MATH 380A HOMEWORK 10

DUE ON NOV.15

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked “optional”.

Notations:

- A denotes a commutative ring.
- k denotes a field.

1. Theorems

Theorem 1. *Tensor product is a right exact functor. More precisely, for A-modules M', M, M'' and N such that $M' \to M \to M'' \to 0$ is exact, $M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$ is also exact.*

(You should use the definition of $M \otimes_A N$ as the object co-representing $\mathrm{Bil}_A(M, N; -)$.)

2. Exercises

2.1. Which of the following maps

$$k[x] \times k[y] \to k[z]$$

are k-bilinear?

(1) $(f, g) \mapsto f(z) + g(z)$;
(2) $(f, g) \mapsto f(z)$;
(3) $(f, g) \mapsto f(z^2)g(z^3)$;
(4) $(f, g) \mapsto f(g(z))$;
(5) $(f, g) \mapsto f(0)g(z)$;
(6) $(f, g) \mapsto f'(z)g'(z)$; $(-)'$ means taking the derivative.

Justify your answer.

2.2. Let M, N be A-modules, and let

$$\mathrm{can}_{M,N} : M \times N \to M \otimes_A N$$

be the canonical map sending (x, y) to the pure tensor $x \otimes y$. Show that for any $a \in A$ and any $x \in M, y \in N$, we have equalities in $M \otimes_A N$

$$a(x \otimes y) = (ax) \otimes y = x \otimes (ay).$$

Here, the first expression refers to the action of $a \in A$ on the A-module $M \otimes_A N$.

2.3. Let B be an A-algebra (recall this means B is a commutative ring equipped with a ring homomorphism $\phi : A \to B$). Let M, N be B-modules (hence also A-modules via ϕ).

(1) Construct a canonical map $f : M \otimes_A N \to M \otimes_B N$ that sends a pure tensor $x \otimes_A y$ to the pure tensor $x \otimes_B y$, for $x \in M$ and $y \in N$.
(2) Show that f is surjective.
(3) Show that if ϕ is surjective, then f is a bijection.

2.4. Let $I, J \subset A$ be ideals. Describe $(A/I) \otimes_A (A/J)$ as an A-module.

2.5. Let A_1, A_2 be commutative rings and M_i be A_i-modules for $i = 1, 2$. We view M_i as an $A := A_1 \times A_2$-module via the projection map $\pi_i : A \to A_i$. What is $M_1 \otimes_A M_2$? Justfiy your answer.