MATH 380A Final Exam

Due on Dec. 20, 2017

There are 7 problems.
You may NOT consult books, Internet or other people regarding the exam.
All rings are assumed to be unital. The letter A always denotes a commutative ring.

1. Classify commutative rings A with cardinality 4.

2. Let k be a field with $\mathrm{char}(k) = p > 0$. Let R be the set of polynomials of the form

$$f(x) = a_0 x + a_1 x^p + a_2 x^{p^2} + \cdots + a_n x^{p^n}$$

for some integer $n \geq 0$ and $a_0, \cdots, a_n \in k$. For $f, g \in k[x]$, define $f \circ g \in k[x]$ to be the composition of polynomials $f(g(x))$.

   a. Show that R is closed under addition and composition, and forms a unital associative ring under these operations.

   b. What are the invertible elements in R?

   c. Exhibit a maximal (two-sided) ideal in R.

3. Let $\mathbb{Z}_p$ be the p-adic integers and let $A = \mathbb{Z}_p[x]$.

   a. Show that A is a UFD, but is not a PID.

   b. Let $\mathfrak{q}$ be a prime ideal of $\mathbb{Z}_p[x]$ such that $\mathbb{Z}_p \cap \mathfrak{q} = 0$. Show that $\mathfrak{q}$ is generated by an irreducible polynomial over $\mathbb{Z}_p$ of positive degree.
   Hint: consider the ideal in $\mathbb{Q}_p[x]$ generated by $\mathfrak{q}$.

   c. Classify all prime ideals in A in terms of irreducible polynomials in $\mathbb{Q}_p[x]$ and in $\mathbb{F}_p[x]$.

4. Let $A = \mathbb{R}[x]$.

   a. Give a complete list of all prime ideals of A (with proof).

   b. Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $T : V \to V$ be an $\mathbb{R}$-linear transformation. State and prove the theorem of rational canonical form for T. (You may use the structure theorem for modules over a PID.)

5. Let $S \subset A$ be a multiplicative subset. Let M be an A-module.

   a. Show that if M is a projective A-module, $S^{-1}M$ is a projective $S^{-1}A$-module.

   b. Assume A is a domain. If $S^{-1}M$ is a projective $S^{-1}A$-module, is M necessarily a projective A-module? Prove it or give a counterexample.

6. Let B be a commutative A-algebra and M be an A-module. Prove the following isomorphism of B-algebras

$$S_B(B \otimes_A M) \cong B \otimes_A S_A(M).$$

Here $S_A(-)$ and $S_B(-)$ denote the symmetric algebras over A and B respectively.

7. Let A be a commutative ring. A sequence of elements $(a_1, a_2, \cdots, a_r)$ in A is called a *regular sequence* if for every $i = 1, \cdots, r$, a_i is not a zero-divisor in $A/(a_1, \cdots, a_{i-1})$ (when $i = 1$ we understand this as saying that a_1 is not a zero-divisor in A).

   a. Suppose $A = k[x, y]$ (where k is a field) and $f(x, y), g(x, y) \in A$ are coprime nonzero polynomials. Show that $(f(x, y), g(x, y))$ is a regular sequence in A.

   b. Let A be any commutative ring and $(a_1, \cdots, a_r)$ be a regular sequence in A. Let N be any A-module. Show that $\mathrm{Ext}^i(A/(a_1, \cdots, a_r), N) = 0$ for $i > r$ and that

   $$\mathrm{Ext}^r_A(A/(a_1, \cdots, a_r), N) \cong N/(a_1, a_2, \cdots, a_r)N.$$

   Hint: induction on r.