Kleinian Groups and Hyperbolic 3-Manifolds Y. Komori, V. Markovic & C. Series (Eds.)
Lond. Math. Soc. Lec. Notes **299**, 3–40 Cambridge Univ. Press, 2003

Combinatorial and geometrical aspects of hyperbolic 3-manifolds

Yair N. Minsky[1]

1. Introduction

This is the edited and revised form of handwritten notes that were distributed with the lectures that I gave at the conference in Warwick on September 11–15 of 2001[2]. The goal of the lectures was to expose some recent work [Min03] on the structure of ends of hyperbolic 3-manifolds, which is part of a program to solve Thurston's Ending Lamination Conjecture (the conclusion of the program, which is joint work with J. Brock and R. Canary, will appear in [BCM]). In the interests of simplicity and the ability to get to the heart of the matter, the notes are quite informal in their treatment of background material, and the main results are often stated in special cases, with detailed examples taking the place of proofs. Thus it is hoped that the reader will be able to extract the main ideas with a minimal investment of effort, and in the event he or she is still interested, can obtain the details in [Min03], which will appear later on.

I would like to thank the organizers of the conference for inviting me and giving me the opportunity to talk for what must have seemed like a very long time.

1.1. Object of Study

If the interior N of a compact 3-manifold $\overline{N}$ admits a complete infinite-volume hyperbolic structure, then there is a multidimensional *deformation space* of such structures. The study of this space goes back to Poincaré and Klein, but the modern theory began with Ahlfors-Bers in the 1960's and received the perspective that we will focus on from Thurston and others in the late 70's. The deformation theory depends deeply on an understanding of the geometry of the *ends* of N (in the sense of Freudenthal [Fre42]), which one can think of as small neighborhoods of the boundary components of $\overline{N}$.

[1]Based on work partially supported by NSF grant DMS-9971596.
[2]The terrible events in New York that coincided with the beginning of this conference overshadow its subject matter in significance, and yet those same events demand of us to continue with our ordinary work.

The interior of the deformation space, as studied by Ahlfors, Bers [AB60, Ber60, Ber70b], Kra [Kra72], Marden [MMa79, Mar74], Maskit [Mak75], and Sullivan [Sul85], can be parametrized using the Teichmüller space of $\partial\overline{N}$ – that is, by choosing a "conformal structure at infinity" for each (non-toroidal) boundary component of $\overline{N}$. (See also [KS93] and [BO01] for other approaches to the study of the interior). The boundary contains manifolds with parabolic cusps [Mak70, McM91], and more generally, with *geometrically infinite ends* [Ber70a, Gre66, Thu79]. The Teichmüller parameter is replaced by Thurston's *ending laminations* for such ends. Thurston conjectured [Thu82] that these invariants are sufficient to determine the geometry of N uniquely – this is known as the Ending Lamination Conjecture (see also [Abi88] for a survey).

In these notes we will consider the special case of *Kleinian surface groups*, for which $\pi_1(N)$ is isomorphic to $\pi_1(S)$ for a surface S. This case suffices for describing the ends of general N, provided $\partial\overline{N}$ is incompressible. (In the compressible case the deeper question of Marden's *tameness conjecture* comes in, and this is beyond the scope of our discussion. See Marden [Mar74] and Canary [Can93b].)

We will show how the the ending laminations, together with the combinatorial structure of the set of simple closed curves on a surface, allows us to build a *Lipschitz model* for the geometric structure of N, which in particular describes the thick-thin decomposition of N. These results, which are proven in detail in [Min03], will later be followed by *bilipschitz* estimates in Brock–Canary–Minsky [BCM], and these will suffice to prove Thurston's conjecture in the case of incompressible boundary.

1.2. Kleinian surface groups

From now on, let S be an oriented compact surface with $\chi(S) < 0$, and let

$$\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$$

be a discrete, faithful representation. If $\partial S \neq \emptyset$ we require $\rho(\gamma)$ to be parabolic for γ representing any boundary component. This is known as a (marked) Kleinian surface group. We name the quotient 3-manifold

$$N = N_\rho = \mathbb{H}^3/\rho(\pi_1(S)).$$

Periodic manifolds Before discussing the general situation let us consider a well-known and especially tractable example.

Let $\varphi : S \to S$ be a pseudo-Anosov homeomorphism (this means that φ leaves no finite set of non-boundary curves invariant up to isotopy). The mapping torus of φ is

$$M_\varphi = S \times \mathbb{R}/\langle (x,t) \mapsto (\varphi(x), t+1) \rangle,$$

a surface bundle over S^1 with fibre S and monodromy φ. Thurston [Thu86b] showed, as part of his hyperbolization theorem, that $int(M_\varphi)$ admits a hyperbolic structure which we'll call N_φ (see also Otal [Ota96] and McMullen [McM96]). Let $N \cong int(S) \times \mathbb{R}$ be the infinite cyclic cover of N_φ, "unwrapping" the circle direction (Figure 1). After identifying S with some lift of the fibre, we obtain an isomorphism $\rho : \pi_1(S) \to \pi_1(N) \subset \mathrm{PSL}_2(\mathbb{C})$, which is a Kleinian surface group.

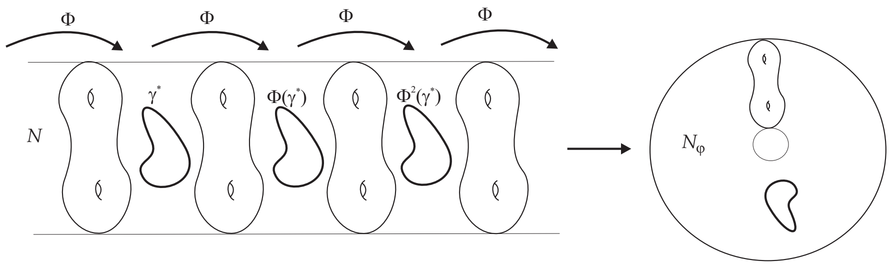


Figure 1: N covers the surface bundle N_φ.

The deck translation $\Phi : N \to N$ of the covering induces $\Phi_* = \varphi_* : \pi_1(S) \to \pi_1(S)$. We next consider the action of φ on the space of *projective measured laminations* $\mathcal{PML}(S)$ (see [FLP79, Bon01], and Lecture 3). For every simple closed curve γ in S, the sequences $[\varphi^n(\gamma)]$ and $[\varphi^{-n}(\gamma)]$ converge to two distinct points ν_+ and ν_- in $\mathcal{PML}(S)$. After isotopy, φ can be represented on S by a map that preserves the leaves of both ν_+ and ν_-, stretching the former and contracting the latter.

We can see $\nu_\pm$ directly in the asymptotic geometry of N: For a curve γ in S, let γ^* be its geodesic representative in N. Now consider $\Phi^n(\gamma^*)$ – these are all geodesics of the same length, marching off to infinity in both directions as $n \to \pm\infty$, and note that $\Phi^n(\gamma^*) = \varphi^n(\gamma)^*$. So, we have a sequence of simple curves in S, converging to ν_+ as $n \to \infty$, whose geodesic representatives "exit the $+$ end" of N (similarly as $n \to \infty$ they converge to ν_- and the geodesics exit the other end).

The laminations $\nu_\pm$ are the ending laminations of ρ in this case. To understand the general case we will have to develop a bit of terminology, and recall the work of Thurston and Bonahon.

Ends Let N_0 denote N minus its cusps (each cusp is an open solid torus, whose boundary in N is a properly embedded open annulus). The relative version of Scott's core theorem (see McCullough [McC86], Kulkarni–Shalen [KSh89] and Scott [Sco73b]) gives us a compact submanifold K in N_0, homeomorphic to $S \times [0,1]$, which meets each cusp boundary in an annulus (including the annuli $\partial S \times [0,1]$). The components of $N_0 \setminus K$ are in one to one correspondence with the topological ends of N_0, and are called neighborhoods of the ends (see Bonahon [Bon86]).

N also has a *convex core* C_N, which is the smallest closed convex submanifold whose inclusion is a homotopy equivalence. Each end neighborhood either meets C_N in a bounded set, in which case the end is called *geometrically finite*, or is contained in C_N, in which case the end is *geometrically infinite*.

From now on, let us assume that N has *no extra cusps*, which means that the cusps correspond only to the components of ∂S. In particular N_0 has exactly two ends, which we label $+$ and $-$ according to an appropriate convention.

Simply degenerate ends In [Thu79], Thurston made the following definition, which can be motivated by the surface bundle example:

Definition 1.1. An end of N is **simply degenerate** if there exists a sequence of simple closed curves α_i in S such that α_i^* exit the end.

Here "exiting the end" means that the geodesics are eventually contained in an arbitrarily small neighborhood of the end, and in particular outside any compact set. Note that a geometrically finite end cannot be simply degenerate, since all closed geodesics are contained in the convex hull.

Thurston then established this theorem (stated in the case without extra cusps):

Theorem 1.2. [Thu79] *If an end e of N is simply degenerate then there exists a unique lamination ν_e in S such that for any sequence of simple closed curves α_i in S,*

$$\alpha_i \to \nu_e \iff \alpha_i^* \text{ exit the end } e.$$

A sequence $\alpha_i \to \nu_e$ can be chosen so that the lengths $\ell_N(\alpha_i^*) \leq L_0$, where L_0 depends only on S.

Furthermore, ν_e fills S – its complement consists of ideal polygons and once-punctured ideal polygons.

(We are being cagey here about just what kind of lamination ν_e is, and what convergence $\alpha_i \to \nu_e$ means. See Lecture 3 for more details.)

Thurston also proved that simply degenerate ends are *tame*, meaning that they have neighborhoods homeomorphic to $S \times (0,\infty)$, and that manifolds obtained as limits of quasifuchsian manifolds have ends that are geometrically finite or simply degenerate. Bonahon completed the picture with his "tameness theorem",

Theorem 1.3. [Bon86] *The ends of N are either geometrically finite or simply degenerate.*

In particular N_0 is homeomorphic to $S \times \mathbb{R}$, and ending laminations are well-defined for each geometrically infinite end.

Geometrically finite ends are the ones treated by Ahlfors, Bers and their coworkers, and their analysis requires a discussion of quasiconformal mappings and Teichmüller theory (see [Ber60, Ber70b, Sul86] for more). In order to simplify our exposition we will limit ourselves, for the remainder of these notes, to Kleinian surface groups ρ with no extra cusps, and with no geometrically finite ends. In particular the convex hull of N_ρ is all of N_ρ, and there are two ending laminations, ν_+ and ν_-. This is called the *doubly degenerate* case.

1.3. Models and bounds

Our goal now is to recover geometric information about N_ρ from the asymptotic data encoded in $\nu_\pm$. The following natural questions arise, for example:

- Thurston's Theorem 1.2 guarantees the existence of a sequence $\alpha_i \to \nu_+$ whose geodesic representatives have bounded lengths $\ell_N(\alpha_i^*)$. How can we determine, from ν_+, which sequences have this property?
- The case of the cyclic cover of a surface bundle is not typical: because it covers a compact manifold (except for cusps), it has "bounded geometry". That is,

  $$\inf_\beta \ell_N(\beta) > 0$$

  where β varies over closed geodesics. The bounded geometry case is considerably easier to understand. In particular the Ending Lamination Conjecture in this category (without cusps) was proven in [Min93, Min94].

  Can we tell from $\nu_\pm$ alone whether N has bounded geometry?
- If N doesn't have bounded geometry, there are arbitrarily short closed geodesics in N, each one encased in a *Margulis tube*, which is a standard collar neighborhood. Such examples were shown to exist by Thurston [Thu86b] and Bonahon-Otal [BO88], and to be generic in an appropriate sense by McMullen [McM91].

  In the unbounded geometry case, can we tell *which* curves in N are short? How are they arranged in N?

We will describe the construction of a "model manifold" M_ν for N, which can be used to answer these questions. M_ν is constructed combinatorially from $\nu_\pm$, and contains for example solid tori that correspond to the Margulis tubes of short curves in N. M_ν comes equipped with a map

$$f : M_\nu \to N$$

which takes the solid tori to the Margulis tubes, is proper, Lipschitz in the complement of the solid tori, and preserves the end structure. This will be the content of the Lipschitz Model Theorem, which will be stated precisely in Lecture 6.

Note that if f is *bilipschitz* then the Ending Lamination Conjecture follows: If N_1, N_2 have the same invariants $\nu_\pm$ then the same model M_ν would admit bilipschitz maps $f_1 : M_\nu \to N_1$ and $f_2 : M_\nu \to N_2$, and $f_2 \circ f_1^{-1} : N_1 \to N_2$ would be a bilipschitz homeomorphism. By Sullivan's Rigidity Theorem [Sul81a], N_1 and N_2 would be isometric.

1.4. Plan

Here is a rough outline of the remaining lectures:

§**2 Hierarchies and model manifolds:** We will show how to build M_ν starting with a geodesic in the *complex of curves* $\mathscr{C}(S)$. The main tool is the *hierarchy of geodesics* developed in Masur-Minsky [MM00]. Much of the discussion will take place in the special case of the 5-holed sphere $S_{0,5}$, where the definitions and arguments are considerably simplified.

§**3 From ending laminations to model manifold:** Using a theorem of Klarreich we will relate ending laminations to *points at infinity* for $\mathscr{C}(S)$, and this will allow us to associate to a pair of ending laminations a geodesic in $\mathscr{C}(S)$, and its associated hierarchy and model manifold.

§**4 The quasiconvexity argument:** We then begin to explore the linkage between geometry of the 3-manifold N_ρ and the curve complex data. We will show that the subset of $\mathscr{C}(S)$ consisting of curves with bounded length in N is *quasiconvex*. The main tool here is an argument using pleated surfaces and Thurston's Uniform Injectivity Theorem.

§**5 Quasiconvexity and projection bounds:** In this lecture we will discuss the Projection Bound Theorem, a strengthening of the Quasiconvexity Theorem that shows that curves that appear in the hierarchy are combinatorially close to the bounded-length curves in N. We will also prove the Tube Penetration Theorem, which controls how deeply certain pleated surfaces can enter into Margulis tubes.

§**6 A priori length bounds and model map:** Applying the Projection Bound Theorem and the Tube Penetration Theorem, we will establish a uniform bound on the lengths of all curves that appear in the hierarchy.

We will then state the Lipschitz Model Theorem, whose proof uses the a priori bound and a few additional geometric arguments. As consequences we will obtain some final statements on the structure of the set of short curves in N.

2. Curve Complex and Model Manifold

In this lecture we will introduce the complex of curves $\mathscr{C}(S)$ and demonstrate how a geodesic in $\mathscr{C}(S)$ leads us to construct a "model manifold". For simplicity we will mostly work with $S = S_{0,5}$, the sphere with 5 holes. (In general let $S_{g,n}$ be the surface with genus g and n boundary components).

2.1. The complex of curves

$\mathscr{C}(S)$ will be a simplicial complex whose vertices are homotopy classes of simple, essential, unoriented closed curves ("Essential" means homotopically nontrivial, and not homotopic to the boundary). Barring the exceptions below, we define the k-simplices to be unordered $k+1$-tuples $[v_0 \ldots v_k]$ such that $\{v_i\}$ can be realized as pairwise disjoint curves. This definition was given by Harvey [Hav81].

Exceptions: If $S = S_{0,4}$, $S_{1,0}$ or $S_{1,1}$ then this definition gives no edges. Instead we allow edges $[vw]$ whenver v and w can be realized with

$$\#v \cap w = \begin{cases} 1 & S_{1,0}, S_{1,1} \\ 2 & S_{0,4}. \end{cases}$$

(see Figure 2). In this case $\mathscr{C}(S)$ is the *Farey graph* in the plane: a vertex is indexed by the slope p/q of its lift to the planar $\mathbb{Z}^2$ cover of S, so the vertex set is $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \infty$. Two vertices p/q, r/s are joined by an edge if $|ps - qr| = 1$ (see e.g. Series [Ser85a] or [Min99]).

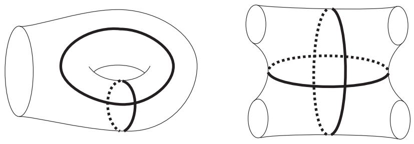


Figure 2: Adjacent vertices in $\mathscr{C}(S_{1,1})$ and $\mathscr{C}(S_{0,4})$

For $S_{0,0}, S_{0,1}, S_{0,2}, S_{0,3}$: $\mathscr{C}(S)$ is empty. (For the annulus $S_{0,2}$ there is another useful construction which we will return to later.)

Let $\mathscr{C}_k(S)$ denote the k-skeleton of $\mathscr{C}(S)$. We will concentrate on $\mathscr{C}_0$ and $\mathscr{C}_1$.

We endow $\mathscr{C}(S)$ with the metric that makes every simplex regular Euclidean of sidelength 1. Thus $\mathscr{C}_1(S)$ is a graph with unit-length edges. Consider a geodesic in $\mathscr{C}_1(S)$ – it is a sequence of vertices $\{v_i\}$ connected by edges (Figure 3), and in particular: v_i, v_{i+1} are disjoint (in the non-exceptional cases), v_i and v_{i+2} intersect but are disjoint from v_{i+1}, and v_i and v_{i+3} *fill* the surface: their union intersects every essential curve. It is harder to characterize topologically the relation between v_i and v_j for $j > i+3$.

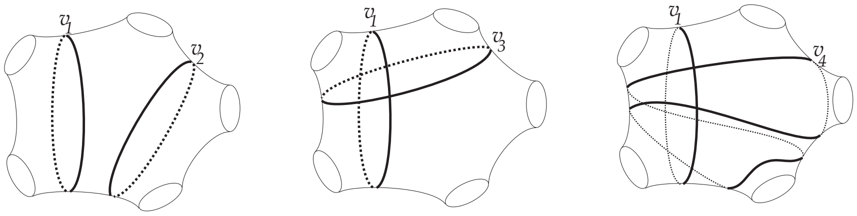


Figure 3: $\{v_1, v_2, v_3, v_4\}$ are the vertices of a geodesic in $\mathscr{C}(S_{0,5})$.

2.2. Model Construction

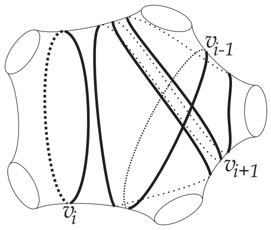
Let $S = S_{0,5}$ – this case is considerably simpler than the general case, while preserving many of the main features.

Starting with a *bi-infinite geodesic* g in $\mathscr{C}_1(S)$ (more about the existence of such geodesics later), we will construct a manifold $M_g \cong S \times \mathbb{R}$, equipped with a piecewise-Riemannian metric. M_g is made of "standard blocks", all isometric, and "tubes", or solid tori of the form (annulus) $\times$ (interval).

Hierarchy We begin by "thickening" g in the following sense: Any vertex $v \in \mathscr{C}_0(S)$ divides S into two components, one $S_{0,3}$ and one $S_{0,4}$. Let W_v denote the second of these. If v_i is a vertex of g then

$$v_{i-1}, v_{i+1} \in \mathscr{C}_0(W_{v_i}).$$

The complex $\mathscr{C}(W_{v_i})$ is just the Farey graph, and we may join v_{i-1} to v_{i+1} by a geodesic in that graph. Name this geodesic h_i, and represent it schematically as in Figure 4.



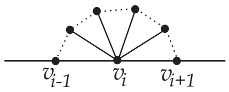


Figure 4: The local configuration at a vertex v_i of g yields a "wheel" in the link of v_i. Note, edges of h_i are not edges of $\mathscr{C}(S)$; call them "rim" edges. The other edges are called "spokes".

We repeat this at every vertex. The resulting system is called a *hierarchy of geodesics*. (In general surfaces, considerable complications arise. Geodesics must satisfy a technical condition called "tightness", and the hierarchy has more levels. This is joint work with Masur [MM99, MM00].)

Note that the construction is not uniquely dependent on g – geodesics are not always unique in the Farey graph, so there are arbitrary choices for each h_i. However what we have to say will work regardless of how the choices are made.

Blocks To each rim edge e we associate a "block" $B(e)$, and then glue these together to form the model manifold. e is an edge of $\mathscr{C}(W_v)$ for some vertex v – denote $W_e \equiv W_v$ for convenience. Let e^-, e^+ be its vertices, ordered from left to right. Let C_+ and C_- be open collar neighborhoods of e^+ and e^-, respectively. We define

$$B(e) = W_e \times [-1,1] - \left(C_+ \times (1/2,1] \cup C_- \times [-1,-1/2)\right).$$

Thus we have removed solid-torus "trenches" from the top and bottom of the product $W_e \times [-1,1]$. Figure 5 depicts this as a gluing construction.

The boundary $\partial B(e)$ divides into four *3-holed spheres*,

$$\partial_\pm B(e) \equiv (W_e - C_\pm) \times \pm 1$$

and some *annuli*. Schematically, we depict this structure in Figure 6.

Gluing Take the disjoint union of all the blocks arising from the hierarchy over g, and glue them along 3-holed sphere, where possible. That is, if $Y \times \{1\}$ appears in $\partial_+ B(e_1)$ and $Y \times \{-1\}$ appears in $\partial_- B(e_2)$, identify them using the identity map in Y.

(A technicality we are eliding is that subsurfaces are determined only up to isotopy; one can select one representative for each isotopy class in a fairly nice and consistent way.)

There are three types of gluings that can occur:

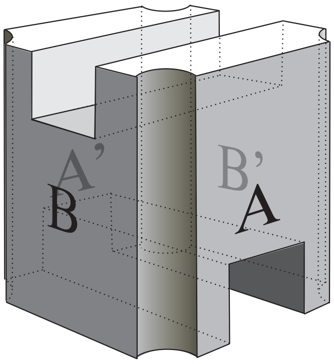


Figure 5: Construct a block $B(e)$ by doubling this object along A, A', B and B'. The curved vertical faces become $\partial W_e \times [-1,1]$.

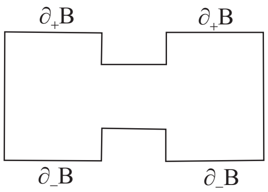


Figure 6: Schematic diagram of the different pieces of the boundary of a block.

1. Both edges occur in the link of the same vertex v; $W_{e_1} = W_{e_2} = W_v$, and $e_1^+ = e_2^-$ (Figure 7). $B(e_1)$ and $B(e_2)$ are glued along $W_v \setminus C_{e_1^+}$, which is composed of three-holed spheres Y_1 and Y_2.

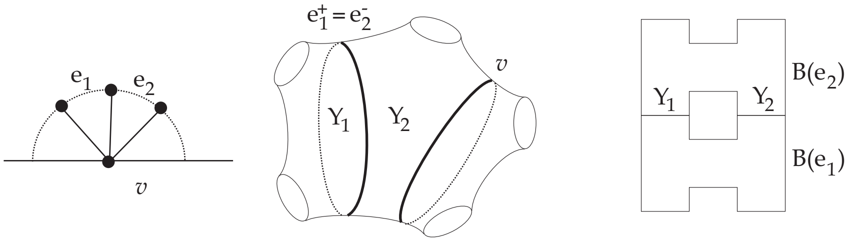


Figure 7

2. $e_1 \subset \mathscr{C}(W_u)$ and $e_2 \subset \mathscr{C}(W_v)$, where u and v are two succesive vertices (Figure 8). Now $e_1^+ = v$ and $e_2^- = u$, and the gluing is along $Y_2 = W_u \cap W_v$, which separates $S_{0,5}$.

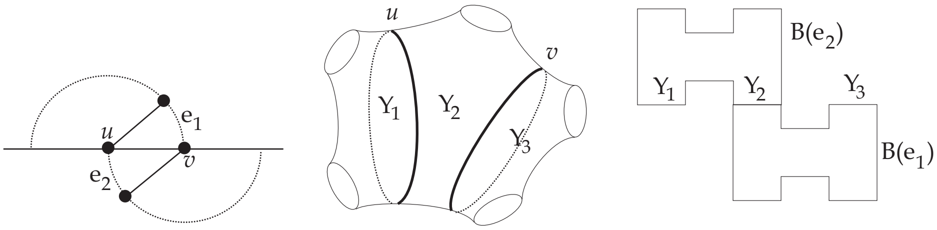


Figure 8

3. $e_1 \subset \mathscr{C}(W_u)$ and $e_2 \subset \mathscr{C}(W_w)$, where u, v, w are three successive vertices (Figure 9). In this case $e_1^+ = e_2^- = v$, and the gluing is along Y_1 which is isotopic to $W_u \cap W_w$ and does not separate $S_{0,5}$. Note that the intersection pattern of u and w is typically more complicated than pictured, as $d_{W_v}(u,w) >> 1$.

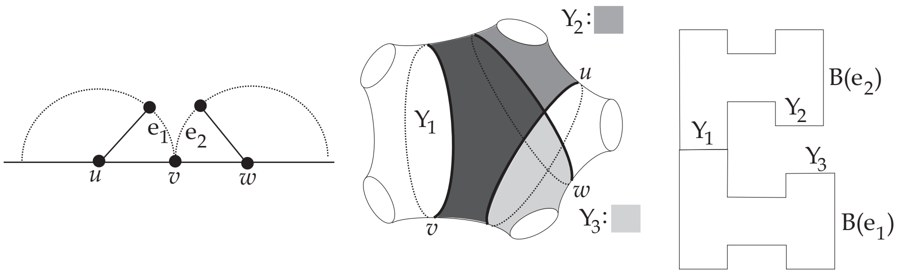


Figure 9

When we fit all the blocks together, the result can be embedded in $S \times \mathbb{R}$, in such a way that any level surface $Y \times \{t\}$ in a block is mapped to $Y \times \{s\}$ in $S \times \mathbb{R}$ by a map that is the identity on the first factor. Call such a map "straight". Note that the blocks can be stretched *vertically* in different ways.

In the gaps between blocks we find solid tori of the form

$$C \times (s,t)$$

where C is one of our collar neighborhoods of a vertex in the hierarchy. Call these the *tubes* of the model.

We should of course verify these claims about the gluing operations. Two things to check are:

1. All the vertices in the hierarchy are distinct (and hence all the tubes are homotopically distinct in $S \times \mathbb{R}$).

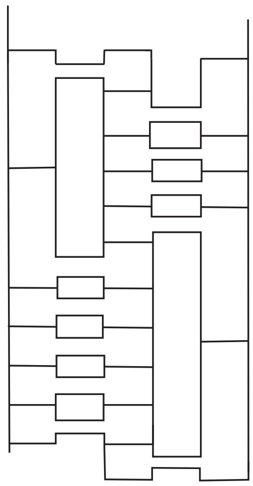


Figure 10: A schematic of the embedding of the blocks in $S\times\mathbb{R}$

2. The gluings we have shown are the only ones.

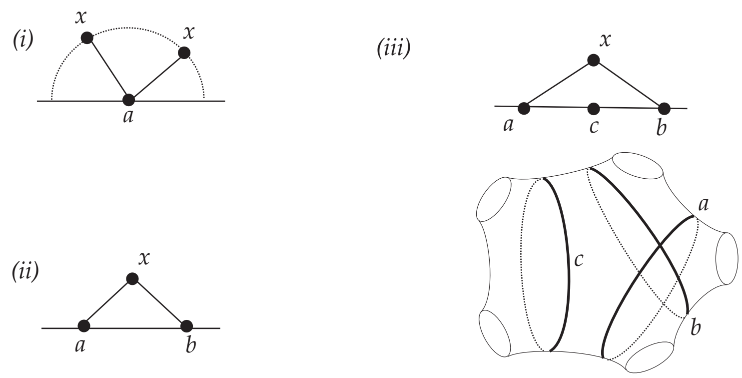


Figure 11: The three ways that a vertex can appear twice

To verify (1), suppose that a vertex x appears in two places in the hierarchy. That is, x is in the wheels (links) of vertices a and b in g. The triangle inequality in $\mathscr{C}(S)$ implies that $d(a,b)\leq 2$, and since g is a geodesic this leaves three possibilities, as in Figure 11.

In case *(i)*, $a=b$. This is not possible since the "rim" path is a geodesic in $\mathscr{C}(W_a)$.

In case *(ii)*, a,b and x make a triangle in $\mathscr{C}(S)$, but $\mathscr{C}(S)$ has no triangles for $S=S_{0,5}$.

In case *(iii)*, a and b "fill" the 4-holed sphere W_c bounded by c, so that if x is represented by a curve disjoint from both, it is equal to c or lies on the complement of W_c. That complement is a 3-holed sphere so the only possibility is that $x = c$. In other words x really only appears once, as a vertex of g.

To prove (2), we must consider how a gluing surface Y (a 3-holed sphere) can occur. There are several possibilities for the curves of ∂Y (Figure 12).

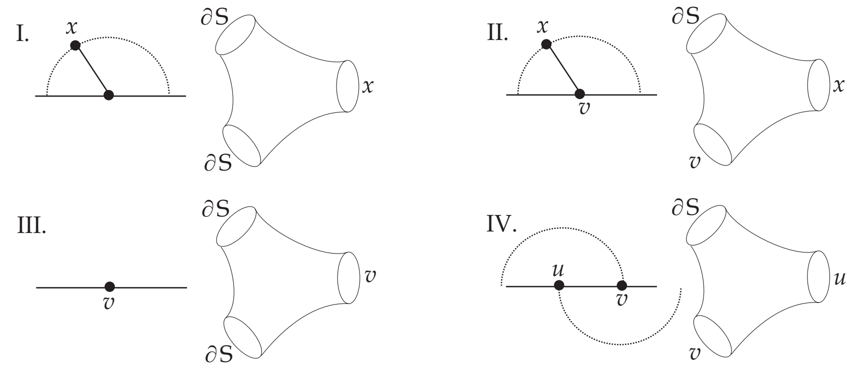


Figure 12

I. ∂Y consists of x and two curves of ∂S, where x is an interior "rim" vertex.

II. ∂Y consists of an interior rim vertex x, a g vertex v, and a curve of ∂S.

III. ∂Y consists of v and two curves of ∂S, where v is a vertex on g.

IV. ∂Y consists of two adjacent vertices u, v of g and a curve of ∂S.

Types I and II occur in pairs as the top and bottom surfaces of two blocks associated to adjacent rim edges meeting at the same x. Types III and IV occur on blocks associated to first and last edges in rim geodesics, and each one occurs in exactly two ways. It is therefore not hard to check all the possibilities and see that the gluings we described indeed produce a manifold.

The embedding of the manifold into $S \times \mathbb{R}$ can be done inductively, by "sweeping" across the hierarchy from left to right.

2.3. Geometry of the model

Fix one *standard block*: Take a copy of W (of type $S_{0,4}$) with two curves α, β that are neighbors in $\mathscr{C}(W)$, collars C_α, C_β, and construct a block B_0 as before out of

$W \times [-1,1]$. Give this block *some* metric with these properties:

- Symmetry of gluing surfaces: Each component of $\partial_\pm B_0$ is isomestric to a *fixed* copy of $S_{0,3}$, which admits a 6-fold orientation-preserving symmetry group permuting the boundary components.
- Flat annuli: All of the annuli of $\partial B_0 \setminus \partial_\pm B_0$ are *flat* – that is, isometric to a circle cross an interval. We assume that all the circles have length 1.

Now given any block $B(e)$, identify it with B_0 so that e^- is identified with α and e^+ is identified with β. Pull back the metric from B_0 to $B(e)$.

The symmetry properties imply that all the gluings can be done by isometries (possibly after isotopy). Thus we obtain a metric on the union of the blocks, and the boundary tori are all Euclidean.

Geometry of the tubes For each vertex v in H we have the associated "tube" $C_v \times (s,t)$ in the complement of the blocks, which we call $U(v)$. The torus $\partial U(v)$ has a *natural marking* by a pair of curves – the core curve γ_v of $C_v \times \{s\}$, and the meridian μ_v of $U(v)$. This marking allows us to record the geometry of the torus via "Teichmüller data": ∂U is a Euclidean torus in the metric inherited from the blocks, and there is a unique number

$$\omega \in \mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

such that $\partial U(v)$ can be identified by an orientation-preserving isometry with the quotient $\mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$, such that $\mathbb{R}$ and $\omega\mathbb{R}$ map to the classes of γ_v and μ_v, respectively. We define $\omega_M(v) \equiv \omega$, the *vertex coefficient* of v. Note that $|\omega_M(v)|$ is the length of the meridian μ_v.

We can then extend the metric on ∂U to make U a "hyperbolic tube" as follows: Given $r > 0$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, let $\mathbb{T}(\lambda, r)$ denote the quotient of an r-neighborhood of a geodesic L in $\mathbb{H}^3$ by a translation γ whose axis is L and whose complex translation distance is λ. The boundary $\partial\mathbb{T}(\lambda, r)$ is a Euclidean torus, on which there is a natural marking by a representative of γ and by a meridian. Hence we obtain a Teichmüller coefficient $\omega(\lambda, r)$ as above. It is a straightforward exercise to show that, given $\omega_M(v)$ there is a unique (λ, r) such that after identifying the markings we have $\omega(\lambda, r) = \omega_M(v)$, and the length of the γ curve in $\partial\mathbb{T}(\lambda, r)$ is 1. We then put a metric on U by identifying it with $\mathbb{T}(\lambda, r)$.

It is not hard to check that as $|\omega_M(v)| \to \infty$, the radius r of the tube goes to ∞, and the length $|\lambda|$ goes to 0.

Let M_g denote the union of blocks and tubes, with the metric we have described, and the identification with $S \times \mathbb{R}$ we have given.

3. From ending laminations to model manifold

Given a doubly degenerate Kleinian surface group $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, Theorem 1.2 gives us a pair of ending laminations ν_+, ν_-. How do these determine a geodesic and a hierarchy from which we can build a model? Roughly speaking, the laminations are "endpoints at ∞" for the hierarchy.

3.1. Background

Hyperbolicity With Masur in [MM99], we proved that

Theorem 3.1. *$\mathscr{C}(S)$ is a δ-hyperbolic metric space.*

We recall the definition, due to Cannon and Gromov [Gro87, Cnn91]: A geodesic metric space S is δ-hyperbolic if all triangles are "δ-thin". That is, given a geodesic triangle $[xy] \cup [yz] \cup [xy]$, each side is contained in a δ-neighborhood of the union of the other two.

This simple synthetic property has many important consequences, and gives X large-scale properties analogous to those of the classical hyperbolic space $\mathbb{H}^n$, and any infinite metric tree. In particular, X has a *boundary at infinity*, ∂X, defined roughly as follows: we fix a basepoint x_0 and endow X with a "contracted" metric d_0 in which $x, y \in X$ are close if

- they are close in the original metric of X, or
- they are "visually close" as seen from x_0 – that is, geodesic segments $[x_0 x]$ and $[x_0 y]$ have large initial segments $[x_0 x']$ and $[x_0 y']$ which are in δ-neighborhoods of each other (figure 13).

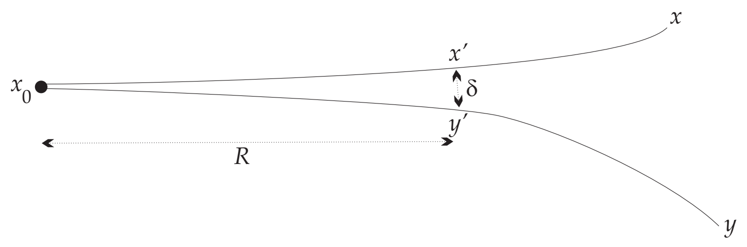


Figure 13

The completion of X in this contracted metric yields new points, which comprise ∂X. The construction does not in fact depend on the choice of x_0. See [ABC$^+$91] for more details. The boundary of $\mathscr{C}(S)$ turns out to be a certain *lamination space*:

Laminations Thurston introduced the space of *measured geodesic laminations* on a surface S, $\mathscr{ML}(S)$. Fixing a complete finite-area hyperbolic metric on $int(S)$, a geodesic lamination is a closed subset foliated by geodesics. A *transverse measure* on a geodesic lamination is a family of Borel measures on arcs transverse to the lamination, invariant by holonomy; that is, by sliding along the leaves. (See Figure 14).

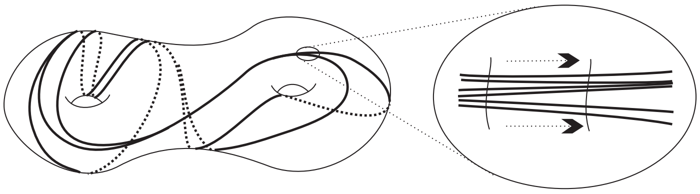


Figure 14: A geodesic lamination with closeup showing a transversal sliding along the leaves.

This space has a natural topology, which makes $\mathscr{ML}(S)$ homeomorphic to $\mathbb{R}^{6g-6+2n}$ when $S = S_{g,n}$. The choice of the hyperbolic metric is not important; all choices yield naturally homeomorphic spaces. See Bonahon [Bon01] for more. Taking the quotient of $\mathscr{ML}(S)$ (minus the empty lamination) by scaling of the measures yields the sphere $\mathscr{PML}(S)$ which was mentioned in Lecture 1. We will actually need to consider a stronger quotient, the space of "unmeasured laminations"

$$\mathscr{UML}(S) = \mathscr{ML}(S)/\text{measures}$$

Thus this is the space of all geodesic laminations which are the supports of measures, with a quotient topology obtained by forgetting the measure. This is different from from $\mathscr{PML}(S)$ because of the existence of non-uniquely ergodic laminations, and its topology is different from the topology on plain geodesic laminations obtained by Hausdorff convergence of compact subsets of S. Note that the simple closed curves, i.e. vertices of $\mathscr{C}(S)$, form a dense subset of $\mathscr{UML}(S)$.

$\mathscr{UML}(S)$ is not a Hausdorff space. However, consider the subset

$$\mathscr{EL}(S) \subset \mathscr{UML}(S)$$

consisting of all "filling" laminations. That is, $\lambda \in \mathscr{EL}(S)$ if and only if all complementary regions of λ in S are ideal polygons or once-punctured ideal polygons. An equivalent condition is that any lamination in $\mathscr{UML}(S)$ different from λ intersects it transversely. We then have (Klarreich [Kla]) that $\mathscr{EL}(S)$ is a Hausdorff space. Furthermore, elements of $\mathscr{EL}(S)$ are exactly those laminations that occur as *ending laminations* for manifolds N_ρ without extra parabolics. (In Theorem 1.2, the convergence to the ending lamination can now be understood as convergence in $\mathscr{UML}(S)$.)

Klarreich showed in [Kla] that:

Theorem 3.2. *There is a homeomorphism*

$$k : \partial\mathscr{C}(S) \to \mathscr{EL}(S)$$

such that a sequence $\beta_i \in \mathscr{C}_0(S)$ converges to $\beta \in \partial\mathscr{C}(S)$ if and only if it converges to $k(\beta)$ in the topology of $\mathscr{UML}(S)$.

Thus, ending laminations are points at infinity for $\mathscr{C}(S)$, and from now on we identify $\partial\mathscr{C}(S)$ with $\mathscr{EL}(S)$.

From lamination to hierarchy Now given a doubly degenerate $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, with ending laminations $\nu_\pm$, we would like to produce a bi-infinite geodesic g in $\mathscr{C}_1(S)$ whose endpoints on $\partial\mathscr{C}(S)$ are $\nu_\pm$.

If $\mathscr{C}(S)$ were *locally finite*, this would be easy: Take a sequence $\{x_i\}_{i=-\infty}^{\infty}$ in $\mathscr{C}_0(S)$ such that

$$\lim_{i\to\pm\infty} x_i = \nu_\pm$$

and note that, by hyperbolicity of $\mathscr{C}(S)$ and the definition of $\partial\mathscr{C}(S)$, the geodesic segments $[x_{-n}, x_n]$ and $[x_{-m}, x_m]$ are 2δ-fellow travelers on larger and larger segments as $n, m \to \infty$. Thus we would expect, after extracting a subsequence, to obtain a limiting geodesic with the endpoints $\nu_\pm$ at infinity.

For $\mathscr{C}(S)$, which is *not* locally finite, the convergence step is not automatic. The machinery in [MM00] gives a way of getting around this, and extracting a convergent subsequence after all. We will leave out this argument, and assume from now on that we have a geodesic with endpoints $\nu_\pm$.

Now, for $S = S_{0,5}$, we are ready to repeat the hierarchy ("wheel") construction of the previous lecture, and build from this our model manifold.

Our discussion so far yields us the following: Given a doubly-degenerate Kleinian surface group $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, we obtain via Bonahon-Thurston its ending laminations $\nu_\pm \in \mathscr{EL}(S)$. Using Klarreich's theorem and the work in [MM00], we produce a geodesic g and a hierarchy H_ν, and a model manifold M_ν – all depending only on $\nu_\pm$ and not on ρ. Our next task will be to connect the geometry of M_ν to the geometry of the hyperbolic 3-manifold N_ρ.

4. The quasiconvexity argument

Our goal, in §6, is to produce a map $f : M_\nu \to N$ which is uniformly Lipschitz on each of the blocks. In particular, if v is a vertex of H_ν, it appears in some block with a *fixed* *length* (independent of v, since all blocks are isometric), and so its image has to have bounded length:

$$\ell_\rho(v) \leq L$$

for some uniform L. (Here $\ell_\rho(v)$ denotes the length in N of the geodesic representative of v via ρ).

To obtain a bound like this, we must exhibit some connection now between the *geometry of* N_ρ and the *combinatorics/geometry of* $\nu_\pm$ *in* $\mathscr{C}(S)$.

Recall that $\nu_\pm$ are by definition limits in $\mathscr{UML}(S)$ of bounded-length curves: that is, there exists a sequence $\{\alpha_i\}_{i=-\infty}^{\infty}$ in $\mathscr{C}_0(S)$ with $\ell_\rho(\alpha_i) \leq L_0$ and $\lim_{i\to\pm\infty} \alpha_i = \nu_\pm$. The geodesic g also accumulates onto $\nu_\pm$ at infinity. However there seems to be no a priori reason for the α_i to be anywhere near g. Define

$$\mathscr{C}(\rho, L) = \{\alpha \in \mathscr{C}_0(S) : \ell_\rho(\alpha) \leq L.\}$$

To understand the relation of g to $\mathscr{C}(\rho, L_0)$, we will begin by proving:

Theorem 4.1 (Quasiconvexity). *For all* $L \geq L_0$ *there exists* K*, so that* $\mathscr{C}(\rho, L)$ *is* K*-quasiconvex.*

(Recall that $A \subset X$ is K-quasiconvex if for any geodesic segment γ with $\partial\gamma \subset A$, $\gamma \subset \mathrm{Nbhd}_K(X)$.)

By hyperbolicity of $\mathscr{C}(S)$ and the definition of the boundary it is not hard to see that, since α_i converge to the endpoints of g as $i \to \pm\infty$, each finite segment G of g is, for large enough i, in a 2δ-neighborhood of $[\alpha_{-i}, \alpha_i]$.

Now $\alpha_i \in \mathscr{C}(\rho, L_0)$, so using the quasiconvexity theorem, this means that all of g is in a K'-neighborhood of $\mathscr{C}(\rho, L_0)$.

We remark that, since $\mathscr{C}$ is locally infinite, a distance bound like this is only a weak sort of control. The Projection Bound Theorem in Lecture 5 will be a considerably stronger generalization.

4.1. The bounded-curve projection

Our main tool will be a "coarsely defined map" from $\mathscr{C}(S)$ to $\mathscr{C}(\rho, L)$:

$$\Pi_{\rho,L} : \mathscr{C}(S) \to \mathscr{P}(\mathscr{C}(\rho, L))$$

where $\mathscr{P}(X)$ is the set of subsets of X. $\Pi_{\rho,L}$ (Π for short) will have the following properties:

1. Coarse Lipschitz:

$$d(x,y) \leq 1 \implies \mathrm{diam}(\Pi(x) \cup \Pi(y)) \leq A$$

2. Coarse Idempotence:

$$x \in \mathscr{C}(\rho, L) \implies x \in \Pi(x)$$

(where A is a constant independent of ρ)

These properties imply that Π is, in a coarse sense, like a Lipschitz projection to the set $\mathscr{C}(\rho, L)$. Together with hyperbolicity of $\mathscr{C}(S)$, this has strong consequences:

Theorem 4.2. *If X is δ-hyperbolic, $Y \subseteq X$ and $\Pi : X \to \mathscr{P}(Y)$ satisfies properties (1) and (2), then Y is quasiconvex.*

The proof is similar to the proof of "stability of quasigeodesics" in Mostow's rigidity theorem. See [Min01] for more details.

4.2. Definition of $\Pi_{\rho,L}$

Pleated surfaces A pleated surface (or pleated map) is a map $f : S \to N$, together with a hyperbolic structure σ_f on S, with the following properties:

- f takes σ_f-rectifiable paths in S to paths in N of the same length.
- There is a σ_f-geodesic lamination λ on S, all of whose leaves are mapped geodesically,
- The complementary regions of λ are mapped totally geodesically.

We call σ_f the *induced metric*, since it is determined uniquely by the map and the first condition. The minimal λ that works in the definition is called the *pleating locus* of f. Informally one can think of the map as "bent" along λ.

This definition is due to Thurston and plays an important role in the synthetic geometry of hyperbolic 3-manifolds. A standard example, which we will be making use of, is the "spun triangulation":

Begin with any set P of curves cutting S into pairs of pants, and fix a hyperbolic metric σ on $int(S)$ of finite area, so that the ends are cusps. On each component of P place one vertex, and then triangulate each pants using only arcs terminating in these vertices and in the cusps. Now "spin" this triangulation around P, by applying a sequence of Dehn twists around each component. If at each stage the triangulation is realized by geodesics in σ, then the geometric limit of the sequence will be a lamination with closed leaves P and a finite number of infinite leaves that spiral on P and/or exit the cusps. (See Figure 15)

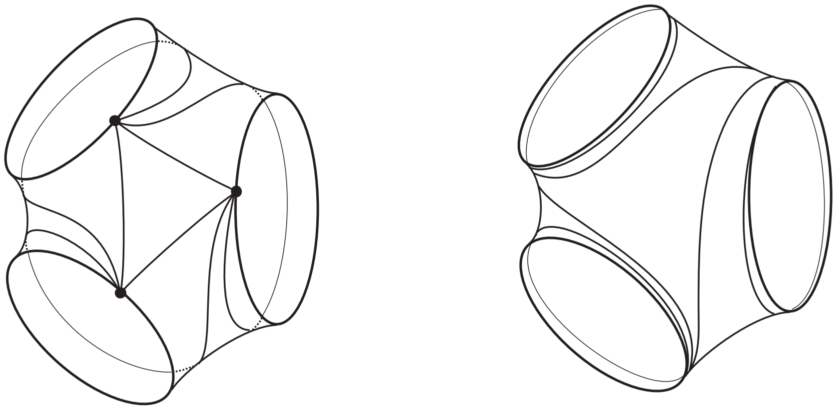


Figure 15: The triangulation and "spun" lamination on a pair of pants, when all boundary components are in P. If some are in ∂S then the leaves go out a cusp instead of spiraling.

In a similar way we can produce a pleated surface, first by mapping the curves of P to their geodesic representatives in N, and then "spinning" the images of the triangulation leaves. Finally when the leaves are in place we fill in the spaces between them with (immersed) totally geodesic ideal triangles, and obtain a surface together with induced metric. (This construction is easier to visualize equivariantly in the universal cover).

It is clear from this example that for any essential curve γ in S there is a pleated map in the homotopy class of ρ that maps γ to its geodesic representative in N. We define

$$\mathbf{pleat}_\rho(\gamma)$$

to be the set of all such pleated maps.

Now for a complete hyperbolic metric σ on $int(S)$, define

$$\mathbf{short}_L(\sigma) = \{v \in \mathscr{C}_0(S) : \ell_\sigma(v) \leq L\}.$$

We can now define:

$$\Pi_{\rho,L}(\alpha) = \bigcup_{f \in \mathbf{pleat}_\rho(\alpha)} \mathbf{short}_L(\sigma_f). \tag{4.1}$$

It is an observation originally of Bers that given S there is a number L_0 so that, for every hyperbolic metric σ on S there is a pants decomposition made up of curves of length at most L_0. We call this number the "Bers constant". Hence for $L \geq L_0$, $\Pi_{\rho,L}(\alpha)$ is always non-empty, and moreover contains a pants decomposition.

Note that if $v \in \mathscr{C}(\rho, L)$ then $v \in \Pi_{\rho,L}(v)$, since if $f \in \mathbf{pleat}_\rho(v)$, $\ell_{\sigma_f}(v) = \ell_\rho(v) \leq L$. Hence property (2) (Coarse Idempotence) is established.

Now our main claim, the Coarse Lipschitz property (1), will follow from the apparently weaker claim:

$$\operatorname{diam}_{\mathscr{C}(S)}\left(\Pi_{\rho,L}(v)\right) \leq b \tag{4.2}$$

for a priori b (depending on L) and any simplex v.

Proof of inequality (4.2) First note that, for any σ,

$$\operatorname{diam}_{\mathscr{C}(S)}(\mathbf{short}_L(\sigma)) \leq C(L). \tag{4.3}$$

This is easy: If two curves have a length bound with respect to the same metric σ, their intersection number is bounded in terms of this, and a bound on the intersection number implies a bound on the $\mathscr{C}(S)$-distance by an inductive argument (see [MM99], or Hempel [Hem01]).

Thus our main point will be to show that, for some a priori constant L_1,

$$\mathbf{short}_{L_1}(\sigma_f) \cap \mathbf{short}_{L_1}(\sigma_g) \neq \emptyset \tag{4.4}$$

for any $f, g \in \mathbf{pleat}_\rho(v)$. This would imply, together with (4.3), that $\operatorname{diam}_{\mathscr{C}(S)}(\Pi_{\rho,L_1}(v)) \leq 2C(L_1)$. In fact since $\mathbf{short}_L$ is increasing with L we can conclude that $\operatorname{diam}_{\mathscr{C}(S)}(\Pi_{\rho,L}(v)) \leq 2C(\max(L, L_1))$ for any L.

To prove inequality (4.4), let us construct a curve γ which has bounded length in both σ_f and σ_g. At first we note that $f(S)$ and $g(S)$ are only guaranteed to agree on the curve v itself, and this curve may be very long. Consider the geodesic representing v in the σ_f metric on S.

Suppose first that there are *no thin parts* on σ_f – that is that $inj(\sigma_f) > \varepsilon$ for some fixed $\varepsilon > 0$. This means that S is a closed surface, and that σ_f has no closed geodesics of length less than 2ε. In this case we can approximate v with a curve of bounded length that is composed of a segment of v concatenated with a very short "jump". That is, for $\varepsilon' < \varepsilon/2$, consider the ε'-neighborhood of a long segment s of v. If this is an embedded rectangle in S then its area is at least $2\varepsilon' l(s)$. Thus the finiteness of the area of S implies that for a certain $l(s)$, this neighborhood (which locally looks like an embedded rectangle because of the injectivity radius lower bound) will fail to embed globally, and where this happens we get a “short cut” of length $2\varepsilon'$ joining a long (but bounded) segment of v to itself (Figure 16). If we are slightly more careful we can arrange for the resulting closed curve β to be simple, and homotopically essential.

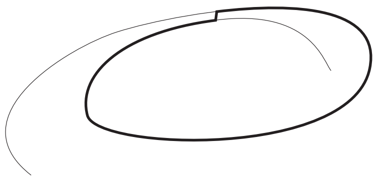


Figure 16

Now to bound the length of β with respect to σ_g requires a bound on the σ_g length of (the homotopy class rel endpoints of) the short jump part of β – the part that runs along v is already the same length in both metrics. In other words we need to prevent a certain kind of “folding” of g, as suggested by Figure 17.

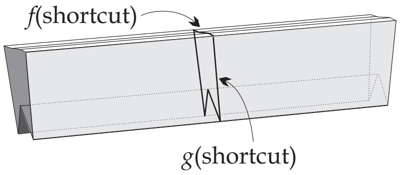


Figure 17: The thin rectangle on the top is in the image of f, whereas the image of g is folded so that the shortcut is not short in σ_g. This is what Uniform Injectivity rules out.

This is prevented by a result of Thurston called the Uniform Injectivity Theorem [Thu86a]. This theorem states, in our setting, that two leaves mapped geodesically by a pleated surface cannot line up too closely in the image unless they are already close in the domain. The two endpoints of the short cut part of β are the midpoints of long subsegments of v that are close to each other in σ_f (we can force them to be as long as we like by taking ε' small enough) and hence the same is true of their images by f, so that the leaves line up nearly parallel in the image at the endpoints of the short cut. Since $f = g$ on v, we can then apply the Uniform Injectivity Theorem to g and conclude that the endpoints of the short cut are close together in σ_g, and in fact (with a bit more care) the short cut itself is homotopic rel endpoints to an arc of length at most $\delta(\varepsilon')$.

When S has nonempty boundary, we must take a bit more care that the closed curve β is in a non-peripheral homotopy class – we may have to use two segments on v and two short cuts.

If we allow σ_f to have very short geodesics, we must consider one more case. If v does not enter any thin part of S with core length less than ε, then the previous argument applies. If v does enter the ε-thin part of a σ_f-geodesic β, then the approximation by a bounded-length curve may fail. However, in this case we see that both f and g, since they agree on v, have images that meet the ε-Margulis tube associated with β in N, and by standard properties of pleated surfaces this implies that β itself has uniformly bounded length in σ_g as well as σ_f.

Inequality (4.2) implies property (1) Now we are ready to establish the coarse Lipschitz property (1). If $d(x,y) \leq 1$ then x and y represent disjoint curves (assume here that S is not a one-holed torus or 4-holed sphere – for those cases there is a very similar argument). Thus the simplex $[xy]$ represents a curve system on S, and $\mathbf{pleat}_\rho([xy])$ is a nonempty set of pleated surfaces. But it is clear that

$$\mathbf{pleat}_\rho([xy]) = \mathbf{pleat}_\rho(x) \cap \mathbf{pleat}_\rho(y)$$

and thus this intersection is nonempty. It follows immediately that

$$\Pi_{\rho,L}(x) \cap \Pi_{\rho,L}(y) \neq \emptyset$$

for any $L \geq L_0$. Hence the diameter bound (4.2) on $\Pi(x)$ and $\Pi(y)$ implies a bound on the union.

This concludes our sketch of the proof of the coarse Lipschitz property for $\Pi_{\rho,L}$ and hence, via Theorem 4.2, of the Quasiconvexity Theorem 4.1.

5. Quasiconvexity and projection bounds

The quasiconvexity of $\mathscr{C}(\rho,L)$ implies that the geodesic g connecting $\nu_\pm$ is a bounded distance from $\mathscr{C}(\rho,L)$ (as we discussed in lecture 4), and furthermore that

$$d_{\mathscr{C}(S)}(v, \Pi_{\rho,L}(v)) \leq B \tag{5.1}$$

for a priori B and all v in g. However we might now wonder *what good is such an estimate*, since $\mathscr{C}(S)$ is locally infinite?

Using a generalization of the Quasiconvexity Theorem, we will obtain a strengthening of the bound (5.1), which will then enable us in §6 to establish the A Priori Bounds Theorem and the Lipschitz Model Theorem.

5.1. Relative bounds for subsurfaces

In order to state our generalization of the projection bound (5.1), we must consider subsurfaces of S and their associated complexes.

The *arc complex* $\mathscr{A}(W)$ of a (non-annular) surface with boundary W is defined as follows: Vertices of $\mathscr{A}(S)$ are homotopy classes of either essential simple closed curves (as for $\mathscr{C}(S)$) or properly embedded arcs. In the latter case the homotopy is taken to keep the endpoints in ∂W. Simplices correspond to disjoint collections of arcs or curves. Hence $\mathscr{C}_0(W) \subset \mathscr{A}_0(W)$, and (except in the sporadic cases $S_{0,4}$ and $S_{1,1}$, which require a separate discussion) $\mathscr{C}(W) \subset \mathscr{A}(W)$.

We also note that $\mathscr{C}_0(W)$ is *cobounded* in $\mathscr{A}(W)$, that is, every point of $\mathscr{A}(W)$ is a bounded distance from $\mathscr{C}_0(W)$, and that distance in $\mathscr{A}(W)$ is estimated by distance in $\mathscr{C}(W)$. Thus the two complexes are quasi-isometric. This is easy to see with a picture (Figure 18).

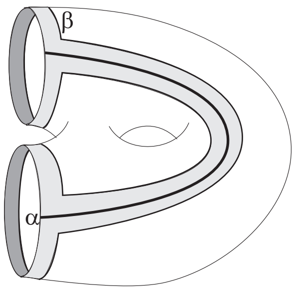


Figure 18: The regular neighborhood of an arc α and ∂W contains an essential curve β in its boundary. This gives a quasi-isometry from $\mathscr{A}_0(W)$ to $\mathscr{C}_0(W)$.

If $W \subset S$ is an essential subsurface, we obtain a map

$$\pi_W : \mathscr{A}(S) \to \mathscr{A}(W) \cup \{\emptyset\}$$

defined by taking a curve system v to the (barycenter of the) simplex formed by the essential intersections $[v \cap W]$, or to $\emptyset$ if there are no essential intersections.

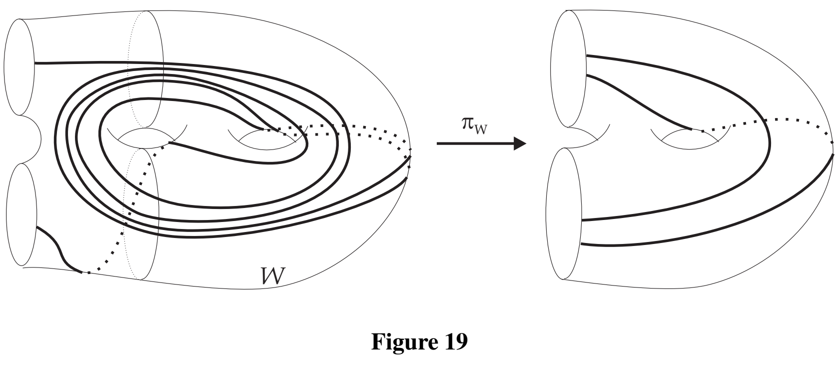


Figure 19

For annuli in S we need a different definition. Let $W \subset S$ be an essential, nonperipheral annulus, let $\widehat{W}$ be the associated annular cover of S, and $\overline{W}$ its natural compactification. (See Figure 20).

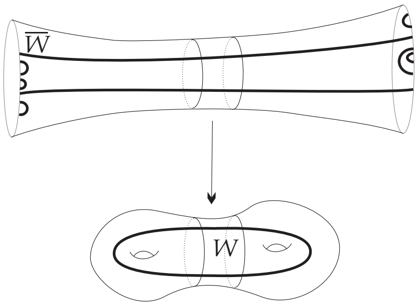


Figure 20

Let $\mathscr{A}(\overline{W})$ be as above, except that vertices are now properly embedded arcs up to homotopy with *fixed endpoints*.

Now, any $\alpha \in \mathscr{A}_0(S)$ lifts to an arc system in $\widehat{W}$, which compactify to arcs in $\overline{W}$. This system contains essential arcs in $\mathscr{A}(\overline{W})$ (those with endpoints on both boundaries) exactly if α intersects W essentially. The set of essential lifted arcs gives us $\pi_W(\alpha)$ in the annulus case.

Let

$$d_W(\alpha,\beta) = dist_{\mathscr{A}(W)}(\pi_W(\alpha),\pi_W(\beta))$$

(replacing $\mathscr{A}(W)$ with $\mathscr{A}(\overline{W})$ in the annulus case). This makes sense provided both projections are nonempty. In the annulus case, this distance measures "relative twisting" of α and β around W. We similarly define $\operatorname{diam}_W(X) = \operatorname{diam}_{\mathscr{A}(W)}(\pi_W(X))$.

We can now state the generalization of (5.1):

Projection Bound Theorem. *If v is in the hierarchy H_v and W is an essential subsurface other than a three-holed sphere, then*

$$\operatorname{diam}_W(v \cup \Pi_{\rho,L}(v)) \leq B$$

provided $\pi_W(v)$ and $\pi_W(\Pi_{\rho,L}(v))$ are nonempty, where $L \geq L_0$ and B depends on S and L.

Note that $\pi_W(\Pi_{\rho,L}(v))$ is *always* nonempty if $L \geq L_0$ and W is not an annulus, since then $\Pi_{\rho,L}(v)$ contains a pants decomposition.

The proof of this theorem is a fancier version of the argument we used for the Quasiconvexity Theorem in the previous lecture. An important ingredient is an adaptation of the "short cut" construction that yielded the curve β of bounded length on two pleated surfaces $f, g \in \mathbf{pleat}_\rho(v)$ (Figure 16). In the context of this theorem we need to make sure that β has essential intersection with the given subsurface W, and so the choice of β has to be carefully guided using Thurston's "train tracks". In addition to this, there is an inductive structure to the argument, using the hierarchy H_ν.

5.2. Penetration in Margulis tubes

We can apply the Projection Bound Theorem to control the way in which a pleated surface enters a Margulis tube in N. This will then play an important role in the A Priori Bound Theorem in §6.

Let $\mathbb{T}_\varepsilon(\alpha)$ denote the ε-*Margulis tube* in N_ρ of $\rho(\alpha)$, for an element α of $\pi_1(S)$ (or a vertex α of $\mathscr{C}(S)$). This is the locus where the translation length of $\rho(\alpha)$ or some power of α is bounded by ε. If ε is less than the Margulis constant ε_0, and $0 < \ell_\rho(\alpha) < \varepsilon$, then $\mathbb{T}_\varepsilon(\alpha)$ is a solid torus, isometric to the hyperbolic tube $\mathbb{T}(r, \lambda)$ (see §2) where λ is the complex translation length of $\rho(\alpha)$ and the radius r goes to ∞ as $\frac{\ell_\rho(\alpha)}{\varepsilon} \to 0$. Our next goal is to detect the presence of Margulis tubes in N, from the structure of the hierarchy.

Tube Penetration Theorem (stated for $S = S_{0,5}$) *There exists $\varepsilon > 0$ depending on S, such that the following holds.*

Let s be a "spoke" of the hierarchy H_ν. If $f \in$ $\mathbf{pleat}_\rho(s)$*, then*

$$f(S) \cap \mathbb{T}_\varepsilon(\alpha) \neq \emptyset$$

only if α is one of the vertices of s.

That is, the only way for f to penetrate deeply into a tube is the "obvious" way – by pleating along the core curve of the tube.

5.3. Proof of the tube penetration theorem

We begin with this standard property of pleated surfaces (observed by Thurston in [Thu86a]): There exists $\varepsilon_1 > 0$ such that, if a pleated surface f in the homotopy class of ρ meets $\mathbb{T}_{\varepsilon_1}(\alpha)$ then $f^{-1}(\mathbb{T}_{\varepsilon_1}(\alpha))$ must be contained in an ε_0-Margulis tube in the metric σ_f – that is, only the thin part of S is mapped into the thin part of N. In particular it follows that $\ell_{\sigma_f}(\alpha) \leq \varepsilon_0$.

Now assume $\varepsilon << \varepsilon_1$. Suppose v is a vertex in g crossing α essentially, and $f \in$ **pleat**$_\rho(v)$ meets $\mathbb{T}_\varepsilon(\alpha)$. Let v^* denote the geodesic representative of v in N, which is in the image of f. Since v crosses α essentially, it must cross the ε_0-collar of α and hence by the previous paragraph v^* must meet an ε_0-neighborhood of $\mathbb{T}_\varepsilon(\alpha)$.

In either the forward or backward direction in g (suppose forward), all vertices of g after v cross α essentially. Number them $v = v_j, v_{j+1}, \ldots$. Eventually, v_i^* for some $i > j$ is outside $\mathbb{T}_{\varepsilon_1}(v)$, since $v_i \to \nu_+$. Let us try to see when this happens.

Lower bound: Let $f \in$ **pleat**$_\rho([v_i v_{i+1}])$. If $f(S)$ meets $\mathbb{T}_{\varepsilon_1}(\alpha)$, then both v_i^* and v_{i+1}^* cross through the ε_0-thin part of S in the metric σ_f. Any point in $v_i^* \cap \mathbb{T}_{\varepsilon_1}(\alpha)$ can be connected, via an arc in $f(S)$ of length bounded by ε_0, to $v_{i+1}^* \cap \mathbb{T}_{\varepsilon_1}(\alpha)$.

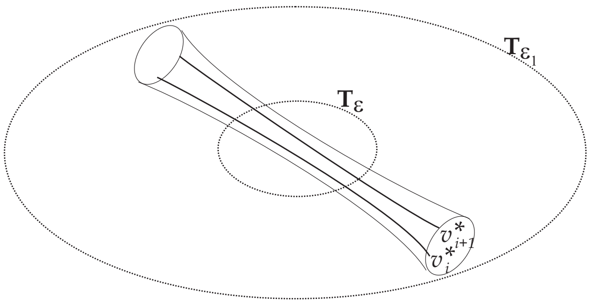


Figure 21

Suppose then that $j + Q$ is the first value of i for which v_i^* fails to meet $\mathbb{T}_{\varepsilon_1}(\alpha)$. Then applying the previous paragraph Q times we have

$$\mathrm{dist}(\mathbb{T}_\varepsilon(\alpha), \partial\mathbb{T}_{\varepsilon_1}(\alpha)) \le Q\varepsilon_0.$$

Since, by the collar lemmas of Brooks-Matelski [BM82] and Meyerhoff [Mey87], this distance is an increasing function of $\varepsilon_1/\varepsilon$, of the form

$$\mathrm{dist}(\mathbb{T}_\varepsilon(\alpha), \partial\mathbb{T}_{\varepsilon_1}(\alpha)) \ge \tfrac{1}{2}\log\frac{\varepsilon_1}{\varepsilon} - C.$$

this gives us a lower bound of the form

$$Q \ge a\log\frac{\varepsilon_1}{\varepsilon} - b. \tag{5.2}$$

Upper bound: If $f \in$ **pleat**(v_i) meets $\mathbb{T}_{\varepsilon_1}(\alpha)$ then, since $\ell_{\sigma_f}(\alpha) \le \varepsilon_0 < L_0$, we have

$$\alpha \in \Pi_{\rho,L_0}(v_i).$$

The Projection Bound Theorem then implies

$$d_{\mathscr{C}(S)}(v_i, \alpha) \leq B$$

So all such v_i's lie in a ball of radius B. Since g is a geodesic, this means that

$$Q \leq 2B. \tag{5.3}$$

Putting the upper and lower bounds (5.2,5.3) together, we obtain an inequality

$$\varepsilon > \varepsilon_2$$

where ε_2 depends on the previous constants. Thus let us assume now that $\varepsilon \leq \varepsilon_2$. Thus if $f \in \mathbf{pleat}_\rho(v)$ meets $\mathbb{T}_\varepsilon(\alpha)$ for $v \in g$, then v and α must *not* intersect essentially. If $v = \alpha$, we are done.

If $v \neq \alpha$ then $\alpha \in \mathscr{C}(W_v)$, and we consider the spokes $\{s_j = [u_j v]\}_{j=0}^m$ around v, and try to mimick the same argument. Suppose that u_j^* meets $\mathbb{T}_\varepsilon(\alpha)$, but that $u_j \neq \alpha$. Now α can be equal to at most one of the u_i so let us assume it occurs for $i < j$ if at all. Then the last vertex u_m crosses α and, since it is just the successor of v in g, the previous argument applies to it and u_m^* must be outside $\mathbb{T}_{\varepsilon_2}(\alpha)$.

Now choose Q to be the first positive number such that u_{j+Q}^* is outside $\mathbb{T}_{\varepsilon_2}(\alpha)$. An upper bound of the form of (5.3) follows using the same argument as before, but applying the relative version of the Projection Bound Theorem,

$$\mathrm{diam}_{W_v}(u_i, \Pi_{\rho,L_0}(u_i)) \leq B.$$

To obtain a lower bound of the form (5.2), but with ε_2 replacing ε_1, we need a construction to replace $\mathbf{pleat}_\rho([v_i v_{i+1}])$, since u_i and u_{i+1} do not represent disjoint curves.

Let λ_i denote the "spun" lamination, as in Lecture 4, whose closed curves are v and u_i. Let $\lambda_{i+1/2}$ be a "halfway lamination", defined as follows. $\lambda_{i+1/2}$ contains the curve v, and agrees with λ_i and λ_{i+1} on the complement of W_v. On W_v itself, $\lambda_{i+1/2}$ is as in Figure 22.

Let f_x be the pleated surface mapping λ_x geodesically, for $x = j + k/2$ ($k = 0, \ldots, 2Q$). Note that $\lambda_{i+1/2}$ has two leaves l_i and l_{i+1}, which must cross α essentially since u_i and u_{i+1} do. l_i is mapped to the same geodesic in N by $f_{i-1/2}, f_i$ and $f_{i+1/2}$ – call this geodesic l_i^*. We can now repeat the lower bound argument for Q, finding a sequence of jumps from u_j^* to u_{j+Q}^* passing through all the l_i^*. We obtain, as before, an inequality of the form

$$\varepsilon > \varepsilon_3$$

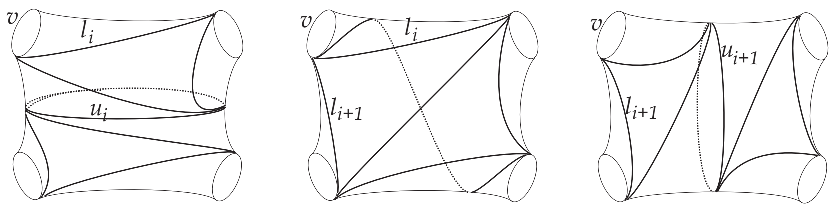


Figure 22: The related laminations λ_i, $\lambda_{i+1/2}$ and λ_{i+1}, restricted to W_v.

where ε_3 depends on the previous constants (and on ε_2). Thus if we choose $\varepsilon \leq \varepsilon_3$ we must have $\alpha = u_j$ after all, and the Tube Penetration Theorem follows.

For general S there is an inductive argument using the structure of the hierarchy, where the halfway surfaces need be used only at the last stage.

6. A priori length bounds and model map

In this last lecture we will sketch the proof of this basic bound:

A Priori Bound Theorem. *If $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ is a doubly degenerate Kleinian surface and v is a vertex in the associated hierarchy $H_{\nu(\rho)}$, then*

$$\ell_\rho(v) \leq B \tag{6.1}$$

where B depends only on the surface S.

We will then state the Lipschitz Model Theorem and indicate how the a priori bound is used in its proof.

Markings and elementary moves A marking of S is a pants decomposition $\{u_i\}$ together with, for each i, a transversal curve t_i that is disjoint from $u_j, j \neq i$, and intersects u_i in the minimal possible way. For $S_{0,5}$, a marking consists of 4 curves, and t_i intersects u_i twice.

An *elementary move* $\mu \to \mu'$ taking one marking to another is one of the following operations:

Twist$_i$ performs one half-twist on t_i around u_i (Figure 24).

Flip$_i$ reverses the roles of u_i and t_i (Figure 25). Note that in this case there has to be an adjustment of the other $t_j, j \neq i$, so that they do not intersect the new u_i' which is the old t_i. There is a finite number of “simplest” ways to do this, and we just pick one.

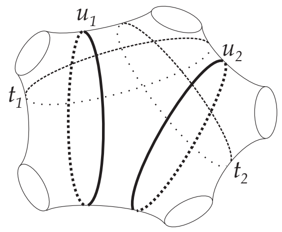


Figure 23

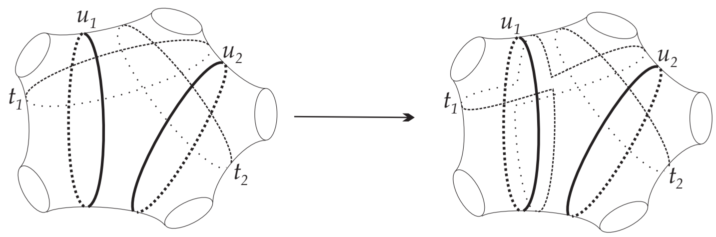


Figure 24: The move Twist_1

The graph whose vertices are markings and whose edges are elementary moves is connected, locally finite, and its quotient by the mapping class group of S is finite. The total length of a marking in a hyperbolic metric σ is just the sum of the lengths of the curves u_i and t_i. Note that a bound on the total length of μ in σ constrains σ to a bounded subset of Teichmüller space. We will need the following observation: If μ_0 is a marking of total length L in a hyperbolic metric σ on S, and

$$\mu_0 \to \mu_1 \to \cdots \to \mu_n$$

is a sequence of elementary moves, then μ_n has total length at most K, where K depends only on L and n.

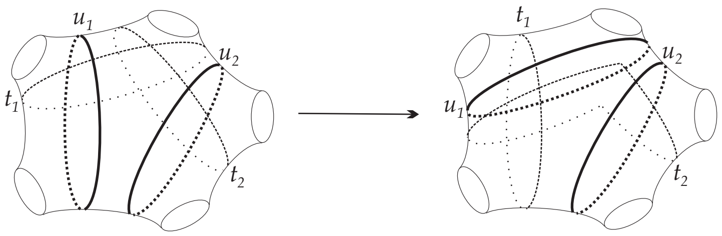


Figure 25: The move Flip_1

We will control elementary moves using this theorem:

Theorem 6.1. (Masur-Minsky [MM00]) *If μ and μ' are two markings and*

$$\sup_{W\subseteq S} d_W(\mu,\mu') \le M$$

then there exists a sequence of elementary moves from μ to μ' with at most

$$CM^{\xi}$$

steps, where C and ξ depend only on S.

Here, $d_W(\mu,\mu')$ is defined as in §5, with the projection $\pi_W(\mu)$ simply being the union of $\pi_W(a)$ over components a of μ. The supremum is over all essential subsurfaces in S, including S itself. The proof of this theorem uses the hierarchy machinery discussed in §2.

6.1. Proving the A Priori Bounds

Again, we are working in the case that $S = S_{0,5}$. Let $s = [u_1 u_2]$ be a spoke of the hierarchy, and $f \in \mathbf{pleat}_\rho(s)$. Let $\varepsilon > 0$ be the constant given by the Tube Penetration Theorem.

Thick case Suppose σ_f has no geodesics of length ε or less. Then σ_f admits a marking μ of total length at most L (depending on ε).

The Projection Bound Theorem implies that

$$d_W([u_1 u_2],\mu) \le B$$

For any W meeting u_1 or u_2 essentially, since $\mu \subset \Pi_{\rho,L}([u_1 u_2])$. This includes all W except for the annuli A_i with cores u_i ($i = 1,2$). We therefore choose transversals t_1, t_2 for u_1, u_2, such that $d_{A_i}(t_i,\mu')$ is at most 2. Thus $\{u_1,t_1,u_2,t_2\}$ give us a marking μ' such that

$$d_W(\mu',\mu) \le B$$

for *all* $W \subseteq S$. Applying Theorem 6.1, we bound the elementary-move distance from μ to μ', and hence obtain a bound on $\ell_{\sigma_f}(\mu')$.

This in turn bounds $\ell_\rho(u_i)$, which gives the a priori bound (6.1) in this case.

Thin case Suppose that σ_f does have some curve α of length less than ε. Then $f(S)$ meets $\mathbb{T}_\varepsilon(\alpha)$. The Tube Penetration Theorem now implies that $\alpha = u_1$ or $\alpha = u_2$. Suppose the former, without loss of generality. Thus, we repeat the argument of case (1) on the subsurface W_α, finding a minimal-length marking (now consisting of just one curve and its transversal) and using the Projection Bound Theorem to bound the length of u_2. Again we have the a priori bound (6.1).

6.2. Constructing the Lipschitz map

We are now ready to state, and summarize the proof of, our main theorem:

It will be convenient to define $\mathscr{U}[k]$ to be the union of tubes $\{U(v) : |\omega_M(v)| \geq k\}$, and to let $M_\nu[k] = M_\nu \setminus \mathscr{U}[k]$. Thus $M_\nu[0]$ is the union of blocks.

If v is a vertex of H_ν, let $\mathbb{T}_{\varepsilon_0}(v)$ be the Margulis tube (if any) of the homotopy class $\rho(v)$ in N_ρ. Let $\mathbb{T}[k]$ denote the union of $\mathbb{T}_{\varepsilon_0}(v)$ over all v with $|\omega_M(v)| \geq k$.

Lipschitz Model Theorem. *There exist $K, k_0 > 0$ such that, if $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ is a doubly degenerate Kleinian surface group with end invariants $\nu(\rho)$, then there is a map*

$$F : M_\nu \to N_\rho$$

with the following properties:

1. *F induces ρ on π_1, is proper, and has degree 1.*
2. *F is K-Lipschitz on $M_\nu[k_0]$.*
3. *F maps $\mathscr{U}[k_0]$ to $\mathbb{T}[k_0]$, and $M_\nu[k_0]$ to $N_\rho \setminus \mathbb{T}[k_0]$.*

Note that the Lipschitz property in part (2) is with respect to the *path metric* on $M_\nu[k_0]$ (the distance function of M_ν restricted to $M_\nu[k_0]$ may be smaller).

The map $F : M_\nu[0] \to N$ can be constructed in each block B_e individually. We first define the map on the gluing boundaries $\partial_\pm B$, which are all isometric to a fixed three-holed sphere; let Y be such a boundary component. As in Lecture 4, there is a pleated map $h : Y \to N$ in the homotopy class determined by ρ, which sends each boundary component of Y to its geodesic representative or to the corresponding cusp if it is parabolic (more accurately h is defined on $int(Y)$, and gives a metric whose completion has geodesic boundaries for the non-parabolic ends of Y and cusps for the parabolic ends).

The a priori bounds give an upper bound on the boundary lengths of this pleated surface. Thus, after excising standard collar neighborhoods of the boundaries, we obtain a surface which can be identified, with uniform bilipschitz distortion, with Y under its original model metric. Composing this identification with h, we obtain the map $F|_Y$.

Our next step is to define F on the "middle" surface of a block, which we can write as $W \times \{0\}$ with W a four-holed sphere (for general S, W can also be a one-holed torus). The "halfway surfaces" from the proof of the Tube Penetration Theorem (defined using the two vertices e^+ and e^-) provide us with a map $W \times \{0\} \to N$ and an induced hyperbolic metric. Another application of the a-priori length bounds together with Thurston's Efficiency of Pleated Surfaces [Thu86b] implies that this metric on W is within uniform bilipschitz distortion of the model metric.

We then extend to the rest of the block by a map that takes vertical lines to geodesics. A Lipschitz bound on this part of the map is then an application of the "figure-8" argument from [Min99]. (In brief, we let X be a wedge of two circles in the gluing boundary Y that generate a nonabelian subgroup of $\pi_1(Y)$ and having bounded length in both $W \times \{0\}$ and Y. The extension gives a map of $X \times [0,1]$ with geodesic tracks $\{x\} \times [0,1]$, and if the track lengths are too long then the images of the two circles in the middle are either both very short, or nearly parallel, violating discreteness either way).

Fixing k_0, the tubes U with $|\omega(U)| < k_0$ fall into some finite set of isometry types, and the map can be extended to those, again with some uniform Lipschitz bounds. If $|\omega(U)| > k_0$ then there is an *upper bound* for the corresponding vertex, $\ell_\rho(v) \leq \varepsilon$ where $\varepsilon \to 0$ as $k_0 \to \infty$. This is the main result of [Min00], which uses similar tools but is slightly different from what we have seen so far. Thus choosing k_0 appropriately, U must correspond to a Margulis tube $\mathbb{T}$ with very large radius (and short core). The map on the blocks cannot penetrate more than a bounded distance into such a tube (by the Lipschitz bounds) and so composition with an additional retraction on a collar of $\partial\mathbb{T}$ yields us a map $F : M_\nu[k_0] \to N \setminus \mathbb{T}[k_0]$ which takes each ∂U with $|\omega(U)| \geq k_0$ to the boundary of the corresonding $\mathbb{T}$.

We fill in the map on the remaining U and the cusps (with no Lipschitz bounds) to obtain a final map $F : M_\nu \to N$ with all the desired properties. The fact that F is proper essentially follows from the fact that every block contains a bounded length curve in a unique homotopy class, and thus their images cannot accumulate in a compact set. That F has degree 1 follows from the fact that blocks far toward the $+$ end of the hierarchy correspond to vertices close to ν_+, and hence their images must go out the $+$ end of N.

6.3. Consequences

It is now easy to obtain the following lower bound on lengths:

Corollary 6.2. *There exists an a priori $\varepsilon > 0$ such that all curves of length less than ε in N must occur as vertices of the hierarchy.*

Proof: If $\ell_N(\gamma) < \varepsilon$ then γ has a Margulis tube $\mathbb{T}_\varepsilon(\gamma)$ in N. Since F has degree 1, this tube is in the image of F. The Lipschitz bound on $F|_{M_\nu[k_0]}$ keeps it out of $\mathbb{T}_\varepsilon(\gamma)$ if ε is sufficiently small. Hence $\mathbb{T}_\varepsilon(\gamma)$ is in the image of some tube $U(v)$, which means that γ corresponds to v.

(Note that this corollary does not follow directly from the Tube Penetration Lemma since that lemma assumes the existence of a pleated surface associated to a spoke which penetrates $\mathbb{T}_\varepsilon(\gamma)$. The global degree argument is necessary).

Another corollary, which requires a bit more work and uses Otal's theorem [Ota95] on the unknottedness of short curves in N, is the following, which describes the topological structure of the set of short curves in N:

Corollary 6.3. *$M_\nu[k_0]$ is homeomorphic to $N \setminus \mathbb{T}[k_0]$.*

For a proof of this see [BCM].

These results give us a complete description of the "short curves" in N, and in particular give a combinatorial criterion (in terms of the hierarchy and the coefficients ω_M) for when the manifold has bounded geometry. This answers the short list of questions posed in the introduction.

We can also obtain somewhat more explicit lower bounds on ρ-lengths of the vertices of the hierarchy, namely:

Corollary 6.4. *If v is a vertex of the hierarchy $H_{\nu(\rho)}$ then*

$$\ell_\rho(v) \geq \frac{c}{|\omega_M(v)|^2}$$

where c depends only on the surface S.

This follows from the fact that, because of the Lipschitz property of the map F, $|\omega_M(v)|$ bounds the meridian length of the corresponding Margulis tube $\mathbb{T}_{\varepsilon_0}(v)$ in N. This gives an upper bound for radius of the tube, and the collar lemmas of [BM82] and [Mey87] then give a lower bound for the core length $\ell_\rho(v)$. An upper bound for $\ell_\rho(v)$ which goes to 0 as $|\omega_M(v)| \to \infty$ also exists: it follows from the main theorems of [Min00], and was already used briefly in the proof of the Lipschitz Model Theorem.

References

[AB60] L. Ahlfors and L. Bers (1960). Riemann's mapping theorem for variable metrics, *Ann. Math.* **72**, 385–404.

[ABC+91] J.M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro and H. Short (1991). Notes on word hyperbolic groups. In *Group Theory from a Geometrical Viewpoint, ICTP Trieste 1990, edited by E. Ghys, A. Haefliger and A. Verjovsky, World Scientific*, 3–63.

[Abi88] W. Abikoff (1988). Kleinian groups – geometrically finite and geometrically perverse, *Geometry of Group Representations, Contemp. Math.* **74**, Amer. Math. Soc., 11–50.

[BCM] J. Brock, R. Canary and Y. Minsky. Classification of Kleinian surface groups II: the ending lamination conjecture, *in preparation*.

[Ber60] L. Bers (1960). Simultaneous uniformization, *Bull. Amer. Math. Soc.* **66**, 94–97.

[Ber70a] L. Bers (1970). On boundaries of Teichmüller spaces and on Kleinian groups: I, *Ann. Math.* **91**, 570–600.

[Ber70b] L. Bers (1970). Spaces of Kleinian groups. In *Maryland conference in Several Complex Variables I, Lec. Notes Math.* **155**, Springer-Verlag, 9–34.

[BM82] R. Brooks and J.P. Matelski (1982). Collars in Kleinian groups, *Duke Math. J.* **49**, no. 1, 163–182.

[BO88] F. Bonahon and J.-P. Otal (1988). Varietes hyperboliques a geodesiques arbitrairement courtes, *Bull. London Math. Soc.* **20**, 255–261.

[BO01] F. Bonahon and J.-P. Otal (2001). Laminations mesurées de plissage des variétés hyperboliques de dimension 3, *preprint*.

[Bon86] F. Bonahon (1986). Bouts des variétés hyperboliques de dimension 3, *Ann. Math.* **124**, 71–158.

[Bon01] F. Bonahon (2001). Geodesic laminations on surfaces. In *Laminations and Foliations in Dynamics, Geometry and Topology (Stony Brook, NY), edited by M. Lyubich, J. Milnor and Y. Minsky, Contemp. Math.* **269**, Amer. Math. Soc., 1–37.

[Can93b] R.D. Canary (1993). Ends of hyperbolic 3-manifolds, *J. Amer. Math. Soc.* **6**, 1–35.

[Cnn91] J. Cannon (1991). The theory of negatively curved spaces and groups. In *Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989)*, Oxford Univ. Press, 315–369.

[FLP79] A. Fathi, F. Laudenbach and V. Poenaru (1979). Travaux de Thurston sur les surfaces, *Astérisque* **66-67**, Soc. Math. de France.

[Fre42] H. Freudenthal (1942). Neuaufbau der Endentheorie, *Ann. Math.* **43**, 261–279.

[Gre66] L. Greenberg (1966). Fundamental polyhedra for Kleinian groups, *Ann. Math.* **84**, 433–441.

[Gro87] M. Gromov (1987). Hyperbolic groups. In *Essays in Group Theory, edited by S. M. Gersten, MSRI Publications* **8**, Springer-Verlag.

[Hav81] W.J. Harvey (1981). Boundary structure of the modular group. In *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, edited by I. Kra and B. Maskit, Ann. Math. Studies* **97**, Princeton.

[Hem01] J. Hempel (2001). 3-manifolds as viewed from the curve complex, *Topology* **40**, no. 3, 631–657.

[Kla] E. Klarreich. The boundary at infinity of the curve complex and the relative Teichmüller space, *preprint*.

[Kra72] I. Kra (1972). On spaces of Kleinian groups, *Comment. Math. Helv.* **47**, 53–69.

[KSh89] R.S. Kulkarni and P.B. Shalen (1989). On Ahlfors' finiteness theorem, *Adv. Math.* **76**, no. 2, 155–169.

[KS93] L. Keen and C. Series (1993). Pleating coordinates for the Maskit embedding of the Teichmüller space of punctured tori, *Topology* **32**, 719–749.

[Mak70] B. Maskit (1970). On boundaries of Teichmüller spaces and on Kleinian groups II, *Ann. Math.* **91**, 607–639.

[Mak75] B. Maskit (1975). Classification of Kleinian groups. In *Proceedings of the International Congress of Mathematicians (Vancouver, 1974)* **2**, Canad. Math. Congress, 213–216.

[Mar74] A. Marden (1974). The geometry of finitely generated Kleinian groups, *Ann. Math.* **99**, 383–462.

[McC86] D. McCullough (1986). Compact submanifolds of 3-manifolds with boundary, *Quart. J. Math. Oxford* **37**, 299–306.

[McM91] C. McMullen (1991). Cusps are dense, *Ann. Math.* **133**, 217–247.

[McM96] C. McMullen (1996). Renormalization and 3-manifolds which fiber over the circle, *Ann. Math. Studies* **142**, Princeton University Press.

[Mey87] R. Meyerhoff (1987). A lower bound for the volume of hyperbolic 3-manifolds, *Canad. J. Math.* **39**, 1038–1056.

[Min93] Y.N. Minsky (1993). Teichmüller geodesics and ends of hyperbolic 3-manifolds, *Topology* **32**, 625–647.

[Min94] Y.N. Minsky (1994). On rigidity, limit sets and end invariants of hyperbolic 3-manifolds, *J. Amer. Math. Soc.* **7**, 539–588.

[Min99] Y.N. Minsky (1999). The classification of punctured-torus groups, *Ann. Math.* **149**, 559–626.

[Min00] Y.N. Minsky (2000). Kleinian groups and the complex of curves, *Geometry and Topology* **4**, 117–148.

[Min01] Y.N. Minsky (2001). Bounded geometry for Kleinian groups, *Invent. Math.* **146**, 143–192.

[Min03] Y.N. Minsky (2003). Classification of Kleinian surface groups I: models and bounds, *arXiv:math.GT/0302208*.

[MM99] H.A. Masur and Y. Minsky (1999). Geometry of the complex of curves I: Hyperbolicity, *Invent. Math.* **138**, 103–149.

[MM00] H.A. Masur and Y. Minsky (2000). Geometry of the complex of curves II: Hierarchical structure, *Geom. and Funct. Anal.* **10**, 902–974.

[MMa79] A. Marden and B. Maskit (1979). On the isomorphism theorem for Kleinian groups, *Invent. Math.* **51**, 9–14.

[Ota95] J.-P. Otal (1995). Sur le nouage des géodésiques dans les variétés hyperboliques, *C. R. Acad. Sci. Paris Sèr. I Math.* **320**, 847–852.

[Ota96] J.-P. Otal (1996). *Le théorème d'hyperbolisation pour les variétés fibrées de dimension trois*, Astérisque No. 235.

[Sco73b] G.P. Scott (1973). Compact submanifolds of 3-manifolds, *J. Lond. Math. Soc.* **7**, 246–250.

[Ser85a] C. Series (1985). The geometry of Markoff numbers, *Math. Intelligencer* **7**, 20–29.

[Sul81a] D. Sullivan (1981). On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. In *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, Ann. Math. Studies* **97**, Princeton.

[Sul85] D. Sullivan (1985). Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity for Kleinian groups, *Acta Math.* **155**, 243–260.

[Sul86] D. Sullivan (1986). Quasiconformal homeomorphisms in dynamics, topology and geometry. In *Proceedings of the International Conference of Mathematicians, Amer. Math. Soc.*, 1216–1228.

[Thu79] W.P. Thurston (1979). The geometry and topology of 3-manifolds, *Princeton University Lecture Notes*. `http://www.msri.org/publications/books/gt3m`

[Thu82] W.P. Thurston (1982). Three dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc.* **6**, 357–381.

[Thu86a] W.P. Thurston (1986). Hyperbolic structures on 3-manifolds I: deformation of acylindrical manifolds, *Ann. Math.* **124**, 203–246.

[Thu86b] W.P. Thurston (1986). Hyperbolic structures on 3-manifolds II: surface groups and 3-manifolds which fiber over the circle, *arXiv:math.GT*.

Yair N. Minsky

Department of Mathematics
SUNY at Stony Brook
Stony Brook, NY 11794-3651
USA

`yair@math.sunysb.edu`

AMS Classification: 30F40, 57M50

Keywords: hyperbolic 3-manifold, Kleinian group, ending lamination, complex of curves