EXTREMAL LENGTH ESTIMATES AND PRODUCT REGIONS IN TEICHMÜLLER SPACE

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ABSTRACT. We study the Teichmüller metric on the Teichmüller space of a surface of finite type, in regions where the injectivity radius of the surface is small. The main result is that in such regions the Teichmüller metric is approximated up to bounded additive distortion by the sup metric on a product of lower dimensional spaces. The main technical tool in the proof is the use of estimates of extremal lengths of curves in a surface based on the geometry of their hyperbolic geodesic representatives.

1. INTRODUCTION

There is a longstanding but imperfect analogy between the geometry of the Teichmüller space $\mathcal{T}(S)$ of a surface and that of a complete, negatively curved space. For example, $\mathcal{T}(S)$ admits a boundary at infinity similar to that of hyperbolic space (See e.g. [8, 23]), and the Teichmüller geodesic flow on its quotient, the moduli space, is ergodic (see [22]). This paper studies one of the strong ways in which this analogy fails, namely the existence of large regions in the space which are closely approximated by products of lower-dimensional metric spaces.

There are several natural metrics on $\mathcal{T}(S)$; we will consider throughout only the *Teichmüller metric* (§2.5). The first indication of positively curved behavior of the Teichmüller metric came from Masur [20], in which examples were given of geodesic rays with a common basepoint, which remain a bounded distance apart for all time. This contradicts non-positive curvature on small scales at every point (though it also involves large-scale behavior of the geodesics in a subtle way). More recently, Masur and Wolf [24] have given examples of geodesic triangles which fail, arbitrarily badly, the "thin triangle" condition for hyperbolicity in the sense of Gromov and Cannon [5, 6, 12]; this implies that the large-scale behavior of the metric is not negatively curved. The main result of this paper describes regions of $\mathcal{T}(S)$ where the large scale behavior exhibits some characteristics of positive curvature.

The main theorem can be summarized as follows (a complete statement appears in §6). Let $\mathcal{T}(S)$ be the Teichmüller space of a surface S of finite type, endowed with

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the Teichmüller metric. Let $\gamma = \{\gamma_1, \dots, \gamma_k\}$ be a system of disjoint, homotopically distinct simple closed curves on S, and let $Thin_\epsilon(S,\gamma)$ denote the set of $\sigma \in \mathcal{T}(S)$ for which $\ell_\sigma([\gamma_i]) \le \epsilon$ for all i, where $\ell_\sigma([\gamma_i])$ denotes hyperbolic length of the σ-geodesic representative of γ_i. Let X_γ denote the product space $\mathcal{T}(S\setminus\gamma)\times\mathbf{H}_1\times\cdots\times\mathbf{H}_k$, where $S\setminus\gamma$ is considered as a punctured surface, and each $\mathbf{H}_i$ is a copy of the hyperbolic plane. Endow X_γ with the *sup metric*, $d_X = \max\{d_{\mathcal{T}(S\setminus\gamma)}, d_{\mathbf{H}_1}, \dots, d_{\mathbf{H}_k}\}$, of the metrics on the factors.

Theorem 6.1 *The Fenchel-Nielsen coordinates on $\mathcal{T}(S)$ give rise to a natural homeomorphism $\Pi : \mathcal{T}(S) \to X_\gamma$, and for ϵ sufficiently small this homeomorphism restricted to $Thin_\epsilon(S,\gamma)$ distorts distances by a bounded* additive amount*. That is,*

$$|d_{\mathcal{T}(S)}(\sigma,\tau) - d_X(\Pi(\sigma),\Pi(\tau))| \le c$$

For $\sigma,\tau \in Thin_\epsilon(S,\gamma)$, and $c = c(\chi(S),\epsilon)$.

Note in particular that the distortion of Π is additive and not multiplicative, as is more commonly the case with quasi-isometries. For this reason it is significant that we endow X_γ with the sup metric, and not the L^2 metric; these are equivalent up to bounded multiplicative, but not additive, distortion. A product of spaces with the sup metric exhibits not just non-negative, but actually positive curvature characteristics in the large. See §7 for a fuller discussion of this point.

The idea of decomposing Teichmüller space as a product using length and twist parameters of simple closed curves dates back to Fenchel-Nielsen [10] (see Abikoff [1, Chap. II] and Fricke-Klein [11]). Such decompositions have particular significance when the curves in question are short, and this has been used by several authors to study the degeneration of Riemann surfaces towards "noded" surfaces, in which a short curve has been replaced by two punctures. An incomplete list of treatments of this phenomenon from complex-analytic, algebraic-geometric and differential-geometric points of view includes Bers [4], Earle-Marden [7], Fay [9], Hejhal [13], Kra [16], Maskit [17, 18], Masur [21], Wolf-Wolpert [27] and Wolpert [29, 30, 28]. In all these cases, the emphasis is on careful local analysis, concentrating on the complex structure of the moduli space in a neighborhood of a noded surface, or on differential geometric structure such as the Weil-Petersson metric in the case of [21, 28], or a precise expression for the hyperbolic metric on the surface in [30]. By contrast, our approach ignores most local analytical details and concentrates on large-scale estimates in the Teichmüller space.

Extremal length estimates. The estimates of Teichmüller distance necessary for theorem 6.1 are obtained using Kerckhoff's theorem (see §2.5), which relates Teichmüller distance to ratios of extremal lengths. Therefore our main tool is a theorem that allows us to estimate extremal lengths of arbitrary curves on a surface, in terms of the behavior of their hyperbolic geodesic representatives.

The relationship between hyperbolic length ℓ_σ and extremal length λ_σ (see §2.3) of curves on a Riemann surface (S,σ) is not completely straightforward. For very short curves, ℓ_σ and λ_σ are nearly proportional. For long curves, the relation depends on the geometry of the surface. In particular an upper bound for λ_σ is given by

$\lambda_\sigma \leq (1/2)\ell_\sigma e^{\ell_\sigma/2}$ (see Maskit [19]), and this is close to sharp in general. But on the other hand $\lambda_\sigma \geq \ell_\sigma^2/2\pi|\chi(S)|$ is the general lower bound that comes from the definition of extremal length, and for any *fixed* conformal structure σ one can show (see e.g. [26] and lemma 4.3) that it is the lower bound which estimates λ_σ to within a bounded factor.

Theorem 5.1 gives an estimate for λ_σ which is correct, up to a bounded factor that depends only on the topological type of the surface, for all cases. The main observation is that, after decomposing a Riemann surface (S,σ) into thick and thin parts, or what we call here an (ϵ_0,ϵ_1) collar decomposition (see §2.4), each component of the decomposition contributes individually to the extremal length of a curve β in S. The contribution $\lambda_{P,\sigma}(\beta)$ of a component P of the thick part (see definition (4.2) and lemma 4.3) is approximately $\ell^2_\sigma(\beta^\sigma \cap P)$, where β^σ is the σ-geodesic representing β. A component A of the thin part, which is an annulus, contributes a quantity $\lambda_{A,\sigma}(\beta)$ which depends on the modulus of A and the amount of "twisting" that β does in A (see §3 and definition (4.3)). The theorem then states:

Theorem 5.1 *Let S be a surface of finite type with a standard hyperbolic metric σ in which boundary lengths are at most ℓ_0, and let $\mathcal{Q}$ be the set of components of the (ϵ_0,ϵ_1) collar decomposition. Then, for any curve system β in S, the extremal length $\lambda_{S,\sigma}(\beta)$ is approximated by*

$$\max_{Q\in\mathcal{Q}} \lambda_{Q,\sigma}(\beta)$$

up to multiplicative factor depending only on $\epsilon_0,\epsilon_1,\ell_0$ and the topological type of S.

In specific cases this theorem recovers the above exponential and quadratic bounds from [19] and [26].

Proofs. The proof of theorem 5.1 is essentially a long exercise in elementary hyperbolic geometry. Given a homotopy class of curves β in S we must arrange it in a sufficiently evenly-spaced way so as to obtain an embedded annulus which gives us an upper bound for extremal length (lower bounds are easier). The part of the argument which is not straightforward has to do with finding an appropriate metric, conformally equivalent to the hyperbolic metric, in which to make the spacing construction. The scaling factors, which vary among components of the thick-thin decomposition, must depend on the way in which β is distributed on S. The construction of this metric is given in lemma 5.2 and the discussion that follows it. (Note that the metric we obtain is in some way mimicking the actual extremal metric, which is given by a quadratic differential. However our construction is explicitly related to the hyperbolic characteristics of β, which makes it more suited to our purposes.)

The proof of theorem 6.1 is an almost direct application of theorem 5.1 and Kerckhoff's theorem 2.1, except for one point. In the passage from a component of $S \setminus \gamma$ with (short) boundary to a punctured surface, we must at some point compare extremal length ratios $\lambda_\sigma/\lambda_\tau$ for properly embedded arcs with endpoints on the boundary to similar ratios for simple closed curves in the interior. That the suprema of these ratios are within a bounded factor of each other is shown in lemma 6.2. The main trick in this lemma is to start with a (possibly non-closed) curve β that maximizes the *reciprocal* ratio $\lambda_\tau/\lambda_\sigma$, and then find a closed curve which is sufficiently close to being "orthogonal" to β.

A different approach to the proof of theorem 6.1 than the one we have taken might involve estimating Teichmüller distances directly by constructing appropriate quasiconformal maps. One might hope that, since a short curve corresponds to a high-modulus annulus in the surface, a quasi-conformal map on the complement of this annulus might be "spliced" across the annulus in a way that takes the appropriate modulus and twist parameters into account, and thus the quasi-conformal dilatation of the spliced map would be estimated by the dilatations of the component maps (and the modulus and twisting information in the annulus). It is therefore of interest that, in our experience, this approach actually runs into quite serious difficulties, since "splicing" quasi-conformal maps is, in general, not very easily done without great loss of control over dilatation. In particular, estimates of dilatation of such spliced maps tend to be, at best, polynomially related to the dilatations of the component maps, and not linearly as we would need for theorem 6.1. It is for this reason that we have adopted the extremal length approach, which circumvents these problems.

Summary of contents. In section 2 we set our notation and recall some well-known facts about conformal structures on surfaces, curves and their lengths, and Teichmüller distance. The only proof we give is an elementary one for lemma 2.2.

In section 3 we define *twisting numbers*, which are a convenient way to measure how many times one geodesic twists around another in a hyperbolic surface, and in particular inside a Margulis annulus. We prove a few elementary facts about them, and relate them to Fenchel-Nielsen twist parameters in lemma 3.5.

Section 4 gives some well-known estimates of extremal lengths, in particular an estimate in terms of hyperbolic length in a component of the thick part (lemma 4.3). Lemma 4.4 describes briefly how to space a curve evenly in a component of the thick part, and this will be used in §5. At the end of the section we define the annulus contribution $\lambda_{A,\sigma}$ to extremal length, in terms of twisting numbers.

Sections 5 and 6 give the proof of theorems 5.1, and 6.1, respectively. Section 7 gives a brief discussion of the geometric consequences of theorem 6.1.

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2. Definitions and background

2.1. Metrics and conformal structures on surfaces. Let S be a smooth orientable surface of finite topological type, by which we mean that S is homeomorphic to a connected, compact surface (possibly with boundary) minus a finite set, which we call punctures or cusps.

By *conformal structure* on S we shall mean a conformal structure on the interior, for which each puncture has a neighborhood conformally equivalent to a punctured disk, and each boundary component has a neighborhood conformally equivalent to an annulus of finite modulus. Let $\mathcal{T}(S)$ denote the Teichmüller space of conformal structures on S considered up to conformal homeomorphisms homotopic to the identity (with no prescribed behavior on the boundary; see [1]). These are called *marked* conformal structures. If S has no boundary (but possibly punctures) then $\mathcal{T}(S)$ is the space of *analytically finite* marked conformal structures.

A conformal structure also determines a class of metrics on the interior, all those that can be described in a local conformal coordinate as a positive function times the Euclidean metric. In this paper we will use metrics which are smooth or mildly singular (non-differentiable on some curves), and which have a completion to the boundary of S. The conformal class of a metric σ will be denoted $[\sigma]$.

Call S *hyperbolic* if its Euler characteristic is negative, which implies that it admits a complete finite-volume hyperbolic metric in which the boundary is geodesic. We will call such a hyperbolic metric on S *standard*. We say a Euclidean metric on an annulus is standard if the annulus can be obtained isometrically from a Euclidean rectangle by identifying a pair of opposite edges. Note that every conformal annulus of finite modulus admits a standard Euclidean metric in the prescribed conformal class, unique up to scaling.

2.2. Curve systems. A *curve system* on S is a disjoint union of homotopically non-trivial properly embedded simple arcs and closed curves (for an arc, non-trivial means not deformable into ∂S with endpoints fixed, and for a closed curve it means not deformable to a point or a puncture). For a hyperbolic S, let $\mathcal{C}(S, \partial S)$ denote the set of homotopy classes of curve systems under homotopies that keep the endpoints (if any) on the boundary. (We admit non-simple curves to the homotopy classes for convenience.) Let $\mathcal{C}(S) \subset \mathcal{C}(S, \partial S)$ denote the subset of curve systems represented by unions of simple closed curves. Note that $\mathcal{C}(S)$ includes curves that are homotopic to components of ∂S. These are called *peripheral*, and to exclude them we define $\mathcal{C}_0(S)$ to be the subset of $\mathcal{C}(S)$ consisting of curve systems with no peripheral components. Also define (for brief use in §3) $\mathcal{C}'(S, \partial S)$ to be the set of homotopy classes of curve systems which are allowed to terminate in punctures as well as boundary components.

For an annulus A, we make a different definition: Let $\mathcal{C}(A, \partial A)$ denote the set of curve systems on A with equivalence under homotopies that *fix* the endpoints.

We call a subsurface $M \subset S$ *hyperbolic* if no component of ∂M is homotopically trivial or homotopic to another component of ∂M. Equivalently, if σ denotes any smooth non-positively curved metric on S in which the boundary is geodesic (or convex), then M is isotopic to a subsurface M^σ whose boundary is geodesic in σ.

Given a hyperbolic subsurface M, there is a natural map

$$\mathcal{R} = \mathcal{R}_M : \mathcal{C}(S, \partial S) \to \mathcal{C}(M, \partial M),$$

defined as follows: Let σ be a standard hyperbolic metric on S. If γ is a curve system on S let γ^σ denote its representative of shortest σ-length (see §2.3). Let $\mathcal{R}([\gamma])$ be the class represented by $\gamma^\sigma \cap M^\sigma$. It is not hard to see that $\mathcal{R}$ does not depend on the choice of σ.

2.3. Lengths and geodesic representatives. For any metric σ on S and any curve α we denote by $\ell_\sigma(\alpha)$ the length of α as measured in σ. If $[\gamma] \in \mathcal{C}(S, \partial S)$ then γ^σ denotes a representative of minimal length – if σ is a standard hyperbolic metric then this is unique, and orthogonal to ∂S. For an annulus A where the homotopies must fix endpoints, there is also a minimal representative γ^σ, but it may not be orthogonal to ∂S. In both cases we may denote $\ell_\sigma(\gamma^\sigma)$ by $\ell_\sigma([\gamma])$. Note also that any $\gamma \in \mathcal{C}'(S, \partial S)$ has a unique geodesic representative γ^σ, though its length may be infinite.

Extremal length. For a metric σ and any class $[\gamma]$ of curves in S, we can define the extremal length $\lambda_\sigma([\gamma])$ as

$$\lambda_\sigma([\gamma]) = \lambda_{S,\sigma}([\gamma]) = \sup_{\sigma' \in [\sigma]} \frac{\inf\limits_{\gamma' \in [\gamma]} \ell^2_{\sigma'}(\gamma')}{\mathrm{Area}(\sigma')}. \tag{2.1}$$

Where the supremum is taken over all metrics σ' conformally equivalent to σ. (See Ahlfors [2].) In particular, λ_σ only depends on the conformal class of σ.

If A is an annulus with conformal structure $[\sigma]$ then its *conformal modulus* $m(A, \sigma)$ (or $m(A)$ if σ is implicit) is the extremal length of the class of arcs joining its boundaries. The extremal length of the class of simple closed curves homotopic to either boundary is $1/m(A)$. A standard Euclidean metric on A is unique in its conformal class up to constant multiple, and realizes the supremum in the definition of the modulus. Thus $m(A) = H/L$ where H is the distance between the boundaries in this metric, and L is the length of either boundary.

2.4. The Thick-Thin decomposition. Fixing a standard hyperbolic metric σ on S, the *ϵ-thin part* of (S, σ) is the subset of S where the injectivity radius is at most ϵ. The closure of its complement is called the ϵ-thick part. If ϵ is less than the Margulis constant for $\mathbf{H}^2$ (see e.g. [14]), then the ϵ-thin part consists of compact annular neighborhoods of short geodesics, and non-compact cuspidal components, or neighborhoods of punctures. Let us call a compact annulus A of the thin part *peripheral* if its core is homotopic to ∂S (in which case one of its boundary components lies on ∂S), and *internal* otherwise. In either case call a boundary component of A *internal* if it lies in the interior of S.

For our later convenience we shall describe the following slight variation on the standard decomposition, which we shall call the (ϵ_0, ϵ_1) *collar decomposition*, for two numbers $0 < \epsilon_1 < \epsilon_0$ less than the Margulis constant. Let $A_1, ..., A_k$ denote the annular neighborhoods of the geodesics of length $\delta_i \leq \epsilon_1$, such that the internal boundary components of A_i have length ϵ_0. It is not hard to see that the A_i are contained in separate components of the ϵ_0-thin part, and that $S - \cup A_i$, minus neighborhoods of the cusps, are in the ϵ_1'-thick part, for some ϵ_1' possibly slightly smaller than ϵ_1.

Let $\mathcal{A}$ denote the set of A_i, and let $\mathcal{P}$ denote the set of closures P of components of $S - \cup A_i$. We call $\mathcal{P}$ the set of *hyperbolic* components of the decomposition. Let $\mathcal{Q} = \mathcal{P} \cup \mathcal{A}$ denote the entire decomposition.

It is well-known that ϵ_0 may be chosen sufficiently small (independently of the surface) that, for any simple geodesic γ in S, the components of $\gamma \cap A_j$ are either segments that traverse A_j from one boundary to the other, or are equal to the geodesic core of A_j. (Roughly, this is because if a geodesic penetrates sufficiently deeply into a Margulis tube and exits on the same side that it entered, it must have gone around the tube more than once, and hence intersects itself).

One can see from an explicit computation (in the upper half-plane, for example) that the modulus of each A_i is

$$m(A_i) = \frac{\pi}{\delta_i} - \frac{2}{\epsilon_0}, \tag{2.2}$$

and we can guarantee by choosing ϵ_1 sufficiently small that $m(A_i)$ is larger than any given constant m_0.

2.5. The Teichmüller metric, and Kerckhoff's theorem. The *Teichmüller distance* on $\mathcal{T}(S)$ is defined for two conformal structures $[\sigma], [\tau]$ as

$$d_{\mathcal{T}(S)}(\sigma, \tau) = \tfrac{1}{2} \log K(\sigma, \tau) \tag{2.3}$$

where $K \geq 1$ is the least number such that there exists a homeomorphism homotopic to the identity on S which is a K-quasiconformal map between the σ and the τ conformal structures. We shall be using a different characterization of $K(\sigma, \tau)$, due to Kerckhoff, namely:

Theorem 2.1. *For a surface of finite topological type S, the following holds:*

$$K(\sigma, \tau) = \sup_{\alpha \in \mathcal{C}(S, \partial S)} \frac{\lambda_{S,\sigma}(\alpha)}{\lambda_{S,\tau}(\alpha)} \tag{2.4}$$

Remarks. Kerckhoff proved this in [15] for a closed surface S. However the same techniques apply to surfaces of finite analytic type, and the case with boundary can also be deduced easily using a doubling argument (see e.g. Abikoff [1, §II.1.5] for an example of such arguments). An interesting feature of the equivalence of the two definitions of K is that the first one is fairly easily seen to be symmetric in σ and τ (since the inverse of a K-quasiconformal map is also K-quasiconformal), but the symmetry of the second is less evident. We shall use this to advantage in section 6.

The Teichmüller space of the torus is isometric to the hyperbolic plane $\mathbf{H}^2$, and Kerckhoff's theorem in this case can be written as a simple formula, after proper interpretation. We will use this formula not for the torus but in conjunction with twisting numbers and the annulus contributions to extremal length, $\lambda_{A,\sigma}$ (definition (4.3)) in section 6. For the sake of exposition we shall give an elementary proof:

Lemma 2.2. *Let $\mathbf{H}$ denote the upper half-plane $\{z : Im(z) > 0\}$, and define for $z_j = x_j + iy_j$*

$$K(z_1, z_2) = \sup_{t \in \mathbf{R}} \frac{y_2 + (t + x_2)^2/y_2}{y_1 + (t + x_1)^2/y_1}.$$

Then

$$d_{\mathbf{H}}(z_1, z_2) = \tfrac{1}{2}\log K(z_1, z_2) \tag{2.5}$$

where $d_{\mathbf{H}}(z_1, z_2)$ *is the hyperbolic distance function.*

(Note: actually the common definition of $d_{\mathbf{H}}$ does not include the factor of a half; however we will retain it in order to be consistent with (2.3).)

Proof. Consider first the case that z_1 and z_2 are purely imaginary. It is an easy exercise that $K(iy_1, iy_2) = \max\{y_2/y_1, y_1/y_2\}$. Since $\frac{1}{2}|\log(y_2/y_1)|$ is exactly the hyperbolic distance function, this completes the proof in this case.

Since for general $z_1, z_2 \in \mathbf{H}$ there is always some hyperbolic isometry $A \in \mathrm{SL}_2(\mathbf{R})$ such that $A(z_1), A(z_2)$ are purely imaginary, it would suffice to show that K is invariant under the action of $\mathrm{SL}_2(\mathbf{R})$.

One can see this conveniently by reducing it to a linear algebra problem. Let $\widehat{\mathbf{H}}$ denote $\{(\alpha,\beta) \in \mathbf{C}^2 : Im(\beta/\alpha) > 0\}$. For $(\alpha,\beta) \in \widehat{\mathbf{H}}$ and $u, v \in \mathbf{R}$, define

$$\lambda_{\alpha,\beta}(u,v) = \frac{|u\alpha + v\beta|^2}{Im(\beta\overline{\alpha})}.$$

Then define $\widehat{K}$ on $\widehat{\mathbf{H}} \times \widehat{\mathbf{H}}$ by

$$\widehat{K}(\alpha_1, \beta_1, \alpha_2, \beta_2) = \sup_{u,v\in\mathbf{R}} \frac{\lambda_{\alpha_2,\beta_2}(u,v)}{\lambda_{\alpha_1,\beta_1}(u,v)}.$$

One can check that the left action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathbf{C}^2$ (viewed as column vectors) preserves $Im(\beta\bar{\alpha}) = |\alpha|^2 Im(\beta/\alpha)$, and therefore in particular preserves $\widehat{\mathbf{H}}$. Furthermore note that the denominator of $\lambda_{\alpha,\beta}(u,v)$ is the squared norm of an inner product $(u,v)\cdot(\alpha,\beta)$, so it is easy to see that $\lambda_{A(\alpha,\beta)}(u,v) = \lambda_{\alpha,\beta}((u,v)A)$. It follows that $A \in \mathrm{SL}_2(\mathbf{R})$ leaves $\widehat{K}$ invariant, since the supremal ratio over all (u,v) is the same as the supremum over all $(u,v)A$.

It is an easy computation to show that $\lambda_{1,z}(t,1) = y + (t+x)^2/y$ if $z = x + iy$, and that $\lambda_{1,\beta/\alpha}(u,v) = \lambda_{\alpha,\beta}(u,v)$. Moreover, $\lambda_{\alpha,\beta}(cu,cv) = c^2\lambda_{\alpha,\beta}(u,v)$. It follows that

$$\widehat{K}(\alpha_1, \beta_1, \alpha_2, \beta_2) = K(\beta_1/\alpha_1, \beta_2/\alpha_2).$$

The map $(\alpha,\beta) \mapsto \beta/\alpha$ is a fibration of $\widehat{\mathbf{H}}$ over $\mathbf{H}$, which takes the linear action of $\mathrm{SL}_2(\mathbf{R})$ upstairs to its action by hyperbolic isometries downstairs. We conclude that K is also invariant under $\mathrm{SL}_2(\mathbf{R})$. $\square$

Remark. One can give a geometric interpretation to this proof by associating to $(\alpha,\beta) \in \widehat{\mathbf{H}}$ a torus, obtained by identifying opposite sides on a parallelogram spanned by α and β in $\mathbf{C}$. The quantity $Im(\beta\overline{\alpha})$ is the area of this parallelogram. For $u, v \in \mathbf{Z}$, the quantity $\lambda_{\alpha,\beta}(u,v)$ is the extremal length of the homotopy class in the torus represented by (u,v) in the generators α and β (for general $u, v \in \mathbf{R}$, one can give an interpretation in terms of measured foliations on the torus). Now $\widehat{K}$ can be seen as a supremum of extremal length ratios, just as in Kerckhoff's theorem. The fibration from $\widehat{\mathbf{H}}$ to $\mathbf{H}$ is just identification under conformal equivalence.

3. Twisting numbers and Fenchel-Nielsen coordinates

In this section we will describe a measure, called $t_{\gamma,\sigma}(\alpha)$, for the amount that a simple curve α in a hyperbolic surface (S, σ) "twists" around another curve γ. In particular if γ is short then this number will approximate the amount of twisting that α undergoes in the thin annulus corresponding to γ, and this will determine the contribution of this annulus to the extremal length of α. At the end of the section we will relate this construction to the familiar Fenchel-Nielsen twist coordinates.

Let σ be a standard hyperbolic metric on S. Let γ denote a (homotopically non-trivial) oriented simple closed curve, and let α denote any element of $\mathcal{C}'(S, \partial S)$, which intersects γ. Let x be a point of the intersection $\gamma^\sigma \cap \alpha^\sigma$. In the universal cover $\widetilde{S}$ of S, which is a convex subset of $\mathbf{H}^2$, let L_γ be a geodesic in the lift of γ^σ, and L_α be a geodesic in the lift of α^σ (extended to an infinite geodesic if it meets $\partial\widetilde{S}$), such that L_γ and L_α meet in a point that projects to x. Identify L_γ isometrically with $\mathbf{R}$, consistent with the orientation of γ, and let let a_R, a_L be the endpoints of L_α to the right and left, respectively, of L_γ. Let $p_\gamma : \mathbf{H}^2 \cup S^1 \to L_\gamma$ be the orthogonal projection, and define the (signed) twisting number

$$t'_{\gamma,\sigma}(\alpha, x) = \frac{p_\gamma(a_R) - p_\gamma(a_L)}{\ell_\sigma(\gamma)}. \tag{3.1}$$

(see figure 1). It is clear that the definition does not depend on the particular choice of lifts.

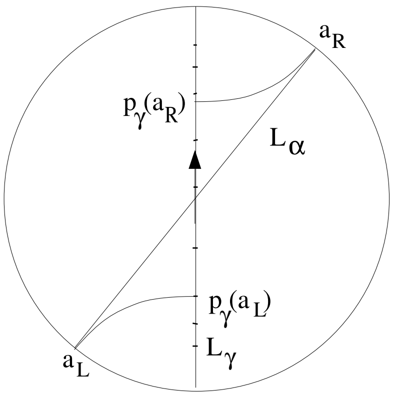

Figure 1

Lemma 3.1. *Let γ be a homotopically non-trivial oriented simple closed curve, $\alpha \in \mathcal{C}(S, \partial S)$, and σ a standard hyperbolic metric on S. If x and x' are two points of $\gamma^\sigma \cap \alpha^\sigma$, then*

$$|t'_{\gamma,\sigma}(\alpha, x) - t'_{\gamma,\sigma}(\alpha, x')| \leq 1.$$

Proof. Let g be the indivisible element of $\pi_1(S)$ whose axis is L_γ. Choosing appropriate lifts L_α and L'_α corresponding to x and x', we may assume that L'_α is trapped between L_α and $g(L_\alpha)$. The proof is completed by figure 2, and the fact that if $a < s < a+1$ and $b < t < b+1$ then $(b-a)-1 < (t-s) < (b-a)+1$. □

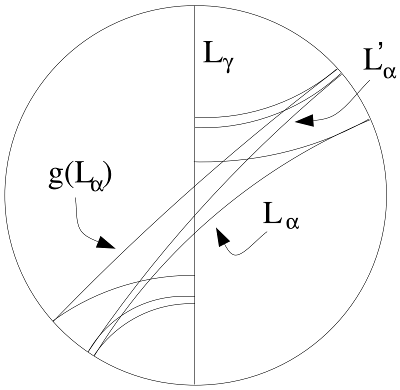

FIGURE 2

Thus it is useful to define the quantity

$$t_{\gamma,\sigma}(\alpha) = \min_x t'_{\gamma,\sigma}(\alpha, x) \tag{3.2}$$

where the minimum is taken over all x in $\gamma^\sigma \cap \alpha^\sigma$.

We can also measure twisting inside an annulus. If A is an annulus with conformal structure $[\sigma]$ then let σ be a standard Euclidean metric on A in this conformal class. The univeral cover of A is isometric to a strip $[0, L] \times \mathbf{R}$ in the plane, with fundamental domain of height H. If α is a curve connecting the boundaries of A, it lifts to a curve with endpoints $(0, y_0)$ and (L, y_1), and we may define

$$t_{A,\sigma}(\alpha) = (y_1 - y_0)/H$$

(the sign of t is again determined after a choice of orientation of A).

Now consider again a standard hyperbolic metric σ on S, and an annulus A of the (ϵ_0, ϵ_1) collar decomposition of (S, σ). If $\alpha \in \mathcal{C}(S)$ intersects A essentially then its geodesic representative α^σ intersects A in several disjoint components, whose twists therefore differ by no more than one. Thus we may take $t_{A,\sigma}(\alpha)$ to be the minimum of these. Note in particular that if A is peripheral then $t_{A,\sigma}(\alpha) = 0$ since α^σ is perpendicular to ∂A.

Lemma 3.2. *There exist ϵ_0, ϵ_1 less than the Margulis constant such that, for any standard hyperbolic metric σ on a surface S, if A is an (ϵ_0, ϵ_1) collar with (non-peripheral) core γ and α is any element of $\mathcal{C}(S)$ that crosses γ,*

$$|t_{A,\sigma}(\alpha) - t_{\gamma,\sigma}(\alpha)| \le 1.$$

Proof. In case A is peripheral both twists are 0, so let us assume A is internal. Consider figure 3, where in the upper half-plane model of $\mathbf{H}^2$ we place a lift L_γ of γ^σ, and the corresponding lift $\widetilde{A}$ of A. Pick $x \in \gamma^\sigma \cap \alpha^\sigma$ and choose a component L_α of the lift of α^σ which intersects L_γ in a point projecting to x. Let α_x denote the component of $\alpha^\sigma \cap A$ which contains x. We may arrange the figure so that L_α exits $\widetilde{A}$ as shown, terminating at infinity in the interval J.

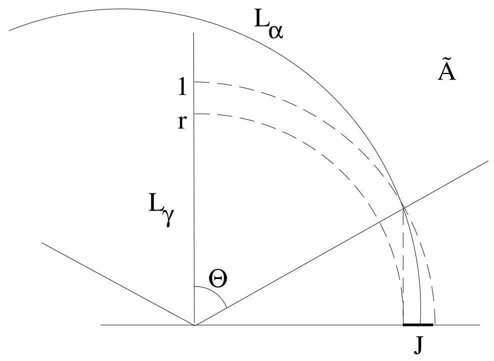

FIGURE 3

Let $\epsilon \leq \epsilon_1$ be the translation length of γ along its axis. The condition that A is an (ϵ_0, ϵ_1) collar implies that the angle θ between L_γ and $\partial\widetilde{A}$ satisfies $\cos\theta = \epsilon/\epsilon_0$. The left endpoint of J is determined by the limiting case where L_α is vertical, so its projection to L_γ has height $r = \sin\theta$. It follows that the distance between the projection to L_γ of the endpoint of α_x and the endpoint at infinity of L_α is at most $|\log\sin\theta|$. The quantity $|t'_{\gamma,\sigma}(\alpha, x) - t_{A,\sigma}(\alpha_x)|$ is equal to exactly double this distance (counting the other endpoint), scaled by $1/\epsilon$. We obtain:

$$|t'_{\gamma,\sigma}(\alpha, x) - t_{A,\sigma}(\alpha_x)| < \frac{1}{\epsilon}\left|\log(1 - \epsilon^2/\epsilon_0^2)\right|.$$

Thus given ϵ_0 we may choose ϵ_1 and hence ϵ sufficiently small that this is at most 1. Since both $t'_{\gamma,\sigma}(\alpha, x)$ and $t_{A,\sigma}(\alpha_x)$ are minimized for the same x (both depend monotonically on the angle at which γ^σ and α_x cross), we have the desired inequality. □

The following lemma observes that for a thin annulus A any twisting number can be achieved, up to an additive constant, by some curve that spends "most" of its time in A. This fact will be used in the proof of theorem 6.1.

Lemma 3.3. *Let (Q, τ) be a surface of finite topological type with standard hyperbolic metric τ. Let $\epsilon_0 > \epsilon_1 > 0$ be as in lemma 3.2, and fix $\ell_1 > 0$. Suppose that $\ell_\tau(\gamma) \leq \ell_1$ for all boundary components γ of Q, and let $A \subset Q$ be an internal annulus of the (ϵ_0, ϵ_1) collar decomposition. Given $t \in \mathbf{R}$ there exists $\alpha \in \mathcal{C}(Q)$ such that*

$$|t_{A,\tau}(\alpha) - t| \leq c_1,$$

and α consists of one or two arcs traversing A, together with one or two segments of hyperbolic length at most c_2. The constants c_1, c_2 depend only on $\epsilon_0, \epsilon_1, \ell_1$ and $\chi(Q)$.

Proof. We claim that, for some r_1 depending on $\epsilon_0, \epsilon_1, \ell_1$ and $\chi(Q)$, there is, for each boundary component of A, a geodesic arc outside A of length at most r_1, which either joins that component to the other, or to itself. To see this, let A_r denote the r-neighborhood of A in Q; A_r is the union of A with two pieces, each adjacent to a boundary component. If at least one of them is a collar, i.e. an embedded annulus of radius r in the interior of Q, we have $\mathrm{Area}(A_r) > \epsilon_1 e^r$, and since $\mathrm{Area}(Q) \leq 2\pi|\chi(Q)|$ we have an upper bound $r < r_0 = \log 2\pi|\chi(Q)|/\epsilon_1$. Thus, if $r \geq r_0$ then both added pieces fail to be collars. If for some $r \leq r_0$ a collar meets itself or the other collar we have the desired geodesic arc of length $r_1 = 2r_0$. The other possibility is for a collar to hit the boundary at some $r \leq r_0$. In this case there is an arc of length at most $r_1 = 2r_0 + \ell_1$ which runs once around the boundary and back to A.

Now if there is an arc joining both components, we can take α_0 to be the union of this short arc with an arc that runs through A. If there are two arcs, joining each boundary to itself, we can join these arcs to a union of two parallel arcs through A. Applying n Dehn twists around the core of A to α_0 will clearly increase (or decrease) its twisting number by about n (see also lemma 3.5), so that by an approriate twist we may obtain α with the desired twisting number. $\square$

In the proof of corollary 5.4, we will need to measure twisting using a hyperbolic metric that is not quite standard. In particular, suppose that σ is a standard hyperbolic metric on S and that P is a subsurface consisting of a union of components of the (ϵ_0, ϵ_1) collar decomposition, such that any annular components in P are non-peripheral in P. Let $\hat{\sigma}$ be the standard hyperbolic metric on P which is conformally equivalent to σ. Now if $\gamma \subset P$ is a non-peripheral geodesic of length $\ell_\sigma(\gamma) < \epsilon_1$, we can measure twisting around γ with respect to both metrics. The following lemma compares the two:

Lemma 3.4. *Let P, S, σ and $\hat{\sigma}$ be as above. There exist ϵ_0, ϵ_1 sufficiently small and $c > 0$ such that, if $\ell_\sigma(\gamma^\sigma) < \epsilon_1$ for a non-peripheral simple closed curve $\gamma \subset P$, then*

$$|t_{\gamma,\sigma}(\beta) - t_{\gamma,\hat{\sigma}}(\mathcal{R}_P(\beta))| \leq c$$

for any $\beta \in \mathcal{C}(S, \partial S)$.

Proof. We shall give the proof in the case that S has no boundary. The general case can be reduced to this by means of a doubling argument. Let β^σ be the σ-geodesic representative of β. Let P_0 be the complement in P of $\epsilon_{\mathcal{M}}$-thin annulus neighborhoods of ∂P (measured in $\hat{\sigma}$), where $\epsilon_{\mathcal{M}} > \epsilon_0$ is the Margulis constant. We claim that, given $\nu > 0$ there is a choice of ϵ_0 so that in P_0 the ratio $\sigma/\hat{\sigma}$ is ν-close to 1, and its derivatives (in normal $\hat{\sigma}$ coordinates) are bounded by ν. This will imply that $\beta^\sigma \cap P_0$ is a curve system with very low ($O(\nu)$) curvature in the metric $\hat{\sigma}$. Thus the $\hat{\sigma}$-geodesic $\hat{\beta}_0$ homotopic to $\beta^\sigma \cap P_0$ rel endpoints is $O(\nu)$-close to it. Now, we are interested in $\mathcal{R}_P(\beta)^{\hat{\sigma}}$, the $\hat{\sigma}$-geodesic representative of $\beta^\sigma \cap P$, rel ∂P. But by lemma 4.5 applied in $(P, \hat{\sigma})$, the intersection $\mathcal{R}_P(\beta)^{\hat{\sigma}} \cap P_0$ is obtained by a bounded homotopy from $\hat{\beta}_0$. It follows that the twisting number around γ, which is certainly in P_0, has not changed by more than a bounded amount.

It remains to prove our claim about $\sigma/\hat{\sigma}$, but this is an elementary argument in complex analysis. If we consider the universal coverings of P in (subsets of) the unit disk which are obtained from the two metrics σ and $\hat{\sigma}$, we see that the identity on P lifts to a conformal map Φ between these covers. For $x \in P_0$, the hyperbolic distance to ∂P, in either metric, is bounded below by roughly $\log \epsilon_{\mathcal{M}}/\epsilon_0$, which we can make as large as we please. If x lifts to 0 in both covers we see (using the formulas for hyperbolic distance) that Φ must map a disk of radius roughly $(\epsilon_{\mathcal{M}} - \epsilon_0)/(\epsilon_{\mathcal{M}} + \epsilon_0)$ univalently over a disk of similar radius. The Schwarz lemma applied to Φ and Φ^{-1} implies that $|\Phi'(z)|$ is close to 1 in a neighborhood of 0, and an application of the Cauchy integral formula serves to bound $\Phi''(0)$. Our claim about $\sigma/\hat{\sigma}$ follows. □

Recall now the construction of Fenchel-Nielsen coordinates on a Teichmüller space $\mathcal{T}(S)$ (see e.g. [1]) Let $\gamma_1, \ldots, \gamma_n$ denote a system of oriented simple closed curves which cuts S into a union of pairs of pants. Suppose, to start, that S has no punctures and p boundary components, so that $n = -3\chi(S)/2$ and each pair of pants is compact. For any standard hyperbolic structure σ on S we immediately obtain n positive numbers $\{\ell_\sigma(\gamma_i^\sigma)\}$, the *length parameters.* Another $n - p$ real numbers describe "twisting" parameters for the gluings of the pairs of pants. Let us make this concrete as follows.

A pair of pants P with boundary curves $\gamma_1, \gamma_2, \gamma_3$ contains three unique homotopy classes of simple properly embedded arcs $\alpha_{12}, \alpha_{23}, \alpha_{13}$ (the "seams"), such that α_{ij} joins γ_i to γ_j. Fix a set of representatives of the seams which match on opposite sides of each non-peripheral γ_i. This determines a homotopy class of curve systems, μ in $\mathcal{C}(S, \partial S)$. We call the (oriented) γ_i together with μ a *marking*, and note that it is purely topological information.

Now for any standard hyperbolic structure σ on S, the seams have geodesic representatives α_{ij}^σ which are orthonal to γ_i^σ and γ_j^σ. The seams cut each pair of pants into two congruent right-angled hexagons, and in particular they bisect the boundary components. Now for each γ_j and each pair of corresponding seam endpoints there is a unique geodesic path along γ_j that must be spliced between the endpoints, so that the resulting curve is in the homotopy class μ. The length $m_j(\sigma)$ of this path (which can be given a sign, since γ_j is oriented) is the same for both pairs of seam endpoints on γ_j. We define our *twist parameter* $s_j(\sigma)$ to be $m_j(\sigma)/\ell_\sigma(\gamma_j)$.

This construction gives a homeomorphism $F : \mathcal{T}(S) \to \mathbf{R}_+^n \times \mathbf{R}^{n-p}$, taking σ to $(\ell_\sigma(\gamma_1), \ldots, \ell_\sigma(\gamma_n), s_1(\sigma), \ldots, s_{n-p}(\sigma))$. We note also that a positive Dehn twist on γ_j has the effect of incrementing $s_j(\sigma)$ by one and leaving the other coordinates invariant.

In the case with punctures the construction is similar, except that one or two ends of a pair of pants may be a puncture, and we allow the seams to be non-compact arcs terminating in boundaries or punctures, so that μ is in $\mathcal{C}'(S, \partial S)$.

To conclude this section, we give a comparison between our twisting numbers and the Fenchel-Nielsen twist coordinates.

Lemma 3.5. *Let $\gamma_1, \ldots, \gamma_n$ and μ be a marking for S, and $s_j : \mathcal{T}(S) \to \mathbf{R}$ be the associated twist coordinates. Then for any standard hyperbolic metric σ on S,*

$$|s_j(\sigma) - t_{\gamma_j,\sigma}(\mu)| \le 1.$$

Furthermore, given any $\alpha \in \mathcal{C}(S, \partial S)$ and two standard hyperbolic metrics σ and σ',

$$\left|(t_{\gamma_j,\sigma}(\alpha) - t_{\gamma_j,\sigma'}(\alpha)) - (s_j(\sigma) - s_j(\sigma'))\right| \le 4$$

Proof. To obtain the first inequality, lift γ_j^σ to a geodesic L_j in $\mathbf{H}^2$, and consider a component $\widetilde{\mu}$ of the preimage of μ^σ in $\mathbf{H}^2$ that intersects L_j. This geodesic is homotopic to a curve that follows two (lifts of) seam curves $\alpha^\sigma_{jk}, \alpha^\sigma_{jm}$ whose endpoints on L_j are separated by $s_j(\sigma)\ell_\sigma(\gamma_j)$ (see figure 4). The other endpoints of these seams meet lifts L_k and L_m of $\gamma_k^\sigma, \gamma_m^\sigma$. Sinc L_k (resp. L_m) is disjoint from its translates, its projection to L_j is a segment of length at most $\ell_\sigma(\gamma_j)$ centered on an endpoint of α^σ_{jk} (resp. α^σ_{jm}). The endpoints of $\widetilde{\mu}$ at infinity must be separated from L_j by L_k and L_m, respectively. It follows that the length of the projection of μ to L_j, divided by $\ell_\sigma(\gamma_j)$, differs from $s_j(\sigma)$ by at most 1.

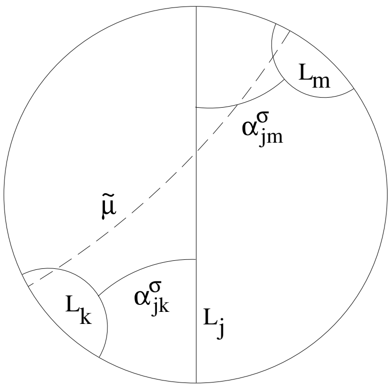

FIGURE 4

For the rest of the lemma we note the fact that, for any two curve systems $\alpha, \beta \in \mathcal{C}'(S, \partial S)$, the *difference* $t_{\gamma_j,\sigma}(\alpha) - t_{\gamma_j,\sigma}(\beta)$ is independent of σ up to a bounded error (of 1). This is because we can detect this difference topologically: lifting to the universal cover, let L_α be a lift of a component of α which crosses a lift L_j of γ_j. The translates of L_α along L_j divide the plane into a sequence of consecutive strips, and a component of a lift of β that crosses L_j meets some number δ of these strips. (The number obtained from a different lift of β may differ by at most 1). It is easy to see that δ, appropriately signed, gives this twist difference, up to an error of 1.

The second inequality now follows from this fact applied to α and μ, together with the first inequality. $\square$

4. Basic extremal length estimates

In this section we shall discuss and derive some fairly well-known estimates for extremal lengths, which will be the building blocks of the proof in section 5.

Here and in the rest of the paper, we shall use "$a \asymp b$" to mean that the ratio a/b is bounded above and below by positive constants, which usually depend on topological data, and on previously chosen constants. Similarly, "$a \prec b$" and "$b \succ a$" mean that a/b is bounded above. Where necessary we will indicate explicitly the dependence of the constants.

4.1. Spacing and upper bounds. Lower bounds on extremal lengths are relatively easy to obtain, by exhibiting some metric and computing the length in that metric. The following lemma is helpful for obtaining upper bounds.

For a representative curve $\hat\beta$ of a class β in $C(S, \partial S)$ and a metric σ on S, define the *spacing* $\nu_{S,\sigma}(\hat\beta)$ as the supremum of numbers ν such that the ν-neighborhood $\mathcal{N}_\nu(\hat\beta)$ is a standard product $\hat\beta \times [0,1]$. We have:

Lemma 4.1. *Given a metric σ on S and a representative $\hat\beta$ of $\beta \in C(S, \partial S)$, the extremal length of β is bounded by*

$$\lambda_{S,\sigma}(\beta) \le \frac{\operatorname{Area}(S,\sigma)}{4\nu^2_{S,\sigma}(\hat\beta)}.$$

Proof. Let $\beta_1, \dots, \beta_n$ denote the components of $\hat\beta$. The neighborhoods $R_i = \mathcal{N}_\nu(\beta_i)$ for any $\nu < \nu_{S,\sigma}(\hat\beta)$ are, by assumption, either annuli or rectangles. In either case, the extremal length of the family of transverse arcs α_i connecting the boundaries $\beta_i \times \{0,1\}$ is, applying the definition,

$$\lambda_{R_i,\sigma}(\alpha_i) \ge \frac{4\nu^2}{\operatorname{Area}(R_i,\sigma)}. \tag{4.1}$$

Let $[\beta_i]$ denote the family of curves in R_i freely homotopic to β_i when β_i is a closed curve, and homotopic with endpoints on the edges $\partial\beta_i \times [0,1]$ when β_i is a segment. Then it is a standard fact that

$$\lambda_{R_i,\sigma}([\beta_i]) = 1/\lambda_{R_i,\sigma}(\alpha_i).$$

This together with (4.1) bounds $\lambda_{R_i,\sigma}([\beta_i])$ from above. Now let β_R denote the class of all curve systems that are a union of representatives from each $[\beta_i]$. Since the R_i are all disjoint it is easy to see that

$$\lambda_{\cup R_i,\sigma}(\beta_R) = \sum_{i=1}^{n} \lambda_{R_i,\sigma}([\beta_i])$$

by an application of the Schwarz inequality (actually only the "$\le$" direction is needed, but the other direction is even easier).

The proof is completed by a standard monotonicity argument: Since the class β in $\mathcal{C}(S, \partial S)$ is bigger than the class β_R (it has representatives not limited to the R_i), its length in any metric is smaller. Furthermore the area of S in any metric is at least that of $\cup R_i$. Thus $\lambda_{S,\sigma}(\beta) \le \lambda_{\cup R_i,\sigma}(\beta_R)$, and the lemma follows. $\square$

4.2. Extremal length in hyperbolic subsurfaces. If a hyperbolic subsurface M of S is "insulated" from S by sufficiently thick annuli on its boundary, then extremal lengths in $\mathcal{C}(S)$ and $\mathcal{C}_0(M)$ (curves that are non-peripheral in M) are approximately the same:

Lemma 4.2. [25, lemma 8.4] *Let (S,σ) be any Riemann surface. There exist m_0 and C, depending only on the topological type of S, such that if $M \subset S$ is a hyperbolic subsurface for which each component γ of ∂M bounds an annulus $A_\gamma \subset M$ with modulus $m(A_\gamma) \ge m_0$, then for any $\alpha \in \mathcal{C}_0(M)$,*

$$1 \le \frac{\lambda_{M,\sigma}(\alpha)}{\lambda_{S,\sigma}(\alpha)} \le C.$$

More generally, a curve in S may cut through M, and if M is a nicely chosen subsurface then we can say something about the extremal length of the intersection. For a standard hyperbolic metric σ, recall the (ϵ_0, ϵ_1) collar decomposition $\mathcal{Q} = \mathcal{P} \cup \mathcal{A}$. For a hyperbolic component $P \in \mathcal{P}$, define

$$\lambda_{P,\sigma}(\beta) = \lambda_{P,\sigma}(\mathcal{R}_P(\beta)). \tag{4.2}$$

Note that, provided ϵ_0 is chosen sufficiently small, $\mathcal{R}_P(\beta)$ is represented by $\beta^\sigma \cap P$. This hyperbolic geodesic also gives a good estimate for extremal length on P, via the following:

Lemma 4.3. *Suppose that σ is a standard hyperbolic metric for S, and suppose that $\ell_\sigma(\gamma) \le \ell_0$ for any component γ of ∂S. For any $\beta \in \mathcal{C}(S, \partial S)$ and a hyperbolic component P of the (ϵ_0, ϵ_1) collar decomposition,*

$$\lambda_{P,\sigma}(\beta) \asymp \ell_\sigma^2(\beta^\sigma \cap P)$$

Where the multiplicative constants for $\asymp$ depend only on the topological type of S and on ϵ_0, ϵ_1 and ℓ_0.

Proof. Because of the absence of thin parts in P we can rearrange $\beta^\sigma \cap P$ to have sufficiently large spacing in the hyperbolic metric σ. Indeed,

Lemma 4.4. *There is a curve system β' in P homotopic to $\beta^\sigma \cap P$ by a homotopy that moves no point of $\beta^\sigma \cap \partial P$ more than halfway around the boundary, and such that*

$$\nu_{P,\sigma}(\beta') \ge \frac{c}{\ell_\sigma(\beta^\sigma \cap P)}$$

where the constant c depends only on the topological type of S, and on ϵ_0, ϵ_1 and ℓ_0

Proof. Subdivide P into pairs of pants $Q_1, \ldots, Q_p$, using the shortest collection of pairwise disjoint simple geodesics in P. The number p of pieces is given by $p = |\chi(P)|$, and we note that some pairs of pants may have one or two punctures rather than boundaries. The lengths of all boundary components of Q_i are bounded below by ϵ_1, and above by a constant ℓ_1 depending on ℓ_0 and the topological type of S. (This is a standard fact, whose proof is similar to the proof that the injectivity radius of (S, σ) is bounded above in terms of the hyperbolic area of S).

Begin by performing a homotopy on $\beta^\sigma \cap Q_j$, for each j, which moves the points of $\beta^\sigma \cap \partial Q_j$ around ∂Q_j until they are evenly spaced, and keeps the rest of the curve geodesic. Call the new curve β_1. Clearly this can be done so that no endpoint moves more than halfway around its boundary component. Furthermore, provided ϵ_0 has been chosen sufficiently small, β_1 will still be contained in P.

The bound ℓ_1 on boundary lengths means that the length of no arc has changed by more than an additive constant by this homotopy. It also means that there is a lower bound ℓ_2 on the length of any path joining two boundaries of any Q_j. Furthermore no closed component of $\beta^\sigma \cap P$ has length less than ϵ_1. We conclude that

$$\ell_\sigma(\beta_1) \leq c_1 \ell_\sigma(\beta^\sigma \cap P)$$

where c_1 depends on the previous constants.

Now we deform β_1 in each Q_k separately to obtain a curve β_2 with maximal spacing. Consider first the case where $\beta_1 \cap Q_k$ has no closed curve components, and where Q_k has three boundaries and no cusps.

One way to make the construction concrete is to form the following Voronoi diagram: For $i = 1, 2, 3$ let C_i be the collar defined by

$$C_i = \{x \in Q_k : d(x, \gamma_i) \leq d(x, \gamma_j) \text{ for } j \neq i\}.$$

Then each C_i is an annulus whose whose width at any point is bounded between two constants c_2, c_3, and whose boundary in the interior of Q_k consists of two geodesic arcs $\alpha_{ij} = C_i \cap C_j$ (for $j \neq i$) of length $s_{ij} \in [c_4, c_5]$ (The estimates here are uniform since the lengths of γ_i are bounded above and below, so we have a compact set of possible pairs of pants. Alternatively one can obtain explicit estimates using hyperbolic trigonometry.)

In fact, one can use (for example) the foliation of each C_i by geodesics perpendicular to γ_i to obtain a Euclidean metric $\hat\sigma$ on C_i which is bilipschitz equivalent to σ, with a uniform constant.

Now for each arc τ of $\beta_1 \cap Q_k$, associate a point $x(\tau)$ on α_{ij} if the endpoints of τ lie on γ_i and γ_j. Arrange the points $\{x(\tau)\}$ so that they are uniformly spaced on each α_{ij}. There is a unique arc τ', homotopic to τ, which spirals in C_i from γ_i to $x(\tau)$, and then in C_j from $x(\tau)$ to γ_j, and is a $\hat\sigma$-geodesic in each C_i. The homotopy from τ to τ' will move no point more than a bounded distance (this is easiest to see in the universal cover – see figure 5). It is also easy to see that the resulting arcs will be disjoint

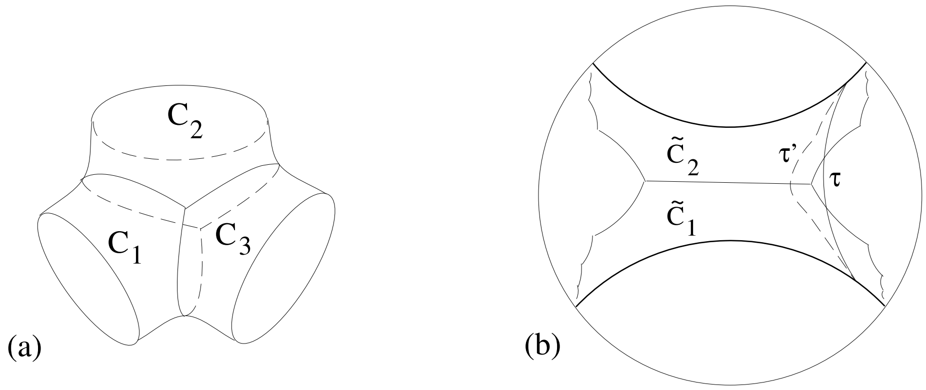

FIGURE 5. (a) The decomposition of a pair of pants into three annuli. (b) In the lifts of two adjacent annuli to $\mathbf{H}^2$, an example of an arc τ and the homotopic arc τ'.

We can treat the case with cusps similarly, except that we first excise a horospherical neighborhood of each puncture with boundary length ϵ_0, and work in the complement.

If β_1 has a closed component in Q_k, it must be homotopic to a boundary component γ_j, and no part of β_1 can cross that component. Thus we can take a collar neighborhood of γ_j of definite width and space the corresponding components evenly (and maximally) there.

Now for maximally spaced curves in a Euclidean annulus it is clear that the spacing is approximately the area divided by the total length. Where two collars meet along a common boundary segment, the local picture is, up to uniform bilipschitz distortion, as in figure 6, and it is an exercise in Euclidean geometry to see that the spacing is equal to the minimum of the spacings on either side.

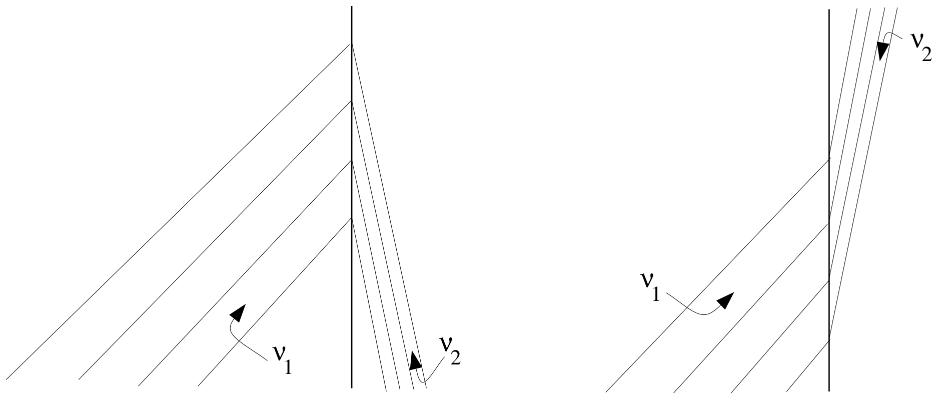

FIGURE 6

Thus we have, since the area of each Q_k is bounded,

$$\nu_{Q_k,\sigma}(\beta_2 \cap Q_k) \geq \frac{C}{\ell_\sigma(\beta^\sigma \cap Q_k)}$$

for each Q_k, and for all of P we obtain

$$\nu_{P,\sigma}(\beta_2) \geq C' \min_k \nu_{Q_k,\sigma}(\beta_2 \cap Q_k)$$

since again the spacing is not reduced by more than a constant factor at the interfaces between subsurfaces.

Setting $\beta' = \beta_2$ and observing that $\ell_\sigma(\beta^\sigma \cap P) \geq \max_k \ell_\sigma(\beta^\sigma \cap Q_k)$, the statement of lemma 4.4 follows. □

Now we can apply the spacing lemma 4.1 and the fact that the area of P is bounded in terms of its Euler characteristic to conclude that

$$\lambda_{P,\sigma}(\beta) \leq \frac{C}{\nu^2_{P,\sigma}(\beta')} \leq C\ell^2_\sigma(\beta^\sigma \cap P).$$

To obtain the inequality in the other direction we observe that the minimal length in the homotopy class $\mathcal{R}_P(\beta)$ is nearly attained by $\beta^\sigma \cap P$. In fact, the shortest representative β^P can be obtained by sliding the endpoints of $\beta^\sigma \cap P$ around ∂P until an orthogonal arc is obtained. The sliding distance is uniformly bounded by lemma 4.5 which appears below. Thus, $\ell_\sigma(\beta^P) \geq c\ell_\sigma(\beta^\sigma \cap P)$ for uniform c, and we conclude by the definition of extremal length that

$$\lambda_{P,\sigma}(\beta) \geq c^2 \frac{\ell^2_\sigma(\beta^\sigma)}{\mathrm{Area}(S,\sigma)}.$$

Since $\mathrm{Area}(S,\sigma)$ is determined by the Euler characteristic of S, we are done with lemma 4.3. □

Lemma 4.5. *Let A_1 and A_2 be annuli (possibly the same) of the (ϵ_0, ϵ_1) collar decomposition of (S,σ), and let α,β be geodesic arcs in $S - (A_1 \cup A_2)$ with endpoints on $\partial A_1 \cup \partial A_2$, such that α is homotopic to β keeping the endpoints on the boundary. Then the homotopy can be taken to move the endpoints of α by no more than $c = c(\epsilon_0, \epsilon_1)$.*

Proof. Lifting to $\widetilde{A}_1, \widetilde{A}_2, \widetilde{\alpha}$ and $\widetilde{\beta}$ in $\mathbf{H}^2$, we see that the endpoints of $\widetilde{\alpha}$ and $\widetilde{\beta}$ must lie in the arcs a_1 and a_2 in figure 7. It is fairly easy to see that the extremal case (longest a_i) occurs when $\widetilde{A}_1$ and $\widetilde{A}_2$ are actually tangent, and each is the r-neighborhood of a geodesic, where $r = \cosh^{-1}(\epsilon_0/\epsilon_1)$. The length of a_1 and a_2 in this case gives the constant c. □

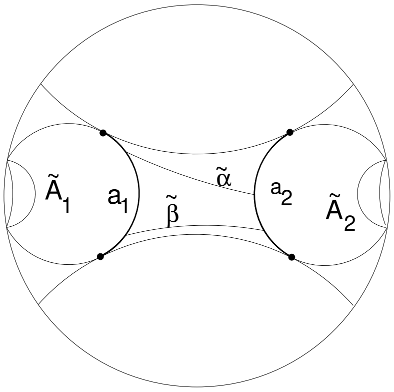

FIGURE 7

4.3. Extremal length on annuli. For annuli A in S we will define a quantity $\lambda_{A,\sigma}(\beta)$ which is not exactly extremal length, but will quantify the extremal length contribution of A to the whole curve $\beta \in \mathcal{C}(S, \partial S)$.

In particular, if A is an annulus of the (ϵ_0, ϵ_1) collar decomposition of (S, σ) with core curve γ, then for $\beta \in \mathcal{C}(S, \partial S)$ let $i(\beta, A)$ or $i(\beta, \gamma)$ denote the (minimal) intersection number of β with γ (note, this includes the case that A is peripheral, $\gamma \subset \partial S$ and β has $i(\beta, \gamma)$ endpoints on γ). The geodesic β^σ can meet A either in a collection of $i(\beta, A)$ properly embedded arcs, or if $i(\beta, A) = 0$, in a union of closed curves homotopic to γ. In either case let $n(\beta, A)$ denote the number of components of $\beta^\sigma \cap A$. We define

$$\lambda_{A,\sigma}(\beta) = \begin{cases} i(\beta, A)^2 \left(m(A) + \frac{t_{A,\sigma}(\beta)^2}{m(A)}\right) & \text{if } i(\beta, A) > 0, \\ n(\beta, A)^2 / m(A) & \text{otherwise.} \end{cases} \tag{4.3}$$

It is worth nothing that, if $t_{A,\sigma}(\beta)$ is an integer and a gluing together (without twisting) of the boundaries of A takes β to a closed curve on the torus, then (4.3) gives the extremal length of the homotopy class of this curve. Compare also to lemma 2.2.

5. The main length estimate

For the remainder of the paper we shall fix a choice of $\epsilon_0 > \epsilon_1 > 0$ so that the following hold for any standard hyperbolic metric σ on S.

(1) ϵ_0 is less than $\epsilon_{\mathcal{M}}$, the Margulis constant of $\mathbf{H}^2$.
(2) Each annulus of the (ϵ_0, ϵ_1) collar decomposition has modulus at least 1. It is furthermore contained in a slightly larger annulus, so that each internal boundary component bounds an annulus in the thick part of modulus m_0, where m_0 is the modulus bound in lemma 4.2.
(3) A simple geodesic in (S, σ), if it intersects an annulus of the (ϵ_0, ϵ_1) collar decomposition, either does so in arcs that connect the boundaries or it is the geodesic core of the annulus (see §2.4).

(4) Lemma 3.2 holds.

Let σ denote a fixed standard hyperbolic metric on S, and let $\mathcal{Q} = \mathcal{P} \cup \mathcal{A}$ denote the (ϵ_0, ϵ_1) collar decomposition. of section 2.4.

Theorem 5.1. *Let S be a surface of finite type with a standard hyperbolic metric σ in which boundary lengths are at most ℓ_0, and let $\mathcal{Q}$ be the set of components of the (ϵ_0, ϵ_1) collar decomposition. Then, for any $\beta \in \mathcal{C}(S, \partial S)$ we have*

$$\lambda_{S,\sigma}(\beta) \asymp \max_{Q \in \mathcal{Q}} \lambda_{Q,\sigma}(\beta)$$

where the constants for $\asymp$ depend only on $\epsilon_0, \epsilon_1, \ell_0$ and the topological type of S.

Proof. Fix, for the remainder of the section, a homotopy class $\beta \in \mathcal{C}(S, \partial S)$. Throughout the rest of the section we shall make the abbreviations $\lambda_Q = \lambda_{Q,\sigma}(\beta)$, $t_A = t_{A,\sigma}(\beta)$, $m_A = m(A, \sigma)$, and $n(A) = n(\beta, A)$. On each annulus $A \in \mathcal{A}$, let σ_e denote the unique standard Euclidean metric on A which is conformally equivalent to σ, and agrees with σ on ∂A.

The upper bound. To bound $\lambda_{S,\sigma}(\beta)$ from above, we shall need to construct a metric on S, in the conformal class of σ, which is sufficiently similar to the extremal metric for β that the spacing estimate in lemma 4.1 can be applied. We remark that it is of course known that an extremal metric exists, and in fact is given by a quadratic differential. Our construction, by contrast, is not extremal (in fact we make no explicit comparison to the extremal metric), but gives concrete estimates which are explicitly related to the hyperbolic geometry of (S, σ).

Let us make some new definitions. Let $h_1 : \mathcal{Q} \to [0, \infty)$ be given by

$$h_1(Q) = \begin{cases} \sqrt{\lambda_Q} & Q \in \mathcal{P} \\ \sqrt{\lambda_Q / m_Q} & Q \in \mathcal{A}. \end{cases} \tag{5.1}$$

This quantity has the property that the scaled metrics $h_1(P)\sigma$ on $P \in \mathcal{P}$ and $h_1(A)\sigma_e$ on $A \in \mathcal{A}$ have areas comparable to λ_P and λ_A.

However, in order to scale the metric in a consistent way on the entire surface we shall need to adjust h_1 to obtain a function h, as follows:

Lemma 5.2. *There exists a positive function $h : \mathcal{Q} \to \mathbf{R}_+$ such that the following hold.*

(1) $h(Q) \geq h_1(Q)$ *for all* $Q \in \mathcal{Q}$,

(2) $\max_Q h(Q) = \max_Q h_1(Q)$,

and, whenever $A \in \mathcal{A}$ is adjacent to $P \in \mathcal{P}$,

(3) $h(A) \leq h(P)$.

Furthermore, for any internal annulus $A \in \mathcal{A}$ which is adjacent to components $P_1, P_2 \in \mathcal{P}$:

(4) $h(A)^2 = \max\left(h_1(A)^2, h(P_1)h(P_2)e^{-\pi m_A}\right)$,

and if A is peripheral and adjacent to $P \in \mathcal{P}$,

(5) $h(A) = \max\left(h_1(A), h(P)e^{-\pi m_A/2}\right)$

Remark: This function will be used to rescale σ on each P and σ_e on each annulus A, in such a way that we can interpolate between the different scaling factors in a controlled fashion. Conditions 3, 4 and 5 bound the ratios between scaling factors on adjacent components for this purpose.

Proof. Define $h_2(A) = h_1(A)$ for $A \in \mathcal{A}$. For $P \in \mathcal{P}$, let $A_1, \ldots, A_k \in \mathcal{A}$ be the annuli adjacent to P. Define

$$h_2(P) = \max\{h_1(P), h_1(A_1), \ldots, h_1(A_k)\}. \tag{5.2}$$

This function satisfies conditions (1), (2) and (3) of the lemma, and note also that $h_2(P) > 0$ for at least one P. Next, let us adjust h_2 to a function h_n that satisfies the following additional condition: for each internal $A \in \mathcal{A}$ adjacent to $P_1, P_2 \in \mathcal{P}$, we require that $h_n(P_1)h_n(P_2) > 0$, and

$$\left|\log \frac{h_n(P_1)}{h_n(P_2)}\right| \leq \pi m_A. \tag{5.3}$$

If the condition fails for h_2, then it must fail with some A, P_1, P_2 for which $h_2(P_1) > 0$ and $h_2(P_2) < h_2(P_1)e^{-\pi m_A}$. Define

$$h_3(P_2) = h_2(P_1)e^{-\pi m_A}, \tag{5.4}$$

and $h_3 = h_2$ elsewhere. Clearly $h_3 \geq h_2$, and the maximal value of h_2 hasn't been changed. We may repeat this process to obtain a sequence of functions $h_1, h_2, h_3, \ldots$, which clearly terminates after a finite number of steps in a function h_n that satisfies conditions (1),(2),(3) and (5.3). Note also that h_n is positive on $\mathcal{P}$, since if $h_n(P) = 0$ after these adjustments, then P must be adjacent only to peripheral annuli on which $h_1 = 0$, and in that case β is empty (we are assuming that S is connected).

At this point we may set $h = h_n$ for every $P \in \mathcal{P}$. For each internal annulus $A \in \mathcal{A}$, we set $h(A)$ to the maximum given in condition (4). For each peripheral annulus A we set $h(A)$ according to condition (5). Note that h still satisfies conditions (1)-(3). In particular, condition (3) is satisfied by virtue of the inequality (5.3). This concludes the proof of lemma 5.2. $\square$

The next step is to define a new metric σ' on S, conformally equivalent to σ. For $P \in \mathcal{P}$ we have simply

$$\sigma'|_P = h(P)\sigma. \tag{5.5}$$

For an internal annulus A, let P_1 and P_2 be the components of $\mathcal{P}$ which adjoin A. For each P_i, if $h(P_i) > h(A)$ we must interpolate between the natural scalings for P_i and A. Let $m = m_A$ and let

$$\mu_i = \frac{1}{2\pi}\log(h(P_i)/h(A)). \tag{5.6}$$

This is the modulus of an annular region in the plane between two concentric circles whose radii are in the ratio $h(P_i)/h(A)$. We would like to reserve subannuli on A of appropriate modulus on which σ' is chosen isometric to such concentric plane annuli. In addition we want to reserve a sufficiently large (modulus at least $m/2$) middle section of A for other purposes. It is for these reasons that condition (4) was imposed in lemma 5.2.

Condition (4) implies that $\mu_1 + \mu_2 \leq m/2$. Thus, let B_1 and B_2 be annulus neighorhoods of the boundary components of A, whose radii in the Euclidean metric σ_e are $\epsilon_0\mu_1, \epsilon_0\mu_2$. Let C denote the closure of the complementary annulus $A - B_1 - B_2$; the modulus of C is at least $m/2$.

On B_1 and B_2 we can make σ' the metric conformally equivalent to σ such that (B_i, σ') is a concentric planar annulus as above, with boundary lengths $\epsilon_0 h(P_i)$ (for the boundary between A and P_i) and $\epsilon_0 h(A)$ on the inner boundary. On the remaining annulus C between B_1 and B_2 let $\sigma' = h(A)\sigma_e$. Note that this makes σ' a continous metric on S. (See figure 8.)

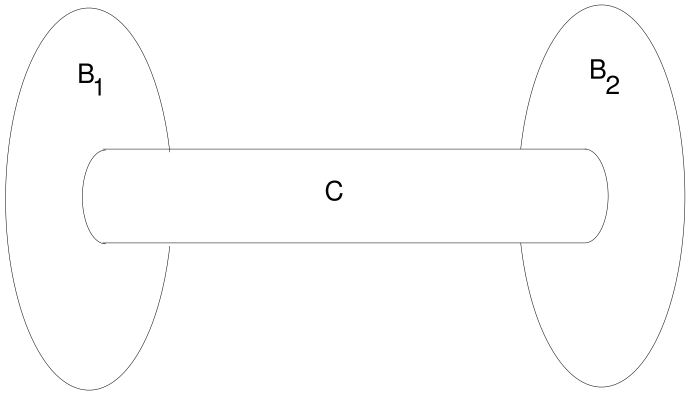

FIGURE 8. The metric σ' on the annulus A decomposed into B_1, C and B_2.

Similarly if A is a peripheral annulus adjacent to P, we can apply condition (5) of lemma 5.2 which implies that $\mu = \frac{1}{2\pi}\log(h(P)/h(A)) \leq m/4$. Thus we place a planar annulus metric on the annulus neighborhood B of $P \cap A$, of σ_e-radius $\epsilon_0\mu$. On $A - B$ we again let $\sigma' = h(A)\sigma_e$.

Finding a good representative. Having defined σ', we shall now construct a representative of β whose spacing in σ' is as uniform as possible.

Lemma 5.3. *There is a representative β_2 of β whose spacing in the metric σ' is bounded by*

$$\nu_{S,\sigma'}(\beta_2) \geq c.$$

The constant c depends only on $\epsilon_0, \epsilon_1, \ell_0$ and $\chi(S)$.

Proof. We begin by applying lemma 4.4 to deform $\beta \cap P$ (if non-empty) on each $P \in \mathcal{P}$ to a curve system β_1 whose intersection with each boundary component of P is evenly spaced, and whose spacing on the interior satisfies $\nu_{P,\sigma} \geq c/\ell_\sigma(\beta^\sigma \cap P)$.

(The portions of β in the annuli $A \in \mathcal{A}$ can be moved along with the homotopy on $P \in \mathcal{P}$, and we will examine these separately). By lemma 4.3, this implies

$$\nu_{P,\sigma}(\beta_1) \geq c'/\sqrt{\lambda_P} = c'/h_1(P)$$

and therefore

$$\nu_{P,\sigma'}(\beta_1) \geq c'h(P)/h_1(P) \geq c'.$$

Now consider the case of an internal annulus $A \in \mathcal{A}$, partitioned as before into boundary annuli B_1, B_2 and a central annulus C. Suppose first that $i(\beta, A) > 0$. Deform $\beta_1 \cap A$, fixing its intersection with ∂A, so that each resulting arc intersects (B_i, σ') in a radial segment, and (C, σ') in a σ'-geodesic arc that twists several times around C. Call this new curve β_2. We have

$$t_{C,\sigma'}(\beta_2 \cap C) = t_A + \delta \tag{5.7}$$

where the error satisfies $|\delta| \leq 1$, since by lemma 4.4 the endpoints have moved at most halfway around the boundary in the homotopy from β to β_1. It follows that the length of $\beta_2 \cap C$ is, if $i(\beta, A) > 0$,

$$\begin{aligned} \ell_{\sigma'}(\beta_2 \cap C) &= n(A)h(A)\epsilon_0\sqrt{m(C)^2 + (t_A + \delta)^2} \\ &\leq c\epsilon_0 h(A)\sqrt{m_A \lambda_A} \end{aligned} \tag{5.8}$$

Here we are using the formula (4.3) for $\lambda_A = \lambda_{A,\sigma}(\beta)$. Note that the error δ can be replaced by the multiplicative factor c because $m_A \geq 1$ (by choice of ϵ_0, ϵ_1).

If $i(\beta, A) = 0$ and $n(A) > 0$ we can make $\beta_2 \cap A$ a disjoint union of closed σ'-geodesics uniformly spaced in C. Then we have:

$$\begin{aligned} \ell_{\sigma'}(\beta_2 \cap C) &= n(A)h(A)\epsilon_0 \\ &= \epsilon_0 h(A)\sqrt{m_A \lambda_A} \end{aligned} \tag{5.9}$$

again using (4.3). Now, since β_2 is uniformly spaced in C with respect to the standard Euclidean metric σ', its spacing is equal to the area divided by its length. Applying either (5.8) or (5.9), and the fact that $m(C) \geq m_A/2$, we obtain:

$$\begin{aligned} \nu_{C,\sigma'}(\beta_2 \cap C) &= \frac{\text{Area}(C, \sigma')}{\ell_{\sigma'}(\beta_2 \cap C)} \\ &\geq c\frac{\epsilon_0^2 h(A)^2 m_A/2}{\epsilon_0 h(A)\sqrt{m_A \lambda_A}} \\ &= c'\frac{h(A)}{h_1(A)}. \end{aligned} \tag{5.10}$$

The spacing in the boundary annuli B_1, B_2 is no smaller than this, and $h(A) \geq h_1(A)$ by lemma 5.2, so we conclude

$$\nu_A \geq c'. \tag{5.11}$$

The case when A is peripheral is dealt with in a similar way, and (5.11) still holds. In fact it is slightly easier since $t_A = 0$ in this case.

Thus we have a curve β_2 with a uniform lower bound on $\nu_{Q,\sigma'}$ for all $Q \in \mathcal{Q}$. We can now make the same observation as in the proof of lemma 4.4 to conclude that

$$\nu_{S,\sigma'}(\beta_2) \geq c \min_{Q\in\mathcal{Q}} \nu_{Q,\sigma'}(\beta_2).$$

Namely, since on a uniformly sized neighborhood of an interface between two components the situation is uniformly bilipschitz equivalent to the Euclidean situation depicted in figure 6, the spacing of the combined curve is no less than a constant times the minimum of the spacings on either side. This concludes the proof of lemma 5.3. $\square$

We can now apply lemma 4.1 to obtain:

$$\begin{aligned} \lambda_{S,\sigma}(\beta) &= \lambda_{S,\sigma'}(\beta) \\ &\leq c\,\mathrm{Area}(S,\sigma') = c \sum_{Q\in\mathcal{Q}} \mathrm{Area}(Q,\sigma'). \end{aligned} \tag{5.12}$$

The desired upper bound on $\lambda_{S,\sigma}$ will be obtained if we show that

$$\max_Q \mathrm{Area}(Q,\sigma') \leq c \max_Q \lambda_Q \tag{5.13}$$

for a uniform constant c, since the number of terms in the sum (5.12) is bounded in terms of the topology of S.

For any $P \in \mathcal{P}$, $\mathrm{Area}(P,\sigma') \leq ch^2(P)$ for a constant c, by (5.5) and the fact that $\mathrm{Area}(P,\sigma)$ is bounded. An internal annulus $A \in \mathcal{A}$ is divided into three subannuli B_1, B_2 and C whose areas are $\mathrm{Area}(C,\sigma') \leq \epsilon_0 h^2(A) m_A$, and $\mathrm{Area}(B_i,\sigma') = \epsilon_0^2(h^2(P_i) - h^2(A))/4\pi$, where P_1, P_2 are the neighboring components of $\mathcal{P}$. A peripheral annulus is treated similarly. Thus, it suffices to bound the quantities $h^2(P)$ and $h^2(A)m_A$.

Note first that, by definition (5.1), $h_1^2(Q) \leq \lambda_Q$, since $m_A \geq 1$ for all $A \in \mathcal{A}$. Thus we have

$$\max_{Q\in\mathcal{Q}} h_1^2(Q) \leq \max_Q \lambda_Q. \tag{5.14}$$

It follows from this and part (2) of lemma 5.2 that, for any $P \in \mathcal{P}$,

$$h^2(P) \leq \max_Q \lambda_Q. \tag{5.15}$$

Now for $A \in \mathcal{A}$, suppose first that $h(A) = h_1(A)$. In this case

$$\begin{aligned} h^2(A) m_A &= h_1^2(A) m_A \\ &= \lambda_A \leq \max_Q \lambda_Q \end{aligned} \tag{5.16}$$

If $h(A) > h_1(A)$ then, if A is internal we use parts (4) and (2) of lemma 5.2 to obtain

$$\begin{aligned} h^2(A)m_A &= h(P_1)h(P_2)m_A e^{-\pi m_A} \\ &\le \max_Q h_1^2(Q) m_A e^{-\pi m_A} \\ &\le \frac{1}{\pi e} \max_Q \lambda_Q \end{aligned} \tag{5.17}$$

where the last line follows from (5.14) and the fact that $xe^{-\pi x} \le 1/\pi e$ for all x. In case A is peripheral we apply parts (2) and (5), to similar effect. The bounds (5.15-5.17) together imply (5.13), so that with (5.12) we have the desired upper bound

$$\lambda_{S,\sigma}(\beta) \le c \max_{Q\in\mathcal{Q}} \lambda_{Q,\sigma}(\beta).$$

The lower bound. For each $Q \in \mathcal{Q}$, we shall find a metric σ'' conformal to σ such that $\ell^2_{\sigma''}([\beta])/\operatorname{Area}(S,\sigma'')$ is approximately λ_Q. This will give the desired lower bound.

If $Q = P \in \mathcal{P}$, let $\sigma'' = \sigma$. Note that $\ell_\sigma([\beta]) \ge \ell_\sigma(\beta^\sigma \cap P) \asymp \sqrt{\lambda_Q}$, and that $\operatorname{Area}(S,\sigma)$ is a constant, $2\pi|\chi(S)|$. Thus:

$$\lambda_{S,\sigma}(\beta) \ge \frac{\ell^2_\sigma(\beta)}{\operatorname{Area}(S,\sigma)} \succ \lambda_Q \tag{5.18}$$

If $Q = A \in \mathcal{A}$, define the metric σ^A to be

$$\sigma^A = \begin{cases} \sigma & \text{in } S - A \\ \sigma_e & \text{in } A \end{cases} \tag{5.19}$$

where σ_e is the Euclidean metric defined as in the beginning of this section.

If $i(\beta, A) = 0$ but $n(A) > 0$, there are $n(A)$ copies in β of the core curve γ of A. It is easy to see that $\ell_{\sigma^A}(\gamma) = \epsilon_0$ in this case, since σ^A is non-positively curved so that the σ^A-geodesic cores of A are the shortest representatives of γ. Thus,

$$\lambda_{S,\sigma}(\beta) \ge \frac{n^2(A)\epsilon_0^2}{\epsilon_0^2 m_A + 2\pi|\chi(S)|} \succ \lambda_A.$$

where for the last step we use the fact that $m_A \ge 1$ to replace an additive constant with a multiplicative one.

Now consider the case where $i(\beta, A) > 0$. Let β^A be a representative of β of minimal σ^A-length (in fact β^A is unique, but we do not need to use this), and note that each arc of $\beta^A \cap (S - A)$ is geodesic, and homotopic with endpoints on ∂A to a corresponding arc of $\beta^\sigma \cap (S - A)$ (this is most clearly seen in the universal cover). Thus by lemma 4.5, the homotopy between β^σ and β^A only moves the intersection with ∂A a bounded amount c. In particular we may conclude that

$$|t_{A,\sigma}(\beta^A) - t_A| \le 2c/\epsilon_0,$$

and therefore

$$\begin{aligned} \ell_{\sigma^A}(\beta^A \cap A) &\geq n(A)\epsilon_0\sqrt{m^2(A) + (t_A - 2c/\epsilon_0)^2} \\ &\succ n(A)\epsilon_0\sqrt{m^2(A) + t_A^2} \\ &\succ \sqrt{\lambda_A m_A}. \end{aligned} \tag{5.20}$$

(Note that we have again used the fact that $m_A \geq 1$.)

We conclude that

$$\lambda_{S,\sigma}(\beta) \succ \frac{\lambda_A m_A}{\epsilon_0 m_A + 2\pi|\chi(S)|} \succ \lambda_A. \tag{5.21}$$

This completes the proof of the lower bound and therefore of theorem 5.1. □

We can generalize theorem 5.1 slightly as follows. Suppose that $A_1, \ldots, A_k$ are a subset of the annuli of the (ϵ_0, ϵ_1) collar decomposition of (S, σ) and $P_1, \ldots, P_m$ are the components of $S - \cup_i A_i$. We call $\mathcal{Q} = \{A_1, \ldots, A_k, P_1, \ldots, P_m\}$ a *partial* (ϵ_0, ϵ_1) decomposition. We can then state:

Corollary 5.4. *If $\mathcal{Q}$ is a partial (ϵ_0, ϵ_1) collar decomposition with ϵ_0, ϵ_1 chosen as above, then for any $\beta \in \mathcal{C}(S, \partial S)$*

$$\lambda_{S,\sigma} \asymp \max_{Q \in \mathcal{Q}} \lambda_{Q,\sigma}(\beta).$$

Proof. Let $\bar{\mathcal{Q}} = \bar{\mathcal{P}} \cup \bar{\mathcal{A}}$ be the full (ϵ_0, ϵ_1) collar decomposition. Let $\bar{\mathcal{Q}}_i = \bar{\mathcal{P}}_i \cup \bar{\mathcal{A}}_i$ be the subset of $\bar{\mathcal{Q}}$ consisting of components that lie in P_i. Let $\hat{\sigma}_i$ be the standard hyperbolic metric on P_i which is conformally equivalent to the restriction of σ. We first show that, for any $\bar{Q} \in \bar{\mathcal{Q}}_i$,

$$\lambda_{\bar{Q},\sigma}(\beta) \asymp \lambda_{\bar{Q},\hat{\sigma}}(\mathcal{R}_{P_i}(\beta)). \tag{5.22}$$

For a hyperbolic component $\bar{Q} \in \bar{\mathcal{P}}_i$, these two quantities are the same by definition (4.2), since only the conformal class of the metric matters for extremal length.

For an annulus $\bar{Q} = \bar{A} \in \bar{\mathcal{A}}_i$, the definition (4.3) depends in particular on the measure of twisting $t_{\bar{A},\sigma}$, which depends on the actual geometry. However, lemma 3.4 assures us that the difference between twisting in σ and $\hat{\sigma}$ is no more than a constant. It follows that (5.22) holds for annuli as well, with uniform constants that depend only on ϵ_0, ϵ_1 and the topology of S.

The corollary now follows by applying theorem 5.1 separately to each P_i.

□

6. The product region theorem

In this section S denotes an oriented finite genus surface without boundary, but possibly with finitely many punctures.

Let γ be an element of $\mathcal{C}(S)$ all of whose components $\gamma_1, \ldots, \gamma_k$ are homotopically distinct, and fix an orientation on each component of γ. For any $\epsilon > 0$, denote by $Thin_\epsilon(S, \gamma)$ the subset of $\mathcal{T}(S)$ consisting of all $[\sigma]$ such that $\ell_\sigma(\gamma_i) \leq \epsilon$.

Let $\gamma_{k+1},\ldots,\gamma_n$ denote a completion of γ to an oriented pair-of-pants decomposition $\hat\gamma$. Fix also a curve system μ composed of seam arcs, giving a marking of S as in §3. This gives rise to length and twist coordinates on $\mathcal{T}(S)$, denoted in §3 by the map $F:\mathcal{T}(S)\to\mathbf{R}^n_+\times\mathbf{R}^n$. Let us write the coordinates of F as F_i, where $F_1,\ldots,F_n$ are length coordinates, and $F_{n+1},\ldots,F_{2n}$ are twist coordinates. Separating out the coordinates of the first k curves from the rest decomposes $\mathcal{T}(S)$ as a product.

Let S_γ denote the punctured (or "noded") surface obtained from S by deleting $\gamma_1,\ldots,\gamma_k$ and replacing each curve by a pair of punctures. Let $\mathcal{T}(S_\gamma)$ be the space of (analytically finite) conformal structures on S_γ. Let $\mathbf{H}$ denote the upper half-plane $\{(x,y):y>0\}$ endowed with the hyperbolic metric $d_{\mathbf{H}}$. Let X_γ denote the product space $X_\gamma=\mathcal{T}(S_\gamma)\times\mathbf{H}_1\times\cdots\times\mathbf{H}_k$ where each $\mathbf{H}_i$ is a copy of $\mathbf{H}$, endowed with the sup metric $d_X=\max(d_{\mathcal{T}(S_\gamma)},d_{\mathbf{H}_1},\ldots,d_{\mathbf{H}_k})$.

The marking by $\hat\gamma$ and μ induces a marking on each component of S_γ, and a map $G:\mathcal{T}(S_\gamma)\to\mathbf{R}^{n-k}_+\times\mathbf{R}^{n-k}$, given by $G(\rho)=(\ell_\rho(\gamma_{k+1}),\ldots,\ell_\rho(\gamma_n),s_{k+1}(\rho),\ldots,s_n(\rho))$ for any complete hyperbolic metric ρ on S_γ.

We can therefore define a map

$$\Pi:\mathcal{T}(S)\to\mathcal{T}(S_\gamma)\times\mathbf{H}_1\times\cdots\times\mathbf{H}_k \tag{6.1}$$

as follows. Let Π_0 denote the component of Π mapping to $\mathcal{T}(S_\gamma)$, and for $i>0$ let Π_i denote the component mapping to $\mathbf{H}_i$. We define Π_0 via

$$\Pi_0(\sigma)=G^{-1}(F_{k+1}(\sigma),\ldots,F_n(\sigma),F_{n+k+1}(\sigma),\ldots,F_{2n}(\sigma)). \tag{6.2}$$

In other words, we build a punctured surface by forgetting the length and twist coordinates of the first k curves, and using the remaining coordinates to build new pairs of pants. We then use the forgotten coordinates for the rest of Π, by defining

$$\Pi_i(\sigma)=(s_i(\sigma),1/\ell_\sigma(\gamma_i)) \tag{6.3}$$

for $i=1,\ldots,k$.

Let ϵ_0 and ϵ_1 be chosen as in the previous section. Our main theorem is the following.

Theorem 6.1. *Let $\gamma\in\mathcal{C}(S)$ be an oriented curve system as above, extended to a marking $\hat\gamma,\mu$ of S. Given $\epsilon\in(0,\epsilon_1)$, there is a contant a_0 depending on ϵ_0,ϵ_1 and the topological type of S, such that the homeomorphism $\Pi:\mathcal{T}(S)\to X_\gamma$ defined above, when restricted to $Thin_\epsilon(S,\gamma)$, satisfies*

$$|d_{\mathcal{T}(S)}(\sigma,\tau)-d_X(\Pi(\sigma),\Pi(\tau))|\le a_0$$

for any $[\sigma],[\tau]\in Thin_\epsilon(S,\gamma)$.

Proof. For a given $[\sigma]\in Thin_\epsilon(S,\gamma)$, let σ denote its hyperbolic representative. Let $A_1,\ldots,A_k$ be the annuli of the (ϵ_0,ϵ_1) collar decomposition which correspond to the curves $\gamma_1,\ldots,\gamma_k$. The complement $S-\cup A_i$ is also naturally identified with S_γ. Let $P_1,\ldots,P_m$ denote the connected components of S_γ, and let $\mathcal{Q}=\{A_j\}\cup\{P_i\}$.

Note that for another point $[\tau] \in Thin_\epsilon(S,\gamma)$ we obtain a similar decomposition, which we also call $\mathcal{Q}$.

Our strategy will be to estimate distances using Kerckhoff's theorem, so we shall need to be able to control extremal length ratios of the type $\lambda_{Q,\sigma}(\alpha)/\lambda_{Q,\tau}(\alpha)$. Note that, for Q a subsurface of S, if $\lambda_{Q,\sigma}(\alpha) = 0$ then $\lambda_{Q,\tau}(\alpha) = 0$ as well. Thus we adopt the convention that, if $\lambda_{Q,\sigma}(\alpha) = 0$ then the expression $\lambda_{Q,\sigma}(\alpha)/\lambda_{Q,\tau}(\alpha)$ is defined to be 1.

The first step is the following:

Lemma 6.2. *Let Q be any hyperbolic surface of finite type with boundary, and $\lambda_1 > \lambda_0 > 0$ two given constants. Let $[\sigma], [\tau] \in \mathcal{T}(Q)$ be two conformal structures such that $\lambda_{Q,\sigma}(\gamma) \in [\lambda_0, \lambda_1]$ for any boundary component γ, and similarly for τ. Then*

$$\sup_{\alpha\in\mathcal{C}_0(Q)} \frac{\lambda_{Q,\sigma}(\alpha)}{\lambda_{Q,\tau}(\alpha)} \asymp \sup_{\alpha\in\mathcal{C}(Q,\partial Q)} \frac{\lambda_{Q,\sigma}(\alpha)}{\lambda_{Q,\tau}(\alpha)} \tag{6.4}$$

where the constants for "$\asymp$" depend only on $\chi(S), \lambda_0$ and λ_1.

(Recall that $\mathcal{C}_0(Q)$ is the set of homotopy classes of closed curve systems in Q with no peripheral components.)

Proof. The left side is clearly no larger than the right, since $\mathcal{C}_0(Q) \subset \mathcal{C}(Q,\partial Q)$.

The supremum on the right side is equal to $\exp 2d_{\mathcal{T}(Q)}(\sigma,\tau)$, by Kerckhoff's theorem 2.1, where $d_{\mathcal{T}(Q)}$ is the distance in the Teichmüller space of Q. In particular, this distance is symmetric, so that the supremum of the reciprocal ratio is the same quantity. Choose $\beta \in \mathcal{C}(Q,\partial Q)$ such that $\lambda_{Q,\tau}(\beta)/\lambda_{Q,\sigma}(\beta)$ is nearly equal to the supremum on the right side of (6.4).

By lemma 6.3, whose proof appears below, there exists $\alpha \in \mathcal{C}_0(S)$ such that, for a uniform $c > 0$,

$$i(\alpha,\beta)^2 \geq c\lambda_{Q,\tau}(\alpha)\lambda_{Q,\tau}(\beta). \tag{6.5}$$

(One should think of α as approximately "orthogonal" to β.)

On the other hand, the following inequality is elementary for any two curves $\alpha, \beta \in \mathcal{C}(Q)$ and $[\sigma] \in \mathcal{T}(Q)$ (see e.g. [26, Lemma 5.1]).

$$\lambda_{Q,\sigma}(\alpha)\lambda_{Q,\sigma}(\beta) \geq i(\alpha,\beta)^2. \tag{6.6}$$

Combining (6.5) with(6.6), we obtain

$$\frac{\lambda_{Q,\sigma}(\alpha)}{\lambda_{Q,\tau}(\alpha)} \geq c\frac{\lambda_{Q,\tau}(\beta)}{\lambda_{Q,\sigma}(\beta)}$$

which gives the remaining direction of lemma 6.2. $\square$

To complete lemma 6.2 we need to prove the following:

Lemma 6.3. *Let Q be a hyperbolic surface of finite type, with a conformal structure $[\tau]$. Suppose that for $0 < \lambda_0 < \lambda_1$ we have $\lambda_{Q,\tau}(\gamma) \in [\lambda_0, \lambda_1]$ for any boundary component γ of Q. Then for any $\beta \in \mathcal{C}(Q, \partial Q)$ there exists $\alpha \in \mathcal{C}_0(Q)$ for which*

$$i(\alpha, \beta)^2 \geq c\lambda_{Q,\tau}(\alpha)\lambda_{Q,\tau}(\beta)$$

where c depends only on λ_0, λ_1 and $\chi(Q)$.

Proof. Represent $[\tau]$ by a standard hyperbolic metric τ, and let $T_1, \dots, T_j$ be the (ϵ_0, ϵ_1) collar decomposition of Q, with respect to τ. Note that ϵ_1 may be chosen (depending on λ_0) so that no annuli in the decomposition are peripheral in Q. Applying theorem 5.1, there exists one component, say T_1, such that $\lambda_{Q,\tau}(\beta) \asymp \lambda_{T_1,\tau}(\beta)$.

Suppose first that T_1 is a non-annular component. We claim that there exist constants c_0, λ_2 such that there is an $\alpha \in \mathcal{C}_0(T_1)$ with $\lambda_{T_1,\tau}(\alpha) \leq \lambda_2$, and $i(\alpha, \beta)^2 \geq c_0\lambda_{T_1,\tau}(\beta)$. We can see this as follows: Since T_1 has no thin parts we can cut it along moderate-length curves to obtain bounded-diameter pieces. More precisely, note that the components of ∂T_1 are either boundaries of thin parts of length ϵ_0, or boundaries of Q, which have τ-length in some interval $[\ell_0, \ell_1]$ (depending on λ_0, λ_1). We can find a pair-of-pants decomposition for T_1 of minimal total boundary lengths, and then add sufficiently many simple closed geodesics, as short as possible, to cut T_1 into disks and boundary-parallel annuli. The result is a collection of non-peripheral simple closed geodesics $\alpha_1, \dots, \alpha_q$ in T_1 (q dependent only on $\chi(T_1)$) such that $\ell_\tau(\alpha_i) \leq \ell_2$ (depending on $\epsilon_0, \lambda_0, \lambda_1, \chi(T_1)$) and which cut T_1 up into components that are either disks, or annuli which share a boundary with T_1, or annular neighborhoods of cusps. In the first two cases the diameters of the components are uniformly bounded. In the third case we may remove a neighborhood of the cusp avoided by β (since β is simple) and so that what remains has bounded diameter. Furthermore note that any component of the intersection of the geodesic β^τ with one of these regions has length at most some δ_0, since β^τ meets ∂Q orthogonally, and meets the other components of ∂T_1 nearly orthogonally (compare lemma 4.5).

It follows, adding over the pieces of β^τ in these components, that $\sum_j \delta_0 i(\beta, \alpha_j) \geq \ell_\tau(\beta^\tau \cap T_1)$. By lemma 4.3, $\ell^2_\tau(\beta^\tau \cap T_1) \asymp \lambda_{T_1,\tau}(\beta)$. Thus, there must be some $\alpha = \alpha_j$ such that

$$i(\alpha, \beta)^2 \geq C\lambda_{T_1,\tau}(\beta).$$

We also know that there is an upper bound

$$\lambda_{Q,\tau}(\alpha) \leq \lambda_3$$

for a uniform λ_3, since there is an upper bound on the hyperbolic length of α by construction. The statement of the lemma follows in this case.

Now let $T_1 = A$ be an annular component with modulus $m = m(A, \tau)$, and consider first the case when $i(\beta, A) > 0$, and $t_1 = t_{A,\tau}(\beta) \neq 0$. Let $t_2 = -m^2/t_1$ – we choose this number because two geodesics with twists t_1, t_2 in a standard Euclidean annulus of modulus m are orthogonal when $t_1 t_2 = -m^2$. Lemma 3.3

guarantees the existence of an element $\alpha \in \mathcal{C}_0(Q)$ with $|t_{A,\tau}(\alpha) - t_2| \leq c_1$ for a uniform c_1.

Now using the definition of $\lambda_{A,\tau}$ in (4.3), we can see that

$$\begin{aligned} \lambda_{A,\tau}(\alpha)\lambda_{A,\tau}(\beta) &\asymp n(\alpha, A)n(\beta, A)(m + t_1^2/m)(m + t_2^2/m) \\ &= n(\alpha, A)n(\beta, A)(t_1 - t_2)^2. \end{aligned} \tag{6.7}$$

where the additive error c_1 becomes a multiplicative error since $m \geq 1$.

On the other hand it is easy to see that, if two properly embedded arcs α_1, α_2 in A have twisting numbers $t_i = t_{A,\tau}(\alpha_i)$, then their intersection number is estimated by

$$|t_1 - t_2| - 1 \leq i(\alpha_1, \alpha_2) \leq |t_1 - t_2| + 1. \tag{6.8}$$

Combining (6.8) with (6.7) we obtain

$$\lambda_{A,\tau}(\alpha)\lambda_{A,\tau}(\beta) \succ i(\alpha, \beta)^2 \tag{6.9}$$

(again an additive error has become multiplicative). By lemma 3.3, α may be chosen to have bounded length outside A, and thus by theorem 5.1, $\lambda_{A,\tau}(\alpha) \asymp \lambda_{Q,\tau}(\alpha)$. Since also $\lambda_{A,\tau}(\beta) \asymp \lambda_{Q,\tau}(\beta)$ by choice of A, the statement of lemma 6.3 follows.

We have left the case where $t_1 = 0$, and the case where $i(\beta, A) = 0$ but $n(\beta, A) > 0$. In the former case, We can pick α to be the core of A (essentially this is $t_2 = \infty$), and note that $\lambda_{Q,\sigma}(\alpha) \asymp 1/m$, so that the lemma holds automatically. In the latter case we may choose α with $t_2 = 0$, and the roles of α and β are reversed. $\square$

We now resume the proof of theorem 6.1. Let us establish the following for $[\sigma], [\tau] \in Thin_\epsilon(S, \gamma)$:

$$\sup_{\alpha \in \mathcal{C}(S)} \frac{\lambda_{S,\sigma}(\alpha)}{\lambda_{S,\tau}(\alpha)} \asymp \max_{Q \in \mathcal{Q}} \sup_{\alpha \in \mathcal{C}(S)} \frac{\lambda_{Q,\sigma}(\alpha)}{\lambda_{Q,\tau}(\alpha)}. \tag{6.10}$$

One direction is a direct application of the main extremal length estimate. Note that $\mathcal{Q}$ is what we called a "partial (ϵ_0, ϵ_1) collar decomposition" in the previous section. Thus by corollary 5.4 of theorem 5.1, we have

$$\lambda_{S,\sigma}(\alpha) \asymp \max_{Q \in \mathcal{Q}} \lambda_{Q,\sigma}(\alpha) \tag{6.11}$$

for each $\alpha \in \mathcal{C}(S)$. Letting Q_α maximize $\lambda_{Q,\sigma}(\alpha)$ over all $Q \in \mathcal{Q}$, we apply (6.11) to obtain

$$\begin{aligned} \sup_{\alpha \in \mathcal{C}(S)} \frac{\lambda_{S,\sigma}(\alpha)}{\lambda_{S,\tau}(\alpha)} &\asymp \sup_{\alpha \in \mathcal{C}(S)} \frac{\max_{Q \in \mathcal{Q}} \lambda_{Q,\sigma}(\alpha)}{\max_{Q \in \mathcal{Q}} \lambda_{Q,\tau}(\alpha)} \\ &\leq \sup_{\alpha \in \mathcal{C}(S)} \frac{\lambda_{Q_\alpha,\sigma}(\alpha)}{\lambda_{Q_\alpha,\tau}(\alpha)} \\ &\leq \max_{Q \in \mathcal{Q}} \sup_{\alpha \in \mathcal{C}(S)} \frac{\lambda_{Q,\sigma}(\alpha)}{\lambda_{Q,\tau}(\alpha)} \end{aligned} \tag{6.12}$$

For the inequality in the other direction, consider first a non-annular component $Q \in \mathcal{Q}$. By definition (4.2) of $\lambda_{Q,\sigma}$ the supremum $\sup_{\alpha\in\mathcal{C}(S)} \lambda_{Q,\sigma}(\alpha)/\lambda_{Q,\tau}(\alpha)$ may be obtained by letting α vary over $\mathcal{C}(Q,\partial Q)$. By lemma 6.2, we may further restrict to $\alpha \in \mathcal{C}_0(Q)$, at the expense of a bounded factor. By lemma 4.2, $\lambda_{Q,\sigma}(\alpha) \asymp \lambda_{S,\sigma}(\alpha)$ for $\alpha \in \mathcal{C}_0(Q)$ and any $\sigma \in Thin_\epsilon(S,\gamma)$. Since $\mathcal{C}_0(Q) \subset \mathcal{C}(S)$, the supremal ratio over all of $\mathcal{C}(S)$ can only be bigger, so in this case we have the desired inequality

$$\sup_{\alpha\in\mathcal{C}(S)} \frac{\lambda_{S,\sigma}(\alpha)}{\lambda_{S,\tau}(\alpha)} \geq \sup_{\alpha\in\mathcal{C}(S)} \frac{\lambda_{Q,\sigma}(\alpha)}{\lambda_{Q,\tau}(\alpha)}. \tag{6.13}$$

Now consider an annulus $A \in \mathcal{Q}$, and let α be a curve nearly maximizing the ratio $\lambda_{A,\sigma}(\alpha)/\lambda_{A,\tau}(\alpha)$. By lemma 3.3, there exists $\alpha' \in \mathcal{C}(S)$ such that

$$|t_{A,\tau}(\alpha') - t_{A,\tau}(\alpha)| \leq c_1, \tag{6.14}$$

and such that (applying theorem 5.1)

$$\lambda_{S,\tau}(\alpha') \asymp \lambda_{A,\tau}(\alpha'). \tag{6.15}$$

Applying lemmas 3.2 and 3.5, we see that $t_{A,\tau}(\alpha') - t_{A,\sigma}(\alpha')$ and $t_{A,\tau}(\alpha) - t_{A,\sigma}(\alpha)$ are both approximated, up to bounded additive error, by the twist-parameter difference $s_j(\tau) - s_j(\sigma)$ associated to A. This together with inequality (6.14) implies that $|t_{A,\sigma}(\alpha') - t_{A,\sigma}(\alpha)|$ is uniformly bounded as well. Since $\lambda_{A,\sigma}$ and $\lambda_{A,\tau}$ are determined by the twisting numbers, we conclude that

$$\frac{\lambda_{A,\sigma}(\alpha')}{\lambda_{A,\tau}(\alpha')} \asymp \frac{\lambda_{A,\sigma}(\alpha)}{\lambda_{A,\tau}(\alpha)}.$$

Now we may apply (6.15) together with the inequality $\lambda_{A,\sigma}(\alpha') \prec \lambda_{S,\sigma}(\alpha')$, which is a consequence of theorem 5.1, to conclude that

$$\frac{\lambda_{S,\sigma}(\alpha')}{\lambda_{S,\tau}(\alpha')} \succ \frac{\lambda_{A,\sigma}(\alpha')}{\lambda_{A,\tau}(\alpha')}. \tag{6.16}$$

Inequalities (6.12), (6.13) and (6.16) together imply (6.10).

The length ratio on the left side of (6.10) is, by Kerckhoff's theorem 2.1, equal to $\exp 2d_{\mathcal{T}(S)}(\sigma,\tau)$. It remains to connect the length ratios on the right of (6.10) to the distances $d_i \quad (i = 0,\ldots,k)$, so that we may complete the proof of theorem 6.1.

Consider first d_0, the Teichmüller distance on $\mathcal{T}(S_\gamma)$. Given a $Q \in \mathcal{Q}$ which is a component of S_γ, with the conformal structure $\Pi_0(\sigma)$ or $\Pi_0(\tau)$ (restricted from S_γ to Q), we have by Kerckhoff's theorem:

$$\sup_{\alpha\in\mathcal{C}_0(Q)} \frac{\lambda_{Q,\Pi_0(\sigma)}(\alpha)}{\lambda_{Q,\Pi_0(\tau)}(\alpha)} = \exp 2d_{\mathcal{T}(Q)}(\Pi_0(\sigma),\Pi_0(\tau)).$$

We now claim that the extremal length ratio on the left can be approximated by a ratio taken with respect to σ and τ. In fact we claim that (Q,σ) embeds K-quasi-conformally, with uniform K, in $(Q,\Pi_0(\sigma))$ (note that the embedding will not be onto). Recall the construction: we measure the length and twist parameters for $\gamma_{k+1},\ldots,\gamma_n$ for σ, and then construct $\Pi_0(\sigma)$ with the same parameters, but with the boundary curves $\gamma_1, \ldots, \gamma_k$ replaced by punctures. Thus the embedding of (Q,σ) in $(Q,\Pi_0(\sigma))$ can be done separately on each pair of pants. After splitting each pair of pants along seam curves into right-angled hexagons, the problem reduces to the following: Let H_1 be a hyperbolic right-angled hexagon with lengths ℓ_1, ℓ_2, ℓ_3 on alternating sides, where $\ell_1 \leq \epsilon_1$, and let H_2 be a hyperbolic right-angled hexagon with lengths $0, \ell_2, \ell_3$ (really a right-angled pentagon with one ideal vertex). There is a K-quasiconformal embedding (in fact bilipschitz) of H_1 into H_2 which is an isometry on sides 2 and 3, where K depends on ϵ_1, but not on the ℓ_i. This fact follows easily from the hexagon cosine law (see e.g. [3]). A similar computation applies when a pair of pants has more than one short boundary which is to be replaced by punctures. The maps on the pairs of pants fit together across the boundaries, because the marking and twist parameters are the same for both surfaces.

It now follows (applying also lemma 4.2) that $\lambda_{Q,\sigma}(\alpha) \asymp \lambda_{Q,\Pi_0(\sigma)}(\alpha)$ uniformly for all $\sigma \in Thin_\epsilon(S,\gamma)$ and $\alpha \in \mathcal{C}_0(Q)$. Thus

$$\sup_{\alpha\in\mathcal{C}_0(Q)} \frac{\lambda_{Q,\Pi_0(\sigma)}(\alpha)}{\lambda_{Q,\Pi_0(\tau)}(\alpha)} \asymp \sup_{\alpha\in\mathcal{C}_0(Q)} \frac{\lambda_{Q,\sigma}(\alpha)}{\lambda_{Q,\tau}(\alpha)}.$$

Applying again lemma 6.2, we conclude

$$\exp 2d_{\mathcal{T}(Q)}(\Pi_0(\sigma),\Pi_0(\tau)) \asymp \sup_{\alpha\in\mathcal{C}(Q)} \frac{\lambda_{Q,\sigma}(\alpha)}{\lambda_{Q,\tau}(\alpha)}.$$

Since $d_0 = d_{\mathcal{T}(S_\gamma)}$ is the supremum of the distances in the Teichmüller spaces of the components, we have shown that the contribution to the right side of (6.10) coming from the non-annular components is, up to bounded factor, just $\exp 2d_0$.

We next consider an annular component A_j in $\mathcal{Q}$, and estimate its extremal length contribution $\lambda_{A_j,\sigma}(\alpha)$, for $\alpha \in \mathcal{C}(S)$ and $\sigma \in Thin_\epsilon(S,\gamma)$, in terms of the coordinates $\Pi_j(\sigma) = (s_j(\sigma), 1/\ell_\sigma(\gamma_j))$. Rewrite $\Pi_j(\sigma) = (x_j(\sigma), y_j(\sigma))$, and note first that (by (2.2)) $y_j(\sigma)$ is closely approximated by $m(A_j,\sigma)/\pi$.

Fix some arbitrary $\sigma_0 \in Thin_\epsilon(S,\gamma)$ and define $b_j(\alpha) = t_{\gamma_j,\sigma_0}(\alpha) - s_j(\sigma_0)$. By lemma 3.5, $t_{\gamma_j,\sigma}(\alpha)$ is estimated by $b_j(\alpha) + s_j(\sigma)$, up to bounded additive error, for all σ and α. Applying also lemma 3.2, this gives an estimate for $t_{A_j,\sigma}(\alpha)$. Now applying definition (4.3) when $i(\alpha,\gamma_j) > 0$, we obtain this approximation for the length ratio contribution of the annulus:

$$\frac{\lambda_{A_j,\sigma}(\alpha)}{\lambda_{A_j,\tau}(\alpha)} \asymp \frac{y_j(\sigma) + (b_j(\alpha) + x_j(\sigma))^2/y_j(\sigma)}{y_j(\tau) + (b_j(\alpha) + x_j(\tau))^2/y_j(\tau)}, \tag{6.17}$$

Note that the case where $i(\alpha,\gamma_j) = 0$ and $n(\alpha,\gamma_j) > 0$ can be obtained by the limit of (6.17) as $b_j \to \infty$.

By lemma 3.3, we know that $t_{\gamma_j,\sigma_0}(\alpha)$, and therefore $b_j(\alpha)$, can take on any value in $\mathbf{R}$ up to bounded difference. Since $y_j(\sigma), y_j(\tau)$ are at least $1/\epsilon$, we may once again convert the additive error to a multiplicative one. Thus, taking a supremum of (6.17) over all $\alpha \in \mathcal{C}(S)$, and applying lemma 2.2, we obtain

$$\sup_{\alpha\in\mathcal{C}(S)} \frac{\lambda_{A_j,\sigma}(\alpha)}{\lambda_{A_j,\tau}(\alpha)} \asymp \exp 2d_{\mathbf{H}_j}(\Pi_j(\sigma),\Pi_j(\tau)).$$

It follows that the right side of (6.10) is, up to bounded multiple,

$$\max_{j=0,\dots,k} \exp 2d_j(\Pi_j(\sigma),\Pi_j(\tau)).$$

Taking logarithms, we obtain the desired statement of theorem 6.1. □

7. Geometric remarks

Having shown that regions of the form $Thin_\epsilon(S,\gamma)$ in $\mathcal{T}(S)$ are approximated up to bounded additive distortion by products with the sup metric, we discuss briefly here the implications to "non-hyperbolic" behavior of the Teichmüller space. Indeed, a product with the sup metric is not just non-negatively curved, but exhibits some definite positive-curvature behavior, as we shall see.

Let X be a metric space, and let $x, y, z \in X$ denote three points. We say that z is "δ-between" x and y if

$$d(x,z) + d(z,y) - d(x,y) < \delta$$

for $\delta > 0$. In particular if X is a length space and $[xy]$ denotes a shortest path (geodesic) between x and y, then when z is δ-between x and y, the path $[xz] \cup [zy]$ is a quasigeodesic with only additive distortion δ.

In a negatively curved length space in the sense of Gromov and Cannon, if z is δ-between x and y then it is contained in an $R(\delta)$-neighborhood of $[xy]$, where $R(\delta)$ is independent of x and y. This is called stability of quasi-geodesics.

Let us define an "instability function" s for X as follows: For $\delta, L > 0$ define $s(\delta, L)$ by

$$s(\delta, L) = \sup\{d(z,[xy]) : z \text{ is } \delta\text{-between } x \text{ and } y \text{ and } d(x,y) \le L\}.$$

In order to consider only the large-scale behavior, let us denote by $\bar{s}(\delta)$ the asymptotic growth rate of $s(\delta, L)$; that is,

$$\bar{s}(\delta) = \limsup_{L\to\infty} \frac{\log s(\delta, L)}{\log L}.$$

It is easy to see that $\bar{s}(\delta) \in [0,1]$, and $\bar{s}$ is a non-decreasing function of δ.

In a negatively curved space, $s(\delta, L)$ is bounded independently of L. Thus, $\bar{s} \equiv 0$. In Euclidean space it is not hard to see that $s(\delta, L) = \sqrt{2L\delta + \delta^2}/2$. In particular $\bar{s}(\delta) \equiv 1/2$. In a space which contains large positively curved regions such as hemispheres, s could be made to grow roughly proportionally to L, for fixed δ. Thus $\bar{s}(\delta)$ can be made identically 1 in such a space.

Let X denote the product $\mathbf{R} \times \mathbf{R}$ with the sup metric. Then for any δ, $s(\delta, L)$ in this space is at least $L/2$, as one can see by considering $x = (0,0), y = (L,0)$ and $z = (L/2, L/2)$. Thus the growth rate $\bar{s} \equiv 1$, as in the case of positive curvature.

Let X, Y be two length spaces with instability functions s_X, s_Y. If $f : X \to Y$ is a map such that $|d_X - d_Y \circ f| \leq c$, it is easy to show that

$$s_X(\delta, L) \leq 3c + 4s_Y(\delta + 3c, L + c), \tag{7.1}$$

and vice versa if f is invertible. Here is a sketch of the proof: given $x, y, z \in X$ with $d(x, y) \leq L$ and z δ-between x and y, let $[xy]$ be a geodesic and let z' denote the point in $[xy]$ for which $d(x, z') = d(x, z)$. Then both $f(z)$ and $f(z')$ are , at worst, $(\delta + 3c)$-between $f(x)$ and $f(y)$. This bounds the distance from $f(z)$ and $f(z')$ to their nearest points ζ and ζ' on a geodesic $[f(x)f(y)]$. We can also bound $d(\zeta, \zeta')$ using the choice of z', and the desired bound on $d(z, z')$ follows.

Applying (7.1) to the homeomorphism Π of theorem 6.1, we may conclude that, for $\mathcal{T}(S)$, $\bar{s} \equiv 1$. This is the sense in which we say that the large scale behavior of $\mathcal{T}(S)$ has characteristics of positive curvature.

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