# MATH 380A HOMEWORK 3

DUE ON SEP 20

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notation:

- $A$: a commutative and associative ring with unit;
- $k$: a field.

## 1. Theorems

**Theorem 1** (Cayley-Hamilton)**.** *Let $T : A^n \to A^n$ be an $A$-linear map. Let $P(x) \in A[x]$ be the characteristic polynomial of $T$. Then $P(T) = 0$ as an endomorphism of $A^n$.*

## 2. Exercises

**2.1.** Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $A$-modules. Show that if $M'$ and $M''$ are finitely generated as $A$-modules, so is $M$.

**2.2.** This exercise gives an example of a submodule of a finitely generated module that is not finitely generated. Let $A = k[x_1, x_2, \cdots]$ be the polynomial ring over a field $k$ with infinitely many variables. Let $\mathfrak{m} = (x_1, x_2, \cdots)$ be the ideal generated by all the variables $x_i$. Show that $\mathfrak{m}$ is not finitely generated as an $A$-module.

**2.3.** Show the following generalization of the Chinese Remainder Theorem. Let $A$ be a commutative ring and $M$ be an $A$-module. Let $I_1, \cdots, I_n$ be ideals of $A$ such that $I_i + I_j = A$ for $i \neq j$. Then the natural map (given by projections onto each factor)

$$M/(I_1 \cap \cdots \cap I_n)M \to M/I_1M \oplus \cdots \oplus M/I_nM$$

is an isomorphism of $A$-modules.

**2.4.** Let $M$ be an $A[x]$-module generated by one element. Moreover, suppose $M$ is free as an $A$-module of rank $n$. Show that there is a *monic* polynomial $f(x) \in A[x]$ of degree $n$ such that $M \cong A[x]/(f(x))$ as an $A$-module. (Hint: you may need to use a previous homework problem.)

**2.5.** Let $A$ be a PID. An element $x = (x_1, \cdots, x_n) \in A^n$ is *primitive* if the gcd of $x_1, \cdots, x_n$ is 1. Show that an element $x \in A^n$ extends to a basis of $A^n$ if and only if it is primitive.

**2.6.** Let $A$ be a PID. Let $M$ be a finitely generated $A$-module of the form

$$M \cong A^r \oplus A/(f_1) \oplus \cdots \oplus A/(f_n) \tag{2.1}$$

for some nonzero elements $f_1, \cdots, f_n$ (note that we are **not** requiring $f_1|f_2$ etc). The *characteristic ideal* $\chi_M$ of $M$ is the ideal generated by $f_1 f_2 \cdots f_n$ in $A$.

1. Show that $\chi_M$ depends only on $M$ and not on the decomposition (2.1).
2. Show that the formation of $\chi_M$ is additive in $M$ when $M$ is torsion: this means if we have a short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated torsion $A$-modules, then $\chi_M = \chi_{M'}\chi_{M''}$ (as ideals on $A$).
3. When $A = k[x]$ and $V$ is an $A$-module which is finite-dimension over $k$, in other words the action of $x$ on $V$ is a linear transformation $T : V \to V$. What is $\chi_M$ in linear algebra terms?