REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE: EXERCISES

ZHIWEI YUN

Notation

Fix a prime number p and a power q of p.
$k = \mathbb{F}_q$;
$k_d = \mathbb{F}_{q^d}$.
$\nu \vdash n$ means ν is a partition of n.

1. GL_n

Conjugacy classes

1.1. Let $A \subset \mathrm{GL}_n(k)$ be a maximal abelian subgroup whose order is prime to p. We call such A a *maximal torus.*

(1) Show that A is isomorphic to $\prod_{i=1}^r k_{\nu_i}^\times$ for a partition $\nu = (\nu_1, \cdots, \nu_r)$ of n. We say A is *a maximal torus of type* ν.
(2) If two such A, A' correspond to the same partition of n, then they are conjugate to each other inside $\mathrm{GL}_n(k)$.

1.2. For a partition $\nu \vdash n$, let $T_\nu \subset \mathrm{GL}_n(k)$ be a maximal torus of type ν. Describe the normalizer $N(T_\nu)$ of T_ν inside $\mathrm{GL}_n(k)$, and the Weyl group $W_\nu = N(T_\nu)/T_\nu$. Show that two elements in T_ν are conjugate in $\mathrm{GL}_n(k)$ if and only if they are in the same W_ν-orbit (acting by conjugation on T_ν).

1.3. Show that there is a bijection

$$\coprod_{\nu \vdash n} T_\nu / W_\nu \leftrightarrow \{\text{conjugacy classes in } \mathrm{GL}_n(k)\}$$

such that if the W_ν-orbit of $t \in T_\nu$ corresponds to the conjugacy class of $g \in \mathrm{GL}_n(k)$, then t and g have the same characteristic polynomial.

1.4. Suppose $g \in \mathrm{GL}_n(k)$ has characteristic polynomial $\prod_f f^{m_f}$ (where f runs over monic irreducible polynomials in $k[t]$), and the sizes of the Jordan blocks for each f is given by a partition $\nu(f)$ of m_f. Let $Z(g)$ be the centralizer of g in $\mathrm{GL}_n(k)$. Show that

$$\#Z(g) = \prod_f \left(q_f^{m_f + 2n(\nu(f))} \prod_{i \geq 1} \phi_{\lambda_i(f) - \lambda_{i+1}(f)}(q_f^{-1}) \right).$$

Here

- $q_f = q^{\deg(f)}$.
- For a partition $\nu = (\nu_1 \geq \cdots \nu_s)$, we define
$$n(\nu) = \sum_{i=1}^{s} (i-1)\nu_i.$$
- $\lambda(f) = (\lambda_1(f) \geq \lambda_2(f) \cdots)$ is the partition conjugate to $\nu(f)$.

- For $r \geq 1$ we define $\phi_r(t)$ to be the polynomial

$$\phi_r(t) = (1-t)(1-t^2)\cdots(1-t^r). \tag{1.1}$$

and, for $r=0$, we define $\phi_0(t)=1$.

Hall algebras

In the following exercises, let X be a smooth algebraic curve over k. Let $\mathbb{H}_{X,\mathrm{tor}}$ be the Hall algebra of torsion coherent sheaves on X, with $\mathbb{Q}$-*coefficients* (in fact it suffices to consider $\mathbb{Z}[p^{-1}]$-coefficients).

1.5. Show that $\mathbb{H}_{X,\mathrm{tor}}$ is commutative and cocommutative. (Hint: use Serre duality.)

1.6. Show that the Hall algebra $\mathbb{H}_{X,\mathrm{tor}}$ is a bialgebra, i.e., the comultiplication $\Delta : \mathbb{H}_{X,\mathrm{tor}} \to \mathbb{H}_{X,\mathrm{tor}} \otimes \mathbb{H}_{X,\mathrm{tor}}$ is an algebra map.

1.7. Define a map $S : \mathbb{H}_{X,\mathrm{tor}} \to \mathbb{H}_{X,\mathrm{tor}}$ by sending $\mathbf{1}_M$ (the characteristic function of a torsion $\mathcal{O}_X$-module M) to

$$S(\mathbf{1}_M) := \sum_{F_*M} (-1)^{\ell(F_*M)} \frac{\prod \#\mathrm{Aut}(\mathrm{Gr}_i^F M)}{\#\mathrm{Aut}(M)} (\prod \mathbf{1}_{\mathrm{Gr}_i^F M}) \tag{1.2}$$

where the sum runs over all finite *strict* filtrations

$$F_*M : 0 = F_0M \subsetneq F_1M \subsetneq \cdots \subsetneq F_\ell M = M$$

with $\ell(F_*M) = \ell$, and the last $\prod$ in (1.2) means Hall multiplication.

Show that S is an antipode for $\mathbb{H}_{X,\mathrm{tor}}$, so that $\mathbb{H}_{X,\mathrm{tor}}$ is a Hopf algebra. Recall that this means

- S is an involutive automorphism of $\mathbb{H}_{X,\mathrm{tor}}$ as a bi-algebra;
- $m \circ (1 \otimes S) \circ \Delta = i \circ \epsilon = m \circ (S \otimes 1) \circ \Delta$, where i and ϵ are the unit and counit maps.

1.8. The above construction of the antipode holds in greater generality. Let $H = \oplus_{n \geq 0} H_n$ be a graded bi-algebra over a commutative ring A and that $H_0 = A$ (with multiplication m, comultiplication Δ, unit $i : A \to H$ given by the inclusion of $A = H_0$ and the counit $\epsilon : H \to A$ given by projection to $H_0 = A$). Then

(1) There is a unique graded map $S : H \to H$ such that $m \circ (1 \otimes S) \circ \Delta = i \circ \epsilon$. (Hint: inductively define S on each H_n.)

(2) Suppose H is commutative and cocommutative, show that S is a bialgebra automorphism and that $S^2 = \mathrm{id}$. Therefore in this case H has a unique structure of a Hopf algebra.

(3) We have the following explicit formula for S. For a strict composition $c = (n_1, \cdots, n_\ell)$ of n (strict means each $n_i > 0$), let $\Delta_c : H_n \to H_{n_1} \otimes \cdots \otimes H_{n_\ell}$ be the projection of the iterated comultiplication to the corresponding graded pieces; similarly, let $m_c : H_{n_1} \otimes \cdots \otimes H_{n_\ell} \to H_n$ be the multiplication map. Then we have for each $a_n \in H_n$,

$$S(a_n) = \sum_c (-1)^{\ell(c)} m_c(\Delta_c(a_n))$$

where the sum is over all strict compositions of n.

(One can try to use this explicit formula to prove (2).)

1.9. Let $x \in |X|$, and let $\mathbf{1}_{\mathrm{Coh}_x}$ be the constant function on $\mathrm{Coh}_x(X)$ (torsion sheaves supported at x), viewed as an element in the completion $\widehat{\mathbb{H}}_{X,\mathrm{tor}}$ of $\mathbb{H}_{X,\mathrm{tor}}$ (by degree). What is $S(\mathbf{1}_{\mathrm{Coh}_x})$?

Symmetric functions

Let $\Lambda = \mathbb{Z}[x_1, x_2, \cdots]^{S_\infty}$ be the graded ring of symmetric functions in the variables $x_1, x_2, \cdots$. We denote the elementary symmetric functions, complete symmetric functions and power sums by e_n, h_n and p_n. For example,

$$h_3 = \sum_i x_i^3 + \sum_{i<j} (x_i^2 x_j + x_i x_j^2) + \sum_{i<j<k} x_i x_j x_k.$$

1.10. Show that as a commutative ring, Λ is freely generated by the set $\{e_n\}_{n\geq 1}$ over $\mathbb{Z}$; Λ is freely generated by the set $\{h_n\}_{n\geq 1}$ over $\mathbb{Z}$; $\Lambda_{\mathbb{Q}}$ is freely generated by the set $\{p_n\}$ over $\mathbb{Q}$.

1.11. Define the following elements in $\Lambda[[t]]$:

$$E(t) = \sum_{n\geq 0} e_n t^n,$$
$$H(t) = \sum_{n\geq 0} h_n t^n,$$
$$PS(t) = \sum_{n\geq 1} \frac{p_n}{n} t^n.$$

Show that

$$E(-t)^{-1} = H(t) = \exp(PS(t)).$$

1.12. Recall Λ has a graded Hopf algebra structure determined by the condition that p_n be primitive. Denote the corresponding comultiplication by Δ. Show that

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i},$$
$$\Delta(h_n) = \sum_{i=0}^{n} h_i \otimes h_{n-i}.$$

1.13. Show that the antipode S of Λ maps e_n to $(-1)^n h_n$, for all n.

Symmetric groups

1.14. Let $\mathbf{R} = \oplus_{n\geq 0} R(S_n)$, the direct sum of Grothendieck groups of $\mathbb{C}$-representations of all symmetric groups S_n (where S_0 is by definition the trivial group).

(1) Equip $\mathbf{R}$ with the structure of a Hopf algebra using induction along $S_n \times S_m \hookrightarrow S_{n+m}$.

(2) Let S be the antipode on $\mathbf{R}$. Show that, for an irreducible representation ρ of S_n, we have

$$S(\rho) = \pm\rho \otimes \mathrm{sgn}_n \tag{1.3}$$

and figure out the sign above. Here sgn_n is the sign character of S_n.

Let $\Phi_1 : \Lambda \to \mathbf{R}$ be the graded ring homomorphism sending h_n to the trivial representation of S_n.

1.15. Show that Φ_1 is an isomorphism of Hopf algebras.

1.16. Show that

$$\Phi_1(e_n) = \mathrm{sgn}_n \in R(S_n), \quad n \geq 1. \tag{1.4}$$

1.17. Show that $\Phi_1(p_n) \in R(S_n)$ has nonzero character value only on the class of cyclic permutations $(12\cdots n)$, and compute this value.

Hall algebra and symmetric functions

Let $\mathbb{H}_q$ be the Hall algebra (with $\mathbb{Q}$-coefficients) of torsion coherent sheaves on the algebraic curve $\mathbb{G}_m$ supported at the point 1. For a partition λ, let $\mathbf{1}_\lambda \in \mathbb{H}_q$ be the characteristic function on the torsion modules with Jordan type λ. For example, $\lambda = (1^n)$ means k^n and $\lambda = (n)$ means $k[t, t^{-1}]/(t-1)^n$.

Define a ring homomorphism

$$\Phi_q : \Lambda_{\mathbb{Q}} \to \mathbb{H}_q \tag{1.5}$$

by

$$\Phi_q(e_n) = q^{n(n-1)/2}\mathbf{1}_{(1^n)}.$$

1.18. Show that Φ_q is an isomorphism of Hopf algebras over $\mathbb{Q}$.

1.19. Show that

$$\Phi_q(h_n) = \sum_{\lambda\vdash n} \mathbf{1}_\lambda;$$
$$\Phi_q(p_n) = \sum_{\lambda\vdash n} \phi_{r(\lambda)-1}(q)\mathbf{1}_\lambda.$$

Here $r(\lambda)$ denotes the number of parts in the partition λ; see (1.1) for the definition of $\phi_r(t)$.

Groupoids and inertia

Let X be a groupoid and let $\mathcal{F} \in \mathrm{Sh}(X)$, the category of sheaves on X in abelian groups (i.e., such a sheaf is the same as a functor from X to the category of abelian groups). Define the global sections and global sections with compact support of $\mathcal{F}$ to be

$$\Gamma(X, \mathcal{F}) := \prod_{x\in \mathrm{Ob}(X)/\cong} (\mathcal{F}_x)^{\mathrm{Aut}(x)};$$
$$\Gamma_c(X, \mathcal{F}) := \bigoplus_{x\in \mathrm{Ob}(X)/\cong} (\mathcal{F}_x)_{\mathrm{Aut}(x)}.$$

Let $f : X \to Y$ be a map between groupoids, define $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ by the formula

$$(f_*\mathcal{F})_y = \Gamma(X_y, \mathcal{F}|_{X_y})$$

where $X_y = f^{-1}(y)$ is the fiber of f over y (a groupoid), and $\mathcal{F}|_{X_y}$ is the restriction of $\mathcal{F}$ to X_y. Similarly define $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ by the formula

$$(f_!\mathcal{F})_y = \Gamma_c(X_y, \mathcal{F}|_{X_y}).$$

We also have the obvious pullback functor

$$f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X).$$

1.20. Let $\varphi : G \to H$ be a group homomorphism, and let $f : \mathrm{pt}/G \to \mathrm{pt}/H$ be the induced map of groupoids. Identify $\mathrm{Sh}(\mathrm{pt}/G)$ with $\mathrm{Rep}(G)$, and $\mathrm{Sh}(\mathrm{pt}/H)$ with $\mathrm{Rep}(H)$.

(1) Show that f^* is the restriction functor $\mathrm{Rep}(H) \to \mathrm{Rep}(G)$.

(2) Show that f_* is the composition

$$\mathrm{Rep}(G) \xrightarrow{(-)^K} \mathrm{Rep}(\varphi(G)) \xrightarrow{\mathrm{Ind}^H_{\varphi(G)}} \mathrm{Rep}(H)$$

where $K = \ker(\varphi)$.

(3) Show that $f_!$ is the composition

$$\mathrm{Rep}(G) \xrightarrow{(-)_K} \mathrm{Rep}(\varphi(G)) \xrightarrow{\mathrm{ind}^H_{\varphi(G)}} \mathrm{Rep}(H)$$

Here $\mathrm{ind}^H_{\varphi(G)} = \mathbb{Z}[H] \otimes_{\mathbb{Z}[\varphi(G)]} (-)$.

1.21. If $f : X \to Y$ and $g : Y \to Z$ are maps between groupoids, then there are canonical isomorphisms of functors

$$(g \circ f)_* = g_* \circ f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Z),$$
$$(g \circ f)_! = g_! \circ f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Z).$$

1.22. Let $f : X \to Y$ be a map between groupoids. Show that $f_!$ is left adjoint to f^*, and f_* is right adjoint to f^*.

1.23. Let X be a groupoid and let I_X be its inertia groupoid. Let $\mathrm{Sh}_f(X, \mathbb{Q})$ be the category of $\mathbb{Q}$-sheaves on X with finite-dimensional stalks. Define a natural additive map

$$\chi_X : \mathrm{Sh}_{fd}(X, \mathbb{Q}) \to \mathrm{Fun}(I_X, \mathbb{Q})$$

where $\mathrm{Fun}(I_X, \mathbb{Q})$ denotes vector space of $\mathbb{Q}$-valued functions on the set of isomorphism classes of I_X.

1.24. Let $f : X \to Y$ is a finite map of groupoids.

(1) Show that the induced map $I_f : I_X \to I_Y$ is also finite, so that the "integration along the fibers" is defined
$$f_! : \mathrm{Fun}(I_X, \mathbb{Q}) \to \mathrm{Fun}(I_Y, \mathbb{Q}).$$

(2) Show that the following diagram is commutative
$$\begin{tikzcd} \mathrm{Sh}_{fd}(X,\mathbb{Q}) \arrow[r, "\chi_X"] \arrow[d, "f_!"] & \mathrm{Fun}(I_X,\mathbb{Q}) \arrow[d, "f_!"] \\ \mathrm{Sh}_{fd}(Y,\mathbb{Q}) \arrow[r, "\chi_Y"] & \mathrm{Fun}(I_Y,\mathbb{Q}) \end{tikzcd}$$

Hecke algebras

1.25. Let Ind_n be the induction of the trivial rep from $B_n(k)$ (upper triangular matrices) to $\mathrm{GL}_n(k)$. Define the Hecke algebra to be
$$H_q(S_n) := \mathrm{End}_{\mathrm{GL}_n(k)}(\mathrm{Ind}_n).$$
The notation suggests that $H_q(S_n)$ is a q-deformation of the group algebra of S_n.

(1) Show that $H_q(S_n)$ is a semisimple algebra (over $\mathbb{C}$).
(2) Show that the irreducible $H_q(S_n)$-modules are in natural bijection with irreducible $\mathrm{GL}_n(k)$-representations that appear in Ind_n.
(3) There is a natural embedding of algebras
$$H_q(S_n) \otimes H_q(S_m) \hookrightarrow H_q(S_{n+m})$$
deforming the standard embedding $S_n \times S_m \hookrightarrow S_{n+m}$.

1.26. Let $R_{\mathrm{uni},n} \subset R(\mathrm{GL}_n(k))$ be the $\mathbb{Z}$-span of irreducible summands of Ind_n for various n. Then by the previous exercise, we may identify $R_{\mathrm{uni},n}$ with $R(H_q(S_n))$, the Grothendieck group of finite-dimensional $H_q(S_n)$-modules. The Hopf algebra structure on $R_{\mathrm{uni}} = \oplus_{n\geq 0} R_{\mathrm{uni},n}$ induces a Hopf algebra structure on $\oplus_{n\geq 0} R(H_q(S_n))$. Can you describe this latter Hopf algebra structure directly in terms of $H_q(S_n)$-modules?

Semisimple functions

We are in the setting of Hall algebra $\mathbb{H}_{X,\mathrm{tor}}$ for torsion coherent sheaves on a curve X. Let $\mathrm{Div}^+(X)$ be the set of effective divisors on X. Then we have the support map (with finite fibers)
$$\pi : \mathrm{Coh}_{\mathrm{tor}}(X) \to \mathrm{Div}^+(X).$$
Hence a pullback map on functions
$$\pi^* : C_c(\mathrm{Div}^+(X)) \to \mathbb{H}_{X,\mathrm{tor}}.$$
The elements in the image of π^* are called *semisimple functions.*

We sometimes also consider pullback of functions without support conditions
$$\pi^* : C(\mathrm{Div}^+(X)) \to \widehat{\mathbb{H}}_{X,\mathrm{tor}} = C(\mathrm{Coh}_{\mathrm{tor}}(X)).$$
and we also call its image semisimple functions.

1.27. Let $\mathrm{add} : \mathrm{Div}^+(X) \times \mathrm{Div}^+(X) \to \mathrm{Div}^+(X)$ be the addition map on effective divisors. Show that for $f \in C_c(\mathrm{Div}^+(X))$, we have
$$\Delta(\pi^* f) = (\pi\times\pi)^*\mathrm{add}^*(f) \in C_c(\mathrm{Coh}_{\mathrm{tor}}(X) \times \mathrm{Coh}_{\mathrm{tor}}(X)) = \mathbb{H}_{X,\mathrm{tor}} \otimes \mathbb{H}_{X,\mathrm{tor}}.$$
In particular, semisimple functions form a sub-coalgebra of $\mathbb{H}_{X,\mathrm{tor}}$.

1.28. Show, on the other hand, that semisimple functions do not form a subalgebra of $\mathbb{H}_{X,\mathrm{tor}}$.

1.29. Suppose $f \in \widehat{\mathbb{H}}_{X,\mathrm{tor}}$ is an *additive function*, namely, for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathrm{Coh}_{\mathrm{tor}}(X)$, we have $f(M) = f(M') + f(M'')$.

(1) Show that f is a semisimple function.
(2) Show that
$$\Delta(f) = f \otimes \mathbf{1}_{\mathrm{Coh}} + \mathbf{1}_{\mathrm{Coh}} \otimes f.$$
Here $\mathbf{1}_{\mathrm{Coh}}$ is the constant function 1 on $\mathrm{Coh}_{\mathrm{tor}}(X)$ (Warning: $\mathbf{1}_{\mathrm{Coh}}$ is not the unit of $\mathbb{H}_{X,\mathrm{tor}}$).
(3) Show that $fS(\mathbf{1}_{\mathrm{Coh}})$ is a primitive element in $\widehat{\mathbb{H}}_{X,\mathrm{tor}}$. Here S is the antipode of $\mathbb{H}_{X,\mathrm{tor}}$.
(4) Let $x \in |X|$ (with residue field $k(x)$) and let $\ell_x \in \mathbb{H}_x = C(\mathrm{Coh}_x(X))$ be the additive function given by $\ell_x(M) = \dim_{k(x)} M$. Let $\mathbf{1}_{\mathrm{Coh}_x}$ be the constant function on $\mathrm{Coh}_x(X)$. Recall the isomorphism (see (1.5))
$$\Phi_x : \Lambda_{\mathbb{Q}} \cong \mathbb{H}_x$$
sending e_n to $q_x^{n(n-1)/2}\mathbf{1}_{k(x)^n}$.
Show that $\Phi_x(p_n)$ is the degree $nd(x)$ piece of $\ell_x S(\mathbf{1}_{\mathrm{Coh}_x}) \in \widehat{\mathbb{H}}_x$.

2. Deligne-Lusztig theory

2.1. Let $G = \mathrm{SL}_2(k)$ for $k = \mathbb{F}_q$ with odd q. Let T be a non-split maximal torus in G (so that $T(k) = (\mathbb{F}_{q^2}^\times)^{\mathrm{Nm}=1}$). Let $\epsilon : T(k) \to \{\pm 1\}$ be the quadratic character of $T(k)$. Consider Deligne-Lusztig curve $C \subset \mathbb{A}^2$ defined by
$$C : xy^q - x^q y = 1.$$

(1) Describe the action of $T(k)$ on C.
The Delgine-Lusztig representation R_T^ϵ attached to T and ϵ is
$$V = \mathrm{H}^1_c(C, \overline{\mathbb{Q}}_\ell)[\epsilon]$$
(the ϵ-isotypic part of the $T(k)$-action.) We will decompose V into the direct sum of two irreducible representations of $G(k)$ by studying how the Frobenius F acts on it.
(2) Let $\overline{C}$ be the closure of C in $\mathbb{P}^2$. Show that $T(k)$ acts trivially on the boundary $\overline{C} - C$, and conclude that
$$V \cong \mathrm{H}^1(\overline{C}, \overline{\mathbb{Q}}_\ell)[\epsilon]$$
as $G(k)$-modules.
(3) Let C' be the base change of C to $\mathbb{F}_{q^2}$ and let $\overline{C'}$ be the corresponding projective curve. Show that $\overline{C'}$ is isomorphic to the Fermat curve of degree $q+1$ in $\mathbb{P}^2$ (over $\mathbb{F}_{q^2}$):
$$\overline{C'} : u^{q+1} + v^{q+1} + w^{q+1} = 0.$$
(4) Prove, by counting the number of points in $\overline{C'}$, that the eigenvalues of F^2 on V are all equal to $-q$.
(5) Let α and β be two square roots of $-q$ in $\overline{\mathbb{Q}}_\ell$, and let V_α and V_β be the corresponding generalized eigenspaces. Show that under the cup product pairing on $\mathrm{H}^1(\overline{C}, \overline{\mathbb{Q}}_\ell)[\epsilon]$, V_α and V_β are isotropic and are paired perfectly with each other.
(6) Show that $\dim V_\alpha = \dim V_\beta = \frac{1}{2}\dim V = \frac{1}{2}(q-1)$.
(7) By computing the inner product $\langle V, V\rangle_{G(k)}$, show that V_α and V_β are irreducible and non-isomorphic to each other.

References

E-mail address: `zhiwei.yun@yale.edu`