MATH 380A HOMEWORK 11

DUE ON NOV.29

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notations:

- A denotes a commutative ring.
- k denotes a field.

1. Theorems

Theorem 1. *Let B be an A-algebra. The functor*

$$\begin{array}{rcl} A-\underline{\mathrm{mod}} & \to & B-\underline{\mathrm{mod}} \\ N & \mapsto & B \otimes_A N \end{array}$$

is left adjoint to the forgetful functor $F : B-\underline{\mathrm{mod}} \to A-\underline{\mathrm{mod}}$.

2. Exercises

2.1. Let L, M and N be A-modules.

(1) Show that the set $\mathrm{Hom}_A(M, N)$ carries a natural A-module structure.

(2) Show that there is an isomorphism of A-modules

$$\mathrm{Hom}_A(L \otimes_A M, N) \cong \mathrm{Hom}_A(L, \mathrm{Hom}_A(M, N)).$$

2.2. Let I be a small category and $I \ni i \mapsto M_i$ be an I-diagram of A-modules. Let N be another A-module. Show that there is a canonical isomorphism of A-modules

$$(\varinjlim_{i \in I} M_i) \otimes_A N \cong \varinjlim_{i \in I} (M_i \otimes_A N).$$

2.3. Let M and N be A-modules. Show that there is an A-algebra isomorphism

$$S(M \oplus N) \cong S(M) \otimes_A S(N).$$

(The best way to do it is not by writing down the maps on elements.)

2.4. Let $\mathfrak{p}$ be a prime ideal of A. We know that $\mathfrak{p}A_{\mathfrak{p}}$ is the unique maximal ideal of $A_{\mathfrak{p}}$. Show that $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is isomorphic to the fractional field of the domain $A/\mathfrak{p}$.

2.5. Let M be an A-module. A *quadratic form on M* is a map

$$q : M \to A$$

such that

- $q(ax) = a^2 q(x)$ for all $a \in A, x \in M$.
- the map $b_q : M \times M \to A$ given by $(x, y) \mapsto q(x + y) - q(x) - q(y)$ is A-bilinear.

(1) Show that if 2 is invertible in A, the assignment $q \mapsto b_q$ gives a bijection between quadratic forms on M and symmetric bilinear forms on M with values in A.

(2) Let k be a field of characteristic 2 and $M = k^n$. Write $n = m + 2\ell$ for integers $m, \ell \geq 0$, and consider the function q_m on k^n defined by

$$q_m(x_1, \cdots, x_n) = \sum_{i=1}^{m} x_i^2 + \sum_{j=m+1}^{m+\ell} x_j x_{j+\ell}.$$

Show that q_m is a quadratic form on k^n. Moreover, show that for different m, q_m are not equivalent to each other. (Two quadratic forms q, q' on a k-vector space M are called equivalent if there exists a linear automorphism $T : M \xrightarrow{\sim} M$ such that $q(Tx) = q'(x)$ for all $x \in M$.)

(3) Let k be a perfect field of characteristic 2 (perfect in this context means every element $a \in k$ has a square root). Let M be a k-vector space. We let

$$Q(M) = (\wedge^2 M) \times M$$

as a set. For $(p, x), (q, y) \in Q(M)$, where $p, q \in \wedge^2 M$ and $x, y \in M$, we define

$$(p, x) + (q, y) = (p + q + x \wedge y, x + y).$$

For $a \in k$, $p \in \wedge^2 M$ and $x \in M$, define

$$a(p, x) = (ap, a^{1/2}x).$$

Show that under these operations, $Q(M)$ becomes a k-vector space.

(4) In the situation of (3), show that there is a canonical bijection between quadratic forms on M and $\mathrm{Hom}_k(Q(M), k)$, the set of k-linear functions on $Q(M)$.