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Goresky–MacPherson Calculus for the Affine Flag Varieties

Zhiwei Yun

Abstract. We use the fixed point arrangement technique developed by Goresky and MacPherson to calculate the part of the equivariant cohomology of the affine flag variety $\mathcal{F}\ell_G$ generated by degree 2. We use this result to show that the vertices of the moment map image of $\mathcal{F}\ell_G$ lie on a paraboloid.

1 Introduction

1.1 Goresky–MacPherson Calculus

Goresky and MacPherson [2] considered a complex algebraic torus A acting on a connected complex projective variety X. Let $\widehat{H}^*_A(X)$ be the part of the equivariant cohomology $H^*_A(X)$ generated by degree 0 and degree 2. The affine scheme $\operatorname{Spec}\widehat{H}^*_A(X)$ is naturally a closed subscheme of $H^A_2(X)$, because $\widehat{H}^*_A(X)$ is a quotient of $\operatorname{Sym}(H^2_A(X))$. If A acts with finitely many fixed points and X is *equivariantly formal* (for a definition, see [3, §1.2]), Goresky and MacPherson show that $\operatorname{Spec}\widehat{H}^*_A(X)$ is reduced and consists of an arrangement of linear subspaces $V_x \subset H^A_2(X)$, one for each fixed point $x \in X$. Here V_x is the image of the natural map induced by the inclusion $i_x\colon \{x\} \hookrightarrow X$,

$$\mu_x = i_{x,*}\colon H^A_2(\{x\}) \cong \mathfrak{a} \to H^A_2(X),$$

where $\mathfrak{a}$ is the Lie algebra of A. We call $V = \bigcup_{x\in X^A} V_x \subset H^A_2(X)$ the *fixed point arrangement* of the action A on X.

1.2 Main Results

In this note we extend this result to certain ind-schemes X. In the examples in which we are interested, $H^A_2(X)$ is still finite dimensional but there are infinitely many (still discrete) fixed points. The fixed point arrangement $V \subset H^A_2(X)$ still makes sense in this situation. In Theorem 2.1, we show that the subscheme $\operatorname{Spec}\widehat{H}^*_A(X)$ of $H^A_2(X)$ is the Zariski closure of V. We then apply this result to $X = \mathcal{F}\ell_G$, the *affine flag variety* associated with a simple reductive group G and $A = \widetilde{T}$, the *augmented torus*, which is the product of a maximal torus T of G and the loop-rotation torus $\mathbb{G}^{\mathrm{rot}}_m$. We calculate

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explicitly the fixed point arrangement of the $\widetilde{T}$-action on $\mathcal{F}\ell_G$. We show in Theorem 3.4 that $\operatorname{Spec}\widehat{H}^*_{\widetilde{T}}(\mathcal{F}\ell_G)$ is a quadric cone in $H_2^{\widetilde{T}}(\mathcal{F}\ell_G)$ cut out by the one-dimensional kernel of the cup product

$$\bigcup\colon \operatorname{Sym}^2(H^2_{\widetilde{T}}(\mathcal{F}\ell_G)) \to H^4_{\widetilde{T}}(\mathcal{F}\ell_G). \tag{1.1}$$

In Theorem 3.5, we describe $H_2^{\widetilde{T}}(\mathcal{F}\ell_G)$ in more intrinsic terms using Kac–Moody algebras. We show that, by analogy with the finite flag varieties, the fixed point arrangement is essentially the union of graphs of the affine Weyl group action on the affine Cartan algebra.

The results here have some overlaps with the results of Kostant–Kumar [6]. However, we approach the problem from a different perspective.

The proofs are largely omitted in this note. Details and more results on the full equivariant cohomology ring of $\mathcal{F}\ell_G$ appear in [8].

1.3 Application

One application of our results is a proof in Section 4.3 of the following folk-theorem: *the moment map images of the T-fixed points of $\mathcal{F}\ell_G$ lie on a paraboloid in* $\widetilde{\mathfrak{t}}^*_{\mathbb{R}}$. A similar statement for the affine Grassmannians was proved by Atiyah and Pressley [1, Theorem 1].

1.4 Notation and Conventions

Throughout this note, all (ind-)schemes are over $\mathbb{C}$ and all (co-)homologies are taken with $\mathbb{C}$-coefficients unless otherwise specified. For a torus A, we denote its character and cocharacter lattices by $X^\bullet(A)$ and $X_\bullet(A)$, respectively.

2 Equivariant Cohomology of Ind-Schemes

Let $X = \bigcup X_n$ be a strict ind-projective scheme, *i.e.*, each $\iota_n\colon X_n \to X_{n+1}$ is a closed embedding of projective schemes. Let A be an algebraic torus acting on X. We make the following assumptions:

(i) Each X_n is stable and equivariantly formal under the A-action.
(ii) Each X_n has only finitely many fixed points.
(iii) The vector space $H_2(X)$ is finite dimensional.

Following the idea of R. MacPherson, we extend (part of) the main theorem of [2] to the ind-scheme case.

Theorem 2.1 *Suppose the A-action on a connected ind-projective scheme X satisfies the above assumptions. Then* $\operatorname{Spec}\widehat{H}^*_A(X)$ *is the Zariski closure of the fixed point arrangement V inside the affine space $H_2^A(X)$ with the reduced scheme structure.*

Proof For each n consider the natural commutative diagram

$$\begin{tikzcd} H_A^*(X) \arrow[r, "\rho"] \arrow[d] & \prod_{x\in X^A} H_A^*(\{x\}) \arrow[d] \\ H_A^*(X_n) \arrow[r, "\rho_n"] & \prod_{x\in X_n^A} H_A^*(\{x\}) \end{tikzcd}$$

Here the arrows are the obvious restriction maps. By assumption (i), we can apply the localization theorem (see [3, Theorem 1.2.2]) to conclude that ρ_n is injective for all n. This implies that ρ is also injective. In particular, $H_A^*(X)$, hence $\widehat{H}_A^*(X)$, is reduced.

Since X is connected, we have a surjection followed by an injection:

$$\mathrm{Sym}(H_A^2(X)) \twoheadrightarrow \widehat{H}_A^*(X) \overset{\rho}{\hookrightarrow} \prod_{x\in X^T} H_A^*\big(\{x\}\big)\,,$$

which induces maps on spectra

$$\coprod_{x\in X^A} \mathfrak{a} \xrightarrow{\coprod \mu_x} \mathrm{Spec}\,\widehat{H}_A^*(X) \hookrightarrow H_2^A(X).$$

Now the first arrow has dense image, because ρ is injective. This shows that the support of $\mathrm{Spec}\,\widehat{H}_A^*(X)$ is the Zariski closure of $\bigcup_{x\in X^A}\mu_x(\mathfrak{a}) = V$. ∎

3 The Fixed Point Arrangement of Affine Flag Varieties

3.1 The Affine Flag Varieties

Let G be a simple and simply-connected group over $\mathbb{C}$. Let r be the rank of G. Let $F = \mathbb{C}((z))$ and $\mathcal{O} = \mathbb{C}[[z]]$ be the formal Laurent series and formal power series, respectively. Choose a maximal torus T of G and a Borel subgroup B containing T. Let $I \subset G(\mathcal{O})$ be the Iwahori subgroup corresponding to B. The *affine flag variety* is the ind-scheme $\mathcal{F}\ell_G = G(F)/I$. Basic facts about affine flag varieties can be found in [4, §§5, 14].

Let Φ be the based root system determined by (G, T, B) with Weyl group W. Let $\theta \in \Phi$ be the highest root. We fix a Killing form $(\cdot\,|\,\cdot)$ on $\mathfrak{t}^*$ such that $(\theta|\theta) = 2$. This induces a Killing form $(\cdot\,|\,\cdot)$ on $\mathfrak{t}$. We use $|\cdot|^2$ to denote the quadratic forms on $\mathfrak{t}^*$ or $\mathfrak{t}$ associated with the Killing forms.

3.2 Notations Concerning the Affine Kac–Moody Algebra

Consider the affine Kac–Moody algebra $L(\mathfrak{g})$ associated with the Lie algebra $\mathfrak{g}$ of G. We will follow the notations of [5, Ch. 6], which we briefly recall here. The affine Cartan algebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ have decompositions

$$\mathfrak{h} = \mathbb{C}d \oplus \mathfrak{t} \oplus \mathbb{C}K, \quad \mathfrak{h}^* = \mathbb{C}\delta \oplus \mathfrak{t}^* \oplus \mathbb{C}\Lambda_0, \tag{3.1}$$

where K is the *canonical central element*, d the *scaling element*, and δ the positive generator of the imaginary roots. We extend the Killing forms to $\mathfrak{h}$ and $\mathfrak{h}^*$ as in [5].

Let $\widetilde{\Phi}$ be the affine root system of $L(\mathfrak{g})$ with simple roots $\{\alpha_0 = \delta - \theta, \alpha_1, \dots, \alpha_r\}$ and simple coroots $\{\check{\alpha}_0 = K - \check{\theta}, \check{\alpha}_1, \dots, \check{\alpha}_r\}$. Let $\widetilde{W} = X_\bullet(T) \rtimes W$ be the affine Weyl group with simple reflections $\{s_0, \dots, s_r\}$ corresponding to the simple roots of $\widetilde{\Phi}$. The group $\widetilde{W}$ acts on $\mathfrak{h}$ via the formula

$$\widetilde{w}(u, \xi, v) = \left(u, w\xi + u\lambda, v - (w\xi|\lambda) - \frac{u}{2}|\lambda|^2 \right), \tag{3.2}$$

where (u, ξ, v) are the coordinates in terms of the decomposition (3.1) and $\widetilde{w} = (\lambda, w) \in \widetilde{W}$.

3.3 The Torus Action on $\mathcal{F}\ell_G$

The torus T acts on $\mathcal{F}\ell_G$ by left translations. The one-dimensional torus $\mathbb{G}_m^{\mathrm{rot}}$ acts on $\mathcal{O}$ by dilation $s\colon z \mapsto sz, s \in \mathbb{G}_m^{\mathrm{rot}}$. This induces an action on $\mathcal{F}\ell_G$, which is the so-called *loop rotation*. Let $\widetilde{T} = \mathbb{G}_m^{\mathrm{rot}} \times T$ be the *augmented torus*. In the sequel, we will concentrate on the $\widetilde{T}$-action on $\mathcal{F}\ell_G$.

We identify the Lie algebra $\widetilde{\mathfrak{t}}$ of $\widetilde{T}$ with $\mathbb{C}d \oplus \mathfrak{t}$ such that the scaling element d corresponds to the positive generator of $X_\bullet(\mathbb{G}_m^{rot})$.

3.4 The $\widetilde{W}$-Symmetry

For each $\lambda \in X_\bullet(T)$, let $z^{-\lambda}$ be the image of z under $-\lambda\colon \mathbb{G}_m(F) \to T(F)$. The assignment $\lambda \mapsto z^{-\lambda}$ identifies $X_\bullet(T)$ as a subgroup of $T(F)$ (We put a minus sign here in order to make some formulas appear nicer in the sequel). Let $N_G(T)$ be the normalizer of T in G. Let $\tilde{N} \subset G(F)$ be the product $X_\bullet(T) \rtimes N_G(T) \subset G(F)$. Clearly, $\tilde{N}$ acts on $\mathcal{F}\ell_G$ via left translation. We have the following.

Lemma 3.1 *The induced action of $\widetilde{N}$ on $H_*^{\widetilde{T}}(\mathcal{F}\ell_G)$ factors through the quotient $\widetilde{N} \twoheadrightarrow \widetilde{W}$. The natural exact sequence*

$$0 \to H_2(\mathcal{F}\ell_G) \to H_2^{\widetilde{T}}(\mathcal{F}\ell_G) \xrightarrow{\pi} \widetilde{\mathfrak{t}} \to 0 \tag{3.3}$$

is $\widetilde{W}$-equivariant. Here the action of $\widetilde{W}$ on $H_2(\mathcal{F}\ell_G)$ is trivial, and the action on $\widetilde{\mathfrak{t}}$ is induced from the action (3.2).

The action of $\widetilde{W}$ on the ordinary homology is trivial because the $\widetilde{N}$-action comes from an action of the connected group $G(F)$, the rest of the lemma is an easy calculation.

3.5 The Fixed Points

Let $\mathcal{F}\ell_G^T$ be the T-fixed point subset of $\mathcal{F}\ell_G$. Note that it coincides with the set of $\widetilde{T}$-fixed points. The group $\tilde{N}$ acts on $\mathcal{F}\ell_G^T$ via the quotient $\widetilde{W}$ so that $\mathcal{F}\ell_G^T$ becomes a $\widetilde{W}$-torsor. We denote the base point of $\mathcal{F}\ell_G$ corresponding to I by x_0. By our convention of the minus sign in Section 3.4, if $\widetilde{w} = (\lambda, w)$, then $\widetilde{w} \cdot x_0 = z^{-\lambda} n_w x_0$ for any lift $n_w \in N_G(T)$ of w. From Lemma 3.1, we get the following.

Corollary 3.2 *For $\widetilde{w} \in \widetilde{W}$, we have $\mu_{\widetilde{w}\cdot x_0} = \widetilde{w} \circ \mu_{x_0} \circ \widetilde{w}^{-1}$.*

3.6 Description of $H_2(\mathcal{F}\ell_G)$

Each I-orbit of $\mathcal{F}\ell_G$ contains a unique fixed point $\widetilde{w} \cdot x_0$ and has dimension $\ell(\widetilde{w})$, the length of $\widetilde{w}$ [4, §14]. Therefore the one-dimensional orbits are in one-to-one correspondence with simple reflections in $\widetilde{W}$. Let C_i be the closure of $Is_i \cdot x_0$. We have a unique isomorphism

$$\phi_* : X_\bullet(T) \oplus \mathbb{Z}K = \mathrm{Span}_{\mathbb{Z}}\{\check{\alpha}_0, \dots, \check{\alpha}_r\} \xrightarrow{\sim} H_2(\mathcal{F}\ell_G, \mathbb{Z}), \tag{3.4}$$

sending $\check{\alpha}_i$ to the cycle class $[C_i]$.

The main computational result is the following.

Proposition 3.3 (i) *There is a unique isomorphism*

$$\psi_* \colon \mathbb{C}d \oplus \mathfrak{t} \oplus \mathfrak{t} \oplus \mathbb{C}K \xrightarrow{\sim} H_2^{\widetilde{T}}(\mathcal{F}\ell_G)$$

such that $\mu_{x_0}(u, \xi) = \psi_*(u, \xi, \xi, 0)$ and the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{t} \oplus \mathbb{C}K \arrow[r, "i_{34}"] \arrow[d, "\phi_{*,\mathbb{C}}"] & \mathbb{C}d \oplus \mathfrak{t} \oplus \mathfrak{t} \oplus \mathbb{C}K \arrow[r, "p_{12}"] \arrow[d, "\psi_*"] & \mathbb{C}d \oplus \mathfrak{t} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_2(\mathcal{F}\ell_G) \arrow[r] & H_2^{\widetilde{T}}(\mathcal{F}\ell_G) \arrow[r, "\pi"] & \widetilde{\mathfrak{t}} \arrow[r] & 0
\end{tikzcd}$$

is an isomorphism of exact sequences. Here i_{34} is the inclusion of the last two factors and p_{12} is the projection to the first two factor. The bottom sequence is (3.3).

(ii) *For $\widetilde{w} = (\lambda, w) \in \widetilde{W}$, we have*

$$\mu_{\widetilde{w}\cdot x_0}(u, \xi) = \psi_*\Big(u, \xi, w^{-1}(\xi - u\lambda), (\xi|\lambda) - \frac{u}{2}|\lambda|^2 \Big). \tag{3.5}$$

The isomorphism ψ_* is given by a particular choice of the splitting of the sequence (3.3). To prove (3.5), we first determine $\mu_{s_i \cdot x_0}$ by reducing to a $\mathbb{P}^1$-calculation, then apply Corollary 3.2 to get the general formula.

Now we can apply Theorem 2.1 to our situation: we take X to be the affine flag variety $\mathcal{F}\ell_G$ and A to be the augmented torus $\widetilde{T}$. It is easy to verify the assumptions in Section 2 in this situation. Moreover, since G is simply-connected, $\mathcal{F}\ell_G$ is connected.

Theorem 3.4 *Under the isomorphism ψ_*, the affine subscheme*

$$\mathrm{Spec}\, \widehat{H}^*_{\widetilde{T}}(\mathcal{F}\ell_G) \subset H_2^{\widetilde{T}}(\mathcal{F}\ell_G) \xrightarrow{\psi_*^{-1}} \mathbb{A}^1_d \times \mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1_K$$

is the nondegenerate quadric cone Q given by the equation

$$q(u,\xi,\eta,v) = |\xi|^2 - |\eta|^2 - 2uv = 0.$$

Here (u,ξ,η,v) are the coordinates for $\mathbb{A}^1_d \times \mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1_K$.

Proof Using the explicit formula (3.5), one easily checks that the image $V_{\widetilde{w}\cdot x_0}$ of any $\mu_{\widetilde{w}\cdot x_0}$ lies on the quadric cone Q. According to Theorem 2.1, it is enough to check that the fixed point arrangement $V = \bigcup_{\widetilde{w}\in\widetilde{W}} V_{\widetilde{w}\cdot x_0}$ is Zariski dense in Q. Since the projection to the first three factors $\pi_Q\colon Q \to \mathbb{G}_m \times \mathfrak{t} \times \mathfrak{t}$ is birational, it suffices to check that $\pi(V)$ is Zariski dense in $\mathbb{G}_m \times \mathfrak{t} \times \mathfrak{t}$. Now by (3.5), $\pi(V)$ contains points of the form $(u,\xi,\xi - u\lambda)$ for all $\xi \in \mathfrak{t}, \lambda \in X_\bullet(T)$ and $u \neq 0$. For any fixed $\xi, u \neq 0$, the set $\{\xi - u\lambda \mid \lambda \in X_\bullet(T)\}$, being a shifted lattice in $\mathfrak{t}$, is clearly Zariski dense in $\mathfrak{t}$. Therefore $\pi(V)$ is Zariski dense in $\mathbb{G}_m \times \mathfrak{t} \times \mathfrak{t}$. This proves the theorem. ■

3.7 An Intrinsic Description of $H_2^{\widetilde{T}}(\mathcal{F}\ell_G)$

We first list the structures that $H_2^{\widetilde{T}}(\mathcal{F}\ell_G)$ naturally carries. By Theorem 3.4, the cup product (1.1) has a one-dimensional kernel, which gives a quadratic form q on $H_2^{\widetilde{T}}(\mathcal{F}\ell_G)$, canonical up to scalar. By Lemma 3.1, $H_2^{\widetilde{T}}(\mathcal{F}\ell_G)$ carries a natural $\widetilde{W}$-action. Moreover, if we view $H^*_{\widetilde{T}}(\mathcal{F}\ell_G)$ as the $\mathbb{G}_m^{\mathrm{rot}}$-equivariant cohomology of the double coset stack $[I\backslash G(F)/I]$, it carries an involution τ induced by the inversion map $g \mapsto g^{-1}$ of $G(F)$.

Next, consider $\mathfrak{h} \oplus \mathfrak{h}$ with the quadratic form $(x,y) \mapsto |x|^2 - |y|^2$ and the involution $(x,y) \mapsto (-y,-x), \forall x,y \in \mathfrak{h}$. The group $\widetilde{W}$ acts on the first factor $\mathfrak{h}$ according to (3.2). The vector $(K,K) \in \mathfrak{h}\oplus\mathfrak{h}$ is isotropic and $\widetilde{W}$-invariant. We define a subquotient $\mathfrak{h}_2$ of $\mathfrak{h} \oplus \mathfrak{h}$ by $\mathfrak{h}_2 := (K,K)^\perp/\mathbb{C}(K,K)$. By construction, $\mathfrak{h}_2$ inherits a quadratic form and an involution from those of $\mathfrak{h}\oplus\mathfrak{h}$, which we denote by q_2 and τ_2. Moreover, $\mathfrak{h}_2$ inherits a $\widetilde{W}$-action. For each $\widetilde{w} \in \widetilde{W}$, its graph $\Gamma(\widetilde{w}) \subset \mathfrak{h} \oplus \mathfrak{h}$ actually lies in $(K,K)^\perp$.

Theorem 3.5 (i) *Up to a scalar, there is a unique isomorphism $H_2^{\widetilde{T}}(\mathcal{F}\ell_G) \cong \mathfrak{h}_2$ sending q to q_2, τ to τ_2 and intertwining the $\widetilde{W}$-actions.*
(ii) *Under any such isomorphism, the fixed point subspace $V_{\widetilde{w}\cdot x_0}$ becomes the image of the graph $\Gamma(\widetilde{w}^{-1})$ in $\mathfrak{h}_2$.*

The isomorphism is, up to a scalar, the composition of ψ_* with the isomorphism

$$\begin{aligned}\mathfrak{h}_2 \xleftarrow{\sim} \mathbb{C}(d,d) \oplus \mathfrak{t} \oplus \mathfrak{t} \oplus \mathbb{C}(0,K) &\xrightarrow{\sim} \mathbb{C}d \oplus \mathfrak{t} \oplus \mathfrak{t} \oplus \mathbb{C}K,\\ u(d,d) + (\xi,0) + (0,\eta) + v(0,K) &\mapsto (u,\xi,\eta,v).\end{aligned}$$

We remark that part (ii) of the above theorem explains the similarity between the formulas (3.2) and (3.5).

4 Application to Moment Maps

In general, for a torus A acting on an (ind-)scheme X, describing the moment map images of fixed points is dual to describing the fixed point arrangement: the duals of

μ_x give a map $\mathfrak{M}_X\colon X^A \times H^2_A(X,\mathbb{R}) \to \mathfrak{a}^*_{\mathbb{R}}$. For an A-equivariant line bundle $\mathcal{L}$ with equivariant Chern class $c_1(\mathcal{L}) \in H^2_A(X,\mathbb{R})$, the associated moment map $m_{\mathcal{L}}$ will send the fixed point x to $\mathfrak{M}_X(x, c_1(\mathcal{L}))$.

4.1 Line Bundles on $\mathcal{F}\ell_G$

For each weight $\chi \in X^\bullet(T)$, we have a line bundle $\mathcal{L}_\chi = G(F) \overset{I}{\times} \mathbb{C}(-\chi)$. Here $\mathbb{C}(-\chi)$ is the one-dimensional representation of I through the quotient $I \twoheadrightarrow B \twoheadrightarrow T$ with character χ. Moreover, we have the *determinant bundle* $\mathcal{L}_{\det}$ which is the pull-back of the positive generator of $\mathrm{Pic}(\mathcal{G}r_G)$, where $\mathcal{G}r_G$ is the affine Grassmannian $G(F)/G(\mathcal{O})$ (*cf.* [7], however the notion of the determinant bundle there is slightly different from ours).

We have a natural *degree pairing*

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon\ \mathrm{Pic}(\mathcal{F}\ell_G) \times H_2(\mathcal{F}\ell_G,\mathbb{Z}) \to \mathbb{Z}, \tag{4.1}$$

which sends $(\mathcal{L},[C])$ to $\deg(\mathcal{L}|_{[C]}) = \langle c_1(\mathcal{L}), [C]\rangle$.

Lemma 4.1 *We have a unique isomorphism $\phi^*\colon X^\bullet(T) \oplus \mathbb{Z}\Lambda_0 \xrightarrow{\sim} \mathrm{Pic}(\mathcal{F}\ell_G)$ which sends $\chi \in X^\bullet(T)$ to $\mathcal{L}_\chi$ and Λ_0 to $\mathcal{L}_{\det}$. Moreover, the isomorphisms ϕ_* (see* (3.4)*) and ϕ^* intertwine the degree pairing* (4.1) *on the right and the natural pairing induced from that between $\mathfrak{h}$ and $\mathfrak{h}^*$ on the left.*

Consider a line bundle $\mathcal{L} = \mathcal{L}_\chi \otimes \mathcal{L}_{\det}^{\otimes\kappa}$. It admits a $\widetilde{T}$-equivariant structure and different choices of equivariant structures only result in a translation of the moment map image. We have the associated moment map $m_{\mathcal{L}}\colon \mathcal{F}\ell_G \to \widetilde{\mathfrak{t}}^*_{\mathbb{R}} = \mathbb{R}\delta \oplus \mathfrak{t}^*_{\mathbb{R}}$. Using the duals of the results in Proposition 3.3, we can write down the image $m_{\mathcal{L}}(\widetilde{w}\cdot x_0)$ explicitly.

Lemma 4.2 *For $\widetilde{w} = (\lambda, w) \in \widetilde{W}$, the image of $\widetilde{w}\cdot x_0$ under the moment map $m_{\mathcal{L}}$ is*

$$m_{\mathcal{L}}(\widetilde{w}\cdot x_0) = \left(-\langle w\chi, \lambda\rangle - \tfrac{\kappa}{2}|\lambda|^2, w\chi + \kappa\sigma(\lambda)\right),$$

where $\sigma\colon \mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^*$ is induced by the Killing form $(\,\cdot\,|\,\cdot\,)$.

It is then straightforward to see the following.

Theorem 4.3 *The image of $\mathcal{F}\ell_G^T$ under the moment map $m_{\mathcal{L}}$ lies on the paraboloid*

$$\mathfrak{P}_{\chi,\kappa} = \left\{ (m_0, m_1) \in \mathbb{R}\delta \oplus \mathfrak{t}^*_{\mathbb{R}} | m_0 = -\tfrac{1}{2\kappa}(|m_1|^2 - |\chi|^2) \right\}.$$

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References

[1] M. F. Atiyah and A. N. Pressley, *Convexity and Loop Groups. Arithmetic and Geometry.* Progr. Math. 36, Birkhäuser Boston, Bostohn, MA, 1983, pp. 33-64.

[2] M. Goresky and R. MacPherson, *On the spectrum of the equivariant cohomology ring.* Canad. J. Math. **62**(2010), 262–283 (this issue). doi:10.4153/CJM-2010-016-4

[3] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality and the localization theorem.* Invent. Math. **131**(1998), no. 1, 25–83. doi:10.1007/s002220050197

[4] ———, *Homology of affine Springer fibers in the unramified case.* Duke Math. J. **121**(2004), no. 3, 509–561. doi:10.1215/S0012-7094-04-12135-9

[5] V. Kac, *Infinite-dimensional Lie algebras.* Third edition. Cambridge University Press, Cambridge, 1990.

[6] B. Kostant and S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G.* Adv. in Math. **62**(1986), no. 3, 187–237. doi:10.1016/0001-8708(86)90101-5

[7] Y. Laszlo and C. Sorger, *The line bundles on the moduli of parabolic G-bundles over curves and their sections.* Ann. Sci. École Norm. Sup. **30**(1997), no. 4, 499–525.

[8] Z. Yun, *Goresky-MacPherson calculus for the affine flag varieties.* arXiv:0712.4395.

Princeton University, Princeton, New Jersey 08544, USA
e-mail: zyun@math.princeton.edu