MATH 380A HOMEWORK 6

DUE ON OCT. 11

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notation:

- Suppose $\mathcal{C}$ is a category, its opposite category $\mathcal{C}^{\mathrm{op}}$ is the category with the same collection of objects as $\mathcal{C}$, but for any two objects X, Y, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ is defined to be $\mathrm{Hom}_{\mathcal{C}}(Y, X)$.

1. Theorems

Theorem 1 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category and let $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}})$ be the category of functors from the opposite category $\mathcal{C}^{\mathrm{op}}$ to the category of sets. Then the functor*

$$\begin{aligned} h : \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}}) \\ X &\mapsto (h_X : Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)) \end{aligned}$$

is fully faithful.

2. Exercises

2.1. Let $\mathcal{C}$ be the category of pointed sets: the objects of $\mathcal{C}$ are pairs (X, x) where X is a set and $x \in X$ is an element in X (called the base point of X). Morphisms in $\mathcal{C}$ are maps between sets that send base point to base point. Describe in concrete terms the direct sum and direct product of two objects (X, x) and (Y, y) in $\mathcal{C}$ and justify your answer.

2.2. Let $\{M_i\}_{i\in I}, \{M_i'\}_{i\in I}$ and $\{M_i''\}_{i\in I}$ be direct systems of A-modules indexed by some directed index set I. Suppose for each $i \in I$ we have a short exact sequence of A-modules $0 \to M_i' \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} M_i'' \to 0$. Suppose for each pair $i \le j$ in I, the short exact sequences for i and j fit into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M_i' \arrow[r, "\alpha_i"] \arrow[d] & M_i \arrow[r, "\beta_i"] \arrow[d] & M_i'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_j' \arrow[r, "\alpha_j"] & M_j \arrow[r, "\beta_j"] & M_j'' \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are the transition maps defining the direct systems.

(1) Show that $\{\alpha_i\}_{i\in I}$ induce an A-linear map $\alpha : \varinjlim_{i\in I} M_i' \to \varinjlim_{i\in I} M_i$, and $\{\beta_i\}_{i\in I}$ induce an A-linear map $\beta : \varinjlim_{i\in I} M_i \to \varinjlim_{i\in I} M_i''$.

(2) Show that

$$0 \to \varinjlim_{i\in I} M_i' \xrightarrow{\alpha} \varinjlim_{i\in I} M_i \xrightarrow{\beta} \varinjlim_{i\in I} M_i'' \to 0$$

is a short exact sequence.

2.3. Let $\mathcal{C}$ be a category. Let $E \in \mathcal{C}$ be an object which is both initial and final. Let $\{X_i\}$ be a collection of objects in $\mathcal{C}$ indexed by a set I, and suppose the direct sum $\coprod_{i\in I} X_i$ and direct product $\prod_{i\in I} X_i$ exist. Construct a canonical morphism $\coprod_{i\in I} X_i \to \prod_{i\in I} X_i$

Hint: you are going to use E at some point!

2.4. Let G be a group. Let I be the category with one object i_0 and morphisms $\mathrm{Hom}_I(i_0, i_0) = G$. Let $\underline{\mathrm{Ab}}$ be the category of abelian groups (with group homomorphisms as morphisms).

(1) What is an I-diagram in $\underline{\mathrm{Ab}}$ in more concrete terms?
(2) Let $X : I \to \underline{\mathrm{Ab}}$ be an I-diagram in $\underline{\mathrm{Ab}}$. Describe the colimit $\varinjlim_I X$.