

Rigidity in automorphic representations and local systems

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ABSTRACT. We introduce the notion of rigidity for automorphic representations of groups over global function fields. We construct the Langlands parameters of rigid automorphic representations explicitly as local systems over open curves. We expect these local systems to be rigid. Examples of rigid automorphic representations from previous work are reviewed and more examples for GL_2 are discussed in details.

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1. Introduction

In this article, we shall study rigid local systems over algebraic curves from the point of view of the Langlands correspondence.

1.1. The goal. Fix a smooth, projective and connected algebraic curve X over an algebraically closed field k . Let $S \subset X$ be a finite set of closed points. A local system \mathcal{F} over $X - S$ is *physically rigid* if it is determined up to isomorphism by its local monodromy around points $x \in S$. A local

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system \mathcal{F} over $X - S$ is *cohomologically rigid* if $H^1(X, j_* \text{End}^\circ(\mathcal{F})) = 0$, where $\text{End}^\circ(\mathcal{F})$ is the local system of trace-free endomorphisms of \mathcal{F} , and j_* means the sheaf (not derived) push-forward along $j : X - S \hookrightarrow X$. These notions were defined and studied in depth by N. Katz [24]. The main result of [24] is an algorithmic description of tame local systems.

We are interested in local systems in a broader sense. Let H be a connected reductive algebraic group over $\overline{\mathbb{Q}_\ell}$. An H -local system on $U = X - S$ is a continuous homomorphism of the étale fundamental group $\pi_1(U, u)$ (for some base point u) into $H(\overline{\mathbb{Q}_\ell})$. This specializes to the notion of rank n local systems when $H = \text{GL}_n$. Both notions of rigidity can be easily extended to H -local systems. Details are discussed in §3.

We would like to construct many examples of H -local systems that are rigid, or, at least, expected to be rigid. The tool we use for our construction is the Langlands correspondence over function fields.

Now let X be a smooth, projective and geometrically connected curve over a finite field k . Let G be a connected reductive group over the function field F of X , which for simplicity is assumed to be split. In this case, Langlands philosophy predicts that there should be a finite-to-one correspondence $\pi \mapsto \rho_\pi$ from automorphic representations π of $G(\mathbb{A}_F)$ to continuous representations $\rho : W_F \rightarrow \widehat{G}(\overline{\mathbb{Q}_\ell})$, where W_F is the Weil group of the function field F and \widehat{G} is the Langlands dual group to G . When the automorphic representation π is unramified outside a finite set of places S , ρ_π should also be unramified outside S , and the datum of ρ_π is the same as a \widehat{G} -local system \mathcal{F}_π over the open curve $U = X - S$. The correspondence should satisfy the following property: for each closed point $x \notin S$ of X , the Satake parameter of the spherical representation π_x of $G(F_x)$ should coincide with the conjugacy class of $\rho_\pi(\text{Frob}_x)$.

The main strategy of our construction may be summarized as follows.

- There should be a notion of rigidity for automorphic representations of $G(\mathbb{A}_F)$. For a rigid automorphic representation π of $G(\mathbb{A}_F)$, the corresponding \widehat{G} -local system \mathcal{F}_π should also be rigid.
- Rigid automorphic representations should be *easier* to construct than rigid local systems. Once a rigid automorphic representations is known, there should be a way to construct the corresponding local system via the *geometric* Langlands correspondence.

1.2. Applications. Before describing how we implement these ideas, let us list a few applications of our construction. In fact, it is these applications that convinced the author that a systematic study of rigid local system from the point of view of Langlands correspondence was meaningful.

1.2.1. *Local systems with exceptional monodromy groups.* Deligne showed in [5] that the classical Kloosterman sums (or hyper Kloosterman sums) are obtained as the Frobenius trace function of a local system over $\mathbb{P}_{\mathbb{F}_p}^1 - \{0, \infty\}$, the Kloosterman sheaf Kl_n . Katz showed that for $p > 2$, the Zariski closure

of the monodromy of Kl_n is either SL_n (when n is odd) or Sp_n (when n is even). In joint work with Heinloth and Ngô [19], we construct a \widehat{G} -local system $\mathrm{Kl}_{\widehat{G}}$ on $\mathbb{P}_{\mathbb{F}_p}^1 - \{0, \infty\}$ for every almost simple group \widehat{G} . The local monodromy of $\mathrm{Kl}_{\widehat{G}}$ resembles that of the classical Kloosterman sheaf Kl_n , and the Zariski closure of their global monodromy is a large subgroup of \widehat{G} . For example, when \widehat{G} is of type E_7, E_8, F_4 or G_2 , the monodromy is Zariski dense in \widehat{G} . These give the first examples of motivic local systems with Zariski dense monodromy in exceptional groups other than G_2 (the G_2 case was constructed earlier by Katz [23]). In [44] we give further generalizations of Kloosterman sheaves.

Our construction in [19] was inspired by the construction of simple supercuspidal representations by Gross and Reeder [13], an observation of Gross [14] on the global realization of such representations and the work of Frenkel and Gross [10] on rigid irregular connections. Gross showed in [14] that when G is simply-connected, the automorphic representations for G over the rational function field $F = k(t)$ which is Steinberg at 0 and simple supercuspidal at ∞ (and unramified elsewhere) should be unique. He then conjectures that when $G = \mathrm{GL}_n$, the Satake parameters of this automorphic representation should give the classical Kloosterman sums. Our work [19] confirms this conjecture and generalizes it to other reductive groups.

1.2.2. *Motives over number fields with exceptional motivic Galois groups.* In early 1990s, Serre asked the following question [38]: Is there a motive over a number field whose motivic Galois group is of exceptional type such as G_2 or E_8 ?

A motive M over a number field K is, roughly speaking, part of the cohomology $H^i(X)$ for some (smooth projective) algebraic variety X over K and some integer i , which is cut out by geometric operations (such as group actions). The definition of the motivic Galois group of M relies on the validity of standard conjectures in algebraic geometry. However, one can use the following alternative definition which is believed to give the same group. For each prime ℓ , the motive M has the associated ℓ -adic cohomology $H_\ell(M) \subset H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$, which admits a Galois action:

$$\rho_{M,\ell} : \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{GL}(H_\ell(M))$$

The ℓ -adic motivic Galois group $G_{M,\ell}$ of M is the Zariski closure of the image of $\rho_{M,\ell}$. This is an algebraic group over \mathbb{Q}_ℓ . Classical groups appear as ℓ -adic motivic Galois groups of abelian varieties (see [32]). However, it is proved in [32] that abelian varieties do not have exceptional motivic Galois groups; nor is there a Shimura variety of type G_2 or E_8 . This is why Serre raised the question for exceptional groups, and remarked that it was “plus hasardeuse”. Until recently, the only known case of Serre’s question was G_2 , by the work of Dettweiler and Reiter [8].

In [42], we give a *uniform* construction of local systems on $\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\}$ with Zariski dense monodromy in exceptional groups E_7, E_8 and G_2 , which

come from cohomology of families of varieties over $\mathbb{P}^1 - \{0, 1, \infty\}$. As a consequence of this construction, we give an affirmative answer to the ℓ -adic version of Serre’s question for E_7, E_8 and G_2 : these groups can be realized as the ℓ -adic motivic Galois groups for motives over number fields (in fact the number field is either \mathbb{Q} or $\mathbb{Q}(i)$). With a bit more work, one can also realize F_4 as a motivic Galois group over \mathbb{Q} .

1.2.3. Inverse Galois Problem. The inverse Galois problem over \mathbb{Q} asks whether every finite group can be realized as the Galois group of some Galois extension K/\mathbb{Q} . The problem is still open for many finite simple groups, especially those of Lie type. The same rigid local systems over $\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\}$ constructed to answer Serre’s question can be used to solve new cases of the inverse Galois problem. We show in [42] that for sufficiently large primes ℓ , the finite simple groups $G_2(\mathbb{F}_{\ell})$ and $E_8(\mathbb{F}_{\ell})$ can be realized as Galois groups over \mathbb{Q} . With a bit more work, one can also prove the same statement for $F_4(\mathbb{F}_{\ell})$.

In inverse Galois theory, people use the “rigidity method” to prove certain finite groups H are Galois groups over \mathbb{Q} . The rigidity method is an analog of rigid local systems with finite monodromy group. Although the idea of rigidity has long been used in the inverse Galois theory, the connection with Langlands correspondence and automorphic forms has not been explored before. Our result shows that this connection can be useful in solving the inverse Galois problem, and it even sheds some light to the rigidity method itself. In fact, our construction of the local system over $\mathbb{P}^1 - \{0, 1, \infty\}$ suggests a triple in $E_8(\mathbb{F}_{\ell})$ which should be a rigid triple (see [42, Conjecture 5.16]). This has been confirmed by Guralnick and Malle [17], where they used this triple to show that $E_8(\mathbb{F}_{\ell})$ is a Galois group over \mathbb{Q} as long as $\ell \geq 7$.

1.3. Main results.

1.3.1. Rigid automorphic data. Let X be a smooth, projective and geometrically connected curve over a finite field k . For simplicity we assume G is split over F and simply-connected. Fix a finite set S of closed points of X . By an *automorphic datum* for G with respect to S we mean a triple (ω, K_S, χ_S) where ω is a central character $Z(F) \backslash Z(\mathbb{A}_F) \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$, K_S is a collection of compact open subgroups $K_x \subset G(F_x)$ for each $x \in S$, and χ_S is a collection of characters $\chi_x : K_x \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$. An automorphic representation π of $G(\mathbb{A}_F)$ is called (ω, K_S, χ_S) -typical if its central character is ω , it is unramified outside S and for each $x \in S$, π_x has an eigenvector under K_x on which K_x acts through the character χ_x .

We also introduce the notion of a *geometric automorphic datum* $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ in §2.6. The idea is to give more structure to an automorphic datum so that it makes sense to base change to an extension of the ground field k . In particular, we replace K_x by a pro-algebraic subgroup \mathbf{K}_x of the loop group of G at x , and replace the character χ_x by a rank one character sheaf \mathcal{K}_x on \mathbf{K}_x . The geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$

not only recovers the automorphic datum by the sheaf-to-function correspondence, but also gives an automorphic datum $(\omega', K_{S'}, \chi_{S'})$ for G over $F \otimes_k k'$ for any finite extension k'/k .

We introduce the notion of rigidity for a geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ in Definition 2.7.2. The datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is called *strongly rigid* if for every finite extension k'/k , there is a unique $(\omega', K_{S'}, \chi_{S'})$ -typical automorphic representation π' of $G(\mathbb{A}_{F \otimes_k k'})$ for the automorphic datum $(\omega', K_{S'}, \chi_{S'})$ obtained from base change. We also introduce the notion of weak rigidity. We expect that strong (resp. weak) rigidity of geometric automorphic data should correspond to the physical (resp. cohomological) rigidity of \widehat{G} -local systems under the Langlands correspondence.

The first main result is Theorem 2.7.10, which gives a sheaf-theoretic criterion for rigidity of geometric automorphic data, using “relevant” points on certain moduli stack $\text{Bun}_G(\mathbf{K}_S)$ of G -torsors over X with level structures.

1.3.2. *Construction of the local system.* The Langlands correspondence $\pi \mapsto \mathcal{F}_\pi$ for cuspidal automorphic representations π has been established by recent work of V.Lafforgue [26]. However, the construction in [26] does not give an explicit description of the \widehat{G} -local system \mathcal{F}_π . We would like to construct the \widehat{G} -local system \mathcal{F}_π explicitly in the rigid situation.

The main results in this direction are Theorem 4.4.2 and Proposition 4.5.2. The former guarantees that under a certain rigidity assumption on the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$, the \widehat{G} -local system \mathcal{F}_π for an (ω, K_S, χ_S) -typical automorphic representation π can be constructed. The latter gives a concrete description of \mathcal{F}_π as a direct summand of the direct image sheaf of a family of varieties over $X - S$.

We also make a Conjecture 4.1.2 which relates the Artin conductor of \mathcal{F}_π at $x \in S$ and certain relative dimension of \mathbf{K}_x in a precise way. This conjecture has been verified for many examples in §5 and §6.

The above discussion is over-simplified. In the main body of the paper we consider quasi-split groups G that are not assumed to be simply-connected. Complications arise in the general case because the moduli stack $\text{Bun}_G(\mathbf{K}_S)$ has several connected components. The definition of rigidity for geometric automorphic data needs to be modified to take care of unramified twists of automorphic representations. In this generality, Theorem 4.4.2 only guarantees the existence of \mathcal{F}_π as a \widehat{G} -local system in a weakened sense. With a bit extra structure we can construct an actual \widehat{G} -local system \mathcal{F}_π from $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$, see Theorem 4.7.3.

1.3.3. *The method.* The method we use to construct \widehat{G} -local systems is the geometric Langlands correspondence, a program initiated by Drinfeld and Laumon. The geometric Langlands correspondence can be set up over a general field, and can be viewed as an upgraded version of the Langlands correspondence, in which functions are replaced by sheaves. The starting point is an observation of Weil which says that the double coset $G(F) \backslash G(\mathbb{A}_F) / \prod_{x \in |X|} G(\mathcal{O}_x)$ may be interpreted as the set of G -torsors over

X . Let Bun_G be the moduli stack of G -torsors over X . By the sheaf-to-function correspondence of Grothendieck, $\overline{\mathbb{Q}}_\ell$ -sheaves on Bun_G (for the étale topology) should be thought of as an upgraded version of automorphic forms, which are $\overline{\mathbb{Q}}_\ell$ -valued functions on the automorphic space $G(F)\backslash G(\mathbb{A}_F)/\prod_{x \in |X|} G(\mathcal{O}_x)$ (in the everywhere unramified case).

Given a geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$, assuming each \mathcal{K}_x descends to a finite-dimensional quotient \mathbf{L}_x of \mathbf{K}_x , then the automorphic sheaves for the situation are sheaves over $\text{Bun}_G(\mathbf{K}_S^+)$ (the moduli stack of G -torsors over X with $\mathbf{K}_x^+ = \ker(\mathbf{K}_x \rightarrow \mathbf{L}_x)$ level structures at $x \in S$), together with certain equivariance structures dictated by Ω and \mathcal{K}_S . Such sheaves form a derived category $D_{G,\Omega}(\mathbf{K}_S, \mathcal{K}_S)$. Rigidity roughly means that the category $D_{G,\Omega}(\mathbf{K}_S, \mathcal{K}_S)$ contains a unique irreducible perverse sheaf \mathcal{A} .

The advantage of the geometric Langlands correspondence is that it allows us to apply geometric Hecke operators to \mathcal{A} , and in this way we can construct the Hecke eigen \widehat{G} -local system \mathcal{F} explicitly. Details will be explained in §4.2-4.3.

1.3.4. *Examples.* There are three classes of examples of geometric automorphic data in the case $G = \text{GL}_2$ that we work out in details at various places of the paper. The corresponding local systems in these examples are exactly the three types of hypergeometric sheaves of rank two constructed by Katz, of which the Kloosterman sheaf of rank two is a special case. We prove in §2.8 that these geometric automorphic data are strongly rigid. We show in §3.3 that the hypergeometric sheaves of rank two essentially exhaust all rigid local systems of rank two on \mathbb{P}^1 ramified at at least two points. In §4.5 we explain how these geometric automorphic data are related to hypergeometric sheaves via the geometric Hecke operators.

In §5 and §6 we review the work [19], [44] and [42], which are the main examples that lead to the theory of rigid automorphic representations.

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2. Rigidity for automorphic representations

§2.1-§2.3 contain background material on reductive groups and automorphic representations. §2.4-§2.5 provide background on the sheaf-theoretic interpretation of automorphic forms. More results on rank one character sheaves are contained in Appendix A. The key definitions appear in §2.6 and §2.7, where we introduce geometric automorphic data and the notion of rigidity for them. The main result is Theorem 2.7.10 which gives a criterion for the rigidity of geometric automorphic data. Examples in GL_2 are worked out in details in §2.8.

2.1. Function field. Let k be a perfect field and fix an algebraic closure \bar{k} of k . Let X be a projective, smooth and geometrically connected curve X over k . Let $F = k(X)$ be the field of rational functions on X . Let $\Gamma_F = \text{Gal}(F^s/F)$ be the absolute Galois group of F , where F^s is a separable closure of F . The field $F \otimes_k \bar{k}$ has absolute Galois group $I_F := \text{Gal}(F_k^s/F \otimes_k \bar{k}) \triangleleft \Gamma_F$ and $\Gamma_F/I_F \cong \text{Gal}(\bar{k}/k)$.

Let $|X|$ be the set of closed points of X . For each $x \in |X|$, let F_x denote the completion of F at the place x . The valuation ring and residue field of F_x are denoted by \mathcal{O}_x and k_x . The maximal ideal of \mathcal{O}_x is denoted by \mathfrak{m}_x . The absolute Galois group of F_x is denoted by Γ_x , inside of which we have the inertia group $I_x = \text{Gal}(F_x^s/F_x^{\text{ur}}) \triangleleft \Gamma_x$ where F_x^{ur} is the maximal unramified extension of F_x inside a separable closure F_x^s . The quotient group Γ_x/I_x is $\text{Gal}(\bar{k}/k_x)$. Fixing an embedding $F^s \hookrightarrow F_x^s$ for each x gives an embedding of the Galois groups $\Gamma_x \hookrightarrow \Gamma_F$ and $I_x \hookrightarrow I_F$.

The ring of adèles of F is the restricted product

$$\mathbb{A}_F := \prod'_{x \in |X|} F_x$$

where for almost all x , the x -component of an element $a \in \mathbb{A}_F$ lies in \mathcal{O}_x . It is equipped with a natural topology: a neighborhood basis of 0 is given by $K_D = \prod_{x \in |X|} \mathfrak{m}_x^{d_x}$ where $D = \sum_{x \in |X|} d_x \cdot x$ is an effective divisor on X .

When k is a finite field, F is a global function field, and $|X|$ can be identified with the set of places of F . The ring of adèles \mathbb{A}_F is locally compact. The Galois group $\text{Gal}(\bar{k}/k)$ is topologically generated by the geometric Frobenius element Frob_k . Recall the Weil group $W_F \subset \Gamma_F$ is the preimage of $\text{Frob}_k^{\mathbb{Z}}$ under the quotient $\Gamma_F \twoheadrightarrow \text{Gal}(\bar{k}/k)$.

2.2. Groups over a function field.

2.2.1. *Tori.* For any diagonalizable group T over any field K , we denote the character and cocharacter lattices of $T \otimes_K \bar{K}$ by $\mathbb{X}^*(T)$ and $\mathbb{X}_*(T)$, where \bar{K} is an algebraic closure of K . They are discrete abelian groups with continuous actions of $\text{Gal}(\bar{K}/K)$.

2.2.2. *Quasi-split groups.* We start with a split connected reductive group \mathbb{G} over k . Let $\mathbb{Z}\mathbb{G}$ be the center of \mathbb{G} and $\mathbb{G}^{\text{ad}} = \mathbb{G}/\mathbb{Z}\mathbb{G}$ be the adjoint form of \mathbb{G} . Fix a pinning of \mathbb{G} , i.e., a split maximal torus \mathbb{T} , a Borel subgroup \mathbb{B} containing \mathbb{T} (hence a based root system $\Phi \subset \mathbb{X}_*(\mathbb{T})$ with simple roots Δ) and an isomorphism $\mathbb{U}_\alpha \cong \mathbb{G}_a$ for each simple root subgroup $\mathbb{U}_\alpha \subset \mathbb{G}$ ($\alpha \in \Delta$). Let $\text{Aut}^\dagger(\mathbb{G})$ be the pinned automorphism group of \mathbb{G} , i.e., automorphisms of \mathbb{G} preserving the pinning. Then $\text{Aut}^\dagger(\mathbb{G})$ is canonically isomorphic to the outer automorphism group $\text{Out}(\mathbb{G}) = \text{Aut}(\mathbb{G})/\mathbb{G}^{\text{ad}}$.

Fix a homomorphism $\theta : \Gamma_F \rightarrow \text{Aut}^\dagger(\mathbb{G})$. The image of θ is of the form $\Gamma = \text{Gal}(F'/F)$ for a finite Galois extension F'/F inside F^s . The extension of F'/F corresponds to a Galois branched cover $\theta_X : X' \rightarrow X$ of X . We assume that π is tamely ramified. Note that we allow base field extension,

for example X' can be $X \otimes_k k'$ for some finite extension k'/k . For any F -algebra R , we define

$$(2.1) \quad G(R) = \{g \in \mathbb{G}(F' \otimes_F R) \mid \gamma g = \theta(\gamma)g, \forall \gamma \in \Gamma\}.$$

Here $g \mapsto \gamma g$ denotes the action of Γ on $\mathbb{G}(F' \otimes_F R)$ induced from its action on F' . The functor (2.1) is represented by a connected reductive group G over F which is a quasi-split form of \mathbb{G} and which becomes split over F' .

The torus $\mathbb{T} \subset \mathbb{G}$ gives rise to a maximal torus $T \subset G$. We use Z , G^{der} , G^{ad} , D to denote the center, the derived group, the adjoint quotient and the maximal torus quotient of G .

2.2.3. The Langlands dual of G . Let $\widehat{T} = \mathbb{X}^*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{G}_{m, \overline{\mathbb{Q}}_\ell}$ be a torus over $\overline{\mathbb{Q}}_\ell$. Let \widehat{G} be the connected split reductive group over $\overline{\mathbb{Q}}_\ell$ with maximal torus \widehat{T} and based root system $\Delta(\widehat{G}) \subset \Phi(\widehat{G})$ identified with based coroot system $\Delta^\vee \subset \Phi^\vee$ of \mathbb{G} with respect to \mathbb{T} . We extend the based root system of \widehat{G} into a pinning, and denote the pinned automorphism of \widehat{G} by $\text{Aut}^\dagger(\widehat{G})$. Then there is a canonical isomorphism $\text{Aut}^\dagger(\widehat{G}) \cong \text{Aut}^\dagger(\mathbb{G})$. The homomorphism $\theta : \Gamma_F \rightarrow \text{Aut}^\dagger(\mathbb{G})$ then induces $\hat{\theta} : \Gamma_F \rightarrow \text{Aut}^\dagger(\widehat{G})$ that factors through the finite quotient $\Gamma = \text{Gal}(F'/F)$.

2.2.4. Integral models of G . An integral model of G over X is a smooth group scheme \mathcal{G} over X together with an isomorphism of group schemes $\iota : \mathcal{G}|_{\text{Spec } F} \xrightarrow{\sim} G$ over F (here we identify $\text{Spec } F$ with the generic point of X). One can construct an integral model of G as follows. Let $U_\theta = X - S_\theta$ where S_θ is the ramification locus of θ_X . Then the same formula (2.1), now with R a commutative k -algebra with a morphism $\text{Spec } R \rightarrow U_\theta$, gives a reductive group scheme \mathcal{G}_{U_θ} over U_θ . We then need to extend \mathcal{G}_{U_θ} to a smooth group scheme over X . One can do this locally at every closed point $x \in S_\theta$ and patch the results together. At $x \in S_\theta$, the theory of Bruhat and Tits gives several choices of smooth group schemes \mathcal{G}_x over $\text{Spec } \mathcal{O}_x$ extending $G|_{\text{Spec } F_x}$ (the parahoric subgroups), and we may choose any of them. Once we choose a model \mathcal{G} for G , we can talk about $\mathcal{G}(\mathcal{O}_x)$ for any $x \in |X|$.

For the maximal torus T of G and the center Z of G , we may use formulae similar to (2.1) to define their integral models \mathcal{T} and \mathcal{Z} . Then \mathcal{T} is the finite-type version of the Néron model of T over X . We make the following technical assumption (which may turn out to hold always, and which holds in all examples we consider):

The group scheme \mathcal{Z} fits into an exact sequence $1 \rightarrow \mathcal{Z} \rightarrow \mathcal{T}_1 \rightarrow \mathcal{T}_2 \rightarrow 1$, where \mathcal{T}_1 and \mathcal{T}_2 are smooth group schemes over X whose generic fibers are tori.

2.2.5. Loop groups. For each $x \in |X|$, we may view $G(F_x)$ as the set of k -points of a group indscheme $L_x G$ called *the loop group of G at x* . The loop group of G at x represents the functor $R \mapsto G(R \widehat{\otimes}_k F_x)$ where R is any k -algebra and $R \widehat{\otimes}_k F_x$ is the completion of $R \otimes_k F_x$ with respect to the $R \otimes \mathfrak{m}_x$ -adic topology. Similarly we may define a pro-algebraic group over k called the positive loop group $L_x^+ \mathcal{G}$, representing the functor $R \mapsto \mathcal{G}(R \widehat{\otimes}_k \mathcal{O}_x)$.

More generally, for a parahoric subgroup $\mathbf{P}_x \subset G(F_x)$ we may define the associated positive loop group which we still denote by \mathbf{P}_x (but viewed as a subgroup scheme of $L_x G$).

Warning: $L_x G$ and $L_x^+ G$ are both over k and not over k_x : alternatively we may define versions of loop groups over k_x and take restriction of scalars to k .

2.3. Automorphic representations and automorphic data. In this subsection we assume k to be finite. We also fix a prime $\ell \neq \text{char}(k)$, and choose an algebraic closure $\overline{\mathbb{Q}}_\ell$ of \mathbb{Q}_ℓ . We will consider automorphic representations of $G(\mathbb{A}_F)$ on $\overline{\mathbb{Q}}_\ell$ -vector spaces. At this point one could replace $\overline{\mathbb{Q}}_\ell$ by \mathbb{C} , but it will be convenient to use $\overline{\mathbb{Q}}_\ell$ directly because later we will relate automorphic forms with $\overline{\mathbb{Q}}_\ell$ -sheaves.

The group $G(\mathbb{A}_F)$ of \mathbb{A}_F -points of G is the restricted product of $G(F_x)$ over $x \in |X|$ with respect to the subgroups $\mathcal{G}(\mathcal{O}_x)$. Then $G(\mathbb{A}_F)$ is a locally compact topological group with a neighborhood basis of the identity given by $\prod_{x \in S} K_x \times \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x)$, where $S \subset |X|$ is finite, $K_x \subset G(F_x)$ is a compact open subgroup for all $x \in S$. With this topology on $G(\mathbb{A}_F)$ we may talk about locally constant $\overline{\mathbb{Q}}_\ell$ -valued functions on $G(\mathbb{A}_F)$. Let $C^\infty(G(F) \backslash G(\mathbb{A}_F))$ be the space of locally constant $\overline{\mathbb{Q}}_\ell$ -valued functions on $G(\mathbb{A}_F)$ that are left invariant under $G(F)$. By definition, we have

$$C^\infty(G(F) \backslash G(\mathbb{A}_F)) = \varinjlim_{(S, \{K_x\}_{x \in S})} \text{Fun}(G(F) \backslash G(\mathbb{A}_F) / (\prod_{x \in S} K_x \times \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x))),$$

where $\text{Fun}(-)$ means the space of $\overline{\mathbb{Q}}_\ell$ -valued functions. The group $G(\mathbb{A}_F)$ acts on $C^\infty(G(F) \backslash G(\mathbb{A}_F))$ via right translation: $(g \cdot f)(x) = f(xg)$ where $g \in G(\mathbb{A}_F)$, $f \in C^\infty(G(F) \backslash G(\mathbb{A}_F))$ and $x \in G(F) \backslash G(\mathbb{A}_F)$.

2.3.1. **DEFINITION** (See [3, Definition 5.8]). (1) A function $f \in C^\infty(G(F) \backslash G(\mathbb{A}_F))$ is called an *automorphic form* if for some (equivalently any) $x \in |X|$, the $G(F_x)$ -module spanned by right $G(F_x)$ -translations of f is admissible. Denote the space of automorphic forms by \mathcal{A}_G . This is a $G(\mathbb{A}_F)$ -module under right translation.

(2) An *automorphic representation* of $G(\mathbb{A}_F)$ is an irreducible subquotient of \mathcal{A}_G .

2.3.2. *Kottwitz homomorphism.* For the local field F_x , Kottwitz defined a homomorphism [25, §7]

$$(2.2) \quad \kappa_{G, F_x} : G(F_x) \rightarrow (\mathbb{X}^*(Z\widehat{G})_{I_x})^{\text{Frob}_x}.$$

Here $Z\widehat{G}$ is the center of \widehat{G} , and $I_x \triangleleft \Gamma_F$ acts on \widehat{G} , and hence also on $\mathbb{X}^*(Z\widehat{G})$ via $\hat{\theta}$. For the global field F one can similarly define a homomorphism

$$(2.3) \quad \kappa_{G, F} : G(\mathbb{A}_F) \rightarrow (\mathbb{X}^*(Z\widehat{G})_{I_F})^{\text{Frob}_k}$$

which is trivial on $G(F)$ and is compatible with (2.2) in the obvious sense. Since [25] works in the local setting, we sketch the construction of (2.3) along the same lines.

To construct $\kappa_{G,F}$, we first work with $F \otimes_k \bar{k}$ instead of F and try to construct $\kappa_{G,F \otimes_k \bar{k}} : G(\mathbb{A}_{F \otimes_k \bar{k}}) \rightarrow \mathbb{X}^*(Z\widehat{G})_{I_F}$, then take Frob_k -invariants to get $\kappa_{G,F}$.

In the sequel we assume $k = \bar{k}$. To construct $\kappa_{G,F}$, we follow these steps (see [25, §7.2-7.4]):

- When $G = T$ is a torus, it can be fit into an exact sequence $T_1 \rightarrow T_2 \rightarrow T \rightarrow 1$ where T_1 and T_2 are induced tori and the corresponding sequence $T_1(F) \backslash T_1(\mathbb{A}_F) \rightarrow T_2(F) \backslash T_2(\mathbb{A}_F) \rightarrow T(F) \backslash T(\mathbb{A}_F) \rightarrow 1$ is also exact. For an induced torus $T' = \text{Res}_F^E \mathbb{G}_m$ (where E is a finite Galois F -algebra), the map $\kappa_{T',F} : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{Z} = \mathbb{Z}[\text{Gal}(E/F)]_{I_F}$ is given by the degree map on idèles. We then define $\kappa_{T,F}$ to be the unique arrow making the following diagram with exact rows commutative

$$\begin{array}{ccccccc} T_1(F) \backslash T_1(\mathbb{A}_F) & \longrightarrow & T_2(F) \backslash T_2(\mathbb{A}_F) & \longrightarrow & T(F) \backslash T(\mathbb{A}_F) & \longrightarrow & 1 \\ \downarrow \kappa_{T_1,F} & & \downarrow \kappa_{T_2,F} & & \downarrow \kappa_{T,F} & & \\ \mathbb{X}_*(T_1)_{I_F} & \longrightarrow & \mathbb{X}_*(T_2)_{I_F} & \longrightarrow & \mathbb{X}_*(T)_{I_F} & \longrightarrow & 0 \end{array}$$

- When G^{der} is simply-connected, $Z\widehat{G}$ is a torus and $\mathbb{X}^*(Z\widehat{G}) = \mathbb{X}_*(D)$. The map $\kappa_{G,F}$ is defined as the composition of $G(\mathbb{A}_F) \rightarrow D(\mathbb{A}_F)$ and $\kappa_{D,F}$.
- In general, one can find a central extension $1 \rightarrow C \rightarrow \widetilde{G} \rightarrow G \rightarrow 1$ such that $\widetilde{G}^{\text{der}}$ is simply-connected and C is a torus. We have already defined $\kappa_{C,F}$ and $\kappa_{\widetilde{G},F}$ in the previous steps. We define $\kappa_{G,F}$ to be the unique arrow making the following diagram with exact rows commutative

$$\begin{array}{ccccccc} C(\mathbb{A}_F) & \longrightarrow & \widetilde{G}(\mathbb{A}_F) & \longrightarrow & G(\mathbb{A}_F) & \longrightarrow & 1 \\ \downarrow \kappa_{C,F} & & \downarrow \kappa_{\widetilde{G},F} & & \downarrow \kappa_{G,F} & & \\ \mathbb{X}_*(\widehat{C})_{I_F} & \longrightarrow & \mathbb{X}_*(Z\widehat{G})_{I_F} & \longrightarrow & \mathbb{X}_*(Z\widehat{G})_{I_F} & \longrightarrow & 0 \end{array}$$

2.3.3. DEFINITION. An *unramified character* of $G(\mathbb{A}_F)$ is a smooth character $\chi : G(\mathbb{A}_F) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ that factors through $\kappa_{G,F}$. Given an irreducible representation π of $G(\mathbb{A}_F)$ and an unramified character χ of $G(\mathbb{A}_F)$, we obtain another irreducible representation $\pi \otimes \chi$, called an *unramified twist* of π .

2.3.4. REMARK. Unramified characters of $G(\mathbb{A}_F)$ are in bijection with $(Z\widehat{G})_{\text{Frob}_k}^{I_F}$. Langlands correspondence for function fields predicts that to an automorphic representation π of $G(\mathbb{A}_F)$ one should attach a continuous cocycle $\rho : W_F \rightarrow \widehat{G}(\mathbb{Q}_\ell)$ (with respect to the action $\widehat{\theta}$ of Γ_F on \widehat{G}) up to \widehat{G} -conjugacy, i.e., a class in $H_{\text{cont}}^1(W_F, \widehat{G})$. Since $W_F/I_F = \text{Frob}_k^{\mathbb{Z}}$, there is a natural map $(Z\widehat{G})_{\text{Frob}_k}^{I_F} = H^1(\text{Frob}_k^{\mathbb{Z}}, (Z\widehat{G})^{I_F}) \rightarrow H_{\text{cont}}^1(W_F, Z\widehat{G})$,

and the latter acts on $H_{\text{cont}}^1(W_F, \widehat{G})$. Therefore there is a natural action of $(Z\widehat{G})_{\text{Frob}_k}^{I_F}$ on $H_{\text{cont}}^1(W_F, \widehat{G})$. The Langlands correspondence is expected to intertwine unramified twists of automorphic representations and the action of $(Z\widehat{G})_{\text{Frob}_k}^{I_F}$ on $H_{\text{cont}}^1(W_F, \widehat{G})$.

2.3.5. Central characters and restricted central characters. Let $Z \subset G$ be the center. For a continuous character $\omega : Z(F) \backslash Z(\mathbb{A}_F) \rightarrow \overline{\mathbb{Q}}_\ell^\times$, let $C_\omega^\infty(G(F) \backslash G(\mathbb{A}_F)) \subset C^\infty(G(F) \backslash G(\mathbb{A}_F))$ be the subspace on which $Z(\mathbb{A}_F)$ acts through ω by right translation. Similarly let

$$\mathcal{A}_{G,\omega} = \mathcal{A}_G \cap C_\omega^\infty(G(F) \backslash G(\mathbb{A}_F)).$$

Let $Z(\mathbb{A}_F)^\natural \subset Z(\mathbb{A}_F)$ be the kernel of the composition $Z(\mathbb{A}_F) \rightarrow G(\mathbb{A}_F) \xrightarrow{\kappa_{G,F}} \mathbb{X}^*(Z\widehat{G})_{I_F}^{\text{Frob}_k}$. For each $x \in |X|$, let $Z(F_x)^\natural \subset Z(F_x)$ be the kernel of $Z(F_x) \rightarrow G(F_x) \xrightarrow{\kappa_{G,F_x}} \mathbb{X}^*(Z\widehat{G})_{I_x}^{\text{Frob}_x}$. Then $Z(F_x)^\natural$ is a compact open subgroup of $Z(F_x)$, and for almost all $x \in |X|$, $Z(F_x)^\natural = \mathcal{Z}(\mathcal{O}_x)$. Therefore, for sufficiently large S and sufficiently small compact open subgroup $K_{Z,S} \subset \prod_{x \in S} Z(F_x)$, $\prod_{x \notin S} \mathcal{Z}(\mathcal{O}_x) \times K_{Z,S}$ is contained in $(G(F) \backslash G(\mathbb{A}_F))^\natural$ and the quotient $Z(F) \backslash Z(\mathbb{A}_F)^\natural / \prod_{x \notin S} \mathcal{Z}(\mathcal{O}_x) \times K_{Z,S}$ is finite. We call a continuous character $\omega^\natural : Z(F) \backslash Z(\mathbb{A}_F)^\natural \rightarrow \overline{\mathbb{Q}}_\ell^\times$ a *restricted central character*. We can similarly define the spaces $C_{\omega^\natural}^\infty(G(F) \backslash G(\mathbb{A}_F))$ and $\mathcal{A}_{G,\omega^\natural}$, which are stable under unramified twists of $G(\mathbb{A}_F)$. For a restricted central character ω^\natural , we may restrict it to $Z(F_x)^\natural$ and get a continuous character $\omega_x^\natural : Z(F_x)^\natural \rightarrow \overline{\mathbb{Q}}_\ell^\times$.

2.3.6. Cusp forms. A function $f \in C^\infty(G(F) \backslash G(\mathbb{A}_F))$ is called a *cuspidal form* if for every parabolic subgroup $P \subset G$ defined over F with unipotent radical N_P , we have

$$\int_{N_P(F) \backslash N_P(\mathbb{A}_F)} f(ng) dn = 0$$

for all $g \in G(\mathbb{A}_F)$. For a central character ω , the space of cuspidal forms in $\mathcal{A}_{G,\omega}$ form a sub- $G(\mathbb{A}_F)$ -module $\mathcal{A}_{G,\omega}^{\text{cusp}} \subset \mathcal{A}_{G,\omega}$. It is known that $\mathcal{A}_{G,\omega}^{\text{cusp}}$ decomposes discretely into a direct sum of irreducible admissible $G(\mathbb{A}_F)$ -modules, called *cuspidal automorphic representations*.

2.3.7. DEFINITION. Let $S \subset |X|$ be finite.

(1) An *automorphic datum* for G is a triple (ω, K_S, χ_S) where

- $\omega : Z(F) \backslash Z(\mathbb{A}_F) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is a central character;
- K_S is a collection of compact open subgroups $K_x \subset G(F_x)$, one for each $x \in S$;
- χ_S is a collection of smooth characters $\chi_x : K_x \rightarrow \overline{\mathbb{Q}}_\ell^\times$, one for each $x \in S$.

Such a triple should satisfy the following compatibility conditions.

- For each $x \in S$, $\omega_x|_{Z(F_x) \cap K_x} = \chi_x|_{Z(F_x) \cap K_x}$;
- For each $x \notin S$, ω_x is trivial on $Z(F_x) \cap \mathcal{G}(\mathcal{O}_x)$.

(2) A *restricted automorphic datum* is a triple $(\omega^\natural, K_S, \chi_S)$, where K_S and χ_S are as above, $\omega^\natural : Z(F) \backslash Z(\mathbb{A}_F)^\natural \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is a restricted central character, and the compatibility conditions become

- For each $x \in S$, $\omega_x^\natural|_{Z(F_x)^\natural \cap K_x} = \chi_x|_{Z(F_x)^\natural \cap K_x}$;
- For each $x \notin S$, ω_x^\natural is trivial on $Z(F_x)^\natural \cap \mathcal{G}(\mathcal{O}_x) = Z(F_x) \cap \mathcal{G}(\mathcal{O}_x)$.

2.3.8. DEFINITION. Let (ω, K_S, χ_S) be an automorphic datum. An automorphic representation $\pi = \otimes'_x \pi_x$ of $G(\mathbb{A}_F)$ is called (ω, K_S, χ_S) -*typical* if

- The central character of π is ω ;
- For each $x \in S$, the local component π_x contains a nonzero vector on which K_x acts through the character χ_x ;
- For each $x \notin S$, π_x is spherical; i.e., $\pi_x^{\mathcal{G}(\mathcal{O}_x)} \neq 0$.

When $(\omega^\natural, K_S, \chi_S)$ is a restricted automorphic datum, we define $(\omega^\natural, K_S, \chi_S)$ -*typical automorphic representations* in a similar way, except that the first condition changes to that the central character of π restricted to $Z(F) \backslash Z(\mathbb{A}_F)^\natural$ is ω^\natural . Similarly, we define the notion of (K_S, χ_S) -*typical automorphic representations* by dropping the central character condition.

Let

$$(2.4) \quad C_{\mathcal{G}, \omega}(K_S, \chi_S) := C_\omega(G(F) \backslash G(\mathbb{A}_F) / \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x) \times \prod_{x \in S} (K_x, \chi_x)).$$

denote the space of functions on $G(F) \backslash G(\mathbb{A}_F)$ which are eigenvectors under $Z(\mathbb{A}_F)$ with eigenvalue ω , invariant under $\prod_{x \notin S} \mathcal{G}(\mathcal{O}_x)$ and are eigenvectors under the action of each K_x with eigenvalue χ_x , $x \in S$. Similarly one defines $C_{\mathcal{G}, \omega^\natural}(K_S, \chi_S)$ for a restricted automorphic datum $(\omega^\natural, K_S, \chi_S)$.

2.3.9. LEMMA. *Let (ω, K_S, χ_S) be an automorphic datum. If there is a subset $\Sigma \subset G(F) \backslash G(\mathbb{A}_F)$, compact modulo $Z(\mathbb{A}_F)$, such that all functions in $C_{\mathcal{G}, \omega}(K_S, \chi_S)$ vanish outside Σ , then any (ω, K_S, χ_S) -typical automorphic representation π of $G(\mathbb{A}_F)$ is cuspidal. Moreover,*

$$(2.5) \quad C_{\mathcal{G}, \omega}(K_S, \chi_S) = \bigoplus_{\pi} \left(\bigotimes_{x \in S} \pi_x^{(K_x, \chi_x)} \right) \otimes \left(\bigotimes_{x \notin S} \pi_x^{\mathcal{G}(\mathcal{O}_x)} \right)$$

where π runs over (ω, K_S, χ_S) -typical automorphic representations, and $\pi_x^{(K_x, \chi_x)}$ denotes the eigenspace of π_x under K_x on which it acts through χ_x .

PROOF. If every function in $C_{\mathcal{G}, \omega}(K_S, \chi_S)$ is supported on some Σ compact modulo $Z(\mathbb{A}_F)$, then the space $C_{\mathcal{G}, \omega}(K_S, \chi_S) \subset \text{Fun}(G(F) \backslash G(\mathbb{A}_F) / \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x) \times \prod_{x \in S} \ker(\chi_x))$ is finite-dimensional and stable under the action of the spherical Hecke algebra at all places $x \notin S$. Then we appeal to [26, Lemme 7.16] to conclude that $C_{\mathcal{G}, \omega}(K_S, \chi_S)$ consists of cusp forms.

Now let π be a (ω, K_S, χ_S) -typical automorphic representation. By definition π is a subquotient of the space \mathcal{A}_G , therefore we may assume π is a

quotient of a $G(\mathbb{A}_F)$ -submodule $\mathcal{A}' \subset \mathcal{A}_{G,\omega}$. Let $\mathcal{B} = \mathcal{A}' \cap C_{\mathcal{G},\omega}(K_S, \chi_S)$, then \mathcal{B} consists of cusp forms. By the semisimplicity of the action of the compact group $\prod_{x \notin S} \mathcal{G}(\mathcal{O}_x) \times \prod_{x \in S} K_x$, we have $\mathcal{B} \rightarrow (\otimes_{x \in S} \pi_x^{(K_x, \chi_x)}) \otimes (\otimes_{x \notin S} \pi_x^{\mathcal{G}(\mathcal{O}_x)})$. The $G(\mathbb{A}_F)$ -module $\tilde{\mathcal{B}}$ generated by \mathcal{B} is then contained in the space of cusp forms, and we have $\tilde{\mathcal{B}} \rightarrow \pi$. Therefore π is also cuspidal. The equality (2.5) follows from the fact that the cuspidal spectrum is discrete. \square

2.4. Moduli of G -torsors. A. Weil observed that the double coset $G(F) \backslash G(\mathbb{A}_F) / \prod_{x \in |X|} \mathcal{G}(\mathcal{O}_x)$ may be interpreted as the set of k -points of a certain moduli stack. We recall this interpretation in this subsection.

2.4.1. *Moduli of \mathcal{G} -torsors.* Let k be any field. For an integral model \mathcal{G} of G over X , we may talk about \mathcal{G} -torsors on X : these are schemes $\mathcal{E} \rightarrow X$ with a right action of \mathcal{G} such that étale locally over X , \mathcal{E} is \mathcal{G} -equivariantly isomorphic to \mathcal{G} for the right translation of \mathcal{G} on itself. Similarly one can define \mathcal{G} -torsors over $X \otimes_k R$ for any k -algebra R . Let $\text{Bun}_{\mathcal{G}}$ be the moduli stack of \mathcal{G} -torsors over X , i.e., $\text{Bun}_{\mathcal{G}}(R)$ is the groupoid of \mathcal{G} -torsors over $X \otimes_k R$. Then $\text{Bun}_{\mathcal{G}}$ is an algebraic stack locally of finite type.

2.4.2. **EXAMPLE.** When $G = \text{GL}_n$, there is a natural way to assign a vector bundle \mathcal{V} of rank n to a GL_n -torsor \mathcal{E} and vice versa: $\mathcal{V} = \mathcal{E} \times^{\text{GL}_n} \mathbb{A}_k^n$ and $\mathcal{E} = \underline{\text{Isom}}(\mathcal{O}_X^n, \mathcal{V})$. Therefore we have an isomorphism of stacks

$$\text{Bun}_{\text{GL}_n} \cong \text{Bun}_n$$

the latter being the moduli stack of vector bundles of rank n on X . In particular, for $n = 1$, $\text{Bun}_{\text{GL}_1} \cong \text{Bun}_1 \cong \text{Pic}_X$.

Similarly, Bun_{SL_n} is equivalent to the groupoid of pairs (\mathcal{V}, ι) where \mathcal{V} is a vector bundle of rank n on X and $\iota : \wedge^n \mathcal{V} \xrightarrow{\sim} \mathcal{O}_X$ is a trivialization of the determinant of \mathcal{V} .

When $G = \text{PGL}_n$, $\text{Bun}_{\text{PGL}_n}$ is isomorphic to the quotient stack $\text{Bun}_n / \text{Pic}_X$ with Pic_X acting on Bun_n via tensor product. The content of this statement is Tsen's theorem on the vanishing of the Brauer group for curves over an algebraically closed field.

2.4.3. *The set $\text{H}_{\mathcal{G}}^1(F, G)$.* For each $x \in |X|$, we have a map $\text{H}^1(\text{Spec } \mathcal{O}_x, \mathcal{G}) \rightarrow \text{H}^1(F_x, G)$ obtained by restricting a \mathcal{G} -torsor over $\text{Spec } \mathcal{O}_x$ to $\text{Spec } F_x$. Define $\text{H}_{\mathcal{G}}^1(F, G) \subset \text{H}^1(F, G)$ using the Cartesian diagram

$$\begin{array}{ccc} \text{H}_{\mathcal{G}}^1(F, G) & \longrightarrow & \prod_{x \in |X|} \text{Im}(\text{H}^1(\text{Spec } \mathcal{O}_x, \mathcal{G}) \rightarrow \text{H}^1(F_x, G)) \\ \downarrow & & \downarrow \\ \text{H}^1(F, G) & \longrightarrow & \prod_{x \in |X|} \text{H}^1(F_x, G) \end{array}$$

Note that $\text{H}^1(F \otimes_k \bar{k}, G)$ hence $\text{H}_{\mathcal{G}}^1(F \otimes_k \bar{k}, G)$ is a singleton (see [2, §8.6], [39, II, 2.3, Remarque 1]). When k is finite and \mathcal{G} has connected geometric fibers, $\text{H}^1(\text{Spec } \mathcal{O}_x, \mathcal{G})$ is trivial and $\text{H}_{\mathcal{G}}^1(F, G)$ is the same as $\ker^1(F, G) :=$

$\ker(\mathrm{H}^1(F, G) \rightarrow \prod_{x \in |X|} \mathrm{H}^1(F_x, G))$. It is also known that when G is split and k is finite, $\ker^1(F, G)$ is a singleton ([26, Remark 7.13]). In general, $\mathrm{H}_G^1(F, G)$ measures how a \mathcal{G} -torsor over X can fail to be trivializable at the generic point of X . Any \mathcal{G} -torsor \mathcal{E} over X determines a class in $\mathrm{H}_G^1(F, G)$ by restricting it to the generic point $\eta \in X$.

For each class $\xi \in \mathrm{H}_G^1(F, G)$, we choose a \mathcal{G} -torsor \mathcal{E}_ξ with generic class ξ (this is possible since the class ξ gives a \mathcal{G} -torsor \mathcal{E}_U over some nonempty Zariski open subset $U \subset X$ and we can extend \mathcal{E}_U across the missing points $x \in X - U$ because the restriction of ξ to $\mathrm{H}^1(F_x, G)$ lies in the image of $\mathrm{H}^1(\mathrm{Spec} \mathcal{O}_x, \mathcal{G})$). Define $\mathcal{G}_\xi = \underline{\mathrm{Aut}}_G(\mathcal{E}_\xi)$, the group scheme over X of \mathcal{G} -automorphisms of \mathcal{E}_ξ . This is an inner form of \mathcal{G} . We denote by G_ξ the generic fiber of \mathcal{G}_ξ , which is an inner form of G over F .

2.4.4. *Weil's interpretation.* Weil observed a natural bijection of groupoids

$$(2.6) \quad e : \bigsqcup_{\xi \in \mathrm{H}_G^1(F, G)} G_\xi(F) \backslash G_\xi(\mathbb{A}_F) / \prod_{x \in |X|} \mathcal{G}_\xi(\mathcal{O}_x) \xrightarrow{\sim} \mathrm{Bun}_G(k).$$

In other words, not only are the isomorphism classes of both sides in bijection, but for any coset $g = (g_x) \in \prod'_{x \in |X|} G_\xi(F_x) / \mathcal{G}_\xi(\mathcal{O}_x)$, the automorphism group of $e(g)$ (as a \mathcal{G}_ξ -torsor) is isomorphic to the stabilizer of g under the left action of $G(F)$.

We give the definition of the map e on the part where ξ is the trivial class, i.e., $G(F) \backslash G(\mathbb{A}_F) / \prod_{x \in |X|} \mathcal{G}(\mathcal{O}_x) \rightarrow \mathrm{Bun}_G(k)$. We have

$$G(F) \backslash G(\mathbb{A}_F) / \prod_{x \in |X|} \mathcal{G}(\mathcal{O}_x) = \bigcup_{S \subset |X|, \text{finite}} \mathcal{G}(\mathcal{O}_{X-S}) \backslash \prod_{x \in S} G(F_x) / \mathcal{G}(\mathcal{O}_x).$$

Thus it suffices to construct maps

$$e_S : \mathcal{G}(\mathcal{O}_{X-S}) \backslash \prod_{x \in S} G(F_x) / \mathcal{G}(\mathcal{O}_x) \rightarrow \mathrm{Bun}_G(k)$$

compatible with inclusions $S \hookrightarrow S'$. To $g = (g_x) \in \prod_{x \in S} G(F_x)$, we assign a \mathcal{G} -torsor \mathcal{E}_g by gluing the trivial torsor $\mathcal{E}_{X-S}^{\mathrm{triv}} = \mathcal{G}|_{X-S}$ over $X - S$ with the trivial \mathcal{G} -torsor $\mathcal{E}_x^{\mathrm{triv}} = \mathcal{G}|_{\mathrm{Spec} \mathcal{O}_x}$ over $\mathrm{Spec} \mathcal{O}_x$ for each $x \in S$ along $\mathrm{Spec} F_x$. The gluing datum at x is given by the isomorphism $G|_{\mathrm{Spec} F_x} = \mathcal{E}_x^{\mathrm{triv}}|_{\mathrm{Spec} F_x} \xrightarrow{\sim} \mathcal{E}_{X-S}^{\mathrm{triv}}|_{\mathrm{Spec} F_x} = G|_{\mathrm{Spec} F_x}$ which is given by left multiplication by g_x . Changing the trivializations of $\mathcal{E}_{X-S}^{\mathrm{triv}}$ and $\mathcal{E}_x^{\mathrm{triv}}$ results in a right multiplication of g_x by an element in $\mathcal{G}(\mathcal{O}_x)$ and left multiplication by an element in $\mathcal{G}(\mathcal{O}_{X-S})$. Thus the isomorphism type of the resulting torsor \mathcal{G} -torsor \mathcal{E}_g only depends on the double coset $\mathcal{G}(\mathcal{O}_{X-S})g \prod_{x \in S} \mathcal{G}(\mathcal{O}_x)$. This defines the map e_S .

Working with the base field \bar{k} , noting that $\mathrm{H}^1(F \otimes_k \bar{k}, G)$ vanishes, (2.6) becomes an equivalence of groupoids

$$(2.7) \quad G(F \otimes_k \bar{k}) \backslash G(\mathbb{A}_{F \otimes_k \bar{k}}) / \prod_{x \in X(\bar{k})} \mathcal{G}(\mathcal{O}_x^{\mathrm{ur}}) \cong \mathrm{Bun}_G(\bar{k}).$$

2.4.5. EXAMPLE. Unitary similitude groups. Assume $\text{char}(k) \neq 2$. Let $\pi : X' \rightarrow X$ be a double cover (possibly ramified) of X which is generically étale with X' smooth. Let $\sigma \in \text{Aut}_X(X')$ be the nontrivial involution. Fix a line bundle \mathcal{L} on X . An \mathcal{L} -Hermitian vector bundle of rank n on X' is a pair (\mathcal{E}', h) where \mathcal{E}' is a vector bundle of rank n on X' and h is a pairing

$$h : \mathcal{E}' \otimes_{\mathcal{O}_{X'}} \sigma^* \mathcal{E}' \rightarrow \pi^* \mathcal{L}$$

such that

- (1) $\sigma^* h = h \circ s$ where $s : \mathcal{E}' \otimes_{\mathcal{O}_{X'}} \sigma^* \mathcal{E}' \rightarrow \sigma^* \mathcal{E}' \otimes_{\mathcal{O}_{X'}} \mathcal{E}'$ switches the two factors. (Here we use the canonical isomorphism $\sigma^* \pi^* \mathcal{L} \cong \pi^* \mathcal{L}$) to identify the targets of $\sigma^* h$ and $h \circ s$.
- (2) The map $\mathcal{E}' \rightarrow \underline{\text{Hom}}_{X'}(\sigma^* \mathcal{E}', \pi^* \mathcal{L}) = \sigma^* \mathcal{E}'^\vee \otimes \pi^* \mathcal{L}$ induced by h is an isomorphism.

Let $\text{Bun}_{n, X'/X}^{\text{Herm}}$ be the moduli stack classifying triples $(\mathcal{L}, \mathcal{E}', h)$ as above, i.e., \mathcal{L} is a line bundle over X and (\mathcal{E}', h) is an \mathcal{L} -Hermitian vector bundles of rank n over X' .

Let $\mathcal{E}'_0 = \mathcal{O}_{X'}^{\oplus n}$ and $h_0 : \mathcal{E}'_0 \otimes \sigma^* \mathcal{E}'_0 \rightarrow \mathcal{O}_{X'}$ be the direct sum of n copies of rank one \mathcal{O}_X -Hermitian bundles. We define a group scheme \mathcal{G} over X in the following way: for each k -algebra R and $x \in X(R)$, we denote the preimage of x in X' by $\text{Spec } R'$, then the fiber $\mathcal{G}_x(R) = \{(g, \lambda) \in \text{GL}_n(R') \times R^\times \mid h_0(g e_1, g e_2) = \lambda h_0(e_1, e_2) \text{ for all } e_1, e_2 \in R'^n\}$. The generic fiber G of \mathcal{G} is the unitary similitude group $\text{GU}(n, h_0)$ over F defined using the Hermitian form $h_0|_{\text{Spec } F}$ on F'^n .

The set $\text{H}^1(F, G)$ parametrizes F^\times -homothety classes of (F' -valued) Hermitian forms h on F'^n up to change of bases. The subset $\text{H}_{\mathcal{G}}^1(F, G) \subset \text{H}^1(F, G)$ parametrizes those Hermitian forms h on F'^n whose base change to F'_x for each $x \in |X|$ contains an \mathcal{O}'_x -lattice Λ_x such that $\Lambda_x^\perp = c \Lambda_x$ for some $c \in F_x^\times$ (here $\Lambda_x^\perp = \{v \in F'_x \mid h(v, \Lambda_x) \subset \mathcal{O}'_x\}$). We call such a lattice *essentially self-dual under h* . Let $\text{Herm}_{\text{esd}}(n, F)$ be the set of Hermitian forms on F'^n whose base change to each F'_x contains an essentially self-dual lattice. Then $\text{H}_{\mathcal{G}}^1(F, G) = \text{Herm}_{\text{esd}}(n, F) / (\text{GL}_n(F') \times F^\times)$, where $\text{GL}_n(F')$ acts by changing bases for F'^n and F^\times acts by scaling the form.

For each $h \in \text{Herm}_{\text{esd}}(n, F)$, there exists a nonempty Zariski open subset $U \subset X$ such that (F'^n, h) extends to an \mathcal{O}_U -Hermitian bundle $(\mathcal{O}_{U'}^n, h_{U'})$ where $U' = \pi^{-1}(U)$. For each $x \in X - U$ we may choose an essentially self-dual lattice $\Lambda_x^h \subset F'_x$ and $c_x \in F_x^\times / \mathcal{O}_x^\times$ such that $\Lambda_x^{h, \perp} = c_x \Lambda_x^h$. Define $c_x = 1$ for $x \in |U|$. Let \mathcal{L}_c be the line bundle on X defined by the idèle class $c = (c_x)_{x \in |X|}$. We may glue $(\mathcal{O}_{U'}^n, h_{U'})$ with the lattices Λ_x^h to form an \mathcal{L}_c -Hermitian vector bundle (\mathcal{E}'_h, h) . Then we may define the unitary similitude group scheme over X in a similar way as we defined \mathcal{G} , using (\mathcal{E}'_h, h) instead of $(\mathcal{O}_{X'}^n, h_0)$, and its generic fiber now becomes $\text{GU}(n, h)$. The equivalence

(2.6) in this case states an equivalence of groupoids

$$(2.8) \quad \bigsqcup_{[h] \in \text{Herm}_{\text{esd}}(n, F) / (\text{GL}_n(F') \times F^\times)} \text{GU}(n, h, F) \backslash \text{GU}(n, h, \mathbb{A}_F) \\ / \prod_{x \in |X|} \text{GAut}_{\mathcal{O}'_x}(\Lambda_x^h, h) \xrightarrow{\sim} \text{Bun}_n^{\text{Herm}}(k).$$

Here $\text{GAut}_{\mathcal{O}'_x}(\Lambda_x^h, h)$ is the group of \mathcal{O}'_x -linear automorphisms of Λ_x^h that preserve h up to a scalar in \mathcal{O}'_x^\times . Note that the left side of (2.8) maps to $F^\times \backslash \mathbb{A}_F^\times / \prod_x \mathcal{O}_x^\times \cong \text{Pic}_X(k)$ by taking the similitude character, which corresponds to the morphism $\text{Bun}_n^{\text{Herm}} \rightarrow \text{Pic}_X$ sending $(\mathcal{L}, \mathcal{E}', h)$ to \mathcal{L} .

2.4.6. *Level structures.* One can generalize Bun_G to \mathcal{G} -torsors with level structures. Fix a finite set $S \subset |X|$, and for each $x \in S$ let $\mathbf{K}_x \subset L_x G$ be a pro-algebraic subgroup that is contained in some parahoric subgroup of $L_x G$, and has finite codimension therein. Then there is a group scheme $\mathcal{G}(\mathbf{K}_S)$ obtained by modifying \mathcal{G} at each $x \in S$ such that $L_x^+ \mathcal{G}(\mathbf{K}_S) = \mathbf{K}_x$. A \mathcal{G} -torsors over X with \mathbf{K}_S -level structures is the same as a $\mathcal{G}(\mathbf{K}_S)$ -torsor. We shall denote the corresponding moduli stack by $\text{Bun}_G(\mathbf{K}_S)$. Let $K_x = \mathbf{K}_x(k) \subset G(F_x)$. Then Weil's equivalence (2.6) can be generalized to an equivalence of groupoids

$$(2.9) \quad \bigsqcup_{\xi \in \text{H}_{\mathcal{G}, \mathbf{K}_S}^1(F, G)} G_\xi(F) \backslash G_\xi(\mathbb{A}_F) / \left(\prod_{x \notin S} \mathcal{G}_\xi(\mathcal{O}_x) \times \prod_{x \in S} K_x \right) \xrightarrow{\sim} \text{Bun}_G(\mathbf{K}_S)(k).$$

Here $\text{H}_{\mathcal{G}, \mathbf{K}_S}^1(F, G)$ is the set defined similarly as $\text{H}_G^1(F, G)$, replacing \mathcal{G} with $\mathcal{G}(\mathbf{K}_S)$.

2.4.7. *Connected components of Bun_G .* The construction of the Kottwitz map $\kappa_{G, F}$ (2.3) can be adapted to give a map on the level of moduli stacks. For S containing S_θ and \mathbf{K}_x contained in some parahoric subgroup of $L_x G$, there is a morphism

$$(2.10) \quad \kappa : \text{Bun}_G(\mathbf{K}_S) \rightarrow \underline{\mathbb{X}^*(Z\widehat{G})}_{I_F}.$$

Here $\underline{\mathbb{X}^*(Z\widehat{G})}_{I_F}$ means the finite étale group scheme over k with geometric points $\mathbb{X}^*(Z\widehat{G})_{I_F}$ and the obvious Frob_k -action on it. This map is always surjective. In [18, Theorem 6], J. Heinloth showed that when G is semisimple or a torus (and \mathbf{K}_x are contained in parahoric subgroups), the map κ exhibits $\underline{\mathbb{X}^*(Z\widehat{G})}_{I_F}$ as the set of geometric connected components of the stack $\text{Bun}_G(\mathbf{K}_S)$.

2.4.8. *The action of Bun_Z .* Recall from §2.2.4 that Z admits an integral model \mathcal{Z} over X . We may define the moduli functor Bun_Z classifying \mathcal{Z} -torsors over X . The assumption in §2.2.4 guarantees that Bun_Z is representable by an algebraic stack. In fact, present \mathcal{Z} as the kernel of a surjection $\mathcal{T}_1 \twoheadrightarrow \mathcal{T}_2$ of smooth models of tori $T_1 \twoheadrightarrow T_2$ over X , Bun_Z is represented by the kernel of the morphism $\text{Bun}_{\mathcal{T}_1} \rightarrow \text{Bun}_{\mathcal{T}_2}$ between Picard stacks. Concretely, Bun_Z can be identified with the moduli stack of pairs (\mathcal{E}_1, τ) where

\mathcal{E}_1 is a \mathcal{T}_1 -torsor over X and τ is a trivialization of the associated \mathcal{T}_2 -torsor $\mathcal{E}_1 \times^{\mathcal{T}_1} \mathcal{T}_2$.

More generally, for a pro-algebraic subgroup $\mathbf{K}_{Z,x} \subset L_x^+ \mathcal{Z}$ of finite codimension (one for each $x \in S$), we may modify the integral model \mathcal{Z} at S to get another integral model $\mathcal{Z}(\mathbf{K}_{Z,S})$. We can form the moduli stack $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ of $\mathcal{Z}(\mathbf{K}_{Z,S})$ -torsors over X (i.e., \mathcal{Z} -torsors with $\mathbf{K}_{Z,x}$ -level structures at $x \in S$). This is a $\prod_{x \in S} L_x^+ \mathcal{Z} / \mathbf{K}_{Z,x}$ -torsor over $\text{Bun}_{\mathcal{Z}}$.

Now consider a certain level structure $\mathbf{K}_S = \{\mathbf{K}_x\}_{x \in S}$ for G . Let $\mathcal{G}(\mathbf{K}_S)$ be the group scheme defined in §2.4.6. Let $\mathbf{K}_{Z,x} = L_x \mathcal{Z} \cap \mathbf{K}_x$, which is necessarily contained in $L_x^+ \mathcal{Z}$. Therefore $\mathcal{Z}(\mathbf{K}_{Z,S})$ and $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ are defined. For every point $\mathcal{E} \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$, the automorphism group scheme $\underline{\text{Aut}}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E})$ is a group scheme over X which is an inner form of $\mathcal{G}(\mathbf{K}_S)$. Since $\mathcal{Z}(\mathbf{K}_{Z,S}) \subset \mathcal{G}(\mathbf{K}_S)$, $\mathcal{Z}(\mathbf{K}_{Z,S})$ is canonically a central subgroup scheme of $\underline{\text{Aut}}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E})$. It therefore makes sense to twist \mathcal{E} by a $\mathcal{Z}(\mathbf{K}_{Z,S})$ -torsor. In conclusion we get an action of the Picard stack $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ on $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$.

- 2.4.9. EXAMPLE. (1) In Example 2.4.5, the center $Z \subset \text{GU}(n, h_0)$ is the induced torus $\text{Res}_F^{F'} \mathbb{G}_m$ with the integral model $\mathcal{Z} = \text{Res}_X^{X'} \mathbb{G}_m$. Therefore $\text{Bun}_{\mathcal{Z}} \cong \text{Pic}_{X'}$. The action of $\mathcal{L}' \in \text{Pic}_{X'}$ on $\text{Bun}_n^{\text{Herm}}$ is given by $(\mathcal{L}, \mathcal{E}', h) \mapsto (\text{Nm}_{X'/X}(\mathcal{L}') \otimes \mathcal{L}, \mathcal{E}' \otimes \mathcal{L}', h')$, where h' is the obvious $\text{Nm}_{X'/X}(\mathcal{L}') \otimes \mathcal{L}$ -Hermitian structure on $\mathcal{E}' \otimes \mathcal{L}'$ induced from h by tensoring with the natural isomorphism $\mathcal{L}' \otimes \sigma^* \mathcal{L}' \cong \pi^* \text{Nm}_{X'/X}(\mathcal{L}')$.
- (2) Suppose G is split (i.e., $G = \mathbb{G} \otimes_k F$) and $X = \mathbb{P}^1$. In this case, $\mathcal{G} = \mathbb{G} \times X$ and $\mathcal{Z} = \mathbb{Z}\mathbb{G} \times X$. Then $\text{Bun}_{\mathcal{Z}}$ is simply the classifying space $\mathbb{B}(\mathbb{Z}\mathbb{G})$, because any $\mathbb{Z}\mathbb{G}$ -torsor over \mathbb{P}^1 is trivial. The fact that there exists an action of $\text{Bun}_{\mathcal{Z}} = \mathbb{B}(\mathbb{Z}\mathbb{G})$ on $\text{Bun}_{\mathcal{G}}$ simply says that the automorphism group of every point of $\text{Bun}_{\mathcal{G}}$ contains $\mathbb{Z}\mathbb{G}$.

In general, let $\mathbf{A}_{Z,S}$ be the automorphism group of points in $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$, then $\mathbf{A}_{Z,S}$ is contained in the automorphism group of every point of $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$.

2.5. The sheaf-to-function correspondence.

2.5.1. *Convention.* We temporarily use X to denote an algebraic stack locally of finite type over a field k . Let $D_c^b(X)$ denote the bounded derived category of constructible complex of $\overline{\mathbb{Q}}_\ell$ -sheaves for the étale topology of X . For detailed definition see [6] for the case of schemes and [27] and [28] for the case of stacks. All complexes of sheaves we consider will have $\overline{\mathbb{Q}}_\ell$ -coefficients unless otherwise stated. All sheaf-theoretic functors are understood to be derived unless otherwise stated.

2.5.2. *The correspondence.* For the rest of this subsection, k is a finite field. Let $\mathcal{F} \in D_c^b(X)$. For each $x \in X(k')$, $\text{Gal}(\bar{k}/k')$ acts on the geometric stalk $\mathcal{F}_{\bar{x}}$ (a complex of $\overline{\mathbb{Q}}_\ell$ -vector spaces). Let $\text{Frob}_{k'} \in \text{Gal}(\bar{k}/k')$ be the

geometric Frobenius element. Define a function

$$\begin{aligned} f_{\mathcal{F},k'} : X(k') &\rightarrow \overline{\mathbb{Q}}_\ell \\ x &\mapsto \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\mathrm{Frob}_{k'}, H^i \mathcal{F}_x) \end{aligned}$$

We get a family of $\overline{\mathbb{Q}}_\ell$ -valued functions $\{f_{\mathcal{F},k'}\}_{k'/k}$ defined on $X(k')$ for varying finite extensions k'/k . Let $\mathrm{Fun}(S)$ denote the vector space of $\overline{\mathbb{Q}}_\ell$ -valued functions on a set S . The map

$$\begin{aligned} \mathrm{Ob} D_c^b(X) &\rightarrow \prod_{k'/k} \mathrm{Fun}(X(k')) \\ \mathcal{F} &\mapsto \{f_{\mathcal{F},k'}\}_{k'/k} \end{aligned}$$

is called the *sheaf-to-function* correspondence.

2.5.3. *Functorial properties of the sheaf-to-function correspondence.* If $\phi : X \rightarrow Y$ is a schematic morphism of finite type between algebraic stacks over k which induces a map $\phi(k') : X(k') \rightarrow Y(k')$ between finite sets, then for any $\mathcal{F} \in D_c^b(X)$, we have

$$(2.11) \quad f_{\phi_! \mathcal{F},k'} = \phi(k')_! f_{\mathcal{F},k'}.$$

Here $\phi_!$ is the derived push-forward functor with compact support, while $\phi(k')_! : \mathrm{Fun}(X(k')) \rightarrow \mathrm{Fun}(Y(k'))$ is “summation along the fibers” (in the case where ϕ is not schematic, summation along the fibers should be the weighted by the cardinalities of the automorphism groups of the k -points on the fiber).

A special case of (2.11) is the Lefschetz trace formula. Namely if we take $Y = \mathrm{Spec} k$, X a scheme over k with ϕ the structure morphism and $k' = k$, we have on the one hand

$$(2.12) \quad f_{\phi_! \mathcal{F},k} = \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\mathrm{Frob}_k, H_c^i(X \otimes_k \overline{k}, \mathcal{F}))$$

and on the other

$$(2.13) \quad \phi(k)_! f_{\mathcal{F},k} = \sum_{x \in X(k)} f_{\mathcal{F},k}(x) = \sum_{x \in X(k)} \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{Tr}(\mathrm{Frob}_k, H^i \mathcal{F}_x).$$

The equality of the right hand sides of (2.12) and (2.13) is the Lefschetz trace formula for the Frobenius morphism of X .

For any $\mathcal{G} \in D_c^b(Y)$, we have

$$f_{\phi^* \mathcal{G},k'} = \phi(k')^* f_{\mathcal{G},k'}$$

where $\phi(k')^* : \mathrm{Fun}(Y(k')) \rightarrow \mathrm{Fun}(X(k'))$ is the pullback of functions.

If $\mathcal{F}, \mathcal{G} \in D_c^b(X)$, then we have

$$f_{\mathcal{F} \otimes \mathcal{G},k'} = f_{\mathcal{F},k'} f_{\mathcal{G},k'}.$$

where the right hand side means pointwise multiplication of functions on $X(k')$.

2.5.4. *Character sheaves.* In general, there is no canonical way to go from functions back to sheaves. However in many cases, starting from a function f on $X(k)$, there is a natural candidate sheaf \mathcal{F} for which $f_{\mathcal{F},k} = f_k$. When $X = L$ is an algebraic group over k and $f : L(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is a character, we will see in Appendix A that often times one can construct a *rank one character sheaf* on L whose associated function is f . The correspondence between rank one character sheaves on L and characters of $L(k)$ will play an important role in defining geometric automorphic data in the next subsection, and we shall frequently refer to the notations and results in Appendix A.

2.6. Geometric automorphic data and automorphic sheaves.

Let k be any field. We fix an integral model \mathcal{G} of G over X constructed as in §2.2.4. Recall that $S_\theta \subset |X|$ is the locus where θ_X is ramified. The group scheme $\mathcal{G}|_{X-S_\theta}$ is reductive with connected fibers. The exact choice of the model \mathcal{G} at $x \in S_\theta$ is not important at this point because we will impose level structures at these points. Let $S \subset |X|$ be a finite set containing S_θ .

2.6.1. *Geometric (restricted) central character.* Let \mathcal{Z} be the integral model of Z over X as in §2.2.4. For each $x \in S$, let $\mathbf{K}_{Z,x}^+ \subset L_x^+ \mathcal{Z}$ be a *connected* pro-algebraic subgroup. We may form the stack $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}^+)$ of \mathcal{Z} -torsors over X with $\mathbf{K}_{Z,x}^+$ -level structures at x for each $x \in S$, as in §2.4.8. Restricting the Kottwitz morphism $\kappa_{\text{Bun}_{\mathcal{G}}}$ in §2.4.7 to $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}^+)$, we get a homomorphism $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}^+) \rightarrow \mathbb{X}^*(Z\widehat{G})_{I_F}$. Let $\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S}^+)$ be the kernel of this homomorphism. When $S \neq \emptyset$ and $\mathbf{K}_{Z,x}^+$ is chosen small enough, $\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S}^+)$ is an algebraic group over k (i.e., \mathcal{Z} -torsors with $\mathbf{K}_{Z,S}^+$ -level structures have trivial automorphism groups, and $\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S}^+)$ is of finite type over k). We consider the category $\mathcal{CS}_1(\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S}^+))$ of rank one character sheaves on $\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S}^+)$.

If we shrink $\mathbf{K}_{Z,x}^+$ to a smaller level $\mathbf{K}_{Z,x}^{++}$ for each $x \in S$, then we have a natural projection $\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_S^{++}) \rightarrow \text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_S^+)$ which is a $\prod_{x \in S} \mathbf{K}_{Z,x}^+ / \mathbf{K}_{Z,x}^{++}$ -torsor. The pullback functor $\mathcal{CS}_1(\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S}^+)) \rightarrow \mathcal{CS}_1(\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S}^{++}))$ is fully faithful since $\mathbf{K}_{Z,x}^+ / \mathbf{K}_{Z,x}^{++}$ is geometrically connected. We may form the colimit over smaller and smaller $\mathbf{K}_{Z,S}^+$ with respect to the fully faithful embeddings of categories

$$\mathcal{CS}_1^{\natural}(Z; S) := \varinjlim_{\mathbf{K}_{Z,S}^+} \mathcal{CS}_1(\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S}^+)).$$

An object in the category $\mathcal{CS}_1^{\natural}(Z; S)$ is called a *geometric (restricted) central character*.

When k is a finite field, let $K_{Z,x}^+ = \mathbf{K}_{Z,x}^+(k)$. Via the sheaf-to-function correspondence and the homomorphism

$$Z(F) \backslash Z(\mathbb{A}_F)^\natural / \left(\prod_{x \notin S} \mathcal{Z}(\mathcal{O}_x) \times \prod_{x \in S} K_{Z,x}^+ \right) \hookrightarrow \text{Bun}_{\mathcal{Z}}^\natural(\mathbf{K}_{Z,S})(k)$$

as in (2.9), we get a homomorphism

$$\text{CS}_1(\text{Bun}_{\mathcal{Z}}^\natural(\mathbf{K}_{Z,S}^+)) \rightarrow \text{Hom}(Z(F) \backslash Z(\mathbb{A}_F)^\natural / \left(\prod_{x \notin S} \mathcal{Z}(\mathcal{O}_x) \times \prod_{x \in S} K_{Z,x}^+ \right), \overline{\mathbb{Q}}_\ell^\times).$$

Here $\text{CS}_1(-)$ means the group of isomorphism classes of objects in $\mathcal{CS}_1(-)$. Passing to the colimit, we get a homomorphism

$$\text{CS}_1^\natural(Z; S) := \text{Ob } \mathcal{CS}_1^\natural(Z; S) \rightarrow \text{Hom}_{\text{cont}}(Z(F) \backslash Z(\mathbb{A}_F)^\natural / \prod_{x \notin S} \mathcal{Z}(\mathcal{O}_x), \overline{\mathbb{Q}}_\ell^\times).$$

i.e., each geometric restricted central character gives a restricted central character.

For any choice of $\mathbf{K}_{Z,S}^+$ and each $x \in S$, there is a homomorphism $i_x : L_x Z \rightarrow L_x Z / \mathbf{K}_{Z,x}^+ \rightarrow \text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}^+)$ by locally modifying the \mathcal{Z} -torsors at x . Let $(L_x Z)^\natural \subset L_x Z$ be the kernel of $L_x Z \rightarrow L_x G \rightarrow \widehat{\mathbb{X}}^*(Z\widehat{G})_{L_x}$ (the local Kottwitz homomorphism (2.2)). Then i_x restricts to a homomorphism $i_x^\natural : (L_x Z)^\natural \rightarrow (L_x Z)^\natural / \mathbf{K}_{Z,x}^+ \rightarrow \text{Bun}_{\mathcal{Z}}^\natural(\mathbf{K}_{Z,S}^+)$. Suppose $\Omega \in \mathcal{CS}_1^\natural(Z; S)$ comes from an object in $\text{CS}_1(\text{Bun}_{\mathcal{Z}}^\natural(\mathbf{K}_{Z,S}^+))$, we let

$$\Omega_x := i_x^{\natural,*} \Omega \in \mathcal{CS}_1((L_x Z)^\natural),$$

which is a rank one character sheaf that descends to the algebraic group $(L_x Z)^\natural / \mathbf{K}_{Z,x}^+$. The object Ω_x is independent of the choice of $\mathbf{K}_{Z,S}^+$. Let $\Omega_S = \boxtimes_{x \in S} \Omega_x \in \mathcal{CS}_1(\prod_{x \in S} (L_x Z)^\natural)$.

2.6.2. DEFINITION. A quadruple $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is a *geometric automorphic datum* with respect to S if

- $\Omega \in \mathcal{CS}_1^\natural(Z; S)$ is a geometric restricted central character;
- \mathbf{K}_S is a collection $\{\mathbf{K}_x\}_{x \in S}$, where $\mathbf{K}_x \subset L_x G$ is a pro-algebraic group contained in some parahoric subgroup of $L_x G$ with finite codimension. We require that each \mathbf{K}_x is generated by $\mathbf{K}_x \cap L^x Z$ and $\mathbf{K}_x \cap L_x G^{\text{der}}$. We often use \mathbf{K}_S also to denote the product $\prod_{x \in S} \mathbf{K}_x$.
- \mathcal{K}_S is a collection $\{\mathcal{K}_x\}_{x \in S}$ where each $\mathcal{K}_x \in \mathcal{CS}_1(\mathbf{K}_x)$ is a rank one character sheaf on \mathbf{K}_x that is the pullback of a rank one character sheaf from some finite dimensional quotient $\mathbf{K}_x \rightarrow \mathbf{L}_x$. We often use \mathcal{K}_S to denote the tensor product $\boxtimes_{x \in S} \mathcal{K}_x \in \mathcal{CS}_1(\mathbf{K}_S)$.
- Note that $\mathbf{K}_{Z,x} := \mathbf{K}_x \cap L_x Z$ is automatically contained in $(L_x Z)^\natural$, since \mathbf{K}_x was assumed to be contained in a parahoric subgroup. Let $\mathbf{K}_{Z,S} := \prod_{x \in S} \mathbf{K}_{Z,x}$. Then ι_S is an isomorphism $\iota_S : \Omega_S|_{\mathbf{K}_{Z,S}} \cong \mathcal{K}_S|_{\mathbf{K}_{Z,S}}$ in the category $\mathcal{CS}_1(\mathbf{K}_{Z,S})$ (which is the same as a collection of isomorphisms $\iota_x : \Omega_x|_{\mathbf{K}_{Z,x}} \cong \mathcal{K}_x|_{\mathbf{K}_{Z,x}}$, one for each $x \in S$).

When k is a finite field, a geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ gives rise to a restricted automorphic datum $(\omega^\natural, K_S, \chi_S)$ as in Definition 2.3.7, by setting $K_x := \mathbf{K}_x(k)$ and using the sheaf-to-function correspondence to turn Ω and \mathcal{K}_x into ω^\natural and χ_x . However, passage from a geometric automorphic datum to a restricted automorphic datum loses information, especially that carried by ι_S , as we shall see in §2.6.4.

2.6.3. *A smaller level \mathbf{K}_S^+ .* Let $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ be a geometric automorphic datum. For each $x \in S$ we choose a pro-algebraic normal subgroup $\mathbf{K}_x^+ \triangleleft \mathbf{K}_x$ of finite codimension such that

- The character sheaf \mathcal{K}_x descends to the finite-dimensional quotient $\mathbf{L}_x := \mathbf{K}_x / \mathbf{K}_x^+$.
- Let $\mathbf{K}_{Z,x}^+ = \mathbf{K}_x^+ \cap L_x Z$. Then $\mathbf{K}_{Z,x}^+$ is connected and $\text{Bun}_{Z,S}(\mathbf{K}_{Z,S}^+)$ is a scheme (i.e., the automorphic group of every point is trivial), and the geometric central character Ω descends to $\text{Bun}_Z^{\natural}(\mathbf{K}_{Z,S}^+)$.

By shrinking \mathbf{K}_S^+ these conditions can always be satisfied. With this choice of \mathbf{K}_S^+ , we shall view \mathcal{K}_x as an object in $\mathcal{CS}_1(\mathbf{L}_x)$, and view Ω as an object in $\mathcal{CS}_1(\text{Bun}_Z^{\natural}(\mathbf{K}_{Z,S}^+))$.

For each $x \in S$, let $\mathbf{K}_{Z,x} = \mathbf{K}_x \cap L_x Z$. Then $\mathbf{L}_{Z,x} := \mathbf{K}_{Z,x} / \mathbf{K}_{Z,x}^+$ is a central subgroup of \mathbf{L}_x . Let

$$\mathbf{L}_S := \prod_{x \in S} \mathbf{L}_x; \quad \mathcal{K}_S := \boxtimes_{x \in S} \mathcal{K}_x \in \mathcal{CS}_1(\mathbf{L}_S); \quad \mathbf{L}_{Z,S} := \mathbf{L}_{Z,x}$$

The morphism $\text{Bun}_G(\mathbf{K}_S^+) \rightarrow \text{Bun}_G(\mathbf{K}_S)$ is an \mathbf{L}_S -torsor. Similarly, $\text{Bun}_Z^{\natural}(\mathbf{K}_{Z,S}^+) \rightarrow \text{Bun}_Z^{\natural}(\mathbf{K}_{Z,S})$ is an $\mathbf{L}_{Z,S}$ -torsor. The group scheme $\text{Bun}_Z(\mathbf{K}_{Z,S}^+)$ acts on $\text{Bun}_G(\mathbf{K}_S^+)$ by the discussion in §2.4.8. Summarizing the situation we have a diagram

$$(2.14) \quad \begin{array}{ccccc} \text{Bun}_Z^{\natural}(\mathbf{K}_{Z,S}^+) & \hookrightarrow & \text{Bun}_Z(\mathbf{K}_{Z,S}^+) & \curvearrowright & \text{Bun}_G(\mathbf{K}_S^+) \\ \mathbf{L}_{Z,S} \downarrow & & \mathbf{L}_{Z,S} \downarrow & & \downarrow \mathbf{L}_S \\ \text{Bun}_Z^{\natural}(\mathbf{K}_{Z,S}) & \hookrightarrow & \text{Bun}_Z(\mathbf{K}_{Z,S}) & \curvearrowright & \text{Bun}_G(\mathbf{K}_S) \end{array}$$

2.6.4. *Secondary central character.* The isomorphism ι_S is an extra piece of structure in the geometric automorphic datum that do not appear in the restricted automorphic datum $(\omega^\natural, K_S, \chi_S)$. Choose \mathbf{K}_S^+ as in §2.6.3, then ι_S is an isomorphism $\Omega|_{\mathbf{L}_{Z,S}} \cong \mathcal{K}_S|_{\mathbf{L}_{Z,S}}$. Let $\mathbf{A}_{Z,S}$ be the automorphism group of the identity point of the Picard stack $\text{Bun}_Z(\mathbf{K}_{Z,S})$. Then we have a canonical homomorphism $i_Z : \mathbf{A}_{Z,S} \rightarrow \mathbf{K}_{Z,S} \hookrightarrow \mathbf{K}_S$. On the other hand, $\mathbf{A}_{Z,S}$ is the kernel of the homomorphism $\mathbf{L}_{Z,S} = \mathbf{K}_{Z,S} / \mathbf{K}_{Z,S}^+ \hookrightarrow \prod_{x \in S} L_x Z / \mathbf{K}_{Z,S}^+ \xrightarrow{(i_x)_{x \in S}} \text{Bun}_Z(\mathbf{K}_{Z,S}^+)$ for $\mathbf{K}_{Z,S}^+$ as in §2.6.3. Since Ω is a local system on $\text{Bun}_Z^{\natural}(\mathbf{K}_{Z,S}^+)$, the restriction $\Omega_S|_{\mathbf{A}_{Z,S}}$ admits a canonical trivialization. Composing this trivialization with the restriction of the isomorphism

$\iota_S : \Omega_S|_{\mathbf{L}_{Z,S}} \cong \mathcal{K}_S|_{\mathbf{L}_{Z,S}} \in \mathcal{CS}_1(\mathbf{L}_{Z,S})$ to $\mathbf{A}_{Z,S}$, we get a trivialization

$$\tau : \mathcal{K}_S|_{\mathbf{A}_{Z,S}} \cong \overline{\mathbb{Q}}_\ell \in \mathcal{CS}_1(\mathbf{A}_{Z,S}).$$

We would like to call τ the *secondary central character* attached to the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$. By Remark A.1.2(3), the trivializations of $\mathcal{K}_S|_{\mathbf{A}_{Z,S}}$ form a torsor under $\mathrm{Hom}(\pi_0(\mathbf{A}_{Z,S})_{\mathrm{Gal}(\bar{k}/k)}, \overline{\mathbb{Q}}_\ell^\times)$, so the secondary central character can add as much as $\#\pi_0(\mathbf{A}_{Z,S})_{\mathrm{Gal}(\bar{k}/k)}$ extra constraints in addition to the restricted automorphic datum $(\omega^\natural, K_S, \chi_S)$.

The above discussion also shows that when the coarse moduli space of $\mathrm{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ is a point, the secondary central character determines (Ω, ι_S) . We summarize this into a lemma.

2.6.5. LEMMA. *Let (\mathbf{K}_S, χ_S) be as in Definition 2.6.2. Let \mathbf{K}_S^+ be chosen as in §2.6.3. Suppose the coarse moduli space of $\mathrm{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ is a point, i.e., $\mathrm{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}) \cong \mathbb{B}\mathbf{A}_{Z,S}$. Then the pairs (Ω, ι_S) such that $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ form a geometric automorphic datum correspond bijectively to the descents of \mathcal{K}_S to $\mathbf{L}_S/\mathbf{A}_{Z,S}$.*

In particular, if $\mathrm{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}) \cong \mathbb{B}\mathbf{A}_{Z,S}$ and $\mathbf{A}_{Z,S}$ is connected, there is up to isomorphism a unique pair (Ω, ι_S) making $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ a geometric automorphic datum.

2.6.6. EXAMPLE. Suppose G is split and semisimple and $X = \mathbb{P}^1$. Then $\mathcal{G} = \mathbb{G} \times X$ and $\mathcal{Z} = \mathbb{Z}\mathbb{G} \times X$ are constant group schemes over X . Assume each \mathbf{K}_x contains $\mathbb{Z}\mathbb{G}$, then $\mathrm{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}) = \mathbb{B}(\mathbb{Z}\mathbb{G})$ and $\mathbf{A}_{Z,S} = \mathbb{Z}\mathbb{G}$. Given \mathcal{K}_S , the datum of (Ω, ι_S) correspond bijectively to descents $\overline{\mathcal{K}}_S \in \mathcal{CS}_1(\mathbf{K}_S/\mathbb{Z}\mathbb{G})$ of \mathcal{K}_S . Via the sheaf-to-function correspondence, $\overline{\mathcal{K}}_S$ gives a character of $(\mathbf{K}_S/\mathbb{Z}\mathbb{G})(k)$, which is in general a larger group than $\mathbf{K}_S(k)/\mathbb{Z}\mathbb{G}(k)$. On the other hand, the corresponding restricted automorphic datum $(\omega^\natural, K_S, \chi_S)$ only gives a character on the smaller group $\mathbf{K}_S(k)/\mathbb{Z}\mathbb{G}(k)$. Therefore in general geometric automorphic data contain more information than the corresponding restricted automorphic data.

2.6.7. Base change of geometric automorphic data. Let k'/k be a field extension. Let S' be the preimage of S in $X \otimes_k k'$ (a closed point in S may split into several in S'). Given a geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ for G over F with respect to S , we may define a corresponding geometric automorphic datum $(\Omega', \mathbf{K}_{S'}, \mathcal{K}_{S'}, \iota_{S'})$ for the function field $F \otimes_k k'$ with respect to S' as follows.

The base change $\mathbf{K}_x \otimes_k k'$ naturally decomposes as a product $\prod_{y \mapsto x} \mathbf{K}_y$ where y runs over the preimages of x in S' . The pullback of \mathcal{K}_x to $\mathcal{K}_x \otimes_k k'$ then takes the form $\boxtimes_y \mathcal{K}_y$, whose tensor factors give the base change character sheaves $\mathcal{K}_y \in \mathcal{CS}_1(\mathbf{K}_y)$.

If Ω is a geometric central character coming from $\mathrm{Bun}_{\mathcal{Z}}^\natural(\mathbf{K}_{Z,S}^+)$ for a certain level $\mathbf{K}_{Z,S}^+$, we may similarly define the base change level $\mathbf{K}_{Z,S'}^+$ over k' , so that $\mathrm{Bun}_{\mathcal{Z}_{k'}}^\natural(\mathbf{K}_{Z,S'}^+) = \mathrm{Bun}_{\mathcal{Z}}^\natural(\mathbf{K}_{Z,S}^+) \otimes_k k'$. Then define $\Omega' \in$

$\mathcal{CS}_1(\mathrm{Bun}_{Z/k'}^{\natural}(\mathbf{K}_{Z,S'}^+))$ to be the pullback of Ω via the morphism $\mathrm{Bun}_{Z'}^{\natural}(\mathbf{K}_{Z,S}^+) \otimes_k k' \rightarrow \mathrm{Bun}_{Z'}^{\natural}(\mathbf{K}_{Z,S}^+)$. The isomorphism ι_S induces $\iota_{S'}$.

When k is finite, k'/k a finite extension, and the restricted automorphic datum $(\omega^{\natural}, K_S, \chi_S)$ over F comes from a geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S)$, we may base change it to an automorphic datum $(\omega'^{\natural}, K_{S'}, \chi_{S'})$ over $F \otimes_k k'$ coming from the base change $(\Omega', \mathbf{K}_{S'}, \mathcal{K}_{S'})$ of $(\Omega, \mathbf{K}_S, \mathcal{K}_S)$ via the sheaf-to-function correspondence. For example, ω'^{\natural} is the composition $Z(F \otimes_k k') \backslash Z(\mathbb{A}_{F \otimes_k k'})^{\natural} \xrightarrow{\mathrm{Nm}} Z(F) \backslash Z(\mathbb{A}_F)^{\natural} \xrightarrow{\omega^{\natural}} \overline{\mathbb{Q}}_{\ell}^{\times}$. However, the datum (K_S, χ_S) does not have a well-defined base change unless we specify a geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$ from which it comes.

Let $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ be a geometric automorphic datum giving rise to a restricted automorphic datum $(\omega^{\natural}, K_S, \chi_S)$. We are interested in the sheaf-theoretic analog of the function space (2.4), under the sheaf-to-function correspondence.

2.6.8. *The category of automorphic sheaves.* We form the group scheme

$$\mathbf{M}_S := \mathrm{Bun}_{Z'}^{\natural}(\mathbf{K}_S^+) \times^{\mathbf{L}_{Z,S}} \mathbf{L}_S$$

which acts on $\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)$ preserving each fiber of the geometric Kottwitz morphism (cf. the diagram (2.14)). The quotient $[\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)/\mathbf{M}_S]$ should be thought of as the quotient of $\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S)$ by the Picard stack $\mathrm{Bun}_{Z'}^{\natural}(\mathbf{K}_{Z,S})$, although we do not want to get into the issue of making quotients of a stack by another Picard stack. The isomorphism ι_S in the geometric automorphic datum gives a trivialization of the restriction of $\Omega \boxtimes \mathcal{K}_S \in \mathcal{CS}_1(\mathrm{Bun}_{Z'}^{\natural}(\mathbf{K}_S^+) \times \mathbf{L}_S)$ to the anti-diagonally embedded $\mathbf{L}_{Z,S}$. By Lemma A.1.4, such a trivialization gives a descent of $\Omega \boxtimes \mathcal{K}_S$ to $\mathcal{K}_{S,\Omega} \in \mathcal{CS}_1(\mathbf{M}_S)$. We may then talk about the derived category $D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S^+))$ of $(\mathbf{M}_S, \mathcal{K}_{S,\Omega})$ -equivariant $\overline{\mathbb{Q}}_{\ell}$ -complexes on $\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)$, as we mentioned in §A.4.2.

2.6.9. **REMARK.** There is a subtlety in defining the category $D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S^+))$ because each connected component of $\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)$ is not of finite type in general. To remedy, we can write $\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)$ as a union of \mathbf{M}_S -stable open substack Bun_{ξ} that are of finite type modulo the action of \mathbf{M}_S (depending on a parameter ξ from a filtered set of indices, for example the set of Harder-Narasimhan polygons of \mathcal{G} -torsors). The category $D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S^+), \overline{\mathbb{Q}}_{\ell})$ is defined as the colimit of $D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(\mathrm{Bun}_{\xi}, \overline{\mathbb{Q}}_{\ell})$ as Bun_{ξ} gets larger (with respect to the extension by zero functor). In practice, the categories $D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(\mathrm{Bun}_{\xi}, \overline{\mathbb{Q}}_{\ell})$ will often stabilize when ξ is large enough (this is the sheaf-theoretic manifestation of the cuspidality of automorphic forms).

If we shrink \mathbf{K}_S^+ to another level \mathbf{K}_S^{++} , we have the corresponding algebraic groups $\tilde{\mathbf{L}}_S = \prod_{x \in S} \mathbf{K}_x / \mathbf{K}_x^{++}$, $\mathbf{K}_{Z,x}^{++} = \mathbf{K}_x^{++} \cap L_x Z$ and $\tilde{\mathbf{L}}_{Z,S} =$

$\prod_{x \in S} \mathbf{K}_{Z,x} / \mathbf{K}_{Z,x}^{++}$. We define similarly $\widetilde{\mathbf{M}}_S := \text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}^{++}) \times^{\widetilde{\mathbf{L}}_{Z,S}} \widetilde{\mathbf{L}}_S$ and $\Omega \boxtimes \mathcal{K}_S$ descends to a rank one character sheaf $\widetilde{\mathcal{K}}_{S,\Omega}$ on $\widetilde{\mathbf{M}}_S$. There is a projection $\widetilde{\mathbf{M}}_S \rightarrow \mathbf{M}_S$ whose kernel is $\mathbf{K}_S^+ / \mathbf{K}_S^{++}$, and $\widetilde{\mathcal{K}}_{S,\Omega}$ is the pullback of $\mathcal{K}_{S,\Omega}$. The natural projection $\pi : \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^{++}) \rightarrow \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)$ is also a $\mathbf{K}_S^+ / \mathbf{K}_S^{++}$ -torsor. Therefore the quotient stacks $[\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^{++}) / \widetilde{\mathbf{M}}_S]$ and $[\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+) / \mathbf{M}_S]$ are isomorphic, and the pullback along π induces an equivalence of categories

$$D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)) \xrightarrow{\sim} D_{(\widetilde{\mathbf{M}}_S, \widetilde{\mathcal{K}}_{S,\Omega})}^b(\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^{++})).$$

We define

$$D_{\mathcal{G},\Omega}(\mathbf{K}_S, \mathcal{K}_S) := \varinjlim_{\mathbf{K}_S^+} D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)).$$

where the transition functors are equivalences. Objects in the category $D_{\mathcal{G},\Omega}(\mathbf{K}_S, \mathcal{K}_S)$ are called $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ -typical automorphic sheaves.

Let $K_x^+ := \mathbf{K}_x^+(k)$. For an object $\mathcal{F} \in D_{\mathcal{G},\Omega}(\mathbf{K}_S, \mathcal{K}_S)$ the corresponding function $f_{\mathcal{F},k}$ on $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)(k)$ is an eigenfunction under $\mathbf{M}_S(k)$ with eigenvalues given by the character corresponding to $\mathcal{K}_{S,\Omega}$. In particular, $f_{\mathcal{F},k}$ is $(K_S / K_S^+, \chi_S)$ -equivariant and $(Z(F) \backslash Z(\mathbb{A}_F)^{\natural}, \omega^{\natural})$ -equivariant. According to (2.9), we may identify the double coset $G(F) \backslash G(\mathbb{A}_F) / \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x) \times \prod_{x \in S} K_x^+$ as a sub-groupoid of $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)(k)$. Restricting $f_{\mathcal{F},k}$ to $G(F) \backslash G(\mathbb{A}_F) / \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x) \times \prod_{x \in S} K_x^+$ gives a function in $C_{\mathcal{G},\omega}(K_S, \chi_S)$. Thus we get an additive map

$$\text{Ob} D_{\mathcal{G},\Omega}(\mathbf{K}_S, \mathcal{K}_S) \rightarrow C_{\mathcal{G},\omega^{\natural}}(K_S, \chi_S).$$

2.6.10. Variants of the category of automorphic sheaves. For an extension k'/k , we may take the base-changed geometric automorphic datum $(\Omega', \mathbf{K}_{S'}, \mathcal{K}_{S'}, \iota_{S'})$ for $G(F \otimes_k k')$ and define the corresponding category of automorphic sheaves. We denote this category by $D_{\mathcal{G},\Omega}(k'; \mathbf{K}_S, \mathcal{K}_S)$. Similarly, when k'/k is a finite extension, we may define the function space $C_{\mathcal{G},\omega^{\natural}}(k'; K_S, \chi_S)$.

By the Kottwitz homomorphism (2.3), we may decompose $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$ into open and closed substacks $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)_{\alpha}$ according to the Frob_k -orbits α in $\mathbb{X}^*(Z\widehat{G})_{I_F}$. Each $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)_{\alpha}$ is stable under the action of $\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S})$. Let $D_{\mathcal{G},\Omega}(\mathbf{K}_S, \mathcal{K}_S)_{\alpha} \subset D_{\mathcal{G},\Omega}(\mathbf{K}_S, \mathcal{K}_S)$ be the full subcategory consisting of those sheaves supported on $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)_{\alpha}$. Similarly we may define $C_{\mathcal{G},\omega^{\natural}}(K_S, \chi_S)_{\alpha}$ for any $\alpha \in \mathbb{X}^*(Z\widehat{G})$ (nonzero only when α is fixed by Frob_k).

2.7. Rigidity of geometric automorphic data. In this subsection we introduce several notions of rigidity for geometric automorphic data, and give a criterion for the weak rigidity using the notion of relevant orbits.

2.7.1. DEFINITION. A restricted automorphic datum $(\omega^{\natural}, K_S, \chi_S)$ is called *strongly rigid*, if there is a unique $(\omega^{\natural}, K_S, \chi_S)$ -typical automorphic representation up to unramified twists.

2.7.2. DEFINITION. Let k be a finite field and let $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ be a geometric automorphic datum giving rise to a restricted automorphic datum $(\omega^\natural, K_S, \chi_S)$. For any finite extension k'/k , let $(\omega'^\natural, K_{S'}, \chi_{S'})$ denote the automorphic datum for G over F' obtained from the base-changed geometric automorphic datum $(\Omega', \mathbf{K}_{S'}, \mathcal{K}_{S'}, \iota_{S'})$ from F to $F \otimes_k k'$.

- (1) The geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is called *strongly rigid*, if for *every* finite extension k'/k , the base-changed restricted automorphic datum $(\omega'^\natural, K_{S'}, \chi_{S'})$ is strongly rigid in the sense of Definition 2.7.1.
- (2) The geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is called *weakly rigid*, if there is a constant N such that for *every* finite extension k'/k , we have $\dim C_{\mathcal{G}, \omega^\natural}(k'; K_S, \chi_S)_\alpha \leq N$ for any $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$. For notation see §2.6.10.

We will see several examples of strongly rigid restricted automorphic data for $G = \mathrm{GL}_2$ and $F = k(t)$ in §2.8.

2.7.3. LEMMA. *If $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is weakly rigid, then there exists a number N such that for each finite extension k'/k , there are at most N isomorphism classes of $(\omega'^\natural, K_{S'}, \chi_{S'})$ -typical automorphic representations π' of $G(\mathbb{A}_{F \otimes_k k'})$ up to unramified twists, and all of them are cuspidal (here, $(\omega'^\natural, K_{S'}, \chi_{S'})$ is the restricted automorphic datum for $G(F \otimes_k k')$ obtained from base change).*

PROOF. Let $G(\mathbb{A}_F)_0$ be the preimage of $0 \in \mathbb{X}^*(Z\widehat{G})_{I_F}$ under the Kottwitz homomorphism. The dimension of $C_{\mathcal{G}, \omega^\natural}(K_S, \chi_S)_0$ is the number of double cosets in $Z(\mathbb{A}_F)^\natural G(F) \backslash G(\mathbb{A}_F)_0 / \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x) \times \prod_{x \in S} K_x$ that support functions with prescribed eigenproperties under the action of $Z(F) \backslash Z(\mathbb{A}_F)^\natural$ and K_S . Since $\dim C_{\mathcal{G}, \omega^\natural}(K_S, \chi_S)_0 \leq N$, all functions in $C_{\mathcal{G}, \omega^\natural}(K_S, \chi_S)_0$ are supported on finitely many such double cosets, hence they are all supported on some compact-modulo- $Z(\mathbb{A}_F)^\natural$ subset of $G(\mathbb{A}_F)_0$. Extending ω^\natural to a central character ω , then functions in $C_{\mathcal{G}, \omega}(K_S, \chi_S)$ are all supported on compact-modulo- $Z(\mathbb{A}_F)$ subset of $G(\mathbb{A}_F)$. We then apply Lemma 2.3.9 to conclude that all (ω, K_S, χ_S) -typical automorphic representations are cuspidal, and there are at most N of them. Since every $(\omega'^\natural, K_{S'}, \chi_{S'})$ -typical automorphic representation can be made (ω, K_S, χ_S) -typical by an unramified twist, they are also cuspidal, and the number of them up to unramified twists are bounded by $\dim C_{\mathcal{G}, \omega^\natural}(K_S, \chi_S)_0 \leq N$. Replacing k with a finite extension k' , the same estimate holds. \square

2.7.4. *Relevant points.* One can talk about the stabilizer of $\mathrm{Bun}_{\mathbb{Z}}^\natural(\mathbf{K}_{Z,S})$ at \mathcal{E} : it is the Picard groupoid $A_{\mathcal{E}}$ whose objects are pairs (b, β) where $b \in \mathrm{Bun}_{\mathbb{Z}}^\natural(\mathbf{K}_{Z,S})$ and $\beta \in \mathrm{Isom}_{\mathcal{G}, \mathbf{K}_S}(b \cdot \mathcal{E}, \mathcal{E})$, with the obvious definition of composition of and isomorphisms between such pairs. Since the automorphism group of points in $\mathrm{Bun}_{\mathbb{Z}}^\natural(\mathbf{K}_{Z,S})$ is a subgroup of the automorphism group of every point in $\mathrm{Bun}_{\mathcal{G}}(\mathbf{K}_S)$, it turns out that $A_{\mathcal{E}}$ is a group scheme.

Choosing a smaller level \mathbf{K}_S^+ as in §2.6.3, a preimage $\mathcal{E}^+ \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)$ of \mathcal{E} gives an isomorphism $i_{\mathcal{E}^+} : A_{\mathcal{E}} \cong \mathbf{M}_{S,\mathcal{E}^+}$, the latter being the stabilizer of \mathcal{E}^+ under \mathbf{M}_S . There is a canonical homomorphism $\mathbf{M}_{S,\mathcal{E}^+} \rightarrow \mathbf{M}_S$ which may not be an embedding. Consider the composition

$$(2.15) \quad \text{ev}_{\mathbf{K}_S^+,\mathcal{E}^+} : A_{\mathcal{E}} \xrightarrow{i_{\mathcal{E}^+}} \mathbf{M}_{S,\mathcal{E}^+} \rightarrow \mathbf{M}_S.$$

We then define $\mathcal{K}_{\mathcal{E}} \in \mathcal{CS}_1(A_{\mathcal{E}})$ to be the pullback of $\mathcal{K}_{S,\Omega}$ along $\text{ev}_{\mathbf{K}_S^+,\mathcal{E}^+}$. The object $\mathcal{K}_{\mathcal{E}}$ a priori depends on the choices of \mathbf{K}_S^+ and \mathcal{E}^+ , but we have

- 2.7.5. LEMMA. (1) *The isomorphism type of $\mathcal{K}_{\mathcal{E}} \in \mathcal{CS}_1(A_{\mathcal{E}})$ is independent of the choice of \mathbf{K}_S^+ and \mathcal{E}^+ .*
(2) *For $\mathcal{E}_1, \mathcal{E}_2 \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S)(\bar{k})$ in the same $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ -orbit, the pairs $(A_{\mathcal{E}_1}, \mathcal{K}_{\mathcal{E}_1})$ and $(A_{\mathcal{E}_2}, \mathcal{K}_{\mathcal{E}_2})$ are (non-canonically) isomorphic.*

PROOF. (1) We denote the $\mathcal{K}_{\mathcal{E}}$ defined in (2.15) by $\mathcal{K}_{\mathbf{K}_S^+,\mathcal{E}^+}$ to emphasize its a priori dependence on auxiliary choices. For fixed choice of \mathbf{K}_S^+ , changing \mathcal{E}^+ will change the map $\text{ev}_{\mathbf{K}_S^+,\mathcal{E}^+}$ by an inner automorphism of \mathbf{M}_S . Since inner automorphisms act trivially on $\mathcal{CS}_1(\mathbf{M}_S)$, the isomorphism type of $\mathcal{K}_{\mathbf{K}_S^+,\mathcal{E}^+}$ is independent of \mathcal{E}^+ .

Changing \mathbf{K}_S^+ to an even smaller level \mathbf{K}_S^{++} , and choosing a preimage \mathcal{E}^{++} of \mathcal{E}^+ in $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^{++})$, we get a surjection $\widetilde{\mathbf{M}}_S \rightarrow \mathbf{M}_S$ (here $\widetilde{\mathbf{M}}_S$ is the counterpart of \mathbf{M}_S for \mathbf{K}_S^{++}), and a similar isomorphism $i_{\mathcal{E}^{++}} : A_{\mathcal{E}} \cong \widetilde{\mathbf{M}}_{S,\mathcal{E}^{++}}$ making a commutative diagram

$$\begin{array}{ccc} A_{\mathcal{E}} & \xrightarrow{i_{\mathcal{E}^{++}}} \widetilde{\mathbf{M}}_{S,\mathcal{E}^{++}} & \longrightarrow \widetilde{\mathbf{M}}_S \\ & \searrow i_{\mathcal{E}^+} & \downarrow \\ & & \mathbf{M}_{S,\mathcal{E}^+} \longrightarrow \mathbf{M}_S \end{array}$$

Since $\mathcal{K}_{S,\widetilde{\Omega}}$ (the counterpart of $\mathcal{K}_{S,\Omega}$ for \mathbf{K}_S^{++}) is the pullback of $\mathcal{K}_{S,\Omega}$ along $\widetilde{\mathbf{M}}_S \rightarrow \mathbf{M}_S$, we have $\mathcal{K}_{\mathbf{K}_S^+,\mathcal{E}^+} = \text{ev}_{\mathbf{K}_S^+,\mathcal{E}^+}^* \mathcal{K}_{S,\Omega} \cong \text{ev}_{\mathbf{K}_S^{++},\mathcal{E}^{++}}^* \mathcal{K}_{S,\widetilde{\Omega}} = \mathcal{K}_{\mathbf{K}_S^{++},\mathcal{E}^{++}}$. Therefore $\mathcal{K}_{\mathcal{E}}$ does not change under shrinking \mathbf{K}_S^+ either.

(2) Fix a level \mathbf{K}_S^+ as in §2.6.3. For $\mathcal{E}_1, \mathcal{E}_2$ in the same $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ -orbit, their arbitrary preimages \mathcal{E}_1^+ and \mathcal{E}_2^+ in $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)$ are in the same $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}^+) \times \mathbf{L}_S$ -orbit. Therefore their stabilizers $\mathbf{M}_{S,\mathcal{E}_1}$ and $\mathbf{M}_{S,\mathcal{E}_2}$ are conjugate to each other in \mathbf{M}_S . We then argue as in the first part of the proof of (1). \square

2.7.6. DEFINITION. A point $\mathcal{E} \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S)(\bar{k})$ is called (Ω, \mathcal{K}_S) -relevant if the restriction of $\mathcal{K}_{\mathcal{E}}$ to the neutral component of $A_{\mathcal{E}}$ is isomorphic to the constant sheaf. Otherwise \mathcal{E} is called (Ω, \mathcal{K}_S) -irrelevant.

By Lemma 2.7.5(2), a $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ -orbit on $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)(\bar{k})$ either consists entirely of (Ω, \mathcal{K}_S) -relevant points or entirely of (Ω, \mathcal{K}_S) -irrelevant points.

Therefore we may talk about (Ω, \mathcal{K}_S) -relevant and irrelevant $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ -orbits.

In practice we introduce a weaker relevance condition by ignoring the central character. Let $\mathcal{E} \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S)(\bar{k})$. Then we have an affine algebraic group $\text{Aut}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E})$ over \bar{k} , the automorphism group of \mathcal{E} preserving the \mathbf{K}_x -level structures. For each $x \in S(\bar{k})$, the restriction $\mathcal{E}|_{\text{Spec } \mathcal{O}_x}$ can be viewed as a \mathbf{K}_x -torsor. Let $\text{Aut}_x(\mathcal{E})$ be the pro-algebraic group of automorphisms of $\mathcal{E}|_{\text{Spec } \mathcal{O}_x}$ as a \mathbf{K}_x -torsor. Choosing a trivialization of $\mathcal{E}|_{\text{Spec } \mathcal{O}_x}$ as a \mathbf{K}_x -torsor, we get an isomorphism $\epsilon_x : \text{Aut}_x(\mathcal{E}) \cong \mathbf{K}_x$; changing the trivialization changes the isomorphism ϵ_x by an inner automorphism of \mathbf{K}_x . For each $x \in S(\bar{k})$, restricting an automorphism of \mathcal{E} to $\text{Spec } \mathcal{O}_x$ gives a homomorphism of pro-algebraic groups

$$\text{ev}_{S, \mathcal{E}} : \text{Aut}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E}) \rightarrow \prod_{x \in S(\bar{k})} \text{Aut}_x(\mathcal{E}) \cong \mathbf{K}_S \otimes_k \bar{k}$$

which is well-defined up to conjugacy. We can pullback the character sheaf \mathcal{K}_S to $\text{Aut}(\mathcal{E})$ via $\text{ev}_{S, \mathcal{E}}$ and get a rank one character sheaf

$$\mathcal{K}_{S, \mathcal{E}} := \text{ev}_{S, \mathcal{E}}^* \mathcal{K}_S \in \mathcal{CS}_1(\text{Aut}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E})).$$

This is well-defined because inner automorphisms act trivially on the isomorphism classes of rank one character sheaves.

2.7.7. DEFINITION. A point $\mathcal{E} \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S)(\bar{k})$ is called \mathcal{K}_S -relevant if the restriction of $\mathcal{K}_{S, \mathcal{E}}$ to the neutral component of $\text{Aut}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E})$ is isomorphic to the constant sheaf. Otherwise the point \mathcal{E} is called \mathcal{K}_S -irrelevant. Again \mathcal{K}_S -relevance is a property of a $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ -orbit.

2.7.8. LEMMA. *If a point $\mathcal{E} \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S)(\bar{k})$ is (Ω, \mathcal{K}_S) -relevant, then it is also \mathcal{K}_S -relevant. Moreover, if $\mathbf{A}_{Z,S}$ is connected, then a point $\mathcal{E} \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S)(\bar{k})$ is \mathcal{K}_S -relevant if and only if it is (Ω, \mathcal{K}_S) -relevant.*

PROOF. If $\mathcal{E}^+ \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)$ is a lifting of \mathcal{E} , then $\mathbf{L}_{S, \mathcal{E}^+}$ is canonically identified with $\text{Aut}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E})$, hence a homomorphism $\lambda : \text{Aut}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E}) \cong \mathbf{L}_{S, \mathcal{E}^+} \rightarrow \mathbf{M}_{S, \mathcal{E}^+} \cong A_{\mathcal{E}}$. We have $\lambda^* \mathcal{K}_{\mathcal{E}} \cong \mathcal{K}_{S, \mathcal{E}}$, hence (Ω, \mathcal{K}_S) -relevance implies \mathcal{K}_S -relevance.

Next we assume $\mathbf{A}_{Z,S}$ is connected and deduce that \mathcal{K}_S -relevance implies (Ω, \mathcal{K}_S) -relevance. Note that $\mathbf{A}_{Z,S} \subset \mathbf{L}_{Z,S} \subset \mathbf{L}_S$ is the kernel of the homomorphism $\mathbf{L}_S \rightarrow \mathbf{M}_S$, and is always contained in $\text{Aut}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E})$. Therefore, we have an exact sequence

$$1 \rightarrow \mathbf{A}_{Z,S} \rightarrow \text{Aut}_{\mathcal{G}, \mathbf{K}_S}(\mathcal{E}) \xrightarrow{\lambda} A_{\mathcal{E}}.$$

Since $\mathbf{A}_{Z,S}$ is connected, the fact that $\mathcal{K}_{S, \mathcal{E}}$ is trivial on $\text{Aut}_{\mathcal{G}, \mathbf{K}_S}^{\circ}(\mathcal{E})$ implies that $\mathcal{K}_{\mathcal{E}}$ is trivial when restricted to $\text{Aut}_{\mathcal{G}, \mathbf{K}_S}^{\circ}(\mathcal{E})/\mathbf{A}_{Z,S}$. To show that $\mathcal{K}_{\mathcal{E}}|_{A_{\mathcal{E}}}$ is trivial, it suffices to show that $\text{Aut}_{\mathcal{G}, \mathbf{K}_S}^{\circ}(\mathcal{E})/\mathbf{A}_{Z,S} = A_{\mathcal{E}}^{\circ}$, or to show that $\text{coker}(\lambda)$ is finite.

Since $\text{coker}(\mathbf{L}_S \rightarrow \mathbf{M}_S)$ is the coarse moduli space of $\text{Bun}_Z^{\mathfrak{h}}(\mathbf{K}_{Z,S})$, $\text{coker}(\lambda)$ is the coarse moduli space of the stabilizer of \mathcal{E} under $\text{Bun}_Z^{\mathfrak{h}}(\mathbf{K}_{Z,S})$. Recall D is the maximal torus quotient of G . Let $\mathbf{K}_{D,S} = \prod_{x \in S} \mathbf{K}_{D,x}$ where $\mathbf{K}_{D,x}$ is the image of \mathbf{K}_S in $L_x D$. By our assumption, $\mathbf{K}_x = \mathbf{K}_{Z,x} \cdot (\mathbf{K}_x \cap L_x G^{\text{der}})$, hence $\mathbf{K}_{D,x}$ is also the image of $\mathbf{K}_{Z,x}$ under $L_x Z \rightarrow L_x D$. We have maps $\text{Bun}_Z(\mathbf{K}_{Z,S}) \rightarrow \text{Bun}_G(\mathbf{K}_S) \rightarrow \text{Bun}_{\mathcal{D}}(\mathbf{K}_{D,S})$. For $b \in \text{Bun}_Z(\mathbf{K}_{Z,S})$ and $\mathcal{E} \in \text{Bun}_G(\mathbf{K}_S)$, we denote their images in $\text{Bun}_{\mathcal{D}}(\mathbf{K}_{D,S})$ by b_D and \mathcal{E}_D . Suppose $b \cdot \mathcal{E} \cong \mathcal{E}$, then $b_D \mathcal{E}_D \cong \mathcal{E}_D$ in $\text{Bun}_{\mathcal{D}}(\mathbf{K}_{D,S})$, which implies that b lies in the kernel of $\zeta : \text{Bun}_Z(\mathbf{K}_{Z,S}) \rightarrow \text{Bun}_{\mathcal{D}}(\mathbf{K}_{D,S})$. Since $Z \rightarrow D$ is an isogeny, we have a factorization $[n] : Z \rightarrow D \rightarrow Z$ for some positive integer n . This induces a factorization $[n] : \text{Bun}_Z(\mathbf{K}_{Z,S}) \xrightarrow{\zeta} \text{Bun}_{\mathcal{D}}(\mathbf{K}_{D,S}) \rightarrow \text{Bun}_Z(\mathbf{K}_{Z,S})$. Therefore $\ker(\zeta)^c \subset \ker([n])^c$ (where $(-)^c$ means taking the coarse moduli space). Hence $\text{coker}(\lambda) \subset \ker([n])^c$, which is finite. This completes the proof. \square

- 2.7.9. LEMMA. (1) Let $\mathcal{E} \in \text{Bun}_G(\mathbf{K}_S)(\bar{k})$ be a (Ω, \mathcal{K}_S) -irrelevant point. Then for any object $\mathcal{F} \in D_{G,\Omega}(\mathbf{K}_S, \mathcal{K}_S)$, $i_{\mathcal{E}}^* \mathcal{F} = 0$ and $i_{\mathcal{E}}^! \mathcal{F} = 0$. Here $i_{\mathcal{E}}$ denotes the inclusion map of the fiber of \mathcal{E} in $\text{Bun}_G(\mathbf{K}_S^+)$.
- (2) Let $[g] \in G(F) \backslash G(\mathbb{A}_F) / \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x) \times \prod_{x \in S} K_x$ be a double coset whose corresponding point $\mathcal{E} \in \text{Bun}_G(\mathbf{K}_S)(k)$ is \mathcal{K}_S -irrelevant. Then any function $f \in C_G(K_S, \chi_S)$ vanishes on the double coset $G(F)g(\prod_{x \notin S} \mathcal{G}(\mathcal{O}_x) \times \prod_{x \in S} K_x)$. Similar statement holds when k is replaced by a finite extension k' .

PROOF. (1) We work over \bar{k} without changing notation. Choose a smaller level \mathbf{K}_S^+ as in §2.6.3 and a preimage $\mathcal{E}^+ \in \text{Bun}_G(\mathbf{K}_S^+)(\bar{k})$ of \mathcal{E} . Let $i_{\mathcal{E}^+} : \text{Spec } \bar{k} \rightarrow \text{Bun}_G(\mathbf{K}_S^+)$ be the inclusion of \mathcal{E}^+ . It suffices to prove that $i_{\mathcal{E}^+}^* \mathcal{F} = 0$ and $i_{\mathcal{E}^+}^! \mathcal{F} = 0$. Since $D_{G,\Omega}(\mathbf{K}_S, \mathcal{K}_S) \cong D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}(\text{Bun}_G(\mathbf{K}_S^+))$, the stalks $i_{\mathcal{E}^+}^* \mathcal{F}$ and $i_{\mathcal{E}^+}^! \mathcal{F}$ are $\mathbf{M}_{S,\mathcal{E}^+}$ -equivariant complexes over $\text{Spec } \bar{k}$. By Lemma A.4.4, this category is zero if $\mathcal{K}_{S,\Omega}|_{\mathbf{M}_{S,\mathcal{E}^+}^\circ}$ is not the constant sheaf, i.e., if \mathcal{E} is a (Ω, \mathcal{K}_S) -irrelevant point of $\text{Bun}_G(\mathbf{K}_S)(\bar{k})$.

(2) Let $A := \text{Aut}_G(\mathcal{E})$. Then $A(k)$ is the automorphism group of the double coset $[g]$ if we view $G(F) \backslash G(\mathbb{A}_F) / \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x) \times \prod_{x \in S} K_x$ as a groupoid. There is a function $f \in C_G(K_S, \chi_S)$ nonzero on the double coset represented by \mathcal{E} if and only if the character $\chi_{S,[g]} : A(k) \rightarrow \prod_{x \in S} K_x \xrightarrow{\prod \chi_x} \overline{\mathbb{Q}}_\ell^\times$ is trivial. Under the sheaf-to-function correspondence, $\chi_{S,[g]}$ corresponds to the rank one character sheaf $\mathcal{K}_{S,\mathcal{E}}$ on A . Since \mathcal{E} is irrelevant, $\mathcal{K}_{S,\mathcal{E}}|_{A^\circ}$ is nontrivial. By Theorem A.3.9, the character $\chi_{S,[g]}|_{A^\circ(k)}$ is also nontrivial, and therefore f must vanish on $[g]$. \square

2.7.10. THEOREM. Let $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ be a geometric automorphic datum. Consider the following statements.

- (1) $\text{Bun}_G(\mathbf{K}_S)$ has only finitely many \mathcal{K}_S -relevant $\text{Bun}_Z(\mathbf{K}_{Z,S})$ -orbits over \bar{k} .

- (2) $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$ has only finitely many (Ω, \mathcal{K}_S) -relevant $\text{Bun}_Z(\mathbf{K}_{Z,S})$ -orbits over \bar{k} .
- (3) For each $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$, $D_{\mathcal{G},\Omega}(\bar{k}; \mathbf{K}_S, \mathcal{K}_S)_\alpha$ (see §2.6.10 for notation) contains only finitely many irreducible perverse sheaves up to isomorphism.
- (4) The geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is weakly rigid.

Then (1) \Rightarrow (4) \Rightarrow (2) \Leftrightarrow (3). When $\mathbf{A}_{Z,S}$ is connected, all the statements are equivalent.

PROOF. (2) \Rightarrow (3) We base change all stacks to \bar{k} without changing notation. Clearly (2) is equivalent to that $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_\alpha$ has only finitely many (Ω, \mathcal{K}_S) -relevant \mathbf{M}_S -orbits over \bar{k} for each α . If this is the case, let $\mathcal{E}_1^+, \dots, \mathcal{E}_N^+$ be representatives of the relevant \mathbf{M}_S -orbits in $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)(\bar{k})_\alpha$, whose image in $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$ we denote by $\mathcal{E}_1, \dots, \mathcal{E}_N$. By Lemma 2.7.9, any object $\mathcal{F} \in D_{\mathcal{G},\Omega}(\mathbf{K}_S, \mathcal{K}_S)_\alpha \cong D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_\alpha)$ can only have nonzero stalk and costalk along the \mathbf{M}_S -orbits of \mathcal{E}_i^+ . If such an \mathcal{F} is an irreducible $(\mathbf{M}_S, \mathcal{K}_{S,\Omega})$ -equivariant perverse sheaf, it is the middle extension of some $(\mathbf{M}_S, \mathcal{K}_{S,\Omega})$ -equivariant local system on the \mathbf{M}_S -orbit $O_i := \mathbf{M}_S \cdot \mathcal{E}_i^+$. By Lemma A.4.4, the category of $(\mathbf{M}_S, \mathcal{K}_{S,\Omega})$ -equivariant local system on $O_i \cong \mathbf{M}_S/A_{\mathcal{E}_i}$ is equivalent to the category of twisted representations of $\pi_0(A_{\mathcal{E}_i})$ under some cocycle in $H^2(\pi_0(A_{\mathcal{E}_i}), \overline{\mathbb{Q}}_\ell^\times)$ determined by $\mathcal{K}_{\mathcal{E}_i}$, which only has finitely many irreducible objects. Therefore there are only finitely many irreducible perverse sheaves in $D_{\mathcal{G},\Omega}(\bar{k}; \mathbf{K}_S, \mathcal{K}_S)$ up to isomorphism.

(3) \Rightarrow (2) For each (Ω, \mathcal{K}_S) -relevant \mathbf{M}_S -orbit $O \cong \mathbf{M}_S/A_{\mathcal{E}} \subset \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_\alpha$, the category $\text{Loc}_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}(O) \cong \text{Rep}_\xi(\pi_0(A_{\mathcal{E}}))$ for some cocycle $\xi \in H^2(\pi_0(A_{\mathcal{E}}), \overline{\mathbb{Q}}_\ell^\times)$. One can find a central extension $1 \rightarrow C \rightarrow \widetilde{A}_{\mathcal{E}} \rightarrow A_{\mathcal{E}} \rightarrow 1$ and a character $\chi_C : C \rightarrow \overline{\mathbb{Q}}_\ell^\times$ such that $\text{Rep}_\xi(\pi_0(A_{\mathcal{E}}))$ is equivalent to the category of finite-dimensional representations of $\widetilde{A}_{\mathcal{E}}$ on which C acts through χ_C . For the argument see §4.5.1. In particular, $\text{Rep}_\xi(\pi_0(A_{\mathcal{E}}))$ is never the zero category. Take any nonzero irreducible object therein, we get an irreducible $(\mathbf{M}_S, \mathcal{K}_{S,\Omega})$ -equivariant local system \mathcal{L} on O , and the middle extension of a suitable shift of \mathcal{L} is an irreducible perverse sheaf \mathcal{F} . For different (Ω, \mathcal{K}_S) -relevant \mathbf{M}_S -orbits we obtain different irreducible perverse sheaves in this way. Therefore, if there are only finitely many irreducible perverse sheaves in $D_{\mathcal{G},\Omega}(\bar{k}; \mathbf{K}_S, \mathcal{K}_S)_\alpha$, $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_\alpha$ can only have finitely many (Ω, \mathcal{K}_S) -relevant \mathbf{M}_S -orbits, i.e., (2) holds.

(1) \Rightarrow (4) We will show that $\dim C_{\mathcal{G},\omega^\natural}(K_S, \chi_S)_\alpha$ is bounded by a constant independent of the base field k and $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$. Choose the level \mathbf{K}_S^+ small enough as in §2.6.3 so that \mathbf{K}_S^+ is pro-unipotent, hence $K_S/K_S^+ = L_S := \mathbf{K}_S(k)$. Let $G(\mathbb{A}_F)_\alpha$ be the preimage of α under the Kottwitz homomorphism $\kappa_{G,F}$. Then $\dim C_{\mathcal{G},\omega^\natural}(K_S, \chi_S)_\alpha$ is the number of $Z(F) \backslash Z(\mathbb{A}_F)^\natural \times L_S$ -orbits on $G(F) \backslash G(\mathbb{A}_F)_\alpha / \prod_{x \in S} K_x^+ \times \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x)$ that can support $(Z(F) \backslash Z(\mathbb{A}_F)^\natural, \omega^\natural)$ and (L_S, χ_S) -eigenfunctions. By Lemma 2.7.9(2), only

those cosets that are \mathcal{K}_S -relevant can support such functions. By (1), there are only a finite number of $\text{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S})$ -orbits in $\text{Bun}_G(\mathbf{K}_S)_\alpha$ over \bar{k} (any this bound can be chosen to be independent of α). Take such an orbit and denote its preimage in $\text{Bun}_G(\mathbf{K}_S^+)_\alpha$ by O . Suppose O contains a k -point (otherwise it does not contribute to the function space $C_{\mathcal{G}, \omega^{\natural}}(K_S, \chi_S)_\alpha$). We shall bound the number of $Z(F) \backslash Z(\mathbb{A}_F)^{\natural} \times L_S$ -orbits on $O(k)$ by a number that is unchanged upon passing k to a finite extension.

Fix $\mathcal{E}^+ \in O(k)$, and let $S_{\mathcal{E}^+}$ be the stabilizer of \mathcal{E}^+ under $\text{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S}^+) \times \mathbf{L}_S$. Let $P := \text{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S}^+)(k)$. Since O is a single orbit under $\text{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S}^+) \times \mathbf{L}_S$ over \bar{k} , the number $P \times L_S$ -orbits in $O(k)$ is bounded by $\#\mathbb{H}^1(k, S_{\mathcal{E}^+}) \leq \#\pi_0(S_{\mathcal{E}^+})$.

Note that $\text{Bun}_G(\mathbf{K}_S^+)_\alpha(k)$ is decomposed according to classes $\zeta \in \mathbb{H}_{G, \mathbf{K}_S^+}^1(F, G)$ as in (2.9), and only the trivial class corresponds to $G(F) \backslash G(\mathbb{A}_F)_\alpha / \prod_{x \in S} K_x^+ \times \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x)$, which is all we care about. Let $O(k)^{\heartsuit}$ be the subset of $O(k)$ that belongs to $G(F) \backslash G(\mathbb{A}_F)_\alpha / \prod_{x \in S} K_x^+ \times \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x)$. We also have a map $P \rightarrow \mathbb{H}_{\mathbb{Z}, \mathbf{K}_{Z,S}^+}^1(F, Z)$, the latter is a subgroup of $\mathbb{H}^1(F, Z)$ and it acts on $\mathbb{H}_{G, \mathbf{K}_S^+}^1(F, G)$ compatibly with the action of P on $\text{Bun}_G(\mathbf{K}_S^+)_\alpha(k)$. Let $P_0 \subset P$ be the subset which maps to $\ker_{S, \mathbf{K}_S^+}(Z, G) := \ker(\mathbb{H}_{\mathbb{Z}, \mathbf{K}_{Z,S}^+}^1(F, Z) \rightarrow \mathbb{H}_{G, \mathbf{K}_S^+}^1(F, G))$. Then each $P \times L_S$ -orbit on $O(k)$ intersects $O(k)^{\heartsuit}$ in a single $P_0 \times L_S$ -orbit. Therefore the number of $P_0 \times L_S$ -orbits on $O(k)^{\heartsuit}$ is also bounded by $\#\pi_0(S_{\mathcal{E}^+})$.

Finally we bound the number of $Z(F) \backslash Z(\mathbb{A}_F)^{\natural} \times L_S$ -orbits on $O(k)^{\heartsuit}$. For this we only need to bound the cokernel of $Z(F) \backslash Z(\mathbb{A}_F)^{\natural} \rightarrow P_0$, which is a subgroup of $\ker_{S, \mathbf{K}_S^+}(Z, G)$. Therefore it suffices to bound $\#\ker_{S, \mathbf{K}_S^+}(Z, G)$. For a group H over F we denote $\mathbb{H}_{S\text{-ram}}^1(F, H)$ to be the kernel of $\mathbb{H}^1(F, H) \mapsto \prod_{x \in |U|} \mathbb{H}^1(F_x^{\text{ur}}, H)$. Then $\ker_{S, \mathbf{K}_S^+}(Z, G) \subset \ker(\mathbb{H}_{S\text{-ram}}^1(F, Z) \rightarrow \mathbb{H}_{S\text{-ram}}^1(F, G)) =: \ker_S(Z, G)$. It suffices to bound the size of $\ker_S(Z, G)$.

Over the finite Galois extension F'/F , Z become constant diagonalizable groups (i.e., $\text{Gal}(F^s/F')$ acts trivially on their character groups), and we may write $Z \otimes_F F'$ as $\mathbb{G}_m^r \times Z^{\text{fin}}$ for some finite constant diagonalizable group over F' . Let $\Gamma = \text{Gal}(F'/F)$, then we have exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{H}^1(\Gamma, Z(F')) & \longrightarrow & \mathbb{H}^1(F, Z) & \longrightarrow & \mathbb{H}^1(F', Z)^\Gamma \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{H}^1(\Gamma, G(F')) & \longrightarrow & \mathbb{H}^1(F, G) & \longrightarrow & \mathbb{H}^1(F', G)^\Gamma \end{array}$$

The image of $\ker_S(Z, G)$ in $\mathbb{H}^1(F', Z)^\Gamma$ certainly lies in $\mathbb{H}_{S\text{-ram}}^1(F', Z)^\Gamma$. Since $Z \otimes_F F' \cong \mathbb{G}_m^r \times Z^{\text{fin}}$, $\mathbb{H}_{S\text{-ram}}^1(F', Z)^\Gamma \subset \mathbb{H}_{S\text{-ram}}^1(F', Z^{\text{fin}})^\Gamma$. On the other hand, the kernel of $\ker_S(Z, G) \rightarrow \mathbb{H}^1(F', Z)^\Gamma$ lies in the kernel of $\mathbb{H}^1(\Gamma, Z(F')) \rightarrow \mathbb{H}^1(\Gamma, G(F'))$, which is surjected by $\mathbb{H}^1(\Gamma, C(F'))$ where

$C = \ker(Z \rightarrow G)$ is a finite diagonalizable group over F . The conclusion is that

$$\#\ker_S(Z, G) \leq \#\mathrm{H}_{S-\mathrm{ram}}^1(F', Z^{\mathrm{fn}})^\Gamma \cdot \#\mathrm{H}^1(\Gamma, C(F')).$$

Both groups on the right side are finite and their cardinalities are bounded independent of extensions of k . This gives a bound on $\#\ker_{S, \mathbf{K}_S^+}(Z, G) \leq \#\ker_S(Z, G)$, and hence on the number of $Z(F) \backslash Z(\mathbb{A}_F)^\natural \times L_S$ -orbits on $O^\heartsuit(k)$ for any \mathcal{K}_S -relevant $\mathrm{Bun}_Z^\natural(\mathbf{K}_{Z,S}^+)$ -orbit on $\mathrm{Bun}_G(\mathbf{K}_S^+)_\alpha$ for any α . Therefore the dimension of $C_{G, \omega^\natural}(K_S, \chi_S)_\alpha$ is bounded independent of α and the field extension k'/k .

(4) \Rightarrow (2) Suppose $C_{G, \omega^\natural}(k'; K_S, \chi_S)_\alpha \leq N$ for all finite extensions k'/k , we shall show that $\mathrm{Bun}_G(\mathbf{K}_S^+)$ contains at most N (Ω, \mathcal{K}_S) -relevant \mathbf{M}_S -orbits over \bar{k} . If not, let O_1, \dots, O_{N+1} be distinct (Ω, \mathcal{K}_S) -relevant \mathbf{M}_S -orbits, and let $\mathcal{E}_i^+ \in O_i(\bar{k})$. Replacing k by a large enough finite extension, we may assume that all \mathcal{E}_i^+ are defined over k . However, recall from (2.9) that not all k -points of $\mathrm{Bun}_G(\mathbf{K}_S^+)$ correspond to points in the double coset $G(F) \backslash G(\mathbb{A}_F) / \prod_{x \in S} K_x^+ \times \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x)$. For each point \mathcal{E}_i^+ there is a class $\zeta_i \in \mathrm{H}^1(F, G)$ recording the isomorphism class of \mathcal{E}_i^+ at the generic point of X . Since $\mathrm{H}^1(F \otimes_k \bar{k}, G)$ vanishes, by enlarging k to a finite extension, we may kill the classes ζ_i , and hence make sure that \mathcal{E}_i^+ do represent certain double cosets $[g_i] \in G(F) \backslash G(\mathbb{A}_F) / \prod_{x \in S} K_x^+ \times \prod_{x \notin S} \mathcal{G}(\mathcal{O}_x)$ (which are necessarily distinct). By the argument of (3) \Rightarrow (2), there is a nonzero local system $\mathcal{L}_i \in \mathrm{Loc}(\mathbf{M}_S, \mathcal{K}_{S, \Omega})(O_i)$ with support on the closure of the orbit $\mathbf{M}_S \cdot \mathcal{E}_i^+$ (enlarge k if necessarily to make sure that \mathcal{L}_i is also defined over k). Then for large enough finite extensions k'/k , the function $f_{\mathcal{L}_i, k'}$ attached to \mathcal{L}_i via the sheaf-to-function correspondence lies in $C_{G, \omega^\natural}(k'; K_S, \chi_S)_\alpha$, and is nonzero at $[g_i]$ and vanishing outside the $\mathbf{M}_S(k')$ -orbit of $[g_i]$. Then $\{f_{\mathcal{L}_i, k'}\}_{i=1, \dots, N+1}$ gives at least $(N+1)$ linearly independent functions in $C_{G, \omega^\natural}(k'; K_S, \chi_S)_\alpha$, contradicting our original assumption.

Finally, when $\mathbf{A}_{Z,S}$ is connected, we know from Lemma 2.7.8 that (1) \Rightarrow (2), hence all four statements are equivalent to each other. \square

2.7.11. *Numerical condition for weak rigidity.* Let $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ be a geometric automorphic datum. For each $x \in S$ we define an integer $d(\mathbf{K}_x^{\mathrm{ad}})$ as follows. Let $\mathbf{K}_x^{\mathrm{ad}} = \mathbf{K}_x / \mathbf{K}_{Z,x} \subset L_x G^{\mathrm{ad}}$. The Lie algebra $\mathfrak{k}_x^{\mathrm{ad}}$ of $\mathbf{K}_x^{\mathrm{ad}}$ is an \mathcal{O}_x -lattice in $\mathfrak{g}^{\mathrm{ad}}(F_x)$. On the other hand we have the special parahoric subgroup $L_x^+ \mathcal{G}^{\mathrm{ad}}$ whose Lie algebra is another lattice $\mathfrak{g}^{\mathrm{ad}}(\mathcal{O}_x) \subset \mathfrak{g}^{\mathrm{ad}}(F_x)$. It makes sense to consider the relative dimension $\dim_k(\mathfrak{g}^{\mathrm{ad}}(\mathcal{O}_x) : \mathfrak{k}_x^{\mathrm{ad}})$ between the two lattices $\mathfrak{g}^{\mathrm{ad}}(\mathcal{O}_x)$ and $\mathfrak{k}_x^{\mathrm{ad}}$: take another lattice Λ contained in both of them and define $\dim_k(\mathfrak{g}^{\mathrm{ad}}(\mathcal{O}_x) : \mathfrak{k}_x^{\mathrm{ad}})$ as $\dim_k(\mathfrak{g}^{\mathrm{ad}}(\mathcal{O}_x) / \Lambda) - \dim_k(\mathfrak{k}_x^{\mathrm{ad}} / \Lambda)$. Note that the inertia group I_x acts on \mathbb{G}^{ad} (as restricted from the homomorphism $\Gamma_F \xrightarrow{\theta} \mathrm{Aut}^\dagger(\mathbb{G}) \rightarrow \mathrm{Aut}^\dagger(\mathbb{G}^{\mathrm{ad}})$). Define

$$d(\mathbf{K}_x^{\mathrm{ad}}) := \dim \mathbb{G}^{\mathrm{ad}} - \dim \mathbb{G}^{\mathrm{ad}, I_x} + 2 \dim_k(\mathfrak{g}^{\mathrm{ad}}(\mathcal{O}_x) : \mathfrak{k}_x^{\mathrm{ad}}).$$

- 2.7.12. LEMMA. (1) For any choice of \mathbf{K}_S^+ as in §2.6.3, the stack $[\mathbf{M}_S \backslash \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_{\alpha}]$ is smooth of pure dimension $(g_X - 1) \dim \mathbb{G}^{\text{ad}} + \frac{1}{2} \sum_{x \in S} d(\mathbf{K}_x^{\text{ad}})$.
- (2) Suppose there exists a point on $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$ whose stabilizer under $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ is finite. If $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is weakly rigid, then

$$(2.16) \quad \frac{1}{2} \sum_{x \in S} d(\mathbf{K}_x^{\text{ad}}) = (1 - g_X) \dim \mathbb{G}^{\text{ad}}.$$

PROOF. (1) Since $\dim[\mathbf{M}_S \backslash \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_{\alpha}] = \dim \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_{\alpha} - \dim \mathbf{M}_S = \dim \text{Bun}_{\mathcal{G}}(\mathbf{K}_S) - \dim \text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$, we shall compute the dimension of $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$ and $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ separately. One can define $d(\mathbf{K}_x)$ and $d(\mathbf{K}_{Z,x})$ in a way analogous to $d(\mathbf{K}_x^{\text{ad}})$, with the group G^{ad} changed to G or Z (correspondingly \mathbb{G}^{ad} changes to \mathbb{G} or $\mathbb{Z}\mathbb{G}$). Clearly $d(\mathbf{K}_x) - d(\mathbf{K}_{Z,x}) = d(\mathbf{K}_x^{\text{ad}})$. We shall prove that

$$(2.17) \quad \text{Bun}_{\mathcal{G}}(\mathbf{K}_S) = (g_X - 1) \dim \mathbb{G} + \frac{1}{2} \sum_{x \in S} d(\mathbf{K}_x),$$

$$(2.18) \quad \text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}) = (g_X - 1) \dim \mathbb{Z}\mathbb{G} + \frac{1}{2} \sum_{x \in S} d(\mathbf{K}_{Z,x}).$$

Taking the difference of these equations gives (1).

It is well-known that $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$ is smooth of dimension $-\chi(X, \text{Lie } \mathcal{G}_{\mathbf{K}_S})$, where $\text{Lie } \mathcal{G}_{\mathbf{K}_S}$ is the coherent sheaf on X obtained from $\text{Lie } \mathcal{G}$ by changing its local sections over $\text{Spec } \mathcal{O}_x$ from $\mathfrak{g}(\mathcal{O}_x)$ to $\mathfrak{k}_x = \text{Lie } \mathbf{K}_x$. A local calculation shows that $\deg(\text{Lie } \mathcal{G}) = -\frac{1}{2} \sum_{x \in S} \dim \mathbb{G} / \dim \mathbb{G}^{I_x}$. Moreover, $\deg \text{Lie } \mathcal{G} - \deg \text{Lie } \mathcal{G}_{\mathbf{K}_S} = \sum_{x \in S} \dim_k(\mathfrak{g}(\mathcal{O}_x) : \mathfrak{k}_x)$. Therefore $\deg \text{Lie } \mathcal{G}_{\mathbf{K}_S} = -\sum_{x \in S} (\frac{1}{2} \dim \mathbb{G} / \dim \mathbb{G}^{I_x} + \dim_k(\mathfrak{g}(\mathcal{O}_x) : \mathfrak{k}_x)) = -\frac{1}{2} \sum_{x \in S} d(\mathbf{K}_x)$. By Riemann-Roch theorem, $-\chi(X, \text{Lie } \mathcal{G}_{\mathbf{K}_S}) = -\deg \text{Lie } \mathcal{G}_{\mathbf{K}_S} + (g_X - 1) \text{rank}(\text{Lie } \mathcal{G}_{\mathbf{K}_S}) = \frac{1}{2} \sum_{x \in S} d(\mathbf{K}_x) + (g_X - 1) \dim \mathbb{G}$. This proves (2.17). The proof of (2.18) is similar. This finishes the proof of (1).

(2) The stabilizer of a point $\mathcal{E} \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$ under $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ is the same as the stabilizer of any of its preimage $\mathcal{E}^+ \in \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_{\alpha}$ under \mathbf{M}_S . Let $\mathcal{U} \subset \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_{\alpha}$ be the locus where the stabilizers under \mathbf{M}_S are finite. The given condition ensures that \mathcal{U} is nonempty for some α . Also \mathcal{U} is stable under \mathbf{M}_S and consists of (Ω, \mathcal{K}_S) -relevant points (because the condition for relevance only has to do with the neutral component of stabilizers). Moreover \mathcal{U} is open because the dimension of the stabilizers under \mathbf{M}_S is upper semicontinuous. By Theorem 2.7.10, $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ being weakly rigid implies that \mathcal{U} consists of finitely many \mathbf{M}_S -orbits. Therefore $\dim[\mathbf{M}_S \backslash \text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)_{\alpha}] = \dim[\mathbf{M}_S \backslash \mathcal{U}] = 0$, and we get (2.16) from (1). \square

2.8. Rigid automorphic data for GL_2 . Let k be a finite field. Consider the split group $G = \text{GL}_2$ over $F = k(t)$, the function field of $X = \mathbb{P}_k^1$. Since G is split, we take the constant group scheme over X as the integral model \mathcal{G} , and denote \mathcal{G} also by G .

2.8.1. *S consists of three points.* Let $S = \{0, 1, \infty\} \subset |\mathbb{P}_k^1| = |X|$. For $x \in S$, let \mathbf{I}_x be the Iwahori subgroup of $L_x G$ such that

$$\mathbf{I}_x(k_x) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \in \mathfrak{m}_x, a, b, d \in \mathcal{O}_x \right\}.$$

The reductive quotient of \mathbf{I}_x is \mathbb{G}_m^2 given by sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $a \pmod{\mathfrak{m}_x}$ and $d \pmod{\mathfrak{m}_x}$. For each $x \in S$, we choose a character $\chi_x = (\chi_x^{(1)}, \chi_x^{(2)}) : k^\times \times k^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$, which corresponds to a rank one character sheaf $\mathcal{K}_x = \mathcal{K}_x^{(1)} \boxtimes \mathcal{K}_x^{(2)}$ on \mathbb{G}_m^2 . We may view \mathcal{K}_x as a rank one character sheaf on \mathbf{I}_x by pullback along $\mathbf{I}_x \rightarrow \mathbb{G}_m^2$. We temporarily use I_x to denote $\mathbf{I}_x(k_x)$ (not to be confused with the inertia group I_x in §2.1). Then (I_S, χ_S) comes from $(\mathbf{I}_S, \mathcal{K}_S)$ via the sheaf-to-function correspondence.

2.8.2. PROPOSITION. (1) *There is an (I_S, χ_S) -typical automorphic representation of $G(\mathbb{A}_F)$ only if*

$$(2.19) \quad \prod_{x \in S} \chi_x^{(1)} \chi_x^{(2)} = 1.$$

In this case there is unique way (up to isomorphism) to extend $(\mathbf{I}_S, \mathcal{K}_S)$ into a geometric automorphic datum $(\Omega, \mathbf{I}_S, \mathcal{K}_S, \iota_S)$.

(2) *If, moreover, for any map $\epsilon : S \rightarrow \{1, 2\}$ we have*

$$\prod_{x \in S} \chi_x^{(\epsilon(x))} \neq 1 \text{ (the trivial character),}$$

then the geometric automorphic datum $(\Omega, \mathbf{I}_S, \mathcal{K}_S, \iota_S)$ is strongly rigid.

PROOF. (1) Let π be an (I_S, χ_S) -typical automorphic representation of $G(\mathbb{A}_F)$. Then its central character of $\omega : F^\times \backslash \mathbb{A}_F^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is compatible with $(I_S, \chi_S) : \omega|_{\mathcal{O}_x^\times}$ is trivial for $x \notin S$ and $\omega_x|_{\mathcal{O}_x^\times}$ is equal to $\mathcal{O}_x^\times \rightarrow k_x^\times \xrightarrow{\chi_x^{(1)} \chi_x^{(2)}} \overline{\mathbb{Q}}_\ell^\times$ for $x \in S$. Since $\otimes \omega_x$ has to be trivial on F^\times , and in particular k^\times , we must have (2.19). When (2.19) is satisfied, the central character ω is also unique up to an unramified twist, and in particular the restricted part ω^\natural is unique determined by the characters $\{\chi_x\}_{x \in S}$.

Since in our case $\text{Bun}_Z^{\natural}(\mathbf{I}_{Z,S}) = \text{Pic}^0(X) \cong \mathbb{B}\mathbb{G}_m$, Lemma 2.6.5 implies that when (2.19) holds, there is a unique pair (Ω, ι_S) making $(\Omega, \mathbf{I}_S, \mathcal{K}_S, \iota_S)$ into a geometric automorphic datum.

(2) We shall show that (I_S, χ_S) -typical automorphic representations are unique up to unramified twists. For this we may fix a central character ω compatible with (I_S, χ_S) , and argue that (ω, I_S, χ_S) -typical automorphic representations are unique up to unramified twists.

By (2.6), we have an equivalence of groupoids (note that $H^1(F, \mathrm{GL}_2)$ is trivial)

$$G(F) \backslash G(\mathbb{A}_F) / \prod_{x \notin S} G(\mathcal{O}_x) \times \prod_{x \in S} I_x \xrightarrow{\sim} \mathrm{Bun}_2(\mathbf{I}_S)(k).$$

Here $\mathrm{Bun}_2(\mathbf{I}_S)$ is the moduli stack classifying (\mathcal{V}, ℓ_S) , where \mathcal{V} is a vector bundle of rank two on X and ℓ_S is a collection of lines ℓ_x in the two-dimensional fiber \mathcal{V}_x , one for each $x \in S$. The Kottwitz map in this case is $\kappa : \mathrm{Bun}_2(\mathbf{I}_S) \rightarrow \mathbb{Z}$ sending (\mathcal{V}, ℓ_S) to the degree of \mathcal{V} . We denote $\kappa^{-1}(d)$ by $\mathrm{Bun}_2^d(\mathbf{I}_S)$. We shall determine the \mathcal{K}_S -relevant points on each $\mathrm{Bun}_2^d(\mathbf{I}_S)$.

Evaluating an automorphisms of $(\mathcal{V}, \ell_S) \in \mathrm{Bun}_2(\mathbf{I}_S)(\bar{k})$ at $x \in S$ gives a map

$$\alpha_x = (\alpha_x^{(1)}, \alpha_x^{(2)}) : \mathrm{Aut}(\mathcal{V}, \ell_S) \rightarrow \mathbb{G}_m^2.$$

which sends an automorphism $\varphi \in \mathrm{Aut}(\mathcal{V}, \ell_S)$ to the scalars by which φ acts on ℓ_x and on \mathcal{V}_x/ℓ_x . Then (\mathcal{V}, ℓ_S) is \mathcal{K}_S -relevant if and only if $\mathcal{K}_{S, \mathcal{V}} := \bigotimes_{x \in S} \alpha_x^{(1),*} \mathcal{K}_x^{(1)} \otimes \alpha_x^{(2),*} \mathcal{K}_x^{(2)}$ is trivial on the neutral component of $\mathrm{Aut}(\mathcal{V}, \ell_S)$.

There is one situation in which we may immediately conclude that (\mathcal{V}, ℓ_S) is irrelevant, that is when $\mathcal{V} = \mathcal{L}' \oplus \mathcal{L}''$ is a sum of two line bundles such that for each $x \in S$, either $\ell_x = \mathcal{L}'_x$ or $\ell_x = \mathcal{L}''_x$. We call such a point (\mathcal{V}, ℓ_S) *decomposable*. In this case, consider the subgroup $\mathbb{G}'_m \hookrightarrow \mathrm{Aut}(\mathcal{V}, \ell_S)$ given by scaling only the direct summand \mathcal{L}' . The map $\alpha_x|_{\mathbb{G}'_m}$ is given by the inclusion of \mathbb{G}_m into the first or second factor of \mathbb{G}_m^2 . Therefore the sheaf $\mathcal{K}_{S, \mathcal{V}}|_{\mathbb{G}'_m}$ takes the form

$$\mathcal{K}_{S, \mathcal{V}}|_{\mathbb{G}'_m} \cong \bigotimes_{x \in S} \mathcal{K}_x^{(\epsilon(x))}$$

where $\epsilon(x) = 1$ if $\ell_x = \mathcal{L}'_x$ and $\epsilon(x) = 2$ if $\ell_x = \mathcal{L}''_x$. Under the sheaf-to-function correspondence, $\mathcal{K}_{S, \mathcal{V}}|_{\mathbb{G}'_m}$ goes to the character $\prod_{x \in S} \chi_x^{(\epsilon(x))} : k^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$, which is nontrivial by our assumption. Therefore, $\mathcal{K}_{S, \mathcal{V}}|_{\mathbb{G}'_m}$ is nontrivial, and (\mathcal{V}, ℓ_S) is irrelevant.

To determine the relevant points, we need to look for indecomposable (\mathcal{V}, ℓ_S) 's. Any rank two vector bundles \mathcal{V} over $X = \mathbb{P}^1$ is of the form $\mathcal{L}_1 \oplus \mathcal{L}_2$ where $\mathcal{L}_1 \cong \mathcal{O}(a)$, $\mathcal{L}_2 \cong \mathcal{O}(b)$ for integers $a \geq b$. If $a > b$, the automorphism group of \mathcal{V} consists of 2×2 matrices $\begin{pmatrix} \lambda & \phi \\ 0 & \mu \end{pmatrix}$ where $\phi \in \mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1)$. If $\ell_x \neq \mathcal{L}_{1,x}$ then there is a unique value $n_x \in \mathrm{Hom}(\mathcal{L}_{2,x}, \mathcal{L}_{1,x})$ such that $\begin{pmatrix} 1 & n_x \\ 0 & 1 \end{pmatrix} \cdot \mathcal{L}_{2,x} = \ell_x$.

If $a - b \geq 2$, there exists $\phi \in \mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1) \cong H^0(\mathbb{P}^1, \mathcal{O}(a - b))$ such that $\phi(x) = n_x$ for all $x \in S$ whenever $\ell_x \neq \mathcal{L}_{1,x}$ (since $\deg S = 3$). Therefore, letting $\mathcal{L}' = \mathcal{L}_1$ and $\mathcal{L}'' = \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} \cdot \mathcal{L}_2$, we get a new decomposition $\mathcal{V} = \mathcal{L}' \oplus \mathcal{L}''$ such that for each $x \in S$, ℓ_x is either \mathcal{L}'_x or \mathcal{L}''_x , i.e., (\mathcal{V}, ℓ_S) is decomposable hence \mathcal{K}_S -irrelevant.

The remaining possibilities are

- (1) If $a - b = 1$, for (\mathcal{V}, ℓ_S) to be indecomposable, none of the ℓ_x should be contained in \mathcal{L}_1 by the above discussion. When this is the case, we may use an automorphism of \mathcal{V} as above to obtain another decomposition $\mathcal{V} = \mathcal{L}' \oplus \mathcal{L}''$ such that ℓ_0 and ℓ_∞ are contained in the fibers of \mathcal{L}'' while ℓ_1 is spanned by $(1, 1)$ under trivializations of \mathcal{L}' and \mathcal{L}'' at 1. This gives a unique \mathcal{K}_S -relevant point $\star_{2a-1} \in \text{Bun}_2^{2a-1}(\mathbf{I}_S)$ (which is also a k -point).
- (2) If $a = b$, trivializing $\mathbb{P}(\mathcal{V})$ we may view ℓ_x as a point in \mathbb{P}^1 . The point $(\mathcal{V}, \ell_S \in (\mathbb{P}^1)^S)$ is indecomposable if and only if the three points $\ell_x \in \mathbb{P}^1$ are distinct. Since $\text{Aut}(\mathcal{V}) = \text{GL}_2$, all such (\mathcal{V}, ℓ_S) give the same point $\star_{2a} \in \text{Bun}_2^{2a}(\mathbf{I}_S)$. This is a k -point, and is the unique \mathcal{K}_S -relevant point on $\text{Bun}_2^{2a}(\mathbf{I}_S)$.

To conclude, for each $d \in \mathbb{Z}$, $\text{Bun}_2^d(\mathbf{I}_S)$ contains a unique \mathcal{K}_S -relevant point \star_d , which is defined over k . For each $d \in \mathbb{Z}$, let $G(\mathbb{A}_F)_d$ be the subset consisting of $g = (g_x)$ with $\deg \det(g) = d$, and let $C_G(I_S, \chi_S)_d = C(G(F) \backslash G(\mathbb{A}_F)_d / \prod_{x \notin S} G(\mathcal{O}_x) \times \prod_{x \in S} (I_x, \chi_x))$. The above discussion shows that $\dim C_G(I_S, \chi_S)_d = 1$, in which all functions are supported on the double coset given by \star_d .

The action of the center $F^\times \backslash \mathbb{A}_F^\times$ on $G(F) \backslash G(\mathbb{A}_F)$ identifies $G(F) \backslash G(\mathbb{A}_F)_d$ for all d with the same parity. Therefore $\dim C_{G, \omega}(I_S, \chi_S) = 2$, and the restriction map $C_{G, \omega}(I_S, \chi_S) \rightarrow C_G(I_S, \chi_S)_0 \oplus C_G(I_S, \chi_S)_1$ is an isomorphism. Formula (2.5) implies that there are at most two (ω, I_S, χ_S) -typical automorphic representations, and they are cuspidal. All statements so far hold when k is replaced by any finite extension k' (with respect to the base-changed automorphic datum).

If there is only one (ω, I_S, χ_S) -typical automorphic representation π , there is nothing else to show. Suppose there are two (ω, I_S, χ_S) -typical automorphic representations π and π' . By multiplicity one for $G = \text{GL}_2$, π is not isomorphic to π' as $G(\mathbb{A}_F)$ -modules. Consider the unramified quadratic character $\eta : G(F) \backslash G(\mathbb{A}) / \prod_x G(\mathcal{O}_x) \xrightarrow{\deg \circ \det} \mathbb{Z} \twoheadrightarrow \{\pm 1\}$. Twisting by η sends one (ω, I_S, χ_S) -typical automorphic representation to another. We claim that $\pi \otimes \eta = \pi'$. If not, then $\pi \otimes \eta = \pi$. By the global Langlands correspondence for GL_2 , there is a Galois representation $\rho : W_F \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_\ell)$ attached to π , and it satisfies $\rho \otimes \chi_\eta \cong \rho$, where χ_η is the unramified character $W_F \rightarrow \mathbb{Z} \twoheadrightarrow \{\pm 1\}$. This implies that $\rho = \text{Ind}_{W_E}^{W_F} \zeta$ for some character $\zeta : W_E \rightarrow \overline{\mathbb{Q}}_\ell^\times$, where $E = F \otimes_k k'$ and k' is the quadratic extension of k . In particular $\rho|_{W_E}$ is reducible. This means that the base change $\text{BC}_{E/F}(\pi)$, as a $(\omega', I_{S'}, \chi_{S'})$ -typical automorphic representation of $G(\mathbb{A}_E)$, is not cuspidal, which is a contradiction. Therefore up to unramified twists, there is only one (ω, I_S, χ_S) -typical automorphic representation of $G(\mathbb{A}_F)$. The same argument applies when k is replaced by any finite extension k' . Therefore $(\Omega, \mathbf{I}_S, \mathcal{K}_S, \iota_S)$ is strongly rigid. \square

2.8.3. *S consists of two points–Case I.* Let $S = \{0, \infty\}$ and $\text{char}(k) \neq 2$. We consider the following geometric automorphic datum. At $x = 0$, let \mathbf{K}_0 be the Iwahori subgroup \mathbf{I}_0 . Let $\chi_0 : \mathbb{T}(k) = k^\times \times k^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a character, viewed as a character of $I_0 = \mathbf{I}_0(k)$ via $I_0 \rightarrow \mathbb{T}(k)$. Let $\mathcal{K}_0 \in \mathcal{CS}_1(\mathbb{T})$ be the corresponding character sheaf, also viewed as a character sheaf on $\mathbf{K}_0 = \mathbf{I}_0$.

On the other hand, fix a nontrivial additive character $\psi : k \rightarrow \overline{\mathbb{Q}}_\ell^\times$. At $x = \infty$, let $\mathbf{K}_\infty = \mathbf{I}_\infty^+$, the pro-unipotent radical of the Iwahori subgroup at ∞ , i.e.,

$$\mathbf{K}_\infty(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \in t^{-1}k[[t^{-1}]], a, b, d \in k[[t^{-1}]], a, b \equiv 1 \pmod{t^{-1}} \right\}.$$

Consider the homomorphism

$$(2.20) \quad \begin{aligned} \mathbf{K}_\infty &\rightarrow \mathbb{G}_a^2 \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (b_0, c_{-1}). \end{aligned}$$

Here b_0 is the constant term of b and c_{-1} is the coefficient of t^{-1} in c . Let $\phi : k^2 \rightarrow k$ be a linear function. Then the character $\psi\phi : \mathbb{G}_a^2(k) = k \times k \rightarrow \overline{\mathbb{Q}}_\ell^\times$ corresponds to an Artin-Schreier sheaf AS_ϕ . Let $\mathcal{K}_\infty = \text{AS}_\phi$, viewed as a rank one character sheaf on \mathbf{K}_∞ via the map (2.20). Since $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{\mathcal{Z}, S})$ is a point in this case, by Lemma 2.6.5 there is a unique way (up to isomorphism) to extend $(\mathbf{K}_S, \mathcal{K}_S)$ into a geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$.

2.8.4. PROPOSITION. *Suppose the linear function $\phi : k^2 \rightarrow k$ is nontrivial on each coordinate, then the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is strongly rigid.*

PROOF. Let $(\omega^\natural, K_S, \chi_S)$ be the restricted automorphic datum attached to $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$. To show strong rigidity, we need to show that there is a unique $(\omega^\natural, K_S, \chi_S)$ -typical automorphic representation of $G(\mathbb{A}_F)$ up to unramified twists (and the argument will work when k is replaced with any finite extension k').

Let π be a $(\omega^\natural, K_S, \chi_S)$ -typical automorphic representation. The moduli stack we should consider is $\text{Bun}_G(\mathbf{K}_S) = \text{Bun}_2(\mathbf{I}_0, \mathbf{I}_\infty^+)$. It classifies quadruples $(\mathcal{V}, \ell_0, v_\infty^{(1)}, v_\infty^{(2)})$, where \mathcal{V} is a rank two vector bundle over $X = \mathbb{P}^1$, ℓ_0 is a line in the fiber \mathcal{V}_0 , $v_\infty^{(1)}$ is a nonzero vector in \mathcal{V}_∞ and $v_\infty^{(2)}$ is a nonzero vector in the quotient line $\mathcal{V}_\infty / \langle v_\infty^{(1)} \rangle$. Again the Kottwitz map gives a decomposition of $\text{Bun}_2(\mathbf{I}_0, \mathbf{I}_\infty^+)$ into $\text{Bun}_2^d(\mathbf{I}_0, \mathbf{I}_\infty^+)$ according to the degree of \mathcal{V} . We shall show that each $\text{Bun}_2^d(\mathbf{I}_0, \mathbf{I}_\infty^+)$ contains a unique \mathcal{K}_S -relevant point, and the rest of the argument is similar to that of Proposition 2.8.2.

Fix a \bar{k} -point $(\mathcal{V}, \ell_0, v_\infty^{(1)}, v_\infty^{(2)})$ of $\text{Bun}_2(\mathbf{I}_0, \mathbf{I}_\infty^+)$. Let $A = \text{Aut}(\mathcal{V}, \ell_0, v_\infty^{(1)}, v_\infty^{(2)})$. For an automorphism $\varphi \in A$, we may evaluate it at ∞ to get $\varphi_\infty \in \text{GL}(\mathcal{V}_\infty)$, which fixes $v_\infty^{(1)}$ and preserves $v_\infty^{(2)}$ up to a multiple of $v_\infty^{(1)}$. Then $\varphi_\infty(v_\infty^{(2)}) - v_\infty^{(2)} = f_1(\varphi)v_\infty^{(1)}$ for some $f_1(\varphi) \in \bar{k}$. On the other hand, extending $v_\infty^{(1)}$ to a section $\tilde{v}_\infty^{(1)}$ of \mathcal{V} on the first infinitesimal neighborhood of

∞ , then $\varphi(\tilde{v}_\infty^{(1)}) - \tilde{v}_\infty^{(1)} = f_2(\varphi)t^{-1}v_\infty^{(2)}$ modulo the span of $t^{-1}v_\infty^{(1)}$, for some $f_2(\varphi) \in \bar{k}$ independent of the choice of $\tilde{v}_\infty^{(1)}$ lifting $v_\infty^{(1)}$. This way we get a homomorphism

$$f = (f_1, f_2) : A \rightarrow \mathbb{G}_a^2.$$

The point $(\mathcal{V}, \ell_0, v_\infty^{(1)}, v_\infty^{(2)})$ is \mathcal{K}_S -relevant only if $f^*\mathcal{K}_\infty$ is trivial on A° (in fact A is connected).

The vector bundle \mathcal{V} is isomorphic to $\mathcal{L}' \oplus \mathcal{L}''$, where $\mathcal{L}' \cong \mathcal{O}(a)$, $\mathcal{L}'' \cong \mathcal{O}(b)$ for integers $a \geq b$. When $a > b$, as in the proof of Proposition 2.8.2, we may adjust the decomposition by an automorphism of \mathcal{V} so that ℓ_0 is either \mathcal{L}'_0 or \mathcal{L}''_0 , and at the same time either $v_\infty^{(1)} \in \mathcal{L}'_\infty$ or $v_\infty^{(1)} \in \mathcal{L}''_\infty$.

If $v_\infty^{(1)} \in \mathcal{L}'_\infty$, there is an automorphism $\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} \in A$ where $\phi \in \text{Hom}_X(\mathcal{L}'', \mathcal{L}') = \text{H}^0(\mathbb{P}^1, \mathcal{O}(a-b))$ such that $\phi_0 = 0$ and $\phi_\infty \neq 0$. This means that there is a subgroup $\mathbb{G}_a \subset A$ such that $f|_{\mathbb{G}_a}$ identifies it with the first factor of \mathbb{G}_a^2 . Since \mathcal{K}_∞ is nontrivial on the first factor of \mathbb{G}_a^2 , $f^*\mathcal{K}_\infty|_{\mathbb{G}_a}$ is nontrivial, and $(\mathcal{V}, \ell_0, v_\infty^{(1)}, v_\infty^{(2)})$ is \mathcal{K}_S -irrelevant.

If $v_\infty^{(1)} \in \mathcal{L}''_\infty$ and $a - b \geq 2$, then we may consider the similar automorphism with $\phi_0 = 0, \phi_\infty = 0$ but the coefficient of t^{-1} in the local expansion of ϕ at ∞ is nonzero. This way we get a subgroup $\mathbb{G}_a \subset A$ such that $f|_{\mathbb{G}_a}$ identifies it with the second factor of \mathbb{G}_a^2 , and since \mathcal{K}_∞ is nontrivial on the second factor of \mathbb{G}_a^2 , we again conclude that $(\mathcal{V}, \ell_0, v_\infty^{(1)}, v_\infty^{(2)})$ is \mathcal{K}_S -irrelevant. When $a - b = 1$ and $\ell_0 = \mathcal{L}'_0$ we may drop the requirement $\phi_0 = 0$ and the above argument still works.

The remaining cases are

- (1) If $a - b = 1, \ell_0 = \mathcal{L}''_0$ and $v_\infty^{(1)} \in \mathcal{L}''_\infty$. This is in fact a single point in $\text{Bun}_2^{2a-1}(\mathbf{I}_0, \mathbf{I}_\infty^+)$ defined over k with trivial automorphism, and it is the unique \mathcal{K}_S -relevant point therein.
- (2) If $a = b$, after twisting by $\mathcal{O}(-a)$ we may reduce to the case $a = 0$ and we have $\mathcal{V} = V \otimes_{\bar{k}} \mathcal{O}_X$ for the two-dimensional vector space $V = \text{H}^0(X, \mathcal{V})$. The lines ℓ_0 and $\langle v_\infty^{(1)} \rangle$ determine two points in $\mathbb{P}(V)$. If these two points are the same, then A is the unipotent radical U of the Borel subgroup of $\text{GL}(V)$ preserving ℓ_0 , which maps isomorphically to the first factor of \mathbb{G}_a^2 under f . Therefore in this case $(\mathcal{V}, \ell_0, v_\infty^{(1)}, v_\infty^{(2)})$ is \mathcal{K}_S -irrelevant. The only possibility for $(\mathcal{V}, \ell_0, v_\infty^{(1)}, v_\infty^{(2)})$ to be relevant is when the lines ℓ_0 and $\langle v_\infty^{(1)} \rangle$ are distinct. This is in fact a single point in $\text{Bun}_2^{2a}(\mathbf{I}_0, \mathbf{I}_\infty^+)$ defined over k with trivial automorphism, and it is the unique \mathcal{K}_S -relevant point therein.

□

2.8.5. *S consists of two points—Case II.* Let $S = \{0, \infty\}$ and $\text{char}(k) \neq 2$. We consider the following geometric automorphic datum. At $x = 0$, let \mathbf{K}_0

be the Iwahori subgroup \mathbf{I}_0 with reductive quotient $\mathbb{T} \cong \mathbb{G}_m^2$. Choose a characters $\chi_0 = (\chi_0^{(1)}, \chi_0^{(2)}) : \mathbb{T}(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$, which correspond to character sheaves $\mathcal{K}_0^{(1)}$ and $\mathcal{K}_0^{(2)}$ on \mathbb{G}_m . Let $\mathcal{K}_0 = \mathcal{K}_0^{(1)} \boxtimes \mathcal{K}_0^{(2)}$ be the character sheaf on \mathbb{T} , viewed also as a character sheaf on \mathbf{I}_0 .

On the other hand, fix a nontrivial additive character $\psi : k \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Let $\mathbf{K}_\infty^1 = \ker(L_\infty^+ G \xrightarrow{\text{mod } \mathfrak{m}_\infty} \mathbb{G})$ be the first congruence subgroup of $L_\infty^+ G$, and let $\mathbf{K}_\infty = \mathbf{K}_\infty^1 \cdot \mathbb{T}$. There is a natural projection $\mathbf{K}_\infty^1 \rightarrow \mathfrak{g}$ (where $\mathfrak{g} = \text{Lie } \mathbb{G}$) given by reduction modulo \mathfrak{m}_∞^2 . Let $\mathfrak{t} = \text{Lie } \mathbb{T}$. Choose $\phi = (\phi^{(1)}, \phi^{(2)}) \in k^2$, which gives an additive character $\psi\phi : \mathfrak{t} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ sending $\text{diag}(x, y)$ to $\psi(\phi^{(1)}x + \phi^{(2)}y)$. We have the corresponding Artin-Schreier sheaf AS_ϕ on \mathfrak{t} . Also choose characters $(\chi_\infty^{(1)}, \chi_\infty^{(2)}) : \mathbb{T}(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$, which correspond to a character sheaf $\mathcal{K}_\infty^{(1)} \boxtimes \mathcal{K}_\infty^{(2)}$ on \mathbb{T} . Let $\mathcal{K}_\infty = \mathcal{K}_\infty^{(1)} \boxtimes \mathcal{K}_\infty^{(2)} \boxtimes \text{AS}_\phi \in \mathcal{CS}_1(\mathbb{T} \times \mathfrak{t})$. Via the projection $\mathbf{K}_\infty \rightarrow \mathbb{T} \times \mathfrak{g} \rightarrow \mathbb{T} \times \mathfrak{t}$ (the last projection $\mathfrak{g} \rightarrow \mathfrak{t}$ is the projection onto the diagonal matrices), we view \mathcal{K}_∞ as a character sheaf on \mathbf{K}_∞ . The corresponding character of $K_\infty = \mathbf{K}_\infty(k)$ is

$$\chi_\infty : K_\infty \rightarrow \mathbb{T}(k) \times \text{Mat}_2(k) \rightarrow k^\times \times k^\times \times k \times k \xrightarrow{(\chi_\infty^{(1)}, \chi_\infty^{(2)}, \psi\phi^{(1)}, \psi\phi^{(2)})} \overline{\mathbb{Q}}_\ell^\times.$$

2.8.6. PROPOSITION. (1) *There is a (K_S, χ_S) -typical automorphic representation of $G(\mathbb{A}_F)$ only if*

$$\prod_{x \in S} \chi_x^{(1)} \chi_x^{(2)} = 1.$$

In this case, there is a unique way (up to isomorphism) to extend $(\mathbf{K}_S, \mathcal{K}_S)$ into a geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$.

(2) *If the following extra conditions are satisfied*

- *For any map $\epsilon : S \rightarrow \{1, 2\}$ we have*

$$(2.21) \quad \prod_{x \in S} \chi_x^{(\epsilon(x))} \neq 1;$$

- $\phi^{(1)} \neq \phi^{(2)}$.

Then the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is strongly rigid.

SKETCH OF PROOF. The basic idea of the proof is similar to that of Proposition 2.8.2. The only nontrivial part in proving (2) is to show that each $\text{Bun}_2^d(\mathbf{K}_S)$ contains a unique \mathcal{K}_S -relevant point. Let us sketch the argument.

The moduli stack $\text{Bun}_2(\mathbf{K}_S) = \text{Bun}_2(\mathbf{I}_0, \mathbf{K}_\infty)$ in this situation classifies the data $(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)})$ where \mathcal{V} is a rank two vector bundle over $X = \mathbb{P}^1$, ℓ_0 is a line in \mathcal{V}_0 , and $\ell_\infty^{(1)}$ and $\ell_\infty^{(2)}$ are independent lines in \mathcal{V}_∞ . Fix a point $(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)}) \in \text{Bun}_2(\mathbf{I}_0, \mathbf{K}_\infty)(\bar{k})$ and let $A = \text{Aut}(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)})$. Evaluating an automorphism $\varphi \in A$ at 0 we get

$$\alpha_0 = (\alpha_0^{(1)}, \alpha_0^{(2)}) : \text{Aut}(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)}) \rightarrow \mathbb{G}_m^2 = \mathbb{T}$$

where the two coordinates are the scalars by which φ_0 acts on ℓ_0 and on \mathcal{V}_0/ℓ_0 . Let $v^{(i)} \in \ell_\infty^{(i)} - \{0\}$ for $i = 1, 2$. Expanding φ near ∞ to the first order, we may write $\varphi(v^{(1)}) = \alpha_\infty^{(1)}(\varphi)v^{(1)} + t^{-1}\beta^{(1)}(\varphi)v^{(1)}$ modulo $t^{-1}v^{(2)}$ and t^{-2} ; similarly $\varphi(v^{(2)}) = \alpha_\infty^{(2)}(\varphi)v^{(2)} + t^{-1}\beta^{(2)}(\varphi)v^{(2)}$ modulo $t^{-1}v^{(1)}$ and t^{-2} . The functions $\alpha_\infty^{(1)}, \alpha_\infty^{(2)}, \beta^{(1)}, \beta^{(2)}$ are well-defined (independent of the choices of $v^{(1)}$ and $v^{(2)}$). They define homomorphisms

$$\begin{aligned} \alpha_\infty &= (\alpha_\infty^{(1)}, \alpha_\infty^{(2)}) & : & \quad A \rightarrow \mathbb{G}_m^2 = \mathbb{T} \\ \beta &= (\beta^{(1)}, \beta^{(2)}) & : & \quad A \rightarrow \mathbb{A}^2 = \mathfrak{t}. \end{aligned}$$

The point $(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)})$ is \mathcal{K}_S -relevant if and only if both character sheaves

$$\mathcal{K}_\alpha := \bigotimes_{x \in S} \alpha_x^{(1),*} \mathcal{K}_x^{(1)} \otimes \alpha_x^{(2),*} \mathcal{K}_x^{(2)}; \text{ and } \mathcal{K}_\beta := \beta^* \text{AS}_\phi$$

are trivial on the neutral component A° .

We write $\mathcal{V} = \mathcal{L}' \oplus \mathcal{L}''$ where $\mathcal{L}' \cong \mathcal{O}(a)$, $\mathcal{L}'' \cong \mathcal{O}(b)$ with $a \geq b$. If $a > b$, we may use automorphisms of \mathcal{V} to arrange so that one of the following happens

- $\ell_\infty^{(1)} = \mathcal{L}'_\infty$, $\ell_\infty^{(2)} = \mathcal{L}''_\infty$, $\ell_0 = \mathcal{L}'_0$. Then the diagonal torus $\text{diag}(\lambda, 1)$ belongs to A , and \mathbf{K}_α is nontrivial on this torus by assumption (2.21). Therefore $(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)})$ is \mathcal{K}_S -irrelevant.
- $\ell_\infty^{(1)} = \mathcal{L}'_\infty$, $\ell_\infty^{(2)} = \mathcal{L}''_\infty$, $\ell_0 = \mathcal{L}''_0$. Similar argument for the torus $\text{diag}(1, \lambda)$ shows that $(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)})$ is \mathcal{K}_S -irrelevant.
- $\ell_\infty^{(1)} = \mathcal{L}''_\infty$, $\ell_\infty^{(2)} = \mathcal{L}'_\infty$, $\ell_0 = \mathcal{L}'_0$ or \mathcal{L}''_0 . Similar argument as in the previous cases shows that $(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)})$ is \mathcal{K}_S -irrelevant.
- $\ell_\infty^{(1)} = \mathcal{L}''_\infty$, $\ell_\infty^{(2)} \neq \mathcal{L}'_\infty$ or \mathcal{L}''_∞ . If $a \geq b + 2$, then there is an additive subgroup $\varphi : \mathbb{G}_a \hookrightarrow A$ which is the identity when evaluated at 0 (so that ℓ_0 is preserved) and takes the form $\varphi(u) = \begin{pmatrix} 1 & ut^{-1} \\ 0 & 1 \end{pmatrix}$ modulo t^{-2} , $u \in \mathbb{G}_a$. Calculation shows that $\beta(\varphi(u)) = (us, -us)$ for some $s \neq 0$. Therefore $\mathcal{K}_\beta|_{\mathbb{G}_a} \cong m_s^* \text{AS}_{\phi^{(1)} - \phi^{(2)}}$ (where $m_s : \mathbb{G}_a \rightarrow \mathbb{G}_a$ is the multiplication by s map). Since $\phi^{(1)} - \phi^{(2)} \neq 0$, \mathcal{K}_β is nontrivial on $\varphi(\mathbb{G}_a) \subset A$, hence $(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)})$ is \mathcal{K}_S -irrelevant. If $a - b = 1$ and $\ell_0 \in \mathcal{L}'_0$, then ℓ_0 is preserved by any automorphism of \mathcal{V} , and we still have an additive subgroup $\varphi : \mathbb{G}_a \hookrightarrow A$ given by $\varphi(u) = \begin{pmatrix} 1 & ut^{-1} \\ 0 & 1 \end{pmatrix}$. The argument above still works to show that $(\mathcal{V}, \ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)})$ is \mathcal{K}_S -irrelevant.

The remaining cases are

- $a = b$. Up to twisting by $\mathcal{O}(-a)$ we may assume $\mathcal{V} \cong V \otimes_{\bar{k}} \mathcal{O}_X$ for some two-dimensional vector space V . Then we may view $\ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)}$ as lines in V . To avoid possible \mathbb{G}_m in A on which \mathcal{K}_α is

nontrivial, the three lines $\ell_0, \ell_\infty^{(1)}, \ell_\infty^{(2)}$ must be distinct. This defines the unique \mathcal{K}_S -relevant point in $\text{Bun}_2^{2a}(\mathbf{K}_S)$, which is defined over k with automorphism group equal to the central \mathbb{G}_m .

- $a = b + 1$ and none of the three lines $\ell_0, \ell_\infty^{(1)}$ and $\ell_\infty^{(2)}$ are in \mathcal{L}'_∞ . This defines the unique \mathcal{K}_S -relevant point in $\text{Bun}_2^{2a-1}(\mathbf{K}_S)$, which is defined over k with automorphism group equal to the central \mathbb{G}_m .

The rest of the argument is similar to that of Proposition 2.8.2. \square

We will see in §3.3 that the three types of rigid automorphic data we considered in this subsection for GL_2 correspond, under the Langlands correspondence, to the three types of hypergeometric local systems of rank two over punctured \mathbb{P}^1 .

3. Rigidity for local systems

In this section, we will review some basic facts about local systems in étale topology in §3.1, and introduce the notion of rigidity for them in §3.2, following Katz [24]. Rigid local systems of rank two are studied in details in §3.3. We also review the notion of rigid tuples in Inverse Galois Theory in §3.4 for comparison.

3.1. Local systems. Let k be any field.

3.1.1. *Local systems in étale topology.* Let U be a scheme of finite type over k . We recall some definitions from [20, §1.2, §1.4.2, §1.4.3]. A \mathbb{Z}_ℓ -local system on U is a projective system $(\mathcal{F}_n)_{n \geq 1}$ of locally constant locally free $\mathbb{Z}/\ell^n\mathbb{Z}$ -sheaves \mathcal{F}_n of finite rank on X under the étale topology such that the natural map $\mathcal{F}_n \otimes \mathbb{Z}/\ell^{n-1}\mathbb{Z} \rightarrow \mathcal{F}_{n-1}$ is an isomorphism for all n . Denote the category of \mathbb{Z}_ℓ -local systems on U by $\text{Loc}(U, \mathbb{Z}_\ell)$, which is a \mathbb{Z}_ℓ -linear abelian category. The category of \mathbb{Q}_ℓ -local systems is by definition the abelian category $\text{Loc}(U, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$ obtained by inverting ℓ in the Hom groups in $\text{Loc}(U, \mathbb{Z}_\ell)$. Similar definition gives $\text{Loc}(U, \mathcal{O}_L)$ and $\text{Loc}(U, L)$ for any finite extension L of \mathbb{Q}_ℓ . Finally define $\text{Loc}(U) := \text{Loc}(U, \overline{\mathbb{Q}_\ell})$ to be the colimit $\varinjlim_L \text{Loc}(U, L)$ over all finite extensions L of \mathbb{Q}_ℓ .

In the sequel we assume U is normal and connected. Fix a geometric point $u \in U$. Let \mathcal{F} be a $\overline{\mathbb{Q}_\ell}$ -local system on U of rank n . The stalk \mathcal{F}_u is a $\overline{\mathbb{Q}_\ell}$ -vector space of dimension n which carries an action of the étale fundamental group $\pi_1(U, u)$ defined in [16, V, §7]. Thus \mathcal{F} determines a *continuous* homomorphism

$$\rho_{\mathcal{F}} : \pi_1(U, u) \rightarrow \text{Aut}_{\overline{\mathbb{Q}_\ell}}(\mathcal{F}_u) \cong \text{GL}_n(\overline{\mathbb{Q}_\ell}),$$

where $\overline{\mathbb{Q}_\ell}$ is topologized as the colimit of finite extensions E of \mathbb{Q}_ℓ , each with the ℓ -adic topology. The last isomorphism above depends upon the choice of a basis of \mathcal{F}_u . This way we get a functor

$$(3.1) \quad \omega_u : \text{Loc}(U) \rightarrow \text{Rep}_{\text{cont}}(\pi_1(U, u))$$

where $\text{Rep}_{\text{cont}}(\pi_1(U, u))$ is the category of continuous representation of $\pi_1(U, u)$ on finite-dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces. Both sides of (3.1) carry tensor structures and ω_u is in fact an equivalence of tensor categories ([20, Proposition 1.2.5]).

Let $\pi_1^{\text{geom}}(U, u) := \pi_1(U \otimes_k \overline{k}, u)$, and call it the *geometric fundamental group* of U (with respect to the base point u). This is a normal subgroup of $\pi_1(U, u)$ which fits into an exact sequence ([16, V, Proposition 6.13])

$$1 \rightarrow \pi_1^{\text{geom}}(U, u) \rightarrow \pi_1(U, u) \rightarrow \text{Gal}(k^s/k) \rightarrow 1.$$

3.1.2. *H-local systems.* Let H be an affine algebraic group over $\overline{\mathbb{Q}}_\ell$. We may define the notion of H -local system on a connected normal scheme U over k . There are two ways to do this.

First definition. Fix a geometric point $u \in U$. An H -local system on U is a continuous homomorphism

$$(3.2) \quad \rho : \pi_1(U, u) \rightarrow H(\overline{\mathbb{Q}}_\ell).$$

Such homomorphisms form a category $\text{Loc}_H(U)$, in which isomorphisms are given by $H(\overline{\mathbb{Q}}_\ell)$ -conjugacy of representations. For example, $\text{Loc}_{\text{GL}_n}(U)$ is equivalent to the full subcategory of $\text{Loc}(U)$ consisting of local systems of rank n .

Second (and more canonical) definition. Let $\text{Rep}(H)$ be the tensor category of algebraic representations of H on finite-dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces. We define an H -local system to be a tensor functor $\mathcal{F} : \text{Rep}(H) \rightarrow \text{Loc}(U)$ (notation: $V \mapsto \mathcal{F}_V$). We form the category of H -local system on U by taking the category of such tensor functors

$$\text{Loc}_H(U) := \text{Fun}^\otimes(\text{Rep}(H), \text{Loc}(U)).$$

The two notions of H -local systems are equivalent. Given a representation ρ as in (3.2) and for $V \in \text{Rep}(H)$, the composition

$$\rho_V : \pi_1(U, u) \xrightarrow{\rho} H(\overline{\mathbb{Q}}_\ell) \rightarrow \text{GL}(V)$$

is an object in $\text{Loc}(U)$ of rank equal to $\dim V$. The assignment $V \mapsto \rho_V$ gives a tensor functor $\mathcal{F} : \text{Rep}(H) \rightarrow \text{Loc}(U)$. Conversely, given a tensor functor $\mathcal{F} : \text{Rep}(H) \rightarrow \text{Loc}(U)$, using the equivalence (3.1), it can be viewed as a tensor functor $\text{Rep}(H) \rightarrow \text{Rep}_{\text{cont}}(\pi_1(U, u))$. The Tannakian formalism [7] then implies that such a tensor functor comes from a group homomorphism ρ as in (3.2), well-defined up to conjugacy.

3.1.3. DEFINITION. Let $\mathcal{F} \in \text{Loc}_H(U)$ be given by a continuous homomorphism $\rho : \pi_1(U, u) \rightarrow H(\overline{\mathbb{Q}}_\ell)$. The *global geometric monodromy group* $H_{\mathcal{F}}^{\text{geom}}$ of \mathcal{F} is the Zariski closure of $\rho(\pi_1^{\text{geom}}(U, u))$ in H .

Recall the following fundamental result of Deligne on the nature of the global geometric monodromy groups of local systems when k is a finite field.

3.1.4. THEOREM (Deligne). *Let k be a finite field and \mathcal{F} be an H -local system on U (defined over k). Suppose that for some faithful representation*

$H \hookrightarrow \mathrm{GL}(V)$, the associated local system \mathcal{F}_V is ι -pure with respect to an embedding $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. Then the neutral component of $H_{\mathcal{F}}^{\mathrm{geom}}$ is semisimple.

In fact, purity of \mathcal{F}_V implies that \mathcal{F}_V is semisimple over $U_{\bar{k}}$, and we then apply [6, Corollaire 1.3.9] to get the above result. For the definition of ι -purity, see [6, §1.2.6].

3.1.5. *Local monodromy.* Let X be a projective, smooth and geometrically connected curve over a perfect field k . We shall use the notation from §2.1. Fix a finite set of closed points $S \subset |X|$ and let $U = X - S$. An algebraic closure \overline{F}_x of F_x gives a geometric generic point $\eta_x \in X$. The morphism $\mathrm{Spec} \overline{F}_x \rightarrow U$ then induces an injective homomorphism of fundamental groups

$$(3.3) \quad \Gamma_x = \mathrm{Gal}(\overline{F}_x/F_x) \hookrightarrow \pi_1(U, \eta_x) \cong \pi_1(U, u),$$

where the second map is well-defined up to conjugacy. Since F_x is a complete discrete valuation field with perfect residue field k_x , we have an exact sequence

$$1 \rightarrow I_x \rightarrow \mathrm{Gal}(\overline{F}_x/F_x) \rightarrow \mathrm{Gal}(\overline{k}/k_x) \rightarrow 1.$$

Under (3.3), I_x is contained in the normal subgroup $\pi_1^{\mathrm{geom}}(U, u) \triangleleft \pi_1(U, u)$.

When $\mathrm{char}(k) = p > 0$, there is a normal subgroup $I_x^w \triangleleft I_x$ called the *wild inertia group* such that the quotient $I_x^t := I_x/I_x^w$ is the maximal prime-to- p quotient of I_x , called the *tame inertia group*. We have a canonical isomorphism of $\mathrm{Gal}(\overline{k}/k_x)$ -modules

$$I_x^t \xrightarrow{\sim} \varprojlim_{(n,p)=1} \mu_n(\overline{k}).$$

Let $\rho : \pi_1(U, u) \rightarrow H(\overline{\mathbb{Q}}_\ell)$ be an H -local system. The *local monodromy of ρ at $x \in S$* is the homomorphism $\rho_x := \rho|_{I_x} : I_x \rightarrow H(\overline{\mathbb{Q}}_\ell)$. The local system ρ is said to be *tame at $x \in S$* if ρ_x factors through the tame inertia group I_x^t .

3.1.6. *Conductor.* For a continuous linear representation of the inertia group on a $\overline{\mathbb{Q}}_\ell$ -vector space V

$$\sigma : I_x \rightarrow \mathrm{GL}(V)$$

one can define its Swan conductor and Artin conductor (see [22, Chapter 1]). We recall their definitions in the case $\sigma(I_x)$ is finite (see [37, §VI.2], [13, §2]). Let $D = \sigma(I_x) = \mathrm{Gal}(L/F_x^{\mathrm{ur}})$ for some finite Galois representation L/F_x^{ur} inside F_x^s . There is a filtration of D

$$D = D_0 \triangleright D_1 \triangleright D_2 \triangleright \cdots .$$

For $i \geq 0$, D_i is the subgroup of D which acts trivially on $\mathcal{O}_L/\mathfrak{m}_L^i$ (\mathfrak{m}_L is the maximal ideal of \mathcal{O}_L). This is called the *lower numbering filtration* on D . In particular, $D_1 = \sigma^{ss}(I_x^w) = \sigma(I_x^w)$. The *Swan conductor* of the representation σ is

$$\mathrm{Sw}(\sigma) := \sum_{i \geq 1} \frac{\dim(V/V^{D_i})}{[D : D_i]}.$$

This turns out to be an integer. The Swan conductor of σ is zero if and only if σ is tame. The *Artin conductor* of the representation σ is

$$a(\sigma) := \dim(V/V^{I_x}) + \text{Sw}(\sigma).$$

If $\sigma = \rho_x : I_x \rightarrow \text{GL}(V)$ is the local monodromy of a local system $\mathcal{F} \in \text{Loc}(U)$, we also denote $\text{Sw}(\sigma)$ by $\text{Sw}_x(\mathcal{F})$ and $a(\sigma)$ by $a_x(\mathcal{F})$.

3.1.7. *The twisted situation.* Let $\theta_H : \pi_1(U, u) \rightarrow \text{Aut}(H)$ be a homomorphism that factors through a finite quotient $\Gamma = \text{Gal}(U'/U)$ for some finite étale Galois cover $U' \rightarrow U$. We define a θ_H -twisted H -local system on U to be a pair (\mathcal{F}, δ) where

- \mathcal{F} is an H -local system on U' , viewed as a tensor functor $\text{Rep}(H) \rightarrow \text{Loc}(U')$;
- δ is a collection of isomorphisms $\delta_{\gamma, V} : \mathcal{F}_{V^\gamma} \cong \gamma^* \mathcal{F}_V$, one for each $\gamma \in \Gamma, V \in \text{Rep}(H)$, where V^γ is the representation of H given by the composition $H \xrightarrow{\gamma} H \rightarrow \text{GL}(V)$. The isomorphisms $\{\delta_{\gamma, V}\}$ are required to satisfy the usual cocycle relations with respect to the multiplication in Γ , and to be compatible with the tensor structure of \mathcal{F} .

This definition a priori depends on the choice of Γ through which θ_H factors. However, enlarging Γ gives an equivalent notion of θ_H -twisted H -local systems, therefore the definition is independent of the choice of the finite quotient Γ of $\pi_1(U, u)$. We denote the category of θ_H -twisted H -local systems on U by $\text{Loc}_{H, \theta_H}(U)$.

It is easy to check that there is an equivalence of groupoids

$$(3.4) \quad \text{Loc}_{H, \theta_H}(U) \cong Z_{\text{cont}}^1(\pi_1(U, u), H(\overline{\mathbb{Q}}_\ell)) / H(\overline{\mathbb{Q}}_\ell)$$

where $Z_{\text{cont}}^1(\pi_1(U, u), H(\overline{\mathbb{Q}}_\ell))$ is the set of continuous 1-cocycles on $\pi_1(U, u)$ with values in $H(\overline{\mathbb{Q}}_\ell)$ which carries an action of $\pi_1(U, u)$ via θ_H . Note that $Z_{\text{cont}}^1(\pi_1(U, u), H(\overline{\mathbb{Q}}_\ell))$ is in bijection with liftings of $\pi_1(U, u) \rightarrow \Gamma$ to $\pi_1(U, u) \rightarrow H(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma$, and $H(\overline{\mathbb{Q}}_\ell)$ acts on such liftings by conjugation. In particular, isomorphism classes in $\text{Loc}_{H, \theta_H}(U)$ are in bijection with $H_{\text{cont}}^1(\pi_1(U, u), H(\overline{\mathbb{Q}}_\ell))$.

In the twisted situation, the local geometric monodromy of $\mathcal{F} \in \text{Loc}_{H, \theta_H}(U)$ at a point $x \in S$ is a cohomology class $H_{\text{cont}}^1(I_x, H(\overline{\mathbb{Q}}_\ell))$, where I_x acts on $H(\overline{\mathbb{Q}}_\ell)$ through $I_x \hookrightarrow \pi_1(U, u) \xrightarrow{\theta_H} \text{Aut}(H)$. Moreover, the above construction is functorial with respect to $\pi_1(U, u)$ -equivariant homomorphisms $H \rightarrow H'$.

3.2. Rigidity for local systems. Rigidity of a local system is a geometric property, therefore we assume the base field k to be algebraically closed in this subsection. Let X be a complete smooth connected algebraic curve over k . Let X be a smooth, projective and connected curve over k . Fix an open subset $U \subset X$ with finite complement S .

Let H be an algebraic group over $\overline{\mathbb{Q}}_\ell$ with an action $\theta_H : \pi_1(U, u) \rightarrow \text{Aut}(H)$ that factors through a finite quotient $\Gamma = \text{Gal}(U'/U)$.

3.2.1. DEFINITION (extending Katz [24, §1.0.3]). Let $\mathcal{F} \in \text{Loc}_{H, \theta_H}(U)$. Then \mathcal{F} is *physically rigid* if, for any other object $\mathcal{F}' \in \text{Loc}_{H, \theta_H}(U)$ such that for each $x \in S$, the local geometric monodromy of \mathcal{F}' is isomorphic to that of \mathcal{F} (i.e., the two local systems give the same class in $H_{\text{cont}}^1(I_x, H(\overline{\mathbb{Q}}_\ell))$), we have $\mathcal{F} \cong \mathcal{F}'$ as objects in $\text{Loc}_{H, \theta_H}(U)$.

Although the definition uses U as an input, the notion of physical rigidity is in fact independent of the open subset U : for any nonempty open subset $V \subset U$, \mathcal{F} is rigid over U if and only if $\mathcal{F}|_V$ is rigid over V . Therefore, physical rigidity of an H -local system is a property of the Galois representation $\text{Gal}(F^s/F) \rightarrow H(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma$ obtained by restricting ρ to a geometric generic point of the X .

3.2.2. *Motivation for cohomological rigidity.* Another notion of rigidity can be thought of as an infinitesimal version of physical rigidity. We motivate the definition by considering local systems in the complex analytic topology. Suppose the algebraic curve X is defined over \mathbb{C} . Let $U = X - S \subset X$ be a Zariski open dense subset and let U^{an} denote the Riemann surface structure on $U(\mathbb{C})$. Let H be a connected reductive group over $\overline{\mathbb{Q}}_\ell$. An H -local system \mathcal{F} in this case is simply a group homomorphism

$$\rho_{\mathcal{F}} : \Gamma^{\text{top}} := \pi_1^{\text{top}}(U^{\text{an}}, u) \rightarrow H.$$

without continuity conditions. For $x \in S$, the local monodromy of \mathcal{F} around x is the image of a counterclockwise loop around x (with starts and ends at u) in H .

In this case, the objects of the category $\text{Loc}_H(U^{\text{an}})$ are the $\overline{\mathbb{Q}}_\ell$ -points of an algebraic stack $\mathcal{L}oc_H(U^{\text{an}})$ defined over $\overline{\mathbb{Q}}_\ell$. In fact, for simplicity we consider the case where $S \neq \emptyset$, then Γ^{top} is simply a free group F_r with r generators. Then $\mathcal{L}oc_H(U^{\text{an}})$ is isomorphic to the quotient stack $[H^r/H]$ where H acts diagonally on H^r by conjugation. From this description we readily see that $\mathcal{L}oc_H(U^{\text{an}})$ is a smooth algebraic stack.

It is convenient to work with a Riemann surface with boundary that is homotopy equivalent to U^{an} . There is a canonical such Riemann surface: the real blow-up \tilde{X}^{an} of X^{an} along the points in S . The preimage of $x \in S$ in \tilde{X}^{an} is a circle ∂_x which is identified with the circle of unit tangent vectors in $T_x X$. The boundary $\partial \tilde{X}^{\text{an}}$ of \tilde{X}^{an} is the disjoint union of ∂_x for $x \in S$. The moduli stack $\mathcal{L}oc_H(\tilde{X}^{\text{an}})$ which is canonically isomorphic to $\mathcal{L}oc_H(U^{\text{an}})$. We also have the moduli stack $\mathcal{L}oc_H(\partial \tilde{X}^{\text{an}})$: it is simply isomorphic to $[H/H]^S$, one factor of the adjoint quotient $[H/H]$ for each boundary component ∂_x . Let H^{ab} be the quotient torus of H . We can similarly define $\mathcal{L}oc_{H^{\text{ab}}}(\tilde{X}^{\text{an}})$ and $\mathcal{L}oc_{H^{\text{ab}}}(\partial \tilde{X}^{\text{an}})$. We have a commutative diagram relating these moduli

stacks

$$(3.5) \quad \begin{array}{ccc} \mathcal{L}oc(\tilde{X}^{\text{an}}) & \xrightarrow{\text{res}} & \mathcal{L}oc(\partial\tilde{X}^{\text{an}}) \\ \downarrow & & \downarrow \\ \mathcal{L}oc_{H^{\text{ab}}}(\tilde{X}^{\text{an}}) & \longrightarrow & \mathcal{L}oc_{H^{\text{ab}}}(\partial\tilde{X}^{\text{an}}) \end{array}$$

where the horizontal maps are given by restricting local systems to the boundary, and vertical maps are induced from the projection $H \rightarrow H^{\text{ab}}$.

Now we fix a monodromy datum (\mathcal{A}, C_S) :

- An object $\mathcal{A} \in \mathcal{L}oc_{H^{\text{ab}}}(\tilde{X}^{\text{an}})$.
- $C_S = \{C_x\}_{x \in S}$, where C_x is a conjugacy class in H for each $x \in S$, such that the local monodromy of $\mathcal{A}|_{\partial_x}$ is the image C_x^{ab} of C_x in H^{ab} .

Define a substack $\mathcal{C}_S = \prod_{x \in S} [C_x/H]$ of $\mathcal{L}oc_H(\tilde{X}^{\text{an}})$, and similarly $\mathcal{C}_S^{\text{ab}} = \prod_x [C_x^{\text{ab}}/H^{\text{ab}}] \subset \mathcal{L}oc_{H^{\text{ab}}}(\partial\tilde{X}^{\text{an}})$. We define

$$\begin{aligned} \mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S) &:= \mathcal{C}_S \times_{\mathcal{L}oc_H(\partial\tilde{X}^{\text{an}})} \mathcal{L}oc_H(\tilde{X}^{\text{an}}) \\ \mathcal{L}oc_{H^{\text{ab}}}(\tilde{X}^{\text{an}}, C_S^{\text{ab}}) &:= \mathcal{C}_S^{\text{ab}} \times_{\mathcal{L}oc_{H^{\text{ab}}}(\partial\tilde{X}^{\text{an}})} \mathcal{L}oc_{H^{\text{ab}}}(\tilde{X}^{\text{an}}). \end{aligned}$$

The object \mathcal{A} lies in $\mathcal{L}oc_{H^{\text{ab}}}(\tilde{X}^{\text{an}}, C_S^{\text{ab}})$. We form the fiber

$$\mathcal{L}oc_H(\tilde{X}^{\text{an}}, \mathcal{A}, C_S) := \mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S) \times_{\mathcal{L}oc_{H^{\text{ab}}}(\tilde{X}^{\text{an}}, C_S^{\text{ab}})} \{\mathcal{A}\}.$$

Since \tilde{X}^{an} is homotopy equivalent to U^{an} , the stack $\mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S)$ classifies H -local systems \mathcal{F} on U^{an} with monodromy around x lying in the conjugacy class C_x and an isomorphism $\mathcal{F}^{\text{ab}} \cong \mathcal{A}$.

We are interested in the infinitesimal deformations of a point $\mathcal{F} \in \mathcal{L}oc_H(\tilde{X}^{\text{an}}, \mathcal{A}, C_S)$. The tangent complex $T_{\mathcal{F}}\mathcal{L}oc_H(\tilde{X}^{\text{an}})$ of $\mathcal{L}oc_H(\tilde{X}^{\text{an}})$ at \mathcal{F} is given by

$$T_{\mathcal{F}}\mathcal{L}oc_H(\tilde{X}^{\text{an}}) \cong \mathbb{H}^*(\tilde{X}^{\text{an}}, \text{Ad}(\mathcal{F}))[1]$$

where the local system $\text{Ad}(\mathcal{F})$ is the tensor functor \mathcal{F} evaluated at the adjoint representation of H on its Lie algebra \mathfrak{h} . Let $\mathcal{F}_{\partial} := \mathcal{F}|_{\partial\tilde{X}^{\text{an}}} \in \mathcal{L}oc_H(\partial\tilde{X}^{\text{an}})$. The same calculation shows

$$T_{\mathcal{F}_{\partial}}\mathcal{L}oc_H(\partial\tilde{X}^{\text{an}}) \cong \mathbb{H}^*(\partial\tilde{X}^{\text{an}}, \text{Ad}(\mathcal{F}_{\partial}))[1].$$

The tangent complex of \mathcal{C}_S is concentrated in degree -1 and is equal to $\mathbb{H}^0(\partial\tilde{X}^{\text{an}}, \text{Ad}(\mathcal{F}_{\partial}))[1]$. Therefore we have a distinguished triangle

$$(3.6) \quad T_{\mathcal{F}_{\partial}}\mathcal{C}_S \rightarrow T_{\mathcal{F}_{\partial}}\mathcal{L}oc_H(\partial\tilde{X}^{\text{an}}) \rightarrow \mathbb{H}^1(\partial\tilde{X}^{\text{an}}, \text{Ad}(\mathcal{F}_{\partial})) \rightarrow$$

Since $\mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S)$ is defined as the preimage of \mathcal{C}_S under the morphism res in (3.5), the map $T_{\mathcal{F}}\mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S) \rightarrow T_{\mathcal{F}}\mathcal{L}oc_H(\tilde{X}^{\text{an}})$ has isomorphic

cone with the map $T_{\mathcal{F}\partial} \mathcal{C}_S \rightarrow T_{\mathcal{F}\partial} \mathcal{L}oc_H(\partial \tilde{X}^{\text{an}})$. By (3.6), we get a distinguished triangle

$$(3.7) \quad T_{\mathcal{F}} \mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S) \rightarrow H^*(\tilde{X}^{\text{an}}, \text{Ad}(\mathcal{F}))[1] \rightarrow H^1(\partial \tilde{X}^{\text{an}}, \text{Ad}(\mathcal{F}\partial)) \rightarrow$$

We may express $T_{\mathcal{F}} \mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S)$ in more complex-geometric terms, without appealing to the boundary $\partial \tilde{X}^{\text{an}}$. Let $j : U \hookrightarrow X$ and $i_x : \{x\} \hookrightarrow X$ ($x \in S$) be the inclusions. We define $j_{!*} \text{Ad}(\mathcal{F})$ to be the *non-derived* direct image of $\text{Ad}(\mathcal{F})$ along j . Concretely, the stalk of $j_{!*} \text{Ad}(\mathcal{F})$ at $x \in S$ is $(\mathfrak{h})^{I_x}$ where I_x is the inertia group at x . We have a distinguished triangle of complexes of sheaves on X^{an}

$$j_{!*} \text{Ad}(\mathcal{F}) \rightarrow j_* \text{Ad}(\mathcal{F}) \rightarrow \bigoplus_{x \in S} i_{x,*} H^1(\partial_x, \text{Ad}(\mathcal{F}\partial))[-1] \rightarrow$$

which then gives a distinguished triangle after taking cohomology

$$H^*(X^{\text{an}}, j_{!*} \text{Ad}(\mathcal{F}))[1] \rightarrow H^*(\tilde{X}^{\text{an}}, \text{Ad}(\mathcal{F}))[1] \rightarrow H^1(\partial \tilde{X}^{\text{an}}, \text{Ad}(\mathcal{F}\partial)) \rightarrow$$

Comparing with (3.7), we get a quasi-isomorphism

$$(3.8) \quad T_{\mathcal{F}} \mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S) \cong H^*(X^{\text{an}}, j_{!*} \text{Ad}(\mathcal{F}))[1].$$

We use the notation $j_{!*}$ because in this case $j_{!*} \text{Ad}(\mathcal{F})[1]$ is also the middle extension of the perverse sheaf $\text{Ad}(\mathcal{F})[1]$ from U to X .

Similarly,

$$(3.9) \quad T_{\mathcal{A}} \mathcal{L}oc_{H^{\text{ab}}}(\tilde{X}^{\text{an}}, C_S^{\text{ab}}) \cong H^*(X^{\text{an}}, j_{!*} \text{Ad}(\mathcal{A}))[1].$$

Since H^{ab} is abelian, $\text{Ad}(\mathcal{A})$ is the constant sheaf $\mathfrak{h}^{\text{ab}} = \text{Lie } H^{\text{ab}}$, and $j_{!*} \text{Ad}(\mathcal{A})$ is the constant sheaf \mathfrak{h}^{ab} on X^{an} . Since $\mathcal{L}oc_H(\tilde{X}^{\text{an}}, \mathcal{A}, C_S)$ is defined as the fiber of $\mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S)$ over \mathcal{A} , we have a distinguished triangle

$$T_{\mathcal{F}} \mathcal{L}oc_H(\tilde{X}^{\text{an}}, \mathcal{A}, C_S) \rightarrow T_{\mathcal{F}} \mathcal{L}oc_H(\tilde{X}^{\text{an}}, C_S) \rightarrow T_{\mathcal{A}} \mathcal{L}oc_{H^{\text{ab}}}(\tilde{X}^{\text{an}}, C_S^{\text{ab}}) \rightarrow$$

Comparing (3.8) and (3.9) we conclude that

$$T_{\mathcal{F}} \mathcal{L}oc_H(\tilde{X}^{\text{an}}, \mathcal{A}, C_S) \cong H^*(X^{\text{an}}, j_{!*} \text{Ad}^{\text{der}}(\mathcal{F}))[1]$$

where $\text{Ad}^{\text{der}}(\mathcal{F}) = \ker(\text{Ad}(\mathcal{F}) \rightarrow \text{Ad}(\mathcal{A}))$ is the local system obtained by evaluating the functor \mathcal{F} on the adjoint representation of H on $\mathfrak{h}^{\text{der}} = \ker(\mathfrak{h} \rightarrow \mathfrak{h}^{\text{ab}})$.

As usual, classes in $H^2(X^{\text{an}}, j_{!*} \text{Ad}^{\text{der}}(\mathcal{F}))$ are obstructions to infinitesimally deform \mathcal{F} with prescribed local monodromy around S and abelianization $\mathcal{F}^{\text{ab}} = \mathcal{A}$. When $H^2(X^{\text{an}}, j_{!*} \text{Ad}^{\text{der}}(\mathcal{F})) = 0$, the moduli stack $\mathcal{L}oc_H(\tilde{X}^{\text{an}}, \mathcal{A}, C_S)$ is smooth at \mathcal{F} and $H^1(X^{\text{an}}, j_{!*} \text{Ad}^{\text{der}}(\mathcal{F}))$ is its Zariski tangent space at \mathcal{F} .

Since $\mathfrak{h}^{\text{der}}$ carries an $\text{Ad}(H)$ -invariant symmetric bilinear form, $j_{!*} \text{Ad}^{\text{der}}(\mathcal{F})$ is Verdier self-dual and $H^1(X^{\text{an}}, j_{!*} \text{Ad}^{\text{der}}(\mathcal{F}))$ is a symplectic space. Moreover, the vanishing of $H^2(X^{\text{an}}, j_{!*} \text{Ad}^{\text{der}}(\mathcal{F}))$ is equivalent to the vanishing of $H^0(X^{\text{an}}, j_{!*} \text{Ad}^{\text{der}}(\mathcal{F}))$, i.e., if \mathcal{F} does not have infinitesimal automorphisms then there is no obstruction to its infinitesimal deformation.

3.2.3. *Cohomological rigidity.* Now back to ℓ -adic local systems. We keep the notations from the beginning of this subsection. Let H be a connected reductive group over $\overline{\mathbb{Q}}_\ell$ and $\theta_H : \pi_1(U, u) \rightarrow \text{Aut}(H)$ be a homomorphism that factors through a finite quotient Γ . Let $\mathcal{F} \in \text{Loc}_{H, \theta_H}(U)$, which correspond to a homomorphism $\rho : \pi_1(U, u) \rightarrow H(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma$. Let $\mathfrak{h}^{\text{der}} = \text{Lie } H^{\text{der}}$. Consider the composition

$$\text{Ad}^{\text{der}}(\rho) : \pi_1(U, u) \xrightarrow{\rho} H(\overline{\mathbb{Q}}_\ell) \rtimes \Gamma \xrightarrow{\xi} \text{GL}(\mathfrak{h}^{\text{der}})$$

where $\xi(h, \gamma) = \text{Ad}(h) \circ \theta_H(\gamma)$. This defines a local system on U of rank equal to $\dim H^{\text{der}}$, which we denote by $\text{Ad}^{\text{der}}(\mathcal{F})$. Let $j : U \hookrightarrow X$ be the open inclusion. The middle extension $j_{!*}\text{Ad}^{\text{der}}(\mathcal{F})$ still makes sense in the ℓ -adic setting. We arrive at the following definition.

3.2.4. DEFINITION (extending Katz [24, §5.0.1]). An object $\mathcal{F} \in \text{Loc}_{H, \theta_H}(U)$ is called *cohomologically rigid*, if

$$\text{Rig}(\mathcal{F}) := \text{H}^1(X, j_{!*}\text{Ad}^{\text{der}}(\mathcal{F})) = 0.$$

- 3.2.5. REMARK. (1) The space $\text{Rig}(\mathcal{F})$ only depends on the generic stalk of \mathcal{F} : if we shrink U we get the same middle extension $j_{!*}\text{Ad}^{\text{der}}(\mathcal{F})$ and hence the same vector space $\text{Rig}(\mathcal{F})$.
- (2) The Killing form on $\mathfrak{h}^{\text{der}}$ induces a symplectic form on $\text{Rig}(\mathcal{F})$ with values in $\text{H}^2(X, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell(-1)$. In particular, the dimension of $\text{Rig}(\mathcal{F})$ is always even.
- (3) According to the discussion preceding §3.2.3, assuming $\text{H}^2(X, j_{!*}\text{Ad}^{\text{der}}(\mathcal{F}))$ vanishes as well, we should think of the vanishing of $\text{Rig}(\mathcal{F})$ as saying that there are no nontrivial infinitesimal deformation of \mathcal{F} with prescribed local monodromy around S and prescribed abelianization \mathcal{F}^{ab} . However, defining the moduli stack of ℓ -adic local systems is much subtler, and this interpretation only serves as a heuristic.

The following lemma is easily verified, using the natural transformations $j! \rightarrow j_{!*} \rightarrow j_*$.

3.2.6. LEMMA. *For any local system \mathcal{L} on U , we have an exact sequence*

$$\begin{aligned} 0 \rightarrow \text{H}^0(U, \mathcal{L}) \rightarrow \bigoplus_{x \in S} (\mathcal{L}_x)^{I_x} \rightarrow \text{H}_c^1(U, \mathcal{L}) \rightarrow \text{H}^1(U, \mathcal{L}) \\ \rightarrow \bigoplus_{x \in S} (\mathcal{L}_x)_{I_x}(-1) \rightarrow \text{H}_c^2(U, \mathcal{L}) \rightarrow 0. \end{aligned}$$

Moreover, we have canonical isomorphisms

$$\begin{aligned} \text{H}^0(X, j_{!*}\mathcal{L}) &\cong \text{H}^0(U, \mathcal{L}) \cong (\mathcal{L}_u)^{\pi_1(U, u)} \\ \text{H}^1(X, j_{!*}\mathcal{L}) &\cong \text{Im}(\text{H}_c^1(U, \mathcal{L}) \rightarrow \text{H}^1(U, \mathcal{L})) \\ \text{H}^2(X, j_{!*}\mathcal{L}) &\cong \text{H}_c^2(U, \mathcal{L}) \cong (\mathcal{L}_u)_{\pi_1(U, u)}(-1). \end{aligned}$$

We now give a numerical criterion for cohomological rigidity.

3.2.7. PROPOSITION. *Let $\mathcal{F} \in \text{Loc}_{H,\theta_H}(U)$. Then \mathcal{F} is cohomologically rigid if and only if*

$$(3.10) \quad \frac{1}{2} \sum_{x \in S} a_x(\text{Ad}^{\text{der}}(\mathcal{F})) = (1 - g_X) \dim \mathfrak{h}^{\text{ad}} - \dim H^0(U, \text{Ad}^{\text{der}}(\mathcal{F})).$$

Here $a_x(\text{Ad}^{\text{der}}(\mathcal{F}))$ is the Artin conductor of $\text{Ad}^{\text{der}}(\mathcal{F})$ at x (see §3.1.6) and g_X is the genus of X .

PROOF. Let $\mathcal{L} = \text{Ad}^{\text{der}}(\mathcal{F})$. Recall the Grothendieck-Ogg-Shafarevich formula

$$\chi_c(U, \mathcal{L}) := \sum_{i=0}^2 (-1)^i \dim H_c^i(U, \mathcal{L}) = \chi_c(U) \text{rank}(\mathcal{L}) - \sum_{x \in S} \text{Sw}_x(\mathcal{L}).$$

where $\chi_c(U) = -2g_X + 2 - \#S$. On the other hand, by the exact sequence in Lemma 3.2.6 we have

$$\dim H_c^1(X, j_{!*}\mathcal{L}) = \dim H_c^1(U, \mathcal{L}) - \sum_{x \in S} \dim(\mathcal{L}_x)^{I_x} + \dim H^0(U, \mathcal{L}).$$

Combining these facts we get

$$\begin{aligned} \dim H_c^1(X, j_{!*}\mathcal{L}) &= \sum_{x \in S} (\dim \mathcal{L}_x / \mathcal{L}_x^{I_x} + \text{Sw}_x(\mathcal{L})) + (2g_X - 2) \text{rank}(\mathcal{L}) \\ &\quad + \dim H_c^2(U, \mathcal{L}) + \dim H^0(U, \mathcal{L}). \end{aligned}$$

Using the relation between Swan and Artin conductors, and using the self-duality of $\mathcal{L} = \text{Ad}^{\text{der}}(\mathcal{F})$, we get

$$\dim H_c^1(X, j_{!*}\mathcal{L}) = \sum_{x \in S} a_x(\mathcal{L}) + (2g_X - 2) \text{rank}(\mathcal{L}) + 2 \dim H^0(U, \mathcal{L}).$$

Therefore the vanishing of $H_c^1(X, j_{!*}\mathcal{L})$ is equivalent to the equality (3.10). \square

From (3.10) we see that cohomologically rigid H -local systems exist only when $g_X \leq 1$. When $g_X = 1$ and $\mathcal{F} \in \text{Loc}_{H,\theta_H}(U)$ is cohomologically rigid, $\text{Ad}(\mathcal{F})$ must be everywhere unramified. There are very few such examples (see [24, §1.4]). Most examples of rigid local systems are over open subsets of \mathbb{P}^1 .

When $H = \text{GL}_n$, the two notions of rigidity are related by the following theorem.

3.2.8. THEOREM (Katz [24, Theorem 5.0.2]). *For $X = \mathbb{P}^1$, cohomological rigidity GL_n -local systems (i.e., rank n local systems) are also physical rigidity.*

3.2.9. REMARK. Let H be semisimple. An alternative approach to define the notion of rigidity for a θ_H -twisted local system \mathcal{F} on U over a finite field k is by requiring the adjoint L -function of the Galois representation $\rho_{\mathcal{F}} : \Gamma_F \rightarrow H(\overline{\mathbb{Q}}_{\ell}) \rtimes \Gamma$ to be trivial (constant function 1). This is the approach

taken by Gross in [15]. When $H^0(U_{\bar{k}}, \text{Ad}^{\text{der}}(\mathcal{F})) = 0$, triviality of the adjoint L -function of $\rho_{\mathcal{F}}$ is equivalent to the cohomological rigidity of \mathcal{F} .

3.3. Rigid local systems of rank two. Let $X = \mathbb{P}_k^1$ where k is an algebraically closed field with $\text{char}(k) \neq 2$. We shall classify cohomologically rigid $H = \text{GL}_2$ -local systems (or rank two local systems) over $U = X - S$ for some finite $S \subset |X|$. Let $\mathcal{F} \in \text{Loc}_2(U)$ be *irreducible* and cohomologically rigid. We may assume that S is taken to be minimal in the sense that \mathcal{F} does ramify at all points $x \in S$. We have $\text{Ad}^{\text{der}}(\mathcal{F}) = \text{End}^0(\mathcal{F})$, the local system of traceless endomorphisms of \mathcal{F} . Irreducibility of \mathcal{F} implies that $H^0(U, \text{End}^0(\mathcal{F})) = 0$. The formula (3.10) then reads

$$\sum_{x \in S} a_x(\text{End}^0(\mathcal{F})) = 6.$$

3.3.1. LEMMA. *Let $\rho_x : I_x \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_\ell)$ be the local monodromy representation of \mathcal{F} at x .*

- (1) *If $\text{End}^0(\mathcal{F})$ is tame at x , then $a_x(\text{End}^0(\mathcal{F})) = 2$. If moreover $\det(\mathcal{F})$ is tame at x , then \mathcal{F} is tame at x .*
- (2) *If $\text{End}^0(\mathcal{F})$ is wildly ramified at x , then $a_x(\text{End}^0(\mathcal{F})) \geq 4$. If $a_x(\text{End}^0(\mathcal{F})) = 4$ and $\det(\mathcal{F})$ is tame at x , then one of the two cases happens.*
 - *Both breaks of \mathcal{F} at x are $1/2$, hence $\text{Sw}_x(\mathcal{F}) = 1$. In this case, the local monodromy $\rho_x : I_x \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_\ell)$ of \mathcal{F} is isomorphic to the induction $\text{Ind}_{J_x}^{I_x}(\chi)$ where $J_x \triangleleft I_x$ is the unique subgroup of index two and $\chi : J_x \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is a character.*
 - *The local monodromy $\rho_x : I_x \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_\ell)$ is the direct sum of two characters $\chi_1, \chi_2 : I_x \rightarrow \overline{\mathbb{Q}}_\ell^\times$ with $\text{Sw}(\chi_1) = \text{Sw}(\chi_2) = 1$.*

PROOF. Let $\rho_x^{\text{der}} : I_x \xrightarrow{\rho_x} \text{GL}_2(\overline{\mathbb{Q}}_\ell) \rightarrow \text{GL}(\mathfrak{sl}_2)$ be the local monodromy representation of $\text{End}^0(\mathcal{F})$ at x . The image of the wild inertia I_x^w under ρ_x is a p -group, hence lies in a maximal torus of GL_2 . Therefore $\rho_x|_{I_x^w} \cong \alpha \oplus \beta$ for characters $\alpha, \beta : I_x^w \rightarrow \overline{\mathbb{Q}}_\ell^\times$, and $\rho_x^{\text{der}}|_{I_x^w} \cong \alpha\beta^{-1} \oplus \alpha^{-1}\beta \oplus 1$.

(1) If ρ_x^{der} is tame, then $\alpha = \beta$. A topological generator $\zeta \in I_x^t$ maps to a non-identity element in PGL_2 , whose centralizer is necessarily one-dimensional. Hence $\dim(\mathfrak{sl}_2)^{I_x} = 1$ and of course $\text{Sw}_x(\text{End}^0(\mathcal{F})) = 0$. Therefore $a_x(\text{End}^0(\mathcal{F})) = 3 - \dim(\mathfrak{sl}_2)^{I_x} + \text{Sw}_x(\text{End}^0(\mathcal{F})) = 2$.

If we assume $\det(\rho_x)$ is tame, then $\alpha\beta = 1$. But we also have $\alpha = \beta$, which forces α and β both have order at most two, hence trivial. Therefore ρ_x is tame.

(2) If ρ_x^{der} is wildly ramified, then $\alpha \neq \beta$. Therefore there is a unique maximal torus $T \subset \text{GL}_2$ containing $\rho_x(I_x^w)$. We have $\dim(\mathfrak{sl}_2)^{I_x^w} = 1$. The image of $\rho_x(I_x)$ should normalize $\rho_x(I_x^w)$, hence it lies in the normalizer $N_{\text{GL}_2}(T)$, which gives a map $I_x^t \rightarrow N_{\text{GL}_2}(T)/T = \{\pm 1\}$.

Suppose I_x^t maps onto $N_{\mathrm{GL}_2}(T)/T$. Then $(\mathfrak{sl}_2)^{I_x} = 0$ and since $\mathrm{Sw}_x(\mathrm{End}^0(\mathcal{F})) \geq 1$, we have $a_x(\mathrm{End}^0(\mathcal{F})) = 3 - \dim(\mathfrak{sl}_2)^{I_x} + \mathrm{Sw}_x(\mathrm{End}^0(\mathcal{F})) \geq 4$. When equality holds, we have $\mathrm{Sw}_x(\mathrm{End}^0(\mathcal{F})) = 1$. The action of I_x^w on \mathfrak{sl}_2 is the sum of a trivial character and $\alpha\beta^{-1}$ and $\alpha^{-1}\beta$. Therefore $\alpha\beta^{-1}$ has break $1/2$. Suppose further that $\det(\mathcal{F})$ is tame, then $\alpha\beta = 1$. Hence $\alpha\beta^{-1} = \alpha^2$ has break $1/2$, which means α and β both have break $1/2$. The unique subgroup $J_x \triangleleft I_x$ of index two stabilizes the two eigenlines of $\rho_x|_{I_x^w}$, and therefore $\rho_x \cong \mathrm{Ind}_{J_x}^{I_x^w}(\chi)$ where χ is the character of J_x by which it acts on the α -eigenline of I_x^w .

Suppose I_x^t maps trivially to $N_{\mathrm{GL}_2}(T)/T$. This means $\rho_x(I_x) \subset T$ hence $\rho_x \cong \chi_1 \oplus \chi_2$ for characters $\chi_1, \chi_2 : I_x \rightarrow \overline{\mathbb{Q}}_\ell^\times$ extending α and β . In this case $\dim(\mathfrak{sl}_2)^{I_x} = 1$, and $\mathrm{Sw}_x(\mathrm{End}^0(\mathcal{F})) = \mathrm{Sw}(\chi_1\chi_2^{-1}) + \mathrm{Sw}(\chi_1^{-1}\chi_2) \geq 2$ (since $\alpha \neq \beta$). Therefore $a_x(\mathrm{End}^0(\mathcal{F})) = 3 - \dim(\mathfrak{sl}_2)^{I_x} + \mathrm{Sw}_x(\mathrm{End}^0(\mathcal{F})) \geq 4$. When equality holds, $\alpha\beta^{-1}$ has break 1. If moreover $\det(\mathcal{F})$ is tame, we have $\alpha = \beta^{-1}$, therefore the break of α^2 is 1, hence the breaks of both α and β are 1. \square

3.3.2. PROPOSITION. *Let $\mathcal{F} \in \mathrm{Loc}_2(U)$ be an irreducible cohomologically rigid rank two local systems on $U = \mathbb{P}^1 - S$ (and S is chosen minimally). Assume $0, \infty \in S$. Then up to an operation $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{L}$ for some rank one local system on U , and up to an automorphism of $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$, one of the following situations happens.*

- (1) $S = \{0, 1, \infty\}$, \mathcal{F} is tamely ramified at each $x \in S$ and its monodromy at 1 is a pseudo-reflection.
- (2) $S = \{0, \infty\}$, \mathcal{F} is tame at 0 and $\mathrm{Sw}_\infty(\mathcal{F}) = 1$ with two breaks equal to $1/2$.
- (3) $S = \{0, \infty\}$, \mathcal{F} is tame at 0 and $\mathrm{Sw}_\infty(\mathcal{F}) = 1$ with one break equal to 1 and another break equal to 0.

PROOF. The determinant $\det \mathcal{F}$ is a continuous character of $\chi : \pi_1(U) \rightarrow \overline{\mathbb{Q}}_\ell^\times$, and we can write it as $\chi_p \chi^p$ where χ_p is the p -power part of χ and χ^p is tame. Since $p > 2$, χ_p has a unique square root which gives a rank one local system \mathcal{L} over U . The local system $\mathcal{F}' = \mathcal{F} \otimes \mathcal{L}^{-1}$ is still irreducible and cohomologically rigid, and now its determinant is tame. Therefore we may assume that $\det(\mathcal{F})$ is tame.

By Lemma 3.3.1, when $\#S \geq 2$, there two cases.

(1) $\#S = 3$ and $a_x(\mathrm{End}^0(\mathcal{F})) = 2$ for all $x \in S$. Up to an automorphism of \mathbb{G}_m we may assume $S = \{0, 1, \infty\}$. Since $\det(\mathcal{F})$ is tame, Lemma 3.3.1 implies that \mathcal{F} is tame. We may find a rank one local system \mathcal{L} on $\mathbb{P}^1 - \{0, 1\}$ whose local monodromy at 1 is one of the eigenvalues of the local monodromy of \mathcal{F} at 1. Then $\mathcal{F} \otimes \mathcal{L}^{-1}$ is tame and its local monodromy at 1 is a pseudo-reflection.

(2) $\#S = 2$ (hence $S = \{0, \infty\}$). By Lemma 3.3.1, up to switching 0 and ∞ , we may assume that \mathcal{F} is $a_0(\mathrm{End}^0(\mathcal{F})) = 2$ and $a_\infty(\mathrm{End}^0(\mathcal{F})) = 4$. By

Lemma 3.3.1(1), \mathcal{F} is tame at 0. There are two cases for \mathcal{F} at ∞ according to Lemma 3.3.1(2):

- Both breaks of \mathcal{F} at ∞ are $1/2$ and hence $\text{Sw}_\infty(\mathcal{F}) = 1$.
- Both breaks of \mathcal{F} at ∞ are 1. The action of I_∞ on $\overline{\mathbb{Q}}_\ell^2$ is a direct sum of two characters $\chi_1, \chi_2 : I_\infty \rightarrow \overline{\mathbb{Q}}_\ell^\times$. The character χ_1 extends to a rank one local system \mathcal{L}_1 on \mathbb{G}_m with tame local monodromy at ∞ (Katz’s canonical extension, see [21]). Then $\mathcal{F} \otimes \mathcal{L}_1^{-1}$ is still tame at 0, with breaks 0 and 1 at ∞ .

□

3.3.3. REMARK. The three cases in Proposition 3.3.2 are the irreducible hypergeometric sheaves of rank two constructed by Katz [23, §8.2]. In fact, if \mathcal{F} is one of the local systems in Proposition 3.3.2, it is easy to see that $\chi_c(\mathbb{G}_m, j!_*\mathcal{F}[1]) = 1$ in all three cases (where $j : U \hookrightarrow \mathbb{G}_m$). By [23, Theorem 8.5.3], \mathcal{F} is a hypergeometric sheaf.

As we shall see in §4.5, the three local systems in Proposition 3.3.2 are the Langlands parameters (restricted to I_F) of the automorphic representations considered in §2.8.1, §2.8.3 and §2.8.5 respectively.

3.4. Rigidity in Inverse Galois Theory. It is instructive to compare the notion of rigidity for local systems with the notion of a rigid tuple in inverse Galois theory. We give a quick review following [40, Chapter 8].

3.4.1. DEFINITION. Let H be a finite group with trivial center. A tuple of conjugacy classes (C_1, C_2, \dots, C_n) in H is called (*strictly*) *rigid*, if

- The equation

$$(3.11) \quad g_1 g_2 \cdots g_n = 1$$

has a solution with $g_i \in C_i$, and the solution is unique up to simultaneous H -conjugacy;

- For any solutions (g_1, \dots, g_n) of (3.11), $\{g_i\}_{i=1, \dots, n}$ generate H .

The connection between rigid tuples and local systems is given by the following theorem. Let $S = \{P_1, \dots, P_n\} \subset \mathbb{P}^1(\mathbb{Q})$, and let $U = \mathbb{P}^1_{\mathbb{Q}} - S$.

3.4.2. THEOREM (Belyi, Fried, Matzat, Shih, and Thompson). *Let (C_1, \dots, C_n) be a rigid tuple in H . Then up to isomorphism there is a unique connected unramified Galois H -cover $\pi : Y \rightarrow U \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ such that a topological generator of the (tame) inertia group at P_i acts on Y as an element in C_i .*

Furthermore, if each C_i is rational (i.e., C_i takes rational values for all irreducible characters of H), then the H -cover $Y \rightarrow U \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is defined over \mathbb{Q} .

From the above theorem we see that the notion of a rigid tuple is an analog of physical rigidity for H -local systems when the algebraic group H is a finite group.

Rigid tuples combined with the Hilbert irreducibility theorem solves the inverse Galois problem for H .

3.4.3. COROLLARY. *Suppose there exists a rational rigid tuple in H , then H can be realized as $\text{Gal}(K/\mathbb{Q})$ for some Galois number field K/\mathbb{Q} .*

For a comprehensive survey on finite groups that are realized as Galois groups over \mathbb{Q} using rigid tuples, we refer the readers to the book [30] by Malle and Matzat.

4. Calculus of geometric Hecke operators

In this section, we will connect the notion of rigidity for automorphic data and the notion of rigidity for local systems together. Guided by the Langlands correspondence, we formulate a conjectures in §4.1. Then we review some basic techniques from the geometric Langlands program in §4.2-§4.3 such as the geometric Hecke operators. The main results are Theorem 4.4.2 and Proposition 4.5.2 which constructs the Hecke eigen local system under the rigidity assumption. Examples in GL_2 from §2.8 and §3.3 are finally connected to one another in §4.6.

4.1. Langlands correspondence for rigid objects. We are in the situation of §2, with k a finite field.

4.1.1. *The Langlands correspondence for groups over function fields.* Let $S \subset |X|$ be a finite set of places containing the ramification locus of θ_X , and $U = X - S$. Let U' be the preimage of U in X' . Let $W(U, u) \subset \pi_1(U, u)$ be the Weil group of U with respect to the base point u , i.e., it is the preimage of $\text{Frob}^{\mathbb{Z}}$ under the homomorphism $\pi_1(U, u) \rightarrow \text{Gal}(\bar{k}/k)$.

Recall that the Langlands correspondence predicts that to an automorphic representation π of $G(\mathbb{A}_F)$, unramified outside S , one should be able to attach a continuous cocycle $\rho_\pi : W(U, u) \rightarrow \widehat{G}(\mathbb{Q}_\ell)$ (with respect to the pinned action $\widehat{\theta}$ of $\Gamma = \text{Gal}(U'/U)$ on \widehat{G}) up to \widehat{G} -conjugacy. Base change to \bar{k} , one should be able attach a $\widehat{\theta}$ -twisted \widehat{G} -local system $\mathcal{F}_\pi \in \text{Loc}_{\widehat{G}, \widehat{\theta}}(U_{\bar{k}})$ to π . The assignment $\pi \mapsto \rho_\pi$ should be such that the Satake parameter of π_x for any $x \notin S$ matches the semisimple part of the conjugacy class of $\rho_\pi(\text{Frob}_x)$ in \widehat{G} . By the work of V.Lafforgue [26], when π is cuspidal, such a continuous cocycle ρ_π (hence \mathcal{F}_π) exists.

We expect that when π is rigid, the local system \mathcal{F}_π is also rigid. More precisely we propose the following conjecture.

4.1.2. CONJECTURE. *Let $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ be a weakly rigid geometric automorphic datum, with the corresponding restricted automorphic datum $(\omega^\natural, K_S, \chi_S)$. Then*

- (1) *For any $(\omega^\natural, K_S, \chi_S)$ -typical automorphic representation π , the local system $\mathcal{F}_\pi \in \text{Loc}_{\widehat{G}, \widehat{\theta}}(U_{\bar{k}})$ attached to it via the Langlands correspondence is cohomologically rigid.*
- (2) *If moreover $\text{Ad}^{\text{der}}(\mathcal{F}_\pi)$ (a local system over $U_{\bar{k}}$ of rank equal to $\dim \widehat{G}^{\text{der}}$) does not have nonzero global sections, and that the generic*

stabilizers of $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S})$ on $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S)$ are finite, then

$$(4.1) \quad d(\mathbf{K}_x^{\text{ad}}) = a_x(\text{Ad}^{\text{der}}(\mathcal{F}_\pi)).$$

For the definition of $d(\mathbf{K}_x^{\text{ad}})$, see §2.7.11; for the Artin conductor $a_x(\text{Ad}^{\text{der}}(\mathcal{F}_\pi))$ see §3.1.6.

4.1.3. REMARK. In the situation of Conjecture 4.1.2(2), the equality (4.1) at all $x \in S$ implies the cohomological rigidity of \mathcal{F}_π by the numerical criterion Proposition 3.2.7. In fact, Conjecture 4.1.2(2) was motivated by the similarity between the equations (2.16) and (3.10), the first one being a numerical condition for the weak rigidity of a geometric automorphic datum and the second a numerical criterion for the cohomological rigidity of a local system. For many known examples of weakly rigid geometric automorphic data, Conjecture 4.1.2(2) can be proved for those points $x \in S$ at which \mathcal{F}_π is tamely ramified, using the techniques in [44, §4]. See also [44, §9], where Conjecture 4.1.2 is fully verified for the local systems constructed in [42]. When \mathcal{F}_π is wildly ramified at x , [19, §5] also verifies (4.1) in the examples known as Kloosterman sheaves.

4.2. Geometric Hecke operators. From now on until §4.7.4, k is an algebraically closed field.

4.2.1. The geometric Satake equivalence. Let $L\mathbb{G}$ be the loop group of \mathbb{G} : this is an ind-scheme representing the functor $R \mapsto \mathbb{G}(R[[t]])$. Let $L^+\mathbb{G}$ the positive loops of \mathbb{G} : this is a scheme (not of finite type) representing the functor $R \mapsto \mathbb{G}(R[[t]])$. The fppf quotient $\text{Gr} = L\mathbb{G}/L^+\mathbb{G}$ is called the *affine Grassmannian* of \mathbb{G} . Then $L^+\mathbb{G}$ acts on Gr via left translation. The $L^+\mathbb{G}$ -orbits on Gr are indexed by dominant coweights $\lambda \in \mathbb{X}_*(\mathbb{T})^+$. The orbit containing the element $t^\lambda \in \mathbb{T}(k((t)))$ is denoted by Gr_λ and its closure is denoted by $\text{Gr}_{\leq \lambda}$. We have $\dim \text{Gr}_\lambda = \langle 2\rho, \lambda \rangle$, where 2ρ denotes the sum of positive roots in \mathbb{G} . The reduced scheme structure on $\text{Gr}_{\leq \lambda}$ is a projective variety called the affine Schubert variety attached to λ . We denote the intersection complex of $\text{Gr}_{\leq \lambda}$ by IC_λ : this is the middle extension of the shifted constant sheaf $\overline{\mathbb{Q}}_\ell[\langle 2\rho, \lambda \rangle]$ on Gr_λ .

The *Satake category* $\text{Sat} = \text{Perv}_{L^+\mathbb{G}}(\text{Gr}, \overline{\mathbb{Q}}_\ell)$ is the category of $L^+\mathbb{G}$ -equivariant perverse $\overline{\mathbb{Q}}_\ell$ -sheaves on Gr supported on $\text{Gr}_{\leq \lambda}$ for some λ . In [29], [12] and [33], it is shown that Sat carries a natural tensor structure such that the global cohomology functor $h = \text{H}^*(\text{Gr}, -) : \text{Sat}^{\text{geom}} \rightarrow \text{Vec}$ is a fiber functor. It is also shown that the Tannaka dual group of the tensor category Sat is isomorphic to the Langlands dual group \widehat{G} . The Tannakian formalism gives the *geometric Satake equivalence* of tensor categories

$$\text{Sat} \cong \text{Rep}(\widehat{G}).$$

4.2.2. Hecke correspondence. We are in the situation of §2.6. Recall an integral model \mathcal{G} of G was fixed as in §2.2.4. Let $S \subset |X|$ be a finite set containing the ramification locus S_θ of θ_X . Let $U = X - S$ and $U' = \theta_X^{-1}(U)$.

Then $U' \rightarrow U$ is a finite étale Galois cover with Galois group Γ . The base change $\mathcal{G} \times_U U'$ is the constant group scheme $\mathbb{G} \times U'$.

Fix a geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$. Let \mathbf{K}_S^+ be a smaller level as in §2.6.3. To alleviate notation, we write

$$\mathrm{Bun} := \mathrm{Bun}_{\mathbb{G}}(\mathbf{K}_S); \quad \mathrm{Bun}^+ := \mathrm{Bun}_{\mathbb{G}}(\mathbf{K}_S^+).$$

Consider the following diagram

$$(4.2) \quad \begin{array}{ccc} & \mathrm{Hk}_{U'} & \xrightarrow{\pi} U' \\ \overleftarrow{h} \swarrow & & \searrow \overrightarrow{h} \\ \mathrm{Bun}^+ & & \mathrm{Bun}^+ \end{array}$$

We explain the meaning of $\mathrm{Hk}_{U'}$ and the various morphisms. The stack $\mathrm{Hk}_{U'}$ classifies the data $(x', \mathcal{E}_1, \mathcal{E}_2, \tau)$ where $x' \in U'$ has image $x \in U$, $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}^+$ and $\tau : \mathcal{E}_1|_{X-\{x\}} \xrightarrow{\sim} \mathcal{E}_2|_{X-\{x\}}$ is an isomorphism of \mathcal{G} -torsors over $X - \{x\}$ preserving the \mathbf{K}_y^+ -level structures at $y \in S$. The morphisms \overleftarrow{h} , \overrightarrow{h} and π' send $(x', \mathcal{E}_1, \mathcal{E}_2, \tau)$ to $\mathcal{E}_1, \mathcal{E}_2$ and x' respectively.

Let R be a k -algebra and $x' \in U'(R)$ with image $x \in U(R)$. Let $\mathcal{O}_{x'}$ be the formal completion of $X' \otimes_k R$ along the graph of x' . After localizing R we may choose a continuous isomorphism $\alpha : \mathcal{O}_{x'} \cong R[[t]]$. Denote the preimage of x' under π' by $\mathrm{Hk}_{x'}$, which in fact only depends on x . For a point $(x', \mathcal{E}_1, \mathcal{E}_2, \tau) \in \mathrm{Hk}_{x'}$, if we fix trivializations of \mathcal{E}_1 and \mathcal{E}_2 over $\mathrm{Spec} \mathcal{O}_{x'} \cong \mathrm{Spec} R[[t]]$, the isomorphism τ restricted to $\mathrm{Spec} R((t))$ is an isomorphism between the trivial \mathbb{G} -torsors over $\mathrm{Spec} R((t))$, hence given by a point $g_\tau \in G(R((t)))$. Changing the trivializations of $\mathcal{E}_1|_{\mathrm{Spec} \mathcal{O}_x}$ and $\mathcal{E}_2|_{\mathrm{Spec} \mathcal{O}_x}$ will result in left and right multiplication of g_τ by an element in $\mathbb{G}(R[[t]])$ (here we use the fact that \mathcal{G} becomes a constant group scheme over U'). Thus we have an evaluation morphism

$$(4.3) \quad \mathrm{ev}_{x'} : \mathrm{Hk}_{x'} \rightarrow (L^+\mathbb{G} \backslash L\mathbb{G} / L^+\mathbb{G}) \otimes_k R.$$

Changing the isomorphism α will change the morphism $\mathrm{ev}_{x'}$ by composing with the action of an element in $\mathrm{Aut}(R[[t]])$ on the target. This construction gives a morphism

$$(4.4) \quad \mathrm{ev} : \mathrm{Hk}_{U'} \rightarrow \left[\frac{L^+\mathbb{G} \backslash L\mathbb{G} / L^+\mathbb{G}}{\mathrm{Aut}_k[[t]]} \right].$$

Here $\mathrm{Aut}_k[[t]]$ is the pro-algebraic group over k whose R -points is the group of continuous R -linear ring automorphisms of $R[[t]]$ (with the t -adic topology), and it acts on $L\mathbb{G}$ and $L^+\mathbb{G}$.

4.2.3. *The geometric Hecke operators.* For each object $V \in \mathrm{Rep}(\widehat{G})$, the corresponding object $\mathrm{IC}_V \in \mathrm{Sat}$ under the geometric Satake equivalence is automatically $\mathrm{Aut}_k[[t]]$ -equivariant, and it defines a complex on the quotient stack $\left[\frac{L^+\mathbb{G} \backslash L\mathbb{G} / L^+\mathbb{G}}{\mathrm{Aut}_k[[t]]} \right]$ which we still denote by IC_V . Let $\mathrm{IC}_V^{\mathrm{Hk}}$ be the pullback

of IC_V to $\mathrm{Hk}_{U'}$ along the morphism ev in (4.4). We define a functor

$$\begin{aligned} \mathfrak{T}_V : D_{(\mathbb{M}_S, \mathcal{K}_{S, \Omega})}^b(\mathrm{Bun}^+) &\rightarrow D_{(\mathbb{M}_S, \mathcal{K}_{S, \Omega})}^b(U' \times \mathrm{Bun}^+) \\ \mathcal{A} &\mapsto (\pi \times \overrightarrow{h})_!(\overleftarrow{h}^* \mathcal{A} \otimes \mathrm{IC}_V^{\mathrm{Hk}}). \end{aligned}$$

The formation $V \mapsto \mathfrak{T}_V$ is also $\Gamma := \mathrm{Gal}(U'/U)$ -equivariant in the following sense. The group Γ acts on $\mathrm{Hk}_{U'}$ because $\mathrm{Hk}_{U'}$ descends to U . It also acts on $\mathrm{Rep}(\widehat{G})$: $\gamma \in \Gamma$ sends a representation $\rho : \widehat{G} \rightarrow \mathrm{GL}(V)$ to the representation $\widehat{G} \xrightarrow{\widehat{\theta}(\gamma)} \widehat{G} \xrightarrow{\rho} \mathrm{GL}(V)$ which we denote by V^γ . Then there is a natural isomorphism of functors

$$\alpha_{\gamma, V} : \mathfrak{T}_{V^\gamma} \cong (\mathrm{id}_{\mathrm{Bun}^+} \times \gamma)^* \circ \mathfrak{T}_V.$$

For $\gamma_1, \gamma_2 \in \Gamma$, we have

$$\alpha_{\gamma_1 \circ \gamma_2, V} = (\mathrm{id}_{\mathrm{Bun}^+} \times \gamma_2)^* \alpha_{\gamma_1, V} \circ \alpha_{\gamma_2, V^{\gamma_1}}.$$

The isomorphisms $\alpha_{\gamma, V}$ are obtained from the fact that the morphism ev in (4.4) is Γ -equivariant, and the fact that the Satake equivalence $\mathrm{Sat} \cong \mathrm{Rep}(\widehat{G})$ is also Γ -equivariant (where Γ acts on both \mathbb{G} and \widehat{G} by pinned automorphisms).

To spell out how \mathfrak{T}_V is compatible with the tensor structure on $\mathrm{Rep}(\widehat{G})$, it is best to consider more general Hecke operators.

4.2.4. Iterated Hecke operators. We refer to [11, §2.4-2.7] for details. For each finite set I we may consider the stack Hk_{U^I} over $\mathrm{Bun}^+ \times \mathrm{Bun}^+ \times (U')^I$ classifying the data $(x, \mathcal{E}_1, \mathcal{E}_2, \tau)$ where $x : I \rightarrow U'$ with graph $x_I \subset U$, $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}^+$ and $\tau : \mathcal{E}_1|_{X-x_I} \cong \mathcal{E}_2|_{X-x_I}$ is an isomorphism of \mathcal{G} -torsors preserving level structures. Let $\overleftarrow{h}_I, \overrightarrow{h}_I : \mathrm{Hk}_{U^I} \rightarrow \mathrm{Bun}^+$ and $\pi_I : \mathrm{Hk}_{U^I} \rightarrow (U')^I$ be the projections. For any representation $V_I \in \mathrm{Rep}(\widehat{G}^I)$, we have an object $\mathrm{IC}_{V_I}^{\mathrm{Hk}}$ on Hk_{U^I} defined using evaluation maps similar to (4.4) indexed by I . For any scheme Y over k , we may introduce the functor

$$\begin{aligned} \mathfrak{T}_{V_I, Y} : D_{(\mathbb{M}_S, \mathcal{K}_{S, \Omega})}^b(Y \times \mathrm{Bun}^+) &\rightarrow D_{(\mathbb{M}_S, \mathcal{K}_{S, \Omega})}^b((U')^I \times Y \times \mathrm{Bun}^+) \\ \mathcal{A} &\mapsto (\pi_I \times p_Y \times \overrightarrow{h}_I)_!((p_Y \times \overleftarrow{h}_I)^* \mathcal{A} \otimes \mathrm{IC}_{V_I}^{\mathrm{Hk}}) \end{aligned}$$

where $p_Y : \mathrm{Hk}_{U^I} \times Y \rightarrow Y$ is the projection. These functors are functorial in V_I , and satisfy the following factorization properties.

- (1) For the trivial representation $\mathrm{triv} \in \mathrm{Rep}(\widehat{G}^I)$, there is a canonical isomorphism $\mathfrak{T}_{\mathrm{triv}, Y}(\mathcal{A}) \cong \mathcal{A} \boxtimes \overline{\mathbb{Q}}_\ell$.
- (2) For two finite sets I, J and $V_I \in \mathrm{Rep}(\widehat{G}^I), V_J \in \mathrm{Rep}(\widehat{G}^J)$, we have a natural isomorphism of functors

$$(4.5) \quad \mathfrak{T}_{V_I \boxtimes V_J, Y} = \mathfrak{T}_{V_I, U^J \times Y} \circ \mathfrak{T}_{V_J, Y}$$

satisfying obvious associativity and unit conditions.

- (3) For any surjection $\varphi : J \twoheadrightarrow I$ of finite sets, we have a corresponding diagonal map $\Delta(\varphi) : U^I \hookrightarrow U^J$. We also have the diagonal

map $\Delta(\varphi) : \widehat{G}^I \hookrightarrow \widehat{G}^J$ which allows us to view V_J as a representation $\Delta(\varphi)^*V_J \in \text{Rep}(\widehat{G}^J)$. Then there is a natural isomorphism of functors

$$(4.6) \quad (\text{id}_{\text{Bun}^+} \times \Delta(\varphi) \times \text{id}_Y)^* \mathfrak{I}_{V_J, Y} \cong \mathfrak{I}_{\Delta(\varphi)^*V_J, Y}$$

compatible with the composition of surjections.

The first property above is almost a tautology and the second one uses the definition of the convolution product in the Satake category. There are also compatibilities among (1)-(3), which we do not spell out.

Moreover, the functors $V_I \mapsto \mathfrak{I}_{V_I, Y}$ are Γ -equivariant as in the case of \mathfrak{I}_V .

When Y is a point, we write $\mathfrak{I}_{V_I, Y}$ as \mathfrak{I}_{V_I} . Taking Y to be a point, $J = \{1, 2\} \rightarrow \{1\}$, letting $V_J = V_1 \boxtimes V_2 \in \text{Rep}(\widehat{G}^2)$, and combining the isomorphisms (4.5) and (4.6), we get a canonical isomorphism

$$(4.7) \quad (\mathfrak{I}_{V_1, U'} \circ \mathfrak{I}_{V_2})|_{\text{Bun}^+ \times \Delta(U')} \cong \mathfrak{I}_{V_1 \otimes V_2} : \\ D_{(\mathbf{M}_S, \mathcal{K}_{S, \Omega})}^b(\text{Bun}^+) \rightarrow D_{(\mathbf{M}_S, \mathcal{K}_{S, \Omega})}^b(\text{Bun}^+ \times U')$$

which is compatible with the associativity of the tensor product $V_1 \otimes V_2 \otimes V_3$ in $\text{Rep}(\widehat{G})$ in the obvious sense.

4.2.5. *Compatibility with the Kottwitz homomorphism.* For $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$, let Bun_α (resp. Bun_α^+) be the preimage of α under the Kottwitz morphism $\kappa : \text{Bun} \rightarrow \mathbb{X}^*(Z\widehat{G})_{I_F}$ (resp. $\kappa^+ : \text{Bun}^+ \rightarrow \mathbb{X}^*(Z\widehat{G})_{I_F}$). In the definition of $D_{\mathcal{G}, \Omega}(\mathbf{K}_S, \mathcal{K}_S)$ in terms of $D_{(\mathbf{M}_S, \mathcal{K}_{S, \Omega})}^b(\text{Bun}^+)$, we may work with Bun_α^+ instead of the whole Bun^+ (because \mathbf{M}_S preserves each Bun_α^+), and define

$$D_\alpha := D_{(\mathbf{M}_S, \mathcal{K}_{S, \Omega})}^b(\text{Bun}_\alpha^+).$$

Then $D_{\mathcal{G}, \Omega}(\mathbf{K}_S, \mathcal{K}_S)$ is the Cartesian product of D_α over $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$. For any scheme Y over k , we also define

$$D_\alpha(Y) := D_{(\mathbf{M}_S, \mathcal{K}_{S, \Omega})}^b(Y \times \text{Bun}_\alpha^+).$$

where \mathbf{M}_S acts trivially on Y .

For each $x \in X$, we have the local Kottwitz homomorphism $\kappa_x : L_x G \rightarrow \mathbb{X}^*(Z\widehat{G})_{I_x}$ as in (2.2). In particular, when $x \in U$, I_x acts trivially on $\mathbb{X}^*(Z\widehat{G})$, and the map κ_x factors as $L_x^+ G \backslash L_x G / L_x^+ G \rightarrow \mathbb{X}^*(Z\widehat{G})$. However this map is canonical only after choosing a point $x' \in U'$ over x . Therefore there is a well-defined map $\kappa_{\text{Hk}} : \text{Hk}_{U'} \rightarrow \mathbb{X}^*(Z\widehat{G})$ (while for Hk_U only the map to $\mathbb{X}^*(Z\widehat{G})_{I_F}$ is well-defined). Similarly, for the iterated Hecke correspondence, we have a map $\kappa_{\text{Hk}, I} : \text{Hk}_{U^I} \rightarrow \mathbb{X}^*(Z\widehat{G})^I \rightarrow \mathbb{X}^*(Z\widehat{G})$, the last map being the addition. For $\nu \in \mathbb{X}^*(Z\widehat{G})$, let $\text{Hk}_{U^I, \nu}$ be the preimage of ν under $\kappa_{\text{Hk}, I}$.

For each $\nu \in \mathbb{X}^*(Z\widehat{G})$, let $\text{Rep}(\widehat{G}; \nu)$ be the full subcategory of $\text{Rep}(\widehat{G})$ consisting of those $V \in \text{Rep}(\widehat{G})$ with central character $\nu \in \mathbb{X}^*(Z\widehat{G})$. Similarly we define $\text{Rep}(\widehat{G}^I; \nu)$ by looking at how the diagonal copy of $Z\widehat{G}$ acts.

For $V \in \text{Rep}(\widehat{G}; \nu)$, the support of $\text{IC}_V \in \text{Sat}$ is on the preimage of ν under $L^+\mathbb{G} \backslash L\mathbb{G}/L^+\mathbb{G} \rightarrow \mathbb{X}^*(Z\widehat{G})$. From this we conclude the following.

4.2.6. LEMMA. For $V_I \in \text{Rep}(\widehat{G}^I; \nu)$, $\text{IC}_{V_I}^{\text{Hk}}$ is supported on $\text{Hk}_{U^I, \nu}$, and the geometric Hecke operators restricts to functors

$$(4.8) \quad \mathfrak{T}_{V_I, \alpha, Y} : D_\alpha(Y) \rightarrow D_{\alpha + \bar{\nu}}(U^I \times Y) \quad \forall \alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$$

where $\bar{\nu}$ is the image of ν in the quotient $\mathbb{X}^*(Z\widehat{G})_{I_F}$.

4.3. Hecke eigen properties.

4.3.1. DEFINITION. A *Hecke eigensheaf* for the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is a triple $(\mathcal{A}, \mathcal{F}, \varphi)$ where

- $\mathcal{A} \in D_{\mathcal{G}, \Omega}(\mathbf{K}_S, \mathcal{K}_S)$.
- $\mathcal{F} \in \text{Loc}_{\widehat{G}, \widehat{\theta}}(U)$ called the *Hecke eigen local system*. We think of \mathcal{F} as a \widehat{G} -local system over U' together with a descent datum $\{\delta_{\gamma, V} : \mathcal{F}_{V\gamma} \xrightarrow{\sim} \gamma^* \mathcal{F}_V\}_{\gamma \in \Gamma, V \in \text{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)}$ as in §3.1.7.
- φ is a collection of isomorphisms, one for each $V \in \text{Rep}(\widehat{G})$ and functorial in V

$$\varphi_V : \mathfrak{T}_V(\mathcal{A}) \cong \mathcal{F}_V \boxtimes \mathcal{A}.$$

The triple $(\mathcal{A}, \mathcal{F}, \varphi)$ must satisfy the following conditions.

- (1) For the trivial representation $\text{triv} = \overline{\mathbb{Q}}_\ell$ of \widehat{G} , we have the canonical isomorphism $\mathfrak{T}_{\text{triv}}(\mathcal{A}) \cong \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}$ from §4.2.4(1) and also $\mathcal{F}_{\text{triv}} = \overline{\mathbb{Q}}_\ell$. We require that φ_V be the identity map of $\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}$.
- (2) For $V_1, V_2 \in \text{Rep}(\widehat{G})$, $\varphi_{V_1 \otimes V_2}$ is equal to the composition

$$\begin{aligned} \mathfrak{T}_{V_1 \otimes V_2}(\mathcal{A}) &\xrightarrow{(4.7)} (\mathfrak{T}_{V_1, U'} \circ \mathfrak{T}_{V_2}(\mathcal{A}))|_{\text{Bun}^+ \times \Delta(U')} \\ &\xrightarrow{\mathfrak{T}_{V_2, U'}(\varphi_{V_2})} \mathfrak{T}_{V_1, U'}(\mathcal{F}_{V_2} \boxtimes \mathcal{A})|_{\text{Bun}^+ \times \Delta(U')} \\ &\xrightarrow{\text{id}_{\mathcal{F}_{V_2}} \boxtimes \varphi_{V_1}} (\mathcal{F}_{V_1} \boxtimes \mathcal{F}_{V_2} \boxtimes \mathcal{A})|_{\text{Bun}^+ \times \Delta(U')} = (\mathcal{F}_{V_1} \otimes \mathcal{F}_{V_2}) \boxtimes \mathcal{A} \cong \mathcal{F}_{V_1 \otimes V_2} \boxtimes \mathcal{A}. \end{aligned}$$

The last step uses the tensor structure on the functor \mathcal{F} .

- (3) For any $\gamma \in \Gamma = \text{Gal}(U'/U)$ and $V \in \text{Rep}(\widehat{G})$, we have a commutative diagram

$$(4.9) \quad \begin{array}{ccc} \mathfrak{T}_{V\gamma}(\mathcal{A}) & \xrightarrow{\varphi_{V\gamma}} & \mathcal{F}_{V\gamma} \boxtimes \mathcal{A} \\ \downarrow \alpha_{\gamma, V} & & \downarrow \delta_{\gamma, V} \otimes \text{id}_{\mathcal{A}} \\ (\text{id}_{\text{Bun}^+} \times \gamma)^* \mathfrak{T}_V(\mathcal{A}) & \xrightarrow{(\text{id} \times \gamma)^* \varphi_V} & \gamma^* \mathcal{F}_V \boxtimes \mathcal{A} \end{array}$$

These properties imply that $\mathfrak{T}_{V_1 \boxtimes \dots \boxtimes V_n}(\mathcal{A}) \cong \mathcal{F}_{V_1} \boxtimes \dots \boxtimes \mathcal{F}_{V_n} \boxtimes \mathcal{A}$, compatible with the factorization structures in §4.2.4.

We also axiomatize a weak version of Definition 4.3.1. For this we introduce a variant of the notion of \widehat{G} -local systems.

4.3.2. DEFINITION. Let A be a set with an action of $\mathbb{X}^*(Z\widehat{G})$ (denoted by $\nu : \alpha \mapsto \nu \cdot \alpha$, for $\nu \in \mathbb{X}^*(Z\widehat{G})$ and $\alpha \in A$). A (\widehat{G}, A) -weak local system over U' is the following data.

- (1) For each finite set I and $\alpha \in A$, there is an additive functor

$$\begin{aligned} \mathcal{F}_{(-),\alpha} : \text{Rep}(\widehat{G}^I) &\rightarrow \text{Loc}(U'^I) \\ V_I &\mapsto \mathcal{F}_{V_I,\alpha}. \end{aligned}$$

- (2) For $\text{triv} \in \text{Rep}(\widehat{G}^I)$ the trivial representation, there is a canonical isomorphism $\mathcal{F}_{\text{triv},\alpha} \cong \overline{\mathbb{Q}}_\ell$ (the constant sheaf on U'^I).
- (3) For finite sets I and J , $V_I \in \text{Rep}(\widehat{G}^I; \mu)$ and $V_J \in \text{Rep}(\widehat{G}^J; \nu)$, there is an isomorphism

$$\mathcal{F}_{V_I,\nu \cdot \alpha} \boxtimes \mathcal{F}_{V_J,\alpha} \cong \mathcal{F}_{V_I \boxtimes V_J,\alpha}.$$

satisfying obvious associativity for three finite sets I, J, K and for the unit constraint in (2).

- (4) For a surjection $\varphi : J \twoheadrightarrow I$ of finite sets, and $V_J \in \text{Rep}(\widehat{G}^J; \nu)$ restricting to $\Delta(\varphi)^*V_J \in \text{Rep}(\widehat{G}^I; \nu)$ via the diagonal $\Delta(\varphi) : \widehat{G}^I \hookrightarrow \widehat{G}^J$, there is an isomorphism

$$\Delta(\varphi)^* \mathcal{F}_{V_J,\alpha} \cong \mathcal{F}_{\Delta(\varphi)^*V_J,\alpha}$$

Here $\Delta(\varphi)$ also denotes the diagonal map $U'^I \hookrightarrow U'^J$. These isomorphisms should be compatible with compositions of surjections and with the isomorphisms given in (2) and (3).

If, moreover, the action of $\mathbb{X}^*(Z\widehat{G})$ on A factors through the quotient $\mathbb{X}^*(Z\widehat{G})_{I_F}$, and the functor $\mathcal{F}_{(-),\alpha}$ is equipped with a descent datum $\{\delta_{\gamma,V_I}\}_{\gamma \in \Gamma, V_I \in \text{Rep}(\widehat{G}^I)}$ as in §3.1.7, which is compatible with the other isomorphisms we introduced above, $\{\mathcal{F}_{(-),\alpha}\}$ is called a $\widehat{\theta}$ -twisted (\widehat{G}, A) -weak local system.

The following lemma records some easy consequences of the definition above.

4.3.3. LEMMA. *Let $\{\mathcal{F}_{(-),\alpha}\}$ be a (\widehat{G}, A) -weak local system on U' . Then*

- (1) *For $V_i \in \text{Rep}(\widehat{G}; \nu_i)$, $i = 1, \dots, n$, we have a functorial isomorphism*

$$(4.10) \quad \mathcal{F}_{V_1 \boxtimes \dots \boxtimes V_n,\alpha} \cong \mathcal{F}_{V_1,(\nu_2 + \dots + \nu_n) \cdot \alpha} \boxtimes \dots \boxtimes \mathcal{F}_{V_{n-1},\nu_n \cdot \alpha} \boxtimes \mathcal{F}_{V_n,\alpha}.$$

- (2) *For any $V_1 \in \text{Rep}(\widehat{G}; \mu)$, $V_2 \in \text{Rep}(\widehat{G}; \nu)$, we have a functorial isomorphism*

$$(4.11) \quad \mathcal{F}_{V_1,\alpha} \boxtimes \mathcal{F}_{V_2,\mu \cdot \alpha} \cong \mathcal{F}_{V_1,\nu \cdot \alpha} \boxtimes \mathcal{F}_{V_2,\alpha}.$$

- (3) *For any $V_1 \in \text{Rep}(\widehat{G}; \mu)$, $V_2 \in \text{Rep}(\widehat{G}; \nu)$, we have functorial isomorphisms*

$$\mathcal{F}_{V_1,\alpha} \otimes \mathcal{F}_{V_2,\mu \cdot \alpha} \cong \mathcal{F}_{V_1 \otimes V_2,\alpha} \cong \mathcal{F}_{V_2 \otimes V_1,\alpha} \cong \mathcal{F}_{V_2,\alpha} \otimes \mathcal{F}_{V_1,\nu \cdot \alpha}.$$

- (4) For $V \in \text{Rep}(\widehat{G}; \nu)$ with dual $V^* \in \text{Rep}(\widehat{G}; -\nu)$ and any $\alpha \in A$, $\mathcal{F}_{V,\alpha}$ and $\mathcal{F}_{V^*,\nu\alpha}$ are dual local systems in a canonical way.
- (5) For $V \in \text{Rep}(\widehat{G})$ and any $\alpha \in A$, the rank of $\mathcal{F}_{V,\alpha}$ is $\dim V$.
- (6) Let $(Z\widehat{G})' \subset Z\widehat{G}$ be the subgroup with character group equal to the image of $\mathbb{X}^*(Z\widehat{G}) \rightarrow \text{Aut}(A)$. Let $\widehat{G}' = \widehat{G}/(Z\widehat{G})'$. Then for each $\alpha \in A$, the functor $\mathcal{F}_{(-),\alpha}$ restricted to $\text{Rep}(\widehat{G}')$ gives a \widehat{G}' -local system over U' . Same holds in the $\widehat{\theta}$ -twisted situation.

PROOF. (1) and (6) are easy and we omit the proof.

(2) By (4.10), $\mathcal{F}_{V_1,\nu\alpha} \boxtimes \mathcal{F}_{V_2,\alpha} \cong \mathcal{F}_{V_1 \boxtimes V_2,\alpha}$ and $\mathcal{F}_{V_2,\mu\alpha} \boxtimes \mathcal{F}_{V_1,\alpha} \cong \mathcal{F}_{V_2 \boxtimes V_1,\alpha}$. Applying (3) to the transposition $\sigma : \{1, 2\} \rightarrow \{1, 2\}$ we get $\sigma^* \mathcal{F}_{V_2 \boxtimes V_1,\alpha} \cong \mathcal{F}_{V_1 \boxtimes V_2,\alpha}$. Combining these facts we get (4.11).

(3) Restricting (4.11) to the diagonal $U' \hookrightarrow U' \times U'$.

(4) We define $\text{ev} : \mathcal{F}_{V^*,\nu\alpha} \otimes \mathcal{F}_{V,\alpha} \rightarrow \overline{\mathbb{Q}}_\ell$ as the composition $\mathcal{F}_{V^*,\nu\alpha} \otimes \mathcal{F}_{V,\alpha} \cong \mathcal{F}_{V^* \otimes V,\alpha} \rightarrow \mathcal{F}_{\text{triv},\alpha} \cong \overline{\mathbb{Q}}_\ell$ where we use the evaluation map $\text{ev} : V^* \otimes V \rightarrow \text{triv}$. We define $\text{coev} : \overline{\mathbb{Q}}_\ell \rightarrow \mathcal{F}_{V,\alpha} \otimes \mathcal{F}_{V^*,\nu\alpha}$ to be the composition $\overline{\mathbb{Q}}_\ell \cong \mathcal{F}_{\text{triv},\nu\alpha} \rightarrow \mathcal{F}_{V \otimes V^*,\nu\alpha} \cong \mathcal{F}_{V,\alpha} \otimes \mathcal{F}_{V^*,\nu\alpha}$ where we use the coevaluation map $\text{coev} : \text{triv} \rightarrow V \otimes V^*$. We need to check that these maps exhibit $\mathcal{F}_{V^*,\nu\alpha}$ as the dual object to $\mathcal{F}_{V,\alpha}$ in the tensor category $\text{Loc}(U')$. The map $(\text{id} \otimes \text{ev}) \circ (\text{coev} \otimes \text{id}) : \mathcal{F}_{V,\alpha} \rightarrow \mathcal{F}_{V,\alpha}$ is the composition

$$\begin{aligned} \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_{V,\alpha} &\xrightarrow{\text{coev} \otimes \text{id}} \mathcal{F}_{V \otimes V^*,\nu\alpha} \otimes \mathcal{F}_{V,\alpha} \cong \mathcal{F}_{V,\alpha} \otimes \mathcal{F}_{V^*,\nu\alpha} \otimes \mathcal{F}_{V,\alpha} \cong \mathcal{F}_{V,\alpha} \otimes \mathcal{F}_{V^* \otimes V,\alpha} \\ &\xrightarrow{\text{id} \otimes \text{ev}} \mathcal{F}_{V,\alpha} \otimes \overline{\mathbb{Q}}_\ell \end{aligned}$$

which is the identity map as part of the compatibility conditions we alluded to in Definition 4.3.2. Similarly one checks that the map $(\text{ev} \otimes \text{id}) \circ (\text{id} \otimes \text{coev}) : \mathcal{F}_{V^*,\nu\alpha} \rightarrow \mathcal{F}_{V^*,\nu\alpha}$ is also the identity.

(5) We may assume $V \in \text{Rep}(\widehat{G}; \nu)$. By (4), $\mathcal{F}_{V,\alpha}$ and $\mathcal{F}_{V^*,\nu\alpha}$ are dual local systems, hence have the same rank. On the other hand, $V^* \otimes V \in \text{Rep}(\widehat{G}^{\text{ad}})$, and by (6), $\mathcal{F}_{V^* \otimes V,\alpha}$ has rank $\dim(V^* \otimes V) = (\dim V)^2$. Hence $\mathcal{F}_{V^*,\nu\alpha} \otimes \mathcal{F}_{V,\alpha} \cong \mathcal{F}_{V^* \otimes V,\alpha}$ has rank $(\dim V)^2$, and therefore $\mathcal{F}_{V,\alpha}$ has rank $\dim V$. \square

4.3.4. COROLLARY. Let $\mathcal{F} = (\mathcal{F}_{(-),\alpha})_{\alpha \in A}$ be a (\widehat{G}, A) -weak local system on U' . Then

- (1) For any $V \in \text{Rep}(\widehat{G})$, if $\alpha, \beta \in A$ are in the same orbit of $\mathbb{X}^*(Z\widehat{G})$, then the semisimplifications of $\mathcal{F}_{V,\alpha}$ and $\mathcal{F}_{V,\beta}$ are isomorphic (in a non-canonical way).
- (2) Suppose all $\mathcal{F}_{V,\alpha}$ are semisimple local systems, then for any $V_1, V_2 \in \text{Rep}(\widehat{G})$ and $\alpha \in A$ there is a (non-canonical) isomorphism of local systems $\mathcal{F}_{V_1,\alpha} \otimes \mathcal{F}_{V_2,\alpha} \cong \mathcal{F}_{V_1 \otimes V_2,\alpha}$.

PROOF. (1) We may assume $V \in \text{Rep}(\widehat{G}; \mu)$, $\beta = \nu \cdot \alpha$ for some $\nu \in \mathbb{X}^*(Z\widehat{G})$. Pick any $W \in \text{Rep}(\widehat{G}; \nu)$. Then we have $\mathcal{F}_{V,\alpha} \boxtimes \mathcal{F}_{W,\mu\alpha} \cong \mathcal{F}_{V,\beta} \boxtimes \mathcal{F}_{W,\alpha}$ by Lemma 4.3.3(2). Restricting to $U' \times \{x'\}$ for some geometric point $x' \in U'$,

we get $\mathcal{F}_{V,\alpha}^{\oplus \dim W} \cong \mathcal{F}_{V,\beta}^{\oplus \dim W}$ using the rank formula in Lemma 4.3.3(5). From this we conclude that the semisimplifications of $\mathcal{F}_{V,\alpha}$ and $\mathcal{F}_{V,\beta}$ are isomorphic.

(2) Again we may assume $V_2 \in \text{Rep}(\widehat{G}; \nu)$. By (1), $\mathcal{F}_{V_1, \nu \cdot \alpha} \cong \mathcal{F}_{V_1, \alpha}$. Therefore the desired isomorphism follows from Lemma 4.3.3(3). \square

4.3.5. DEFINITION. A *weak Hecke eigensheaf* for the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is a triple $(\mathcal{A}, \mathcal{F}, \varphi)$ where

- $\mathcal{A} \in D_{\mathcal{G}, \Omega}(\mathbf{K}_S, \mathcal{K}_S)$, which means an object $\mathcal{A}_\alpha \in D_\alpha$ for each $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$.
- \mathcal{F} is a $\widehat{\theta}$ -twisted $(\widehat{G}, \mathbb{X}^*(Z\widehat{G})_{I_F})$ -weak local system with respect to the obvious action of $\mathbb{X}^*(Z\widehat{G})$ on $\mathbb{X}^*(Z\widehat{G})_{I_F}$ by addition. In other words, \mathcal{F} is a collection $\{\mathcal{F}_{V,\alpha}\}$ of local systems over U' , for $V \in \text{Rep}(\widehat{G})$ and $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$, together with the extra structures as in Definition 4.3.2.
- φ is a collection of natural isomorphisms of functors (for each $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$ and $\nu \in \mathbb{X}^*(Z\widehat{G})$ with image $\bar{\nu} \in \mathbb{X}^*(Z\widehat{G})_{I_F}$):

$$\varphi_{(-), \alpha} : \mathfrak{F}_{(-), \alpha}(\mathcal{A}_\alpha) \cong \mathcal{F}_{(-), \alpha} \boxtimes \mathcal{A}_{\alpha + \bar{\nu}} : \text{Rep}(\widehat{G}^I; \nu) \rightarrow D_{\alpha + \bar{\nu}}(U'^I).$$

The natural isomorphisms $\varphi_{(-), \alpha}$ should intertwine the factorization structures (1)(2)(3) in §4.2.4 for the iterated Hecke operators and the factorization structures (2)(3)(4) of \mathcal{F} in Definition 4.3.2.

4.4. Rigid Hecke eigensheaves. The main purpose of this subsection is to establish the existence of Hecke eigensheaves in a special situation.

4.4.1. *Assumptions.* We make the following assumptions on the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$.

- (1) The affine part of the coarse moduli space of $\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S})$ is finite (recall every commutative algebraic group over k is canonically an extension of an abelian scheme by an affine commutative algebraic group; the latter is called the affine part). For example, this is true if G is semisimple, or more generally $\mathbf{K}_{Z,x}$ has finite index in $L_x^+ \mathcal{Z}$ for each $x \in S$.
- (2) We assume that for each $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$, Bun_α contains a unique (Ω, \mathcal{K}_S) -relevant $\text{Bun}_{\mathcal{Z}}^{\natural}(\mathbf{K}_{Z,S})$ -orbit O_α (see Definition 2.7.6).
- (3) Choose a point $\mathcal{E}_\alpha \in O_\alpha$. We assume that the group $A_\alpha := A_{\mathcal{E}_\alpha}$ (see §2.7.4) is finite.
- (4) Let $\mathcal{K}_\alpha := \mathcal{K}_{\mathcal{E}_\alpha} \in \mathcal{CS}_1(A_\alpha)$ (see §2.7.4). Then \mathcal{K}_α gives a class $\xi_\alpha \in \text{H}^2(A_\alpha, \overline{\mathbb{Q}}_\ell^\times)$ (see Lemma A.4.1). We assume that the category $\text{Rep}_{\xi_\alpha}(A_\alpha)$ (see §A.4.3) contains a unique irreducible object W_α up to isomorphism.

Here is the main result of this section.

4.4.2. THEOREM. *Under the assumptions in §4.4.1, the category D_α contains a unique irreducible perverse sheaf \mathcal{A}_α up to isomorphism. Let*

$\mathcal{A} \in D_{\mathcal{G},\Omega}(\mathbf{K}_S, \mathcal{K}_S)$ be an object whose restriction to Bun_α^+ is \mathcal{A}_α . Then \mathcal{A} can be extended to a weak Hecke eigensheaf $(\mathcal{A}, \mathcal{F}, \varphi)$ in which $\mathcal{F}_{V,\alpha}$ is a semisimple local system for all $V \in \mathrm{Rep}(\widehat{G})$ and $\alpha \in \mathbb{X}^*(Z\widehat{G})_{IF}$.

By Corollary 4.3.4(2), each functor $\mathcal{F}_{(-),\alpha} : \mathrm{Rep}(\widehat{G}) \rightarrow \mathrm{Loc}(U')$ is weak tensor in the sense that there exist isomorphisms $\mathcal{F}_{V_1,\alpha} \otimes \mathcal{F}_{V_2,\alpha} \cong \mathcal{F}_{V_1 \otimes V_2,\alpha}$ which do not necessarily satisfy the associativity constraint.

The proof of Theorem 4.4.2 occupies the rest of the subsection.

Fix a smaller level $\mathbf{K}_S^+ \triangleleft \mathbf{K}_S$ as in §2.6.3. For each $\alpha \in \mathbb{X}^*(Z\widehat{G})_{IF}$, let O_α^+ be the preimage of O_α in Bun_α^+ , which is an \mathbf{M}_S -orbit. Choose $\mathcal{E}_\alpha^+ \in O_\alpha^+$ over \mathcal{E}_α . Then the stabilizer $\mathbf{M}_{S,\mathcal{E}_\alpha^+}$ is identified with A_α .

4.4.3. LEMMA. *The embedding $j_\alpha : O_\alpha \hookrightarrow \mathrm{Bun}_\alpha$ is open.*

PROOF. Equivalently we need to show that $j_\alpha^+ : O_\alpha^+ \hookrightarrow \mathrm{Bun}_\alpha^+$ is open. Let $\mathcal{U} \subset \mathrm{Bun}_\alpha^+$ be the locus where the stabilizers under \mathbf{M}_S are finite. Then \mathcal{U} is an open substack of Bun_α^+ because the dimension of stabilizers is upper semicontinuous. On the other hand, any \bar{k} -point in \mathcal{U} is (Ω, \mathcal{K}_S) -relevant because the relevance condition is vacuous there. Since O_α^+ is the only relevant \mathbf{M}_S -orbit, we have $\mathcal{U} = O_\alpha^+$, hence O_α^+ is open. \square

We have

$$D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(O_\alpha^+) \cong D_{(A_\alpha, \mathcal{K}_\alpha)}^b(\mathrm{pt}) \cong D^b(\mathrm{Rep}_{\xi_\alpha}(A_\alpha)).$$

The first equivalence follows from the fact that $[\mathbf{M}_S \backslash O_\alpha^+] \cong A_\alpha \backslash \mathrm{pt}$; the second equivalence follows from Lemma A.4.4. The unique irreducible object $W_\alpha \in \mathrm{Rep}_{\xi_\alpha}(A_\alpha)$ gives an irreducible $(\mathbf{M}_S, \mathcal{K}_{S,\Omega})$ -equivariant local system \mathcal{L}_α over O_α^+ .

4.4.4. LEMMA. (1) *We have $j_{\alpha,!}^+ \mathcal{L}_\alpha \xrightarrow{\sim} j_{\alpha,*}^+ \mathcal{L}_\alpha \in D_\alpha$. We denote both objects by \mathcal{A}_α .*

(2) *For any scheme Y over k , the functor*

$$(4.12) \quad \begin{aligned} D^b(Y) &\rightarrow D_\alpha(Y) = D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(Y \times \mathrm{Bun}_\alpha^+) \\ \mathcal{A}' &\mapsto \mathcal{A}' \boxtimes \mathcal{A}_\alpha \end{aligned}$$

is an equivalence of categories. In particular, when $Y = \mathrm{Spec} k$, a shift of \mathcal{A}_α is the unique irreducible perverse sheaf in D_α .

PROOF. (1) Since O_α^+ is the unique (Ω, \mathcal{K}_S) -relevant \mathbf{M}_S -orbit, any object in D_α must have vanishing stalks and costalks outside O_α^+ by Lemma 2.7.9. Therefore $j_{\alpha,!}^+ \mathcal{L}_\alpha \xrightarrow{\sim} j_{\alpha,*}^+ \mathcal{L}_\alpha$.

(2) We shall construct an inverse to the functor (4.12). Since O_α^+ is the unique (Ω, \mathcal{K}_S) -relevant \mathbf{M}_S -orbit, any object in $D_\alpha(Y)$ must have vanishing stalks and costalks outside $Y \times O_\alpha^+$, by the same argument as Lemma 2.7.9. Therefore the restriction functor

$$(4.13) \quad (\mathrm{id}_Y \times j_\alpha^+)^* : D_\alpha(Y) = D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(Y \times \mathrm{Bun}_\alpha^+) \rightarrow D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(Y \times O_\alpha^+)$$

is an equivalence of categories.

Let $i_\alpha^+ : \text{Spec } k \rightarrow O_\alpha^+$ be the point \mathcal{E}_α^+ . The restriction functor

$$(\text{id}_Y \times i_\alpha^+)^* : D_{(\mathbf{M}_S, \mathcal{K}_{S, \Omega})}^b(Y \times O_\alpha^+) \rightarrow D_{(A_\alpha, \mathcal{K}_\alpha)}^b(Y \times \{\mathcal{E}_\alpha^+\}) = D_{(A_\alpha, \mathcal{K}_\alpha)}^b(Y)$$

is again an equivalence of categories. In the last category, A_α acts trivially on Y . The forgetful functor identifies $D_{(A_\alpha, \mathcal{K}_\alpha)}^b(Y)$ with the category of objects in $D^b(Y)$ together with a ξ_α -twisted action of A_α (for this we need to choose a cocycle representing ξ_α). Since W_α is the unique irreducible object in the semisimple category $\text{Rep}_{\xi_\alpha}(A_\alpha)$, any object $\mathcal{B} \in D_{(A_\alpha, \mathcal{K}_\alpha)}^b(Y)$ is of the form $\mathcal{A}' \otimes W_\alpha$ for $\mathcal{A}' = \text{Hom}_{\text{Rep}_{\xi_\alpha}(A_\alpha)}(W_\alpha, \mathcal{B}) \in D^b(Y)$. Therefore an inverse to (4.12) is given by

$$(4.14) \quad D_\alpha(Y) \ni \mathcal{A} \mapsto \text{Hom}_{\text{Rep}_{\xi_\alpha}(A_\alpha)}(W_\alpha, (\text{id}_Y \times i_\alpha^+ j_\alpha^+)^* \mathcal{A}) \in D^b(Y).$$

□

4.4.5. *Construction of $\mathcal{F}_{V, \alpha}$.* Let $V_I \in \text{Rep}(\widehat{G}^I; \nu)$, $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$ and $\beta = \alpha + \bar{\nu}$. Applying (4.14) from the proof of Lemma 4.4.4 to $Y = U^{II}$ and $\mathfrak{T}_{V_I, \alpha}(\mathcal{A}_\alpha) \in D_\alpha(U^{II})$ (see (4.8)), we get a canonical isomorphism

$$\varphi_{V_I, \alpha} : \mathfrak{T}_{V_I, \alpha}(\mathcal{A}_\alpha) \cong \mathcal{F}_{V_I, \alpha} \boxtimes \mathcal{A}_\beta.$$

where $\mathcal{F}_{V_I, \alpha} \in D^b(U^{II})$ is defined as

$$(4.15) \quad \mathcal{F}_{V_I, \alpha} := \text{Hom}_{\text{Rep}_{\xi_\alpha}(A_\alpha)}(W_\alpha, (\text{id}_{U^{II}} \times i_\alpha^+ j_\alpha^+)^* \mathfrak{T}_{V_I, \alpha}(\mathcal{A}_\alpha)).$$

The properties (1)(2)(3) of the iterated Hecke operators in §4.2.4 implies the properties (2)(3)(4) of $\mathcal{F}_{V, \alpha}$ as in Definition 4.3.2, except that we do not yet know that $\mathcal{F}_{V, \alpha}$ are local systems. Moreover, the Γ -equivariance of the morphisms ev (see (4.4)) gives a $\widehat{\theta}$ -twisted descent datum on $\mathcal{F}_{V_I, \alpha}$:

$$\delta_{\gamma, V_I, \alpha} : \mathcal{F}_{V_I, \alpha}^\gamma \cong \gamma^* \mathcal{F}_{V_I, \alpha}, \forall \gamma \in \Gamma.$$

The descent datum satisfies the analog of condition (3) in Definition 4.3.1 (note for $V_I \in \text{Rep}(\widehat{G}^I; \nu)$, the central character of V^γ is $\nu \circ \gamma$, which has the same image in $\mathbb{X}^*(Z\widehat{G})_{I_F}$, therefore $\mathcal{A}_{\alpha + \bar{\nu}}$ appears in both the upper right and lower right corners of (4.9)). Therefore to prove Theorem 4.4.2, it remains to show that $\mathcal{F}_{V, \alpha}$ is a local system for each $V \in \text{Rep}(\widehat{G})$, because then by (4.10) all $\mathcal{F}_{V_I, \alpha}$ are local systems. We shall do this in two lemmas.

4.4.6. LEMMA. *The complex $\mathcal{F}_{V, \alpha}$ is a semisimple local system on U' for all $V \in \text{Rep}(\widehat{G}; \nu)$ and $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$.*

PROOF. Let $\beta = \alpha + \bar{\nu} \in \mathbb{X}^*(Z\widehat{G})_{I_F}$. Let $\text{Hk}_{U', \alpha \rightarrow \beta}$ be a union of components of $\text{Hk}_{U', \nu}$ which map to Bun_α^+ under \overleftarrow{h} (hence to Bun_β^+ under \overrightarrow{h}). We recall the notion of Universal Local Acyclicity (ULA) from [4, §5.1]. The morphism $\pi \times \overleftarrow{h} : \text{Hk}_{U'} \rightarrow U' \times \text{Bun}^+$ is étale locally a trivial fibration with fibers isomorphic to the affine Grassmannian, and under such a trivialization IC_V^{Hk} becomes IC_V along the fibers. Therefore

$\overleftarrow{h}^* \mathcal{A}_\alpha \otimes \mathrm{IC}_V^{\mathrm{Hk}}$ is ULA with respect to $\pi_{U'} : \mathrm{Hk}_{U', \alpha \rightarrow \beta} \rightarrow U'$. On the other hand, $\pi \times \overrightarrow{h} : \mathrm{Hk}_{U', \alpha \rightarrow \beta} \rightarrow U' \times \mathrm{Bun}^+$ is ind-proper, therefore $\mathfrak{T}_V(\mathcal{A}_\alpha) = (\pi \times \overrightarrow{h})_!(\overleftarrow{h}^* \mathcal{A}_\alpha \otimes \mathrm{IC}_V^{\mathrm{Hk}})$ is also ULA with respect to the projection to U' . Since $\mathfrak{T}_V(\mathcal{A}_\alpha)$ is an external tensor product $\mathcal{F}_{V, \alpha} \boxtimes \mathcal{A}_\beta$ over $U' \times \mathrm{Bun}_\beta^+$, $\mathcal{F}_{V, \alpha}$ is ULA with respect to the identity map on U' , i.e., each cohomology sheaf of $\mathcal{F}_{V, \alpha}$ is a local system. Therefore, to show that $\mathcal{F}_{V, \alpha}$ is a local system, it suffices to show that the geometric stalk of $\mathcal{F}_{V, \alpha}$ at some point $x' \in U'$ is concentrated in degree zero.

Fix a geometric point $x' \in U'$ and let $\mathrm{Hk}_{x', \alpha \rightarrow \beta}$ be the corresponding fiber. Let ${}^\circ \mathrm{Hk}_{x'}$ be the open subset of $\mathrm{Hk}_{x', \alpha \rightarrow \beta}$ which is the preimage of O_α^+ under \overleftarrow{h} ; let $\mathrm{Hk}_{x'}^\circ$ be the preimage of O_β^+ under \overrightarrow{h} ; let ${}^\circ \mathrm{Hk}_{x'}^\circ = {}^\circ \mathrm{Hk}_{x'} \cap \mathrm{Hk}_{x'}^\circ$. Consider the diagram with a Cartesian square on the left

$$\begin{array}{ccccc}
 & & {}^\circ \mathrm{Hk}_{x'}^\circ & \xrightarrow{j_{\mathrm{Hk}}} & \mathrm{Hk}_{x'}^\circ \\
 & \swarrow \overleftarrow{h}^\circ & & \searrow \overleftarrow{h}^\circ & \searrow \overrightarrow{h}^\circ \\
 O_\alpha^+ & \xrightarrow{j_\alpha^+} & \mathrm{Bun}^+ & & O_\beta^+
 \end{array}$$

Let $\overrightarrow{h}^\circ : {}^\circ \mathrm{Hk}_{x'}^\circ \rightarrow O_\beta^+$ be the restriction of \overrightarrow{h} to ${}^\circ \mathrm{Hk}_{x'}^\circ$. To alleviate notation, let $\mathrm{IC}_{x', V}$ denote the restriction of $\mathrm{IC}_V^{\mathrm{Hk}}$ to any open substack of the fiber $\mathrm{Hk}_{x', \alpha \rightarrow \beta}$. Let $i_{x'} : \{x'\} \hookrightarrow U'$ be the inclusion. By the definition of the geometric Hecke operators and proper base change, we have

$$\begin{aligned}
 (4A16)_{\alpha, x'} \otimes \mathcal{L}_\beta &= (i_{x'} \times j_\beta^+)^* \mathfrak{T}_{V, \alpha}(\mathcal{A}_\alpha) \cong \overrightarrow{h}_!^\circ(\overleftarrow{h}^{\circ, *} j_{\alpha, *}^+ \mathcal{L}_\alpha \otimes \mathrm{IC}_{x', V}) \\
 &\cong \overrightarrow{h}_!^\circ j_{\mathrm{Hk}, *}^\circ(\overleftarrow{h}^{\circ, *} \mathcal{L}_\alpha \otimes \mathrm{IC}_{x', V}) = \overrightarrow{h}_!^\circ(\overleftarrow{h}^{\circ, *} \mathcal{L}_\alpha \otimes \mathrm{IC}_{x', V}).
 \end{aligned}$$

On the other hand, we have another sequence of isomorphisms

$$\begin{aligned}
 (4A17)_{\alpha, x'} \otimes \mathcal{L}_\beta &= (i_{x'} \times j_\beta^+)^* \mathfrak{T}_{V, \alpha}(\mathcal{A}_\alpha) \cong \overrightarrow{h}_*^\circ(\overleftarrow{h}^{\circ, *} j_{\alpha, *}^+ \mathcal{L}_\alpha \otimes \mathrm{IC}_{x', V}) \\
 &\cong \overrightarrow{h}_*^\circ j_{\mathrm{Hk}, *}^\circ(\overleftarrow{h}^{\circ, *} \mathcal{L}_\alpha \otimes \mathrm{IC}_{x', V}) = \overrightarrow{h}_*^\circ(\overleftarrow{h}^{\circ, *} \mathcal{L}_\alpha \otimes \mathrm{IC}_{x', V}).
 \end{aligned}$$

Here we have used two facts: one is that \overrightarrow{h}° is ind-proper; the other is that $\overleftarrow{h}^\circ : \mathrm{Hk}_{x'}^\circ \rightarrow \mathrm{Bun}^+$ is étale locally a trivial fibration, therefore $\overleftarrow{h}^{\circ, *} j_{\alpha, *}^+ \cong j_{\mathrm{Hk}, *}^\circ \overleftarrow{h}^{\circ, *}$.

We define $\mathrm{GR}_{x', \beta}$ and its open substack ${}^\circ \mathrm{GR}_{x', \beta}$ by the following Cartesian squares

$$\begin{array}{ccccc}
 & & {}^\circ \mathrm{GR}_{x', \beta} & \hookrightarrow & \mathrm{GR}_{x', \beta} & \longrightarrow & \mathrm{Hk}_{x'}^\circ \\
 & \swarrow \overleftarrow{h}_{\mathrm{GR}} & & \searrow \overleftarrow{h}_{\mathrm{GR}} & & & \searrow \overrightarrow{h}^\circ \\
 O_\alpha^+ & \xrightarrow{j_\alpha^+} & \mathrm{Bun}_\alpha^+ & & & & \{ \mathcal{E}_\beta^+ \} \xrightarrow{j_\beta^+} O_\beta^+
 \end{array}$$

Since $\overrightarrow{h}^\circ : \mathrm{Hk}_{x'}^\circ \rightarrow O_\beta^+$ is \mathbf{M}_S -equivariant and O_β^+ is a single \mathbf{M}_S -orbit, (4.16) and (4.17) can be combined and rewritten as

$$(4.18) \quad \mathcal{F}_{V,\alpha,x'} \otimes W_\beta \cong \mathrm{H}_c^*(\circ\mathrm{GR}_{x',\beta}, \overset{\circ}{\leftarrow} h_{\mathrm{GR}}^* \mathcal{L}_\alpha \otimes \mathrm{IC}_{x',V}) \cong \mathrm{H}^*(\circ\mathrm{GR}_{x',\beta}, \overset{\circ}{\leftarrow} h_{\mathrm{GR}}^* \mathcal{L}_\alpha \otimes \mathrm{IC}_{x',V}).$$

The complex $\mathcal{H} := \overset{\circ}{\leftarrow} h_{\mathrm{GR}}^* \mathcal{L}_\alpha \otimes \mathrm{IC}_{x',V}$ is perverse (on $\circ\mathrm{GR}_{x',\beta}$). By Lemma 4.4.7 below, $\circ\mathrm{GR}_{x',\beta}$ is ind-affine. The support of \mathcal{H} is a closed subscheme of $\circ\mathrm{GR}_{x',\beta}$ of finite type, and is therefore affine. By [1, Théorème 4.1.1], $\mathrm{H}_c^*(\circ\mathrm{GR}_{x',\beta}, \mathcal{H})$ lies in non-negative degrees and $\mathrm{H}^*(\circ\mathrm{GR}_{x',\beta}, \mathcal{H})$ lies in non-positive degrees. By (4.18), these two cohomology groups are isomorphic to each other, hence they are both concentrated in degree zero, and therefore $\mathcal{F}_{V,\alpha,x'}$ is concentrated in degree zero for any $x' \in U'$.

Finally we need to argue that $\mathcal{F}_{V,\alpha}$ is a semisimple local system. Using the usual spreading-out argument we may reduce to the case where k is a finite field, and we may use the theory of weights for complexes of sheaves. We may normalize the local system \mathcal{L}_α to be pure of weight zero, and normalize IC_V to be also pure of weight zero. Then $\mathcal{A}_\alpha = j_{\alpha,!}^+ \mathcal{L}_\alpha = j_{\alpha,*}^+ \mathcal{L}_\alpha$ is pure of weight zero, and $\overleftarrow{h}^* \mathcal{A}_\alpha \otimes \mathrm{IC}_V^{\mathrm{Hk}}$ is pure of weight zero on $\mathrm{Hk}_{U',\alpha \rightarrow \beta}$ (using the fact that $\mathrm{Hk}_{U'}$ is locally a product of $\mathrm{Bun}^+ \times U'$ and the affine Grassmannian). Applying the ind-proper map $\pi \times \overrightarrow{h}$ we conclude that $\mathfrak{T}_V(\mathcal{A}_\alpha) \cong \mathcal{F}_{V,\alpha} \boxtimes \mathcal{A}_\beta$ is a pure complex, hence $\mathcal{F}_{V,\alpha}$ is also a pure complex on U' . By the decomposition theorem [1, Corollaire 5.4.6], $\mathcal{F}_{V,\alpha}$ is a direct sum of shifted simple perverse sheaves. Since $\mathcal{F}_{V,\alpha}$ is a local system, it is a semisimple local system. \square

4.4.7. LEMMA. *For any geometric point $x' \in U'$, the open sub-indscheme $\circ\mathrm{GR}_{x',\beta} \subset \mathrm{GR}_{x',\beta}$ is ind-affine.*

PROOF. We give a reinterpretation of $\circ\mathrm{GR}_{x',\beta}$. Consider the functor $\mathfrak{I}_{x',\alpha \rightarrow \beta}$ classifying pairs (m, ι) where $m \in \mathbf{M}_S$ and $\iota : m \cdot \mathcal{E}_\alpha^+|_{X-\{x\}} \xrightarrow{\sim} \mathcal{E}_\beta^+|_{X-\{x\}}$ (x is the image of x' in U). Then $\mathfrak{I}_{x',\alpha \rightarrow \beta}$ is represented by an indscheme over k . The finite group A_α acts on $\mathfrak{I}_{x',\alpha \rightarrow \beta}$ on the right: $a \in A_\alpha \cong \mathbf{M}_{S,\mathcal{E}_\alpha^+}$ sends (m, ι) to $(ma, \iota \circ a)$. It is easy to show that the map $\mathfrak{I}_{x',\alpha \rightarrow \beta}/A_\alpha \rightarrow \circ\mathrm{GR}_{x',\beta}$ sending (m, ι) to $(m \cdot \mathcal{E}_\alpha^+, \iota)$ is an isomorphism. Therefore it suffices to show that $\mathfrak{I}_{x',\alpha \rightarrow \beta}$ is affine.

The fibers of the natural morphism $\mathfrak{I}_{x',\alpha \rightarrow \beta} \rightarrow \mathbf{M}_S$ ($(m, \iota) \mapsto m$), if nonempty, are torsors under the automorphism group of $\mathcal{E}_\beta^+|_{X-\{x\}}$, which is ind-affine. Therefore the fibers of $\mathfrak{I}_{x',\alpha \rightarrow \beta} \rightarrow \mathbf{M}_S \rightarrow \mathrm{Bun}_Z^{\natural}(\mathbf{K}_{Z,S})$ are ind-affine. Consider the integral model \mathcal{D} of D (the maximal torus quotient of G) as in §2.2.4. Since $Z \hookrightarrow G \twoheadrightarrow D$ is an isogeny, the kernel of $\rho : \mathrm{Bun}_Z^{\natural}(\mathbf{K}_{Z,S}) \rightarrow \mathrm{Bun}_Z \rightarrow \mathrm{Bun}_{\mathcal{D}}$ has affine coarse moduli space. On the other hand, the image of $\mathfrak{I}_{x',\alpha \rightarrow \beta} \rightarrow \mathrm{Bun}_Z^{\natural}(\mathbf{K}_{Z,S}) \xrightarrow{\rho} \mathrm{Bun}_{\mathcal{D}}$ lies in the image of $\circ\mathrm{GR}_{x',\alpha \rightarrow \beta} \subset L_x G/L_x^+ \mathcal{G} \rightarrow L_x D/L_x^+ \mathcal{D} \rightarrow \mathrm{Bun}_{\mathcal{D}}$, and the latter has discrete reduced scheme structure. We conclude that the image

of $\mathfrak{J}_{x',\alpha\rightarrow\beta} \rightarrow \mathrm{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S})$ lies in a disjoint union of translations of its affine part, hence finite by Assumption (1) in §4.4.1. Since the fibers of $\mathfrak{J}_{x',\alpha\rightarrow\beta} \rightarrow \mathrm{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S})$ are ind-affine, $\mathfrak{J}_{x',\alpha\rightarrow\beta}$ is a finite disjoint union of ind-affine schemes, hence ind-affine. \square

4.5. Description of the eigen local system and examples in \mathbf{GL}_2 .

We are in the situation of §4.4.1. We make an extra assumption:

(4.19) The coarse moduli space of the stack $\mathrm{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S})$ is a point.

In other words, $\mathrm{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S})$ is the classifying space of $\mathbf{A}_{Z,S}$. This is satisfied if, for example, $X = \mathbb{P}^1$, $Z = \mathbb{Z}\mathbb{G} \otimes_k F$ and $\mathbf{K}_{Z,x} = L_x^+ \mathbb{Z}\mathbb{G}$ for all $x \in S$. When (4.19) is satisfied, we have $\mathrm{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S}^+) = \mathbf{L}_{Z,S}/\mathbf{A}_{Z,S}$, hence $\mathbf{M}_S = \mathbf{L}_S/\mathbf{A}_{Z,S}$. The character sheaf \mathcal{K}_S descends to a character sheaf $\mathcal{K}_{S,\Omega}$ on \mathbf{M}_S because of the compatibility isomorphism $\iota_S : \Omega|_{\mathbf{K}_{Z,S}} \cong \mathcal{K}_S|_{\mathbf{K}_{Z,S}}$.

Let $V \in \mathrm{Rep}(\widehat{G}; \nu)$, and let $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$, $\beta = \alpha + \bar{\nu} \in \mathbb{X}^*(Z\widehat{G})_{I_F}$. We will give an explicit description of the Hecke-eigen local systems $\mathcal{F}_{V,\alpha}$ defined in (4.15).

Because $\mathrm{Bun}_{\mathbb{Z}}^{\natural}(\mathbf{K}_{Z,S})$ has one geometric point and the assumption (2) of §4.4.1, \mathcal{E}_{α} and \mathcal{E}_{β} are the only relevant points on Bun_{α} and Bun_{β} . Let Aut_{α} and Aut_{β} be the automorphism groups of \mathcal{E}_{α} and \mathcal{E}_{β} , then we have an exact sequence $1 \rightarrow \mathbf{A}_{Z,S} \rightarrow \mathrm{Aut}_{\alpha} \rightarrow A_{\alpha} \rightarrow 1$ and likewise for Aut_{β} .

Let $\mathfrak{G}_{U',\alpha\rightarrow\beta}$ be the ind-scheme over U' whose fiber at $x' \in U'$ classifies isomorphism $\mathcal{E}_{\alpha}|_{X-\{x\}} \xrightarrow{\sim} \mathcal{E}_{\beta}|_{X-\{x\}}$ of \mathcal{G} -torsors preserving \mathbf{K}_S -level structures (here x is the image of x' in U). Let $\mathfrak{J}_{U',\alpha\rightarrow\beta} = \mathfrak{G}_{U',\alpha\rightarrow\beta}/\mathbf{A}_{Z,S}$ where $\mathbf{A}_{Z,S}$ is acting as automorphisms of \mathcal{E}_{α} . The group $A_{\alpha} \times A_{\beta}$ acts on $\mathfrak{J}_{U',\alpha\rightarrow\beta}$ by $(a_{\alpha}, a_{\beta}) \cdot g = a_{\beta} g a_{\alpha}^{-1}$.

Let \mathcal{E}_{α}^+ and \mathcal{E}_{β}^+ in Bun^+ be liftings \mathcal{E}_{α} and \mathcal{E}_{β} . Under such choices, the \mathbf{L}_S -reduction of the \mathbf{K}_S -level structures of \mathcal{E}_{α} and \mathcal{E}_{β} are trivialized. Evaluating at S gives a morphism

$$\mathrm{ev}_{\mathfrak{J},S} : \mathfrak{J}_{U',\alpha\rightarrow\beta} \rightarrow \mathbf{M}_S$$

Evaluating an isomorphism $\mathcal{E}_{\alpha}|_{X'-\{x'\}} \xrightarrow{\sim} \mathcal{E}_{\beta}|_{X'-\{x'\}}$ at the formal disk around $x' \in U'$ gives

$$\mathrm{ev}_{\mathfrak{J}} : \mathfrak{J}_{U',\alpha\rightarrow\beta} \rightarrow \left[\frac{L^+ \mathbb{G} \backslash L\mathbb{G} / L^+ \mathbb{G}}{\mathrm{Aut}_k[[t]]} \right]$$

in the same way ev in (4.4) was defined. Let $\mathrm{IC}_V^{\mathfrak{J}} := \mathrm{ev}_{\mathfrak{J}}^* \mathrm{IC}_V$.

4.5.1. *Central extension.* By the discussion in §A.2, the rank one character sheaf $\mathcal{K}_{S,\Omega}$ on \mathbf{M}_S can be obtained from a central extension

$$1 \rightarrow C \rightarrow \widetilde{\mathbf{M}}_S \xrightarrow{v} \mathbf{M}_S \rightarrow 1$$

by taking $\mathcal{K}_{S,\Omega} \cong (v_! \overline{\mathbb{Q}}_{\ell})_{\chi_C}$. Here $\widetilde{\mathbf{M}}_S$ is connected and C is a finite discrete (necessarily abelian) group scheme over k , and $\chi_C : C \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$ is a character.

Let

$$\tilde{A}_\alpha := A_\alpha \times_{\mathbf{M}_S} \tilde{\mathbf{M}}_S; \quad \tilde{A}_\beta := A_\beta \times_{\mathbf{M}_S} \tilde{\mathbf{M}}_S; \quad \tilde{\mathcal{J}}_{U',\alpha \rightarrow \beta} := \mathcal{J}_{U',\alpha \rightarrow \beta} \times_{\mathbf{M}_S} \tilde{\mathbf{M}}_S.$$

Here all the maps to \mathbf{M}_S are evaluation maps. Then \tilde{A}_α is an extension of A_α by C ; similar remark applies to \tilde{A}_β . Again $\tilde{A}_\alpha \times \tilde{A}_\beta$ acts on $\tilde{\mathcal{J}}_{U',\alpha \rightarrow \beta}$, and the action of the diagonal copy of C is trivial.

Recall that a class $\xi_\alpha \in \mathbf{H}^2(A_\alpha, \overline{\mathbb{Q}}_\ell^\times)$ is given by the restriction of $\mathcal{K}_{S,\Omega}$ to A_α , and we may form the category of twisted representations $\text{Rep}_{\xi_\alpha}(A_\alpha)$. We may identify $\text{Rep}_{\xi_\alpha}(A_\alpha)$ with the subcategory of $\text{Rep}(\tilde{A}_\alpha)$ consisting of representations whose restriction to C is χ_C . The unique irreducible object $W_\alpha \in \text{Rep}_{\xi_\alpha}(A_\alpha)$ (see Assumption (4) in §4.4.1) is viewed as an irreducible representation \tilde{W}_α of \tilde{A}_α in this way.

We denote the pullback of $\text{IC}_V^{\tilde{\mathcal{J}}}$ to $\tilde{\mathcal{J}}_{U',\alpha \rightarrow \beta}$ by $\text{IC}_V^{\tilde{\mathcal{J}}}$. Finally, let $\tilde{\Pi} : \tilde{\mathcal{J}}_{U',\alpha \rightarrow \beta} \rightarrow U'$ be the projection.

4.5.2. PROPOSITION. *We are under the assumptions in §4.4.1 and the extra assumption (4.19). For $V \in \text{Rep}(\widehat{G}; \nu)$, the Hecke-eigen local system $\mathcal{F}_{V,\alpha}$ constructed in Theorem 4.4.2 can be written as*

$$(4.20) \quad \mathcal{F}_{V,\alpha} \cong \text{Hom}_{\tilde{A}_\alpha \times \tilde{A}_\beta}(\tilde{W}_\alpha^\vee \boxtimes \tilde{W}_\beta, \tilde{\Pi}_! \text{IC}_V^{\tilde{\mathcal{J}}}).$$

Here the action of $\tilde{A}_\alpha \times \tilde{A}_\beta$ on $\tilde{\mathcal{J}}_{U',\alpha \rightarrow \beta}$ induces an action on the complex $\tilde{\Pi}_! \text{IC}_V^{\tilde{\mathcal{J}}} \in D^b(U')$, and the right side is the multiplicity of the irreducible representation $\tilde{W}_\alpha^\vee \boxtimes \tilde{W}_\beta$ of $\tilde{A}_\alpha \times \tilde{A}_\beta$ in $\tilde{\Pi}_! \text{IC}_V^{\tilde{\mathcal{J}}}$, which is still a complex on U' .

PROOF. To compute $\mathcal{F}_{V,\alpha}$ we need to restrict $\mathfrak{I}_V(\mathcal{A}_\alpha)$ to $U' \times \{\mathcal{E}_\beta^+\}$. Taking the fiber of $\overleftarrow{h}^{-1}(\mathcal{E}_\beta)$ we get the Beilinson-Drinfeld Grassmannian $\text{GR}_{U',\beta}$ whose fiber over $x' \in U'$ is $\text{GR}_{x',\beta}$ defined in the proof of Lemma 4.4.6. We also have the open subscheme ${}^\circ\text{GR}_{U',\beta}$ which the preimage of O_α^+ under $\overleftarrow{h}_{\text{GR}} : \text{GR}_{U',\beta} \rightarrow \text{Bun}_\alpha^+$. Restricting IC_V^{HK} to ${}^\circ\text{GR}_{U',\beta}$ we get a complex IC_V^{GR} . We have the maps ${}^\circ\overleftarrow{h} : {}^\circ\text{GR}_{U',\beta} \rightarrow O_\alpha^+$ and ${}^\circ\pi_{\text{GR}} : {}^\circ\text{GR}_{U',\alpha \rightarrow \beta} \rightarrow U'$. Then

$$(4.21) \quad \mathcal{F}_{V,\alpha} \otimes W_\beta \cong {}^\circ\pi_{\text{GR},!}({}^\circ\overleftarrow{h}^* \mathcal{L}_\alpha \otimes \text{IC}_V^{\text{GR}}).$$

We claim that there is a canonical isomorphism ${}^\circ\text{GR}_{U',\alpha \rightarrow \beta} \cong \tilde{\mathcal{J}}_{U',\alpha \rightarrow \beta} / \tilde{A}_\alpha = \mathcal{J}_{U',\alpha \rightarrow \beta} / A_\alpha$. We have introduced the indscheme $\mathcal{J}_{x',\alpha \rightarrow \beta}$ in the proof of Lemma 4.4.7. Since O_α is a single point, O_α^+ is a single \mathbf{L}_S -orbit, any point $\mathcal{E}^+ \in O_\alpha^+$ is obtained from \mathcal{E}_α^+ by changing its \mathbf{K}_S^+ -level structure by an element in \mathbf{L}_S modulo the automorphisms Aut_α . Therefore $\mathcal{J}_{x',\alpha \rightarrow \beta}$ is the fiber of $\mathcal{J}_{U',\alpha \rightarrow \beta}$ over $x' \in U'$. We have argued in the proof of Lemma 4.4.7 that $\mathcal{J}_{x',\alpha \rightarrow \beta} / A_\alpha \cong {}^\circ\text{GR}_{x',\alpha \rightarrow \beta}$. That argument works for x' varying over U' and gives the isomorphism ${}^\circ\text{GR}_{U',\alpha \rightarrow \beta} \cong \mathcal{J}_{U',\alpha \rightarrow \beta} / A_\alpha = \tilde{\mathcal{J}}_{U',\alpha \rightarrow \beta} / \tilde{A}_\alpha$. Let $\overleftarrow{i} : \tilde{\mathcal{J}}_{U',\alpha \rightarrow \beta} / \tilde{A}_\alpha \rightarrow \mathbb{B}\tilde{A}_\alpha$ be the canonical projection. Then under

the isomorphism ${}^\circ\text{GR}_{U',\beta} \cong \tilde{\mathcal{I}}_{U',\alpha \rightarrow \beta} / \tilde{A}_\alpha, {}^\circ\overleftarrow{h}^* \mathcal{L}_\alpha \otimes \text{IC}_V^{\text{GR}}$ corresponds to $\overleftarrow{i}^* \tilde{W}_\alpha \otimes \text{IC}_V^{\tilde{\mathcal{I}}}$, viewing $\tilde{W}_\alpha \in \text{Rep}(\tilde{A}_\alpha)$ as a sheaf on $\mathbb{B}\tilde{A}_\alpha$. Therefore we have

$${}^\circ\pi_{\text{GR},!}({}^\circ\overleftarrow{h}^* \mathcal{L}_\alpha \otimes \text{IC}_V^{\text{GR}}) \cong \tilde{\Pi}_!({}^\circ\overleftarrow{i}^* \tilde{W}_\alpha \otimes \text{IC}_V^{\tilde{\mathcal{I}}}) = \text{Hom}_{\tilde{A}_\alpha}(\tilde{W}_\alpha^\vee, \tilde{\Pi}_! \text{IC}_V^{\tilde{\mathcal{I}}}).$$

On the other hand, by (4.21), the above complex is also \tilde{A}_β -equivariantly isomorphic to $\mathcal{F}_{V,\alpha} \otimes \tilde{W}_\beta$. Combing these facts we get (4.20). \square

In the special case where the A_α and A_β are trivial, we have a simpler description using the maps

$$(4.22) \quad \begin{array}{ccc} & \mathcal{I}_{U',\alpha \rightarrow \beta} & \\ \text{ev}_{\mathcal{I},S} \swarrow & & \searrow \Pi \\ \mathbf{M}_S & & U' \end{array}$$

4.5.3. PROPOSITION. *Let $V \in \text{Rep}(\widehat{G}; \nu)$, $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$ and $\beta = \alpha + \bar{\nu}$. Under the assumptions in §4.4.1, (4.19) and the extra assumption that A_α and A_β are trivial. Then the Hecke eigen local system $\mathcal{F}_{V,\alpha}$ constructed in Theorem 4.4.2 can be written as*

$$\mathcal{F}_{V,\alpha} \cong \Pi_!(\text{ev}_{\mathcal{I},S}^* \mathcal{K}_{S,\Omega} \otimes \text{IC}_V^{\mathcal{I}}).$$

The proof is similar to that of Proposition 4.5.2 and we omit it.

4.6. Examples for GL_2 . We shall apply Proposition 4.5.3 to the examples considered in §2.8, and check that the Hecke eigen local systems we get do coincide with the classification of cohomologically rigid rank two local systems in Proposition 3.3.2.

4.6.1. The example from §2.8.1. The unique \mathcal{K}_S -relevant point \star_0 on $\text{Bun}_2^0(\mathbf{I}_S)$ is the trivial bundle $\mathcal{V} := \mathcal{O}_{e_1} \oplus \mathcal{O}_{e_2}$ with lines $\ell_0 = \langle e_1 \rangle$, $\ell_1 = \langle e_1 + e_2 \rangle$ and $\ell_\infty = \langle e_2 \rangle$. Let us lift this point to $\text{Bun}_2^0(\mathbf{I}_S^+)$, i.e., we choose a basis u_x for ℓ_x which is the spanning vector of ℓ_x as written above, and let $v_x \in \mathcal{V}_x / \ell_x$ be given by the image of e_2, e_2 and e_1 at 0, 1 and ∞ respectively. The unique relevant point \star_1 on $\text{Bun}_2^1(\mathbf{I}_S)$ is the bundle $\mathcal{V}' := \mathcal{O}(1)e'_1 \oplus \mathcal{O}e'_2$ with lines $\ell'_0 = \langle e'_2 \rangle$, $\ell'_1 = \langle e'_1 + e'_2 \rangle$ and $\ell'_\infty = \langle e'_2 \rangle$. We similarly lift \star_1 to $\text{Bun}_2^1(\mathbf{I}_S^+)$ by choosing u'_x to be the spanning vector of ℓ'_x as written and $v'_x \in \mathcal{V}'_x / \ell'_x$ given by the image of e'_1, e'_2 and e'_1 at 0, 1 and ∞ respectively.

Let V be the standard representation of $\widehat{G} = \text{GL}_2$. The support \mathfrak{G}_V of the sheaf IC_V on the group scheme $\mathfrak{G}_{U,0 \rightarrow 1}$ can be described as follows: \mathfrak{G}_V classifies embeddings of coherent sheaves $\varphi : \mathcal{V} \rightarrow \mathcal{V}'$ that send ℓ_x to ℓ'_x at each $x \in S$, and is an isomorphism at each $x \in S$. We use the affine coordinate t on \mathbb{P}^1 and as such, $\{1, t\}$ can be viewed as a basis for $\Gamma(\mathbb{P}^1, \mathcal{O}(1))$. A map $\mathcal{V} \rightarrow \mathcal{V}'$ can be written under the bases e_i, e'_i as a matrix $\varphi = \begin{pmatrix} a_0 + a_1 t & b_0 + b_1 t \\ c & d \end{pmatrix}$ for $a_0, a_1, b_0, b_1, c, d \in k$. The conditions that φ should send ℓ_x to ℓ'_x imply that $a_0 = 0, b_1 = 0$ and $a_1 + b_0 = c + d$.

Let us write $a := a_1$ and $b := b_0$. The point where φ has a zero is $y = \frac{bc}{ad}$. Since $y \notin S$, we must have $a, b, c, d \neq 0$ and $ad \neq bc$. Therefore \mathfrak{G}_V is an open subset of \mathbb{A}^3 . The morphism $\pi : \mathfrak{G}_V \hookrightarrow \mathfrak{G}_{U,0 \rightarrow 1} \rightarrow U = \mathbb{P}^1 - \{0, 1, \infty\}$ sends $(a, b, c, d) \mapsto \frac{bc}{ad}$. Evaluating φ at $x \in S$, we get nonzero constants $\alpha_x^{(1)}(\varphi)$ and $\alpha_x^{(2)}(\varphi)$ such that $\varphi_x(u_x) = \alpha_x^{(1)}(\varphi)u'_x$ and $\varphi_x(v_x) = \alpha_x^{(1)}(\varphi)v'_x$. This gives a map $\alpha_x : \mathfrak{G}_V \rightarrow \mathbb{G}_m^2$ for each $x \in S$. These maps are given in coordinates by

$$\alpha_0(a, b, c, d) = (c, b); \quad \alpha_1(a, b, c, d) = (a + b, d - b); \quad \alpha_\infty(a, b, c, d) = (d, a).$$

Since $\mathbf{A}_{Z,S} = \mathbb{G}_m$ in this case, the scheme $\mathfrak{J}_V \subset \mathfrak{J}_{U',0 \rightarrow 1}$ (the support of $\mathrm{IC}_{\mathfrak{J}_V}^{\mathfrak{J}}$) is the quotient $\mathfrak{G}_V/\mathbb{G}_m$ where \mathbb{G}_m acts on (a, b, c, d) by simultaneous scaling. The tensor product characters sheaf $\boxtimes_{x \in S, \epsilon \in \{1,2\}} \alpha_x^{(\epsilon),*} \mathcal{K}_x^{(\epsilon)}$ descends to a character sheaf $\mathcal{K}_{S,\Omega}$ on $\mathbf{M}_S \cong \mathbb{G}_m^6/\Delta(\mathbb{G}_m)$ thanks to the condition (2.19). We may identify \mathfrak{J}_V with a subscheme of \mathfrak{G}_V by setting $a + b = c + d = 1$. Then

$$\mathfrak{J}_V = \{(a, d) \in (\mathbb{G}_m - \{1\})^2 \mid a + d \neq 1\}.$$

The maps α_x restricted to \mathfrak{J}_V become

$$\alpha_0(a, d) = (1 - d, 1 - a); \quad \alpha_1(a, d) = (1, a + d - 1); \quad \alpha_\infty(a, d) = (d, a).$$

The map $\mathfrak{J}_V \rightarrow U' = \mathbb{P}^1 - \{0, 1, \infty\}$ is given by $(a, d) \mapsto (1 - a)(1 - d)/(ad)$.

Let us consider the special case where $\chi_1^{(2)}$ is trivial. In this case, applying the Lefschetz trace formula to the local system $\mathcal{F} := \mathcal{F}_{V,0}$ as described in Proposition 4.5.3, we get the trace of Frob_x on \mathcal{F} at $x \in k^\times - \{1\}$

$$f_{\mathcal{F},k}(x) = - \sum_{(1-a)(1-d)=adx} \chi_0^{(1)}(1-d)\chi_0^{(2)}(1-a)\chi_\infty^{(1)}(d)\chi_\infty^{(2)}(a).$$

Changing variables $a = 1/(1 - v)$ and $d = 1/(1 - u)$ for $u, v \in k^\times - \{1\}$, we rewrite the above sum as

$$(4.23) \quad f_{\mathcal{F},k}(x) = -\chi_\infty^{(1)}(-1)\chi_\infty^{(2)}(-1) \sum_{uv=x} \chi_0^{(1)}(u)(\bar{\chi}_0^{(1)}\bar{\chi}_\infty^{(1)})(u-1)\chi_0^{(2)}(v)(\bar{\chi}_0^{(2)}\bar{\chi}_\infty^{(2)})(v-1).$$

On the other hand, Katz's hypergeometric sheaf $\mathcal{H} := \mathcal{H}(!, \psi; \chi_0^{(1)}, \chi_0^{(2)}; \bar{\chi}_\infty^{(1)}, \bar{\chi}_\infty^{(2)})$ is defined as an iterated multiplicative convolution [23, §8.2]. Its trace function is

$$f_{\mathcal{H},k}(x) = \sum_{x_1 x_2 = y_1 y_2 x} \psi(x_1 - y_1)\chi_0^{(1)}(x_1)\chi_\infty^{(1)}(y_1)\psi(x_2 - y_2)\chi_0^{(2)}(x_2)\chi_\infty^{(2)}(y_2).$$

Making a change of variables $x_1 = uy_1$ and $x_2 = vy_2$ we may rewrite the sum as

$$(4.24) \quad \sum_{uv=x} \chi_0^{(1)}(u)\chi_0^{(2)}(v) \sum_{y_1 \in k^\times} \psi((u-1)y_1)(\chi_0^{(1)}\chi_\infty^{(1)})(y_1) \sum_{y_2 \in k^\times} \psi((v-1)y_2)(\chi_0^{(2)}\chi_\infty^{(2)})(y_2).$$

The two inner sums are equal to $(\overline{\chi}_0^{(1)}\overline{\chi}_\infty^{(1)})(u-1)G(\psi, \chi_0^{(1)}\chi_\infty^{(1)})$ and $(\overline{\chi}_0^{(2)}\overline{\chi}_\infty^{(2)})(v-1)G(\psi, \chi_0^{(2)}\chi_\infty^{(2)})$, where $G(\psi, \chi)$ stands for the Gauss sum. Comparing (4.23) and (4.24) we get

$$f_{\mathcal{F},k}(x) = -\chi_\infty^{(1)}(-1)\chi_\infty^{(2)}(-1)G(\psi, \chi_0^{(1)}\chi_\infty^{(1)})^{-1}G(\psi, \chi_0^{(2)}\chi_\infty^{(2)})^{-1}f_{\mathcal{H},k}(x).$$

This being true for all finite extensions k'/k and $x \in k'^\times - \{1\}$, we conclude that $\mathcal{F} = \mathcal{F}_{V,0}$ and $\mathcal{H} = \mathcal{H}(1, \psi; \chi_0^{(1)}, \chi_0^{(2)}; \overline{\chi}_\infty^{(1)}, \overline{\chi}_\infty^{(2)})$ are isomorphic over $U_{\overline{k}}$.

The case where $\chi_1^{(2)}$ is nontrivial can be reduced to this case by twisting $\mathcal{F}_{V,0}$ by a suitable Kummer sheaf.

4.6.2. *The example from §2.8.3.* The unique \mathcal{K}_S -relevant point $\star_0 \in \text{Bun}_2^0(\mathbf{K}_S)$ is the trivial bundle $\mathcal{V} = \mathcal{O}_{e_1} \oplus \mathcal{O}_{e_2}$ with the line $\ell_0 = \langle e_1 \rangle$ at 0, the vectors $v_\infty^{(1)} = e_2$ and $v_\infty^{(2)} = e_1$ viewed as a basis of $\mathcal{V}_\infty / \langle v_\infty^{(1)} \rangle$. The unique \mathcal{K}_S -relevant point $\star_1 \in \text{Bun}_2^1(\mathbf{K}_S)$ is the vector bundle $\mathcal{V}' = \mathcal{O}(1)e'_1 \oplus \mathcal{O}e'_2$ with the line $\ell'_0 = \langle e'_2 \rangle$ at 0, the vectors $v_\infty'^{(1)} = e'_2$ and $v_\infty'^{(2)} = e'_1$ viewed as a basis of $\mathcal{V}'_\infty / \langle v_\infty'^{(1)} \rangle$.

Let V be the standard representation of $\widehat{G} = \text{GL}_2$. The support \mathfrak{G}_V of the sheaf IC_V on the group scheme $\mathfrak{G}_{U,0 \rightarrow 1}$ can be described as follows: \mathfrak{G}_V classifies embeddings of coherent sheaves $\varphi : \mathcal{V} \rightarrow \mathcal{V}'$ that send ℓ_0 to ℓ'_0 and is an isomorphism at 0, is unipotent upper triangular with respect to the basis $\{v_\infty^{(1)}, v_\infty^{(2)}\}$ of \mathcal{V}_∞ and the basis $\{v_\infty'^{(1)}, v_\infty'^{(2)}\}$ of \mathcal{V}'_∞ . Again we represent such a map $\mathcal{V} \rightarrow \mathcal{V}'$ as a matrix $\varphi = \begin{pmatrix} a_0 + a_1t & b_0 + b_1t \\ c & d \end{pmatrix}$ for $a_0, a_1, b_0, b_1, c, d \in k$. The conditions on φ imply that $\varphi = \begin{pmatrix} t & b \\ c & 1 \end{pmatrix}$ for $b, c \in \mathbb{G}_m$. In other words, $\mathfrak{G}_V = \mathbb{G}_m^2$. The point where φ has a zero is $y = bc$. Evaluating φ at 0, we get the map $\text{ev}_{\mathfrak{G},0} : \mathfrak{G}_V \rightarrow \mathbf{L}_0 = \mathbb{G}_m^2$ given by $(b, c) \mapsto (c, b)$. Evaluating φ at ∞ , we see that $\varphi(v_\infty^{(1)}) = v_\infty'^{(1)} + bt^{-1}v_\infty'^{(2)}$ and $\varphi(v_\infty^{(2)}) = v_\infty'^{(2)} + cv_\infty'^{(1)}$. Therefore the map $\text{ev}_{\mathfrak{G},\infty} : \mathfrak{G}_V \rightarrow \mathbf{L}_\infty = \mathbb{G}_a^2$ is given by $(b, c) \mapsto (b, c)$.

Notice that in this case $\mathbf{A}_{Z,S}$ is trivial, so $\mathfrak{J}_{U,0 \rightarrow 1} = \mathfrak{G}_{U,0 \rightarrow 1}$. The diagram (4.22) now takes the form

$$\begin{array}{ccc} & \mathbb{G}_m^2 & \\ (\sigma, i) \swarrow & & \searrow m \\ \mathbb{G}_m^2 \times \mathbb{G}_a^2 & & \mathbb{G}_m \end{array}$$

where σ is the transposition of the factors of \mathbb{G}_m^2 and i is the natural embedding $\mathbb{G}_m^2 \hookrightarrow \mathbb{G}_a^2$; $m : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m$ is the multiplication map. By Proposition 4.5.3 we get that the eigen local system for the geometric automorphic datum in §2.8.3 is (notice that $\text{IC}_V = \overline{\mathbb{Q}}_\ell[1]$)

$$\mathcal{F}_{V,0} = m_!(\sigma^* \mathcal{K}_{\chi_0} \otimes i^* \text{AS}_\phi)[1]$$

Here $\chi_0 = (\chi_0^{(1)}, \chi_0^{(2)})$ was part of the geometric automorphic datum at 0 and $\phi : k \times k \rightarrow k$ was the linear function defining part of the geometric automorphic datum at ∞ . The Frobenius trace function at $x \in k^\times$ is given by

$$f_{\mathcal{F}_{V,0},k}(x) = - \sum_{b,c \in k^\times, bc=x} \psi(\phi(b,c)) \chi_0^{(2)}(b) \chi_0^{(1)}(c).$$

In the special case $\chi_0 = 1$ and $\phi(b,c) = b+c$, this is the classical Kloosterman sum up to a sign. For generalizations, see §5.

4.6.3. *The example from §2.8.5.* The unique \mathcal{K}_S -relevant $\star_0 \in \text{Bun}_2^0(\mathbf{K}_S)$ is the trivial bundle $\mathcal{V} = \mathcal{O}e_1 \oplus \mathcal{O}e_2$ with the line $\ell_0 = \langle e_1 \rangle$ at 0, the lines $\ell_\infty^{(1)} = \langle e_1 + e_2 \rangle$ and $\ell_\infty^{(2)} = \langle e_2 \rangle$ at ∞ . The unique \mathcal{K}_S -relevant point $\star_1 \in \text{Bun}_2^1(\mathbf{K}_S)$ is the vector bundle $\mathcal{V}' = \mathcal{O}(1)e'_1 \oplus \mathcal{O}e'_2$ with the line $\ell'_0 = \langle e'_2 \rangle$ at 0, the vector $\ell'_\infty^{(1)} = \langle e'_1 + e'_2 \rangle$ and $\ell'_\infty^{(2)} = \langle e'_2 \rangle$.

Let V be the standard representation of $\widehat{G} = \text{GL}_2$. The support \mathfrak{G}_V of the sheaf IC_V on the group scheme $\mathfrak{G}_{U,0 \rightarrow 1}$ can be described as follows: \mathfrak{G}_V classifies embeddings of coherent sheaves $\varphi : \mathcal{V} \rightarrow \mathcal{V}'$ that send the lines $\ell_0, \ell_\infty^{(1)}$ and $\ell_\infty^{(2)}$ to $\ell'_0, \ell'_\infty^{(1)}$ and $\ell'_\infty^{(2)}$ respectively, and that φ is an isomorphism at 0 and ∞ . Again we represent such a map $\mathcal{V} \rightarrow \mathcal{V}'$ as a matrix $\varphi = \begin{pmatrix} a_0 + a_1 t & b_0 + b_1 t \\ c & d \end{pmatrix}$ for $a_0, a_1, b_0, b_1, c, d \in k$. The conditions on φ imply that $\varphi = \begin{pmatrix} (c+d)t & b \\ c & d \end{pmatrix}$ for $b, c, d \in \mathbb{G}_m$ and $c+d \neq 0$. The evaluation map $\text{ev}_{\mathfrak{G},0} : \mathfrak{G}_V \rightarrow \mathbf{L}_0 \cong \mathbb{G}_m^2$ is given by $(b, c, d) \mapsto (c, b)$. The evaluation map $\text{ev}_{\mathfrak{G},\infty} : \mathfrak{G}_V \rightarrow \mathbf{L}_\infty \cong \mathbb{G}_m^2 \times \mathbb{G}_a^2$ is given by $(b, c, d) \mapsto (c+d, d, b, -b)$. The projection $\Pi_{\mathfrak{G}} : \mathfrak{G}_V \rightarrow U = \mathbb{G}_m$ is given by $(b, c, d) \mapsto \frac{bc}{(c+d)d}$.

The scheme $\mathfrak{I}_V = \mathfrak{G}_V/\mathbb{G}_m$ where \mathbb{G}_m acts as simultaneous scaling on b, c and d . We may identify \mathfrak{I}_V as the subscheme of \mathfrak{G}_V with $c+d=1$. Then $\mathfrak{I}_V = \{(b, d) \in \mathbb{G}_m \times (\mathbb{G}_m - \{1\})\}$. The map $\text{ev}_{\mathfrak{I},S} : \mathfrak{I}_V \mapsto \mathbf{M}_S = \mathbf{L}_S/\mathbb{G}_m \cong (\mathbb{G}_m^2 \times \mathbb{G}_m^2 \times \mathbb{G}_a^2)/\mathbb{G}_m$ is given by $(b, d) \mapsto [1-d, b, 1, d, b, -b]$. The projection $\Pi : \mathfrak{I}_V \rightarrow U$ is given by $(b, d) \mapsto b(1-d)/d$. Therefore the Frobenius trace function of $\mathcal{F} = \mathcal{F}_{V,0}$ is given by

$$f_{\mathcal{F},k}(x) = - \sum_{b(1-d)=dx} \chi_0^{(1)}(1-d) \chi_\infty^{(2)}(d) \chi_0^{(2)}(b) \psi(\phi^{(1)}(b) - \phi^{(2)}(b)).$$

Making a change variables $b = -u$ and $d = 1/(1-v)$, we may rewrite the above sum as

(4.25)

$$f_{\mathcal{F},k}(x) = -\chi_0^{(2)}(-1) \chi_\infty^{(2)}(-1) \sum_{uv=x} \chi_0^{(1)}(v) (\bar{\chi}_0^{(1)} \bar{\chi}_\infty^{(2)})(v-1) \chi_0^{(2)}(u) \psi'(u).$$

where $\psi'(y) = \psi(\phi^{(2)}(y) - \phi^{(1)}(y))$ is another nontrivial character of k .

On the other hand, Katz’s hypergeometric sheaf $\mathcal{H} := \mathcal{H}(1, \psi', \chi_0^{(1)}, \chi_0^{(2)}; \bar{\chi}_\infty^{(2)})$ has Frobenius trace function

$$f_{\mathcal{H},k}(x) = \sum_{x_1 x_2 = y_1 x} \psi'(x_1 - y_1) \chi_0^{(1)}(x_1) \chi_\infty^{(2)}(y_1) \psi'(x_2) \chi_0^{(2)}(x_2)$$

Making a change of variables $x_1 = v y_1, x_2 = u$, we get

$$\begin{aligned} f_{\mathcal{H},k}(x) &= \sum_{uv=x} \chi_0^{(1)}(v) \left(\sum_{y_1 \in k^\times} \psi'((v-1)y_1) \chi_0^{(1)}(y_1) \chi_\infty^{(2)}(y_1) \right) \psi'(u) \chi_0^{(2)}(u) \\ &= G(\psi', \chi_0^{(1)}, \chi_\infty^{(2)}) \sum_{uv=x} \chi_0^{(1)}(v) (\bar{\chi}_0^{(1)} \bar{\chi}_\infty^{(2)})(v-1) \chi_0^{(2)}(u) \psi'(u). \end{aligned}$$

Comparing (4.25) and (4.26) we get

$$f_{\mathcal{F},k}(x) = -\chi_0^{(2)}(-1) \chi_\infty^{(2)}(-1) G(\psi', \chi_0^{(1)}, \chi_\infty^{(2)})^{-1} f_{\mathcal{H},k}(x).$$

This implies that \mathcal{F} and \mathcal{H} are isomorphic over $U_{\bar{k}}$.

4.6.4. REMARK. In all three examples, Conjecture 4.1.2 is easily verified, using the knowledge of Artin conductors worked out in §3.3.

4.7. Variants and questions. We are in the situation of (4.4.1).

4.7.1. *A situation where Hecke eigensheaf is guaranteed.* Suppose for some $x \in S$ we are given a pro-algebraic subgroup $\mathbf{K}_x^\sharp \subset L_x G$ such that

- \mathbf{K}_x^\sharp contains \mathbf{K}_x as a normal subgroup.
- The character sheaf \mathcal{K}_x extends to $\mathcal{K}_x^\sharp \in \mathcal{CS}_1(\mathbf{K}_x^\sharp)$;
- The quotient $C_x := \mathbf{K}_x^\sharp / \mathbf{K}_x$ is a discrete group over k , and the local Kottwitz map $L_x G \rightarrow \mathbb{X}^*(Z\hat{G})_{I_F}$ induces an isomorphism $C_x \xrightarrow{\sim} \mathbb{X}^*(Z\hat{G})_{I_F}$.

4.7.2. EXAMPLE. Let $X = \mathbb{P}^1, F = k(t)$ and the quasi-split group G becomes split over $k(t^{1/e})$ for some e prime to $\text{char}(k)$. Then the integral model \mathcal{G} is reductive away from $\{0, \infty\}$. Suppose $0 \in S$. Let $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ be a geometric automorphic datum such that \mathbf{K}_0 is a parahoric subgroup and that $N_{L_0 G}(\mathbf{K}_0) / \mathbf{K}_0$ maps isomorphically to $\mathbb{X}^*(Z\hat{G})_{I_F} = \mathbb{X}^*(Z\hat{G})_{I_0}$. For example we may take \mathbf{K}_0 to be an Iwahori subgroup. Suppose also that the character sheaf \mathcal{K}_0 extends to $N_{L_0 G}(\mathbf{K}_0)$. Then the group $\mathbf{K}_0^\sharp = N_{L_0 G}(\mathbf{K}_0)$ satisfies the conditions in §4.7.1.

4.7.3. THEOREM. *Assuming we are in the situation of Theorem 4.4.2 and the extra assumption in §4.7.1 is satisfied. Then the object $\mathcal{A} \in D_{\mathcal{G}, \Omega}(\mathbf{K}_S, \mathcal{K}_S)$ can be extended to a Hecke eigensheaf $(\mathcal{A}, \mathcal{F}, \varphi)$ in which \mathcal{F}_V is a semisimple local system for all $V \in \text{Rep}(\hat{G})$.*

SKETCH OF PROOF. Let $\mathbf{L}_x^\sharp = \mathbf{K}_x^\sharp / \mathbf{K}_x^+$ and $\mathbf{M}_S^\sharp = \mathbf{M}_S \times^{\mathbf{L}_x^\sharp} \mathbf{L}_x^\sharp$. The stack $\text{Bun}_{\mathcal{G}}(\mathbf{K}_S^+)$ admits an action of \mathbf{M}_S^\sharp . Since the character sheaf \mathcal{K}_x extends to \mathbf{K}_x^\sharp , the character sheaf $\mathcal{K}_{S, \Omega}$ on \mathbf{M}_S also extends to a character sheaf

$\mathcal{K}_{S,\Omega}^\sharp$ on \mathbf{M}_S^\sharp . We may then consider the category $D^\sharp := D_{(\mathbf{M}_S^\sharp, \mathcal{K}_{S,\Omega}^\sharp)}^b(\mathrm{Bun}^+)$.

Since \mathbf{M}_S^\sharp permutes the open and closed substacks Bun_α^+ of Bun^+ transitively with \mathbf{M}_S the stabilizer of each Bun_α^+ ($\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$), restricting to a particular component Bun_0^+ gives an equivalence of categories $D^\sharp \xrightarrow{\sim} D_{(\mathbf{M}_S, \mathcal{K}_{S,\Omega})}^b(\mathrm{Bun}_0^+) = D_0$. Therefore, by Assumptions in 4.4.1, there is (up to isomorphism) a unique irreducible perverse sheaf $\mathcal{A} \in D^\sharp$, whose restriction to Bun_α^+ has to be isomorphic to \mathcal{A}_α for all $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$. The analog of Lemma 4.4.4 then says that for any Y , the analogous category $D^\sharp(Y)$ (with Bun^+ replaced by $Y \times \mathrm{Bun}^+$) is equivalent to $D_c^b(Y)$ via the functor $(-) \boxtimes \mathcal{A}$. Therefore we may define $\mathcal{F}_V \in D_c^b(U')$ by requiring $\mathfrak{T}_V(\mathcal{A}) \cong \mathcal{F}_V \boxtimes \mathcal{A}$. By construction \mathcal{F}_V is canonically identified with $\mathcal{F}_{V,\alpha}$ for all $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$. The $(\widehat{G}, \mathbb{X}^*(Z\widehat{G})_{I_F})$ -weak local system $\{\mathcal{F}_{V,\alpha}\}$, under such a canonical identification, becomes an actual \widehat{G} -local system \mathcal{F}_V . \square

Next we discuss several questions that naturally arise from Theorem 4.4.2.

4.7.4. *Rationality issues.* So far in this section we have been assuming that k is algebraically closed. At various stages of the argument one can in fact work over a more general field k , and work with a smaller coefficient field than $\overline{\mathbb{Q}}_\ell$. In our applications (especially to the inverse Galois problem) it is important that the Hecke eigen local systems \mathcal{F} are defined over a field that is as small as possible.

Suppose we are in the situation of §4.4.1 with the extra assumption (4.19). The description of $\mathcal{F}_{V,\alpha}$ given in Proposition 4.5.2 (which works over \bar{k} as stated) allows us to descend it to a smaller base field and a smaller coefficient field as follows. We use the notation from §4.5. We make the following assumptions:

- For each $\alpha \in \mathbb{X}^*(Z\widehat{G})_{I_F}$, the unique (Ω, \mathcal{K}_S) -relevant point $\mathcal{E}_\alpha \in \mathrm{Bun}_\alpha$ is defined over k . Therefore A_α and its central extension \widetilde{A}_α are finite group schemes over k .
- The representation \widetilde{W}_α of $\widetilde{A}_\alpha(\bar{k})$ can be extended to $\widetilde{A}_\alpha(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$. We choose such an extension for each α and denote it by $\widetilde{W}_\alpha^\heartsuit$. By the uniqueness of \widetilde{W}_α , it always extends to a *projective* representation of $\widetilde{A}_\alpha(\bar{k}) \rtimes \mathrm{Gal}(\bar{k}/k)$. Therefore this assumption means the vanishing of certain class in $\mathrm{H}^2(k, \overline{\mathbb{Q}}_\ell^\times)$.
- The representations $\widetilde{W}_\alpha^\heartsuit$ are defined over some field L where $\mathbb{Q}_\ell \subset L \subset \overline{\mathbb{Q}}_\ell$.

We rewrite the right side of (4.20) as

$$(4.27) \quad \mathcal{F}_{V,\alpha} \cong \mathrm{Hom}_{\widetilde{A}_\alpha(\bar{k}) \times \widetilde{A}_\beta(\bar{k})}(\widetilde{W}_\alpha^{\heartsuit, \vee} \otimes \widetilde{W}_\beta^\heartsuit, \widetilde{\mathrm{H}}_1 \mathrm{IC}_{\widetilde{V}}^\heartsuit).$$

We understand that both $\widetilde{W}_\alpha^{\heartsuit, \vee} \otimes \widetilde{W}_\beta^{\heartsuit}$ and $\widetilde{\Pi}_! \text{IC}_V^{\heartsuit}$ have L -coefficients and view $\widetilde{\Pi}_! \text{IC}_V^{\heartsuit}$ as a complex over U'_k . The above assumptions give a descent datum of the right side of (4.27) to U' over k . Therefore $\mathcal{F}_{V, \alpha}$ is defined over U' over k , and has L -coefficients.

4.7.5. *Rationality of the geometric Satake equivalence.* First of all, to define the Satake category, we only need to work with \mathbb{Q}_ℓ -sheaves, not $\overline{\mathbb{Q}}_\ell$ -sheaves. In the version of the geometric Satake equivalence we reviewed in §4.2.1, we have normalized all IC_V to be pure of weight zero. This involves choosing a square root of the ℓ -adic cyclotomic character $\chi_\ell : \text{Gal}(\overline{k}/k) \rightarrow \mathbb{Z}_\ell^\times$, and there is no natural choice of such. Assume k is either a finite field or a number field. Following a suggestion of Deligne, one can modify the Satake category to avoid choosing a square root of χ_ℓ . One defines a subcategory $\text{Sat}^{\text{Tt}} \subset \text{Sat}$ (the superscript stands for Tate), whose objects are finite direct sums of $\text{IC}_\lambda(n)$ ($n \in \mathbb{Z}$), where (n) denotes tensoring with the one-dimensional representation χ_ℓ^n of $\text{Gal}(\overline{k}/k)$. The category Sat^{Tt} is also a tensor category with $h = \text{H}^*(\text{Gr}, -)$ as a fiber functor. The Tannakian dual of Sat^{Tt} is the reductive group \widehat{G}^{Tt} which is related to \widehat{G} by

$$\widehat{G}^{\text{Tt}} \cong (\widehat{G} \times \mathbb{G}_m^{\text{Wt}}) / \mu_2.$$

Here \mathbb{G}_m^{Wt} is a one-dimensional torus and μ_2 is generated by $((-1)^{2\rho}, -1) \in \widehat{G} \times \mathbb{G}_m^{\text{Wt}}$, where $2\rho \in \mathbb{X}^*(\mathbb{T}) = \mathbb{X}_*(\widehat{T})$ is the sum of positive roots of \mathbb{G} , viewed as a cocharacter of \widehat{T} . For each irreducible representation $V \in \text{Rep}(\widehat{G}^{\text{Tt}})$, if the central torus \mathbb{G}_m^{Wt} acts on V via the w^{th} power, then the corresponding object IC_V in Sat^{Tt} is pure of weight w . The one-dimensional representations of \widehat{G}^{Tt} on which \widehat{G} acts trivially and \mathbb{G}_m^{Wt} acts via the $2n^{\text{th}}$ power corresponds to the sheaf $\overline{\mathbb{Q}}_\ell(-n)$ supported at the point stratum Gr_0 . For details of the construction, see [43, §2.1]. For a quick algebraic account, see [10, §2].

4.7.6. **EXAMPLE.** Let $n \geq 2$ be an integer and $G = \text{PGL}_n$. We have $\widehat{G} = \text{SL}_n$. In this case, $(-1)^{2\rho} = (-1)^{n-1} I_n \in \widehat{G}$. We have

$$\widehat{G}^{\text{Tt}} \cong \begin{cases} \text{SL}_n \times (\mathbb{G}_m^{\text{Wt}} / \mu_2) & \text{if } n \text{ is odd;} \\ (\text{SL}_n \times \mathbb{G}_m^{\text{Wt}}) / \Delta(\mu_2) & \text{if } n \text{ is even.} \end{cases}$$

We define a “standard representation” of dimension n and weight $n - 1$:

$$\begin{aligned} \text{St} & : \widehat{G}^{\text{Tt}} \rightarrow \text{GL}_n \\ (g, t) & \mapsto t^{n-1} g \in \text{GL}_n \text{ (where } g \in \text{SL}_n, t \in \mathbb{G}_m^{\text{Wt}}). \end{aligned}$$

With \widehat{G}^{Tt} replacing \widehat{G} , a Hecke eigen local system \mathcal{F} should be a twisted \widehat{G}^{Tt} -local system over U together with extra compatibility datum with Tate twists. More precisely, \mathcal{F} is a tensor functor $\text{Rep}(\widehat{G}^{\text{Tt}}) \rightarrow \text{Loc}(U')$, $V \mapsto \mathcal{F}_V$ (with $\widehat{\theta}$ -twisted descent datum), together with an isomorphism $\mathcal{F}_{\mathbb{Q}_\ell(1)} \cong \mathbb{Q}_\ell(1)$. Here the $\mathbb{Q}_\ell(1)$ in the subscript means the one-dimensional representation of \widehat{G}^{Tt} on which \widehat{G} acts trivially and \mathbb{G}_m^{Wt} acts through the $(-2)^{\text{th}}$ power, and the other $\mathbb{Q}_\ell(1)$ is the Tate-twisted constant sheaf on U' .

Next we discuss local and global monodromy of Hecke eigen local systems. We are back to the situation where k is algebraically closed.

4.7.7. *Local monodromy of \mathcal{F} .* In certain cases we can describe the local monodromy of the Hecke eigensheaf \mathcal{F} at some point $x \in S$. Suppose \mathbf{K}_x is a parahoric subgroup of $L_x G$ with reductive quotient \mathbf{L}_x , and that \mathcal{K}_x is a character sheaf on \mathbf{K}_x that descends to \mathbf{L}_x . In this case the local system \mathcal{F} is tame at x . For simplicity suppose G is split over F_x , then the restriction of \mathcal{K}_x to a split maximal torus $\mathbb{T} \subset \mathbf{L}_x$ gives a character sheaf on \mathbb{T} , which is a homomorphism $\pi_1^t(\mathbb{G}_m) \otimes \mathbb{X}_*(\mathbb{T}) \rightarrow \overline{\mathbb{Q}}_\ell^\times$, or $\rho'_x : \pi_1^t(\mathbb{G}_m) \rightarrow \widehat{T}(\overline{\mathbb{Q}}_\ell)$. On the other hand, we may canonically identify $\pi_1^t(\mathbb{G}_m)$ with the tame inertia group I_x because both of them are canonically isomorphic to $\varprojlim_{(n, \text{char}(k))=1} \mu_n(k)$. Therefore ρ'_x induces $\rho''_x : I_x \rightarrow \widehat{T}(\overline{\mathbb{Q}}_\ell) \hookrightarrow \widehat{G}(\overline{\mathbb{Q}}_\ell)$, which can be shown to be the *semisimple part* of the local monodromy of \mathcal{F} at x . When G is not necessarily split over F_x , the restriction of \mathcal{K}_x to a maximal split torus gives a section $\rho_x : I_x \rightarrow \widehat{T}(\overline{\mathbb{Q}}_\ell) \rtimes I_x$ up to \widehat{T} -conjugacy, which is again the semisimple part of the monodromy of \mathcal{F} at x .

In [44, §4.9] we give a recipe for the unipotent part of the monodromy using Lusztig’s theory of two-sided cells in affine Weyl groups. In some cases this can be proved using the techniques in [44, §4.10-4.16].

A special case where we know the local monodromy is when \mathbf{K}_x is an Iwahori subgroup and \mathcal{K}_x is trivial. This is the case where the extra condition in §4.7.1 is often satisfied, for example when $\mathbb{X}^*(Z\widehat{G})_{I_x} = \mathbb{X}^*(Z\widehat{G})_{I_F}$. In this case the local monodromy of the Hecke eigen local system \mathcal{F} at x is a regular unipotent element in \widehat{G} .

4.7.8. *Global monodromy of \mathcal{F} .* When we work over a finite field k , the proof of Lemma 4.4.6 shows that the local systems $\mathcal{F}_{V,\alpha}$ are pure. Deligne’s theorem 3.1.4 guarantees that the *neutral component* of $\widehat{G}_{\mathcal{F}}^{\text{geom}}$ (the global monodromy of \mathcal{F}) is a connected semisimple subgroup of \widehat{G} . When we know a local monodromy of \mathcal{F} is a regular unipotent element and \widehat{G} is almost simple, then either $\widehat{G}_{\mathcal{F}}^{\text{geom},\circ} = \widehat{G}$, or $\widehat{G}_{\mathcal{F}}^{\text{geom},\circ} = \widehat{G}^\sigma$ for some pinned automorphism σ of \widehat{G} , or $\widehat{G}_{\mathcal{F}}^{\text{geom},\circ}$ is the image of a principal homomorphism $\text{SL}_2 \rightarrow \widehat{G}$ (see [10, §13]). One can often determine $\widehat{G}_{\mathcal{F}}^{\text{geom}}$ in this case by using information from local monodromy at other points. This method enables us to determine the global monodromy of the examples in the next sections §5 and §6.

5. Kloosterman sheaves as rigid objects over $\mathbb{P}^1 - \{0, \infty\}$

In this section we review the work [19], in which we used rigid automorphic representations ramified at two places to construct generalizations of Kloosterman sheaves.

5.1. Kloosterman automorphic data. In this section, the curve $X = \mathbb{P}^1$ with function field $F = k(t)$, where k is a finite field. We assume that \mathbb{G} is

almost simple. Let e be a positive integer prime to $\text{char}(k)$ such that k contains e^{th} roots of unity. Let $F' = k(t^{1/e})$ and $\theta : \text{Gal}(F^s/F) \rightarrow \text{Gal}(F'/F) \cong \mu_e(k) \hookrightarrow \text{Aut}^\dagger(\mathbb{G})$ be a homomorphism. As in §2.2.2, these data determine a quasi-split form G of \mathbb{G} over F . We have an integral model \mathcal{G} over X defined in §2.2.4.

Let $S = \{0, \infty\}$. We shall define a geometric automorphic datum for G with respect to S , to be called the *Kloosterman automorphic datum*.

Let $\mathbf{K}_0 = \mathbf{I}_0$ be an Iwahori subgroup of L_0G and $\mathbf{K}_\infty = \mathbf{I}_\infty^+$, the pro-unipotent radical of an Iwahori $\mathbf{I}_\infty \subset L_\infty G$.

The reductive quotient of \mathbf{I}_0 is identified with the torus $\mathbb{S} = \mathbb{T}^{I_{F,\circ}}$. A character $\chi : \mathbb{S}(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ corresponds to a rank one character sheaves $\mathcal{K}_0 = \mathcal{K}_\chi$ on \mathbb{S} , which can also be viewed as a character sheaf on $\mathbf{K}_0 = \mathbf{I}_0$.

Let \mathbf{I}_∞^+ be the pro-unipotent radical of \mathbf{I}_∞ . It is generated by the root groups $(L_\infty G)(\tilde{\alpha})$ of all positive affine roots of the loop group $L_\infty G$. Note that $\mathbf{I}_\infty/\mathbf{I}_\infty^+ \cong \mathbb{S}$. Let $\mathbf{I}_\infty^{++} \subset \mathbf{I}_\infty^+$ be the next step in the Moy-Prasad filtration of \mathbf{I}_∞ . There is an \mathbb{S} -equivariant isomorphism

$$(5.1) \quad \mathbf{I}_\infty^+/\mathbf{I}_\infty^{++} \cong \prod_{i=0}^r (L_\infty G)(\alpha_i)$$

where the product runs over simple affine roots. Each $(L_\infty G)(\alpha_i)$ is isomorphic to \mathbb{G}_a over k . A linear function $\phi : (\mathbf{I}_\infty^+/\mathbf{I}_\infty^{++})(k) \rightarrow k$ is said to be *generic* if it does not vanish on any of the factors $(L_\infty G)(\alpha_i)$. Fix such a generic linear function ϕ and fix a nontrivial additive character ψ of k . The composition $\psi \circ \phi$ gives a character of $\mathbf{I}_\infty^+(k)$, and hence a rank one character sheaf $\mathcal{K}_\infty = \mathcal{K}_\phi$ on \mathbf{I}_∞^+ .

Finally, $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{\mathcal{Z},S})$ is a point in this case, and Lemma 2.6.5 implies that there is a unique choice of (Ω, ι_S) (up to isomorphism) making $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ a geometric automorphic datum for G with respect to $S = \{0, \infty\}$. We call it the *Kloosterman automorphic datum*. The situation considered in §2.8.3, when GL_2 is replaced with SL_2 or PGL_2 , is a special case of the situation considered here.

5.1.1. REMARK. The automorphic datum $(\mathbf{I}_\infty^+, \psi \circ \phi)$ at ∞ picks out those representations of $G(F_\infty)$ that contain nonzero eigenvectors of \mathbf{K}_∞ on which \mathbf{K}_∞ acts through the character $\psi \circ \phi$. These representations are first discovered by Gross and Reeder [13, §9.3], and they call them *simple supercuspidal representations*. Such representations appear as direct summands of the compact induction $\text{c-Ind}_{\mathbf{I}_\infty^+(k)}^{G(F_\infty)}(\psi \circ \phi)$.

5.1.2. THEOREM (Gross [14], alternative proof by Heinloth-Ngô-Yun [19]).

- (1) Let ϕ be a generic linear functional on $(\mathbf{I}_\infty^+/\mathbf{I}_\infty^{++})(k)$ as above. Then the Kloosterman automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is strongly rigid.

- (2) Let $(\omega^\natural, K_S, \chi_S)$ be the restricted automorphic datum attached to $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$, and let $\pi(\chi, \phi)$ be a $(\omega^\natural, K_S, \chi_S)$ -typical automorphic representation of $G(\mathbb{A}_F)$, which is unique up to an unramified twist. When $\chi = 1$, the local component at 0 of $\pi(1, \phi)$ is, up to an unramified twist, the Steinberg representation of $G(F_0)$.

The argument for Part (1) is a generalization of the argument of Proposition 2.8.4. In particular, we show that each component of $\text{Bun}_G(\mathbf{K}_S)$ contains a unique \mathcal{K}_S -relevant point (with trivial automorphism group in this case).

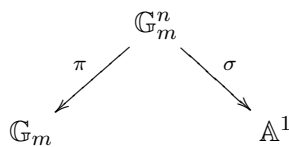
5.2. Kloosterman sheaves. Theorem 4.7.3 is applicable to the Kloosterman automorphic datum. Therefore we have a Hecke eigen local system for the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$, which we denote by $\text{Kl}_{\widehat{G}}(\chi, \phi)$ and call it the *Kloosterman sheaf* attached to $(\widehat{G}, \widehat{\theta})$ and the characters χ and ϕ . This is a $\widehat{\theta}$ -twisted \widehat{G} -local system over $\widetilde{\mathbb{G}_m}$, the e^{th} Kummer covering of $\mathbb{G}_m = \mathbb{P}_k^1 - \{0, \infty\}$. To explain the namesake, we first recall some facts about the classical Kloosterman sheaf defined by Deligne.

5.2.1. *The classical Kloosterman sheaf.* Recall the definition of Kloosterman sums. Let p be a prime number. Fix a nontrivial additive character $\psi : \mathbb{F}_p \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Let $n \geq 2$ be an integer. Then the n -variable Kloosterman sum over \mathbb{F}_p is a function on \mathbb{F}_p^\times whose value at $a \in \mathbb{F}_p^\times$ is

$$\text{Kl}_n(p; a) = \sum_{x_1, \dots, x_n \in \mathbb{F}_p^\times; x_1 x_2 \cdots x_n = a} \psi(x_1 + \cdots + x_n).$$

These exponential sums arise naturally in the study of automorphic forms for GL_n .

Deligne [5] gave a geometric interpretation of the Kloosterman sum. He considered the following diagram of schemes over \mathbb{F}_p



Here π is the morphism of taking the product and σ is the morphism of taking the sum.

5.2.2. DEFINITION (Deligne [5]). The *Kloosterman sheaf* is

$$\text{Kl}_n := \mathbf{R}^{n-1} \pi_! \sigma^* \text{AS}_\psi,$$

over $\mathbb{G}_m = \mathbb{P}_{\mathbb{F}_p}^1 - \{0, \infty\}$. Here AS_ψ is the Artin-Schreier sheaf as in Example A.3.6.

The relationship between the local system Kl_n and the Kloosterman sum $\text{Kl}_n(p; a)$ is explained by the following identity

$$\text{Kl}_n(p; a) = (-1)^{n-1} \text{Tr}(\text{Frob}_a, (\text{Kl}_n)_a).$$

Here Frob_a is the geometric Frobenius operator acting on the geometric stalk $(\text{Kl}_n)_a$ of Kl_n at $a \in \mathbb{G}_m(\mathbb{F}_p) = \mathbb{F}_p^\times$.

In [5, Théorème 7.4, 7.8], Deligne proved:

- (1) Kl_n is a local system of rank n , pure of weight $n - 1$. This implies the Weil-type bound $|\text{Kl}_n(p; a)| \leq np^{(n-1)/2}$.
- (2) Kl_n is tamely ramified at 0, and the monodromy is unipotent with a single Jordan block.
- (3) Kl_n is totally wild at ∞ (i.e., the wild inertia at ∞ has no nonzero fixed vector on the stalk of Kl_n), and the Swan conductor $\text{Sw}_\infty(\text{Kl}_n) = 1$.

5.2.3. REMARK. For $\widehat{G} = \text{SL}_n$ (resp. Sp_{2n}), the Kloosterman sheaf $\text{Kl}_{\widehat{G}}(1, \phi)$, evaluated at the standard representation of \widehat{G} , is the same as Kl_n (resp. Kl_{2n}) of Deligne up to a Tate twist. When $\widehat{G} = \text{SO}_{2n+1}$ or G_2 , $\text{Kl}_{\widehat{G}}(\chi, \phi)$ was constructed by Katz in [23] by different methods (as special cases of hypergeometric sheaves). Our construction of $\text{Kl}_{\widehat{G}}(\chi, \phi)$ using geometric automorphic data treats all \widehat{G} uniformly, and gives the first examples of motivic local systems with geometric monodromy group F_4, E_7 and E_8 (see Theorem 5.2.4(3) below).

In [19] we prove several results on the local and global monodromy of the Kloosterman sheaves $\text{Kl}_{\widehat{G}}(\chi, \phi)$.

5.2.4. THEOREM (Heinloth-Ngô-Yun [19]). *Assume G is split, then $\text{Kl}_{\widehat{G}}(\chi, \phi)$ enjoys the following properties.*

- (1) $\text{Kl}_{\widehat{G}}(\chi, \phi)$ is tame at 0. A generator of the tame inertia I_0^t maps to an element in \widehat{G} with semisimple part given by χ , viewed as an element in \widehat{T} . When $\chi = 1$, a generator of I_0^t maps to a regular unipotent element in \widehat{G} .
- (2) The local monodromy of $\text{Kl}_{\widehat{G}}(\chi, \phi)$ at ∞ is a simple wild parameter in the sense of Gross and Reeder [13, §5]. In particular, $\text{Sw}_\infty(\text{Ad}(\text{Kl}_{\widehat{G}}(\chi, \phi))) = r$ (the rank of \widehat{G}). For more details see §5.2.5.
- (3) If $\chi = 1$, then the global geometric monodromy group of $\text{Kl}_{\widehat{G}}(1, \phi)$ is a connected almost simple subgroup of \widehat{G} of types given by the following table

\widehat{G}	$\widehat{G}_{\text{Kl}_{\widehat{G}}(1, \phi)}^{\text{geom}}$	condition on $\text{char}(k)$
A_{2n}	A_{2n}	$p > 2$
A_{2n-1}, C_n	C_n	$p > 2$
$B_n, D_{n+1} (n \geq 4)$	B_n	$p > 2$
E_7	E_7	$p > 2$
E_8	E_8	$p > 2$
E_6, F_4	F_4	$p > 2$
B_3, D_4, G_2	G_2	$p > 3$

5.2.5. *Local monodromy at ∞ .* Let us explain in more detail what a *simple wild parameter* looks like when G is split, following Gross and Reeder [13, Proposition 5.6]. Assume $p = \text{char}(k)$ does not divide $\#W$ ($W \cong \mathbb{W}$ is the Weyl group of \widehat{G}). Let $\rho_\infty : I_\infty \rightarrow \widehat{G}(\overline{\mathbb{Q}_\ell})$ be the local monodromy of $\text{Kl}_{\widehat{G}}(\chi, \phi)$ at ∞ . Then we have a commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & I_\infty^w & \longrightarrow & I_\infty & \longrightarrow & I_\infty^t \longrightarrow 1 \\
 & & \downarrow \rho_\infty|_{I_\infty^w} & & \downarrow \rho_\infty & & \downarrow \\
 1 & \longrightarrow & \widehat{T} & \longrightarrow & N_{\widehat{G}}(\widehat{T}) & \longrightarrow & W \longrightarrow 1
 \end{array}$$

The image of I_∞^t in W is the cyclic group generated by a Coxeter element $\text{Cox} \in W$, whose order is the Coxeter number h of G . The image $\rho_\infty(I_\infty^w)$ is a \mathbb{F}_p -vector space equipped with the action of Cox . In fact $\rho_\infty(I_\infty^w) \cong \mathbb{F}_p[\zeta_h]$, the extension of \mathbb{F}_p by adjoining h^{th} roots of unity, and the Coxeter element acts by multiplication by a primitive h^{th} root of unity.

When $p \mid \#W$, a simple wild parameter can be more complicated. For example, when $\widehat{G} = \text{PGL}_2$ and $p = 2$, the image of ρ_∞ is isomorphic to the alternating group A_4 embedded in $\text{PGL}_2(\overline{\mathbb{Q}_\ell}) = \text{SO}_3(\overline{\mathbb{Q}_\ell})$ as the symmetry of a regular tetrahedron.

5.3. Generalizations. In [44] we give further generalizations of Kloosterman sheaves. We work with the same class of quasisplit G as in §5.1. We will replace \mathbf{I}_0 and \mathbf{I}_∞ by more general parahoric subgroups.

5.3.1. *Admissible parahoric subgroups and epipelagic representations.* In their construction of supercuspidal representations, Reeder and Yu [35] singled out a class of parahoric subgroups $\mathbf{P}_\infty \subset L_\infty G$ as follows. Let $\mathbf{P}_\infty^{++} \subset \mathbf{P}_\infty^+ \subset \mathbf{P}_\infty$ be the first three steps of the Moy-Prasad filtration on \mathbf{P}_∞ . Then the vector group $\mathbf{V}_\mathbf{P} := \mathbf{P}_\infty^+/\mathbf{P}_\infty^{++}$ is a representation of the reductive group $\mathbf{L}_\mathbf{P} := \mathbf{P}_\infty/\mathbf{P}_\infty^+$. A geometric point in the dual space $\mathbf{V}_\mathbf{P}^*$ is called *stable* if its $\mathbf{L}_\mathbf{P}$ -orbit is closed and its stabilizer under $\mathbf{L}_\mathbf{P}$ is finite. Let $\mathbf{V}_\mathbf{P}^{*,\text{st}}$ be the open subset of stable points. A parahoric subgroup $\mathbf{P}_\infty \subset L_\infty G$ is called *admissible* if $\mathbf{V}_\mathbf{P}^{*,\text{st}} \neq \emptyset$. When $\text{char}(k)$ is large, conjugacy classes of admissible parahorics are in bijection with regular elliptic numbers of the pair (\mathbb{W}, θ) . To each stable point $\phi \in \mathbf{V}_\mathbf{P}^*(k)$, Reeder and Yu construct a class of irreducible supercuspidal representations from the compact induction $c\text{-Ind}_{\mathbf{P}_\infty^+(k)}^{G(F_\infty)}(\psi \circ \phi)$ (ψ is a fixed nontrivial additive character of k), and they call them *epipelagic representations*.

5.3.2. *Generalized Kloosterman automorphic datum.* Let $\mathbf{P}_\infty \subset L_\infty G$ be an admissible parahoric subgroup. Let $\mathbf{P}_0 \subset L_0 G$ be a parahoric of the same type as \mathbf{P}_∞ . Fix a stable point $\phi \in \mathbf{V}_\mathbf{P}^{*,\text{st}}(k)$. The character $\psi \circ \phi$ of $\mathbf{V}_\mathbf{P}(k)$ corresponds to a rank one character sheaf \mathcal{K}_ϕ on $\mathbf{V}_\mathbf{P}$ (and hence on \mathbf{P}_∞^+). Fix another character $\chi : \mathbf{L}_\mathbf{P}(k) \rightarrow \overline{\mathbb{Q}_\ell}^\times$, which corresponds to a rank one character sheaf \mathcal{K}_χ on $\mathbf{L}_\mathbf{P}$ (and hence on \mathbf{P}_0). We consider the geometric

automorphic datum with respect to $S = \{0, \infty\}$ consisting of

$$\mathbf{K}_S = (\mathbf{P}_0, \mathbf{P}_\infty^+); \quad \mathcal{K}_S = (\mathcal{K}_\chi, \mathcal{K}_\phi)$$

and the unique compatible choice of (Ω, ι_S) as in the case of Kloosterman automorphic datum. One can show an analog of Theorem 5.1.2 which says that $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ is strongly rigid.

5.3.3. Generalized Kloosterman sheaves. Theorem 4.7.3 applies to obtain Hecke eigen local systems for the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$. We denote the resulting Hecke eigen local system by $\mathrm{Kl}_{\widehat{G}, \mathbf{P}}(\chi, \phi)$, which is a $\widehat{\theta}$ -twisted \widehat{G} -local system over $\widetilde{\mathbb{G}}_m$. One new feature of this generalization is that when ϕ varies in $\mathbf{V}_{\mathbf{P}}^{*, \mathrm{st}}$, the corresponding local systems $\mathrm{Kl}_{\widehat{G}, \mathbf{P}}(\chi, \phi)$ “glue” together to give a \widehat{G} -local system over $\mathbf{V}_{\mathbf{P}}^{*, \mathrm{st}} \times \widetilde{\mathbb{G}}_m$. We also have an analog of Theorem 5.2.4(1): when G is split, the monodromy of $\mathrm{Kl}_{\widehat{G}, \mathbf{P}}(1, \phi)$ at 0 is tame and unipotent, and the corresponding unipotent class $u_{\mathbf{P}}$ in \widehat{G} can be described purely in terms of \mathbf{P} . For details we refer to [44].

6. Rigid objects over $\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\}$ and applications

In this section, we review the work [42], in which we use rigid automorphic representations to construct local systems on $\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\}$. These local systems are the key objects that lead to the answer to Serre’s question and the solution of the inverse Galois problem for certain finite simple groups of exceptional Lie type.

6.1. The geometric automorphic data. Let k be a field with $\mathrm{char}(k) \neq 2$. Let $X = \mathbb{P}_k^1$ and $S = \{0, 1, \infty\}$. Assume \mathbb{G} is almost simple and simply-connected in this section.

When the longest element w_0 in the Weyl group \mathbb{W} of \mathbb{G} acts by -1 on $\mathbb{X}_*(\mathbb{T})$, we take G to be the split group $\mathbb{G} \otimes_k F$. Otherwise, let $F' = k(t^{1/2})$ and let $\theta : \Gamma_F \twoheadrightarrow \mathrm{Gal}(F'/F) \rightarrow \mathrm{Aut}^\dagger(\mathbb{G})$ map the nontrivial element in $\mathrm{Gal}(F'/F)$ to the unique pinned involution σ that acts by $-w_0$ on $\mathbb{X}_*(\mathbb{T})$. We define G to be the quasi-split form of \mathbb{G} over F using θ as in §2.2.2.

Recall that a Chevalley involution of \mathbb{G} is an involution τ such that $\dim G^\tau$ has the minimal possible dimension, namely $\#\Phi^+$ (the number of positive roots of G). All Chevalley involutions are conjugate to each other under $G^{\mathrm{ad}}(\overline{k})$. The Chevalley involution for \mathbb{G} is inner if and only if w_0 acts by -1 on $\mathbb{X}_*(\mathbb{T})$.

6.1.1. A parahoric subgroup. Up to conjugacy, there is a unique parahoric subgroup $\mathbf{P}_0 \subset L_0G$ such that its reductive quotient \mathbf{L}_0 is isomorphic to the fixed point subgroup \mathbb{G}^τ of a Chevalley involution τ . For example, we can take \mathbf{P}_0 to be the parahoric subgroup corresponding to the facet containing the element $\rho^\vee/2$ in the S -apartment of the building of L_0G , where ρ^\vee is half the sum of positive coroots of \mathbb{G} and S is the maximal split torus of G with $\mathbb{X}_*(S) = \mathbb{X}_*(\mathbb{T})^{\mathrm{Gal}(F'/F)}$.

The Dynkin diagram of the reductive quotient $\mathbf{L}_0 \cong G^\tau$ of \mathbf{P}_0 is obtained by removing one or two nodes from the extended Dynkin diagram of G . We tabulate the type of $L_{\mathbf{P}}$ and the node(s) in the affine Dynkin diagram of L_0G to be removed in each case. When we say a node is long or short, we mean its corresponding affine simple root is long or short. In the following table $n \geq 1$ (and we think of A_1 as C_1).

\mathbb{G}	G split?	$\mathbf{L}_0 \cong G^\tau$	nodes to be removed
A_{2n}	no	B_n	longest node
A_{2n+1}	no	D_{n+1}	longest node
B_{2n}	yes	$B_n \times D_n$	the $(n+1)^{\text{th}}$ counting from the short node
B_{2n+1}	yes	$B_n \times D_{n+1}$	the $(n+1)^{\text{th}}$ counting from the short node
C_n	yes	$A_{n-1} \times \mathbb{G}_m$	the two ends
D_{2n}	yes	$D_n \times D_n$	the middle node
D_{2n+1}	no	$B_n \times B_n$	the middle node
E_6	no	C_4	the long node on one end
E_7	yes	A_7	the end of the leg of length 1
E_8	yes	D_8	the end of the leg of length 2
F_4	yes	$A_1 \times C_3$	second from the long node end
G_2	yes	$A_1 \times A_1$	middle node

Examining all the cases we get

6.1.2. LEMMA. *If \mathbb{G} is not of type C , then $\mathbf{L}_0^{\text{sc}} \rightarrow \mathbf{L}_0$ is a double cover (i.e., the algebraic fundamental group of \mathbf{L}_0 has order two).*

Even if \mathbb{G} is of type C_n , $\mathbf{L}_0 \cong \text{GL}_n$ still admits a unique nontrivial double cover. In all cases, there is a canonical nontrivial double cover $v : \tilde{\mathbf{L}}_0 \rightarrow \mathbf{L}_0$. In particular,

$$\mathcal{K}_0 := (v! \overline{\mathbb{Q}}_\ell)_{\text{sgn}}$$

is a rank one character sheaf on \mathbf{L}_0 (here sgn denotes the nontrivial character of $\ker(v) = \{\pm 1\}$). When k is a finite field, we have an exact sequence

$$1 \rightarrow \{\pm 1\} \rightarrow \tilde{\mathbf{L}}_0(k) \rightarrow \mathbf{L}_0(k) \rightarrow H^1(k, \mu_2) = \{\pm 1\}.$$

The character χ_0 corresponding to \mathcal{K}_0 is given by the last arrow above.

6.1.3. *The automorphic datum.* Let $\mathbf{P}_0 \subset L_0G$ be a parahoric subgroup of the type defined in §6.1.1. Let $\mathbf{P}_\infty \subset L_\infty G$ be a parahoric subgroup of the same type as \mathbf{P}_0 (since L_0G and $L_\infty G$ are isomorphic). Let $\mathbf{I}_1 \subset L_1G$ be an Iwahori subgroup. We consider the geometric automorphic datum given by

$$\begin{aligned} (\mathbf{K}_0, \mathcal{K}_0) &= (\mathbf{P}_0, \mathcal{K}_0); \\ (\mathbf{K}_1, \mathcal{K}_1) &= (\mathbf{I}_1, \overline{\mathbb{Q}}_\ell); \\ (\mathbf{K}_\infty, \mathcal{K}_\infty) &= (\mathbf{P}_\infty, \overline{\mathbb{Q}}_\ell). \end{aligned}$$

Since $\text{Bun}_{\mathcal{Z}}(\mathbf{K}_{Z,S}) = \mathbb{B}(\mathbb{ZG}[2])$, by Lemma 2.6.5, the choices of (Ω, ι_S) to complete $(\mathbf{K}_S, \mathcal{K}_S)$ into a geometric automorphic datum are in bijection with the choices of a descent $\overline{\mathcal{K}}_0$ of \mathcal{K}_0 to $\mathbf{L}_0/\mathbb{ZG}[2]$. For this we require that $\widetilde{\mathbb{ZG}[2]} = v^{-1}(\mathbb{ZG}[2]) \subset \widetilde{\mathbf{L}}_0$ be discrete and we need to choose a character $c : \widetilde{\mathbb{ZG}[2]}(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ extending the sign character on $\mu_2 = \ker(v)$. Therefore, when $\widetilde{\mathbb{ZG}[2]}$ contains a factor μ_4 , we need to assume that $\sqrt{-1} \in k$.

The main technical result of [42] is the following.

6.1.4. THEOREM ([42]). *Assume G is simply-connected, split and is either simply-laced or of type G_2 . Assume that k contains $\sqrt{-1}$ when G is of type A_1, D_{4n+2} or E_7 . Then the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$ satisfies all the assumptions in §4.4.1 (note that $\mathbb{X}^*(Z\widehat{G})$ is trivial in this case). Therefore Theorem 4.4.2 applies to give a Hecke eigen \widehat{G} -local system \mathcal{F} on $U = \mathbb{P}_k^1 - \{0, 1, \infty\}$ attached to the geometric automorphic datum $(\Omega, \mathbf{K}_S, \mathcal{K}_S, \iota_S)$.*

Note that the conditions put on G in the above theorem limits G to be simply-connected of type A_1, D_{2n}, E_7, E_8 and G_2 .

For the unique (Ω, \mathcal{K}_S) -relevant point $\mathcal{E} \in \text{Bun}_G(\mathbf{K}_S)$, we have $\text{Aut}_{G, \mathbf{K}_S}(\mathcal{E}) \cong \mathbb{T}[2]$ and $A_{\mathcal{E}} \cong \mathbb{T}[2]/\mathbb{ZG}[2]$. Here we are identifying \mathbb{T} with a maximal torus of \mathbf{L}_0 . Let $\widetilde{A}_{\mathcal{E}} = v^{-1}(\mathbb{T}[2]) \subset \widetilde{\mathbf{L}}_0$, then we have an exact sequence

$$1 \rightarrow \widetilde{\mathbb{ZG}[2]} \rightarrow \widetilde{A}_{\mathcal{E}} \rightarrow A_{\mathcal{E}} \rightarrow 1.$$

The character sheaf $\overline{\mathcal{K}}_0 \in \mathcal{CS}_1(\mathbf{L}_0/\mathbb{ZG}[2])$ restricts to a cocycle $\xi \in H^2(A_{\mathcal{E}}, \overline{\mathbb{Q}}_\ell^\times)$ and gives the category $\text{Rep}_{\xi}(A_{\mathcal{E}})$. It turns out that $\text{Rep}_{\xi}(A_{\mathcal{E}}) \cong \text{Rep}(\widetilde{A}_{\mathcal{E}}; c)$, the latter being the category of representations of $\widetilde{A}_{\mathcal{E}}$ (a discrete group) on which $\widetilde{\mathbb{ZG}[2]}$ acts through the character c . The commutator pairing $A_{\mathcal{E}} \times A_{\mathcal{E}} \rightarrow \mu_2 = \ker(v)$ is non-degenerate, and $c|_{\ker(v)}$ is nontrivial. These facts imply that $\text{Rep}(\widetilde{A}_{\mathcal{E}}; c)$ contains a unique irreducible object up to isomorphism. This is also a situation where Proposition 4.5.2 applies.

The following theorem summarizes the local and global monodromy of the Hecke eigen \widehat{G} -local system \mathcal{F} .

6.1.5. THEOREM ([42],[44, §9]). *Let G and \mathcal{F} be as in Theorem 6.1.4.*

- (1) *The local system \mathcal{F} is tame.*
- (2) *A topological generator of I_1^t maps to a regular unipotent element in \widehat{G} .*
- (3) *A topological generator of I_∞^t maps to a unipotent element whose centralizer in \widehat{G} has dimension equal to $\#\Phi^+(\widehat{G})$ (unique up to conjugacy).*
- (4) *A topological generator of I_0^t maps to a Chevalley involution in \widehat{G} .*
- (5) *The local system \mathcal{F} is cohomologically rigid.*

- (6) When G is of type A_1, E_7, E_8 and G_2 , the global geometric monodromy of \mathcal{F} is Zariski dense in \widehat{G} . When G is of type D_{2n} , the Zariski closure of the global geometric monodromy of \mathcal{F} contains $\mathrm{SO}_{4n-1} \subset \mathrm{PSO}_{4n} = \widehat{G}$ of $n \geq 3$, and contains $G_2 \subset \mathrm{PSO}_8 = \widehat{G}$ if $n = 2$.

6.2. Applications. By a descent argument (using rigidity), we have the following strengthening of Theorem 6.1.4.

6.2.1. THEOREM ([42]). *Let k be a prime field with $\mathrm{char}(k) \neq 2$ (i.e., \mathbb{F}_p for p an odd prime or \mathbb{Q}). Then the eigen local system \mathcal{F} in Theorem 6.1.4 can be defined over k . Moreover, the monodromy of \mathcal{F} can be conjugated to $\widehat{G}(\mathbb{Q}_\ell)$ inside $\widehat{G}(\overline{\mathbb{Q}}_\ell)$.*

6.2.2. *Application to the construction of motives.* Assume G is of type A_1, E_7, E_8 or G_2 . Applying the above theorem to $k = \mathbb{Q}$, we get a \widehat{G} -local systems $\rho : \pi_1(U_{\mathbb{Q}}) \rightarrow \widehat{G}(\mathbb{Q}_\ell)$ whose geometric monodromy is Zariski dense. For each \mathbb{Q} -point $a \in U(\mathbb{Q}) = \mathbb{Q} - \{0, 1\}$, restricting ρ to the point $a = \mathrm{Spec} \mathbb{Q}$ gives a continuous Galois representation

$$(6.1) \quad \rho_a : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \widehat{G}(\mathbb{Q}_\ell).$$

By Proposition 4.5.2, one sees that for each $V \in \mathrm{Rep}(\widehat{G}, \mathbb{Q}_\ell)$, ρ_V is obtained as part of the middle dimensional cohomology of some family of varieties over U . Using this fact, it can be shown that each ρ_a is obtained from motives over \mathbb{Q} (if G is type E_8 or G_2) or $\mathbb{Q}(i)$ (if G is of type A_1 or E_7).

6.2.3. THEOREM ([42]). *Assume G is of type A_1, E_7, E_8 or G_2 . There are infinitely many $a \in \mathbb{Q} - \{0, 1\}$ such that the ρ_a 's are mutually non-isomorphic and all have Zariski dense image in \widehat{G} . Consequently, there are infinitely many motives over \mathbb{Q} (if G is type E_8 or G_2) or $\mathbb{Q}(i)$ (if G is of type A_1 or E_7) whose ℓ -adic motivic Galois group is isomorphic to \widehat{G} for any prime ℓ .*

This result then gives an affirmative answer to the ℓ -adic analog of Serre question (see §1.2.2) for motivic Galois groups of type E_7, E_8 and G_2 . The case of G_2 was settled earlier by Dettweiler and Reiter [8], using Katz's algorithmic construction of rigid local systems. Our local system \mathcal{F} in the case $G = G_2$ is the same as Dettweiler and Reiter's.

6.2.4. *Application to the inverse Galois problem.* Let ℓ be a prime number. To emphasize on the dependence on ℓ , we denote the Galois representation ρ_a in (6.1) by $\rho_{a,\ell}$. To solve the inverse Galois problem for the groups $\widehat{G}(\mathbb{F}_\ell)$, we would like to choose $a \in \mathbb{Q} - \{0, 1\}$ such that $\rho_{a,\ell}$ has image in $\widehat{G}(\mathbb{Z}_\ell)$ (which is always true up to conjugation), and its reduction modulo ℓ is surjective. This latter condition is hard to satisfy even if we know that the image of $\rho_{a,\ell}$ is Zariski dense in $\widehat{G}(\mathbb{Q}_\ell)$.

To proceed, let us consider the Betti version of Theorem 6.1.4 and Theorem 6.2.1. Namely we consider the base field $k = \mathbb{C}$ and talk about sheaves in \mathbb{Q} -vector spaces on the various complex algebraic moduli stacks. The same argument gives a topological local system

$$\rho^{\text{top}} : \pi_1^{\text{top}}(U_{\mathbb{C}}) \rightarrow \widehat{G}(\mathbb{Q})$$

whose image is Zariski dense. It makes sense to reduce ρ^{top} for large enough primes ℓ

$$\bar{\rho}_{\ell}^{\text{top}} : \pi_1^{\text{top}}(U_{\mathbb{C}}) \rightarrow \widehat{G}(\mathbb{F}_{\ell}).$$

A deep theorem of Matthews, Vaserstein and Weisfeiler [31, Theorem in the Introduction] (see also Nori [34, Theorem 5.1]) says that $\bar{\rho}_{\ell}^{\text{top}}$ is surjective for sufficiently large ℓ , when \widehat{G} is simply-connected. This is the case when G is of type E_8 and G_2 . Using the comparison between Betti cohomology and ℓ -adic cohomology, we conclude that for general $a \in \mathbb{Q} - \{0, 1\}$ (general in the sense of Hilbert irreducibility), the reduction $\bar{\rho}_{a,\ell}$ of $\rho_{a,\ell}$ is also onto $\widehat{G}(\mathbb{F}_{\ell})$. This solves the inverse Galois problem for $E_8(\mathbb{F}_{\ell})$ and $G_2(\mathbb{F}_{\ell})$ for sufficiently large primes ℓ (without an effective bound). The local monodromy of ρ also suggests a triple in $E_8(\mathbb{F}_{\ell})$ that might be rigid, see [42, Conjecture 5.16]. Recent work of Guralnick and Malle [17] establishes the rigidity of this triple, hence solves the inverse Galois problem for $E_8(\mathbb{F}_{\ell})$ for all primes $\ell \geq 7$.

When G is of type A_1 or E_7 , \widehat{G} is the adjoint form. In this case, the result in [31] says that for sufficiently large prime ℓ , the image of $\bar{\rho}_{\ell}^{\text{top}}$ contains the image of $\widehat{G}^{\text{sc}}(\mathbb{F}_{\ell}) \rightarrow \widehat{G}(\mathbb{F}_{\ell})$. We deduce that the same is true for $\rho_{a,\ell}$ for general $a \in \mathbb{Q} - \{0, 1\}$.

Appendix A. Rank one character sheaves

In this appendix, we study rank one local systems on algebraic groups that behave like characters. Most of the results here are well-known to experts, and our proofs are sketchy.

A.1. Definitions and basic properties. In this subsection k is a perfect field. Let L be an algebraic group over k with the multiplication map $m : L \times L \rightarrow L$ and the identity element $e : \text{Spec } k \rightarrow L$.

A.1.1. DEFINITION. A *rank one character sheaf* \mathcal{K} on L is a local system of rank one on L equipped with two isomorphisms

$$\begin{aligned} \mu : m^* \mathcal{K} &\xrightarrow{\sim} \mathcal{K} \boxtimes \mathcal{K}, \\ u : \overline{\mathbb{Q}}_{\ell} &\xrightarrow{\sim} e^* \mathcal{K}. \end{aligned}$$

These isomorphisms should be compatible in the sense that

$$(A.1) \quad \mu|_{L \times \{e\}} = \text{id}_{\mathcal{K}} \otimes u : \mathcal{K} = \mathcal{K} \otimes_{\overline{\mathbb{Q}}_{\ell}} \overline{\mathbb{Q}}_{\ell} \xrightarrow{\sim} \mathcal{K} \otimes e^* \mathcal{K},$$

$$(A.2) \quad \mu|_{\{e\} \times L} = u \otimes \text{id}_{\mathcal{K}} : \mathcal{K} = \overline{\mathbb{Q}}_{\ell} \otimes_{\overline{\mathbb{Q}}_{\ell}} \mathcal{K} \xrightarrow{\sim} e^* \mathcal{K} \otimes \mathcal{K}.$$

Furthermore, μ should make the following diagram commutative (A.3)

$$\begin{array}{ccccc}
 (m \times \mathrm{id}_L)^* m^* \mathcal{K} & \xrightarrow{(m \times \mathrm{id}_L)^* \mu} & (m \times \mathrm{id}_L)^* (\mathcal{K} \boxtimes \mathcal{K}) & \longrightarrow & m^* \mathcal{K} \boxtimes \mathcal{K} \xrightarrow{\mu \boxtimes \mathrm{id}_{\mathcal{K}}} \mathcal{K} \boxtimes \mathcal{K} \boxtimes \mathcal{K} \\
 \parallel & & & & \parallel \\
 (\mathrm{id}_L \times m)^* m^* \mathcal{K} & \xrightarrow{(\mathrm{id}_L \times m)^* \mu} & (\mathrm{id}_L \times m)^* (\mathcal{K} \boxtimes \mathcal{K}) & \longrightarrow & \mathcal{K} \boxtimes m^* \mathcal{K} \xrightarrow{\mathrm{id}_{\mathcal{K}} \boxtimes \mu} \mathcal{K} \boxtimes \mathcal{K} \boxtimes \mathcal{K}
 \end{array}$$

There is an obvious notion of an isomorphism between two rank one character sheaves: it is an isomorphism between local systems intertwining the μ 's and u 's. Let $\mathcal{CS}_1(L)$ be the category (groupoid) whose objects are rank one character sheaves (\mathcal{K}, μ, u) on L , and whose morphisms are isomorphisms between them. Then $\mathcal{CS}_1(L)$ carries a symmetric monoidal structure given by the tensor product of character sheaves with the constant sheaf $\overline{\mathbb{Q}}_\ell$ (equipped with the tautological μ and u) as the unit object. Let $\mathrm{CS}_1(L)$ be set of isomorphism classes of objects in $\mathcal{CS}_1(L)$, which is an abelian group.

- A.1.2. REMARK. (1) Let $\iota : L \rightarrow L$ be the inversion $g \mapsto g^{-1}$. Then any $\mathcal{K} \in \mathcal{CS}_1(L)$ is equipped with a canonical isomorphism $\iota^* \mathcal{K} \cong \mathcal{K}^{-1}$ obtained by restricting μ to the anti-diagonally embedded L .
- (2) When L is connected, the condition (A.3) is automatically satisfied. In this case, the two relations (A.1) and (A.2) also guarantee that μ is commutative, i.e., $\mu = s^* \circ \mu$ where s^* is the pullback map induced by swapping two factors $s : L \times L \rightarrow L \times L$. In this case, a local system \mathcal{K} of rank one being a character sheaf is a property rather than extra structure on \mathcal{K} : \mathcal{K} is a character sheaf if and only if $e^* \mathcal{K}$ is isomorphic to the constant sheaf on $\mathrm{Spec} k$ and that for any point $g \in L(\overline{k})$, the isomorphism type of $\mathcal{K}|_{L_{\overline{k}}}$ is invariant under left and right translation by g . On such a local system \mathcal{K} there is a unique pair (μ, u) up to isomorphism making (\mathcal{K}, μ, u) into a rank one character sheaf.
- (3) The automorphism group of a triple $(\mathcal{K}, \mu, u) \in \mathcal{CS}_1(L)$ is

$$\mathrm{Aut}(\mathcal{K}, \mu, u) \cong \mathrm{Hom}(\pi_0(L_{\overline{k}})_{\mathrm{Gal}(\overline{k}/k)}, \overline{\mathbb{Q}}_\ell^\times).$$

When L is connected, $\mathcal{CS}_1(L)$ is a groupoid with trivial automorphisms, hence is equivalent to the set $\mathrm{CS}_1(L)$.

- (4) Let $\mathcal{CS}_1(L/\overline{k})$ be the category of rank one character sheaves over $L_{\overline{k}}$. We may identify $\mathcal{CS}_1(L)$ as the category of $\mathrm{Gal}(\overline{k}/k)$ -equivariant objects of $\mathcal{CS}_1(L/\overline{k})$ (which may not induce an injection on the isomorphism classes of objects). When L is connected, the base change map $\mathrm{CS}_1(L) \rightarrow \mathrm{CS}_1(L/\overline{k})$ is injective, and identifies $\mathrm{CS}_1(L)$ with the $\mathrm{Gal}(\overline{k}/k)$ -invariants of $\mathrm{CS}_1(L/\overline{k})$.

We record a few functorial properties of rank one character sheaves.

A.1.3. LEMMA. *Let k'/k be a finite extension. Let L be an algebraic group over k' , and let $\mathrm{Res}_k^{k'} L$ be the Weil restriction of L to k . Then there is a*

canonical equivalence of symmetric monoidal categories

$$\mathcal{CS}_1(L) \cong \mathcal{CS}_1(\mathrm{Res}_k^{k'} L).$$

PROOF. Let $\mathrm{Hom}_k(k', \bar{k})$ be the set of k -linear embeddings $k' \hookrightarrow \bar{k}$. The base change of $\mathrm{Res}_k^{k'} L$ to \bar{k} is the Cartesian power $L_{\bar{k}}^{\mathrm{Hom}_k(k', \bar{k})}$, and the action of $\mathrm{Gal}(\bar{k}/k)$ on it permutes the factors according to its action on $\mathrm{Hom}_k(k', \bar{k})$. An object $\mathcal{K} \in \mathcal{CS}_1(\mathrm{Res}_k^{k'} L)$ is a $\mathrm{Gal}(\bar{k}/k)$ -equivariant object in $\mathcal{CS}_1(L_{\bar{k}}^{\mathrm{Hom}_k(k', \bar{k})})$. Any object in $\mathcal{CS}_1(L_{\bar{k}}^{\mathrm{Hom}_k(k', \bar{k})})$ takes the form $\boxtimes_{\iota \in \mathrm{Hom}_k(k', \bar{k})} \mathcal{K}_\iota$ where $\mathcal{K}_\iota \in \mathcal{CS}_1(L/\bar{k})$. The $\mathrm{Gal}(\bar{k}/k)$ -equivariant structure gives isomorphisms $\mathcal{K}_{\gamma^{-1}\circ\iota} \cong \gamma^* \mathcal{K}_\iota$ for all $\gamma \in \mathrm{Gal}(\bar{k}/k)$. Fix an embedding $\iota_0 \in \mathrm{Hom}_k(k', \bar{k})$, then the $\mathrm{Gal}(\bar{k}/k)$ -equivariant structure on $\boxtimes_{\iota \in \mathrm{Hom}_k(k', \bar{k})} \mathcal{K}_\iota$ is the same as a $\mathrm{Gal}(\bar{k}/\iota_0(k'))$ -equivariant structure on \mathcal{K}_{ι_0} , which then gives an object $\mathcal{K}_{\iota_0} \in \mathcal{CS}_1(L)$. One can check that this assignment gives the desired equivalence, and is independent of the choice of \bar{k} and ι_0 . \square

A.1.4. LEMMA. Consider an exact sequence of algebraic groups over k :

$$1 \rightarrow L_1 \xrightarrow{i} L_2 \xrightarrow{\pi} L_3 \rightarrow 1.$$

Let $\mathcal{CS}_1(L_2; L_1)$ be the category of pairs (\mathcal{K}, τ) where $\mathcal{K} \in \mathcal{CS}_1(L_2)$ and $\tau : i^* \mathcal{K} \cong \overline{\mathbb{Q}}_\ell$ is an isomorphism in $\mathcal{CS}_1(L_1)$ (here $\overline{\mathbb{Q}}_\ell$ stands for the constant sheaf on L_1 with the tautological character sheaf structure). Then $\pi^* : \mathcal{CS}_1(L_3) \rightarrow \mathcal{CS}_1(L_2; L_1)$ is an equivalence of symmetric monoidal categories.

PROOF. First assume that L_1 is reduced hence smooth over k . Let $(\mathcal{K}, \tau) \in \mathcal{CS}_1(L_2; L_1)$. The character sheaf structure on \mathcal{K} combined with τ gives a descent datum of \mathcal{K} along the smooth morphism π . By smooth descent of local systems, (\mathcal{K}, τ) gives rise to a local system $\overline{\mathcal{K}}$ on L_3 . It is easy to check that the character sheaf structure of \mathcal{K} induces one on $\overline{\mathcal{K}}$. Hence we get a functor $\mathcal{CS}_1(L_2; L_1) \rightarrow \mathcal{CS}_1(L_3)$ sending (\mathcal{K}, τ) to $\overline{\mathcal{K}}$, and it is straightforward to check that it is inverse to π^* .

When L_1 is not necessarily reduced, let $\tilde{L}_3 = L_2/L_1^{\mathrm{red}}$. We first descend $(\mathcal{K}, \tau) \in \mathcal{CS}_1(L_2; L_1)$ to $\tilde{\mathcal{K}} \in \mathcal{CS}_1(\tilde{L}_3)$ by the above argument, then since $\tilde{L}_3 \rightarrow L_3$ is a homeomorphism for the étale topology, $\tilde{\mathcal{K}}$ further descends to $\overline{\mathcal{K}} \in \mathcal{CS}_1(L_3)$, which gives an inverse functor to π^* . \square

A.2. Relation with Serre's π_1 . Let L be a connected algebraic group over k . Suppose $\mathcal{K} \in \mathcal{CS}_1(L)$ has finite order n , then it corresponds to a $\mu_n(\overline{\mathbb{Q}}_\ell)$ -torsor $L' \rightarrow L$ with L' connected. We shall call L' the associated cover of (L, \mathcal{K}) .

A.2.1. LEMMA. The scheme L' carries a canonical algebraic group structure such that the projection $\pi : L' \rightarrow L$ is a group homomorphism such that $\ker(\pi)$ is central in L' .

PROOF. The associated cover of $(L \times L, \mathcal{K} \boxtimes \mathcal{K})$ is $\pi \times \pi : L' \times L' \rightarrow L \times L$. By the functoriality of the construction of associated covers, the isomorphism $\mu : m^* \mathcal{K} \cong \mathcal{K} \boxtimes \mathcal{K}$ gives a commutative diagram

$$\begin{array}{ccc} L' \times L' & \xrightarrow{m'} & L' \\ \downarrow \pi \times \pi & & \downarrow \pi \\ L \times L & \xrightarrow{m} & L \end{array}$$

We use m' to define the multiplication on L' . The associativity of μ shows that m' is associative. The inversion on L' comes from the isomorphism $e^* \mathcal{K} \cong \mathcal{K}^{-1}$ in Remark A.1.2(1). The fiber of L' at $e \in L$ is the discrete scheme $\mu_n(\overline{\mathbb{Q}}_\ell)$ (over k) under the trivialization $u : \overline{\mathbb{Q}}_\ell \cong e^* \mathcal{K}$. We define the identity element of L' to be the point over $e \in L$ corresponding to $1 \in \mu_n(\overline{\mathbb{Q}}_\ell)$. This completes the construction of the algebraic group structure on L' . From the construction of m' , left and right multiplication of $\ker(\pi) = \mu_n(\overline{\mathbb{Q}}_\ell)$ are both the same as the $\mu_n(\overline{\mathbb{Q}}_\ell)$ -action on L' coming from the $\mu_n(\overline{\mathbb{Q}}_\ell)$ -torsor structure. Therefore $\ker(\pi)$ is central in L' . \square

Let $\text{Cov}(L)$ be the category consisting of central isogenies $\pi : L' \rightarrow L$ of *connected* k -algebraic groups with $\ker(\pi)$ discrete as a k -scheme. The formal (inverse) limit of all objects in $\text{Cov}(L)$ gives the universal central isogeny $\pi^{\text{univ}} : L^{\text{univ}} \rightarrow L$. The kernel $\ker(\pi^{\text{univ}})$ is a pro-finite abelian group, and we denote it by $\pi_1^{\text{Serre}}(L)$. When L is commutative, this is the same as the π_1 defined by Serre in [36, §6.1, Definition 1; §6.2, Proposition 3]. Since each object in $\text{Cov}(L)$ is also a finite étale cover of L , we have a surjection $\pi_1(L)^{\text{ab}} \twoheadrightarrow \pi_1^{\text{Serre}}(L)$. To emphasize the dependence on the base field we write $\text{Cov}(L/k)$ and $\pi_1^{\text{Serre}}(L/k)$.

When $k \hookrightarrow k'$ is a field extension, we have the base change functor $\text{Cov}(L/k) \rightarrow \text{Cov}(L/k')$ which is fully faithful. We have $\text{Cov}(L/\bar{k}) = \varinjlim_{k \subset k' \subset \bar{k}} \text{Cov}(L/k')$ and hence

$$\pi_1^{\text{Serre}}(L/\bar{k}) \cong \varprojlim_{k \subset k' \subset \bar{k}} \pi_1^{\text{Serre}}(L/k').$$

A.2.2. THEOREM. *Let k be any base field and let L be a connected algebraic group over k . Then there is a canonical isomorphism*

$$\text{CS}_1(L) \cong \text{Hom}_{\text{cont}}(\pi_1^{\text{Serre}}(L), \overline{\mathbb{Q}}_\ell^\times)$$

SKETCH OF PROOF. For those $\mathcal{K} \in \text{CS}_1(L)$ of finite order, we have defined an object $(\pi : L' \rightarrow L) \in \text{Cov}(L)$ together with an embedding $\ker(\pi) \hookrightarrow \overline{\mathbb{Q}}_\ell^\times$. This gives a homomorphism $\pi_1^{\text{Serre}}(L) \twoheadrightarrow \ker(\pi) \hookrightarrow \overline{\mathbb{Q}}_\ell^\times$. This construction clearly passes to the limit and defines a homomorphism $\xi : \text{CS}_1(L) \rightarrow \text{Hom}_{\text{cont}}(\pi_1^{\text{Serre}}(L), \overline{\mathbb{Q}}_\ell^\times)$. Conversely, suppose we are given a character $\chi : \pi_1^{\text{Serre}}(L) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ of finite order. The kernel of χ corresponds to a central isogeny $\pi : L' \rightarrow L$ such that $\ker(\pi) \cong \text{Im}(\chi)$. We then define $\mathcal{K}_\chi = (\pi_1 \overline{\mathbb{Q}}_\ell)_\chi$, the χ -isotypical component of $\pi_1 \overline{\mathbb{Q}}_\ell$. Since $\pi^* \mathcal{K}_\chi \cong \overline{\mathbb{Q}}_\ell$ is a

character sheaf, so is \mathcal{K}_χ . This construction $\chi \mapsto \mathcal{K}_\chi$ also passes to the limit to give a homomorphism $\mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{Serre}}(L), \overline{\mathbb{Q}}_\ell^\times) \rightarrow \mathrm{CS}_1(L)$. It is not hard to check that it is inverse to the homomorphism ξ . \square

A.3. Connected groups over a finite field. In this subsection we assume k is a finite field.

A.3.1. *Lang torsor.* For each $\mathcal{K} \in \mathcal{CS}_1(L)$, the sheaf-to-function correspondence gives a function $f_{\mathcal{K}} : L(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$. The isomorphisms $\mu : m^*\mathcal{K} \cong \mathcal{K} \boxtimes \mathcal{K}$ and $u : \overline{\mathbb{Q}}_\ell \cong e^*\mathcal{K}$ imply that $f_{\mathcal{K}}$ is a group homomorphism. This way we obtain a homomorphism

$$f_L : \mathrm{CS}_1(L) \rightarrow \mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times).$$

One the other hand, we have the Lang torsor

$$(A.4) \quad \begin{aligned} \phi_L : L &\rightarrow L \\ g &\mapsto \mathrm{Frob}_{L/k}(g)g^{-1} \end{aligned}$$

where $\mathrm{Frob}_{L/k} : L \rightarrow L$ over k ($\mathrm{Frob}_{L/k}^*$ raises functions on L to the $\#k^{\mathrm{th}}$ power). The morphism ϕ_L is a right $L(k)$ -torsor onto its image: $g \in L(k)$ acts on the source by right multiplication. When L is connected, ϕ_L is surjective and is a right $L(k)$ -torsor. When L is not connected, ϕ_L may not be surjective. The push-forward sheaf $\phi_{L,!}\overline{\mathbb{Q}}_\ell$ carries an action of $L(k)$. For each character $\chi : L(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$, let $\mathcal{K}_\chi = (\phi_{L,!}\overline{\mathbb{Q}}_\ell)_{\chi^{-1}}$ be the χ^{-1} -isotypical direct summand of $\phi_{L,!}\overline{\mathbb{Q}}_\ell$ (we can take the projector in $\overline{\mathbb{Q}}_\ell[L(k)]$ corresponding to χ^{-1} , and apply it to $\phi_{L,!}\overline{\mathbb{Q}}_\ell$ and take the image). When L is connected, we get a homomorphism

$$(A.5) \quad \lambda_L : \mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times) \rightarrow \mathrm{Loc}_1(L)$$

where $\mathrm{Loc}_1(L)$ is the group of isomorphism classes of rank one $\overline{\mathbb{Q}}_\ell$ -local systems on L , with the group structure given by the tensor product.

A.3.2. LEMMA. *Let L be a connected algebraic group over a finite field k . Let f'_L denote the sheaf-to-function correspondence $\mathrm{Loc}_1(L) \rightarrow \mathrm{Fun}(L(k))$. Then the composition $f'_L \lambda_L : \mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times) \rightarrow \mathrm{Loc}_1(L) \rightarrow \mathrm{Fun}(L(k))$ is the natural inclusion of $\mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times)$ into $\mathrm{Fun}(L(k))$.*

PROOF. Let $\chi : L(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a character. We shall calculate the trace of the geometric Frobenius Frob_g acting on the stalk of $\mathcal{K}_\chi = (\phi_{L,!}\overline{\mathbb{Q}}_\ell)_{\chi^{-1}}$ at $g \in L(k)$. Let $x \in L(\overline{k})$ be such that $\mathrm{Frob}_{L/k}(x)x^{-1} = g$. The preimage $\phi_L^{-1}(g)$ consists of points xa where $a \in L(k)$. Since $\mathrm{Frob}_{L/k}(x) = gx$ is also in $\phi_L^{-1}(g)$, we have $gx = xa_0$ for some $a_0 \in L(k)$. The arithmetic Frobenius action on $\phi_L^{-1}(g)$ is given by $xa \mapsto \mathrm{Frob}_{L/k}(xa) = gxa = xa_0a$. We may identify the stalk of $\phi_{L,!}\overline{\mathbb{Q}}_\ell$ at g with $\mathrm{Fun}(\phi_L^{-1}(g))$. Then the stalk of $(\phi_{L,!}\overline{\mathbb{Q}}_\ell)_{\chi^{-1}}$ at g is spanned by the function $e_g : xa \mapsto \chi(a)$. The geometric Frobenius Frob_g on $\mathrm{Fun}(\phi_L^{-1}(g))$ sends e_g to the function $xa \mapsto e_g(\mathrm{Frob}_{L/k}(xa)) =$

$\chi(a_0a) = \chi(a_0)e_g(xa)$. Hence $\text{Tr}(\text{Frob}_g, \mathcal{K}_\chi) = \chi(a_0)$. It remains to show that $\chi(a_0) = \chi(g)$. Note that a_0 and g are conjugate in $L(\bar{k})$ via x ; we will show that a_0 and g are actually conjugate in $L(k)$, and hence have the same value under χ . As usual, a_0 determines a class $[a_0] \in H^1(k, Z_L(g))$ which sends the Frobenius element to $\text{Frob}(x)x^{-1} = g$. Since g lies in the neutral component $Z_L(g)^\circ$ of the centralizer $Z_L(g)$, the class $[a_0]$ in fact comes from a class in $H^1(k, Z_L(g)^\circ)$, which is trivial by Lang's theorem. Therefore a_0 and g are conjugate in $L(k)$ and $\chi(a_0) = \chi(g)$. \square

Next we consider the case where L is commutative and connected.

A.3.3. THEOREM. *Let L be a connected commutative algebraic group over k . Then f_L is an isomorphism of abelian groups*

$$\text{CS}_1(L) \xrightarrow{\sim} \text{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times).$$

PROOF. When L is commutative, the map λ_L in (A.5) has image in $\text{CS}_1(L)$. This follows from the fact the Lang isogeny ϕ_L is a group homomorphism when L is commutative. Thus we have a pair of homomorphisms

$$f_L : \text{CS}_1(L) \rightarrow \text{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times) \text{ and } \lambda_L : \text{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times) \rightarrow \text{CS}_1(L).$$

By Lemma A.3.2, we have $f_L\lambda_L = \text{id}$. Therefore it suffices to show that f_L is injective to conclude that f_L and λ_L are inverse to each other. Let $\mathcal{K} \in \text{CS}_1(L)$. Let $g \in L$ be a closed point with residue field k' . Define the norm $h = \text{Nm}(g) := \prod_{\sigma \in \text{Gal}(k'/k)} \sigma(g) \in L(k)$. We claim that

$$(A.6) \quad \text{Tr}(\text{Frob}_g, \mathcal{K}_g) = \text{Tr}(\text{Frob}_h, \mathcal{K}_h)$$

Once this is proved, the Frobenius trace of \mathcal{K} at any closed point is determined by the function $f_L(\mathcal{K})$, and hence \mathcal{K} is also determined by $f_L(\mathcal{K})$ by the Chebotarev density theorem. Now we prove (A.6). Let $d = [k' : k]$, then g determines a k -point $g_0 \in \text{Sym}^d(L)(k)$. Since L is commutative, the d -fold multiplication map factors as

$$L^d \xrightarrow{s_d} \text{Sym}^d(L) \xrightarrow{m_d} L.$$

By the definition of character sheaves we have $s_d^*m_d^*\mathcal{K} \cong \mathcal{K}^{\boxtimes d}$. By adjunction this gives a nonzero map $\alpha : m_d^*\mathcal{K} \rightarrow (s_{d,*}\mathcal{K}^{\boxtimes d})^{S_d} =: \mathcal{K}^{(d)}$. It is easy to see that $\mathcal{K}^{(d)}$ is also a rank one local system, therefore α has to be an isomorphism. We then compute the stalk of $m_d^*\mathcal{K}$ and $\mathcal{K}^{(d)}$ at g_0 . On one hand, $(m_d^*\mathcal{K})_{g_0} = \mathcal{K}_{m_d(g_0)} = \mathcal{K}_h$ hence $\text{Tr}(\text{Frob}_{g_0}, (m_d^*\mathcal{K})_{g_0}) = \text{Tr}(\text{Frob}_h, \mathcal{K}_h)$. On the other hand, the stalk of $\mathcal{K}^{(d)}$ at g_0 is $\otimes_{i=0}^{d-1} \mathcal{K}_{\text{Frob}^i(g)}$. The Frobenius equivariance structure of \mathcal{K} gives isomorphisms $\mathcal{K}_g \xrightarrow{\iota} \mathcal{K}_{\text{Frob}(g)} \xrightarrow{\iota} \dots \xrightarrow{\iota} \mathcal{K}_{\text{Frob}^{d-1}(g)} \xrightarrow{\iota} \mathcal{K}_g$, and the iteration ι^d of these isomorphisms give the automorphism Frob_g on \mathcal{K}_g . Let $v \in \mathcal{K}_g$ be a basis, then $v \otimes \iota(v) \dots \otimes \iota^{d-1}(v)$ is a basis of $\mathcal{K}_{g_0}^{(d)}$. The action of Frob_{g_0} cyclically permuting the tensor factors, and sends $v \otimes \iota(v) \dots \otimes \iota^{d-1}(v)$ to $\iota^d(v) \otimes \iota(v) \otimes \dots \otimes \iota^{d-1}(v) =$

$\text{Frob}_g(v) \otimes \iota(v) \otimes \cdots \otimes \iota^{d-1}(v)$. Therefore $\text{Tr}(\text{Frob}_{g_0}, \mathcal{K}_{g_0}^{(d)}) = \text{Tr}(\text{Frob}_g, \mathcal{K}_g)$. Combining these calculations we get (A.6). The theorem is proved. \square

A.3.4. COROLLARY. *We have isomorphisms*

$$(A.7) \quad \pi_1^{\text{Serre}}(L/k) \cong L(k),$$

$$(A.8) \quad \pi_1^{\text{Serre}}(L/\bar{k}) \cong \varprojlim_{k \subset k' \subset \bar{k}} L(k')$$

where the projective system is over finite extensions k'/k and the transition maps are the norm maps $\text{Nm}_{k''/k'} : L(k'') \rightarrow L(k')$.

PROOF. (A.7) follows directly from Theorem A.3.3 and Theorem A.2.2. When passing to the limit, we need to compute the transition maps $\nu_{k''/k'} : \pi_1^{\text{Serre}}(L/k'') \rightarrow \pi_1^{\text{Serre}}(L/k')$ for $k' \subset k''$ in terms of $L(k')$ and $L(k'')$. By definition, $\nu_{k''/k'}$ is the one induced from the pullback map $\text{CS}_1(L/k') \rightarrow \text{CS}_1(L/k'')$. During the proof of Theorem A.3.3, the formula (A.6) can be reformulated as a commutative diagram

$$\begin{array}{ccc} \text{CS}_1(L/k') & \xrightarrow{f_{L,k'}} & \text{Hom}(L(k'), \overline{\mathbb{Q}}_\ell^\times) \\ \downarrow \text{pullback} & & \downarrow \text{Nm}_{k''/k'}^* \\ \text{CS}_1(L/k'') & \xrightarrow{f_{L,k''}} & \text{Hom}(L(k''), \overline{\mathbb{Q}}_\ell^\times) \end{array}$$

This implies that $\nu_{k''/k'}$ is given by $\text{Nm}_{k''/k'} : L(k'') \rightarrow L(k')$. This proves completes the proof. \square

A.3.5. EXAMPLE. (Kummer sheaves) Let $L = \mathbb{G}_m$ be the one-dimensional torus over k . For each character $\chi : L(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ we get an object $\mathcal{K}_\chi \in \mathcal{CS}_1(\mathbb{G}_m)$. We claim that the canonical surjection $h : \pi_1(\mathbb{G}_m/\bar{k}) \twoheadrightarrow \pi_1^{\text{Serre}}(L/\bar{k})$ induces an isomorphism

$$(A.9) \quad \pi_1^t(\mathbb{G}_m/\bar{k}) \xrightarrow{\sim} \pi_1^{\text{Serre}}(\mathbb{G}_m/\bar{k})$$

where $\pi_1^t(-)$ denotes the tame fundamental group. On one hand, $\pi_1^{\text{Serre}}(L/\bar{k}) \cong \varprojlim_{k' \subset \bar{k}} k'^\times$ is an inverse limit of finite groups of order prime to p . Therefore $\pi_1(\mathbb{G}_m/\bar{k}) \twoheadrightarrow \pi_1^{\text{Serre}}(L/\bar{k})$ factors through the tame quotient. On the other hand, $\pi_1^t(\mathbb{G}_m/\bar{k}) \cong \widehat{\mathbb{Z}}'(1) := \varprojlim_{(m,p)=1} \mu_m(\bar{k})$. Therefore h induces a homomorphism

$$(A.10) \quad \varprojlim_{(m,p)=1} \mu_m(\bar{k}) \rightarrow \varprojlim_{k' \subset \bar{k}} k'^\times.$$

What is this homomorphism? This is almost the tautological one: we may replace the limit on the left side by $\varprojlim_n \mu_{q^n-1}(\bar{k})$ (where $q = \#k$) because every integer prime to p is divisible by some $q^n - 1$. Clearly $\mu_{q^n-1}(\bar{k}) = \mathbb{F}_{q^n}$, and the norm map $\mathbb{F}_{q^{mn}}^\times \rightarrow \mathbb{F}_{q^n}^\times$ is the same as the power map $[\frac{q^{mn}-1}{q^n-1}] : \mu_{q^{mn}-1}(\bar{k}) \rightarrow \mu_{q^n-1}(\bar{k})$. Therefore the two sides of (A.10) are canonically

isomorphic, and one can check that (A.10) is this canonical isomorphism. This proves (A.9).

The above discussion can easily be generalized to any torus T over k . We define the (prime-to- p -part of the) Tate module of T by

$$\mathbb{T}'(T) := \varprojlim_{(n,p)=1} T[n](\bar{k})$$

as a $\text{Gal}(\bar{k}/k)$ -module. Note that when $T = \mathbb{G}_m$, $\mathbb{T}'(\mathbb{G}_m) = \widehat{\mathbb{Z}}'(1)$. In general we have $\mathbb{T}'(T) = \mathbb{X}_*(T) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}'(1)$ with $\text{Gal}(\bar{k}/k)$ diagonally acting on both factors. Then (A.9) implies a canonical isomorphisms of $\text{Gal}(\bar{k}/k)$ -modules

$$(A.11) \quad \pi_1^t(T/\bar{k}) \cong \mathbb{T}'(T) \cong \varprojlim_{k' \subset \bar{k}} T(k') \cong \pi_1^{\text{Serre}}(T/\bar{k}).$$

In conclusion, any tame local system of rank one on T is a character sheaf. We call these sheaves *Kummer sheaves*.

A.3.6. EXAMPLE. (Artin-Schreier sheaves) When $L = \mathbb{G}_a$ is the additive group over k , for each additive character $\psi : k \rightarrow \overline{\mathbb{Q}}_\ell^\times$ we have a rank one character sheaf which we denote by AS_ψ , the *Artin-Schreier* sheaf. More generally, if V is a vector space over k , viewed as a commutative group scheme, then for every linear function $\phi \in V^\vee$, viewed as a group homomorphism $\phi : V \rightarrow \mathbb{G}_a$, the pullback $\phi^* \text{AS}_\psi$ gives an object in $\mathcal{CS}_1(V)$. Fixing ψ , every object in $\mathcal{CS}_1(V)$ arises this way from a unique $\phi \in V^\vee$.

A.3.7. EXAMPLE. (A non-commutative pathology) When L is not necessarily commutative, the image of the map λ_L may not lie in $\text{CS}_1(L)$. For example, take $k = \mathbb{F}_2$ and $L = \text{SL}_2$. Then $\text{SL}_2(\mathbb{F}_2) \cong S_3$ has a unique character χ of order two. Let $\mathbb{G}_a \subset \text{SL}_2$ denote the upper triangular unipotent matrices. The order two local system $\mathcal{K}_\chi|_{\mathbb{G}_a}$ is the Artin-Schreier sheaf AS_ψ , for the unique nontrivial character ψ of \mathbb{F}_2 . However, when we make $t = \text{diag}(t, t^{-1}) \in \text{SL}_2(\bar{k})$ act on \mathbb{G}_a/\bar{k} by conjugation, the pullback $\text{Ad}(t)^* \mathcal{K}_\chi|_{\mathbb{G}_a}$ (over \mathbb{G}_a/\bar{k}) becomes $[t^2]^* \text{AS}_\psi$ (where $[t^2] : \mathbb{G}_a \rightarrow \mathbb{G}_a$ is multiplication by t^2), which is not isomorphic to AS_ψ (by looking at the functions they define). But if $m^* \mathcal{K}_\chi \cong \mathcal{K}_\chi \boxtimes \mathcal{K}_\chi$, the isomorphism type of \mathcal{K}_χ would be unchanged under the left and right translation of SL_2 on itself, and in particular invariant under the adjoint action of SL_2 . Therefore, $m^* \mathcal{K}_\chi$ is not isomorphic to $\mathcal{K}_\chi \boxtimes \mathcal{K}_\chi$ in this case. In what follows we will see how to remedy this pathology.

Now let L be a connected reductive group over a finite field k . Let L^{sc} be the simply-connected cover of the derived group L^{der} . We have a natural homomorphism $L^{\text{sc}}(k) \rightarrow L(k)$ whose image is a normal subgroup of $L(k)$. We may therefore form the quotient $L(k)/L^{\text{sc}}(k)$ (really quotienting by the image), which is abelian.

A.3.8. THEOREM. *Let L be a connected reductive group over k .*

(1) We have a natural isomorphism of abelian groups

$$(A.12) \quad \text{CS}_1(L) \xrightarrow{\sim} \text{Hom}(L(k)/L^{\text{sc}}(k), \overline{\mathbb{Q}}_\ell^\times).$$

(2) Let T be a maximal torus in L and $T^{\text{sc}} \subset L^{\text{sc}}$ be its preimage in L^{sc} . Then we have isomorphisms

$$(A.13) \quad \pi_1^{\text{Serre}}(L/k) \cong T(k)/T^{\text{sc}}(k) \cong L(k)/L^{\text{sc}}(k),$$

$$(A.14) \quad \pi_1^{\text{Serre}}(L/\bar{k}) \cong \mathbb{T}'(T)/\mathbb{T}'(T^{\text{sc}}).$$

PROOF. First suppose $L = L^{\text{sc}}$. In this case, any central isogeny to L is trivial. Hence $\pi_1^{\text{Serre}}(L/k')$ and $\text{CS}_1(L/k')$ are trivial for any base field k' .

Next consider the general case. Suppose $\mathcal{K} \in \mathcal{CS}_1(L)$. Then its pullback to L^{sc} has to be the trivial local system by the above discussion. Therefore f_L has image in $\text{Hom}(L(k)/L^{\text{sc}}(k), \overline{\mathbb{Q}}_\ell^\times)$.

The inverse map $\text{Hom}(L(k)/L^{\text{sc}}(k), \overline{\mathbb{Q}}_\ell^\times) \rightarrow \text{CS}_1(L)$ is again given by (the restriction of) λ_L . We need to show that if $\chi : L(k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is trivial on $L^{\text{sc}}(k)$, then the corresponding local system \mathcal{K}_χ is a character sheaf. Consider the intermediate $L(k)/L^{\text{sc}}(k)$ -torsor $\phi'_L : L' \rightarrow L$ where $L' = L/\text{Im}(L^{\text{sc}}(k) \rightarrow L(k))$. For $\chi \in \text{Hom}(L(k)/L^{\text{sc}}(k), \overline{\mathbb{Q}}_\ell^\times)$, the local system \mathcal{K}_χ is the corresponding direct summand of $\phi'_{L,*}\overline{\mathbb{Q}}_\ell$. The natural homomorphism $L^{\text{sc}} \rightarrow L$ admits a lift $L^{\text{sc}} \rightarrow L'$, therefore \mathcal{K}_χ is trivial when pulled back to L^{sc} . Also it is easy to see that \mathcal{K}_χ is a Kummer sheaf when restricted to the neutral component of the center Z° , using Theorem A.3.3. Consider the isogeny $Z^\circ \times L^{\text{sc}} \rightarrow L$. In general, whenever we have an isogeny $\pi : \tilde{L} \rightarrow L$, if $\pi^*\mathcal{F}$ is a character sheaf, so is \mathcal{F} (using Remark A.1.2(2)). Therefore $\mathcal{K}_\chi \in \mathcal{CS}_1(L)$. Checking that the two maps between $\text{CS}_1(L)$ and $\text{Hom}(L(k)/L^{\text{sc}}(k), \overline{\mathbb{Q}}_\ell^\times)$ are inverse to each other is left to reader.

The isomorphism (A.12) together with Theorem A.2.2 implies that $\pi_1^{\text{Serre}}(L) \cong L(k)/L^{\text{sc}}(k)$. The central isogeny $Z^\circ \times L^{\text{sc}} \rightarrow L$ restricts to an isogeny of tori $\varphi : Z^\circ \times T^{\text{sc}} \rightarrow T$. Let $A = \ker(\varphi)$, which is a finite group scheme over k of multiplicative type (but may not be discrete). We get a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1^{\text{Serre}}(Z^\circ \times T^{\text{sc}}) & \longrightarrow & \pi_1^{\text{Serre}}(T) & \longrightarrow & A(k) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \pi_1^{\text{Serre}}(Z^\circ \times L^{\text{sc}}) & \longrightarrow & \pi_1^{\text{Serre}}(L) & \longrightarrow & A(k) \longrightarrow 1 \end{array}$$

Note that $\pi_1^{\text{Serre}}(Z^\circ \times L^{\text{sc}}) \cong \pi_1^{\text{Serre}}(Z^\circ)$ since L^{sc} does not admit nontrivial central isogenies. Also $\pi_1^{\text{Serre}}(Z^\circ \times T^{\text{sc}}) \cong \pi_1^{\text{Serre}}(Z^\circ) \times \pi_1^{\text{Serre}}(T^{\text{sc}})$ by the discussion in Example A.3.5. Therefore from the above diagram we conclude that

$$\pi_1^{\text{Serre}}(L) \cong \pi_1^{\text{Serre}}(T)/\pi_1^{\text{Serre}}(T^{\text{sc}}).$$

(A.13) then follows. Using $\mathbb{T}'(T) \cong \varprojlim_{k' \subset \bar{k}} T(k')$ and passing to the limit we get (A.14). □

A.3.9. THEOREM. *Let L be a connected algebraic group over a finite field k .*

- (1) *The homomorphism $f_L : \mathrm{CS}_1(L) \rightarrow \mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times)$ is injective.*
- (2) *There are surjective homomorphisms*

$$\pi_1(L)^{\mathrm{ab}} \twoheadrightarrow L(k)^{\mathrm{ab}} \twoheadrightarrow \pi_1^{\mathrm{Serre}}(L).$$

PROOF. (1) Suppose $1 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 1$ is an exact sequence of connected algebraic groups over k , and that $f_{L'}$ and $f_{L''}$ are known to be injective, we shall show that f_L is also injective. Let $\mathcal{K} \in \mathrm{CS}_1(L)$ be such that $f_L(\mathcal{K}) = 1$. Then $\mathcal{K}|_{L'} \in \mathrm{CS}_1(L')$ also lies in the kernel of $f_{L'}$, hence $\mathcal{K}|_{L'}$ is the constant sheaf by assumption. By Lemma A.1.4, \mathcal{K} descends to $\mathcal{K}'' \in \mathrm{CS}_1(L'')$. The fact that $f_L(\mathcal{K}) = 1$ implies that $f_{L''}(\mathcal{K}'') = 1$, hence \mathcal{K}'' is the constant sheaf by assumption. This shows that \mathcal{K} is also the constant sheaf.

In Theorem A.3.3 we have shown that f_L is an isomorphism when L is connected and commutative; in Theorem A.3.8 we have shown that f_L is injective if L is connected reductive. Since every connected algebraic group L admits a filtration by normal subgroups with associated graded either connected commutative or connected reductive, we conclude that f_L is injective for all connected L .

(2) The surjection $\pi_1(L) \twoheadrightarrow L(k)$ is given by the Lang torsor; the surjection $L(k)^{\mathrm{ab}} \twoheadrightarrow \pi_1^{\mathrm{Serre}}(L)$ comes from the injection $f_L : \mathrm{CS}_1(L) \cong \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{Serre}}(L), \overline{\mathbb{Q}}_\ell^\times) \hookrightarrow \mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times)$. \square

A.4. Case of a finite group scheme. Let k be any perfect field, and A be a finite group scheme over k .

A.4.1. LEMMA. (1) *There is a canonical isomorphism*

$$\mathrm{CS}_1(A/\bar{k}) \cong \mathrm{H}^2(A(\bar{k}), \overline{\mathbb{Q}}_\ell^\times).$$

(2) *There is an exact sequence*

$$(A.15) \quad 1 \rightarrow \mathrm{H}^1(k, \mathrm{Hom}(A(\bar{k}), \overline{\mathbb{Q}}_\ell^\times)) \rightarrow \mathrm{CS}_1(A) \rightarrow \mathrm{H}^2(A(\bar{k}), \overline{\mathbb{Q}}_\ell^\times)^{\mathrm{Gal}(\bar{k}/k)}.$$

PROOF. (1) We temporarily assume $k = \bar{k}$. We may assume A is reduced (passing to A^{red} does not change the category of rank one character sheaves), and hence identify A with the discrete group $A(\bar{k})$. A rank one character sheaf on A is a collection of 1-dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces \mathcal{K}_a , one for each $a \in A$. The datum of μ is a collection of linear isomorphism $\mu_{a,b} : \mathcal{K}_{ab} \cong \mathcal{K}_a \otimes \mathcal{K}_b$, one for each pair $(a, b) \in A^2$, that satisfy the cocycle relation

$$(A.16) \quad \mu_{ab,c} \circ (\mu_{a,b} \otimes \mathrm{id}_{\mathcal{K}_c}) = \mu_{a,bc} \circ (\mathrm{id}_{\mathcal{K}_a} \otimes \mu_{b,c}), \forall a, b, c \in A.$$

The datum of u gives a basis $e_1 \in \mathcal{K}_1$ such that $\mu_{a,1} = \mathrm{id}_{\mathcal{K}_a} \otimes e_1$ and $\mu_{1,a} = e_1 \otimes \mathrm{id}_{\mathcal{K}_a}$ for all $a \in A$. If we choose a basis $e_a \in \mathcal{K}_a$ for every a (keep the same choice for e_1 given by u), then $\mu_{a,b}(e_{ab}) = \xi_{a,b} e_a \otimes e_b$ for some $\xi_{a,b} \in \overline{\mathbb{Q}}_\ell^\times$. The cocycle relation (A.16) implies that $\{\xi_{a,b}\}$ is a 2-cocycle of

A with values in $\overline{\mathbb{Q}}_\ell^\times$ satisfying $\xi_{1,a} = \xi_{a,1} = 1$. Changing the choices of $\{e_a\}$ changes $\xi_{a,b}$ by a coboundary of a map $\eta : A \rightarrow \overline{\mathbb{Q}}_\ell^\times$ with $\eta_1 = 1$. Therefore we have a well-defined homomorphism $\alpha : \text{CS}_1(A/\overline{k}) \rightarrow \text{H}^2(A, \overline{\mathbb{Q}}_\ell^\times)$.

Conversely, from a cocycle $\xi \in Z^1(A, \overline{\mathbb{Q}}_\ell^\times)$, we may first multiply it by a coboundary to make $\xi_{a,1} = \xi_{1,a} = 1$ for all $a \in A$. We then define \mathcal{K} to be the constant sheaf on A , and define the map $\mu_{a,b} : \overline{\mathbb{Q}}_\ell^\times = \mathcal{K}_{ab} \rightarrow \mathcal{K}_a \otimes \mathcal{K}_b = \overline{\mathbb{Q}}_\ell^\times$ as multiplication by $\xi_{a,b}$. We also define $u : \overline{\mathbb{Q}}_\ell \cong \mathcal{K}_1$ as the identity map. The fact that ξ is a cocycle guarantees that (\mathcal{K}, μ, u) thus defined is a rank one character sheaf on A . Changing ξ by the coboundary of $\eta : A \rightarrow \overline{\mathbb{Q}}_\ell^\times$ with $\eta_1 = 1$ does not change the isomorphism type of (\mathcal{K}, μ, u) . This way we have defined a homomorphism $\beta : \text{H}^2(A, \overline{\mathbb{Q}}_\ell^\times) \rightarrow \text{CS}_1(A/\overline{k})$. It is easy to check that α and β are inverse to each other.

(2) We have an equivalence of categories $\text{CS}_1(A) \cong \text{CS}_1(A/\overline{k})^{\text{Gal}(\overline{k}/k)}$, the latter being the category of objects in $\text{CS}_1(A/\overline{k})^{\text{Gal}(\overline{k}/k)}$ together with a $\text{Gal}(\overline{k}/k)$ -equivariant structure. Since $\text{CS}_1(A/\overline{k})$ is a Picard category with the group of isomorphism classes of objects $\text{CS}_1(A/\overline{k}) \cong \text{H}^2(A/\overline{k}, \overline{\mathbb{Q}}_\ell^\times)$ by (1), and automorphism group $\text{Hom}(A/\overline{k}, \overline{\mathbb{Q}}_\ell^\times)$ by Remark A.1.2(3), the exact sequence (A.15) follows. \square

A.4.2. Equivariant sheaves. Let L be an algebraic group acting on a scheme X of finite type. Let $\mathcal{K} \in \text{CS}_1(L)$. Let $a : L \times X \rightarrow X$ be the action map. Then an (L, \mathcal{K}) -equivariant perverse sheaf \mathcal{F} on X is a pair (\mathcal{F}, α) where \mathcal{F} is a perverse sheaf on X and α is an isomorphism on $L \times X$

$$\alpha : a^* \mathcal{F} \cong \mathcal{K} \boxtimes \mathcal{F}$$

that restricts to $u \boxtimes \text{id}_{\mathcal{F}}$ on $\{e\} \times X$ and such that the composition

$$\begin{aligned} m^* \mathcal{K} \boxtimes \mathcal{F} &= (m \times \text{id}_X)^*(\mathcal{K} \boxtimes \mathcal{F}) \xrightarrow{(m \times \text{id}_X)^* \alpha^{-1}} (m \times \text{id}_X)^* a^* \mathcal{F} \\ &= (\text{id}_L \times a)^* a^* \mathcal{F} \xrightarrow{(\text{id}_L \times a)^* \alpha} (\text{id}_L \times a)^*(\mathcal{K} \boxtimes \mathcal{F}) \\ &= \mathcal{K} \boxtimes a^* \mathcal{F} \xrightarrow{\text{id} \boxtimes \alpha} \mathcal{K} \boxtimes \mathcal{K} \boxtimes \mathcal{F} \end{aligned}$$

is equal to $\mu \boxtimes \text{id}_{\mathcal{F}}$.

In the main body of the paper we need a generalization of the notion of (L, \mathcal{K}) -equivariant perverse sheaves. One can define, using the theory of ℓ -adic sheaves on Artin stacks as developed by Laszlo and Olsson [27] or Liu and Zheng [28], a derived category $D_{(L, \mathcal{K})}^b(X)$ of (L, \mathcal{K}) -equivariant $\overline{\mathbb{Q}}_\ell$ -complexes on X . This construction works also when X is an Artin stack of finite type.

A.4.3. Twisted representations. To describe equivariant sheaves on a homogeneous variety, we need the notion of twisted representations of a group which we now recall. Let Γ be a group and $\xi \in Z^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$ be a cocycle such that $\xi_{1,1} = 1$. A ξ -twisted representation of Γ is a finite-dimensional

$\overline{\mathbb{Q}}_\ell$ -vector space V with automorphisms $T_\gamma : V \rightarrow V$, one for each $\gamma \in \Gamma$, such that $T_1 = \text{id}_V$ and

$$(A.17) \quad T_{\gamma\delta} = \xi_{\gamma,\delta} T_\gamma T_\delta, \forall \gamma, \delta \in \Gamma.$$

A ξ -twisted representation of Γ on V gives a projective representation $\overline{T} : \Gamma \rightarrow \text{PGL}(V)$ whose image under the connecting homomorphism $\text{H}^1(\Gamma, \text{PGL}(V)) \rightarrow \text{H}^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$ (associated with $1 \rightarrow \overline{\mathbb{Q}}_\ell^\times \rightarrow \text{GL}(V) \rightarrow \text{PGL}(V) \rightarrow 1$) is the class of ξ .

Let $\text{Rep}_\xi(\Gamma)$ be the category of finite-dimensional ξ -twisted representations of Γ . It is an $\overline{\mathbb{Q}}_\ell$ -linear abelian category. If ξ' is another such cocycle in the same cohomology class as ξ , then $\xi' = \xi \cdot d\eta$ for some map $\eta : \Gamma \rightarrow \overline{\mathbb{Q}}_\ell^\times$ with $\eta_1 = 1$, and η induces an equivalence $\text{Rep}_\xi(\Gamma) \xrightarrow{\sim} \text{Rep}_{\xi'}(\Gamma)$ sending $(V, \{T_\gamma\}_{\gamma \in \Gamma})$ to $(V, \{\eta_\gamma T_\gamma\}_{\gamma \in \Gamma})$. Therefore the category $\text{Rep}_\xi(\Gamma)$ up to equivalence only depends on the cohomology class $[\xi] \in \text{H}^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$.

We consider the situation where X is a homogeneous L -scheme.

A.4.4. LEMMA. *Let $k = \bar{k}$. Let X be a scheme over k with a transitive action of an algebraic group L , and let $\mathcal{K} \in \mathcal{CS}_1(L)$. Fix a point $x \in X(k)$ and let L_x be its stabilizer with neutral component L_x° .*

- (1) *If $\mathcal{K}|_{L_x^\circ}$ is not isomorphic to the constant sheaf, then $\text{Loc}_{(L,\mathcal{K})}(X)$ consists only of the zero object.*
- (2) *If $\mathcal{K}|_{L_x^\circ}$ is isomorphic to the constant sheaf, then $\mathcal{K}|_{L_x}$ defines a class $\xi \in \text{H}^2(\pi_0(L_x), \overline{\mathbb{Q}}_\ell^\times)$, such that there is an equivalence of categories*

$$\text{Loc}_{(L,\mathcal{K})}(X) \cong \text{Rep}_\xi(\pi_0(L_x)).$$

PROOF. (1) Restricting an object $\mathcal{F} \in \text{Loc}_{(L,\mathcal{K})}(X)$ to x we get an $(L_x, \mathcal{K}|_{L_x})$ -equivariant local system \mathcal{F}_x over the point x . The action map of L_x on x becomes the structure map $L_x \rightarrow \text{Spec } k$. The equivariance condition gives an isomorphism between the constant sheaf $\mathcal{F}_x \otimes \overline{\mathbb{Q}}_\ell$ on L_x and $\mathcal{F}_x \otimes \mathcal{K}|_{L_x}$. In particular, if $\mathcal{F}_x \neq 0$, $\mathcal{K}|_{L_x}$ must be the constant sheaf, and therefore $\mathcal{K}|_{L_x^\circ}$ is also isomorphic to the constant sheaf.

(2) Since $\mathcal{K}|_{L_x^\circ}$ is trivial, $\mathcal{K}|_{L_x}$ descends to a rank one character sheaf $\overline{\mathcal{K}}$ on $\pi_0(L_x)$, hence giving a class $\xi \in \text{H}^2(\pi_0(L_x), \overline{\mathbb{Q}}_\ell^\times)$ by Lemma A.4.1. Restricting a local system to x gives an equivalence of categories $\text{Loc}_{(L,\mathcal{K})}(X) \cong \text{Loc}_{(\pi_0(L_x), \overline{\mathcal{K}})}(\text{pt})$. Therefore it suffices to show that $\text{Loc}_{(\Gamma,\mathcal{K})}(\text{pt}) \cong \text{Rep}_\xi(\Gamma)$ for any finite group Γ (viewed as a discrete group scheme over $k = \bar{k}$) and any rank one character sheaf $\mathcal{K} \in \mathcal{CS}_1(\Gamma)$ giving the class $\xi \in \text{H}^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$. Given $\mathcal{F} \in \text{Loc}_{(\Gamma,\mathcal{K})}(\text{pt})$, its stalk is a vector space V equipped with isomorphisms $\varphi_\gamma : \mathcal{K}_\gamma \otimes V \rightarrow V$, one for each $\gamma \in \Gamma$, satisfying the associativity condition and $\varphi_1 = \text{id}$. Choosing a trivialization of \mathcal{K}_γ , the cocycle ξ is defined by the recipe given in Lemma A.4.1(1), and φ_γ is identified with an automorphism T_γ of V . The associativity of $\{\varphi_\gamma\}$ becomes the property (A.17) for $\{T_\gamma\}$; i.e., the $\{T_\gamma\}$ give a ξ -twisted representation of Γ on V . Therefore $\text{Loc}_{(\Gamma,\mathcal{K})}(\text{pt}) \cong \text{Rep}_\xi(\Gamma)$. \square

References

- [1] Beilinson, A.; Bernstein, J.; Deligne, P. Faisceaux pervers. In *Analysis and topology on singular spaces, I*, 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.
- [2] Borel, A.; Springer, T. Rationality properties of linear algebraic groups. II. *Tôhoku Math. J.* 20, 443–497 (1968)
- [3] Borel, A.; Jacquet, H. Automorphic forms and automorphic representations. With a supplement “On the notion of an automorphic representation” by R. P. Langlands. *Proc. Sympos. Pure Math.*, XXXIII, Automorphic forms, representations and L -functions, Part 1, pp. 189–207, Amer. Math. Soc., Providence, R.I., 1979.
- [4] Braverman, A.; Gaitsgory, D. Geometric Eisenstein series. *Invent. math.* 150 (2002), 287–384.
- [5] Deligne, P. Applications de la formule des traces aux sommes trigonométriques. In *Cohomologie étale*. SGA 4 $\frac{1}{2}$. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin-New York, 1977.
- [6] Deligne, P. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.* No. 52 (1980), 137–252.
- [7] Deligne, P.; Milne, J. S. Tannakian categories. In *Hodge cycles, motives, and Shimura varieties*, 101–228. Lecture Notes in Mathematics, Vol. 900. Springer-Verlag, Berlin-New York, 1982.
- [8] Dettweiler, M.; Reiter, S. Rigid local systems and motives of type G_2 . With an appendix by M. Dettweiler and N. M. Katz. *Compositio Math.* 146 (2010), no. 4, 929–963.
- [9] Feit, W.; Fong, P. Rational rigidity of $G_2(p)$ for any prime $p > 5$. *Proceedings of the Rutgers group theory year, 1983–1984* (New Brunswick, N.J., 1983–1984), 323–326, Cambridge Univ. Press, Cambridge, 1985.
- [10] Frenkel, E.; Gross, B. A rigid irregular connection on the projective line. *Ann. of Math.* (2), 170 (2009), no. 3, 1469–1512.
- [11] Gaitsgory, D. On de Jong’s conjecture. *Israel J. Math.* 157 (2007), 155–191.
- [12] Ginzburg, V. Perverse sheaves on a Loop group and Langlands’ duality. arXiv:math/9511007.
- [13] Gross, B.; Reeder, M. Arithmetic invariants of discrete Langlands parameters. *Duke Math. J.* 154 (2010), no. 3, 431–508.
- [14] Gross, B. Irreducible cuspidal representations with prescribed local behavior. *Amer. J. Math.* 133 (2011), no. 5, 1231–1258.
- [15] Gross, B. Trivial L -functions for the rational function field. *J. Number Theory* 133 (2013), no. 3, 970–976.
- [16] Grothendieck, A. Revêtements étales et groupe fondamental. *Séminaire de Géométrie Algébrique, 1960/61*. Institut des Hautes Études Scientifiques, Paris 1963.
- [17] Guralnick, R.; Malle, G. Rational rigidity for $E_8(p)$. *Compositio Math.*, to appear. arXiv:1207.1464.
- [18] Heinloth, J. Uniformization of \mathcal{G} -bundles. *Math. Ann.* 347 (2010), 499–528
- [19] Heinloth, J.; Ngô, B.-C.; Yun, Z. Kloosterman sheaves for reductive groups. *Ann. of Math.* (2) 177 (2013), no. 1, 241–310.
- [20] Jouanolou, J.-P. Cohomologie ℓ -adique. Exposé VI in *Séminaire de Géométrie Algébrique du Bois-Marie 1965–66* (SGA 5), Lecture Notes in Mathematics, Vol. 589. Springer-Verlag, Berlin-New York, 1977.
- [21] Katz, N. M. Local-to-global extensions of representations of fundamental groups. *Ann. Inst. Fourier (Grenoble)* 36 (1986), no. 4, 69–106.
- [22] Katz, N. M. *Gauss sums, Kloosterman sums, and monodromy groups*. *Annals of Mathematics Studies*, 116. Princeton University Press, Princeton, NJ, 1988. x+246 pp.
- [23] Katz, N. M. *Exponential Sums and Differential Equations*. *Ann. of Math. Studies*, 124, Princeton Univ. Press, Princeton, NJ, 1990.

- [24] Katz, N. M. *Rigid local systems*. Annals of Mathematics Studies, 139. Princeton University Press, Princeton, NJ, 1996. viii+223 pp.
- [25] Kottwitz, R. Isocrystals with additional structure. II. *Compositio Math.* 109 (1997), no. 3, 255-339.
- [26] Lafforgue, V. Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale. arXiv:1209.5352.
- [27] Laszlo, Y.; Olsson, M. The six operations for sheaves on Artin stacks. I. Finite coefficients. II. Adic coefficients. *Publ. Math. Inst. Hautes Études Sci.* No. 107 (2008), 109-168, 169-210.
- [28] Liu, Y.; Zheng, W. Enhanced six operations and base change theorem for sheaves on Artin stacks. arXiv:1211.5948.
- [29] Lusztig, G. Singularities, character formulas, and a q -analog of weight multiplicities. *Analysis and topology on singular spaces, II, III* (Luminy, 1981), 208-229, *Astérisque*, 101-102, Soc. Math. France, Paris, 1983.
- [30] Malle, G.; Matzat, B. H. *Inverse Galois theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.
- [31] Matthews, C. R.; Vaserstein, L. N.; Weisfeiler, B. Congruence properties of Zariski-dense subgroups. I. *Proc. London Math. Soc.* (3) 48 (1984), no. 3, 514-532.
- [32] Milne, J. S. Shimura varieties and motives. In *Motives* (Seattle, WA, 1991), 447-523, Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., Providence, RI, 1994.
- [33] Mirković, I.; Vilonen, K. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math.* (2) 166 (2007), no. 1, 95-143.
- [34] Nori, M. On subgroups of $GL_n(\mathbb{F}_p)$. *Invent. Math.* 88 (1987), no. 2, 257-275.
- [35] Reeder, M.; Yu, J-K. Epipelagic representations and invariant theory. *J. Amer. Math. Soc.*, electronically published on August 5, 2013
- [36] Serre, J.P. Groupes proalgébriques. *Inst. Hautes Études Sci. Publ. Math.* No. 7, 5-67, 1960.
- [37] Serre, J.P. *Corps locaux*. (Deuxième édition) Publications de l'Université de Nancago, No. VIII. Hermann, Paris, 1968. 245 pp.
- [38] Serre, J.P. Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ -adiques. In *Motives* (Seattle, WA, 1991), 377-400, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.
- [39] Serre, J.P. Cohomologie Galoisienne. Fifth edition. *Lecture Notes in Mathematics*, 5. Springer-Verlag, Berlin, 1994. x+181 pp.
- [40] Serre, J.P. *Topics in Galois theory*. Second edition. With notes by Henri Darmon. *Research Notes in Mathematics*, 1. A K Peters, Ltd., Wellesley, MA, 2008. xvi+120 pp.
- [41] Thompson, J. G. Rational rigidity of $G_2(5)$. *Proceedings of the Rutgers group theory year, 1983-1984* (New Brunswick, N.J., 1983-1984), 321-322, Cambridge Univ. Press, Cambridge, 1985.
- [42] Z. Yun, Motives with exceptional Galois groups and the inverse Galois problem. *Invent. Math.*, 196 (2014), Issue 2, 267-337.
- [43] Z. Yun, Galois representations attached to moments of Kloosterman sums and conjectures of Evans (with an appendix by C.Vincent). *Compositio Math.*, to appear. arXiv: 1308.3920.
- [44] Z. Yun, Epipelagic representations and rigid local systems. arXiv:1401.7647.

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