MATH 380A HOMEWORK 12

NO NEED TO SUBMIT

Notations:

- A denotes a commutative ring.
- k denotes a field.

1. Theorems

Theorem 1 (Snake lemma)**.** *Consider the following diagram of A-linear maps between A-modules*

$$\begin{tikzcd} & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "s"] & N \arrow[r, "t"] & N'' & \end{tikzcd} \tag{1.1}$$

where the top row is right exact, the bottom row is left exact, and the squares are commutative. Then there is an A-linear map $\delta : \ker(f'') \to \operatorname{coker}(f')$ making the following a long exact sequence

$$\ker(f') \xrightarrow{u} \ker(f) \xrightarrow{v} \ker(f'') \xrightarrow{\delta} \operatorname{coker}(f') \xrightarrow{s} \operatorname{coker}(f) \xrightarrow{t} \operatorname{coker}(f'').$$

Here u, v, s and t are induced by the same-named maps in the diagram (1.1)*.*

2. Exercises

2.1. Consider a diagram of A-modules (where dotted arrows do not exist)

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed] & P_1 \arrow[r] \arrow[d, dashed] & P_0 \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r] & C_2 \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

The top row is a projective resolution of M; the bottom row is exact. Show that

1. It is possible to fill in the dotted arrows so that the diagram becomes a morphism between complexes (i.e., all squares are commutative).
2. Any two ways of filling in the dotted arrows are chain homotopic to each other.

2.2. Let $M = [\cdots \to M_i \xrightarrow{f_i} M_{i-1} \to \cdots]$ be a complex of A-modules. Let $N = [\cdots \to N_i \xrightarrow{g_i} N_{i-1} \to \cdots]$ be another complex of A-modules. Let $\phi = (\phi_i)$ and $\psi = (\psi_i)$ be morphisms of complexes $M \to N$. Suppose there is a chain homotopy between ϕ and ψ (recall this means that there exist A-linear maps $h_i : M_i \to N_{i+1}$ such that $\phi_i - \psi_i = h_i f_i + g_{i+1} h_i$ for all i), show that ϕ and ψ induce the same map $H_i(M) \to H_i(N)$ for all $i \in \mathbb{Z}$.

2.3. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of A-modules. Let M be another A-module. Show that we have a long exact sequence

(2.1)

$$0 \to \operatorname{Hom}_A(M, N') \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, N'') \to \operatorname{Ext}^1_A(M, N') \to \operatorname{Ext}^1_A(M, N) \to \operatorname{Ext}^1_A(M, N'') \to \operatorname{Ext}^2_A(M, N') \cdots$$

2.4. Let M be an A-module. Show that the following are equivalent:

(1) M is a projective A-module.
(2) M is a direct summand of a free A-module.
(3) For any A-module N and $i > 0$, $\mathrm{Ext}^i_A(M, N) = 0$.
(4) For any A-module N, $\mathrm{Ext}^1_A(M, N) = 0$.

2.5. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Show that

(1) If M' and M'' are projective A-modules, so is M.
(2) If M and M'' are projective A-modules, so is M'.

2.6. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

2.7. Let $A = \mathbb{Z}[\sqrt{-5}]$. Show that the ideal $(2, 1 + \sqrt{-5})$ is a projective A-module.

2.8. Let $A = B[x, y]$. Compute $\mathrm{Ext}^i_A(B, B)$. Here B is viewed as an A-module by letting x and y act as zero.