MATH 380A HOMEWORK 2

DUE ON SEP 13

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notation:

- R: an associative ring with unit (not necessarily commutative);
- A: a commutative and associative ring with unit;
- k: a field.
- An ideal in R always means a two-sided ideal unless we explicitly say left ideal or right ideal.
- For an ideal $I \subset A$ and an A-module M, $I \cdot M := \{a \cdot m | a \in I, m \in M\} \subset M$.
- For two ideals $I, J \subset A$, $IJ := \{ab | a \in I, b \in J\}$ is another ideal of A.

1. Theorems

Theorem 1 (Chinese Remainder Theorem, if you haven't done it last week)**.** *Let R be a ring and $I_1, \cdots, I_n$ be ideals. Suppose for $1 \le i \ne j \le n$ we have $I_i + I_j = R$. Then for any given $b_i \in R/I_i$ $(i = 1, \cdots, n)$, the system of congruence equations*

$$x \equiv b_i (\mathrm{mod}\ I_i), i = 1, \cdots, n \tag{1.1}$$

has a solution $x \in R$, and the solution is unique modulo $I_1 \cap I_2 \cap \cdots \cap I_n$.

Theorem 2. *If A is a UFD, the polynomial ring $A[x_1, \cdots, x_n]$ over A is also a UFD.*

2. Exercises

2.1. Let A be a commutative ring and let M, N be A-modules. Let $\mathrm{Hom}_A(M, N)$ be the set of A-linear maps from M to N. Define an A-module structure on $\mathrm{Hom}_A(M, N)$.

2.2. Let A be a domain. For an A-module M, let $M^\vee = \mathrm{Hom}_A(M, A)$.

1. Show that $M^\vee$ is torsion-free (i.e., if $a \in A - \{0\}$ and $x \in M^\vee - \{0\}$, then $a \cdot x \ne 0$).
2. Define a canonical A-linear map $\iota_M : M \to (M^\vee)^\vee$.
3. Give an example of (A, M) where ι_M is not injective.
4. Give an example of (A, M) where ι_M is not surjective.

2.3. Let A be a commutative ring and M be an A-module generated by m elements. Let $N \subset M$ be an A-submodule. Should N be generated by at most m elements? Justify your answer.

2.4. Let $\mathfrak{m}$ be a maximal ideal of A. Let $k = A/\mathfrak{m}$ be the residue field.

1. Let M be a finitely generated A-module which is killed by a power of $\mathfrak{m}$ (i.e., $\mathfrak{m}^\ell \cdot M = 0$ for some $\ell > 0$). Show that there exists a finite sequence of A-submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M \tag{2.1}$$

such that M_i/M_{i-1} is killed by $\mathfrak{m}$ for each i (i.e., M_i/M_{i-1} is a k-vector space).

(2) Let M be as in (1) and define the length of M to be

$$\ell(M) = \sum_{i=1}^{n} \dim_k(M_i/M_{i-1}).$$

Show that $\ell(M)$ is independent of the choice of the filtration (2.1).

(3) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules of the kind in (1). Show that

$$\ell(M) = \ell(M') + \ell(M'').$$

2.5. Let A be a (commutative) k-algebra which is finite-dimensional as a k-vector space.

(1) If A is a domain, show that A is a field.
(2) (without assuming A is a domain) Show that every prime ideal of A is maximal.

2.6. (Optional) Let $f : A^n \to A^n$ be a surjective A-linear map between free A-modules of the same rank n. Show that f is an isomorphism.