MATH 380A HOMEWORK 1

DUE ON SEP 6

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notation:

- R: an associative ring with unit (not necessarily commutative);
- A: a commutative and associative ring with unit;
- k: a field.
- An ideal in R always means a two-sided ideal unless we explicitly say left ideal or right ideal.

1. Theorems

Theorem 1 (Chinese Remainder Theorem)**.** *Let R be a ring and $I_1, \cdots, I_n$ be ideals. Suppose for $1 \leq i \neq j \leq n$ we have $I_i + I_j = R$. Then for any given $b_i \in R/I_i$ ($i = 1, \cdots, n$), the system of congruence equations*

$$x \equiv b_i (\text{mod } I_i), i = 1, \cdots, n \tag{1.1}$$

has a solution $x \in R$, and the solution is unique modulo $I_1 \cap I_2 \cap \cdots \cap I_n$.

2. Exercises

2.1. A map $f : R \to R'$ between rings is called a *non-unital homomorphism* if it satisfies all conditions for a ring homomorphism, except that $f(1_R)$ is not necessarily $1_{R'}$.

(1) Give an example of a *non-unital homomorphism* f such that $f(1_R) \neq 1_{R'}$.
(2) If a non-unital homomorphism $f : R \to R'$ is surjective, show that f is a ring homomorphism.
(3) If under a non-unital homomorphism $f : R \to R'$, $f(1_R)$ is invertible in R', show that f is a ring homomorphism.

2.2. An element a in a ring R is called *nilpotent* if $a^n = 0$ for some n.

(1) If R is commutative, show that nilpotent elements form an ideal.
(2) Show that nilpotent elements are contained in every prime ideal of R.
(3) When R is not commutative, give an example to show that the set of nilpotent elements may not be closed under addition.

2.3. Let k be a field and $R = \text{Mat}_n(k)$. List all left ideals, right ideals and two-sided ideals of R. Justify your answer.

2.4. Let R be a ring and S be a set. We shall define the *free R-algebra $R\langle S\rangle$ generated by S* as the solution to the following universality problem. First of all, $R\langle S\rangle$ is an R-algebra equipped with a map $i_0 : S \to R\langle S\rangle$. Next, for each R-algebra R' and each map $i : S \to R'$ (of sets), we require that there should be a unique R-algebra homomorphism $f : R\langle S\rangle \to R'$ such that $f \circ i_0 = i$.

(1) Explicitly construct the R-algebra $R\langle S\rangle$ which satisfies the above universal property.
(2) When S is a set with one element, show that $R\langle S\rangle \cong R[x]$, the polynomial ring in one variable.
(3) Show, using definition and *not* the construction in (1), that $R\langle S\rangle$ is not commutative when $\#S > 1$.

2.5. Suppose $\mathrm{char}(k) = 0$. A *differential operator* on the polynomial ring $k[x]$ is a map $T : k[x] \to k[x]$ of the form

$$f(x) \mapsto a_0(x)f(x) + a_1(x)f'(x) + \cdots + a_n(x)f^{(n)}(x) \tag{2.1}$$

where $f^{(i)}$ denotes the i-th derivative of f and $a_i(x) \in k[x]$. Let D be the set of differential operators on $k[x]$.

(1) Show that D is a k-algebra, with the multiplication given by composition of differential operators.
(2) Show that D is isomorphic to the k-algebra $k\langle a, b\rangle/I$ where $k\langle a, b\rangle$ is the free k-algebra generated by two variables a, b and I is the two-sided ideal generated by $ab - ba - 1$.
(3) What are the invertible elements in D?

2.6. Let $A = k[[x_1, \cdots, x_n]]$ be the ring of formal power series in n variables.

(1) What are the invertible elements in A?
(2) Show that $I = (x_1, \cdots, x_n)$ is the unique maximal ideal in A.

2.7. Let k be a field. Let $f : k[x] \to k[x]$ be a k-linear ring automorphism. Show that f sends x to a linear function $ax + b$ with $a \neq 0$.

2.8. Let $G = \mathbb{Z}/4\mathbb{Z}$ and let $A = \mathbb{R}[G]$ be the group algebra of G over the real numbers $\mathbb{R}$. Show that $A \cong \mathbb{R} \times \mathbb{R} \times \mathbb{C}$ as $\mathbb{R}$-algebras.