperbolic surface), so that from now on $\xi(S) = 3$ implies S is the thrice-punctured sphere.

The *complex of curves* $\mathcal{C}(S)$, introduced by Harvey in [12], is a finite-dimensional and usually locally infinite simplicial complex defined as follows: A *curve* in S is by definition a nontrivial homotopy class of simple closed curves, not homotopic into a puncture. If $\xi(S) > 3$ then the set of curves is non-empty, and we let these be the vertices of $\mathcal{C}(S)$. If $\xi(S) > 4$ then the k-simplices are the sets $\{v_0, \dots, v_k\}$ of distinct curves that have pairwise disjoint representatives. One easily checks that $\dim(\mathcal{C}(S)) = \xi(S) - 4$.

When $\xi(S) = 4$, S is either a once-punctured torus $S_{1,1}$ or four times punctured sphere $S_{0,4}$, and the complex as defined above has dimension 0. In this case we make an alternate definition: an edge in $\mathcal{C}(S)$ is a pair $\{v, w\}$ where v and w have representatives that intersect once (for $S_{1,1}$) or twice (for $S_{0,4}$). Thus $\mathcal{C}(S)$ is a graph, and in fact is isomorphic to the familiar Farey graph in the plane (see e.g. Bowditch-Epstein [3], Bowditch [4], Hatcher-Thurston [14], and Series [27]). In particular this graph is a triangulation of the 2-disk with vertices on the boundary, and the link of each vertex can be identified with the integers, on which Dehn twists (or half-twists for $S_{0,4}$) act by translation (see Figure 2).

When $\xi(S) = 3$, $\mathcal{C}(S)$ is empty (recall we have excluded the regular torus). When $\xi(S) = 2$, S is the annulus and this case is of interest when S appears as a subsurface of a larger surface. We consider this case further in §2.4.

2.2. **Distance geometry and hyperbolicity.** Let $\mathcal{C}_k(S)$ denote the k-skeleton of $\mathcal{C}(S)$. It is easy to show that $\mathcal{C}_k$ is connected for $k \geq 1$, see e.g. [21, Lemma 2.1]. We can make $\mathcal{C}_k(S)$ into a complete geodesic metric space by giving each simplex the metric of a regular Euclidean simplex with side-length 1 (see Bridson [5]). It is easy to see that the resulting spaces are quasi-isometric for all $k > 0$. In [21] we showed

Theorem 2.1. (Hyperbolicity) *If $\xi(S) \geq 4$ and $k > 0$ then $\mathcal{C}_k(S)$ is an infinite-diameter δ-hyperbolic metric space for some $\delta > 0$.*

See e.g. [6, 11, 2, 10, 1] for background on δ-hyperbolic metric spaces. We recall here just the definition that a geodesic metric space is δ-hyperbolic if for any geodesic triangle each edge is in a δ-neighborhood of the union of the other two edges.

We will usually consider just distances between *vertices* in $\mathcal{C}(S)$, i.e. points in $\mathcal{C}_0(S)$, for which it suffices to consider distances in the graph $\mathcal{C}_1(S)$, which we note are integers. Thus by the notation $d_{\mathcal{C}(S)}(v,w)$, or even $d_S(v,w)$, we will always mean distances as measured in $\mathcal{C}_1(S)$. Writing diam_S to mean diameter in $\mathcal{C}_1(S)$, we define for subsets $A, B \subset \mathcal{C}_0(S)$

$$d_S(A,B) = \operatorname{diam}_S(A \cup B). \tag{2.1}$$

We will also usually think of a *geodesic* in $\mathcal{C}_1(S)$ as a sequence of vertices $\{v_i\}$ in $\mathcal{C}_0(S)$, such that $d_S(v_i,v_j) = |i-j|$. In particular v_i and v_{i+1} are always disjoint (when $\xi(S) > 4$) and v_i and v_{i+3} always fill S, in the sense that the union of the curves they represent, in minimal position, cuts S into a union of disks and once-punctured disks.

A final abuse of notation throughout the paper is in the usage of the term "vertex": when we introduce the notion of tight geodesics in §4.1 we will use "vertex of a geodesic" to denote something more general than a point of $\mathcal{C}_0(S)$, namely a simplex of $\mathcal{C}(S)$, representing a multi-component curve (or multicurve). (One can think of this as a vertex of the first barycentric subdivision). We will also go back and forth freely between vertices or simplices and the (multi)curves they represent.

2.3. **Subdomains, links, arc complexes.** A *domain* (or *subdomain*) Y in S will always be taken to mean an (isotopy class of an) incompressible, non-peripheral, connected open subsurface. Unless we say *proper* subdomain, we include the possibility that $Y = S$. We usually omit the mention of isotopy classes for both surfaces and curves, and to make the discussion clear one might fix a complete hyperbolic metric on S and consider geodesic representatives of curves, and surfaces bounded by them. We also take the word "intersection" to mean *transverse* intersection. However, annuli are an exceptional case in several ways; see below.

In particular note that the boundary curves of a surface do not intersect it.

We immediately obtain an embedding $\mathcal{C}(Y) \subset \mathcal{C}(S)$ except when $\xi(Y) \leq 4$. Another complex of interest is the *arc complex* $\mathcal{C}'(Y)$, which we define as follows: Suppose again that $\xi(Y) > 3$. An *arc* in Y is a homotopy class of properly embedded paths in Y, which cannot be deformed rel punctures to a point or a puncture. The vertices of $\mathcal{C}'(Y)$ are both the arcs and the curves, and simplices as before are sets of vertices that can be realized disjointly. The complex $\mathcal{C}'(Y)$ naturally arises when we try to "project" $\mathcal{C}(S)$ into $\mathcal{C}(Y)$ by taking intersections with Y of curves in S.

Remark: The punctures of Y can come from either punctures of S or from boundary components of Y in S. In fact, it is often useful to think of all the punctures of Y as boundary components, in which case we consider arcs up to homotopy which allows the endpoints to move on the boundary. These points of view are equivalent, and we shall go back and forth between them for convenience.

The next elementary observation is that $\mathcal{C}(Y)$ embeds in $\mathcal{C}'(Y)$ as a co-bounded set. More precisely, letting $\mathcal{P}(X)$ denote the set of finite subsets of X, we have:

Lemma 2.2. *Let $\xi(Y) > 3$. There is a map $\psi = \psi_Y : \mathcal{C}'_0(Y) \to \mathcal{P}(\mathcal{C}_0(Y))$ such that:*

- *$\psi(v) = \{v\}$ for $v \in \mathcal{C}_0(Y)$,*
- *$d_{\mathcal{C}'(Y)}(\alpha, \psi(\alpha)) \leq 1$, and*
- *if $d_{\mathcal{C}'(Y)}(\alpha, \beta) \leq 1$ then $d_{\mathcal{C}(Y)}(\psi(\alpha), \psi(\beta)) \leq 2$.*

Proof. If α is an arc, let $\mathcal{N}$ be a regular neighborhood in Y of the union of α with the component(s) of ∂Y on which its endpoints lie, and consider the frontier of $\mathcal{N}$ in Y. This has either one or two components, and at least one of them must be both nontrivial and nonperipheral, since otherwise Y is a disk, annulus or thrice-punctured sphere, contradicting $\xi(Y) > 3$. We let $\psi(\alpha)$ be the union of the (at most two) nontrivial components (see figure 6). If α is a curve (vertex of $\mathcal{C}_0(Y)$), we define $\psi(\alpha) = \{\alpha\}$.

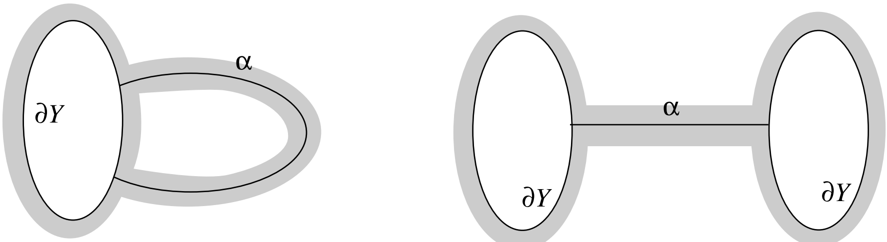

FIGURE 6. The neighborhood $\mathcal{N}$ is shaded. Note that its frontier in Y has two components in the first case and one in the second

Let α and β be adjacent in $\mathcal{C}'(Y)$, so they have disjoint representatives. If either of them is a closed curve then automatically $d(\psi(\alpha), \psi(\beta)) \leq 1$, so assume both are arcs. Similarly if their endpoints lie on disjoint boundary components of Y then $\psi(\alpha)$ and $\psi(\beta)$ have disjoint representatives, so we can assume from now on that there is at least one boundary component which touches both of them. Suppose that the complement of $\alpha \cup \beta$ in Y contains a non-trivial, non-peripheral simple closed curve γ. Then γ is also disjoint from $\psi(\alpha)$ and $\psi(\beta)$, and we conclude $d_{\mathcal{C}(Y)}(\psi(\alpha), \psi(\beta)) \leq 2$.

If there is no such γ, then α and β cut Y into a union of (at most 3) disks or punctured disks. The possible cases can therefore be enumerated explicitly.

Let C be a boundary component of Y meeting both α and β. If C meets all the endpoints then there are two possibilities, according to whether the

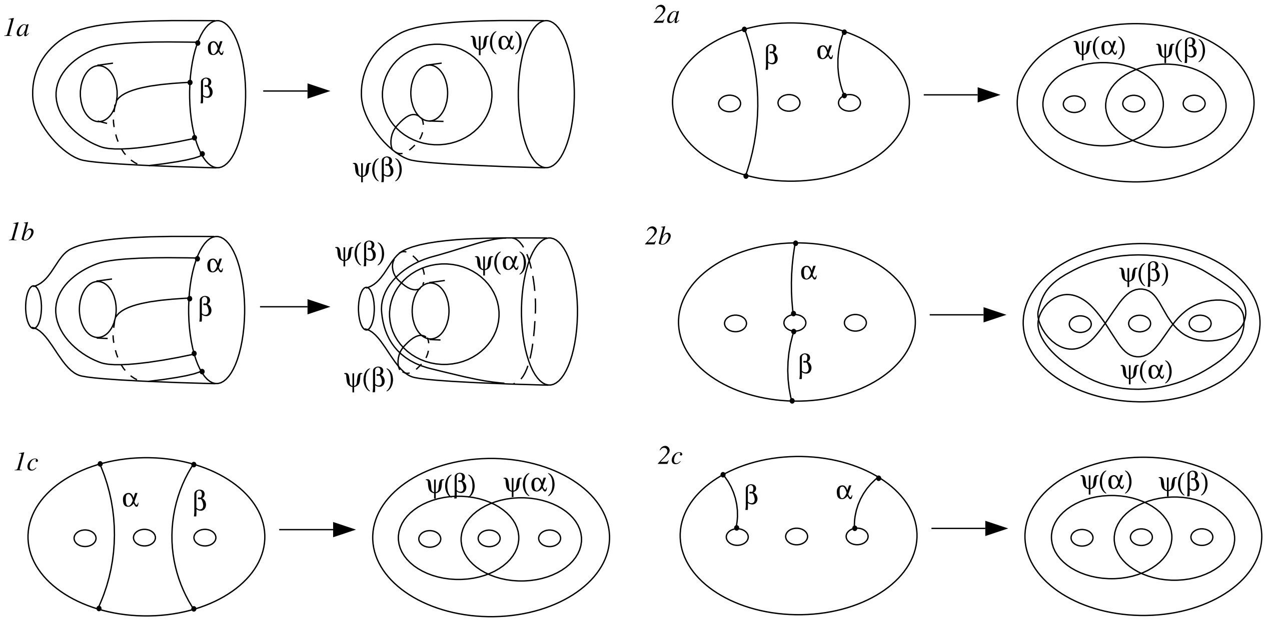

FIGURE 7. The different cases in the proof that $d_{\mathcal{C}(Y)}(\psi(\alpha), \psi(\beta)) \leq 2$.

endpoints separate each other on C. If they separate each other, Y must be a once or twice punctured torus, as in cases 1a and 1b of Figure 7. In case 1a we have $d_{\mathcal{C}(Y)}(\psi(\alpha), \psi(\beta)) = 1$, and in case 1b, $d_{\mathcal{C}(Y)}(\psi(\alpha), \psi(\beta)) = 2$, as shown (note in this case that $\psi(\alpha)$ and $\psi(\beta)$ each have two components). If they do not separate then Y must be a quadruply-punctured sphere (case 1c) and $d_{\mathcal{C}(Y)}(\psi(\alpha), \psi(\beta)) = 1$. (Recall that in the cases where $\xi(Y) = 4$, the definition of $d_{\mathcal{C}(Y)}$ is slightly different).

Suppose that α has one endpoint on C and one on another boundary C'. In all these cases Y turns out to be a quadruply-punctured sphere. If both β's endpoints are on C we get case 2a, where $d_{\mathcal{C}(Y)}(\psi(\alpha), \psi(\beta)) = 1$. If β's other endpoint is on C' we get 2b, where $d_{\mathcal{C}(Y)}(\psi(\alpha), \psi(\beta)) = 2$. If β's other endpoint is on a third component C'', we get case 2c where again $d_{\mathcal{C}(Y)}(\psi(\alpha), \psi(\beta)) = 1$. ☐

Projections to subsurfaces: If Y is a proper subdomain in S with $\xi(Y) \geq 4$ we can define a map $\pi'_Y : \mathcal{C}_0(S) \to \mathcal{P}(\mathcal{C}'_0(Y))$, simply by taking for any curve α the union of (homotopy classes of) its essential arcs of intersection with Y. If α does not meet Y essentially then $\pi'_Y(\alpha) = \emptyset$, and otherwise it is always a simplex of $\mathcal{C}'(Y)$.

Adopting the convention for set-valued maps that $f(A) = \cup_{a\in A} f(a)$, we define π_Y by $\pi_Y(\alpha) = \psi_Y(\pi'_Y(\alpha))$. We also define

$$d_Y(A, B) \equiv d_Y(\pi_Y(A), \pi_Y(B)) \tag{2.2}$$

For sets or elements A and B in $\mathcal{C}_0(S)$, and similarly we let $diam_Y(A)$ denote $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(A))$.

This construction, and the equivalent of Lemma 2.2, have previously appeared in Ivanov ([16] and [17, §7.8]).

2.4. **Annular domains.** An annular domain is an annulus Y with incompressible boundary in S, which is not homotopic into a puncture of S. The purpose of defining complexes for such annuli is to keep track of Dehn twisting around their cores; hence one would like $\mathcal{C}(Y)$ to be $\mathbf{Z}$. However there seems to be no natural way to do this, and we will be content with something more complicated which is nevertheless quasi-isometric to $\mathbf{Z}$. The statements made in this subsection are all elementary, and we only sketch the proofs.

Let $\widetilde{Y}$ be the annular cover of S to which Y lifts homeomorphically. There is a natural compactification of $\widetilde{Y}$ to a closed annulus $\widehat{Y}$, obtained in the usual way from the compactification of the universal cover $\widetilde{S} = \mathbf{H}^2$ by the closed disk. Define the vertices of $\mathcal{C}(Y)$ to be the paths connecting the two boundary components of $\widehat{Y}$, modulo homotopies that *fix the endpoints*. Put an edge between any two elements of $\mathcal{C}_0(Y)$ which have representatives with disjoint interiors. As before we can make $\mathcal{C}(Y)$ into a metric space with edge lengths 1. If $\alpha \in \mathcal{C}_0(S)$ is the core curve of Y we also write $\mathcal{C}(\alpha) = \mathcal{C}(Y)$, and similarly $d_Y = d_\alpha$.

Fixing an orientation on S and an ordering on the components of $\partial\widehat{Y}$, we can define algebraic intersection number $\alpha \cdot \beta$ for $\alpha, \beta \in \mathcal{C}_0(Y)$ (only interior intersections count). It is easy to see by an inductive argument that

$$d_Y(\alpha, \beta) = 1 + |\alpha \cdot \beta| \tag{2.3}$$

whenever $\alpha \neq \beta$. Let us also observe the convenient identity

$$\gamma \cdot \alpha = \gamma \cdot \beta + \beta \cdot \alpha + j \tag{2.4}$$

where $j = 0, 1$ or -1 (the value of j depends on the exact arrangement of endpoints on $\partial\widehat{Y}$).

We claim that $\mathcal{C}(Y)$ is quasi-isometric to $\mathbf{Z}$ with the standard metric. In fact define a map $f : \mathcal{C}_0(Y) \to \mathbf{Z}$ by fixing some $\alpha \in \mathcal{C}_0(Y)$ and letting $f(\beta) = \beta \cdot \alpha$. Then (2.4) and (2.3) imply

$$|f(\gamma) - f(\beta)| \leq d_Y(\gamma, \beta) \leq |f(\gamma) - f(\beta)| + 2. \tag{2.5}$$

In particular this implies $\mathcal{C}(Y)$ is hyperbolic so Theorem 2.1 holds for this complex as well.

Projections to annuli: We can define $\pi_Y : \mathcal{C}_0(S) \to \mathcal{P}(\mathcal{C}_0(Y))$ as follows: If γ is a simple closed curve in S crossing the core of Y transversely, then the lift of γ to $\widetilde{Y}$ has at least one component that connects the two boundaries of $\widehat{Y}$, and together these components make up a (finite) set of diameter 1 in $\mathcal{C}(Y)$. Let $\pi_Y(\gamma)$ be this set. If γ does not intersect Y essentially (including the case that γ is the core of Y!) then $\pi_Y(\gamma) = \emptyset$, as in the previous section.

Finally, for consistency we also define $\pi_Y : \mathcal{C}_0(Y) \to \mathcal{P}(\mathcal{C}_0(Y))$ by $v \mapsto \{v\}$, and define $d_Y(A,B)$ and $\mathrm{diam}_Y(A)$ using the same conventions (e.g. (2.1) and (2.2)) as for larger subdomains. If α is the core of Y we also write diam_α and π_α.

We remark that $\mathcal{C}(Y)$ is not a subcomplex of $\mathcal{C}(S)$, but just as for larger subdomains, any $f \in \mathrm{Mod}(S)$ acts by isomorphism $f : \mathcal{C}(Y) \to \mathcal{C}(f(Y))$, and this fits naturally with the action on $\mathcal{C}(S)$ via $\pi_{f(Y)} \circ f = f \circ \pi_Y$.

With these definitions in place we have the following:

Lemma 2.3. (Lipschitz Projection) *Let Y be a subdomain of Z. For any simplex ρ in $\mathcal{C}(Z)$, if $\pi_Y(\rho) \neq \emptyset$ then $\mathrm{diam}_Y(\rho) \leq 2$. If Y is an annulus and $\xi(Z) > 4$ then the bound is 1.*

Proof. For an annulus Y, if $\xi(Z) > 4$ the bound is immediate, since any two disjoint curves in Z lift to disjoint arcs in $\widetilde{Y}$. If $\xi(Z) = 4$, one easily checks that Farey neighbors in $\mathcal{C}(Z)$ lift to curves that intersect at most once in any annulus cover.

For $\xi(Y) \geq 4$, the bound follows from Lemma 2.2. □

Dehn twists: Let Y be an annulus with core α. Let D_α be a positive Dehn twist in S about α, and let $\widehat{D}_\alpha$ be a positive Dehn twist in the covering annulus $\widehat{Y}$ about its core. Then $\widehat{D}_\alpha$ acts on $\mathcal{C}(Y)$ and it is immediate for any $t \in \mathcal{C}_0(Y)$ that $(\widehat{D}^n_\alpha t) \cdot t = n - 1$ if $n > 0$ and $n + 1$ if $n < 0$. Thus we obtain from (2.3) that $d_Y(\widehat{D}^n_\alpha(t), t) = |n|$ for all $n \in \mathbf{Z}$

With a little more thought one can see that, for any curve β intersecting α transversely,

$$d_Y(D^n_\alpha(\beta), \beta) = 2 + |n| \tag{2.6}$$

for $n \neq 0$. This is because the Dehn twist in S affects every intersection of the lift of β with lifts of α in $\widetilde{Y}$, and this shifts the endpoints on $\partial\widehat{Y}$ enough to enable components of $\pi_Y(D^n_\alpha(\beta))$ and $\pi_Y(\beta)$ to intersect an additional two times.

If β intersects α exactly 2 times with opposite orientation, one can apply a *half twist* to β to obtain a curve $H_\alpha(\beta)$, which is equivalent to taking $\alpha \cup \beta$ and resolving the intersections in a way consistent with orientation (see [19] for a generalization). Then $H^2_\alpha(\beta) = D_\alpha(\beta)$, and one can also see for $n \neq 0$ that

$$d_Y(H^n_\alpha(\beta), \beta) = 2 + \left\lfloor \frac{|n|}{2} \right\rfloor. \tag{2.7}$$

2.5. **Markings.** Assume $\xi(S) \geq 4$ and let $\{\alpha_1, \ldots, \alpha_k\}$ be some simplex in $\mathcal{C}(S)$. A *marking* in S is a set $\mu = \{p_1, \ldots, p_k\}$, where each p_i is either just α_i, or a pair (α_i, t_i) such that t_i is a diameter-1 set of vertices of the annular complex $\mathcal{C}(\alpha_i)$. The α_i are called the base curves and the simplex $\{\alpha_i\}$ is denoted $\mathrm{base}(\mu)$. The (possibly empty) set $\{t_i\}$ is called the set of *transversals* and denoted $\mathrm{trans}(\mu)$. Thus a special case is when $\mathrm{trans}(\mu) = \emptyset$ and then $\mu = \mathrm{base}(\mu)$.

If base(μ) is contained in $\mathcal{C}(Y)$ for some non-annular subsurface in Y, we call μ a marking in Y. If Y is an essential annulus in S then a marking μ in Y is any set of diameter 1 in $\mathcal{C}_0(Y)$ (typically these sets will have at most two elements), and we have $\mu = \text{base}(\mu)$ in this case.

If base(μ) is maximal and every curve has a transversal, the marking is called *complete.*

Markings can be very complicated objects, because the transversals, being arcs in annular covers, can have complicated images in S. Let us therefore define something called a *clean marking*:

Given $\alpha \in \mathcal{C}_0(S)$ a *clean transverse curve* for α is a curve $\beta \in \mathcal{C}_0(S)$ such that a regular neighborhood of $\alpha \cup \beta$ (in minimal position) is a surface F with $\xi(F) = 4$, in which α and β are $\mathcal{C}(F)$-neighbors (note there are only two possible configurations, corresponding to F being a 1-holed torus or 4-holed sphere, and α and β intersect once or twice, respectively).

A marking μ is called *clean* if every pair in μ is of the form $(\alpha_i, \pi_{\alpha_i}(\beta_i))$ where β_i is a clean transverse curve for α_i, which also misses the other curves in base(μ). Note that if μ is clean then the curves β_i are uniquely determined by the transversals $t_i = \pi_{\alpha_i}(\beta_i)$.

We note that, up to homeomorphisms of S, there are only a finite number of clean markings.

If μ is a complete marking, there is an almost canonical way to select a related clean marking. Let us say that a clean marking μ' is *compatible* with a marking μ provided $\text{base}(\mu) = \text{base}(\mu')$, a base curve α has a transversal $t' = \pi_Y(\beta)$ in μ' if and only if it has a transversal t in μ, and $d_\alpha(t, t')$ is minimal among all possible choices of t'.

Lemma 2.4. *Let μ be a complete marking of $Y \subset S$. Then there exist at least 1 and at most n_0^b complete clean markings μ' compatible with μ, where b is the number of base curves of μ, and n_0 is a universal constant. Furthermore, for each $(\alpha, t) \in \mu$ and $(\alpha, t') \in \mu'$ we have $d_\alpha(t, t') \leq n_1$, where n_1 is a universal constant.*

Proof. Fix one clean marking μ_0 with $\text{base}(\mu_0) = \text{base}(\mu)$. All other clean markings with this base are obtained from μ_0 by twists and half-twists, so it follows immediately from (2.6,2.7) and the quasi-isometry (2.5) of an annular complex to $\mathbf{Z}$ that for each $\alpha \in \text{base}(\mu)$ there is a choice of clean transversal β that minimizes $d_\alpha(t, \pi_\alpha(\beta))$, and that there is a uniform bound on this minimum. The fact that the number of choices of β are uniformly bounded for each base curve also follows from (2.6) and (2.7). ☐

(One can in fact show that $n_0 \leq 4$ and $n_1 = 3$, but we will not need this).

Projections of markings: If Y is any subdomain of S and μ any marking in S we can define $\pi_Y(\mu)$ as follows: If Y is an annulus whose core is some $\alpha \in \text{base}(\mu)$, and α has a transversal t, we define $\pi_Y(\mu) = t$. If α has no transversal $\pi_Y(\mu) = \emptyset$. In all other cases, $\pi_Y(\mu) = \pi_Y(\text{base}(\mu))$.

Elementary moves on clean markings: Let μ be a complete clean marking, with pairs $(\alpha_i, \pi_{\alpha_i}(\beta_i))$ as above. There are two types of elementary moves that transform μ into a new clean marking.

1. Twist: Replace β_i by β'_i, where β'_i is obtained from β_i by a Dehn twist or half-twist around α_i.
2. Flip: Replace $(\alpha_i, \pi_{\alpha_i}(\beta_i)) \in \mu$ by $(\beta_i, \pi_{\beta_i}(\alpha_i))$ to get a non-clean marking μ''. Then replace μ'' by a compatible clean marking μ'.

In the first move a twist can be positive or negative. A half-twist is possible when α_i and β_i intersect twice.

The replacement part of the Flip move requires further discussion: The surface F filled by α_i and β_i has $\xi(F) = 4$, and its (non-puncture) boundary components are other elements of base(μ). For each such element α_j there is a transverse β_j which misses α_i but hits β_i. Thus after interchanging α_i and β_i the marking is no longer clean. We must therefore replace β_j by β'_j which misses β_i, subject to the condition that $d_{\alpha_j}(\beta_j, \beta'_j)$ is as small as possible. Lemma 2.4 says that this distance is at most n_1, and there are n_0 possible choices for each β_j. (Actually this is a more special case than Lemma 2.4 and one can get a distance bound of 2).

Thus, given μ there is a finite number of possible elementary moves on it, depending only on the topological type of S.

We conclude with an extension of Lemma 2.3 (Lipschitz Projection).

Lemma 2.5. (Elementary Move Projections) *If μ, μ' are complete clean markings differing by one elementary move, then for any domain Y in S with $\xi(Y) \neq 3$,*

$$d_Y(\mu, \mu') \leq 4$$

If Y is an annulus the bound is 3.

Proof. If Y is an annulus with core curve $\alpha \in$ base(μ), then μ contains $(\alpha, \pi_\alpha(\beta))$ for a clean transversal curve β, and $\pi_Y(\mu) = \pi_\alpha(\beta)$. Then if μ' is obtained by a twist or half-twist on α, a bound of 3 follows from (2.6) and (2.7). If μ' is obtained by a Flip move, replacing $(\alpha, \pi_\alpha(\beta))$ by $(\beta, \pi_\beta(\alpha))$, then $\pi_Y(\mu') = \pi_Y(\text{base}(\mu')) = \pi_\alpha(\beta)$, so the distance is 0.

A similar analysis holds if Y is an annulus with core curve in base(μ').

In all other cases, $\pi_Y(\mu) = \pi_Y(\text{base}(\mu))$ and $\pi_Y(\mu') = \pi_Y(\text{base}(\mu'))$, and by definition $d_Y(\mu, \mu') = \text{diam}_Y(\pi_Y(\text{base}(\mu)) \cup \pi_Y(\text{base}(\mu')))$. If $\pi_Y(\text{base}(\mu))$ and $\pi_Y(\text{base}(\mu'))$ have at least one curve in common, the bound of 4 follows from Lemma 2.3. If not, then the move must be a Flip move, and Y meets only the two base curves α, α' involved in the Flip. Let F be the surface of $\xi = 4$ filled by these curves, which are neighbors in $\mathcal{C}(F)$. If $\xi(Y) = 4$ then $Y = F$ and we are done, with a bound of 1. The remaining possibility is that Y is an essential annulus in F meeting both curves, and then any two lifts of α and α' to $\widetilde{Y}$ intersect at most once, giving a bound of 2. $\square$

3. Projection bounds

Our goal in this section will be to prove Theorem 3.1, which gives strong contraction properties for the subsurface projections π_Y.

Theorem 3.1. (Bounded Geodesic Image) *Let Y be a proper subdomain of Z with $\xi(Y) \neq 3$ and let g be a geodesic segment, ray, or biinifinite line in $\mathcal{C}(Z)$, such that $\pi_Y(v) \neq \emptyset$ for every vertex v of g.*

There is a constant M depending only on $\xi(Z)$ so that

$$\operatorname{diam}_Y(g) \leq M.$$

The intuition behind the statement is this: as we move in one direction in $g = \{...v_1, v_2, ...\}$, we expect the vertices to converge to some foliation in Z. Hence their projections to Y should converge to the intersection with Y of the foliation leaves. Recalling that π_Y identifies parallel arcs, it should follow that eventually $\pi_Y(v_i)$ should stabilize to a finite collection of possible arcs. To make this precise we have to re-introduce the tools of Teichmüller geometry from [21]. We also emphasize that the statements we prove will be strictly weaker than this intuitive description, but will suffice for the diameter bound.

3.1. Quadratic differentials, vertical and horizontal.

Given a finite-type complex structure on Z, recall that a holomorphic quadratic differential q on Z is a tensor of the form $\varphi(z)dz^2$ in local coordinates, with φ holomorphic. Away from zeroes, a coordinate ζ can be chosen so that $q = d\zeta^2$, which determines a Euclidean metric $|d\zeta^2|$ together with a pair of orthogonal foliations parallel to the real and imaginary axes in the ζ plane. These are well-defined globally and are called the *horizontal* and *vertical* foliations, respectively. The zeroes of q are cone points with cone angle $n\pi$, $n \in \mathbf{Z}$, $n \geq 2$. (See Gardiner [9] or Strebel [28].)

For a closed curve or arc α in Z, denote by $|\alpha|_q$ its length in the q metric. Let $|\alpha|_{q,h}$ and $|\alpha|_{q,v}$ denote its horizontal and vertical lengths, respectively, by which we mean the total lengths of the (locally defined) projections of α to the horizontal and vertical directions of q.

Henceforth assume q has finite area, which means that at the punctures it has poles of order 1 or less, and equivalently that its metric completion gives a surface $\hat{Z}$ which is Z with a cone point added at each puncture, with cone angle $n\pi$, $n \in \mathbf{Z}$, $n \geq 1$.

Define a *straight segment* to be a path in Z which meets no punctures or zeroes of q, and is a straight line in the Euclidean metric. A geodesic is composed of a finite number of straight segments, meeting at zeroes with a certain angle condition. We must slightly generalize the notion of "geodesic representative" as follows: If Z has punctures, the incompleteness of $|q|$ means that a non-peripheral homotopy class α may not have a geodesic representative. However, there is a representative in $\hat{Z}$ which goes through the punctures some finite number of times and is geodesic elsewhere, which we can think of as a limit of geodesic representatives in the compact surfaces obtained by deleting open disks of q-radius r around the punctures, for $r \to 0$. Thus by "geodesic representatives" we will in fact mean representatives in this sense.

Let $\epsilon, \theta > 0$ be some fixed (small) constants. We say that a straight segment α is *almost vertical* with respect to q if if it makes an angle of at most θ with the vertical foliation. We say a geodesic is almost vertical if it is composed of straight segments meeting at punctures or zeros, each of which is either almost vertical, or has length at most ϵ. We define *almost horizontal* in the analogous way.

Lemma 3.2. *There is a choice of ϵ, θ depending only on $\xi(Z)$ such that the following holds. Let Y be a domain in Z with $\xi(Y) \neq 3$, q a unit-area quadratic differential on Z, and α a boundary component of Y whose q-geodesic contains an almost-horizontal segment σ of horizontal length 1. Then if β and γ are two almost-vertical curves intersecting Y,*

$$d_Y(\beta, \gamma) \leq 4.$$

Proof. We begin with the case where Y is not an annulus.

For simplicity, suppose first that Y is isotopic to an embedded surface with q-geodesic boundary. Thus we may assume that α is already geodesic. Consider the flow starting from σ and moving along the vertical foliation into Y until it returns to σ or meets a singularity. The points corresponding to flow lines that meet singularities divide σ into at most k_0 intervals $\{I_j\}$, where k_0 depends only on $\xi(Z)$. Each I_j determines a "flow rectangle", which is actually a Euclidean trapezoid or parallelogram with two vertical sides and two almost-horizontal sides which have slope at most $\tan\theta$. The interior of the rectangle is embedded, though its top and bottom edges are segments of σ that may overlap. Since σ has horizontal length 1 there must be an interval I_j of horizontal length at least $1/k_0$. Let R denote the corresponding flow rectangle, and h the average height of R. Thus R has area at least h/k_0, and since q is unit-area, $h \leq k_0$.

Suppose that $\epsilon < 1/k_0$ and that θ is sufficiently small that $\cot\theta > k_0^2 + \tan\theta$.

With these choices, we claim that an almost-vertical geodesic β cannot cross R from one vertical side to the other: Since R has no singularities in its interior, such a crossing would have to be a straight segment τ, and the slope of τ would be at most $hk_0 + \tan\theta \leq k_0^2 + \tan\theta$, which is less than $\cot\theta$ by the choice of θ. Hence τ could not be almost vertical. Thus it would have to have length bounded by ϵ, and hence be shorter than the width of R, again a contradiction.

We conclude that β is disjoint from the interior of some arc a in R connecting the top edge and the bottom edge. Thus, any component of $a \cap Y$ gives an element of $\mathcal{C}'(Y)$ which is distance 1 from each vertex of $\pi'_Y(\beta)$. The same argument applies to γ, with a in the same homotopy class, and we conclude $d_{\mathcal{C}'(Y)}(\pi'_Y(\beta), \pi'_Y(\gamma)) \leq 2$. Lemma 2.2 then gives the desired bound.

Now consider the possiblity that Y is not homotopic to an embedded surface with geodesic boundary α. In particular the geodesic representative of α may traverse one or more geodesic segments more than once, producing arcs of self-tangency. However even in this case we obtain a map of Y into $\hat{Z}$ (Z union its punctures) which is homotopic to the inclusion by a homotopy that is an embedding until the last moment. At that moment families of arcs in Y or its complement, with endpoints on ∂Y, are collapsed to points, producing the arcs of self-tangency. It is easy to see that the same argument holds except that the rectangles R in question may have height zero, with horizontal arcs on the self-tangencies and vertical arcs collapsed.

This concludes the case where Y is not an annulus.

When Y is an annulus, α is in the homotopy class of its core.

Lift α to $\widetilde{\alpha}$ in the universal cover $\widetilde{Z}$. We remind the reader that again the geodesic representative of α may pass through punctures, and as the universal covering is infinitely branched around punctures the topology is easier to keep track of if we keep α outside a small neighborhood of the punctures. At any rate our segment σ can be assumed disjoint from the punctures so we need not worry about this.

If we consider the lines of the vertical flow which start at σ and go in both directions until they hit σ again, we obtain at most $2k_0$ rectangles composed of vertical flow lines with σ passing through them, and we choose R to be one which has width at least $1/2k_0$. Let $\{R_n\}_{n\in\mathbf{Z}}$ denote its lifts corresponding to the lift of α to $\widetilde{\alpha}$, so that $\widetilde{\alpha}$ passes through the interior of each R_n, and the top and bottom edges of R_n lie on translates of α called $\widetilde{\alpha}_n$ and $\widetilde{\alpha}'_n$, respectively. (We include also the degenerate possibility that R has height 0 on one side or the other of α and so the $\widetilde{\alpha}_n$, or $\widetilde{\alpha}'_n$, are each tangent to $\widetilde{\alpha}$ along a segment.)

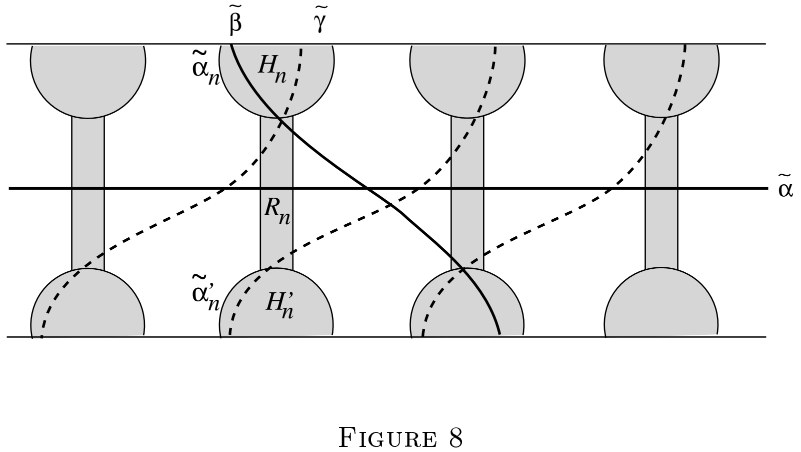

FIGURE 8

After an arbitrary choice of orientation for α, each R_n has a left and a right vertical edge. Let $\widetilde{\beta}$ and $\widetilde{\gamma}$ be components of the lifts of β and γ which cross $\widetilde{\alpha}$. As argued before, and with appropriate choice of ϵ, θ, neither $\widetilde{\beta}$ nor

$\widetilde{\gamma}$ can cross a rectangle R_n from left to right. Let H_n and H'_n denote the halfplanes bounded by $\widetilde{\alpha}_n$ and $\widetilde{\alpha}'_n$, respectively, whose interiors are disjoint from $\widetilde{\alpha}$ (see figure 8). Let U_n denote $R_n \cup H_n \cup H'_n$. Then neither $\widetilde{\beta}$ nor $\widetilde{\gamma}$ can cross through U_n from left to right, because this would involve either crossing R_n from left to right, or entering and exiting the interior of H_n or H'_n, which a geodesic cannot do.

Thus if ρ_n is the right-hand boundary of U_n, $\widetilde{\beta}$ and $\widetilde{\gamma}$ can each cross at most one of the ρ_n. If ρ, $\widehat{\beta}$ and $\widehat{\gamma}$ are the covering projections of ρ_n, $\widetilde{\beta}$ and $\widetilde{\gamma}$, respectively, to the annulus $\widehat{Y}$, then we obtain $|\widehat{\beta} \cdot \rho| \leq 1$ and $|\widehat{\gamma} \cdot \rho| \leq 1$. It follows by (2.4) that $|\widehat{\beta} \cdot \widehat{\gamma}| \leq 3$ and hence $d_Y(\beta, \gamma) \leq 4$ by (2.3). □

3.2. **Teichmüller geodesics and balancing.** A Teichmüller geodesic in $\mathcal{T}(Z)$ "shadows" a $\mathcal{C}(Z)$-geodesic in the following specific sense, which played a crucial role in [21].

Recall that a Teichmüller geodesic $L : \mathbf{R} \to \mathcal{T}(Z)$ can be described in terms of a family of quadratic differentials q_t holomorphic on $L(t)$: Each q_t is obtained from q_0 by scaling the horizontal directions by e^t and the vertical by e^{-t}. This determines the conformal structure $L(t)$.

In [21], we associate to the geodesic L a map $F : \mathbf{R} \to \mathcal{C}_0(Z)$ by letting $F(t)$ be any simple curve of minimal extremal length with respect to $L(t)$. Furthermore we define a map $\pi = \pi_q : \mathcal{C}_0(Z) \to \mathbf{R} \cup \{\pm\infty\}$, called a "balancing projection," as follows: Given any $\alpha \in \mathcal{C}_0(Z)$, its horizontal length $|\alpha|_{q_t,h}$ has the form $|\alpha|_{q_0,h}e^t$ and its vertical length $|\alpha|_{q_t,v}$ has the form $|\alpha|_{q_0,v}e^{-t}$. Thus if both of these are non zero there is a unique point t where they are equal, and we say α is *balanced* at t, and set $\pi_q(\alpha) = t$. If the horizontal lengths are 0 (α is parallel to the vertical foliation) then we let $\pi_q(\alpha) = +\infty$, and if the vertical lengths are 0 we let $\pi_q(\alpha) = -\infty$.

Now suppose we are given $v, w \in \mathcal{C}_0(Z)$ with $d_{\mathcal{C}(Z)}(v, w) \geq 3$. Then v and w fill Z, and so there is a conformal structure and a quadratic differential q_0 for which the horizontal and vertical foliations have closed nonsingular leaves which are isotopic to v and w, respectively. The corresponding Teichmüller geodesic is called the Teichmüller geodesic associated to (v, w). We note immediately that $\pi_q(\alpha) = -\infty$ for $d(\alpha, v) \leq 1$ and similarly that that $\pi_q(\alpha) = +\infty$ for $d(\alpha, w) \leq 1$.

Some basic properties of this projection map are outlined in the following lemma. Here $d()$ and diam() refer to distance and diameter in $\mathcal{C}_1(Z)$. The K_i are constants depending only on $\xi(Z)$. The notation $[s, t]$ refers to the interval with endpoints s and t, regardless of order.

Lemma 3.3. *Let $g = \{v_i\}_{i=M}^N$ be a geodesic segment in $\mathcal{C}(Z)$ with $M - N \geq 3$, and let $L : \mathbf{R} \to \mathcal{T}(Z)$ be the Teichmüller geodesic associated to (v_M, v_N), $F : \mathbf{R} \to \mathcal{C}_0(Z)$ its associated map and $\pi : \mathcal{C}_0(Z) \to \mathbf{R}$ the associated projection. There are constants $K_0, K_1, K_2, m_0 > 0$, depending only on the surface Z, such that:*

1. *(Lipschitz) If* $d(v,w) \le 1$ *for* $v, w \in \mathcal{C}_0(Z)$ *then*
$$\mathrm{diam}(F([\pi(v),\pi(w)])) \le K_0.$$
2. *(Fellow traveling 1) for any* v_i *in* g,
$$d(v_i, F(\pi(v_i))) \le K_1$$
3. *(Fellow traveling 2) For all* $t \in \mathbf{R}$, *there exists some* $v_i \in g$ *such that*
$$\mathrm{diam}(F([t,\pi(v_i)])) \le K_2$$
4. *(Coarse monotonicity) Whenever* v_i, v_j *are in* g *with* $j > i + m_0$,
$$\pi(v_j) > \pi(v_i)$$

Proof. Part (1) is part of Theorem 2.6 of [21], and parts (2) and (3) follow from Theorem 2.6 together with the proof of Lemma 6.1 of [21].

Part (4) is a consequence of parts (1) and (2): Since the last vertex v_N has $\pi(v_N) = +\infty$ by definition, if we have $\pi(v_{i+m}) < \pi(v_i)$ then there is some $m' \ge m$ for which $\pi(v_i) \in [\pi(v_{m'}), \pi(v_{m'+1})]$. By (1), we then have $d(F(\pi(v_i)), F(\pi(v_{i+m'}))) \le K_0$. However since $d(v_i, v_{i+m'}) = m'$, this implies together with (2) and the triangle inequality that $m' \le K_0 + 2K_1$. Setting $m_0 = K_0 + 2K_1$, we have part (4). □

3.3. **Proof of Theorem 3.1.** Let us first consider the case where g is a finite segment $\{v_i\}_{i=M}^N$. Note that at most 3 of the vertices can actually be contained in Y, since they would all be $\mathcal{C}(Z)$-distance 1 from ∂Y. We may assume without loss of generality that $|g| = N - M \ge 3$.

Select a Teichmüller geodesic $L : \mathbf{R} \to \mathcal{T}(Z)$ as above, associated to (v_M, v_N), as well as the associated map F, family of quadratic differentials q_t, and balancing map π.

Let α be any boundary component of Y (non-peripheral in Z). Let $s_0 = \pi(\alpha)$. Note that possibly $s_0 = -\infty$ or $+\infty$, if α is disjoint from v_M or v_N (but not both).

If $s_0 \ne \pm\infty$, then α is balanced at s_0. If $s_0 = -\infty$ then it is horizontal at any q_t. In either case, Lemmas 5.3 and 5.6 of [21] imply that, for $K_3 > 0$ depending only on Z, there is $s_1 \ge s_0$ with

$$\mathrm{diam}(F[s_0, s_1]) \le K_3, \tag{3.1}$$

such that α is almost-horizontal with respect to q_s whenever $s \ge s_1$, and contains an almost horizontal segment of horizontal length ϵ_1, for some fixed $\epsilon_1 > 0$. In fact we may assume $\epsilon_1 = 1$, because horizontal length expands at a definite exponential rate with distance along the Teichmüller geodesic L, and the map F is quasi-Lipschitz by Lemma 5.1 of [21]. (The case $s_0 = \infty$ is treated similarly, interchanging horizontal and vertical).

Lemma 5.7 of [21] implies that, for $K_4 > 0$ depending only on Z, there exists $s_2 > s_1$ such that

$$\mathrm{diam}(F[s_1, s_2]) \le K_4 \tag{3.2}$$

and, for any $\gamma \in \mathcal{C}(Z)$, if $\pi(\gamma) > s_2$ then γ is almost vertical with respect to q_{s_1}. Again, possibly $s_2 = \infty$.

Let j_0 be the index of the vertex of g for which part (3) of Lemma 3.3 gives

$$\operatorname{diam}(F([s_0, \pi(v_{j_0})])) \leq K_2. \tag{3.3}$$

We will now show that for $i > j_0$ sufficiently large, $\pi(v_i) > s_2$.

By the coarse monotonicity (4) of Lemma 3.3, if $i > j_0 + m_0$ then $\pi(v_i) > \pi(v_{j_0})$. Thus if $\pi(v_i) \leq s_2$, we have $d(F(\pi(v_i)), F(\pi(v_{j_0}))) \leq \operatorname{diam}(F([\pi(v_{j_0}), s_2]))$, and the latter is bounded by $K_2 + K_3 + K_4$ because of the bounds (3.1), (3.2) and (3.3). Thus, $i - j_0 = d(v_i, v_{j_0}) \leq K_2 + K_3 + K_4 + 2K_1$ by (2) of Lemma 3.3 and the triangle inequality. Letting $m_1 = 1 + \max(m_0, K_2 + K_3 + K_4 + 2K_1)$, we are therefore assured that if $i \geq j_0 + m_1$ then $\pi(v_i) > s_2$.

Thus, if $i \geq j_0 + m_1$, then v_i is almost vertical with respect to q_{s_1}.

We can now apply Lemma 3.2 using the quadratic differential q_{s_2} and the boundary component α. If $j_0 + m_1 \leq N$ then for any $i, i' \in [j_0 + m_1, N]$ we have by the above that both v_i and $v_{i'}$ are almost vertical with respect to q_{s_2}, and thus $d_Y(v_i, v_i') \leq 4$.

The same argument, with horizontal and vertical interchanged, applies to give a bound for $i, i' \in [M, j_0 - m_1]$, if $M \leq j_0 - m_1$. The remaining segment between $\max(M, j_0 - m_1)$ and $\min(N, j_0 + m_1)$ has a diameter bound of $2m_1$, so its π_Y-image has diameter at most $4m_1$ by Lemma 2.3 (Lipschitz Projection). Thus the image of the full segment g is bounded by $4m_1 + 8$.

Since this bound is independent of N and M, it implies a bound also in the infinite cases, via an exhaustion of g by finite subsegments. This concludes the proof of Theorem 3.1.

4. Tight geodesics and hierarchies

This section describes the main construction of our paper, hierarchies of tight geodesics. After defining these notions in §4.1, we prove some existence results, Lemma 4.5 and Theorem 4.6, in §4.2.

Hierarchies give us the combinatorial framework in which to carry out the link projection arguments first outlined in the examples in §1.5 (and done in generality in Section 6). The main ingredient in this is the *backward and forward sequences* $\Sigma^{\pm}$, whose basic structural properties are stated in Theorem 4.7. The proof of this theorem takes up the rest of Section 4, and along the way we will develop a number of results, notably Theorem 4.20, which describes when a hierarchy is *complete.* We will also define a "time order", which is a partial order on a hierarchy, generalizing the linear order on vertices of a single geodesic, that will serve as a basic organizational principle in the proofs here and in later sections.

4.1. **Definitions.**

Tight geodesics. The non-uniqueness of geodesics in $\mathcal{C}(S)$ is already manifested at a local level, where typically, if $d_{\mathcal{C}}(\alpha,\gamma)=2$ there can be infinitely many choices for a curve β disjoint from both. The notion of *tightness*, defined below, addresses this local problem, but more importantly introduces a crucial ingredient of control that makes our combinatorial description of hierarchies possible. It is worth noting that the only place where we make direct use of tightness is in Lemma 4.10.

A pair of curves or curve systems α,β in a surface Y are said to *fill* Y if all non-trivial non-peripheral curves in Y intersect at least one of α or β. If Y is a subdomain of S then it also holds that any curve γ in S which intersects a boundary component of Y must intersect one of α or β.

Given arbitrary curve systems α,β in $\mathcal{C}(S)$, there is a unique subsurface $F(\alpha,\beta)$ which they fill: Namely, thicken the union of the geodesic representatives, and fill in all disks and once-punctured disks. Note that F is connected if and only if the union of geodesic representatives is connected.

For a subdomain $X\subseteq Z$ let $\partial_Z(X)$ denote the *relative boundary* of X in Z, i.e. those boundary components of X that are non-peripheral in Z.

Definition 4.1. *Let Y be a domain in S. If $\xi(Y)>4$, a sequence of simplices $\{v_0,\dots,v_N\}$ in $\mathcal{C}(Y)$ is called* tight *if*

1. *For any vertices w_i of v_i and w_j of v_j where $i\neq j$, $d_{\mathcal{C}(Y)}(w_i,w_j)=|i-j|$,*
2. *For each $1\le i\le N-1$, v_i represents the relative boundary $\partial_Y F(v_{i-1},v_{i+1})$.*

If $\xi(Y)=4$ then a tight sequence is just the vertex sequence of any geodesic.

If $\xi(Y)=2$ then a tight sequence is the vertex sequence of any geodesic, with the added condition that the set of endpoints on $\partial\widehat{Y}$ of arcs representing the vertices equals the set of endpoints of the first and last arc.

Note that condition (1) of the definition specifies that given any choice of components w_i of v_i the sequence $\{w_i\}$ is a geodesic in the original sense. It also implies that v_{i-1} and v_{i+1} always have connected union.

In the annulus case, the restriction on endpoints of arcs is of little importance, serving mainly to guarantee that there between any two vertices there are only finitely many tight sequences.

With this in mind, a *tight geodesic* will be a tight sequence together with some additional data:

Definition 4.2. *A* tight geodesic *g in $\mathcal{C}(Y)$ consists of a tight sequence $\{v_0,\dots,v_N\}$, and two markings $\mathbf{I}=\mathbf{I}(g)$ and $\mathbf{T}=\mathbf{T}(g)$ (in the sense of §2.5), called its* initial *and* terminal *markings, such that v_0 is a vertex of* base($\mathbf{I}$) *and v_N is a vertex of* base($\mathbf{T}$).

The number N is called the length of g, usually written $|g|=N$. We refer to each of the v_i as vertices *of g (by a slight abuse of notation). Y is called the* domain *or* support *of g and we write $Y=D(g)$. We also say that g is* supported in $D(g)$.

Finally we will also, occasionally, allow tight geodesics to be infinite, in one or both directions. If a tight geodesic g is infinite in the forward direction then $\mathbf{T}(g)$ is not defined, and if it is infinite in the backward direction then $\mathbf{I}(g)$ is not defined.

Subordinacy. We first saw the relations of *forward subordinacy* and *backward subordinacy* in the simple examples in Section 1.5. Let us now introduce a bit more notation and give the general definitions.

Restrictions of markings: If W is a domain in S and μ is a marking in S, then the *restriction* of μ to W, which we write $\mu|_W$, is constructed from μ in the following way: Suppose first that $\xi(W) \geq 3$. Recall that for every $p \in \mu$, either $p = \alpha \in \operatorname{base}(\mu)$ or $p = (\alpha, t)$ with t a transversal to α. We let $\mu|_W$ be the set of those p whose base curve α meets W essentially. (Recall that α meets W essentially if it cannot be deformed away from W – in particular if $\alpha \subset W$ it must be non-peripheral).

If W is an annulus ($\xi(W) = 2$) then $\mu|_W$ is just $\pi_W(\mu)$.

Note in particular that, if all the base curves of μ which meet W essentially are actually contained in W, then $\mu|_W$ is in fact a marking of W. If W is an annulus then $\mu|_W$ is a marking of W whenever it is non-empty.

Component domains: Given a surface W with $\xi(W) \geq 4$ and a curve system v in W we say that Y is a *component domain of* (W, v) if either: Y is a component of $W \setminus v$, or Y is an annulus with core a component of v. Note that in the latter case Y is non-peripheral, and thus satisfies our definition of "domain".

Call a subsurface $Y \subset S$ a *component domain of* g if for some vertex v_j of g, Y is a component domain of $(D(g), v_j)$. We note that this determines v_j uniquely. In such a case, and assuming $\xi(Y) \neq 3$, let

$$\mathbf{I}(Y,g) = \begin{cases} v_{j-1}|_Y & v_j \text{ is not the first vertex} \\ \mathbf{I}(g)|_Y & v_j \text{ is the first vertex} \end{cases}$$

be the *initial marking* of Y relative to g. Similarly let

$$\mathbf{T}(Y,g) = \begin{cases} v_{j+1}|_Y & v_j \text{ is not the last vertex} \\ \mathbf{T}(g)|_Y & v_j \text{ is the last vertex} \end{cases}$$

denote the *terminal marking.* Note in particular that these are indeed markings. If $\xi(Y) = 3$ (Y is a thrice-punctured sphere) then we define $\mathbf{I}(Y,g) = \mathbf{T}(Y,g) = \emptyset$.

Special cases:

1. The motivating case is that in which v_j is neither first nor last, and $\xi(D(g)) > 4$. If Y is the component of $D(g) \setminus v_j$ which is filled by v_{j-1} and v_{j+1}, then $\mathbf{I}(Y,g) = v_{j-1}$ and $\mathbf{T}(Y,g) = v_{j+1}$. If Y is any other component domain of $(D(g), v_j)$ then $\mathbf{I}(Y,g) = \mathbf{T}(Y,g) = \emptyset$.
2. If $\xi(D(g)) > 4$ and Y is an annulus (whose core curve is a component of v_j), then unless $j = 0$ or $j = |g|$, we must have $\mathbf{I}(Y,g) = \mathbf{T}(Y,g) = \emptyset$,

since successive curves in g are disjoint. If e.g. $j = 0$, then the core of Y is a base curve of $\mathbf{I}(g)$, so if this curve has a transversal in the marking $\mathbf{I}(g)$ then $\mathbf{I}(Y, g)$ is nonempty.

3. If $\xi(D(g)) = 4$ then Y must be a thrice-punctured sphere or an annulus. In the former case $\mathbf{I}(Y, g) = \mathbf{T}(Y, g) = \emptyset$, but in the latter case $\mathbf{I}(Y, g)$ and $\mathbf{T}(Y, g)$ may be nonempty because successive curves in g do intersect.

If Y is a component domain of g and $\mathbf{T}(Y, g) \neq \emptyset$ then we say that Y is *directly forward subordinate* to g, or $Y \overset{d}{\searrow} g$. Similarly if $\mathbf{I}(Y, g) \neq \emptyset$ we say that Y is *directly backward subordinate* to g, or $g \overset{d}{\swarrow} Y$.

We can now define subordinacy for geodesics:

Definition 4.3. *If k and g are tight geodesics, we say that k is* directly forward subordinate *to g, or $k \overset{d}{\searrow} g$, provided $D(k) \overset{d}{\searrow} g$ and $\mathbf{T}(k) = \mathbf{T}(D(k), g)$. Similarly we define $g \overset{d}{\swarrow} k$ to mean $g \overset{d}{\swarrow} D(k)$ and $\mathbf{I}(k) = \mathbf{I}(D(k), g)$.*

We denote by *forward-subordinate*, or $\searrow$, the transitive closure of $\overset{d}{\searrow}$, and similarly for $\swarrow$. We let $h \overset{\searrow}{=} k$ denote the condition that $h = k$ or $h \searrow k$, and similarly for $k \overset{\swarrow}{=} h$. We include the notation $Y \searrow f$ where Y is a domain to mean $Y \overset{d}{\searrow} f'$ for some f' such that $f' \overset{\searrow}{=} f$, and similarly define $b \swarrow Y$.

Hierarchies.

Definition 4.4. *A* hierarchy of geodesics *is a collection H of tight geodesics in S with the following properties:*

1. *There is a distinguished* main geodesic *g_H with domain $D(g_H) = S$. The initial and terminal markings of g_H are denoted also $\mathbf{I}(H), \mathbf{T}(H)$.*
2. *Suppose $b, f \in H$, and $Y \subset S$ is a domain such that $b \overset{d}{\swarrow} Y$ and $Y \overset{d}{\searrow} f$. Then H contains a unique tight geodesic k such that $D(k) = Y$, $b \overset{d}{\swarrow} k$ and $k \overset{d}{\searrow} f$.*
3. *For every geodesic k in H other than g_H, there are $b, f \in H$ such that $b \overset{d}{\swarrow} k \overset{d}{\searrow} f$.*

Condition (3) implies that for any k in H, there is a sequence $k = f_0 \overset{d}{\searrow} \ldots \overset{d}{\searrow} f_m = g_H$, and similarly $g_H = b_n \overset{d}{\swarrow} \ldots \overset{d}{\swarrow} b_0 = k$. Later we will prove these sequences are unique.

Infinite hierarchies. An infinite hierarchy is one in which the main geodesic g_H is allowed to be an infinite ray or a line. Note that in this case $\mathbf{I}(H)$ and/or $\mathbf{T}(H)$ may not be defined. Typically a hierarchy will be finite, but most of the machinery of the paper will work for infinite hierarchies, so we will indicate where relevant how each proof works in the infinite case. Infinite hierarchies will arise, as limits, in §6.5, and will be used in Section 7.

4.2. **Existence.** In this section we will prove that hierarchies exist. The first step is the following:

Lemma 4.5. (Tight geodesics exist) *Let u and v be two vertices in $\mathcal{C}(Y)$. There exists a tight sequence $v_0, \ldots, v_N$ such that $v_0 = u$ and $v_N = v$.*

(Note that whereas u and v are single vertices in the complex $\mathcal{C}(Y)$, the interior vertices of the sequence may actually be curve systems, i.e. simplices of $\mathcal{C}(Y)$.)

Proof. If $\xi(Y) = 4$ then the vertex sequence of any geodesic is tight, by definition. If $\xi(Y) = 2$ then the proof is an easy exercise. For example one can start with u and apply Dehn twists in the covering annulus $\widehat{Y}$ to obtain a sequence of curves with the same endpoints as u on $\partial\widehat{Y}$, arriving at one which has one intersection with v and making one final step.

We now assume $\xi(Y) > 4$. To begin, let $h = \{u = u_0, \ldots, u_N = v\}$ be a regular geodesic connecting u and v. We will describe a process that adjusts h until a tight sequence is obtained.

Let v_1, v_2, v_3, v_4 be any sequence of simplices satisfying condition (1) of Definition 4.1, and suppose also that v_3 is the boundary of $F(v_2, v_4)$, so that (2) holds for v_3. If we now replace v_2 by $v_2' = \partial F(v_1, v_3)$, we want to show that v_3 is still $\partial F(v_2', v_4)$. In other words, "fixing" v_2 so that Condition (2) holds for it will not spoil condition (2) for v_3.

Note that (1) still holds for v_1, v_2', v_3, v_4, by the triangle inequality. In particular each component of v_2' intersects each component of v_4, so that their union is connected and so is $F(v_2', v_4)$. Since v_2' is disjoint from $v_3 = \partial F(v_2, v_4)$, v_2' must be contained in $F(v_2, v_4)$ and in particular $F(v_2', v_4) \subseteq F(v_2, v_4)$. Thus it suffices to show that v_2' and v_4 fill $F(v_2, v_4)$. Let α be any curve in $F(v_2, v_4)$. If α doesn't intersect v_4 then it must intersect v_2, and also v_1 since v_1 and v_4 fill S. But since v_2 is not contained in $F(v_1, v_3)$, α must cross $\partial F(v_1, v_3)$, which is just v_2'. We conclude that $F(v_2', v_4) = F(v_2, v_4)$.

Now we can adjust the vertices of h in any order: For any $i \in [1, N-1]$ replace u_i by $\partial F(u_{i-1}, u_{i+1})$. For the new sequence, condition (2) holds for the i-th vertex. Repeating the process for a new value of i in $[1, N-1]$, the previous argument assures us that the condition persists for previously adjusted values of i. Thus after $N - 1$ steps we obtain a tight sequence.

Note that there is no reason to expect a unique tight sequence – the process seems to depend on the order in which the indices are chosen. ☐

We will now show, starting with any two markings in a surface S, how to build a hierarchy connecting them. That is,

Theorem 4.6. (Hierarchies exist) *Let P and Q be two markings in a surface S. There exists a hierarchy H of tight geodesics such that $\mathbf{I}(H) = P$ and $\mathbf{T}(H) = Q$.*

Proof. We say that H is a *partial hierarchy* if it satisfies properties (1) and (3) of Definition 4.4, and the uniqueness part of (2), but not necessarily existence. That is:

(2') Suppose $b, f \in H$, and $Y \subset S$ is a domain such that $b \overset{d}{\swarrow} Y$ and $Y \overset{d}{\searrow} f$. Then H contains *at most one* tight geodesic k such that $D(k) = Y$, $b \overset{d}{\swarrow} k$ and $k \overset{d}{\searrow} f$.

Of course every hierarchy is also a partial hierarchy.

We begin by choosing vertices $v \in \mathrm{base}(P)$ and $w \in \mathrm{base}(Q)$, and connecting them with a tight sequence, which exists by Lemma 4.5. Define a tight geodesic g by letting its sequence be this one, and setting $\mathbf{I}(g) = P$ and $\mathbf{T}(g) = Q$.

Let H_0 be the partial hierarchy $\{g\}$, and let us construct a finite sequence of partial hierarchies H_n, the last of which is a hierarchy.

Call a triple (Y, b, f) with domain Y and $b, f \in H_n$ an *unutilized configuration* if $b \overset{d}{\swarrow} Y \overset{d}{\searrow} f$ but Y is not the support of any geodesic $k \in H_n$ such that $b \overset{d}{\swarrow} k \overset{d}{\searrow} f$.

Choose (Y_n, b_n, f_n) to be any unutilized configuration in H_n. Again use Lemma 4.5 to construct a tight geodesic h_n supported in Y_n, with $\mathbf{I}(h_n) = \mathbf{I}(Y_n, b_n)$ and $\mathbf{T}(h_n) = \mathbf{T}(Y_n, f_n)$. Let $H_{n+1} = H_n \cup \{h_n\}$.

The only thing to check is that the sequence terminates. Define a sequence of tuples $M_n = (M_n(1), M_n(2), \ldots, M_n(\xi(S) - 2))$ by letting $M_n(j)$ denote the number of unutilized configurations (Y_n, b_n, f_n) in H_n with $\xi(Y_n) = \xi(S) - j$. Then, since for each Y_n in the above step, all component domains occurring in the geodesic h_n have complexity ξ strictly smaller than $\xi(Y_n)$, it follows immediately that the sequence M_n is strictly decreasing in lexicographic order as n increases. (Recall that in lexicographic order $(x_1, \ldots, x_k) < (y_1, \ldots, y_k)$ when for some $j \leq k$, $x_i = y_i$ for all $i < j$ and $x_j < y_j$.) Therefore the sequence terminates in a partial hierarchy with no unutilized configurations – that is, a hierarchy. ☐

Note that the uniqueness part of property (2) holds automatically: although the choice of h_n at each stage was arbitrary, we never put in more than one geodesic for a given configuration $b \overset{d}{\swarrow} Y \overset{d}{\searrow} f$.

4.3. **Forward and backward sequences.** Given a domain $Y \subset S$ and a hierarchy H, define

$$\Sigma_H^+(Y) = \{f \in H : Y \subseteq D(f) \ \text{ and } \ \mathbf{T}(f)|_Y \neq \emptyset\}$$

and similarly

$$\Sigma_H^-(Y) = \{b \in H : Y \subseteq D(b) \ \text{ and } \ \mathbf{I}(b)|_Y \neq \emptyset\}$$

which we also abbreviate by omitting the H or Y when they are understood. For infinite hierarchies, we alter the definition by also admitting g_H into Σ^+ whenever g_H is infinite in the forward direction, and into Σ^- whenever it is infinite in the backward direction.

We will call $\Sigma^+(Y)$ the *forward sequence of* Y and $\Sigma^-(Y)$ the *backward sequence of* Y. These names will be justified by the following theorem, which is perhaps the main point of our construction.

Theorem 4.7. (Structure of Sigma) *Let H be a hierarchy, and Y any domain in its support S.*

1. *If $\Sigma^+_H(Y)$ is nonempty then it has the form of a sequence*
$$f_0 \overset{d}{\searrow} \cdots \overset{d}{\searrow} f_n = g_H,$$
where $n \geq 0$. Similarly, if $\Sigma^-_H(Y)$ is nonempty then it has the form of a sequence
$$g_H = b_m \overset{d}{\swarrow} \cdots \overset{d}{\swarrow} b_0,$$
where $m \geq 0$.
2. *If $\Sigma^{\pm}(Y)$ are both nonempty and $\xi(Y) \neq 3$, then $b_0 = f_0$, and Y intersects every vertex of f_0 nontrivially.*
3. *If Y is a component domain in any geodesic $k \in H$ and $\xi(Y) \neq 3$, then*
$$f \in \Sigma^+(Y) \iff Y \searrow f,$$
and similarly,
$$b \in \Sigma^-(Y) \iff b \swarrow Y.$$
If, furthermore, $\Sigma^{\pm}(Y)$ are both nonempty, then in fact Y is the support of $b_0 = f_0$.
4. *Geodesics in H are determined by their supports. That is, if $D(h) = D(h')$ for $h, h' \in H$ then $h = h'$.*

The ingredients for the proof of Theorem 4.7 will be developed throughout the rest of the section, and the proof will be completed in §4.8.

In Section 6, $\Sigma^{\pm}$ will be converted into forward and backward *paths* in $\mathcal{C}(S)$ which will enable us to generalize the projection arguments in the examples of §1.5, and prove Lemmas 6.2, 6.6 and their relatives.

4.4. **Footprints and subordinacy.** We begin with the following basic lemma, which gives one direction of Part (3) of Lemma 4.7:

Lemma 4.8. (Subordinate Intersection 1) *Let H be a hierarchy in a surface S and Y a domain in S. Let h and f denote geodesics in H.*

1. *If $Y \overset{d}{\searrow} h$ then $h \in \Sigma^+_H(Y)$.*
2. *If $h \in \Sigma^+_H(Y)$ and $h \searrow f$, then $f \in \Sigma^+_H(Y)$.*
3. *If $Y \searrow f$ then $f \in \Sigma^+_H(Y)$.*

The same holds with $\searrow$ replaced by $\swarrow$, and Σ^+ replaced by Σ^-.

Footprints. We will first need one new definition, which will be a basic tool in all that follows:

Definition 4.9. *For a domain $Y \subset S$ and a tight geodesic g with non-annular support $D(g) \subset S$, let $\phi_g(Y)$ be the set of vertices of g disjoint from Y. We call this the* footprint *of Y on g.*

If Y is a subdomain of $D(g)$, then immediately

$$\operatorname{diam}(\phi_g(Y)) \leq 2 \tag{4.1}$$

in the curve complex of $D(g)$ (note if Y is an annulus then by definition of subdomain it is nonperipheral in $D(g)$). It is also an immediate consequence of the definition that

$$Y \subseteq Z \implies \phi_g(Z) \subseteq \phi_g(Y). \tag{4.2}$$

Let us record the following elementary but crucial property of footprints, which is the only place where the tightness property is used directly.

Lemma 4.10. *If g is a tight geodesic and $Y \subset D(g)$ is a proper subdomain, then $\phi_g(Y)$ is a sequence of 0,1, 2 or 3 contiguous vertices of g.*

Proof. When $\xi(D(g)) = 4$, $\phi_g(Y)$ is empty except when Y is an annulus whose core is some vertex v of g. In that case every other vertex intersects Y, so $\phi_g(Y)$ is the single vertex v.

Now assume $\xi(D(g)) > 4$. The diameter bound (4.1) implies that the only possibility for $\phi_g(Y)$ other than those mentioned in the lemma is that $\phi_g(Y)$ contains some v_j and v_{j+2} but not v_{j+1}. However, since g is a tight geodesic, if Y intersects v_{j+1} it either intersects v_j or v_{j+2}, since $v_{j+1} = \partial_{D(g)} F(v_j, v_{j+2})$. □

Denote by $\min \phi_g(Y)$ and $\max \phi_g(Y)$ the vertices of $\phi_g(Y)$ with lowest and highest index, respectively.

Proof of Lemma 4.8. Clearly (3) is a consequence of (1) and (2), so we prove them. We will prove the forward-subordinate case. The backward-subordinate case proceeds similarly.

To see (1), suppose $Y \overset{d}{\searrow} h$. Then by definition Y is a component domain of $(D(h), v_i)$ for some vertex v_i of h, and $\mathbf{T}(Y, h) \neq \emptyset$. If v_i is the last vertex then $\mathbf{T}(Y, h) = \mathbf{T}(h)|_Y$, so this is nonempty and $h \in \Sigma^+(Y)$.

If v_i is not the last vertex, we note that $v_i \in \phi_h(Y)$ and v_{i+1} is not in $\phi_h(Y)$. It follows, since the footprint is contiguous (Lemma 4.10), that the last vertex is not in $\phi_h(Y)$, hence $\mathbf{T}(h)|_Y \neq \emptyset$, and again $h \in \Sigma^+(Y)$. If h is infinite in the forward direction ($h = g_H$, and $\mathbf{T}(h)$ undefined) then automatically $h \in \Sigma^+(Y)$.

Now to prove (2), if $h \in \Sigma^+(Y)$ we have $Y \subset D(h)$ and $\mathbf{T}(h)|_Y \neq \emptyset$. Suppose first that $h \overset{d}{\searrow} f$ – then $D(h)$ is a component domain of $(D(f), v_j)$ for some vertex v_j of f. If v_j is the last vertex then $\mathbf{T}(h) = \mathbf{T}(f)|_{D(h)}$, and it follows that $\mathbf{T}(f)|_Y \neq \emptyset$. If v_j is not the last vertex then $\mathbf{T}(h) = v_{j+1}|_{D(h)}$ and since this intersects Y, $v_{j+1} \notin \phi_f(Y)$. Thus also the last vertex of f is not in $\phi_f(Y)$, and we may again conclude that $\mathbf{T}(f)|_Y \neq \emptyset$ (or f is infinite in the forward direction). In each case we have $f \in \Sigma^+(Y)$. Part (2) now follows by induction. □

The proof also gives the following slightly finer statement:

Corollary 4.11. (Footprints) *Let H be a hierarchy, geodesics $h, f, b \in H$ and domain $Y \subset D(h)$. If $h \in \Sigma^+_H(Y)$ and $h \searrow f$, then*

$$\max \phi_f(D(h)) = \max \phi_f(Y).$$

Similarly if $h \in \Sigma^-_H(Y)$ *and* $b \swarrow h$ *then*

$$\min \phi_b(D(h)) = \min \phi_b(Y).$$

Note, a special case of this is that if $h_1 \searrow h_2 \searrow f$ then $\max \phi_f(D(h_1)) = \max \phi_f(D(h_2))$ by letting $Y = D(h_1)$. (The condition $h_2 \in \Sigma^+(D(h_1))$ follows from Lemma 4.8.)

Proof. Since $h \searrow f$ there exists h' such that $h \underset{=}{\searrow} h' \overset{d}{\searrow} f$. Examining the proof of part (2) in Lemma 4.8 above, we note that it shows that $D(h')$ is a component domain of $(D(f), v)$ where $v = \max \phi_f(D(h'))$, and if v is not the last vertex in f then its successor intersects Y and $D(h)$. Hence $v = \max \phi_f(Y) = \max \phi_f(D(h))$. If v is the last vertex then automatically $v = \max \phi_f(Y) = \max \phi_f(D(h))$, since $\phi_f(D(h')) \subseteq \phi_f(D(h)) \subseteq \phi_f(Y)$ by (4.2). The backward case proceeds similarly. ☐

4.5. **Uniqueness of descent.** By virtue of lemma 4.8, we know that $\Sigma^+(Y)$ contains any sequence of geodesics $f_0, \ldots, f_n$ satisfying $f_0 \in \Sigma^+(Y)$ and $f_i \overset{d}{\searrow} f_{i+1}$ (and similarly for Σ^-). The goal of the next lemma is to show that in fact Σ^+ and Σ^- are each just one such sequence, and as a consequence to prove that geodesics in H are determined by their domains.

Lemma 4.12. (Uniqueness of Descent) *Let H be a hierarchy, and Y any domain in its support S.*

1. *If $\Sigma^+_H(Y)$ is nonempty then it has the form of a sequence $f_0 \overset{d}{\searrow} \cdots \overset{d}{\searrow} f_n = g_H$, where $n \geq 0$. Similarly if $\Sigma^-_H(Y)$ is nonempty then it has the form of a sequence $g_H = b_m \overset{d}{\swarrow} \cdots \overset{d}{\swarrow} b_0$, where $m \geq 0$.*
2. *If there is some $h \in H$ with $D(h) = Y$, then there is exactly one such h, and $h = f_0 = b_0$.*

In particular, this gives parts (1) and (4) of Theorem 4.7 (Structure of Sigma).

Proof. Note that (2) is a consequence of (1) for any given Y, since if $Y = D(h)$ then $h \in \Sigma^+$ and must have the smallest domain of any member of Σ^+ – hence $h = f_0$, and similarly with Σ^- we have $h = b_0$. In particular h is unique.

We will prove (1) by induction on $\xi(S) - \xi(Y)$. If $\xi(S) - \xi(Y) = 0$ then $Y = S$ and $\Sigma^+ = \Sigma^- = \{g_H\}$, hence (1) holds.

Let $g \in \Sigma^+(Y)$, and suppose that $f \overset{d}{\searrow} g$ for some $f \in \Sigma^+(Y)$. We claim that $D(f)$ is uniquely determined by Y and g, and in fact if $\xi(D(f)) > \xi(Y)$ then f itself is uniquely determined. Suppose for a moment that Y is not an annulus. By definition of $f \overset{d}{\searrow} g$, $D(f)$ is a component domain for $(D(g), v)$ where the vertex v is $\max \phi_g(D(f))$. By Corollary 4.11 (Footprints), v is also $\max \phi_g(Y)$. Hence, $D(f)$ is the unique component of $(D(g), v)$ containing Y, which depends only on Y and g. Now if $\xi(D(f)) > \xi(Y)$ then by induction (2) holds for $D(f)$, so that f is the unique geodesic in Σ^+ such that $f \overset{d}{\searrow} g$.

If Y is an annulus the same proof goes through verbatim, recalling that if $D(f)$ is not an annulus it must contain Y as a nonperipheral annulus (otherwise $\mathbf{T}(f)|_Y$ would be empty, contradicting $f \in \Sigma^+(Y)$), and is therefore the unique component domain of $(D(g), v)$ with this property. If $D(f)$ is an annulus it must be equal to Y so again it is uniquely determined.

If $\Sigma^+ \neq \emptyset$, then since any $f \in \Sigma^+$ is forward-subordinate to g_H, Lemma 4.8 implies that $g_H \in \Sigma^+$. For any $k \in \Sigma^+$ there exists some f such that $k \stackrel{\searrow}{=} f \stackrel{d}{\searrow} g_H$, and $f \in \Sigma^+$ again by Lemma 4.8. By the previous claim, we know that $D(f)$ is independent of k, and so is f if $D(f) \neq Y$. Thus, replacing g_H with f and repeating this argument inductively, we obtain a single sequence $f_1 \stackrel{d}{\searrow} \cdots \stackrel{d}{\searrow} g_H$ which accounts for all of Σ^+ except possibly those geodesics f with $D(f) = Y$.

Now repeating this for Σ^- we obtain a sequence $g_H \stackrel{d}{\swarrow} \cdots \stackrel{d}{\swarrow} b_1$. If Y does not support any geodesic then we are done (reindexing both sequences to start with 0). If Y supports at least one geodesic h, then $h \in \Sigma^+ \cap \Sigma^-$, and by the same logic as above we have that $h \stackrel{d}{\searrow} f_1$ and $b_1 \stackrel{d}{\swarrow} h$. However, by the uniqueness part of the definition of a hierarchy, there can only be one such h. Setting $f_0 = b_0 = h$, we are done. ☐

The following partial converse of Lemma 4.8 is an immediate corollary of Lemma 4.12:

Lemma 4.13. (Subordinate Intersection 2) *Let k and h be geodesics in a hierarchy H.*

If $h \in \Sigma^+_H(D(k))$ then $k \stackrel{\searrow}{=} h$. Similarly,

If $h \in \Sigma^-_H(D(k))$ then $h \stackrel{\swarrow}{=} k$.

Here is an easy corollary of Lemma 4.13.

Corollary 4.14. *If $D(k)$ is properly contained in $D(h)$ then $\phi_h(D(k))$ is non-empty.*

Proof. If h is infinite then $h = g_H$ and by definition $h \swarrow k$, so $\phi_h(D(k)) \neq \emptyset$. Otherwise $\mathbf{I}(h)$ is defined. If $\mathbf{I}(h)|_{D(k)}$ is empty then $\phi_h(D(k))$ contains the initial vertex. If not, then $h \in \Sigma^-(D(k))$ and, by Lemma 4.13, k is backward-subordinate to h, and hence $D(k)$ is contained in a component domain for some other vertex, so that again $\phi_h(D(k))$ is non-empty. ☐

Let us also record the following consequence of these lemmas:

Lemma 4.15. *Let Y be a domain in S and h in a hierarchy H such that $Y \stackrel{d}{\searrow} h$. Then h is uniquely determined, and in particular, writing $\Sigma^+_H(Y) = \{f_0, \dots\}$ we have either $h = f_1$ and $Y = D(f_0)$, or $h = f_0$ and Y supports no geodesic in H.*

The corresponding statement holds when $h \stackrel{d}{\swarrow} Y$.

Proof. By Lemma 4.8, $h \in \Sigma^+(Y)$. If $h = f_0$ then Y cannot support a geodesic because $D(h)$ has the smallest domain among elements of $\Sigma^+(Y)$. Suppose $h = f_{i+1}$ where $i \geq 0$. Then by Corollary 4.11, $\max \phi_h(Y) =$

$\max \phi_h(D(f_i))$ and hence both Y and $D(f_i)$ are component domains for the same vertex of h. As in the proof of Lemma 4.12 we conclude $Y = D(f_i)$ and so $i = 0$ since there can be no smaller domain in $\Sigma^+(Y)$. The case where $h \overset{d}{\swarrow} Y$ is similar. □

4.6. **Time order.** The vertices of any geodesic admit a linear order from initial to terminal, and the relations $\searrow$ and $\swarrow$ are, by definition, partial orders. It turns out that these can be combined to define a useful partial order $\prec_t$ on a hierarchy, and a related partial order $\prec_p$ on the set of "pointed geodesics" of a hierarchy. In this section we define these relations and study their basic properties. Let a hierarchy H be fixed througout this section.

Definition 4.16. *For any $h, h' \in H$, we say that h* precedes *h'* in time order*, or*

$$h \prec_t h',$$

if there exists a geodesic $m \in H$ such that $D(h), D(h') \subset D(m)$, and

$$\max \phi_m(D(h)) < \min \phi_m(D(h')).$$

(In particular $\phi_m(D(h))$ and $\phi_m(D(h'))$ are disjoint.)

Note that if this occurs then automatically $h \searrow m$ and $m \swarrow h'$: since $\phi_m(D(h))$ must miss the terminal vertex of m, m is in $\Sigma^+(D(h))$ and we may apply Lemma 4.13, and similarly for h' using $\Sigma^-(D(h'))$.

We call m the *geodesic used to compare h and h'*, and note that it is unique: If some m' is also used to obtain $h \prec_t h'$, then both m and m' appear in the forward sequence of h and the backward sequence of h'. In particular either $m \searrow m'$ or $m' \searrow m$; suppose the first, without loss of generality. Then $\phi_{m'}(D(m))$ is non-empty, and by (4.2) is contained in both $\phi_{m'}(D(h))$ and $\phi_{m'}(D(h'))$, contradicting the assumption that they are disjoint.

If either $h \prec_t h'$ or $h' \prec_t h$ then we say h and h' are *time-ordered.* Note, we have not yet shown that these two possibilities are mutually exclusive, or indeed that $\prec_t$ is a partial order. Before we do that let us define a more general relation.

Partial order on pointed geodesics. Let k be a tight geodesic with vertices $v_0, \dots, v_N$. We generalize slightly the notion of vertex to a *position* on k, which is either a vertex v_i, or $\mathbf{I}(k)$ or $\mathbf{T}(k)$. The linear order $v_i < v_j$ when $i < j$ extends to an order on positions where we say $\mathbf{I}(k) < v_0$ if the two are not the same, and similarly $v_N < \mathbf{T}(k)$ if the two are not the same. We can now discuss *pointed geodesics*, which are pairs (k, v) where v is a position in k.

We extend the notion of footprint slightly as follows: Given a pointed geodesic (k, v) and a geodesic h with $D(k) \subseteq D(h)$, we define

$$\hat{\phi}_h(k, v) = \begin{cases} \phi_h(D(k)) & \text{if } D(k) \subset D(h), \\ \{v\} & \text{if } k = h. \end{cases}$$

Note that $\hat{\phi}_h(k,v)$ could be $\{\mathbf{I}(k)\}$ or $\{\mathbf{T}(k)\}$ in the second case, in contrast with regular footprints which can only consist of vertices. We now define a relation $\prec_p$ on pairs (k,v):

Definition 4.17. *We say*

$$(k,v) \prec_p (h,w)$$

if and only if there exists a geodesic m such that $k \stackrel{\searrow}{=} m \stackrel{\swarrow}{=} h$, and

$$\max \hat{\phi}_m(k,v) < \min \hat{\phi}_m(h,w).$$

Again, it is clear that m is unique. It is also immediate from the definitions that

$$k \prec_t h \iff (k, \mathbf{T}(k)) \prec_p (h, \mathbf{I}(h)). \tag{4.3}$$

Indeed, $(k,v) \prec_p (h,w)$ breaks up into four possible, mutually exclusive, cases:

- $k \prec_t h$,
- $k = h$ and $v < w$,
- $k \searrow h$ and $\max \phi_h(D(k)) < w$
- $k \swarrow h$ and $v < \min \phi_k(D(h))$.

We now verify that these relations are partial orders, together with a number of other properties. The following lemma holds for infinite as well as finite hierarchies.

Lemma 4.18. (Time Order)

1. *If $D(h) \subseteq D(h')$ then h and h' are not time-ordered.*
2. *On the other hand if $D(h) \cap D(h') \neq \emptyset$ and neither domain is contained in the other, then h and h' are time-ordered.*
3. *Suppose $b \stackrel{\swarrow}{=} k \stackrel{\searrow}{=} f$. Then either $b = f$, $b \searrow f$, $b \swarrow f$, or $b \prec_t f$.*
4. *Suppose that $k_1 \prec_t k_2$. If $h \stackrel{\searrow}{=} k_1$ then $h \prec_t k_2$. Similarly if $k_2 \stackrel{\swarrow}{=} g$ then $k_1 \prec_t g$.*
5. *The relation $\prec_p$ is a (strict) partial order.*
6. *The relation $\prec_t$ is a (strict) partial order.*

Proof. To prove part (1), suppose $D(h) \subseteq D(h')$. Then in any geodesic m such that $D(m)$ contains $D(h')$, $\phi_m(D(h')) \subseteq \phi_m(D(h))$. In particular the footprints can never be disjoint, and hence neither $h \prec_t h'$ nor $h' \prec_t h$ can hold.

Next let us prove part (2). Suppose $D(h) \cap D(h') \neq \emptyset$. Consider the following assertion: If m is a geodesic such that $D(m)$ contains $D(h) \cup D(h')$, and also $m \swarrow h$, then either $D(h') \subseteq D(h)$, or $D(h) \subseteq D(h')$, or h and h' are time-ordered. We shall prove this by induction on $\xi(D(m))$.

If $\xi(D(m)) = 2$, then $m = h = h'$ and we are done. More generally, if $\xi(D(m)) = \xi(D(h'))$ then $D(h) \subseteq D(h')$, and again we are done. Otherwise, we must have $\xi(D(m)) > \xi(D(h'))$, so consider the footprints $\phi_m(D(h))$ and $\phi_m(D(h'))$ (the former is non-empty since $m \swarrow h$, and the latter is non-empty by Corollary 4.14). If they are disjoint then h and h' are time-ordered and again we are done. If they overlap then, since each is an interval of contiguous vertices, the minimum of one must be contained in the other.

If the minimum v of $\phi_m(D(h))$ is contained in $\phi_m(D(h'))$, let m' be the geodesic such that $m \overset{d}{\swarrow} m' \overset{\swarrow}{=} h$. Then $D(m')$ is a component domain of $(D(m), v)$. Since $v \in \phi_m(D(h'))$, v does not intersect $D(h')$, and since $D(h)$ and $D(h')$ intersect, $D(h')$ must be in the same component domain of $(D(m), v)$, namely $D(m')$. If $h = m'$ then we are done, with $D(h') \subset D(h)$. If not, we may apply the inductive assumption since $\xi(D(m')) < \xi(D(m))$, and again we are done.

If v is not in $\phi_m(D(h'))$, then $v' = \min \phi_m(D(h'))$ must be in $\phi_m(D(h))$, and furthermore v' is *not* the first vertex of m since v lies to its left. Thus $m \in \Sigma^-(D(h'))$ and it follows by Lemma 4.13 that $m \swarrow h'$, and therefore we may reverse the roles of h and h' and apply the previous paragraph. This concludes the proof of the assertion, and (2) follows by applying the assertion when m is the main geodesic g_H.

To prove (3), we may suppose that $b \swarrow k \searrow f$ and $b \neq f$, as the cases of equality are trivial.

By Lemma 4.8, $f \in \Sigma^+(D(k))$. If $D(b) \subset D(f)$ then $f \in \Sigma^+(D(b))$ as well, and by Lemma 4.13, $b \searrow f$. Similarly if $D(f) \subset D(b)$, then $b \swarrow f$.

Now suppose that neither $D(b) \subseteq D(f)$ nor $D(f) \subseteq D(b)$.

Since $D(k) \subset D(b) \cap D(f)$, part (2) shows that b and f are time-ordered, so let $D(m)$ contain both $D(b)$ and $D(f)$ such that their footprints on m are disjoint.

We claim that $\min \phi_m(D(b)) = \min \phi_m(D(k))$. If b is backward-subordinate to m then this is just Corollary 4.11 (Footprints). If b is not backward subordinate to m then neither is k, so that by Lemma 4.13 both minima are equal to the first vertex of m.

Similarly, $\max \phi_m(D(f)) = \max \phi_m(D(k))$. It follows that $\max \phi_m(D(b)) < \min \phi_m(D(f))$ (since we already know they are disjoint). Thus $b \prec_t f$, as desired.

Next we prove (4). Since $k_1 \prec_t k_2$, let m be the geodesic used to compare them. Since $h \overset{\searrow}{=} k_1 \searrow m$, $\max \phi_m(D(h)) = \max \phi_m(D(k_1))$ by Corollary 4.11 (Footprints). Thus $\max \phi_m(D(h)) < \min \phi_m(D(k_2))$, so $h \prec_t k_2$. The proof that $k_1 \prec_t g$ is similar.

To prove (5), we must in particular show that $\prec_p$ is transitive. Suppose $(k, v) \prec_p (k', v')$, and $(k', v') \prec_p (k'', v'')$. Let b be the geodesic used to compare (k, v) and (k', v'), and f be the geodesic used to compare (k', v') and (k'', v''). Then in particular $k \overset{\searrow}{=} b \overset{\swarrow}{=} k' \overset{\searrow}{=} f \overset{\swarrow}{=} k''$.

By (3), either $b = f$, $b \searrow f$, $b \swarrow f$, or $b \prec_t f$.

If $b = f$ then the footprints $\hat{\phi}_b(k, v)$, $\hat{\phi}_b(k', v')$, and $\hat{\phi}_b(k'', v'')$ are disjoint and linearly ordered from left to right, so $(k, v) \prec_p (k'', v'')$ immediately.

If $b \searrow f$ then neither k nor k' can equal f. Thus $\hat\phi_f(k,v) = \phi_f(D(k))$ and $\hat\phi_f(k',v') = \phi_f(D(k'))$. By Corollary 4.11 (Footprints) we have $\max\phi_f(D(k)) = \max\phi_f(D(b))$, and by (4.2) we have $\phi_f(D(b)) \subset \phi_f(D(k'))$. Then since $\max\phi_f(D(k')) < \min\hat\phi_f(k'',v'')$, we conclude $\max\hat\phi_f(k,v) < \min\hat\phi_f(k'',v'')$ so that $(k,v) \prec_p (k'',v'')$. The case where $b \swarrow f$ follows similarly.

If $b \prec_t f$ then since $k \overset{\searrow}{\geqq} b$ part (4) gives $k \prec_t f$, and since $f \overset{\swarrow}{\leqq} k''$ part (4) gives $k \prec_t k''$. It follows that $(k,v) \prec_p (k'',v'')$.

We have therefore proved that $\prec_p$ is transitive. It follows from the definition that $(h,v) \prec_p (h,v)$ can never hold, and so transitivity implies $(h,v) \prec_p (k,w)$ and $(k,w) \prec_p (h,v)$ are mutually exclusive. Thus $\prec_p$ is a strict partial order.

Part (6) follows immediately from the relation (4.3) between $\prec_p$ and $\prec_t$. It is also easy to see it directly by an argument very similar to the above. □

The next lemma gives a sufficient condition for two geodesics with disjoint domains *not* to be time-ordered.

Lemma 4.19. *Let v be a vertex of h and suppose that $D(k)$ and $D(k')$ lie in different component domains of $(D(h),v)$. Then k and k' are not time-ordered.*

Remark: we expect that there *will* be geodesics with disjoint domains which are nevertheless time-ordered. This is in fact one of the serious difficulties in applications of hierarchies.

Proof. Suppose by way of contradiction that $k \prec_t k'$, and let m be the geodesic used to compare them.

Note first $m \neq h$ since the footprints of k and k' on m are disjoint, and on h they both contain v.

Suppose $D(m) \subset D(h)$. Then $\min\phi_h(D(m)) = \min\phi_h(D(k'))$ since $m \swarrow k'$, just as we argued in the proof of Lemma 4.18, part (3). Similarly $\max\phi_h(D(m)) = \max\phi_h(D(k))$. It follows that $\phi_h(D(m))$ contains $\phi_h(D(k)) \cap \phi_h(D(k'))$, which in particular contains v. Thus $D(m)$ is disjoint from v, and since $D(m)$ is connected, it must lie in one component domain of $(D(h),v)$. This contradicts the assumption that $D(k)$ and $D(k')$ lie in different components.

Now suppose $D(h) \subset D(m)$. By Corollary 4.14, $\phi_m(D(h))$ is nonempty, and since $D(h)$ contains both $D(k)$ and $D(k')$, $\phi_m(D(h)) \subset \phi_m(D(k)) \cap \phi_m(D(k'))$. This contradicts the disjointness of footprints of k and k' in m.

Finally if neither $D(h)$ nor $D(m)$ is contained in the other, their intersection is still non-empty since both contain $D(k) \cup D(k')$. Thus by Lemma 4.18 part (2), h and m are time-ordered. Suppose $h \prec_t m$. Since $m \swarrow k'$, by Lemma 4.18 part (4) we have $h \prec_t k'$. However $D(k') \subset D(h)$ so this contradicts Lemma 4.18 part (1). Similarly if $m \prec_t h$ then since $k \searrow m$ we have $k \prec_t h$, again a contradiction. □

4.7. **Complete hierarchies.** A hierarchy H is *complete* if, for every domain Y with $\xi(Y) \neq 3$, which is a component domain in some geodesic $k \in H$, there is a geodesic $h \in H$ with $Y = D(h)$. In this section we will prove:

Theorem 4.20. (Completeness) *If the markings* $\mathbf{I}(H)$ *and* $\mathbf{T}(H)$ *(where defined) are complete, then* H *is complete.*

This will require Lemma 4.21 below, which is also the last and perhaps trickiest piece needed to prove Theorem 4.7 (Structure of Sigma). This lemma addresses the issue of when a component domain appearing in a hierarchy is the support of a geodesic in the hierarchy.

If v is a marking in a non-annular W, a *component domain of* (W, v) is defined to be any component domain of $(W, \mathrm{base}(v))$. This slight generalization will be used below for component domains defined by *positions* in geodesics, including the initial or terminal markings.

Lemma 4.21. (Subordinate Intersection 3) *Let* H *be a hierarchy, and let* Y *be a component domain of* $(D(k), v)$*, where* $k \in H$ *and* v *is a position in* k*. Assume that* $\xi(Y) \neq 3$*.*

If $f \in \Sigma^+(Y)$ *then either* $D(f) = Y$ *or* $Y \searrow f$*.*

Similarly if $b \in \Sigma^-(Y)$ *then either* $D(b) = Y$ *or* $b \swarrow Y$*.*

Note the similarity of this to Lemma 4.13 (Subordinate Intersection 2), the main difference being that Y is not required to be the support of a geodesic. In fact, the conclusions $Y \searrow f$ and $b \swarrow Y$ together imply that Y is the support of a geodesic, so in particular the lemma gives a sufficient condition for this to occur.

Proof. We will prove the forward case. Note that it suffices to show that, if $\Sigma_H^+(Y) \neq \emptyset$ then there exists $h \in H$ such that $Y \overset{d}{\searrow} h$. Lemma 4.15 then implies that, writing $\Sigma^+(Y) = \{f_0, f_1, \dots\}$, either $h = f_0$ and Y is not the support of any domain, or $Y = D(f_0)$ and $h = f_1$. The lemma follows from this.

We argue by induction using the partial order $\prec_p$. In particular, let us write our data as (Y, k, v) where v is a position on k such that Y is a component domain of $(D(k), v)$. Let $N(Y, k, v)$ be the number of such triples (Y', k', v') in H for which $Y \subseteq Y'$ and $(k, v) \prec_p (k', v')$. Note that this number is finite even if the hierarchy is infinite, because $\phi_{g_H}(D(k')) \subset \phi_{g_H}(Y)$, so the candidate triples are limited to a finite subset of H. Clearly if $Y \subseteq Y'$ and $(k, v) \prec_p (k', v')$, then $N(Y, k, v) > N(Y', k', v')$, so we will induct on N.

Let us first show that either $Y \overset{d}{\searrow} k$, in which case we are done, or we can find a certain (Y', k', v') with $Y \subseteq Y'$ and $(k, v) \prec_p (k', v')$ to which we can apply the inductive hypothesis. In particular this will take care of the base case $N(Y, k, v) = 0$. We will then use Y' to deduce the desired conclusion for Y.

For a position $w < \mathbf{T}(k)$ in k let $succ(w)$ denote the next position in the linear order. The following two cases occur:

1: If $v < \mathbf{T}(k)$, let $v' = succ(v)$. If $v'|_Y$ is nonempty then $\mathbf{T}(Y,k)$ is nonempty and $Y \overset{d}{\searrow} k$, as desired. If not, we define (Y', k', v') by letting $k' = k$ and letting Y' be the component domain of $(D(k), v')$ containing Y.

(Remarks: If $v = \mathbf{I}(k)$ or $v' = \mathbf{T}(k)$ then v and v' will share some base curves. In this case it is possible that $Y = Y'$. If Y is an annulus then either $Y' = Y$ or Y' contains Y essentially.)

2: If $v = \mathbf{T}(k)$ (including the case $\mathbf{T}(k)$ is equal to the last vertex), we let k' be the geodesic such that $k \overset{d}{\searrow} k'$. Then $D(k)$ is a component domain of some vertex w in k', with $w < \mathbf{T}(k')$. Letting $w' = succ(w)$, we have $\mathbf{T}(k) = w'|_{D(k)}$. Let $v' = w'$ and let Y' be the component domain of $(D(k'), w')$ containing Y.

Note that in each case, $(k,v) \prec_p (k', v')$: When $k = k'$ we had $v < v'$, and when $k \overset{d}{\searrow} k'$ we had $\phi_{k'}(D(k)) < v'$.

Recall, the assumption that $\Sigma^+(Y) \neq \emptyset$ is equivalent by Lemma 4.12 to $\mathbf{T}(g_H)|_Y \neq \emptyset$ or g_H infinite in the forward direction. Thus, since $Y \subseteq Y'$, $\Sigma^+(Y) \neq \emptyset$ implies $\Sigma^+(Y') \neq \emptyset$. Now we can apply the inductive assumption to (Y', k', v') to conclude that $Y' \overset{d}{\searrow} h'$ for some h'.

Thus if $Y = Y'$, we are done.

From now on let us assume Y is a proper subdomain of Y'. Thus, the relative boundary $\partial_{Y'}(Y)$ is nonempty. We claim that $\partial_{Y'}(Y)$ consists of (base) curves of v in case 1, and of w in case 2:

In case 1, $\partial_{D(k)}(Y)$ is in base(v), and so the claim is immediate. In case 2, $\partial_{D(k)}(Y)$ is in base$(\mathbf{T}(k))$ which is part of w', and $\partial_{D(k')}(D(k))$ is in w. It follows that $\partial_{D(k')}(Y)$ is in $w \cup w'$. Since no curve of w' is an essential curve of Y', $\partial_{Y'}(Y)$ must be in w.

Noting also that $\mathbf{I}(Y', k') = v|_{Y'}$ in case 1 and $\mathbf{I}(Y', k') = w|_{Y'}$ in case 2, we conclude in each case that $\mathbf{I}(Y', k') \neq \emptyset$. Thus, $k' \overset{d}{\swarrow} Y'$. Since we already have $Y' \overset{d}{\searrow} h'$, by definition of a hierarchy there is a geodesic $h \in H$ whose support is Y', and $k' \overset{d}{\swarrow} h$. In fact $(k,v) \prec_p (h, \mathbf{I}(h))$: when $k = k'$ this is because the footprint of h in k has minimum at v'. When $k \overset{d}{\searrow} k'$, it is because $\max \phi_{k'}(D(k)) = w < w' = \min \phi_{k'}(D(h))$, so in fact $k \prec_t h$. Also, Y is a component domain of $(D(h), \mathbf{I}(h))$: In case 1 this is because $\mathbf{I}(h) = v|_{Y'}$ and $\partial_{Y'}(Y) \subseteq v$, and in case 2 it is because $\mathbf{I}(h) = w|_{Y'}$ and $\partial_{Y'}(Y) \subseteq w$. Thus, the lemma holds for $(Y, h, \mathbf{I}(h))$ by induction, and we are done. $\square$

Proof of Completeness Theorem. Let Y be any component domain that is not a thrice-punctured sphere. If $\mathbf{I}(H)$ is defined then it is complete, so $\mathbf{I}(H)|_Y$ is nonempty and hence $g_H \in \Sigma^-(Y)$. If $\mathbf{I}(H)$ is undefined then g_H is infinite in the backward direction so $g_H \in \Sigma^-(Y)$ by definition. Similarly $g_H \in \Sigma^+(Y)$, so that by Lemma 4.21 $g_H \swarrow Y \searrow g_H$. In particular Y must support a geodesic by definition of a hierarchy. $\square$

Note that, even if $\mathbf{I}(H)$ and $\mathbf{T}(H)$ are pants decompositions with no transverals at all, we get a hierarchy which is complete except for the annular domains whose cores are curves of $\mathbf{I}(H)$ or $\mathbf{T}(H)$. If H has a bi-infinite main geodesic then it is automatically complete with no further conditions.

4.8. **Proof of the Structural Theorem.** We now have all the ingredients in place to put together a proof of Theorem 4.7 (Structure of Sigma). Parts (1) and (4) were already shown in Lemma 4.12 (Uniqueness of Descent). For Part (3), one direction of

$$f \in \Sigma^+(Y) \iff Y \searrow f$$

follows from Lemma 4.8 (Subordinate Intersection 1), and the other from Lemma 4.21 (Subordinate Intersection 3); similarly for Σ^-. If $\Sigma^\pm(Y)$ are both nonempty then, again by Lemma 4.21, there must be b and f such that $b \overset{d}{\swarrow} Y \overset{d}{\searrow} f$ and so by definition of a hierarchy Y is the support of a geodesic, which must then be $b_0 = f_0$.

It remains to show Part (2), that if $\Sigma^\pm(Y)$ are both nonempty for any domain Y with $\xi(Y) \neq 3$, then $b_0 = f_0$, and $\phi_{f_0}(Y) = \emptyset$. If $Y = D(h)$ for some geodesic h then we already know $h = b_0 = f_0$, and then $\phi_{f_0}(Y) = \emptyset$ automatically.

In general, if $\phi_{f_0}(Y) \neq \emptyset$, let $v = \max \phi_{f_0}(Y)$. Let W be the component domain of $(D(f_0), v)$ containing Y. (If Y is an annulus with core a component of v, then $W = Y$.) Recall that Lemma 4.12 implies for any domain X that $\Sigma^+(X) \neq \emptyset$ is equivalent to either $\mathbf{T}(H)|_X \neq \emptyset$ or g_H infinite in the forward direction, and similarly for Σ^- and $\mathbf{I}(H)$. Thus $\Sigma^\pm(Y) \neq \emptyset$ implies $\Sigma^\pm(W) \neq \emptyset$. Since W is a component domain and $\xi(W) \neq 3$, Lemma 4.21 then implies that W supports some geodesic h. Letting v' be the successor of v, we note that Y intersects v' nontrivially (this is true even when v is the last vertex of f_0 and $v' = \mathbf{T}(f_0)$). But $v'|_W$ is just $\mathbf{T}(h)$ by definition, and so $h \in \Sigma^+(Y)$. This contradicts the fact that f_0 has the smallest domain in Σ^+. Thus $\phi_{f_0}(Y) = \emptyset$.

In particular, it follows that $f_0 \in \Sigma^-$ since $\mathbf{I}(f_0)|_Y \neq \emptyset$ (or f_0 is infinite in the backward direction). In fact f_0 must be b_0 since Y has nonempty footprint on any other geodesic in Σ^-. This concludes the proof.

5. Slices, resolutions, and markings

In this section we will discuss how to resolve a hierarchy H into a sequence of markings connecting $\mathbf{I}(H)$ to $\mathbf{T}(H)$, so that successive markings are related by elementary moves. Essentially we must somehow combine the vertex sequences of the various geodesics in H, and their partial orders, into one large linearly ordered sequence. This process is by no means unique.

Along the way we will need to develop the notion of a *slice*, which roughly speaking is a marking pieced together from variously nested geodesics in the hierarchy, together with additional organizational structure. These slices will admit a certain partial order, and we will then describe an elementary move on slices, which moves a slice forward in the partial order.

The resulting sequence of slices can then be transformed into a sequence of clean markings of the surface (in a slightly non-unique fashion), and we will prove a lemma bounding the length of this sequence in terms of the size of H.

Slices. Let us assume from now on that the hierarchy H is complete. A *slice* in H is a set τ of pairs (h,v) where $h \in H$ and v is a vertex of h, satisfying the following conditions:

S1: A geodesic h appears in at most one pair in τ.
S2: There is a distinguished pair (h_τ, v_τ) appearing in τ, called the bottom pair of τ. We call h_τ the bottom geodesic.
S3: For every $(k,w) \in \tau$ other than the bottom pair, $D(k)$ is a component domain of $(D(h),v)$ for some $(h,v) \in \tau$.

If in addition this fourth condition holds, we call the slice *complete*:

S4: Given $(h,v) \in \tau$, for every component domain Y of $(D(h),v)$ there is a pair $(k,w) \in \tau$ with $D(k) = Y$.

Most often h_τ will just be the main geodesic of H.

A slice τ is called *initial* if, for each $(h,v) \in \tau$, v is the first vertex of h. Note that a complete initial slice is uniquely determined by its bottom geodesic. The complete initial slice with bottom geodesic g_H is called *the* initial slice of H. We similarly define terminal slices.

Markings associated to a slice: To any slice τ we associate a unique marking μ_τ as follows. It is easy to see by induction that the vertices v appearing in non-annular geodesics in τ are all disjoint and distinct, and hence form a simplex in $\mathcal{C}(S)$. We let this be $\operatorname{base}(\mu_\tau)$. For each base curve v, if τ contains some (k,t) with $D(k)$ the annulus whose core is v, then we let t be the transversal of v in μ_τ. In particular a complete slice determines a complete marking.

Typically μ_τ is not clean, so let us say that a clean marking μ' is *compatible with* τ if it is compatible with μ_τ in the sense of Lemma 2.4. Lemma 2.4 then shows that such a μ' exists, there are at most n_0^b possibilities where $b = \#\operatorname{base}(\mu)$, and any two differ by a bounded number of Twist elementary moves.

Note that, if τ is the initial slice of H, then if the marking $\mathbf{I}(H)$ is clean it is compatible with τ. The same is true for the terminal slice and $\mathbf{T}(H)$.

Partial order on slices: Consider now the set $V(H)$ of complete slices whose bottom geodesic equals the main geodesic of H. This set admits a partial order $\prec_s$ as follows. For $\tau, \tau' \in V(H)$, say that $\tau \prec_s \tau'$ iff $\tau \neq \tau'$ and, for any $(h,v) \in \tau$, either $(h,v) \in \tau'$ or there is some $(h',v') \in \tau'$ such that $(h,v) \prec_p (h',v')$.

Lemma 5.1. *Let H be a complete hierarchy. The relation $\prec_s$ is a strict partial order on $V(H)$.*

Proof. Let us first note the following facts:

1. $\prec_p$ is a strict partial order,
2. Any two elements (h, v) and (k, w) of a slice τ are not $\prec_p$-comparable,
3. If $\tau \subseteq \tau'$ for slices $\tau, \tau' \in V(H)$ then $\tau = \tau'$.

Fact (1) is Lemma 4.18 part (5) , and Fact (2) is an application of Lemma 4.19 and 4.18 part (1). Fact (3) follows from the fact that slices in $V(H)$ are *complete*.

For $\prec_s$ to be a strict partial order it suffices to show that it is transitive, since by definition it is never reflexive. Let $\tau_1 \prec_s \tau_2 \prec_s \tau_3$ for $\tau_i \in V(H)$.

By definition of $\prec_s$, given any $p_i \in \tau_i$ (where $i = 1, 2$ and p_i denotes some pair (h_i, v_i)), there exists $p_{i+1} \in \tau_{i+1}$ such that either $p_i \prec_p p_{i+1}$ or $p_i = p_{i+1}$. By fact (1) this implies either $p_1 \prec_p p_3$ or $p_1 = p_3$. Thus either $\tau_1 \prec_s \tau_3$ or $\tau_1 = \tau_3$. To rule out the latter, note that there is at least one $p_1 \in \tau_1$ which is not in τ_2 (by Fact (3)). Thus $p_1 \prec_p p_2$ so $p_1 \prec_p p_3$, and p_3 cannot lie in τ_1 by Fact (2). □

Forward elementary moves: Roughly, an elementary move on a slice τ consists of incrementing the vertex v of some (h, v) in τ, and making certain adjustments to the other pairs to obtain a new slice τ'.

To begin, let $h \in H$ and let v be a vertex of h, not the last, with successor v'. These will determine two slices σ and σ', not necessarily complete, called the *transition slices* for v and v'. The slices σ, σ' will have the property that (at least when $\xi(D(h)) > 4$) $\mu_\sigma = \mu_{\sigma'} = v \cup v'$. After constructing these we will extend them to complete slices τ, τ', which will constitute our elementary move.

Define σ as the smallest slice with bottom pair (h, v) such that, for any $(k, w) \in \sigma$ and Y a component domain of $(D(k), w)$,

- E1: if $v'|_Y \neq \emptyset$ and Y supports a geodesic q then $(q, u) \in \sigma$ where u is the *last* vertex of q.
- E2: if $v'|_Y = \emptyset$ then no geodesic in Y is included in σ.

Note that σ is easily built inductively from E1 and E2, and is uniquely determined. It is also easy to check that it satisfies the slice properties (S1–S3). We call the domains appearing in E2 "unused domains" for σ. Similarly, define σ' as the smallest slice with bottom pair (h, v'), such that for any $(k, w) \in \sigma'$ and Y a component domain of $(D(k), w))$,

- E1': if $v|_Y \neq \emptyset$ and Y supports a geodesic q then $(q, u) \in \sigma'$ where u is the *first* vertex of q.
- E2': if $v|_Y = \emptyset$ then no geodesic in Y is included in σ.

Before continuing let us consider this construction in several special cases.

1. $D(h)$ is an annulus. Here v and v' are arcs in the closed annulus with disjoint interiors, and $\sigma = \{(h, v)\}$, $\sigma' = \{(h, v')\}$.
2. $D(h)$ is a once-punctured torus. Now v and v' are curves intersecting once in $D(h)$. Let k be the geodesic supported in the annulus Y whose core is v, and let k' be the geodesic supported in the annulus Y' whose

core is v'. Then

$$\sigma = \{(h,v),(k,\pi_Y(v'))\}, \quad \sigma' = \{(h,v'),(k',\pi_{Y'}(v))\}.$$

(If $D(h)$ is a 4-holed sphere then v and v' intersect twice so $\pi_Y(v') = \mathbf{T}(k)$ has two components, only one of which appears as the last vertex of k; and similarly for $\pi_{Y'}(v)$.)

3. $\xi(D(h)) = 5$. Now v and v' are disjoint one-component curves. The complementary domain Y of v with $\xi(Y) = 4$ must contain v', and the complementary domain Y' of v' with $\xi(Y') = 4$ must contain v. Let k and k' be the geodesics supported in Y and Y' respectively. Then

$$\sigma = \{(h,v),(k,v')\}, \quad \sigma' = \{(h,v'),(k',v)\}.$$

Note that the annuli with cores v and v' are not included in these slices, by E2 and E2'. In particular we observe that $\mu_\sigma = \mu_{\sigma'} = v \cup v'$. The general $\xi > 4$ case will be treated in Lemma 5.2 below.

Lemma 5.2. *Let v, v' be successive vertices in a geodesic $h \in H$, where H is a complete hierarchy. Let σ, σ' be the transition slices associated to v, v'. If $\xi(D(h)) > 4$ then no geodesics in σ and σ' have annular domains, the associated markings μ_σ and $\mu_{\sigma'}$ have no transversals and are both equal to $v \cup v'$, and the unused domains in σ and σ' are exactly the component domains of $(D(h), v \cup v')$.*

Thus, the move from σ to σ' in this case involves only a "reorganization", and the underlying curve system is not changed.

Proof. Since $\xi(D(h)) > 4$, v and v' are disjoint curve systems. Consider first a component domain Y of $(D(h), v)$. If Y misses v' then it is an unused domain of σ (case E2) and is also clearly a component domain of $(D(h), v \cup v')$. If Y doesn't miss v', then by E1 (and completeness of H) we have (q, u) in σ where $D(q) = Y$ and u is the last vertex in q.

Since $Y \overset{d}{\searrow} h$, it follows from Lemma 4.15 that $q \overset{d}{\searrow} h$ with $\mathbf{T}(q) = \mathbf{T}(Y,h) = v'|_Y$. Hence in particular $u \subseteq v'$. Note that u need not be all of $v'|_Y$.

Now let Z be any component domain of $(D(q), u)$. By the above, the relative boundary $\partial_{D(h)}(Z)$ consists of some subset of $v \cup v'$. Again if Z misses v' it is unused in σ and a component domain of $(D(h), v \cup v')$, and if $v'|_Z \neq \emptyset$ then $Z \searrow q$ with $\mathbf{T}(Z,q) = \mathbf{T}(q)|_Z = v'|_Z$. Thus the same argument works inductively. The process terminates in an unused domain exactly when this domain is a component domain of $(D(h), v \cup v')$.

The same argument applies to σ' as well, reversing directions as usual.

Every annulus whose core is a component of $v \cup v'$ does not have essential intersection with either v or v'. Thus it is unused, so that the slices σ, σ' have no annulus-domain geodesics, and their markings have no transversals. ☐

We can now define our elementary moves. Let there be given two slices τ and τ', and let h be a geodesic in H with two successive vertices v, v'. We say that τ' is related to τ by a *forward elementary move* along h from v to v' (or $\tau \to \tau'$ for short) provided the following holds: Letting σ, σ' be the transition slices for v, v', we have $\sigma \subset \tau$ and $\sigma' \subset \tau'$, and $\tau \setminus \sigma = \tau' \setminus \sigma'$. The next lemma checks that a forward move in $V(H)$ really moves forward in terms of the partial order:

Lemma 5.3. *Suppose τ and τ' are in $V(H)$, and are related by an elementary move $\tau \to \tau'$. Then $\tau \prec_s \tau'$.*

Proof. First, $\tau \neq \tau'$ since σ and σ' differ in their bottom pair. Now let $(k, w) \in \tau$, such that $(k, w) \notin \tau'$. Then $(k, w) \in \sigma$ and hence $D(k) \subseteq D(h)$, and $v'|_{D(k)} \neq \emptyset$, by construction of σ. If $k = h$ then $(k, w) = (h, v) \prec_p (h, v') \in \sigma'$ and we are done. If not then $\phi_h(D(k))$ contains v and not v', so that $\max \phi_h(D(k)) = v < v'$. We therefore have $(k, w) \prec_p (h, v')$, and again we are done. □

Next, we should show that in fact a sequence of elementary moves does exist connecting the initial to the terminal slice of H, and furthermore give a bound for its length. Let $|H|$ denote the size of the hierarchy H, defined as the sum $\Sigma_{h \in H}|h|$ of the lengths of its geodesics.

Proposition 5.4. *Any complete finite hierarchy H admits a sequence of forward elementary moves $\tau_0 \to \cdots \to \tau_N$ where τ_0 is its initial slice, τ_N is its terminal slice, and*

$$N \leq |H|.$$

Such a sequence is called a resolution *of H.*

Proof. Let us first show that, if τ is not the terminal slice of H, then there exists some τ' such that $\tau \to \tau'$. Indeed, there is at least one $(h, v) \in \tau$ for which v is not the last vertex of h. Choose h minimal in the sense that if $(k, w) \in \tau$ and $D(k) \subset D(h)$ then w is the last vertex of k. Let v' denote the successor of v in h. The subset

$$\sigma = \{(k, w) \in \tau : D(k) \subseteq D(h), v'|_{D(k)} \neq \emptyset\}$$

satisfies conditions (E1,E2), by the minimal choice of h and the fact that τ is complete. Construct σ' via (E1') and (E2'), thus obtaining the transition slices for v, v', and let $\tau' = \sigma' \cup (\tau \setminus \sigma)$. It is easy to check that τ' satisfies conditions (S1–S3) and is hence a slice, with $h_{\tau'} = h_\tau = g_H$. To see that it is a complete slice (S4), consider any $(k, y) \in \tau'$ and let Y be a component domain of $(D(k), y)$. If (k, y) is not in σ' then by definition it is in τ, and since τ is complete, it contains some (l, z) with $D(l) = Y$. If $(l, z) \notin \sigma$ then again by definition $(l, z) \in \tau'$ and we are done. If $(l, z) \in \sigma$ then, since it is a component domain of a pair outside of σ, it can only be the bottom pair (h, v) of σ. But then $Y = D(h)$ and we know that (h, v') is a pair in τ', so again we are done. Now suppose that $(k, y) \in \sigma'$. If Y is a used domain of σ' then by definition it supports some geodesic l' appearing in τ'. If Y is an unused domain then by Lemma 5.2 it is also an unused domain of σ, and hence supports a geodesic q appearing in $\tau \setminus \sigma$. Again we conclude q appears in τ' and we are done.

We thus have a slice τ' in $V(H)$, and an elementary forward move $\tau \to \tau'$. Note that τ' is not uniquely determined by τ, as there may have been more than one h to choose from.

Now if τ_0 is the initial slice of H, the above implies that there is a sequence $\tau_0 \to \tau_1 \to \tau_2 \to \cdots$, which does not terminate at τ_i as long as τ_i is not the terminal slice. On the other hand by Lemma 5.3, the sequence is strictly increasing in $\prec_s$. Since the set of slices is finite, it must terminate for some τ_N, which must then be the terminal slice of H.

All we have left to prove is the bound on the length N of the resolution.

Suppose a pair (h, v) appears in τ_n and (h, w) appears in τ_m for $n < m$. Then $\tau_n \prec_s \tau_m$ as we have seen, and therefore it must be that $v \leq w$. For if not we would have $(h, w) \prec_p (h, v)$, but by definition of $\prec_s$ there is some $(k, u) \in \tau_m$ such that $(h, v) \prec_p (k, u)$. Hence $(h, w) \prec_p (k, u)$, but this contradicts the fact that all pairs in a given slice are not $\prec_p$-comparable (see proof of Lemma 5.1).

By the definition, a forward move $\tau_n \to \tau_{n+1}$ advances exactly one geodesic exactly one step, erases certain pairs of the form (k, u) where u is the last vertex, and creates certain pairs of the form (k', u') where u' is the first vertex, and keeps the rest of the pairs unchanged. Since by the previous paragraph no vertices in any geodesic can be repeated once they have been incremented or erased, it follows that the number of forward moves is bounded by $\Sigma_{h \in H} |h|$, which is $|H|$. ☐

Conversion to a sequence of clean markings. Given a resolution $\tau_0 \to \cdots \to \tau_N$ of H into slices, we may obtain a sequence of clean markings $\mu_0, \ldots, \mu_N$ by requiring that each μ_i be compatible with τ_i. Recall that there may be a finite number of choices for each μ_i. For convenience we also assume that $\mathbf{I}(H)$ and $\mathbf{T}(H)$ are clean, and $\mu_0 = \mathbf{I}(H)$ and $\mu_N = \mathbf{T}(H)$.

What is left to check is the relationship between μ_i and μ_{i+1}. Recall from §2.5 the definition of the elementary moves Flip and Twist on clean markings. We can now establish:

Lemma 5.5. *Let (τ_i) be a resolution of a complete finite hierarchy H, and let (μ_i) be a sequence of complete clean markings compatible with (τ_i). There exists $B > 0$ depending only on the topological type of S, such that μ_i and μ_{i+1} differ by at most B elementary moves.*

In particular, assuming $\mathbf{I}(H)$ and $\mathbf{T}(H)$ are clean, there is a sequence of clean markings $(\hat{\mu}_j)_{j=0}^{M}$, successive ones separated by elementary moves, such that $\hat{\mu}_0 = \mathbf{I}(H)$, $\hat{\mu}_M = \mathbf{T}(H)$, and $M \leq B|H|$.

Proof. We have already seen in the beginning of the section that two clean markings compatible with the same τ_i differ by a bounded number of Twist elementary moves.

Now, recall that $\tau_i \to \tau_{i+1}$ is determined by a transition $v \to v'$ along some geodesic h. If $D(h)$ is an annulus, v and v' differ by distance one in the annular complex $\mathcal{C}(D(h))$, so a bounded number of Twist moves applied to μ_i yields a marking μ'_{i+1} which is compatible with τ_{i+1}. Then as above μ'_{i+1} and μ_{i+1} are related by a bounded number of Twist moves.

Suppose that $\xi(D(h)) = 4$. Then recall that the transition slices σ_i and σ_{i+1} can be written as $\{(h,v),(k,t)\}$ and $\{(h,v'),(k',t')\}$ where k,k' are the geodesics in the complexes of the annuli Y and Y' with cores v and v' respectively, and t and t' are vertices of $\pi_Y(v')$ and $\pi_{Y'}(v)$ respectively. (If $D(h)$ is a 1-holed torus then $t = \pi_Y(v')$ and $t' = \pi_{Y'}(v)$). Thus a clean marking μ'_i can be constructed compatible with τ_i and containing a pair $(v, \pi_Y(v'))$. Now a Flip move on this marking yields a marking μ'_{i+1} with the pair $(v', \pi_{Y'}(v))$, with all other base curves the same, and transversals at distance at most n_1 from those of μ'_i by Lemma 2.4. It follows that, using a bounded number of Twist moves on each base curve, μ'_{i+1} can be made into μ''_{i+1} which is compatible with τ_{i+1}. Since the previous discussion bounds the number of moves to get from μ_i to μ'_i and from μ_{i+1} to μ''_{i+1}, we again have a bound on the number of moves needed to get from μ_i to μ_{i+1}.

Finally when $\xi(D(h)) > 4$, τ_i and τ_{i+1} have exactly the same base curves, and the positions on their annulus geodesics are the same. It follows that any marking compatible with τ_i is also compatible with τ_{i+1}, and hence again μ_i and μ_{i+1} differ by a bounded number of Twist moves. ☐

We remark that explicit bounds for this lemma are straightforward, but somewhat tedious, to compute, so we have elected to leave them out.

6. Comparison and control of hierarchies

In this section, we combine the structural results of the previous two sections with Theorem 3.1 (Bounded Geodesic Image), to prove a number of basic results that allow us to control the higher-order structure of hierarchies, and to compare hierarchies whose main geodesics are close.

As applications we prove Theorem 6.10 (Efficiency of Hierarchies), which shows that hierarchies give rise to sequences of markings separated by elementary moves which are close to shortest possible. These will be used to produce quasi-geodesics in $\mathrm{Mod}(S)$ in Section 7. We also prove Theorem 6.13 (Convergence of Hierarchies) which will allow us to obtain infinite hierarchies as limits of finite ones. Corollary 6.14 (Finite Geodesics) states that between any two points in $\mathcal{C}(S)$ there are only finitely many tight geodesics.

Our basic technical result will be Lemma 6.1 (Sigma Projection), which simplifies and generalizes the "short cut and projection" argument used in the motivating examples in §1.5. Recall how we showed that a large link (long geodesic) in one hierarchy forces a similar large link in a fellow-traveling hierarchy, by producing paths forward and backwards from the given link to its main geodesic, and projecting these back to the domain of the link. The forward and backward sequences $\Sigma^{\pm}$ provide the framework for making this argument work in general.

Lemmas 6.2 (Large Link) and 6.6 (Common Links) will be straightforward applications of Lemma 6.1, and will generalize what we did in the motivating examples. Lemma 6.7 (Slice Comparison) is a more delicate comparison between nearby hierarchies and requires more work.

6.1. **The forward and backward paths.** The "forward path" for a domain Y is built roughly as follows: Starting on the top geodesic in $\Sigma^+(Y)$ we move forward until it ends, at which point we have arrived at the position following the footprint of Y on the next geodesic in $\Sigma^+(Y)$, and we continue in this way until we get to the bottom geodesic g_H. A "backward path" is constructed the same way from $\Sigma^-(Y)$.

More precisely: Let σ denote the set of all pairs (k, v) where $k \in \Sigma^{\pm}(Y)$, and v is a position on k such that $v|_Y \neq \emptyset$. We claim that the partial order $\prec_p$ restricts to a linear order on σ, making it into a sequence:

Indeed, each $f_i \in \Sigma^+(Y)$ for $i > 0$ contributes a segment $\sigma_i^+ = \{(f_i, v_i) \prec_p \cdots \prec_p (f_i, \mathbf{T}(f_i))\}$, where v_i is the position immediately following $\max \phi_{f_i}(Y)$ (if $\max \phi_{f_i}(Y)$ is the last vertex then $\sigma_i^+ = \{(f_i, \mathbf{T}(f_i))\}$). Since $\max \phi_{f_i}(Y) = \max \phi_{f_i}(D(f_{i-1}))$ (Corollary 4.11), we also have $(f_{i-1}, \mathbf{T}(f_{i-1})) \prec_p (f_i, v_i)$. Thus the union of all σ_i^+ are linearly ordered. The same holds for σ_i^-, defined as $\{(b_i, \mathbf{I}(b_i)) \prec_p \cdots \prec_p (b_i, u_i)\}$, where u_i is the last position before $\min \phi_{b_i}(Y)$. Note that the same geodesic may appear in Σ^+ and Σ^-, in which case it can contribute both a σ_j^+ and a σ_i^-, one on each side of the footprint.

The top geodesic $h = b_0 = f_0$ has empty $\phi_h(Y)$ by Theorem 4.7 part (2), and so all its positions are included in σ, and they follow all the σ_i^- and precede all the σ_i^+ pairs, for $i > 0$. We denote the sequence of positions of the top geodesic by σ^0.

We let σ^+ be the concatenation $\sigma_1^+ \cup \cdots \cup \sigma_n^+$ (with the same linear order), and similarly $\sigma^- = \sigma_m^- \cup \cdots \cup \sigma_1^-$. In case clarification is needed we write $\sigma^+(Y), \sigma^-(Y, H)$, etc.

By the definition, for each $(k, v) \in \sigma$ the projection $\pi_Y(v)$ is nonempty. Let $\pi_Y(\sigma)$ denote the union of these projections, and similarly for $\pi_Y(\sigma^+)$ and $\pi_Y(\sigma^-)$. The following property of σ forms the basis of all the proofs in this section:

Lemma 6.1. (Sigma Projection) *There exist constants M_1, M_2 depending only on S such that, for any hierarchy H and domain Y in S,*

$$\operatorname{diam}_Y(\pi_Y(\sigma^+(Y, H))) \leq M_1$$

and similarly for σ^-.

Furthermore, if Y is properly contained in the top domain of $\Sigma(Y)$, then

$$\operatorname{diam}_Y(\pi_Y(\sigma(Y, H))) \leq M_2.$$

Proof. Theorem 3.1 (Bounded Geodesic Image) bounds the diameter of the projection to Y of each $\sigma_i^{\pm}$, and of σ^0 in the case where Y is properly contained in the top domain. The transition from the last position of σ_i^+

to the first of σ^+_{i+1} just consists of adding curves to the marking and so projects to a bounded step in $\mathcal{C}(Y)$ by Lemma 2.3. The same holds for the other transitions between segments of σ. Finally, the number of segments in each of $\sigma^\pm_i$ is bounded by $\xi(S) - \xi(Y)$. These facts together give the desired diameter bounds. □

6.2. **Large links.** The following is an almost immediate consequence of Lemma 6.1:

Lemma 6.2. (Large Link) *If Y is any domain in S and*

$$d_Y(\mathbf{I}(H), \mathbf{T}(H)) > M_2$$

then Y is the support of a geodesic h in H.

Conversely if $h \in H$ is any geodesic with $Y = D(h)$,

$$||h| - d_Y(\mathbf{I}(H), \mathbf{T}(H))| \le 2M_1.$$

Proof. The top geodesic $k = b_0 = f_0$ of $\Sigma(Y)$ has domain $Z = D(k)$ which either equals Y or contains it. If Y does not support a geodesic then Z properly contains Y, and Lemma 6.1 implies

$$d_Y(\mathbf{I}(H), \mathbf{T}(H)) \le \operatorname{diam}_Y(\pi_Y(\sigma)) \le M_2.$$

This proves the first part.

For the second part, if $Y = D(h)$ then by Theorem 4.7 we must have $Z = Y$ and $h = k$. Since σ^+ contains both $\mathbf{T}(h)$ and $\mathbf{T}(H)$, and σ^- contains $\mathbf{I}(h)$ and $\mathbf{I}(H)$, Lemma 6.1 (Sigma Projection) implies that

$$d_Y(\mathbf{I}(h), \mathbf{I}(H)) \le M_1$$

and

$$d_Y(\mathbf{T}(h), \mathbf{T}(H))) \le M_1.$$

The second statement of the lemma follows. □

6.3. **Fellow traveling.** In a δ-hyperbolic metric space, geodesics whose endpoints are near each other must stay together for their whole lengths. Our hierarchies have some similar properties. Before we state them we need some definitions.

Definition 6.3. *We say that two hierarchies H and H' are K*-separated at the ends *if the markings $\mathbf{I}(H)$ and $\mathbf{I}(H')$ are complete and clean, and are separated by at most K elementary moves, and similarly for $\mathbf{T}(H)$ and $\mathbf{T}(H')$.*

Definition 6.4. *Given two geodesics g_1 and g_2 with the same domain, and x_i a vertex in g_i for $i = 1, 2$, we say that g_1 and g_2 are (K, R)*-parallel *at x_1 and x_2 provided $d(x_1, x_2) \le K$ and for at least one of $i = 1$ or 2, x_i is the midpoint of a segment L_i of radius R in g_i, and L_i lies in a K-neighborhood of g_{3-i}.*

Definition 6.5. *We say a hierarchy H is (K, M)-*pseudo-parallel *to a hierarchy H' if, for any geodesic $h \in H$ with $|h| \geq M$ there is a geodesic $h' \in H$ such that $D(h) = D(h')$, and h is contained in a K-neighborhood of h' in $\mathcal{C}(D(h))$.*

(Note that the pseudo-parallel relation is not symmetric)

The following lemma is a generalization of Farb's Bounded Coset Penetration Property.

Lemma 6.6. (Common Links) *Given K there exist K', M such that, if two hierarchies H and H' are K-separated at the ends then each of them is (K', M)-pseudo-parallel to the other.*

Proof. Let h be any geodesic in H. By Lemma 6.2, the hypothesis, and Lemma 2.5, we have

$$|h| - 2M_1 - 4K \leq d_Y(\mathbf{I}(H'), \mathbf{T}(H')) \leq |h| + 2M_1 + 4K.$$

If we assume $|h| > M_2+2M_1+4K$, then the left hand side is greater than M_2, so Lemma 6.2 implies that there is a geodesic $h' \in H'$ with $D(h') = D(h)$. A bound of $2M_1 + 8K$ on $d_Y(\mathbf{I}(h), \mathbf{I}(h'))$ and $d_Y(\mathbf{T}(h), \mathbf{T}(h'))$ follows from Lemmas 6.1 and 2.5. It follows by hyperbolicity of $\mathcal{C}(Y)$ that h and h' remain a bounded distance apart along their whole length. □

In the next lemma we show how to compare slices in a pair of hierarchies that are K-separated at the ends or have parallel segments. The idea is that two such slices can be joined by a hierarchy that only has long geodesics when these are parallel to segments in the original two hierarchies. This is the closest one can come to saying that two hierarchies are fellow-travelers.

Lemma 6.7. (Slice Comparison) *Given K there exist K', M so that the following holds: Let τ and τ' be complete slices in two hierarchies H and H' respectively, with bottom vertices $x \in g_H$ and $x' \in g_{H'}$. Suppose that either*

1. *H and H' are K-separated at the ends, or*
2. *g_H and $g_{H'}$ are $(K, 3K+4)$-parallel at x and x'.*

Let μ and μ' be clean markings compatible with τ and τ' respectively. Then any hierarchy J with $\mathbf{I}(J) = \mu$ and $\mathbf{T}(J) = \mu'$ is (K', M)-pseudo-parallel to both H and H'.

Before giving the proof of this lemma we need the following two results.

Lemma 6.8. *Let H be a complete hierarchy. Let τ be a slice in $V(H)$ and (k, v) a pair where v is a position in $k \in H$. Then exactly one of the following occurs:*

1. *$(k, v) \in \tau$,*
2. *there exists $(h, u) \in \tau$ such that $(k, v) \prec_p (h, u)$,*
3. *there exists $(h, u) \in \tau$ such that $(h, u) \prec_p (k, v)$.*

Furthermore h may be taken so that $D(k) \subseteq D(h)$.

In view of this result, let us write $(k,v) \prec_s \tau$ when case (2) holds, and $\tau \prec_s (k,v)$ when case (3) holds.

Proof. Since $\tau \in V(H)$, it is complete and its bottom geodesic is g_H. We will prove the statement of the lemma inductively for any complete slice whose bottom geodesic g satisfies $D(k) \subseteq D(g)$. Let (g,u) be the bottom pair of τ. If $g = k$ then the statement is immediate – either $v < u$, $u < v$, or $u = v$.

Now suppose $D(k)$ is properly contained in $D(g)$. By Corollary 4.14, $\phi_g(D(k))$ is nonempty. If $\max \phi_g(D(k)) < u$ or $u < \min \phi_g(D(k))$ then $(k,v) \prec_p (g,u)$ or $(g,u) \prec_p (k,v)$, respectively, and we are done. If not then $u \in \phi_g(D(k))$ and there is some component domain Y of $(D(g),u)$ containing $D(k)$. Since τ is complete there is a pair $(h,w) \in \tau$ with $D(h) = Y$. The slice τ' consisting of all $(h',w') \in \tau$ such that $D(h') \subseteq D(h)$ is itself complete, and has bottom pair (h,w). Applying induction to τ', we have the desired statement.

The fact that the three possibilities are mutually exclusive follows directly from the fact that any two elements of a slice are not $\prec_p$-comparable (see proof of Lemma 5.1). $\square$

Lemma 6.9. *Fix a complete hierarchy H and a slice $\tau \in V(H)$. Let Y be any domain in S. Then the path $\sigma(Y)$ contains a pair (k,v) which is in τ.*

Proof. Let (k,v) and (k',v') be succesive pairs in $\sigma(Y)$. We will show that it is not possible for $(k,v) \prec_s \tau$ and $\tau \prec_s (k',v')$ to hold simultaneously. Since the first pair in σ is always $(g_H, \mathbf{I}(g_H))$, for which $(g_H, \mathbf{I}(g_H)) \prec_s \tau$ holds, and the last is $(g_H, \mathbf{T}(g_H))$ for which $\tau \prec_s (g_H, \mathbf{T}(g_H))$ holds, the statement of the lemma follows from Lemma 6.8.

By definition of σ, there are three possibilities for the relation between (k,v) and (k',v'):

1. $k = k'$. Here v' is the position following v.
2. $k \overset{d}{\searrow} k'$. Here $v = \mathbf{T}(k)$, and v' is the position following $\max \phi_{k'}(D(k))$.
3. $k \overset{d}{\swarrow} k'$. Here $v' = \mathbf{I}(k')$, and v is the position preceding $\max \phi_k(D(k'))$.

We will first prove our claim in cases (1) and (2). Suppose that $(k,v) \prec_s \tau$, and let $(h,u) \in \tau$ be a pair such that, as in Lemma 6.8, $(k,v) \prec_p (h,u)$ and $D(k) \subseteq D(h)$.

If $k = h$ then $v < u$ and in particular we must be in case (1) since v is not the last position of k. Thus $k = k' = h$ and v' is the successor of v, so $v' \leq u$. Thus we are done in this case.

If $D(k)$ is properly contained in $D(h)$ then we note that $k \searrow h$. In case (1) we still have $(k',v') = (k,v') \prec_p (h,u)$, so we are done. In case (2), we either have $k' = h$ or $k' \searrow h$. In the first case, v' is the successor of $\max \phi_h(D(k))$ and hence $v' \leq u$ and we are done. In the second, we have $\max \phi_h(D(k')) = \max \phi_h(D(k))$ by Corollary 4.11, and so again $(k',v') \prec_p (h,u)$.

To prove our claim in case (3), we just note that it is equivalent to case (2), with the directions and roles of k and k' reversed. $\square$

Proof of Slice Comparison Lemma. Let $R = 3K+4$. Let m_0 be any geodesic of J with $|m_0| > M$, where the value of M will be determined below, and let $Y = D(m_0)$. We first claim that, up to possibly reversing all the directions in H' (and interchanging $\mathbf{I}(H')$ with $\mathbf{T}(H')$),

$$d_Y(\mathbf{T}(H), \mathbf{T}(H')) \leq M_3, \tag{6.1}$$

$$d_Y(\mathbf{I}(H), \mathbf{I}(H')) \leq M_3 \tag{6.2}$$

for appropriate M_3. (If H is infinite then this holds with $\mathbf{I}(H)$ or $\mathbf{T}(H)$ replaced by any point of g_H on $\sigma^-(Y)$ or $\sigma^+(Y)$, respectively; and similarly for H').

In case (1) this is true by the hypothesis and Lemma 2.5, provided $M_3 \geq 4K$.

In case (2), up to interchanging H and H' we may assume there is an interval L of radius R centered on x which lies in a K-neighborhood of $g_{H'}$. If y, z are the endpoints of L and $y < x < z$, let y', z' be points of $g_{H'}$ closest to y and z respectively. Up to reversing all the directions in H' we may assume $y' < z'$, and then by the triangle inequality we have $y' < x' < z'$.

Since Y is a domain of J, whose main geodesic has length at most K, and x is a curve in $\mathbf{I}(J)$, we have $d_S(\partial Y, x) \leq K+2$ (thinking of ∂Y as a simplex in $\mathcal{C}(S)$). We claim that any curve w on a geodesic from z to z' intersects ∂Y. For if not, $d_S(w, \partial Y) \leq 1$, and so $d(x,z) \leq d(x, \partial Y) + d(\partial Y, w) + d(w,z) \leq 2K+3 < R$, a contradiction. It follows that we can project $[z, z']$ into $\mathcal{C}(Y)$ and conclude $d_Y(z,z') \leq 2K$ by Lemma 2.3.

If $\phi_{g_H}(Y)$ is nonempty, the triangle inequality similarly gives $d_S(x, \phi_{g_H}(Y)) \leq K+3 < R = d(x,z)$ and hence z lies to the right of $\phi_{g_H}(Y)$. Similarly $d_S(x', \phi_{g_{H'}}(Y)) \leq K+3 < R-2K \leq d(x',z')$ and hence z' lies to the right of $\phi_{g_{H'}}(Y)$, if that is nonempty.

It follows that z can be connected to $\mathbf{T}(H)$ by a path lying in $\sigma^+(Y,H)$, and similarly for z' and $\mathbf{T}(H')$. Lemma 6.1 (Sigma Projection) then gives a bound of M_1 on $d_Y(z, \mathbf{T}(H))$ and $d_Y(z', \mathbf{T}(H'))$. Putting these together with the bound on $d_Y(z,z')$ gives (6.1), with $M_3 = 2M_1 + 2K$. The same argument with y and y' gives (6.2).

Next we claim that, for $M_4 = 2M_2 + 4M_1 + 4$,

$$d_Y(\mu, \mathbf{I}(H)) + d_Y(\mu, \mathbf{T}(H)) \leq d_Y(\mathbf{I}(H), \mathbf{T}(H)) + M_4 \tag{6.3}$$

and similarly for μ' and H'.

Begin by observing that, by Lemma 6.9, $\sigma(Y,H)$ contains a pair $(k,v) \in \tau$. If $D(k)$ is an annulus then $D(k) = Y$ and v is a transversal of μ – otherwise it is in $\operatorname{base}(\mu)$. Lemma 2.3 then implies that $\pi_Y(\mu)$ is within distance 2 of $\pi_Y(\sigma(Y,H))$.

By Lemma 6.1 (Sigma Projection), if Y does not support a geodesic in H then $\operatorname{diam}_Y(\sigma(Y,H)) \leq M_2$. Hence the left side of (6.3) is at most $2M_2 + 4$ and the inequality follows by choice of M_4.

If Y supports a geodesic $h \in H$ then Lemma 6.1 implies that $\pi_Y(\sigma(Y,H))$ is Hausdorff distance M_1 from h (i.e. each is in an M_1-neighborhood of the other). We therefore have, for v as above, $d_Y(v,v_0)+d_Y(v,v_{|h|}) \le |h|+2M_1$, where v_0 and $v_{|h|}$ are the first and last vertices of h, and since $\pi_Y(\mathbf{I}(H))$ and $\pi_Y(\mathbf{T}(H))$ are distance M_1 from the respective endpoints of $|h|$, (6.3) again follows with a bound of $6M_1+4$. This is at most M_4 since $M_1 \le M_2$.

Now by the triangle inequality $d_Y(\mu,\mu')$ is bounded by both

$$d_Y(\mu,\mathbf{T}(H)) + d_Y(\mathbf{T}(H),\mathbf{T}(H')) + d_Y(\mu',\mathbf{T}(H'))$$

and

$$d_Y(\mu,\mathbf{I}(H)) + d_Y(\mathbf{I}(H),\mathbf{I}(H')) + d_Y(\mu',\mathbf{I}(H')).$$

Adding these two estimates together and applying (6.1,6.2) and (6.3)), we have

$$2d_Y(\mu,\mu') \le d_Y(\mathbf{I}(H),\mathbf{T}(H)) + d_Y(\mathbf{I}(H'),\mathbf{T}(H')) + 2M_3 + 2M_4.$$

Now again applying (6.1,6.2) and the triangle inequality, we find that $d_Y(\mathbf{I}(H),\mathbf{T}(H))$ and $d_Y(\mathbf{I}(H'),\mathbf{T}(H'))$ differ by at most $2M_3$. This gives

$$d_Y(\mu,\mu') \le d_Y(\mathbf{I}(H),\mathbf{T}(H)) + 4M_3 + 2M_4,$$

and the same inequality for H'.

Since $|m_0| > M$, Lemma 6.2 (Large Link) gives $d_Y(\mu,\mu') > M - 2M_1$, so $d_Y(\mathbf{I}(H),\mathbf{T}(H)) > M - 2M_1 - 4M_3 - 2M_4$. If we set $M = 2M_1 + 4M_3 + 2M_4 + M_2$, Lemma 6.2 again guarantees that Y is the domain of a geodesic $m \in H$. Furthermore, $\mathbf{T}(m_0)$ is within M_1 of $\pi_Y(\mathbf{T}(J)) = \pi_Y(\mu')$, which is within 4 of $\pi_Y(\sigma(Y,H))$ by Lemma 6.9, as above. Applying Lemma 6.1 again we find that this is within M_1 of m. A similar estimate holds for $\mathbf{I}(m_0)$, and by δ-hyperblicity all of m_0 lies within $K' = 2M_1 + 4 + 2\delta$ of m. This establishes that J is (K', M)-pseudo-parallel to H.

The corresponding statement holds for H', and the lemma is proved. □

6.4. **Efficiency.** In Section 5 we saw that a hierarchy H can be resolved into a sequence of markings of length bounded by its size $|H|$. Here we will obtain an estimate in the opposite direction. Let $\widetilde{\mathcal{M}}$ be the graph whose vertices are complete, clean markings in S, and whose edges represent elementary moves. Giving edges length 1, we have for two complete clean markings μ,ν their *elementary move distance* $d_{\widetilde{\mathcal{M}}}(\mu,\nu)$ in this graph. Proposition 5.4 and Lemma 5.5 imply that this graph is connected, but this fact is already well known: it follows for example from a similar connectedness result for the graph of pants decompositions in Hatcher-Thurston [14] (and see proof in Hatcher [13]).

Theorem 6.10. (Efficiency of Hierarchies) *There are constants $c_0, c_1 > 0$ depending only on S so that, if μ and ν are complete clean markings and H is a hierarchy with $\mathbf{I}(H) = \mu$, $\mathbf{T}(H) = \nu$, then*

$$c_0^{-1}|H| - c_1 \le d_{\widetilde{\mathcal{M}}}(\mu,\nu) \le c_0|H|.$$

See Theorem 7.1 (Quasigeodesic Words), in Section 7, for the implication of this to words in the Mapping Class Group.

Proof. The second inequality is an immediate consequence of Proposition 5.4 and Lemma 5.5.

For the first inequality, the idea of the argument is as follows. Consider a shortest path $\{\mu = \mu_0, \ldots, \mu_N = \nu\}$ from μ to ν in $\widetilde{\mathcal{M}}$. Each long geodesic $h \in H$ imposes a lower bound of the form $N \geq c|h|$, because the projection $\pi_{D(h)}(\mu_j)$ moves at bounded speed in $\mathcal{C}(D(h))$ (Lemma 2.5) as j goes from 0 to N, and by Lemma 6.2 (Large Link) it must travel a distance proportional to $|h|$. To obtain our desired statement we must show that the projections of μ_j cannot move far in many different domains at once, and hence the lower bounds for the different geodesics will add. This will be done using Lemma 6.11 below, which relates projections to time-order.

Let $M_5 = 2M_1 + 5$ and $M_6 = 4(M_1 + M_5 + 4)$, and let $\mathcal{G}$ be the set of geodesics $h \in H$ satisfying $|h| \geq M_6$. Let $|\mathcal{G}| = \Sigma_{h\in\mathcal{G}}|h|$. Then we have

$$|\mathcal{G}| \geq d_0|H| - d_1 \tag{6.4}$$

for d_0, d_1 depending only on S (and the choice of M_6). The proof is a simple counting argument, using the fact that the number of component domains of any geodesic is bounded by a constant times its length. Thus the main point will be to bound N below in terms of $|\mathcal{G}|$.

For any $h \in \mathcal{G}$ let us isolate an interval in $[0, N]$ in which the projections to $Y = D(h)$ of the μ_j are "making the transition" between being close to $\pi_Y(\mu_0)$, to being close to $\pi_Y(\mu_N)$.

Let $L = d_Y(\mu_0, \mu_N)$, noting that $L \geq |h| - 2M_1 \geq M_6 - 2M_1$ by Lemma 6.2. The projections of μ_j to $\mathcal{C}(Y)$ are a sequence that moves by bounded jumps $d_Y(\mu_j, \mu_{j+1}) \leq 4$, by Lemma 2.5. Therefore there must be some largest value of $j \in [0, N]$ for which $d_Y(\mu_0, \mu_j) \in [M_5, M_5 + 4]$. Let a_Y be this value. Since $L > 2(M_5 + 4)$, we know that $d_Y(\mu_{a_Y}, \mu_N) > M_5 + 4$. Therefore there is a smallest $j \in [a_Y, N]$ for which $d_Y(\mu_j, \mu_N) \in [M_5, M_5+4]$. Let this be b_Y.

Let J_Y be the interval $[a_Y, b_Y]$. These intervals have the following properties:

1. For any $j \in J_Y$ we have $d_Y(\mu_0, \mu_j) \geq M_5$ and $d_Y(\mu_j, \mu_N) \geq M_5$.
2. $|J_{D(h)}| \geq |h|/8$ for any $h \in \mathcal{G}$.
3. If $h, k \in \mathcal{G}$ are such that $Y = D(h)$, $Z = D(k)$ have nonempty intersection and neither is contained in the other, then J_Y and J_Z are disjoint intervals.

(1) follows immediately from the definition and Lemma 2.5

To prove (2), by the triangle inequality we have $d_Y(\mu_0, \mu_{b_Y}) \geq L - M_5 - 4$. Again by Lemma 2.5, $d_Y(\mu_0, \mu_j)$ changes by at most 4 with each increment of j, so since $d_Y(\mu_0, \mu_{a_Y}) \leq M_5+4$ we conclude that $b_Y - a_Y \geq (L - 2M_5 - 8)/4$. This implies (2), by the choice of constants and the fact that $|h| \geq M_6$.

To prove (3), we will first need the following lemma:

Lemma 6.11. (Order and projections) *Let H be a hierarchy and $h, k \in H$ with $D(h) = Y$ and $D(k) = Z$. Suppose that $Y \cap Z \neq \emptyset$, and neither domain is contained in the other. Then, if $h \prec_t k$ then*

$$d_Y(\partial Z, \mathbf{T}(H)) \leq M_1 + 2 \tag{6.5}$$

and

$$d_Z(\mathbf{I}(H), \partial Y) \leq M_1 + 2. \tag{6.6}$$

Proof of Lemma 6.11. Let m be the geodesic used to compare h and k. It lies in $\Sigma^+(Y)$. Let $v \in \phi_m(Z)$. Since v lies to the right of $\phi_m(Y)$ the pair (m, v) is in the sequence $\sigma^+(Y)$. Lemma 6.1 (Sigma Projection) now implies

$$d_Y(v, \mathbf{T}(H)) \leq M_1.$$

Since ∂Z intersects Y essentially by the assumption on Y and Z, and since ∂Z is disjoint from v, by applying Lemma 2.3 we find that

$$d_Y(\partial Z, \mathbf{T}(H)) \leq M_1 + 2$$

as desired. The second inequality is proved in the same way. □

Returning to the proof of Theorem 6.10, suppose that property (3) is false, so that Y and Z intersect and are non-nested, but J_Y and J_Z overlap. Let $j \in J_Y \cap J_Z$.

Let H_j be a hierarchy such that $\mathbf{I}(H_j) = \mu_0$ and $\mathbf{T}(H_j) = \mu_j$. By property (1) $d_Y(\mu_0, \mu_j)$ and $d_Z(\mu_0, \mu_j)$ are both at least M_5, so Lemma 6.2 implies that Y and Z support geodesics h_j and k_j in H_j. The condition on Y and Z implies that h_j and k_j are time-ordered in H_j by Lemma 4.18, so suppose without loss of generality that $h_j \prec_t k_j$. By Lemma 6.11, we have $d_Y(\partial Z, \mu_j) \leq M_1 + 2$.

However, h and k must also be time-ordered in H, and applying Lemma 6.11 to the hierarchy H, we have either

$$d_Y(\partial Z, \mu_N) \leq M_1 + 2$$

if $h \prec_t k$ by (6.5), or

$$d_Y(\mu_0, \partial Z) \leq M_1 + 2$$

if $k \prec_t h$ by (6.6). Thus, either $d_Y(\mu_j, \mu_N) \leq 2M_1 + 4$ or $d_Y(\mu_0, \mu_j) \leq 2M_1 + 4$. Either one of these contradicts the assumption that $j \in J_Y$, since $2M_1 + 4 < M_5$. This proves (3).

Thus, the intervals $\{J_{D(h)} : h \in \mathcal{G}\}$ cover a subset of $[0, N]$ with multiplicity at most s, where s is the maximal cardinality of a set $D_1, \ldots, D_s$ of domains in S, $\xi(D_i) \neq 3$, for which any two are either disjoint or nested. This number depends only on S (in fact it is easy to show that $s = 2\xi(S) - 6$). It follows that

$$sN \geq \sum_{h \in \mathcal{G}} |J_{D(h)}|.$$

Combining this with (2) which gives $\sum_{h \in \mathcal{G}} |J_{D(h)}| \geq |\mathcal{G}|/8$, and then using (6.4), we obtain

$$N \geq c_0^{-1}|H| - c_1$$

with suitable constants c_0, c_1. □

The following corollary of this theorem can be stated without any mention of hierarchies. It relates elementary-move distance to the sum of all "sufficiently large" projections to subsurfaces in S (including S itself).

Theorem 6.12. (Move distance and projections) *There is a constant $M_6(S)$ such that, given $M \geq M_6$, there are e_0, e_1 for which, if μ and ν are any two complete clean markings then*

$$e_0^{-1} d_{\widetilde{\mathcal{M}}}(\mu,\nu) - e_1 \leq \sum_{\substack{Y \subseteq S \\ d_Y(\mu,\nu) \geq M}} d_Y(\mu,\nu) \leq e_0 d_{\widetilde{\mathcal{M}}}(\mu,\nu) + e_1$$

The proof is simply a rephrasing of the result of Theorem 6.10, together with the inequality (6.4) to restrict consideration to "long" geodesics, and Lemma 6.2 to relate this to projection diameters.

6.5. **Finiteness and limits of hierarchies.** In this section we apply the comparison lemmas to the question of when a sequence of hierarchies converges to a limiting hierarchy. Let us first discuss what we mean by convergence.

Fix a point $x_0 \in \mathcal{C}(S)$ and let $B_R = B_R(x_0)$ denote the R-neighborhood of x_0 in $\mathcal{C}(S)$. For a tight geodesic h with $D(h) \subseteq S$, let $h \cap B_R$ denote the following:

1. If $D(h)$ is a component domain of (S, v) for some v then, if $v \subset B_R$ we let $h \cap B_R = h$, and otherwise $h \cap B_R = \emptyset$.
2. If $D(h) = S$ then $h \cap B_R$ is the set of all positions of h that lie in B_R.

(In (2) this includes $\mathbf{I}(h)$ and/or $\mathbf{T}(h)$ if their bases are in B_R.)

For a hierarchy H, let $H \cap B_R = \{h \cap B_R : h \in H\}$. We say that a sequence $\{H_n\}$ of hierarchies *converges to a hierarchy* H if for all $R > 0$, $H_n \cap B_R = H \cap B_R$ for large enough n. Clearly if H is a finite hierarchy this just means that eventually $H_n = H$.

It is also easy to see the following: Suppose that for all $R > 0$, the sets $H_n \cap B_R$ are eventually constant. Then $\{H_n\}$ converges to a unique hierarchy H. We can now prove the following result:

Theorem 6.13. (Convergence of Hierarchies) *Let $\{H_n\}_{n=1}^{\infty}$ be a sequence of hierarchies such that either*

1. *For a fixed K and any n, m, H_n and H_m are K-separated at the ends, or*
2. *There exists $K > 0$ and a vertex x_n on each g_{H_n} such that, for each $R' > 0$, there exists $n = n_{R'}$ so that for all $m \geq n$, g_{H_n} and g_{H_m} are (K, R')-parallel at x_n and x_m.*

Then $\{H_n\}$ has a convergent subsequence.

Proof. Fix an arbitrary x_0 and let $\mathcal{U}_R$ denote the set of all vertices in $H_n \cap B_R(x_0)$ for all $n > 0$. We claim that $\mathcal{U}_R$ is finite for each $R > 0$. The theorem follows immediately since this implies that $H_n \cap B_R$ varies in a finite set of possibilities for each R, and so the usual diagonalization step extracts a subsequence H_{n_k} for which $H_{n_k} \cap B_R$ is eventually constant.

To show $\mathcal{U}_R$ is finite, consider first case (1). Fix a slice τ_1 in H_1, and note that any vertex v in H_n appears in some complete slice τ of H_n. Consider a hierarchy $J(v)$ joining clean markings compatible with τ and τ_1 respectively. Lemma 6.7 (Slice Comparison), case (1), implies that $J(v)$ is (K', M)-pseudo-parallel to H_1, so each geodesic in $J(v)$ has length bounded either by M or by a constant plus the length of a geodesic in H_1. Since H_1 is finite this gives some uniform bound, so every marking compatible with a slice of H_n can be transformed to a marking compatible with a slice of H_1 in a bounded number of elementary moves. This means the set of all base curves that occur in such markings is finite, and this bounds the set of all vertices occurring in non-annular geodesics in all H_n. The annular geodesics are determined by their initial and terminal markings, up to a finite number of choices (by the definition of tightness), and hence those vertices are finite in number as well. (Note we have actually proved finiteness for all the vertices in all H_n, without mention of R).

In case (2), the condition implies that there is some bound $d(x_0, x_n) \leq R_0$ for all $n > 0$. Given R, choose $R' = R_0 + R + 3K + 8\delta + 5$ and let $n = n_{R'}$. For $m \geq n$ let ℓ_m be the segment of radius R' around x_m. The (K, R')-parallel condition means that $d(x_n, x_m) \leq K$ and either ℓ_m is in a K-neighborhood of g_{H_n} or ℓ_n is in a K-neighborhood of g_{H_m}. In either case, the triangle inequality and δ-hyperbolicity imply that, if $x \in g_{H_m}$ and $d(x, x_m) \leq R_0 + R + 1$ then x is the center of a segment in g_{H_m} of radius $6\delta + 4$ contained in a 2δ-neighborhood of ℓ_n (we should assume that $K > \delta$, which entails no loss of generality).

Now any vertex of $H_m \cap B_R$ occurs in some complete slice τ of H_m with bottom vertex x in B_{R+1} (the slice will be complete because $\mathbf{I}(H_m)$ and $\mathbf{T}(H_m)$ are sufficiently far away from B_R that they have non-trivial restriction to any domain occurring in $H_m \cap B_R$ – so one can apply Lemma 4.21). Thus $d(x, x_m) \leq R_0 + R + 1$ by the triangle inequality, and the previous paragraph implies that, for suitable $x' \in \ell_n$, g_{H_m} and g_{H_n} are $(2\delta, 6\delta + 4)$-parallel at x and x'. By case (2) of Lemma 6.7 we can again conclude that a marking compatible with τ can be connected to some marking compatible with a slice in $H_n \cap B_{R'}$ by a sequence of elementary moves whose length is bounded only in terms of $H_n \cap B_{R'}$. The argument then proceeds as in case (1). ∎

We have the following immediate consequence of this argument:

Corollary 6.14. (Finite Geodesics) *Given a pair of points $x, y \in \mathcal{C}_0(S)$ there are only a finite number of tight geodesics joining them.*

Proof. Fix markings $\mathbf{I}$ and $\mathbf{T}$ containing x and y, respectively. Each tight geodesic connecting x to y can be extended to a hierarchy connecting $\mathbf{I}$ and

T, and the finiteness argument in case (1) of Theorem 6.13 implies this set of hierarchies is finite. □

7. Conjugacy bounds in the Mapping Class Group

In this section we will apply the hierarchy construction to the Mapping Class Group $\mathrm{Mod}(S)$. Our main goals will be Theorem 7.1, stating that hierarchies give rise to quasi-geodesic words in $\mathrm{Mod}(S)$, and Theorem 7.2, which gives a linear upper bound for the length of the minimal word conjugating two pseudo-Anosov mapping classes.

We recall first that any generating set for $\mathrm{Mod}(S)$ induces a *word metric* on the group, denoting by $|g|$ the length of the shortest word in the generators representing $g \in \mathrm{Mod}(S)$, and by $|g^{-1}h|$ the distance between g and h. The metrics associated to any two (finite) generating sets are bilipschitz equivalent.

7.1. **Paths in the mapping class group.** Our first step is to show how a resolution of a hierarchy into a sequence of slices gives rise to a word in the mapping class group. This is a completely standard procedure involving the connection between groupoids and groups. Theorem 6.10 (Efficiency of Hierarchies) will imply that these words are in fact quasi-geodesics.

Let $\widetilde{\mathcal{M}}$ be the graph of complete clean markings of S connected by elementary moves, as in §6.4. The action of $\mathrm{Mod}(S)$ on $\widetilde{\mathcal{M}}$ is not free – a mapping class can permute the components of a marking or reverse their orientations – but it has finite stabilizers. The quotient $\mathcal{M} = \widetilde{\mathcal{M}}/\mathrm{Mod}(S)$ is a finite graph, and we let D denote its diameter. Fix a marking $\mu_0 \in \widetilde{\mathcal{M}}$ and let $\Delta_j \subset \mathrm{Mod}(S)$ denote the set of elements g such that $d_{\widetilde{\mathcal{M}}}(\mu_0, g(\mu_0)) \leq j$. Any marking $\mu \in \widetilde{\mathcal{M}}$ is at most distance D from some $\psi(\mu_0)$, with $\psi \in \mathrm{Mod}(S)$ determined up to pre-composition by elements of Δ_{2D}.

Note that Δ_j is a finite set for any j. Now given any $\psi \in \mathrm{Mod}(S)$, we can write it as a word in Δ_{2D+1} as follows: connect μ_0 to $\psi(\mu_0)$ by a shortest path $\mu_0, \mu_1, \ldots, \mu_N = \psi(\mu_0)$ in $\widetilde{\mathcal{M}}$. For each $i < N$ choose ψ_i such that $\psi_i(\mu_0)$ is within D moves of μ_i, and let $\psi_N = \psi$. Then ψ_i and ψ_{i+1} differ by pre-composition with an element δ_{i+1} in Δ_{2D+1}. Thus we can write $\psi = \delta_1 \cdots \delta_N$.

This gives an upper bound on the word length of ψ, proportional to $N = d_{\widetilde{\mathcal{M}}}(\mu_0, \psi(\mu_0))$. Of course, the word we obtain here can be translated to a word in any other finite generating set in the standard way, and its length will only increase by a bounded multiple (depending on the generating set).

In the other direction, suppose that ψ can be written as $\alpha_1 \cdots \alpha_M$ with α_i in some fixed finite generating set. Then the sequence of markings $\{\mu_j = \alpha_1 \cdots \alpha_j(\mu_0)\}$ satisfies the property that μ_j and μ_{j+1} are separated by some bounded number of elementary moves, with the bound depending on the generating set. This bounds $d_{\widetilde{\mathcal{M}}}(\mu_0, \psi(\mu_0))$ above linearly in terms of $|\psi|$.

Now let H be any hierarchy with $\mathbf{I}(H) = \mu_0$ and $\mathbf{T}(H) = \psi(\mu_0)$. Theorem 6.10 (Efficiency of Hierarchies) gives upper and lower bounds on $d_{\widetilde{\mathcal{M}}}(\mu_0, \psi(\mu_0))$ in terms of $|H|$, and we immediately obtain:

Theorem 7.1. (Quasigeodesic Words) *Fix a complete clean marking μ_0 of S and a set of generators for* $\mathrm{Mod}(S)$*, and let $|\cdot|$ be the word metric with respect to these generators. Then there are $c_2, c_3 > 0$ such that the following holds:*

Given any $\psi \in$ $\mathrm{Mod}(S)$ *let H be a hierarchy such that $\mathbf{I}(H) = \mu_0$ and $\mathbf{T}(H) = \psi(\mu_0)$. Then the words in* $\mathrm{Mod}(S)$ *constructed from resolutions of H via the procedure in this section are quasi-geodesics, and in particular*

$$c_2^{-1}|H| - c_3 \leq |\psi| \leq c_2|H|.$$

(We remark that the additive constant c_3 can be removed if we always choose the hierarchy to have length 0 when $\psi = \mathrm{id}$).

Note also that an estimate on the word length $|\psi|$ can be obtained purely in terms of the quantities $d_Y(\mu_0, \psi(\mu_0))$, using Theorem 6.12.

7.2. **The conjugacy bound.** We are now ready to prove the main theorem of this section:

Theorem 7.2. (Conjugacy Bound) *Fixing a set of generators for* $\mathrm{Mod}(S)$*, there exists a constant K such that if $h_1, h_2 \in$* $\mathrm{Mod}(S)$ *are conjugate pseudo-Anosovs there is a conjugating element w with $|w| \leq K(|h_1| + |h_2|)$.*

Let δ be the hyperbolicity constant for $\mathcal{C}(S)$. Say that two geodesics are c-fellow travelers if each is in a c-neighborhood of the other (Hausdorff distance c) and their endpoints (if any) can be paired to be within distance c of each other. The following three lemmas are standard for any hyperbolic metric space.

Lemma 7.3. *For any K, if β_1, β_2 are two K-fellow traveling bi-infinite geodesics, then they are actually 2δ fellow travelers.*

Proof. Let $\bar{\beta}_1$ be any segment of β_1. Choose points $y_1, y_2 \in \beta_1 \setminus \bar{\beta}_1$ on either side of $\bar{\beta}_1$ whose distance to $\bar{\beta}$ is $2\delta + K + 1$ and points x_i on β_2 such that $d(x_i, y_i) = K$. The quadrilateral $[x_1 x_2 y_2 y_1]$ is 2δ-thin by hyperbolicity: that is, each edge is within 2δ of the union of the other three. By the triangle inequality no point of $\bar{\beta}_1$ can be within 2δ of $[x_i, y_i]$ so each point must be within 2δ of β_2. Since $\bar{\beta}_1$ was arbitrary we are done. □

For the next two lemmas, let β denote any bi-infinite geodesic, and let $\pi = \pi_\beta : \mathcal{C}(S) \to \beta$ be any map which for each x picks out some closest point on β. (Note that π need not be uniquely defined.) Let $[a, b]$ denote any geodesic joining a and b, taken to be a segment of β whenever a and b lie on β.

Lemma 7.4. *Let $x \in \mathcal{C}(S)$ and $z \in \beta$, such that $d(x, z) \leq d(x, \pi_\beta(x)) + k$ for some $k \geq 0$. Then $d(\pi(x), z) \leq k + 4\delta$.*

Proof. We may assume $d(\pi(x), z) > 2\delta$. Let $m \in [\pi(x), z]$ be distance $2\delta + \epsilon$ from $\pi(x)$, for $\epsilon > 0$. By δ-hyperbolicity, m is distance at most δ from either some $m_1 \in [x, \pi(x)]$ or some $m_2 \in [x, z]$. The former case cannot occur, since then $d(m_1, \pi(x)) \leq \delta$ and hence $d(\pi(x), m) \leq 2\delta$, a contradiction. Thus we have $d(x, \pi(x)) \leq d(x, m_2) + \delta$, and $d(m, z) \leq d(m_2, z) + \delta$. Adding these together and using the hypothesis, we conclude that $d(m, z) \leq 2\delta + k$ and hence $d(\pi(x), z) \leq 4\delta + k + \epsilon$. Sending $\epsilon \to 0$ gives the desired result. □

Lemma 7.5. *Let x, y be any two points in $\mathcal{C}(S)$, such that $d(\pi_\beta(x), \pi_\beta(y)) > 8\delta + 2$. Let σ be the subsegment of $[\pi_\beta(x), \pi_\beta(y)]$ on β whose endpoints are distance $4\delta + 1$ from $\pi_\beta(x)$ and $\pi_\beta(y)$ respectively. Then σ is in a 2δ neighborhood of $[x, y]$.*

Proof. Form the quadrilateral whose sides are $[x, \pi(x)], [\pi(x), \pi(y)], [y, \pi(y)]$ and $[x, y]$. By δ-hyperbolicity, any point $z \in [\pi(x), \pi(y)]$ is at most 2δ from one of the other three sides. Suppose that this is the side $[x, \pi(x)]$. Then there is some $m \in [x, \pi(x)]$ such that $d(m, z) \leq 2\delta$, and since $d(x, \pi(x)) \leq d(x, z)$, we must have $d(m, \pi(x)) \leq 2\delta$. It follows that $d(\pi(x), z) \leq 4\delta$. The same argument applies to $[y, \pi(y)]$, and it follows that if $z \in \sigma$ then it must be distance 2δ from $[x, y]$. □

Proposition 7.6. (Axis) *Let h be a pseudo-Anosov element in $Mod(S)$. There exists a bi-infinite tight geodesic β such that for each j, $h^j(\beta)$ and β are 2δ fellow travelers. Moreover there exists a hierarchy H with main geodesic β.*

Proof. Pick any $x \in \mathcal{C}(S)$. Let β_n be a tight geodesic joining $h^{-n}(x)$ and $h^n(x)$ (extend the endpoints in an arbitrary way to complete markings $\mathbf{I}(\beta_n)$ and $\mathbf{T}(\beta_n)$), and let H_n be a hierarchy with main geodesic β_n. By Proposition 3.6 of [21], the sequence $\{h^k(x), k \in \mathbf{Z}\}$ satisfies $d(h^k(x), x) \geq c|k|$ for some $c > 0$ (independent of x or h) so the sequence is a $\frac{d_0}{c}$- quasi-geodesic, where $d_0 = d(h(x), x))$. By δ-hyperbolicity there is a constant $c' = c'(c, d_0, \delta)$ so that β_n and the sequence $\{h^j(x)\}_{|j| \leq n}$ lie in a c'-neighborhood of each other. This implies that there exist $x_n \in \beta_n$, so that given any R, for n, m sufficiently large, β_n and β_m are $(2c', R)$-parallel at x_n, x_m.

We apply Theorem 6.13 (Convergence of Hierarchies) to find a geodesic β and a hierarchy H, which is the limit of a subsequence of H_n. For each j, $h^j(\beta)$ is a $2c'$-fellow traveler to β. Applying Lemma 7.3 gives the result. □

We call β a *quasi-axis* for h. We will need to know the following:

Lemma 7.7. *Given $A > 0$, there is an integer $N > 0$, independent of h, such that for any $x \in \mathcal{C}(S)$ and $n \geq N$,*

$$d(\pi(x), \pi(h^n(x))) \geq A.$$

Proof. We first observe that, if g is any power of h, and β a quasi-axis for h, then

$$d(\pi g(x), g\pi(x)) \leq 10\delta \tag{7.1}$$

for any $x \in \mathcal{C}$. The proof will be given below.

Now using the inequality $d(\pi(x), h^n\pi(x)) \geq c|n|$, with c independent of x and h, from Proposition 3.6 of [21], we simply choose N so that $cN > A+10\delta$.

It remains to prove (7.1): Since β and $g(\beta)$ are 2δ-fellow travelers, we have $d(g(x), g(\beta)) \leq d(g(x), \beta) + 2\delta$, or equivalently, since g is an isometry, $d(g(x), g\pi(x)) \leq d(g(x), \pi g(x)) + 2\delta$. Now $\pi g\pi(x)$ is on β, and again by the fellow traveler property, we have $d(g\pi(x), \pi g\pi(x)) \leq 2\delta$. Thus $d(g(x), \pi g\pi(x)) \leq d(g(x), \pi g(x)) + 4\delta$. Applying Lemma 7.4 with $k = 4\delta$ and $z = \pi g\pi(x)$, we find that $d(\pi g(x), \pi g\pi(x)) \leq 8\delta$. We conclude that $d(\pi g(x), g\pi(x)) \leq 10\delta$, as desired. □

Proof of Conjugacy Theorem. Suppose that $h_2 = w^{-1}h_1w$. Lemma 7.7 guarantees that we can choose N independent of h_1 and h_2 such that $d(\pi(x), \pi h_i^n(x))) \geq 40\delta + 24$ for all $x \in \mathcal{C}$ and $n \geq N$. Let $g_i = h_i^N$ (for $i = 1, 2$). In the proof below, let $C_1, C_2, \ldots$ denote positive constants which are independent of h_1 and h_2.

Fix a complete clean marking μ_0 in S. Let H_i be a hierarchy such that $\mathbf{I}(H_i) = \mu_0$ and $\mathbf{T}(H_i) = g_i(\mu_0)$. We may also assume the main geodesic of H_i is a segment $[v, g_i(v)]$ for a base curve v of μ_0. By Theorem 7.1 (Quasigeodesic Words), we have $|H_i| \leq C_1|g_i| \leq NC_1|h_i|$.

Since w acts by natural isomorphisms on $\mathcal{C}(S)$ and and all the subsurface complexes, we have a hierarchy $w(H_2)$ with main geodesic $[w(v), wg_2(v)] = [w(v), g_1w(v)]$, and $|w(H_2)| = |H_2|$.

Form a quasi-axis β for g_1, together with a hierarchy H, and form the segments

$$I_0 = [\pi(v), \pi g_1(v)]$$

and

$$I'_m = [\pi g_1^m w(v), \pi g_1^{m+1} w(v)]$$

on β. Each of these have length at least $40\delta + 24$. Let $\sigma_0 \subset I_0$ and $\sigma'_m \subset I'_m$ be the subsegments obtained by removing $4\delta + 1$-neighborhoods of the endpoints.

The $\{I'_m\}_{m\in\mathbf{Z}}$ tile β and therefore the gaps between σ'_m and σ'_{m+1} have length $8\delta + 2$. It follows that there exists some $m \in \mathbf{Z}$ such that σ_0 and σ'_m overlap on a segment ζ of length at least $12\delta + 10$.

Let $w' = g_1^m w$, and note that w' also conjugates h_1 and h_2. We will now bound the word length of w', by bounding $d_{\widetilde{\mathcal{M}}}(\mu_0, w'(\mu_0))$.

By Lemma 7.5, the segment ζ is in a 2δ-neighborhood of both $[v, g_1(v)]$ and $[w'(v), g_1w'(v)]$. Let x be a vertex of ζ nearest its midpoint, so that ζ contains an interval L of radius $6\delta + 4$ around x, and let u, u' be vertices on $[v, g_1(v)]$ and $[w'(v), g_1w'(v)]$, respectively, which are nearest to x. Thus the main geodesics of H and H_1 are $(2\delta, 6\delta + 3)$-parallel at x and u, and similarly for those of H and $w'(H_2)$ at x and u'. This will allow us to apply Lemma 6.7 (Slice Comparison) below.

Resolve H_1 into a sequence of slices. One of them must have bottom vertex u (see proof of Proposition 5.4) – let μ_1 be a clean marking compatible with this slice. The resolution gives a bound $d_{\widetilde{\mathcal{M}}}(\mu_0,\mu_1) \le C_2|h_1|$, by Proposition 5.4 and Lemma 5.5. Similarly, resolve $w'(H_2)$, find a slice with bottom vertex u', let μ_2 be a clean marking compatible with this slice, and conclude $d_{\widetilde{\mathcal{M}}}(\mu_2, w'(\mu_0)) \le C_3|h_2|$.

Let μ_3 be a clean marking associated to a slice of H with base vertex x. Let W be a hierarchy with $\mathbf{I}(W) = \mu_1$ and $\mathbf{T}(W) = \mu_3$. Case (2) of Lemma 6.7 (Slice Comparison) tells us that W is (K', M)-pseudo-parallel to H_1 (with K', M depending only on δ). In particular this means $|W| \le C_4|H_1|$. Resolving W, we obtain $d_{\widetilde{\mathcal{M}}}(\mu_1,\mu_3) \le C_5|h_1|$.

Similarly, join μ_3 to μ_2 by a hierarchy W'. The same argument as for W gives us $d_{\widetilde{\mathcal{M}}}(\mu_2,\mu_3) \le C_5|h_2|$.

Adding these bounds, we obtain $d_{\widetilde{\mathcal{M}}}(\mu_0, w'(\mu_0)) \le C_6(|h_1| + |h_2|)$, which as in §7.1 gives the desired bound on $|w'|$. □

8. Hierarchies without annuli

It is useful to observe that most of our discussion can be carried through for hierarchies in which only domains of $\xi \ge 4$ are used (thus excluding annuli, as three-holed spheres were already excluded). The resulting construction is suitable for considering sequences of *pants decompositions*, rather than markings, separated by elementary moves. Let us remark briefly on how this works. The verification consists essentially of looking over the basic constructions with these comments in mind.

In this setting, *markings* have only bases, no transverse curves, and hence there is no distinction between clean and unclean markings. A *complete* marking without transverse curves is just a pants decomposition, and an elementary move between pants decompositions is what remains of a Flip move when the transverse curves are forgotten. That is, one component of the pants decomposition is replaced by a new component disjoint from the others, such that the two of them form an edge in the curve complex of the $\xi = 4$ surface that they fill. Note that there are infinitely many elementary moves one can perform on a given pants decomposition. Lemma 2.5, which gives a bound on $d_Y(\mu,\mu')$ when μ and μ' are related by an elementary move, still goes through when μ and μ' are pants decompositions and Y is not an annulus.

A *hierarchy without annuli* is defined just as in Definition 4.4, except that in part (2) only domains Y that are *not* annuli are considered. With this definition, and given any markings without transverse curves P and Q, a hierarchy without annuli will exist between them just as in Theorem 4.6. In fact this follows from that theorem, just by taking the normal hierarchy and throwing away all geodesics supported on annuli.

Constructions such as $\Sigma(Y)$ still make sense, even for Y an annulus, and all the properties established in Section 4 still hold, with the exception that statements such as the last part of Theorem 4.7 (Structure of Sigma) part (3), where the conclusion is that a subsurface Y supports a geodesic, only hold when Y is not an annulus. Similarly Theorem 4.20 (Completeness) holds, with the amended definition that a hierarchy without annuli is complete if every component domain Y with $\xi(Y) \geq 4$ supports a geodesic.

Slices of course have no annulus geodesics, and resolutions are possible just as before. Conversion of a resolution to a sequence of clean markings is now trivial, since transverse curves posed the only complication. Moves between markings are now elementary moves between pants decompositions, in the sense given above.

Lemma 6.1 (Sigma Projection) goes through unchanged, and Lemma 6.2 (Large Link) holds for non-annular domains. Theorem 6.10 (Efficiency of Hierarchies), estimating distances in $\widetilde{\mathcal{M}}$ via hierarchy sizes, also goes through for hierarchies without annuli, if $\widetilde{\mathcal{M}}$ is replaced by the (locally infinite!) graph of pants decompositions with edges representing elementary moves. Lemma 6.11 (Order and projections) goes through as well. Theorem 6.12 (Move distance and projections) also holds with appropriate reinterpretation; in particular the sum in the middle is only over non-annular domains.

Although we have thrown them away, the annulus geodesics are still "visible" in the following sense: If α is any curve that shows up in a hierarchy without annuli H then the corresponding annulus Y is still a component domain, and the structure $\Sigma(Y)$ still exists. In particular if α is not a curve of either the initial or terminal marking of H then $d_Y(\mathbf{I}(H), \mathbf{T}(H))$ is still well-defined. A resolution of H can be obtained from a resolution of the corresponding regular hierarchy by forgetting the moves that take place on annulus geodesics. Thus, in the resulting sequence of markings $\{\mu_i\}$, what remains from those slices involving Y is a subsequence $\mu_j, \ldots, \mu_k$ consisting of those markings with α as a component. The moves $\mu_{j-1} \to \mu_j$ and $\mu_k \to \mu_{k+1}$ must therefore correspond to replacing some β with α, and then replacing α with some γ. The projection $d_Y(\mathbf{I}(H), \mathbf{T}(H))$ gives an estimate, up to uniform additive error, of $d_Y(\beta, \gamma)$. This follows from Lemma 6.1 (Sigma Projection), since in fact β and γ are the last position on $\sigma_-(Y)$ and the first on $\sigma_+(Y)$, respectively.

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