MATH 380A HOMEWORK 4

DUE ON SEP 27

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notation:

- A: a commutative and associative ring with unit;
- k: a field.

1. Theorems

Theorem 1. *Let A be a PID and M be a finitely generated torsion-free A-module. Then M is a free A-module.*

2. Exercises

2.1. Let A be a domain and M an A-module. A submodule $N \subset M$ is called *saturated* if $am \in N$ $(a \in A, m \in M)$ implies either $m \in N$ or $a = 0$.

(1) Show that N is saturated if and only if M/N is a torsion-free A-module.
(2) Let $N \subset M$ be any submodule. Let $\widetilde{N} = \{m \in M | \text{there exists } a \in A - \{0\} \text{ such that } am \in N\}$. Show that $\widetilde{N}$ is a saturated A-submodule of M containing N, and is the smallest one with this property. The module $\widetilde{N}$ is called the *saturation* of N.
(3) If N is a finitely generated A-submodule of M, should its saturation be finitely generated?

2.2. Let A be a PID, M be a finitely-generated torsion A-module and $T : M \to M$ be an A-linear map.

(1) Show that $\ker(T)$ and $\mathrm{coker}(T)$ have the same characteristic ideal (see Homework 3, Problem 2.6).
(2) Give an example where $\ker(T)$ and $\mathrm{coker}(T)$ are not isomorphic as A-modules.

2.3. Let V be a finite-dimensional vector space over a field k, and $T : V \to V$ be a *nilpotent* linear transformation (i.e., $T^N = 0$ for some $N > 0$). For each $i \geq 1$, express $\dim \ker(T^i)$ and $\dim \mathrm{Im}(T^i)$ in terms of the sizes of the Jordan blocks of T.

2.4. Let V be a finite-dimensional vector space over a field k. Let $T : V \to V$ be a linear transformation. Let $\chi_T \in k[x]$ be the characteristic polynomial of T.

(1) View V as a $k[x]$-module where x acts as T. For each monic irreducible polynomial $\pi \in k[x]$, let $V(\pi)$ be the subspace of $v \in V$ which are killed by a power of $\pi(T)$. Show that $V(\pi)$ is stable under T, $V(\pi) \neq 0$ if and only if π divides the minimal polynomial m_T of T, and that $V = \oplus_{\pi | m_T} V(\pi)$.
(2) Let $\mathrm{End}_T(V)$ be the ring of linear transformations of V that commute with T. Show that $\mathrm{End}_T(V)$ is commutative if and only if $m_T = \chi_T$.
(3) (Optional) In case $\mathrm{End}_T(V)$ is commutative, describe this commutative k-algebra in terms of χ_T.