MATH 380A HOMEWORK 5

DUE ON OCT. 4

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notation:

- A: a commutative and associative ring with unit;
- k: a field.

1. Theorems

Theorem 1. *Let A be a PID, and M and N are free A-modules of finite rank. Let $T : M \to N$ be an A-linear map. Then there exists an A-basis $\{x_1, x_2, \cdots, x_m\}$ for M, an A-basis $\{y_1, \cdots, y_n\}$ for N, and elements $f_1, \cdots, f_r \in A - \{0\}$ ($0 \le r \le \min\{m, n\}$) such that*

1. *$T(x_i) = f_i y_i$ for $1 \le i \le r$.*
2. *$T(x_i) = 0$ for $i > r$.*
3. *$f_i | f_{i+1}$ for $i = 1, \cdots, r-1$.*

2. Exercises

2.1. Describe rational canonical forms for matrices with entries in $\mathbb{R}$.

2.2. Let $T \in \mathrm{Mat}_n(k)$. Consider the matrix $xI_n - T$ with entries in $A = k[x]$ (I_n is the identity matrix), viewed as an A-linear map $A^n \to A^n$. According to the structure theorem for A-modules, we may diagonalize $xI_n - T$ into $\mathrm{diag}(f_1, f_2, \cdots, f_n)$ by invertible row and column operations. Here $f_1 | \cdots | f_n$ are monic polynomials or zero ($f_i = 1$ is allowed).

1. Show that the f_i's are all nonzero.
2. On the other hand, viewing k^n as an A-module with x acting as T, the structure theorem tells that $k^n \cong A/(g_1) \oplus \cdots \oplus A/(g_\ell)$ with monic polynomials $g_1 | \cdots | g_\ell$. What is the relationship between the two sets of polynomials $\{f_i\}$ and $\{g_i\}$?

   (Hint: consider the cokernel of the map $xI_n - T : A^n \to A^n$.)

2.3. Let R be a (not necessarily commutative) ring. Let $\mathcal{C}$ be the category of left R-modules. Describe the natural transformations of the identity functor $\mathrm{id}_{\mathcal{C}}$ to itself in terms of R. What kind of algebraic object do they form (e.g. a group, or a ring)?

2.4. Let M be a free A-module with a countable basis $e_1, e_2, e_3, \cdots$. We write the element $\sum_i a_i e_i \in M$ as $(a_1, a_2, \cdots)$ (only finitely many coordinates are nonzero).

1. For each positive integer n, let $M_n = M$. Define a direct system $\{M_n\}_{n \ge 1}$ using the maps $T_n : M_n \to M_{n+1}$ by $(a_1, a_2, \cdots) \mapsto (a_2, a_3, \cdots)$. What is $\varinjlim M_n$?
2. Again let $M_n = M$ for each positive integer n. Define an inverse system $\{M_n\}_{n \ge 1}$ using the maps $S_n : M_n \to M_{n-1}$ given by the same formula $(a_1, a_2, \cdots) \mapsto (a_2, a_3, \cdots)$. What is $\varprojlim M_n$?