On Thurston's ending lamination conjecture

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Abstract

We describe Thurston's ending lamination conjecture for Kleinian groups, and discuss the relationship of this conjecture to questions of rigidity, deformation and topological dynamics for Kleinian groups. We give a solution for the conjecture in a special case, and conclude with a collection of questions and conjectures raised by our approach.

1 Introduction

Consider a discrete, faithful representation $\rho : G \to \mathrm{PSL}_2(\mathbf{C})$, where G is the fundamental group of a closed surface S. The image group $\Gamma = \rho(G)$ is called a Kleinian group, and acts isometrically on the hyperbolic space $\mathbf{H}^3$ to produce a quotient hyperbolic manifold $N = \mathbf{H}^3/\Gamma$.

Surface groups are the simplest interesting examples of Kleinian groups with infinite-volume quotients, and often appear as subgroups of more general groups. In this capacity they account for a good deal of what is known about the deformation theory of Kleinian groups. Although many of the questions and some of the answers which we will discuss can be formulated more generally, we will restrict ourselves to surface groups in this paper, for ease of exposition. We will later point out which of our results apply to a broader class of groups.

The group $\rho(G)$ also acts by Möbius transformations on the Riemann sphere $\widehat{\mathbf{C}}$, known as the "boundary at infinity" of $\mathbf{H}^3$. The interplay between the "internal" geometry of the action on $\mathbf{H}^3$ and the "external" conformal dynamics of the action on $\widehat{\mathbf{C}}$ is the source of much of the depth and beauty of Kleinian groups. Our discussion is motivated by the following basic questions:

1. Classification: What is a sufficient invariant to determine a group $\rho(G)$ up to conjugacy in $\mathrm{PSL}_2(\mathbf{C})$?

2. Deformation: What can one say about the space of all representations ρ, and its topological structure?

3. Topological dynamics: What is a good topological model for the action of $\rho(G)$ on $\widehat{\mathbf{C}}$?

Terminology. Let us give a few standard definitions. The *limit set* $\Lambda = \Lambda_\Gamma$ of a Kleinian group Γ is the smallest closed Γ-invariant set in $\widehat{\mathbf{C}}$. Its complement $\Omega = \Omega_\Gamma$ is known as the domain of discontinuity, and the action of Γ on Ω is properly discontinuous. Thus the quotient Ω/Γ is a Riemann surface, known sometimes as the conformal boundary of N. In our case N is homotopy-equivalent to S, and by results of Bonahon and Thurston [Bo1, Th1], in fact is actually homeomorphic to $S \times \mathbf{R}$. N has two *ends*, which we may think of in terms of their *neighborhoods* of the form $S \times (r, \infty)$ and $S \times (-\infty, t)$.

A representation ρ (or its image Γ) is *Fuchsian* if it is conjugate in $\mathrm{PSL}_2(\mathbf{C})$ to a subgroup of $\mathrm{PSL}_2(\mathbf{R})$ – in particular it then has an invariant plane $\mathbf{H}^2 \subset \mathbf{H}^3$. We call ρ and Γ *quasi-Fuchsian* if there is a homeomorphism $Q : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ and a Fuchsian representation ρ_0 such that

$$Q \circ \rho_0(g) = \rho(g) \circ Q \tag{1.1}$$

for any $g \in G$. (In fact Q can be taken to be a quasiconformal homeomorphism; see [Mt]).

Of great importance in distinguishing and analyzing the different categories of groups is the *convex hull* of the limit set, $\mathcal{CH}(\Lambda)$. This is the smallest closed convex subset of $\mathbf{H}^3$ that meets Λ at infinity. Its quotient $C(N) = \mathcal{CH}(\Lambda)/\Gamma$ is called the convex core of N, and is the smallest closed convex submanifold of N whose inclusion is a homotopy equivalence.

We shall be considering only representations without parabolics (in the last section the case with parabolics will be briefly mentioned). With this assumption, there are three possibilities for a discrete faithful representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ (where "$\cong$" denotes homeomorphism):

(a) *Quasi-Fuchsian:* Λ_Γ is a Jordan curve, Ω consists of two disks Ω_+ and Ω_-, $\mathcal{CH}(\Lambda_\Gamma) \cong \mathbf{H}^2 \times [0,1]$ and $C(N) \cong S \times [0,1]$ (except when Γ is actually Fuchsian, in which case $\mathcal{CH}(\Lambda_\Gamma) = \mathbf{H}^2$ and $C(N) \cong S$.)

(b) *Singly degenerate:* Ω is a single disk, $\mathcal{CH}(\Lambda_\Gamma) \cong \mathbf{H}^2 \times [0,\infty)$, and $C(N) \cong S \times [0,\infty)$.

(c) *Doubly degenerate:* Ω is the empty set, $\mathcal{CH}(\Lambda_\Gamma) \cong \mathbf{H}^3$, and $C(N) \cong S \times \mathbf{R}$.

Type (a) is a special case of what is called a *geometrically finite* representation. Types (b) and (c) are called *geometrically infinite.* An end of N is called geometrically finite if it has a neighborhood disjoint from $C(N)$ – in that case it corresponds to a component of Ω, and of $\partial\mathcal{CH}(\Lambda)$. Otherwise it is called geometrically infinite. Thus in case (b) there is one geometrically infinite end, and in case (c) there are two. The three-fold classification in terms of the domain of discontinuity was known by Bers, Maskit and Marden (see [Be2, Mt]), whereas the homeomorphism type of $C(N)$ is from Bonahon and Thurston.

Geometrically finite groups. For quasi-Fuchsian groups the questions we listed have fairly complete answers. The correct invariant for classifying the group is the conformal boundary Ω/Γ, which in this case is a pair of Riemann surfaces (one for each end) identified with S.

More precisely, let $QF(S)$ denote the space of quasi-Fuchsian representations of $G = \pi_1(S)$, modulo conjugation by elements of $\mathrm{PSL}_2(\mathbf{C})$. The space of conformal structures on S (modulo isotopy) is known as the Teichmüller space $\mathcal{T}(S)$ (see e.g. [Ga, Ab1]). The following is due to Bers (see [Be1, Be3]):

Theorem 1 *The space $QF(S)$ is homeomorphic to $\mathcal{T}(S) \times \mathcal{T}(S)$ via the correspondence*

$$\rho \leftrightarrow (\Omega_+/\rho(\pi_1(S)), \Omega_-/\rho(\pi_1(S))).$$

This provides an answer to the deformation theory question as well, since $\mathcal{T}(S)$ is known to be an open finite-dimensional cell.

The dynamics question is answered, for our purposes, by the definition – the action of a quasi-Fuchsian group is topologically (in fact quasi-conformally) conjugate to the action of a Fuchsian group, which is fairly well understood.

Some conjectures. One possible type of answer to the classification question is a *rigidity* theorem. For representations with finite-volume quotients, Mostow-Prasad rigidity [Mo, Pr] says that the abstract group is itself a sufficient invariant to determine the representation up to conjugacy. Sullivan rigidity [Su] states that a quasi-conformal homeomorphism of $\widehat{\mathbf{C}}$ conjugating the action of two finitely generated groups must be a Möbius transformation, if it is known to be conformal on the domains of discontinuity. Thus, the structure of the conformal boundary, together with some information embodied in the fact that the homeomorphism is quasiconformal, determines the group. As we shall see, it is reasonable to expect that in fact the quasiconformality condition is unnecessary, and a topological conjugacy is sufficient. Thus we may state the following well-known conjecture (see [Ab2] for a version of it):

Conjecture 1 *If the actions of ρ_1 and ρ_2 on $\widehat{\mathbf{C}}$ are conjugate by a homeomorphism $\mathcal{F}$ whose restriction to the domains of discontinuity is conformal, then $\mathcal{F}$ is in fact a Möbius transformation.*

For quasi-Fuchsian (in fact for geometrically finite) groups this is already known – it follows from Maskit's classification [Mt].

All known examples of Kleinian representations of $\pi_1(S)$ arise as limits of quasi-Fuchsian representations. The following conjecture about the deformation space was first made by Bers in [Be2] for singly-degenerate representations, and is plausible in general (see also [Th2]).

Conjecture 2 *The quasi-Fuchsian representations of $\pi_1(S)$ are dense in the space of all discrete, faithful representations.*

The following topological model has been proposed by Thurston as an answer to question 3 (see [Ab2, Th2]). We say that a representation ρ of G is obtained from a representation ρ_0 by a *collapsing map* if there is a continuous map $\pi : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ of degree 1, such that

$$\rho(g) \circ \pi = \pi \circ \rho_0(g) \tag{1.2}$$

for every $g \in G$, and for every component D of $\Omega_{\rho(G)}$, $\pi^{-1}(D)$ is a component of $\Omega_{\rho_0(G)}$ and is mapped homeomorphically to D.

Conjecture 3 *Any discrete faithful $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ is obtained from a Fuchsian representation ρ_0 by a collapsing map.*

We can motivate conjecture 3 in the following way. Suppose $\{\rho_i\}$ is a sequence of quasi-Fuchsian representations converging to a geometrically infinite representation ρ. In case none of the groups in the sequence or the limit have parabolics (which we are assuming throughout), it is known that the Hausdorff limit of the sequence of limit sets $\Lambda_{\rho_i(G)}$ is the limit set $\Lambda_{\rho(G)}$ (see [JM, KT]), which is either all of $\widehat{\mathbf{C}}$ or the complement of a disk in $\widehat{\mathbf{C}}$. Letting $Q_i : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ denote the maps which conjugate a fixed Fuchsian action ρ_0 to ρ_i (and therefore map $\Lambda_{\rho_0(G)}$ to $\Lambda_{\rho_i(G)}$), it is natural to ask if these maps converge to a collapsing map π that takes ρ_0 to ρ.

Conjecture 1 attempts to classify a group in terms of its topological action on $\widehat{\mathbf{C}}$, and the conformal type of its conformal boundary. There is also a conjectured "internal" answer to the classification question, involving an invariant proposed by Thurston called an "ending lamination", which is heuristically a limiting version of the conformal boundaries, for geometrically infinite ends of the manifold. The main object of this paper is to discuss this invariant, clarify its connection to the above questions, and explain some positive answers to the conjectures in some special cases.

2 Laminations, realizability and the limit set

A geodesic lamination on a hyperbolic surface is a closed set which is the disjoint union of simple geodesics (without endpoints) (see [Th1, CB]). We will also refer to laminations as *topological* objects, which without reference to a hyperbolic metric have many isotopic representatives on a surface (see [Ha]).

Geodesic laminations arise naturally in the study of hyperbolic surfaces, as limits of sequences of simple closed geodesics. They figure prominently in the geometry of hyperbolic three-manifolds as well, as we shall see. Let us now explain how laminations might arise from a collapsing map of the type proposed in conjecture 3.

Let $\pi : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be any continuous map whose restriction to the exterior of the closed unit disk Δ is a homeomorphism to $\pi(\widehat{\mathbf{C}} - \Delta)$. The identifications which π makes on the circle $S^1 = \partial\Delta$ define a lamination in Δ in the following way. For every $z \in \pi(S^1)$ let $E(z) = \pi^{-1}(z) \cap S^1$, and let $C(z)$ denote the convex hull of $E(z)$ in the hyperbolic metric in Δ. We claim that the collection $\{C(z)\}$ is a partition of Δ into disjoint closed sets. This is because, if $C(z) \cap C(w) \neq \emptyset$, we could find two pairs of points in $E(z)$ and $E(w)$ that separate each other on S^1, and it is a simple exercise in plane topology to show that this implies that $z = w$. (This is an argument of Thurston; see [Th4, Th2]).

Each $C(w)$ is a closed polygon with (possibly infinitely many) infinite geodesic sides. Thus the union $\lambda = \bigcup_z \partial C(z)$ is a geodesic lamination on Δ. If π is invariant under some isometric group action, then so is λ, and it will then project to a lamination on the quotient surface. We shall call λ the *collapsing lamination* for π.

Realizability. Let us return to the context of representations of surface groups, and link these ideas to the internal geometry of a 3-manifold. A representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ determines a unique homotopy class $[f_\rho]$ of maps from S to the quotient 3-manifold $N = \mathbf{H}^3/\rho(\pi_1(S))$, namely the one which induces ρ on the fundamental groups. We say a lamination λ on S is *realizable* in $[f_\rho]$ if there is a map $g : S \to N$ which is homotopic to f_ρ and which maps the leaves of λ to geodesics. (In particular every simple closed curve is realizable, by a map that takes it to its unique geodesic representative in N, and extends in any continuous way to the rest of the surface).

Suppose that ρ is quasi-Fuchsian, and let Q denote the quasiconformal homeomorphism conjugating ρ_0 to ρ, as in (1.1). If we realize λ as a geodesic lamination in the Fuchsian representation, and then lift it to $\mathbf{H}^2$ we obtain a lamination $\widetilde{\lambda}$ which is completely specified by the set of pairs of endpoints of its leaves on the circle at infinity $S^1 = \Lambda_{\Gamma_0}$. Each such pair is mapped by Q to a pair of points on Λ_Γ, and after joining the image pairs with geodesics we get a new lamination $\widetilde{\mu}$ in $\mathbf{H}^3$. A continuous map can then be constructed from $\mathbf{H}^2$ to $\mathbf{H}^3$ which takes the leaves of $\widetilde{\lambda}$ to those of $\widetilde{\mu}$, "fills in" somehow in the complement of $\widetilde{\lambda}$, and is equivariant with respect to the group actions of $\rho_0(\pi_1(S))$ on $\mathbf{H}^2$ and $\rho(\pi_1(S))$ on $\mathbf{H}^3$ (see [Th1, CEG]). This descends to a map from S to N that realizes λ. In this way we see that every lamination is realizable in a quasi-Fuchsian representation.

Ending laminations. The attempt to generalize this picture to the case of a geometrically infinite represention of G leads to the formulation of the ending lamination invariant. Conjecturally, the limit set of $\rho(G)$ is still the continuous (but no longer homeomorphic) image of S^1 by a collapsing map π, and a lamination is realizable exactly when all pairs of endpoints of its leaves in the universal cover are mapped to distinct points in Λ_Γ. In particular λ should be unrealizable only if a sublamination of it is contained in the collapsing lamination for π. The trouble, of course, is that π is not known to exist in general.

Thurston circumvented this problem by examining the interior geometry of the manifold $N = \mathbf{H}^3/\Gamma$, and determining the structure of the realizable laminations without reference to the limit set. He considered what happens when one attempts to realize a lamination λ by approximating it with a sequence of simple closed curves $\gamma_i \subset S$. If the sequence γ_i^* of geodesic representatives of $f_\rho(\gamma_i)$ in N remains in a compact set, one may conclude that the geodesics converge to a lamination in N, which is obtained as the image of a realization of λ (this may be done using Thurston's *pleated surfaces* as the realizing maps, since these have nice convergence properties. Pleated surfaces will be discussed further in the next section).

Say that a sequence $\{\gamma_i\}$ of simple closed curves has geodesic representatives that "exit" an end e of N if each neighborhood of e contains all but finitely many γ_i^*. Bonahon and Thurston [Bo1, Th1] showed that, if an end e of ρ is geometrically infinite and has no parabolics, there is associated to it a unique unrealizable lamination μ_e such that a sequence of simple closed curves γ_i on S has geodesics exiting e if and only if γ_i converges to μ_e on S. (The statement is more complicated in the presence of parabolics). μ_e is called the "ending lamination" of e. Thus there are at most two unrealizable laminations, those occurring as ending laminations of one or both ends.

The existence of an ending lamination and the exiting sequences of curves implies some geometrically nice behavior of the end which Thurston called *geometric tameness*, in particular implying that a neighborhood of the end is homeomorphic to $S \times \mathbf{R}$ (topological tameness). (see also [Bo2] and [Ca]).

Thurston's ending lamination conjecture can now be stated for surface groups without parabolics:

Conjecture 4 (Thurston [Th2]) *A representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ without parabolics is determined up to conjugacy in $\mathrm{PSL}_2(\mathbf{C})$ by the conformal boundaries of its geometrically finite ends (if any), and by the ending laminations of its geometrically infinite ends (if any).*

This conjecture was answered in a special case in [Mi3, Mi2]:

Theorem 2 *Let ρ_1 and ρ_2 be representations of $\pi_1(S)$ for which there is a positive ϵ such that injectivity radii in $\mathbf{H}^3/\rho_i(\pi_1(S))$ are at least ϵ at every point, for $i = 1, 2$. If the ending laminations and/or conformal boundaries of the ends of ρ_1 and ρ_2 are the same, then the representations are conjugate in $\mathrm{PSL}_2(\mathbf{C})$.*

(Theorem 2 in fact holds in greater generality; see section 4.)

More on collapsing maps. Let ρ_0 again be a Fuchsian representation, ρ a singly or doubly degenerate representation (without parabolics), and λ_+ the ending lamination for one of the ends of ρ. (If there are two degenerate ends, let λ_- be the ending lamination for the other end). Represent the lifts $\widetilde{\lambda}_\pm$ as laminations on the disks $\Omega_\pm$ comprising the domain of discontinuity of $\rho_0(G)$.

Consider the space obtained by identifying to a point each leaf of $\widetilde{\lambda}_+$, and the closure of every component of the complement of $\widetilde{\lambda}_+$ in Ω_+ (and similarly for $\widetilde{\lambda}_-$, in the doubly degenerate case). Equipped with the natural quotient topology this space is, perhaps surprisingly, homeomorphic again to $\widehat{\mathbf{C}}$ (see [CT]). For the singly degenerate case the quotient map π is the same as in the construction described in the beginning of the section, and $\widetilde{\lambda}_+$ is its collapsing lamination. In the doubly degenerate case something more dramatic happens – the circle $\Lambda_{\rho_0(G)}$ is mapped to a sphere-filling curve.

Cannon and Thurston proved that collapsing maps of this type give rise to the actions of doubly-degenerate groups which arise from hyperbolic surface bundles over the circle, and singly degenerate groups obtained by a similar construction (via Thurston's double limit theorem and iterated pseudo-Anosov maps of S [Th3]). In [Mi2] we proved that this is the case for any surface group satisfying the injectivity radius condition. That is,

Theorem 3 *If ρ is a geometrically infinite representation of $\pi_1(S)$ whose quotient admits a positive lower bound on injectivity radius, then ρ is obtained from a Fuchsian representation by a collapsing map π, and the identifications of the collapse are determined as above by the ending lamination(s) of ρ.*

As an immediate application, this theorem implies a rigidity result which is a positive answer to conjecture 1 for these groups:

Theorem 4 *Let ρ_1 and ρ_2 be representations of $\pi_1(S)$ whose quotients admit a positive lower bound on injectivity radius. If ρ_1 and ρ_2 are conjugate by a homeomorphism $\mathcal{F} : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$, and $\mathcal{F}$ is conformal on the domains of discontinuity, then $\mathcal{F}$ is a Möbius transformation.*

The assumption of quasiconformality of $\mathcal{F}$, as used in Sullivan's rigidity theorem, is now unnecessary, since the existence of $\mathcal{F}$ together with the collapsing result of theorem 3, is sufficient to show that the ending laminations of the two groups are the same, and therefore by theorem 2, the groups are conjugate. We should note that theorem 2 itself is proved by an application of Sullivan's theorem, after constructing a quasi-conformal conjugacy between the representations, as we shall briefly describe in the next section.

The deformation space. The ending lamination conjecture, in the general case, also implies conjecture 2. Given a doubly degenerate representation ρ one can use Thurston's double-limit theorem [Th3] to construct a sequence $\{\rho_i\}$ of quasi-Fuchsian representations which converges to a representation ρ' with the same ending laminations as ρ. (If ρ is singly degenerate, one can use techniques of Bers [Be2] to construct the convergent sequence, and the limit ρ' can be obtained with the same ending lamination on one end, and conformal boundary on the other, as ρ). Conjecture 4 would then imply that ρ' and ρ are conjugate in $\mathrm{PSL}_2(\mathbf{C})$.

If one tries to apply this argument with theorem 2 to prove conjecture 2 for representations satisfying the injectivity radius condition, a difficulty arises: there is no guarantee that the approximating sequence $\{\rho_i\}$ has quotients which admit a uniform lower bound on injectivity radius. The limit representation ρ' may therefore admit no positive lower bound at all, and theorem 2 would not apply. The one case in which the argument carries through is that of manifolds with ending laminations which are stable laminations of pseudo-Anosov homeomorphisms of S. In this case, the sequence of representations produced by Thurston's methods is known to converge to one admitting a positive lower bound on injectivity radius (namely the representations treated in [CT]). Theorem 2 implies that any representation with the same ending lamination is conjugate to one of these, and in particular implies the following:

Corollary 5 *Suppose that $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbf{C})$ has a quotient that admits a positive lower bound on injectivity radii. If ρ has two ending laminations which are* *the stable and unstable laminations of a pseudo-Anosov homeomorphism of S, or a single degenerate end whose ending lamination is a stable lamination of a pseudo-Anosov, then ρ arises on the boundary of the quasi-Fuchsian deformation space.*

3 Teichmüller space and the cross-sections of an end

How does one obtain enough control on the internal geometry of a hyperbolic manifold to prove the above theorems?

Thurston introduced the notion of *pleated surfaces* as a way of using surface geometry to control 3-dimensional geometry. These are maps $g : (S, \sigma) \to N$, where σ is a hyperbolic metric on S, such that g preserves lengths of paths, maps the leaves of some geodesic lamination λ to geodesics, and maps its complementary regions totally geodesically. (Thus pleated surfaces are naturally related to realizable laminations).

Let $\mathcal{P}(\rho)$ denote the set of $[\sigma] \in \mathcal{T}(S)$ for which there exists a pleated surface $g : (S, \sigma) \to N$ in the standard homotopy class $[f_\rho]$. It seems clear that a good understanding of the shape of $\mathcal{P}(\rho)$ should yield information about the geometry of N. Thurston's results give asymptotic information about $\mathcal{P}(\rho)$, relating the ending lamination of N to the accumulation points of $\mathcal{P}(\rho)$ on the various "boundaries at infinity" which one can constuct for $\mathcal{T}(S)$ (see [Mi2] for a discussion).

In [Mi3] we obtain the following:

Theorem 6 *If there is a positive lower bound for the injectivity radii in N, then there exists a Teichmüller geodesic L in $\mathcal{T}(S)$ (which is an biinfinite geodesic if N is doubly degenerate, and a ray if N is singly degenerate) such that $\mathcal{P}(\rho)$ lies in a bounded neighborhood of L and L lies in a bounded neighborhood of $\mathcal{P}(\rho)$.*

The distance function with respect to which we consider geodesics in $\mathcal{T}(S)$ is the Teichmüller distance, which measures minimal quasiconformal distortion between different conformal structures. A Teichmüller geodesic is then a family of conformal structures represented by metrics $\{\sigma_t\}$ on S which, briefly, can be described as follows (see [Ab1] or [Ga]).

The metric σ_0, in the complement of a finite number of singularities, is Euclidean and admits a canonical pair of orthogonal foliations by Euclidean lines (known as the "horizontal" and "vertical" foliations). Choosing local coordinates in which σ_0 is $dx^2 + dy^2$, where the x axis is a leaf of the horizontal foliation, we may write σ_t as

$$e^{2t} dx^2 + e^{-2t} dy^2. \tag{3.1}$$

The parameter t measures arclength along the geodesic.

Theorem 6 is proven with a construction that combines the geometry of Teichmüller space with some results from [Mi1] on the geometry of harmonic maps of surfaces into hyperbolic 3-manifolds. The main idea is to prove that $\mathcal{P}(\rho)$ satisfies a certain "quasi-convexity" property. This is done by exhibiting $\mathcal{P}(\rho)$ as the image of a projection map $\Pi : \mathcal{T}(S) \to \mathcal{T}(S)$, which has similar geometric properties to a closest-point projection in a negatively curved space.

Once this family of metrics on S is given, one can construct a metric

$$e^{2t}dx^2 + e^{-2t}dy^2 + dt^2 \tag{3.2}$$

on $S \times \mathbf{R}$ (or $S \times [0, \infty)$ for the singly degenerate case) which is easy to analyze, and the primary application of theorem 6 is to show that this metric is a good approximation (in the large) to the hyperbolic metric on N.

That is, let M denote $S \times \mathbf{R}$ endowed with this metric (where in the singly degenerate case there is a simpler construction to put a metric on $S \times (-\infty, 0]$ that models the metric on the geometrically finite end). In [Mi2] we show that

Theorem 7 *There is a map $f : M \to N$ whose lift $\widetilde{f} : \widetilde{M} \to \mathbf{H}^3$ is a quasi-isometry.*

For any given σ_t on the geodesic, there is by theorem 6 a nearby metric τ_t and a pleated surface $g : (S, \tau_t) \to N$. A natural candidate for $f|_{S\times\{t\}}$ is

$$f|_{S\times\{t\}} = g \circ \phi_t$$

where $\phi_t : (S, \sigma_t) \to (S, \tau_t)$ is some map with bounded distortion which is homotopic to the identity. One can do this for all integer values of t, say, and interpolate somehow to obtain f on all of $S \times \mathbf{R}$. For this to work one must show, for example, that the separation in N between the pleated surface images of $S \times \{s\}$ and $S \times \{t\}$ is approximately proportional to $|s - t|$. Thus the proof involves an analysis of the way in which pleated surfaces are situated in a 3-manifold with respect to each other, and the relationship between their intrinsic geometry and the ambient three-dimensional geometry.

Theorem 7 implies theorem 2, since two manifolds with the same ending laminations and conformal boundaries are each quasi-isometric to the same "model" manifold M, and hence to each other. This quasi-isometry lifts to the universal cover and extends to a quasiconformal conjugacy of the actions on $\widehat{\mathbf{C}}$, to which Sullivan's rigidity theorem can be applied.

Once the model manifold is known to exist, one may follow the arguments of [CT] to obtain a proof of theorem 3, as well.

4 Conjectures and speculations

Some discussion is in order concerning the scope of our results, and the potential for further generalizations. Although the techniques we have used have some built-in limitations, they suggest a number of interesting questions and ways to proceed onwards.

More general groups. Theorems 2 and 6 are proven in [Mi3, Mi2] not just for surface groups, but for groups G which do not admit a non-trivial free-product decomposition. This condition implies that G is the fundamental group of a manifold with incompressible boundary (which, by Bonahon's theorem, may be chosen so that its interior is homeomorphic to the quotient hyperbolic manifold), and the same techniques apply, essentially by considering the subgroups of G associated to each boundary component. (The condition on injectivity radius still needs to be imposed).

The question of extending theorems 3 and 4 to freely indecomposable groups is still open. Theorem 3 applies separately to each boundary subgroup of such a group, but there are difficulties in extending the result to the action of the whole group.

If we try to apply these techniques to groups which do admit non-trivial free-product decompositions, completely new issues arise. The first problem is that Bonahon's theorem does not apply to these groups, so it is an open question whether in fact such a manifold $\mathbf{H}^3/\rho(G)$ is geometrically tame. Canary has shown (see [Ca]) that a manifold is geometrically tame if and only if it is topologically tame – that is, homeomorphic to the interior of a compact manifold. This result allows one to prove geometric tameness for a variety of new examples, in particular for certain covers of tame manifolds. Once a manifold with compressible boundary is known to be tame, one can apply to it many of the tools used to study incompressible-boundary manifolds. One may therefore hope that our techniques can be applied to the case of general geometrically tame groups which admit a positive lower bound on injectivity radius.

Short geodesics. A hyperbolic manifold not satisfying our injectivity radius restriction can contain a sequence of closed geodesics whose lengths go to zero.

This possibility introduces many technical difficulties into our arguments, and furthermore the results themselves are likely to be different in the general case – examples indicate that the set $\mathcal{P}(\rho)$ does not always lie in a bounded neighborhood of a Teichmüller geodesic, and that the model metric can not take the simple form which it does in (3.2). Although one can understand a number of particular examples fairly well, one can only speculate on whether a variant of our techniques can extend to this general case.

Meanwhile a reasonable question one might ask is, how big is the class of manifolds to which our results do apply?

The known explicit examples of a geometrically infinite manifold satisfying the lower bound on injectivity radius are obtained from the hyperbolic surface bundles over circles constructed by Jorgensen [Jo] and Thurston [Th3]. These constructions give only a countable number of different ending laminations (namely, the stable laminations of pseudo-Anosov homeomorphisms of S).

Examples of closed-surface groups without a lower bound on injectivity radius were constructed by Bohahon and Otal [BO] and Thurston [Th3]. The explicit constructions are quite intricate, but such ends are "generic" in some sense, as we shall shortly see.

Teichmüller rays. In view of the connections between ends of hyperbolic 3-manifolds and rays in Teichmüller space, it is natural to ask a corresponding question for Teichmüller rays. A Teichmüller ray obtained via theorem 6 from an end with a positive lower bound on injectivity radius has the property that there is a positive lower bound on the injectivity radii in the hyperbolic surfaces it parametrizes. We call such rays "precompact" because their images in the moduli space of S lie in compact sets. We can now ask two questions: How many precompact rays are there, and do all precompact rays come from ends of hyperbolic 3-manifolds?

A Teichmüller ray is determined by its basepoint and a *measured foliation* on S, namely the vertical foliation that arises in the canonical metric given in (3.1). This foliation is like a point at infinity for $\mathcal{T}(S)$, and we say the ray *terminates* in the foliation. It follows from results of Masur (see [Ma3, Ma1], and [Mi2] for a discussion) that precompactness of the ray is determined entirely by the topological type of this foliation, so let us call a foliation precompact if all rays terminating in that foliation are precompact. Moreover in the case of a ray arising from a geometrically infinite end, the foliation turns out to be equivalent to the ending lamination, under the standard correspondence between laminations and foliations on a surface (see [Le]).

Every stable foliation of a pseudo-anosov homeomorphism of a surface is precompact – this is because the homeomorphism induces an action on $\mathcal{T}(S)$ whose axis terminates in the foliation. The image of the axis in the moduli space is a closed loop, and in particular compact [Be4, Ab1]. This gives a countable number of examples – the same ones that arise from Thurston's construction.

Masur proved in [Ma2] that the geodesic flow on the moduli space is ergodic.

Thus almost every geodesic enters every part of the moduli space, which implies that the set of precompact rays emanating from a given point is of measure zero.

Still, one expects a bigger than countable set:

Conjecture 5 *The set of precompact foliations on S is uncountable, or equivalently the set of precompact rays based at any point in $\mathcal{T}(S)$ is uncountable.*

As evidence for this conjecture, we may consider the case of the torus, which is well-understood. A foliation on $T^2 = \mathbf{C}/L$ (where L denotes the lattice of Gaussian integers) is described by the slope α of its leaves (after realizing it as an affine foliation). One can prove the following:

Lemma 8 *There are uncountably many precompact rays emanating from a single point in $\mathcal{T}(T^2)$. In fact, such a ray is precompact if and only if its determining foliation has a slope whose continued fraction expansion has bounded coefficients.*

In fact, the stable foliations for pseudo-Anosovs correspond precisely to the periodic continued fraction expansions.

One way to see this is to note that $\mathcal{T}(T^2)$ is just the hyperbolic plane, and the moduli space is the modular surface. This surface has one cusp, and it is easy to construct uncountably many infinite geodesics that avoid a neighborhood of the cusp (by winding around the rest of the surface in non-periodic ways). The continued-fraction interpretation may be obtained, for example, from [Se].

For foliations on higher genus surfaces there are constructions which are analogous to the continued fraction expansions, but it is not clear how to use them to prove conjecture 5.

Returning to the question of how many hyperbolic 3-manifolds satisfy the injectivity radius restriction, we can propose the following:

Conjecture 6 *For every precompact ray R in $\mathcal{T}(S)$, there is a hyperbolic manifold $N \cong S \times \mathbf{R}$ such that an end of N is modelled on R as in theorem 6.*

As we discussed in section 2, it is possible using Thurston's double limit theorem to obtain a sequence of quasi-Fuchsian representations whose limit representation ρ has the endpoint of R as an ending lamination. However, there is no guarantee that N (or any of the approximating manifolds) admit a uniform positive lower bound on injectivity radii. Thus we cannot apply theorem 2 to N.

One is therefore led to make the following conjecture, which implies conjecture 6. (Denote by $qf(\sigma, \tau)$ the manifold associated to $(\sigma, \tau) \in \mathcal{T}(S) \times \mathcal{T}(S)$ by theorem 1.)

Conjecture 7 *Let σ, τ be two points in $\mathcal{T}(S)$ and L a geodesic segment connecting them. If ϵ is a lower bound for the injectivity radius of any surface parametrized by a point of L, then there is some $\delta(\epsilon)$ such that injectivity radii in the quasi-Fuchsian manifold $qf(\sigma, \tau)$ are bounded below by δ.*

Parabolics. We have so far avoided discussing representations with parabolics. In Thurston and Bonahon's theory, parabolics are a nuisance but not a fatal one. In particular, the theory of pleated surfaces still applies for manifolds with parabolic cusps, and Bonahon's result that representations of $\pi_1(S)$ are geometrically tame goes through as well. The curves which are parabolic in a given end can always be represented as disjoint simple closed curves on S (and thus are bounded in number), and the components of the complement of these curves in S essentially function independently as ends, each with its own ending lamination or conformal boundary (one should think of them as ends of the manifold obtained from N by removing standard open neighborhoods of the cusps). The ending lamination conjecture in this context takes the following form:

Conjecture 8 *Suppose that ρ_1 and ρ_2 are discrete faithful representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbf{C})$, which take the same elements of $\pi_1(S)$ to parabolics in each end. If ρ_1 and ρ_2 have the same collection of ending laminations and conformal boundaries then they are conjugate in $\mathrm{PSL}_2(\mathbf{C})$.*

A manifold with cusps automatically fails our injectivity radius restriction, and so is not covered by theorem 2. However one can consider the class of manifolds which may contain cusps but admit a positive lower bound on the lengths of all closed geodesics. There is hope that our techniques can still be extended to this case.

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