QUASI-PROJECTIONS IN TEICHMÜLLER SPACE

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Abstract. We consider a geometric property of the closest-points projection to a geodesic in Teichmüller space: the projection is called contracting if arbitrarily large balls disjoint from the geodesic project to sets of bounded diameter. (This property always holds in sufficiently negatively curved spaces.) It is shown here to hold if and only if the geodesic is precompact, i.e. its image in the moduli space is contained in a compact set. Some applications are given, e.g. to stability properties of certain quasi-geodesics in Teichmüller space, and to estimates of translation distance for pseudo-Anosov maps.

Many parallels have been drawn between geometric properties of the Teichmüller space of a Riemann surface, and those of complete, negatively curved spaces (see for example Bers [2], Kerckhoff [8], Masur [12], Wolpert [17]). This paper investigates one such parallel – the contracting properties of certain projections to geodesics.

We will use the Teichmüller metric throughout the paper (the other famous metric on Teichmüller space, the Weil-Petersson metric, is negatively curved although it is not complete – see Wolpert [16]. The Teichmüller metric is complete, but not negatively curved – see Masur [10].)

Every closed subset C of a metric space X determines a "closest-points" projection, defined as a map $\pi_C : X \to \mathcal{P}(C)$, where $\mathcal{P}(C)$ is the set of closed subsets of C. Namely,

$$\pi_C(x) = \{y \in C : d(x,y) = d(x,C)\}$$

where $d(x,C) = \inf_{y\in C} d(x,y)$. (Provided X is *proper*, i.e. R-balls are compact, $\pi_C(x)$ is never empty.) Suppose now that X is a simply-connected Riemannian manifold with non-positive sectional curvatures, and that C is a geodesic segment, ray or line. In this case $\pi_C(x)$ always consists of a single point. If the sectional curvatures are bounded above by a negative constant, then $||d\pi_C|| \to 0$ as $d(x,C)$ goes to ∞ (where $d\pi_C$ is the differential of π_C). In other words, π is contracting at large distances.

If $X = \mathcal{T}(S)$, the Teichmüller space of a surface S of finite type, this contraction property fails in general, but a coarse form of it can be shown to hold for a special class of geodesics. Call a geodesic (segment, ray, or line) L in $\mathcal{T}(S)$ *precompact* if its projection to the moduli space $\mathcal{M}(S)$ lies in a compact set. We say L is *ϵ-precompact* if $\epsilon > 0$ is a lower bound on the extremal length of any non-peripheral simple closed curve in S, for any conformal structure in L (see section 1 for definitions). Our main theorem is the following:

Contraction Theorem.

(1) *(Contraction property) Given ϵ there is a constant b depending on $\epsilon, \chi(S)$ such that for any ϵ-precompact geodesic L and $\sigma \in \mathcal{T}(S)$,*
$$\operatorname{diam}(\pi_L(\mathcal{N}_{d(\sigma,L)}(\sigma))) \leq b.$$
(2) *(Failure of contraction) Conversely, if L is a non-precompact geodesic then the contraction property does not hold for any b.*

Here $\mathcal{N}_r$ denotes an r-neighborhood in the Teichmüller metric, and we have taken the liberty of denoting by $\pi_L(A)$ the union $\cup_{a\in A}\pi_L(a)$.

Note that, in particular, part (1) implies a uniform bound
$$\operatorname{diam}(\pi_L(\sigma)) \leq b \tag{0.1}$$
for an ϵ-precompact geodesic.

The contraction theorem is a consequence of a characterization of π_L in terms of an extremal-length ratio problem. Minimizing the distance of $\sigma \in \mathcal{T}(S)$ to a geodesic L is equivalent to solving the problem of finding
$$\inf_{\tau\in L} \sup_{\lambda} \frac{E_\tau(\lambda)}{E_\sigma(\lambda)}$$
where the supremum is over λ in $\mathcal{PMF}(S)$, the set of projectivized measured foliations of compact support in S. This is because the supremum of extremal length ratios is just $\exp 2d(\sigma,\tau)$, by a theorem of Kerckhoff. In section 2 we examine the relation of this problem to the "dual" problem
$$\sup_{\lambda} \inf_{\tau\in L} \frac{E_\tau(\lambda)}{E_\sigma(\lambda)},$$
in a slightly generalized situation of families of functions satisfying certain conditions. We prove, in Proposition 2.1, that under these conditions the sup inf and inf sup problems have closely related solutions. In Proposition 2.2 we show that the solutions are also robust under rescalings of the functions.

In section 3 we prove part (1) of the Contraction Theorem by applying the ideas of section 2. In particular, the dependence of $\pi_L(\sigma)$ on σ corresponds to the variation of the inf sup problem under rescalings of the function family.

In section 4 we give some consequences of the contraction property, which are directly analogous to well-known properties of hyperbolic space. Theorem 4.2 is a "stability" property for a class of quasi-geodesics: a quasi-geodesic whose endpoints are connected by an ϵ-precompact geodesic must remain in a bounded neighborhood of the geodesic. Theorem 4.3 bounds the projected images of regions in Teichmüller space which are analogous to horoballs in hyperbolic space. Theorem 4.4 is a lower bound for the translation distance of a pseudo-Anosov automorphism of $\mathcal{T}(S)$, in terms of distance from its axis; a similar estimate holds for arbitrary loxodromic isometries in $\mathbf{H}^n$.

In section 5 we prove Theorem 5.2, which is just part (2) of the Contraction Theorem. This relies on a characterization of the "thin parts" of Teichmüller space as approximately isometric to products of metric spaces, for which a contraction property cannot hold.

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1. Preliminaries

We begin with a brief summary of results and notation. Let S be a surface of finite genus with finitely many punctures, and let $\mathcal{T}(S)$ be the Teichmüller space of conformal structures of finite type on S, where two structures are considered equivalent if there is a conformal isomorphism of one to the other, which is isotopic to the identity on S. By "finite type" we mean that each puncture has a neighborhood conformally equivalent to a punctured disk.

$\mathcal{T}(S)$ has a natural topology, and is homeomorphic to a finite-dimensional Euclidean space. The *Teichmüller distance* between two points $\sigma, \tau \in \mathcal{T}(S)$ is defined as

$$d(\sigma,\tau) = \tfrac{1}{2}\log K(\sigma,\tau),$$

where $K(\sigma,\tau) = K(\tau,\sigma)$ is the smallest possible quasi-conformal dilatation of a homeomorphism from (S,σ) to (S,τ) isotopic to the identity.

Let α be a homotopy class of simple closed curves in S. Given $\sigma \in \mathcal{T}(S)$, one may define the *extremal length* of α in σ, as the inverse of the conformal modulus of the thickest embedded annulus homotopic to α in S:

$$E_\sigma(\alpha) = \inf_{A\sim\alpha} \frac{1}{Mod(A)}.$$

This number is positive provided α is non-peripheral – that is, not deformable into a puncture. We will make heavy use of Kerckhoff's theorem (see [7]), which states that

$$K(\sigma,\tau) = \sup_\alpha \frac{E_\tau(\alpha)}{E_\sigma(\alpha)}. \tag{1.1}$$

Kerckhoff in fact obtains this supremum as a maximum, by using a completion of the space of homotopy classes of non-peripheral simple closed curves known as the space of "measured foliations with compact support", $\mathcal{MF}(S)$ (see [3, 9]). Elements of $\mathcal{MF}(S)$ are equivalence classes of foliations of S with saddle singularities (negative index) in the interior, and singularities of index at most 1/2 at the punctures, equipped with transverse measures. The equivalence is via isotopy, and saddle-collapsing (Whitehead) moves. $\mathcal{MF}(S)$ carries a natural topology, and is homeomorphic to a Euclidean space of the same dimension as $\mathcal{T}(S)$. Multiplication of the measure by positive scalars gives a ray structure to $\mathcal{MF}(S)$, and the quotient of $\mathcal{MF}(S)$ by this multiplication is the projectivized space $\mathcal{PMF}(S)$, which is a sphere.

A homotopy class of simple closed curves, equipped with a real number giving the measure, is represented in $\mathcal{MF}(S)$ as a foliation whose non-singular leaves are in the homotopy class and fit together in a cylinder whose height is the measure. These special foliations are dense in $\mathcal{MF}(S)$, and Kerckhoff shows that the extremal length function extends to a function $E_\sigma(\lambda)$ which is continuous in both σ and λ, and scales quadratically:

$$E_\sigma(a\lambda) = a^2 E_\sigma(\lambda).$$

Moreover, the supremum in (1.1) is realized by a unique projective class in $\mathcal{PMF}(S)$.

Fixing $\sigma \in \mathcal{T}(S)$, let $QD(S,\sigma)$ denote the space of integrable holomorphic quadratic differentials on (S,σ) (see [4]). This space (minus the zero differential) may be identified with $\mathcal{MF}(S)$ by associating to a quadratic differential Φ its horizontal foliation, which we call Φ_h. Hubbard-Masur [6] showed that this correspondence is a homeomorphism. From [7] we also have the equality

$$E_\sigma(\Phi_h) = ||\Phi|| \tag{1.2}$$

where $||\Phi|| = \int_S |\Phi|$. We note also that the vertical foliation Φ_v, which is the orthogonal foliation to Φ_h, is just $(-\Phi)_h$.

Geodesics in $\mathcal{T}(S)$ are determined by holomorphic quadratic differentials (and hence measured foliations) as follows: Fix $\sigma \in \mathcal{T}(S)$ and $\Phi \in QD(S,\sigma)$. The family $\{\sigma_t\}_{t\geq 0}$ of conformal structures obtained by contracting along the leaves of Φ_h by e^t and expanding along Φ_v by e^{-t}, forms a geodesic ray in the Teichmüller metric, where t gives arclength. Geodesics exist and are unique between any two points in $\mathcal{T}(S)$. We also have

$$E_{\sigma_t}(\Phi_h) = e^{-2t} E_\sigma(\Phi_h), \tag{1.3}$$

$$E_{\sigma_t}(\Phi_v) = e^{2t} E_\sigma(\Phi_v). \tag{1.4}$$

Our last group of definitions has to do with intersection number. The standard geometric (unoriented) intersection number $i(\cdot,\cdot)$ between two homotopy classes of simple closed curves may be extended to a continuous function on $\mathcal{MF}(S)\times\mathcal{MF}(S)$, homogenous in both arguments. One may best visualize this in the case when one of the arguments is a simple closed curve and the other is any measured foliation. The intersection number is the smallest possible transverse measure deposited on a homotopic representative of the simple closed curve.

We say a foliation λ is *complete* if for any other foliation μ, $i(\lambda,\mu) = 0$ implies that the underlying leaf structures of λ and μ are the same (up to isotopy and Whitehead moves).

Let L denote a geodesic segment, ray or line in $\mathcal{T}(S)$. We say that L is ϵ-*precompact*, for $\epsilon > 0$, if for all $\tau \in L$ and all non-peripheral simple closed curves γ in S, $E_\tau(\gamma) \geq \epsilon$. Equivalently, let $\mathcal{M}(S)$ denote the moduli space of S, which is the quotient of $\mathcal{T}(S)$ by the induced action of the group of homeomorphisms of S. A geodesic is precompact exactly when its image in $\mathcal{M}(S)$ is contained in a compact set.

We will have need of this observation, which appears in [10] and again in [14]:

Lemma 1.1. *If $\Phi_h \in \mathcal{MF}(S)$ determines an ϵ-precompact Teichmüller ray, then Φ_h is complete.*

2. Two optimization problems

In this section we are going to present the basic line of argument in a mildly abstracted form. Let $\mathcal{F}$ denote a family of positive functions with domain a closed interval $I \subset \mathbf{R}$ (possibly $I = \mathbf{R}$ or I is a half-line). Consider the "dual" pair of optimizations

$$\sup_{f\in\mathcal{F}} \inf_{t\in I} f(t) \tag{2.1}$$

and

$$\inf_{t\in I} \sup_{f\in\mathcal{F}} f(t). \tag{2.2}$$

We will study the relation between solutions of these for families satisfying particular constraints. Assume that the functions in $\mathcal{F}$ are continuous, and that $\mathcal{F}$ is compact in the compact-open topology. Say that (g, s) is a solution pair for problem (2.1) if $g(s) = \inf g$ (or if $s = \pm\infty$ and $\inf g$ is realized by points in I tending to s) and $\inf g = \sup_f \inf f$. Let $\mathrm{Supinf}(\mathcal{F})$ denote the set of solution pairs. Similarly define $\mathrm{Infsup}(\mathcal{F})$ to be the set of solution pairs of problem (2.2). Note that under our assumptions both sets are non-empty.

Define $m(f) \subset I \cup \{\pm\infty\}$ to be the set where f takes on its infimum. From now on we allow $f(\infty)$ and $f(-\infty)$ to denote the appropriate limit, where defined.

Suppose further that $\mathcal{F}$ satisfies the following conditions, for fixed constants $\rho, a_0, a_1, \epsilon_0, D_0, c_0 > 0$.

F1. (Exponential growth) For each f there are constants $b_-, b_+ \geq 0$ such that $\forall t$

$$a_0(b_-\rho^{-t} + b_+\rho^t) \leq f(t) \leq a_1(b_-\rho^{-t} + b_+\rho^t).$$

F2. (Definite supinf) $\sup_f \inf f \geq \epsilon_0$.

F3. (Non-interference) For any $f, g \in \mathcal{F}$ and $t \in m(f), \quad s \in m(g)$,

$$|s - t| \geq D_0 \implies f(t)g(t) \leq c_0.$$

Remarks. It will be easy to see from the rest of this section that the growth condition (F1) could be loosened in various ways; but this form of it will suffice for us. Condition (F3) may seem mysterious, and in the next section we will relate it to intersection-number inequalities for measured foliations.

Proposition 2.1. *Conditions (F1-3) on a family $\mathcal{F}$ imply the following restrictions on the solutions to the two optimization problems, where $d_0, d_1, d_2, d_3 > 0$ depend only on the constants appearing in (F1-3).*

(1) *For all $f \in \mathcal{F}$, $\operatorname{diam} m(f) \leq d_0$.*
(2) *If $(f,t), (f',t') \in \operatorname{Supinf}(\mathcal{F})$ then $|t-t'| \leq d_1$.*
(3) *If $(f,t) \in \operatorname{Supinf}(\mathcal{F})$ then $\sup_{g\in\mathcal{F}} g(t) \leq d_2 f(t)$.*
(4) *If $(f,t) \in \operatorname{Supinf}(\mathcal{F})$ and $(g,s) \in \operatorname{Infsup}(\mathcal{F})$ then $|s-t| \leq d_3$.*

In particular, note that (2) and (4) imply that t-coordinates of solutions of the Supinf and Infsup problems lie in an interval of bounded diameter. Part (3) implies that $\sup\inf f(t)$ gives an upper bound for $\inf\sup f(t)$.

Proof. Part (1) follows obviously from the growth condition (F1). Note that we adopt the convention that $\operatorname{diam}\{\infty\} = \operatorname{diam}\{-\infty\} = 0$.

Let $M = \sup_{f\in\mathcal{F}} \inf_{t\in I} f(t) \geq \epsilon_0$. Continuing to part (2), let (f,t) and (f',t') be in $\operatorname{Supinf}(\mathcal{F})$, so that $f(t) = f'(t') = M$, and $t \in m(f)$ and $t' \in m(f')$. Note we may assume $t, t' \neq \pm\infty$ since an infimum occurring at $\pm\infty$ is always 0, again by (F1). Furthermore, (F1) implies that, for a fixed $c > 0$,

$$f'(t) \geq c\rho^{|t-t'|} f'(t').$$

Now if $|t-t'| \geq D_0$, condition (F3) implies that

$$f(t)f'(t) \leq c_0$$

and we conclude that

$$cM^2\rho^{|t-t'|} \leq c_0.$$

This gives a uniform upper bound on $|t-t'|$.

For part (3), again take $(f,t) \in \operatorname{Supinf}(\mathcal{F})$ and let $g \in \mathcal{F}$ be any other function. We must bound $g(t)$ in terms of $f(t) = M$. Choose $s \in m(g)$. If $|s-t| \geq D_0$ (including the possibility that $s = \pm\infty$), we have again $f(t)g(t) \leq c_0$, which bounds $g(t)$ by $c_0/f(t)$. In particular since $f(t) \geq \epsilon_0$ this implies $g(t) \leq (c_0/\epsilon_0^2) f(t)$. On the other hand if $|s-t| \leq D_0$ then by (F1) $g(t) \leq c'\rho^{D_0} g(s) \leq c'\rho^{D_0} f(t)$, and again we have the desired bound.

For part (4), let $(f,t) \in \operatorname{Supinf}(\mathcal{F})$ and $(g,s) \in \operatorname{Infsup}(\mathcal{F})$. By definition of $\operatorname{Infsup}(\mathcal{F})$, we have

$$g(s) \leq \sup_{h\in\mathcal{F}} h(t)$$

and by part (3)

$$\sup_{h\in\mathcal{F}} h(t) \leq d_2 f(t).$$

On the other hand (again by definition, and by (F1))

$$g(s) \geq f(s) \geq c\rho^{|s-t|} f(t).$$

We conclude that $c\rho^{|s-t|} \leq d_2$, and obtain the desired bound on $|s-t|$. $\square$

Now we consider the way in which solutions of problems (2.1,2.2) vary when the family of functions is varied by rescaling. Let $u : \mathcal{F} \to \mathbf{R}_+$ be any positive function and let $u\mathcal{F}$ denote the family of functions $\{u(f)f : f \in \mathcal{F}\}$. The effect of this change on the Infsup problem is not readily apparent, but the Supinf problem is less affected since the position of the minima of $u(f)f$ and f are the same.

Proposition 2.2. *If $\mathcal{F}$ satisfies (F1-3) and $u : \mathcal{F} \to \mathbf{R}_+$ is a rescaling function, such that*

$$\frac{\sup u}{\inf u} \leq Q(\sup_f \inf_t f(t))^2,$$

then if $(f,t) \in \mathrm{Supinf}(\mathcal{F})$ and $(u(g)g, s) \in \mathrm{Supinf}(u\mathcal{F})$, we have

$$|s - t| \leq \Delta,$$

where Δ depends only on Q and the constants in (F1-3).

Proof. If $|s-t| < D_0$ we are done, so suppose $|s-t| \geq D_0$. In this case (F3) implies $f(t)g(t) \leq c_0$. The growth condition (F1) implies $g(t) \geq c\rho^{|s-t|}g(s)$. Finally, by definition,

$$\begin{aligned} u(g)g(s) &= \sup_{h\in\mathcal{F}} \inf_{t\in I} u(h)h(t) \\ &\geq \left(\inf_{h\in\mathcal{F}} u(h)\right) \sup_{h\in\mathcal{F}} \inf_{t\in I} h(t) \\ &= \left(\inf_{h\in\mathcal{F}} u(h)\right) f(t). \end{aligned}$$

Putting these together we have

$$c\rho^{|s-t|} \frac{\inf u}{u(g)} f(t)^2 \leq c_0$$

and by the assumption on $\sup u / \inf u$ we have

$$|s - t| \leq \log_\rho \frac{c_0 Q}{c}$$

which is the desired bound. $\square$

3. Proof of the contraction property

In this section we will prove part (1) of the Contraction Theorem, using the results of section 2. The family of functions for which we consider the Supinf and Infsup problems will be ratios of extremal lengths, obtained as follows.

Let L be an ϵ-precompact geodesic segment, ray, or line and let $\overline{L}$ denote the complete geodesic line containing L. Choose an arbitrary origin and orientation on $\overline{L}$ and let the parameter t denote signed distance from the origin. Then L corresponds to an interval $I = [x,y]$, where $-\infty \leq x < y \leq \infty$. Denote by $\overline{L}(t)$ the conformal structure on S given by the point parametrized by t; or by $L(t)$ if $t \in [x,y]$.

For each $\alpha \in \mathcal{MF}(S)$ and $t \in \mathbf{R}$, let $E_t(\alpha)$ be short for the extremal length $E_{\overline{L}(t)}(\alpha)$. Now fixing a point $\sigma \in \mathcal{T}(S)$, we obtain the ratio

$$R_{\sigma,\alpha}(t) = \frac{E_t(\alpha)}{E_\sigma(\alpha)}.$$

Note that this quantity is invariant under scaling of α, so it only depends on the class $[\alpha] \in \mathcal{PMF}(S)$. Thus the family of functions

$$\mathcal{R}_\sigma = \{R_{\sigma,\alpha}\}_{[\alpha]\in\mathcal{PMF}(S)}$$

is compact.

The problem of computing $\pi_L(\sigma)$ now reduces to solving Infsup($\mathcal{R}$), by virtue of Kerckhoff's theorem (1.1). Let us therefore show that $\mathcal{R}_\sigma$ satisfies conditions (F1-3).

Condition F1. Let Φ be the holomorphic quadratic differential on $\overline{L}(0)$ that determines the geodesic. The extremal lengths of the horizontal and vertical foliations of Φ vary exponentially with t according to (1.3) and (1.4). We also have $E_0(\Phi_h) = E_0(\Phi_v) = ||\Phi||$ by (1.2), and we may normalize so that $||\Phi|| = 1$.

We will need to know (see e.g. Gardiner-Masur [5]) that Φ_h and Φ_v *fill up* S, in the sense that every non-zero element of $\mathcal{MF}(S)$ has positive intersection with at least one of them. Thus we may define the strictly positive function

$$e_t(\alpha) = \tfrac{1}{2}\left(\frac{i(\alpha,\Phi_h)^2}{E_t(\Phi_h)} + \frac{i(\alpha,\Phi_v)^2}{E_t(\Phi_v)}\right) \tag{3.1}$$

which, by virtue of (1.3,1.4) can be written

$$e_t(\alpha) = \frac{i(\alpha,\Phi_h)^2}{2}e^{2t} + \frac{i(\alpha,\Phi_v)^2}{2}e^{-2t}. \tag{3.2}$$

This will serve as an approximation for $E_t(\alpha)$, by the following:

Lemma 3.1. *There is a fixed $c_0 = c_0(\epsilon, \chi(S))$ such that if L is ϵ-precompact then for any $\alpha \in \mathcal{MF}(S)$ and $t \in [x,y]$,*

$$e_t(\alpha) \le E_t(\alpha) \le c_0 e_t(\alpha).$$

Proof. The left side actually does not depend on the precompactness of L – it follows directly from lemma 3.2 (stated shortly), applied to α and each of Φ_h and Φ_v.

The right side follows from a compactness argument. Let Φ^t denote the image of Φ in $L(t)$ under the Teichmuller map from $\overline{L}(0)$, scaled so that $||\Phi^t|| = 1$. Then as measured foliations we have $\Phi^t_h = \rho^t\Phi_h$ and $\Phi^t_v = \rho^{-t}\Phi_v$. Thus, by (3.2),

$$e_t(\alpha) = \tfrac{1}{2}\left(i(\alpha,\Phi^t_h)^2 + i(\alpha,\Phi^t_v)^2\right).$$

The ratio $E_t(\alpha)/e_t(\alpha)$ is therefore a continuous positive function of the projectivized measured foliation $[\alpha] \in \mathcal{PMF}(S)$ and the pair $(L(t), \Phi^t)$ in the total space of Riemann surfaces equipped with holomorphic quadratic differentials. Fixing the Riemann surface $L(t)$, the space $\mathcal{PMF}(S) \times PQD(L(t))$ is compact; and the function is invariant under the action of the mapping class group. Further, the image of $L(t)$ in the moduli space is restricted to a compact set by our definition of pre-compactness. It follows that $E_t(\alpha)/e_t(\alpha)$ is bounded. $\square$

This lemma and equation (3.2) for $e_t(\alpha)$ establish condition (F1).

Condition (F2). To get a lower bound on $\sup_{R\in\mathcal{R}} \inf R$, recall first that there is a constant ℓ_0 depending on $\chi(S)$ such that the shortest non-peripheral simple closed curve α on S has $E_\sigma(\alpha) \le \ell_0$. Since L is ϵ-precompact, $R_{\sigma,\alpha}(t) \ge \epsilon/\ell_0$ for all $t \in [x,y]$. This gives a uniform lower bound for $\inf R_{\sigma,\alpha}$, and thus on the sup over all functions in $\mathcal{R}_\sigma$.

Condition (F3). This condition is really the central geometric idea of the paper. We must show that if the minima of two functions $R_{\sigma,\alpha}$ and $R_{\sigma,\beta}$ are sufficiently distant, then the product of the two is bounded above at either minimum.

Note first that the minima $m(R_{\sigma,\alpha})$ lie in a uniformly bounded neighborhood of the minimum s_α of $e_t(\alpha)$. This is a direct consequence of lemma 3.1.

The minimum point $s_\alpha \in [x,y]$ can be calculated directly from (3.2); in particular it takes on the value $+\infty$ (resp. $-\infty$) exactly when $y = \infty$ and $i(\alpha,\Phi_h) = 0$ (resp. $x = -\infty$ and $i(\alpha,\Phi_v) = 0$).

We will control the product of length ratios by the intermediary of intersection numbers. The following basic lemma appears for example in [15]:

Lemma 3.2. *If α, β are any two measured foliations on any Riemann surface X, then*

$$E_X(\alpha)E_X(\beta) \ge i(\alpha,\beta)^2.$$

The following lemma obtains bounds in the opposite direction. Together they will allow us to bound the appropriate products.

Lemma 3.3. *There are constants D_1 and c_1, depending only on ϵ and $\chi(S)$, such that for any $\alpha, \beta \in \mathcal{MF}(S)$,*

$$|s_\alpha - s_\beta| \ge D_1 \quad \Longrightarrow \quad i(\alpha,\beta)^2 \ge c_1 E_{s_\alpha}(\alpha)E_{s_\alpha}(\beta).$$

Remark. This lemma is similar to (and was motivated by) lemma 3.2 from [15], in which the role of L is played by a hyperbolic 3-manifold N homeomorphic to $S\times\mathbf{R}$, and the role of the minimum point s_α is played by the geodesic representative of α in N.

Proof. Denote by $I_t(\alpha,\beta)$ the normalized intersection number $i(\alpha,\beta)^2/E_t(\alpha)E_t(\beta)$. Suppose the lemma is false; then there exists a sequence of ϵ-precompact geodesics L_i parametrized by $[x_i,y_i]$, with foliations α_i, β_i such that $|s_{\alpha_i} - s_{\beta_i}| \to \infty$ while $I_{s_\alpha}(\alpha,\beta) \to 0$.

Adjusting if necessary by automorphisms of S, we may assume that the points $L_i(s_{\alpha_i})$ lie in a fixed compact subset of a fundamental domain of the action of the mapping class group on $\mathcal{T}(S)$. Further, the quantity I_t is invariant under scaling of the measures, and so is defined on the compact space $\mathcal{PMF}(S)\times\mathcal{PMF}(S)$. Thus we may take a convergent subsequence and obtain a limit example (L,α,β) in which

$I_{s_\alpha}(\alpha,\beta) = 0$, and $|s_\beta - s_\alpha| = \infty$. Without loss of generality we assume $s_\beta = +\infty$, so that L must contain an ϵ-precompact ray $[s_\alpha, \infty)$.

However, by lemma 1.1, this implies that the foliation Φ_h is complete. Since $s_\beta = +\infty$, we have $i(\beta, \Phi_h) = 0$, so that β must be topologically equivalent to Φ_h. On the other hand, the fact that $s_\alpha \neq +\infty$ implies that $i(\alpha, \Phi_h) > 0$, and therefore that $i(\alpha, \beta) > 0$, a contradiction. □

Combining lemma 3.3 with lemma 3.2 applied to $X = (S, \sigma)$, we immediately obtain

$$|s_\alpha - s_\beta| \geq D_1 \implies R_{\sigma,\alpha}(s_\alpha) R_{\sigma,\beta}(s_\alpha) \leq 1/c_1. \tag{3.3}$$

Condition (F3) is an immediate consequence.

Completion of the proof. The projection $\pi_L(\sigma)$ consists exactly of those points $L(t)$ where, for some α, $(R_{\sigma,\alpha}, t) \in \text{Infsup}(\mathcal{R}_\sigma)$, and furthermore

$$\inf_t \sup_\alpha R_{\sigma,\alpha}(t) = e^{2d(\sigma,L)}$$

By Kerckhoff's theorem.

Having established conditions (F1-3) for the family $\mathcal{R}_\sigma$, Proposition 2.1 applies to show the following: By part (2), the set of t-values appearing in $\text{Supinf}(\mathcal{R}_\sigma)$ has a bounded diameter. By part (4), the t-values appearing in $\text{Infsup}(\mathcal{R}_\sigma)$ are a bounded distance from those appearing in $\text{Supinf}(\mathcal{R}_\sigma)$. Together these imply a uniform bound on $\text{diam}(\pi_L(\sigma))$. Furthermore, by part (3), if $(R_{\sigma,\beta}, s) \in \text{Supinf}(\mathcal{R}_\sigma)$ then

$$\begin{aligned} d_2 R_{\sigma,\beta}(s) &\geq \sup_\alpha R_{\sigma,\alpha}(s) \\ &\geq \inf_t \sup_\alpha R_{\sigma,\alpha}(t) \\ &= e^{2d(\sigma,L)}. \end{aligned} \tag{3.4}$$

Now consider a second point $\tau \in \mathcal{T}(S)$, with $d(\sigma,\tau) \leq d(\sigma, L)$. The family $\mathcal{R}_\tau$ is a rescaling of $\mathcal{R}_\sigma$ by the function $u(\alpha) = E_\sigma(\alpha)/E_\tau(\alpha)$. It follows from Kerckhoff's theorem that

$$\sup_\alpha u(\alpha) = e^{2d(\sigma,\tau)} = \frac{1}{\inf_\alpha u(\alpha)}$$

(where the second equality follows from the symmetry of $d(\sigma,\tau)$.) Thus we have

$$\frac{\sup u}{\inf u} \leq e^{4d(\sigma,\tau)} \leq e^{4d(\sigma,L)}.$$

Now applying (3.4), we obtain

$$\frac{\sup u}{\inf u} \leq d_2^2 \left(\sup_\alpha \inf_t R_{\sigma,\alpha}(t) \right)^2.$$

Therefore we may apply proposition 2.2 to compare $\text{Supinf}(\mathcal{R}_\tau)$ with $\text{Supinf}(\mathcal{R}_\sigma)$: the t values occurring in each differ by at most Δ, which depends only on the previous constants.

The conclusion, since solutions of Supinf and Infsup are within bounded distance, is a uniform bound on $\operatorname{diam}(\pi_L(\sigma)\cup\pi_L(\tau))$. Applying this to all $\tau\in\mathcal{N}_{d(\sigma,L)}(\sigma)$, we obtain the desired bound on $\operatorname{diam}(\pi_L(\mathcal{N}_{d(\sigma,L)}(\sigma)))$. This concludes the proof of the contraction property.

4. Applications

We shall need the following immediate consequence of the contraction theorem, which states that paths a definite distance from a precompact geodesic have projections which are shorter by a definite factor.

Corollary 4.1. *Let L be an ϵ-precompact geodesic in $\mathcal{T}(S)$ and let b be the constant given in the Contraction Theorem. If $R>0$ and points $x,y\in\mathcal{T}(S)$ are connected by a path of length T which remains outside an R-neighborhood of L, then*

$$\operatorname{diam}(\pi_L(x)\cup\pi_L(y))\le\frac{b}{R}T+b. \tag{4.1}$$

Furthermore for any $x,y\in\mathcal{T}(S)$,

$$\operatorname{diam}(\pi_L(x)\cup\pi_L(y))\le d(x,y)+4b. \tag{4.2}$$

Proof. Dividing the path from x to y into at most $1+T/R$ pieces of size R or less, and applying the Contraction Theorem to each one, yields inequality (4.1). To obtain (4.2), let $R=b$ and consider the geodesic G connecting x to y. If G stays outside of an R-neighborhood of L then apply (4.1). If not, let x',y' be the closest points in G to x,y (respectively) such that $d(x',L)=d(y',L)=R$. The triangle inequality together with (4.1) yields

$$\operatorname{diam}(\pi_L(x)\cup\pi_L(y))\le d(x,x')+2R+d(x',y')+d(y,y')+2b$$

which suffices. $\square$

Define a (K,δ)*-quasi-geodesic* in $\mathcal{T}(S)$ as a path $\Gamma:[a,b]\to\mathcal{T}(S)$, parametrized by arclength, such that for any $s,t\in[a,b]$,

$$|s-t|\le Kd(\Gamma(s),\Gamma(t))+\delta.$$

Theorem 4.2. *Let Γ be a (K,δ)-quasi-geodesic path in $\mathcal{T}(S)$, whose endpoints are connected by an ϵ-precompact Teichmüller geodesic L. Then Γ remains in a $B(K,\delta,\epsilon)$-neighborhood of L.*

Proof. The idea of the argument is standard: any segment of Γ which is outside a sufficiently large neighborhood of L is, because of the contractive properties of π_L, much less efficient than the path obtained by going back to L and moving along the projection. Thus large excursions from L violate the quasi-geodesic property.

Fix $R=2Kb$. Let $[s,t]$ be a maximal interval for which $\Gamma((s,t))$ is outside an R-neighborhood of L. Applying corollary 4.1, we have

$$\operatorname{diam}(\pi_L(\Gamma(s))\cup\pi_L(\Gamma(t)))\le\frac{b}{R}|s-t|+b.$$

The maximality of $[s,t]$ implies that $d(\Gamma(s), \pi_L(\Gamma(s))) = R$, and similarly for t. Thus

$$d(\Gamma(s), \Gamma(t)) \leq 2R + \frac{b}{R}|s-t| + b$$

On the other hand, the quasi-geodesic property of Γ gives a lower bound on the distance $d(\Gamma(s), \Gamma(t))$, and together we obtain (using the definition of R), a bound $|s-t| \leq 4KR + 2Kb + 2\delta$. It follows that Γ cannot exit a $(2K+1)R + Kb + \delta$ neighborhood of L. □

Let α be a measured foliation in $\mathcal{MF}(S)$. The set

$$Thin(\alpha, \delta) = \{\sigma \in \mathcal{T}(S) : E_\sigma(\alpha) \leq \delta\}$$

is somewhat analogous to a horoball in hyperbolic space. As in hyperbolic space, it has infinite diameter, but its projection to a precompact geodesic is nevertheless bounded, bar one exceptional case:

Theorem 4.3. *Given ϵ, δ there is a constant B such that, if L is a complete ϵ-precompact geodesic and $\alpha \in \mathcal{MF}(S)$, then*

$$\operatorname{diam}(\pi_L(Thin(\alpha, \delta))) \leq B + \operatorname{diam}(L \cap Thin(\alpha, \delta)).$$

Remarks: (1) A special case of this is when α is a non-peripheral simple closed curve, with weight 1. Then $L \cap Thin(\alpha, \delta)$ is uniformly bounded for all ϵ-precompact L, so we have a uniform bound on $\operatorname{diam}(\pi_L(Thin(\alpha, \delta)))$. (2) If a complete geodesic L is determined by a quadratic differential Φ, let us call $[\Phi_h]$ and $[\Phi_v]$ in $\mathcal{PMF}(S)$ the *endpoints at infinity* for L. Another special case occurs when $[\alpha]$ is equal in $\mathcal{PMF}(S)$ to an endpoint, and then both sides of the inequality are infinite.

Proof. In view of remark (2), let us assume that $[\alpha] \neq [\Phi_h], [\Phi_v]$ in $\mathcal{PMF}(S)$. Masur proved in [11] that, if L is ϵ-precompact, then its endpoints at infinity are *uniquely ergodic* – they support a unique projective class of transverse measures. Thus the assumption that $[\alpha]$ is not an endpoint implies that α is topologically distinct from both foliations, and therefore (by lemma 1.1) it intersects both of them non-trivially. It follows that $E_t(\alpha)$ has a positive infimum, E_0, obtained at some $t_\alpha \neq \pm\infty$.

Let ϵ_0, c_0, D_0 be the constants for which conditions (F2,F3) hold for the families $\mathcal{R}_\sigma$, as in section 3. Fix $\delta_0 = \frac{1}{2}E_0\epsilon_0/c_0$. We will show that $\pi_L(Thin(\alpha, \delta_0))$ lies in a bounded neighborhood of $L(t_\alpha)$.

Given $\sigma \in Thin(\alpha, \delta_0)$, let (λ, s) be a solution pair in $\operatorname{Supinf}(\mathcal{R}_\sigma)$. By Proposition 2.1, it will suffice to bound $|s - t_\alpha|$.

Suppose $|s - t_\alpha| \geq D_0$. Then by condition (F3) we have

$$R_{\sigma,\alpha}(t_\alpha) R_{\sigma,\lambda}(t_\alpha) \leq c_0.$$

On the other hand, $R_{\sigma,\alpha}(t_\alpha) \geq E_0/\delta_0$ by assumption, and $R_{\sigma,\lambda}(t_\alpha) \geq R_{\sigma,\lambda}(s) \geq \epsilon_0$ by condition (F2). Thus we obtain $E_0\epsilon_0/\delta_0 \leq c_0$, which contradicts the choice of δ_0. It follows that $|s - t_\alpha| < D_0$.

To prove the theorem for general $\delta > \delta_0$, we will first bound $\operatorname{diam}(\pi_L(Thin(\alpha,\delta)))$ above by a constant plus $\log \delta/\delta_0$, and then bound $\operatorname{diam}(L \cap Thin(\alpha,\delta))$ from below by a (different) constant plus $\log\delta/\delta_0$.

We first observe that $Thin(\alpha,\delta)$ lies in a neighborhood of $Thin(\alpha,\delta_0)$ of radius $\frac{1}{2}\log\delta/\delta_0$ (since one can move along a Teichmüller geodesic defined by Φ such that $\Phi_h = \alpha$, and apply (1.3)).

Applying corollary 4.1 (inequality 4.2) we conclude immediately that the projection $\pi_L(Thin(\alpha,\delta))$ lies in a $(B + \frac{1}{2}\log\delta/\delta_0)$-neighborhood of $\pi_L(Thin(\alpha,\delta_0))$. Applying the estimate of the previous paragraph this is in a $(C + \frac{1}{2}\log\delta/\delta_0)$-neighborhood of $L(t_\alpha)$, for an appropriate $C(\epsilon)$, so that we have the upper bound.

On the other hand, applying the growth condition, there is a constant c_2 for which $E_t(\alpha) \le c_2 E_0 \exp 2|t - t_\alpha|$. Thus, if

$$|t - t_\alpha| \le \tfrac{1}{2}\log\frac{\epsilon_0}{2c_0c_2}\frac{\delta}{\delta_0}$$

we can conclude (using the definition of δ_0) that $E_t(\alpha) \le \delta$. This bounds $\operatorname{diam}(L \cap Thin(\alpha,\delta))$ from below by a constant plus $\log\delta/\delta_0$. The statement of the theorem follows. $\square$

Our last application is a geometric property of pseudo-Anosov homeomorphisms. Let $f : S \to S$ be pseudo-Anosov, and let f_* denote the induced action on $\mathcal{T}(S)$ (see [3] or [1]). Bers showed in [2] that the infimum $\inf_{\sigma\in\mathcal{T}(S)} d(\sigma, f_*\sigma)$ of translation distance is achieved in $\mathcal{T}(S)$ on a geodesic, namely the axis of f_*, on which f_* acts by translation. The contraction theorem can be used to give lower bounds on how fast the translation distance grows as we consider points far away from the axis (this question was suggested by Feng Luo).

Theorem 4.4. *Let $f : S \to S$ be a pseudo-Anosov homeomorphism, and $L \subset \mathcal{T}(S)$ the axis of f_*. There exists $k_0 > 0$ such that, for all $x \in \mathcal{T}(S)$,*

$$d(x, f_*x) \ge k_0 d(x, L).$$

Remark: it is easy to see that $d(x, f_*x)$ is at most linear with $d(x, L)$, and so in an asymptotic sense the theorem gives maximal growth. However the constant k_0 is completely non-constructive.

Proof. Since the axis L is invariant by f_*, its image in the moduli space is a closed curve, so that L is precompact. Let b be the constant given by the Contraction Theorem for L.

For $x \in \mathcal{T}(S)$, let $R = d(x, L)$ and let $t = d(x, f_*(x))$. Let t_0 denote the translation distance of f_* on L, so that $t \ge t_0$, and $t = t_0$ only for $x \in L$.

For any $n > 0$ we can connect x to $f_*^n x$ by a chain of n geodesics of length t, which remains outside an $R - t/2$ neighborhood of L. By corollary 4.1 we have

$$\operatorname{diam}(\pi_L(x) \cup \pi_L(f_*^n x)) \le \frac{b}{R - t/2} nt + b.$$

(Note that we may assume that $R - t/2 > 0$, because otherwise we would have $t \geq 2R$, and we would be done.) It follows that

$$d(x, f_*^n x) \leq 2R + \frac{b}{R - t/2} nt + b.$$

On the other hand L is also the axis of f_*^n, so that $d(x, f_*^n x) \geq nt_0$. Letting n go to infinity, we conclude that

$$t_0 \leq b \frac{t}{R - t/2}.$$

Solving for t, we obtain the desired inequality. □

5. The case of non-precompact geodesics

In this section we prove part (2) of the Contraction Theorem. Let L be a geodesic which is not precompact. Then L has segments which enter, arbitrarily deeply, into the "thin parts" $Thin(\alpha, \delta)$ defined in the previous section, where α is a simple closed curve with weight 1.

For small δ, the geometry of $Thin(\alpha, \delta)$ is quite far from negatively curved. In [13] we showed that it is approximated by a product space in the following sense.

Let X_α denote $\mathbf{H}^2 \times \mathcal{T}(S \backslash \alpha)$, endowed with the *sup metric* $d_X = \max(d_{\mathbf{H}^2}, d_{\mathcal{T}(S\backslash\alpha)})$ of the metrics on the factors. There is a natural homeomorphism $\Pi : \mathcal{T}(S) \to X_\alpha$, defined using Fenchel-Nielsen coordinates, so that the coordinates in the hyperbolic plane $\mathbf{H}^2$ encode the length and twist parameters associated to α. In particular, the y coordinate in $\mathbf{H}^2$ (in the upper half-plane model) for $\Pi(\sigma)$ is $1/\ell_\sigma(\alpha)$ where ℓ_σ is hyperbolic length, and it follows easily that $\Pi(Thin(\alpha, \delta))$ is contained in $\{y > h_1\} \times \mathcal{T}(S \backslash \alpha)$ and contains $\{y > h_2\} \times \mathcal{T}(S \backslash \alpha)$, where h_1, h_2 are approximately inversely proportional to δ. The main theorem of [13] says that

Theorem 5.1. *(Theorem 6.1 of* [13]*) For sufficiently small δ, the map Π restricted to $Thin(\alpha, \delta)$ has bounded additive distortion; that is,*

$$|d_{\mathcal{T}(S)}(\sigma, \tau) - d_X(\Pi(\sigma), \Pi(\tau))| \leq c$$

for $\sigma, \tau \in Thin(\alpha, \delta)$, where c depends only on δ and the topological type of S.

This product geometry suffices to give part (2) of the Contraction Theorem, namely the failure of part (1) in the non-precompact case:

Theorem 5.2. *If a Teichmüller geodesic L is not precompact, then there is no choice of constant b for which the contraction property holds for π_L.*

Remark: This theorem does not rule out the diameter bound (0.1) for images of single points which is a consequence of the contraction property. In fact it is not ruled out that $\pi_L(\sigma)$ is in all cases a single point; resolving this question seems to require finer techniques than we use here.

Proof. Using the product structure we can show that large segments of L can be replaced by quasi-geodesics that stray arbitrarily far from L. This will contradict theorem 4.2.

Since L is not precompact, for arbitrarily small $\delta > 0$ and large $T > 0$ there exists a segment in L of length T contained in $Thin(\alpha, \delta)$ for some simple closed curve α. (We are using here the fact that the Teichmüller metric on the moduli space is complete, or equivalently that the distance from the boundary of $Thin(\alpha, \delta)$ to $Thin(\alpha, \delta')$ goes to infinity as $\delta/\delta' \to \infty$ – in fact it is $\frac{1}{2}\log(\delta/\delta')$.)

Let σ_1, σ_2 be the endpoints of such a segment, and let $\Pi(\sigma_i) = q_i$ be their images in the product X_α, so that $|d(q_1, q_2) - T| < c$. Using the sup metric on X_α it is easy to construct two $(2,0)$-quasi-geodesics m, m' connecting q_1 to q_2, so that neither quasi-geodesic lies in a $(T - c)$-neighborhood of the other. (Restricting to an $\mathbf{R}^2$ slice, construct an appropriate quadrilateral whose diagonal is the line $[q_1, q_2]$, and let m, m' be the two paths around the perimeter.) Furthermore m, m' can be made to remain in $\Pi(Thin(\alpha, \delta'))$ for δ' slightly bigger than δ. Then applying Π^{-1}, we obtain two $(2, c)$-quasi-geodesics connecting σ_1 to σ_2, neither of which lies in a $(T - 2c)$-neighborhood of the other. It follows that at least one of the quasi-geodesics cannot be in a bounded neighborhood of L, as T is chosen arbitrarily large. This violates the stability of quasi-geodesics proved in Theorem 4.2, and therefore π_L cannot have the contraction property. $\square$

Remark: For a product space with the sup metric the stability of quasi-geodesics is actually violated in a stronger way than used above. See the last section of [13] for a brief discussion of this.

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