MATH 380A HOMEWORK 7

DUE ON OCT. 25

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notations:

- A denotes a commutative ring;
- $S \subset A$ denotes a multiplicative subset.
- For a non-nilpotent element $f \in A$, A_f denotes the localization of A with respect to the multiplicative set $\{f^n | n \geq 0\}$.

1. Theorems

Theorem 1. *Let $\mathcal{C}_S$ be the category whose objects are A-algebras $\varphi : A \to B$ such that $\varphi(S)$ consists of invertible elements in B, and whose morphisms are A-algebra maps. Then $\mathcal{C}_S$ has an initial object.*

2. Exercises

2.1. Show that $A \to S^{-1}A$ is injective if and only if S does not contain zero-divisors in A.

2.2. Let $a \in A$ be non-nilpotent. Denote by $\frac{1}{a} \in A$ the inverse of a in A_a. Consider the A-linear ring homomorphism

$$\varphi_a : A[x]/(ax - 1) \to A_a$$

sending $x \mapsto \frac{1}{a}$. Show that φ_a is an isomorphism of A-algebras.

2.3. Let $C = A \times B$ be the direct product of two commutative rings A and B. Let $S = \{(0, 1)\}$, which is a multiplicative subset of C. Describe the localization $S^{-1}C$.

2.4. Let A, S be as in Notations.

1. Show that if $f | g$ (i.e., $g = fa$ for some $a \in A$) then there is a unique A-algebra map $A_f \to A_g$.
2. (Optional) Let I_S be the category whose objects are elements in S, and $\mathrm{Hom}_{I_S}(f, g)$ is either a one element set if $f | g$ or empty if f does not divide g, for all $f, g \in S$. Show that $\{A_f\}_{f \in S}$ form an I_S-diagram in the category of A-algebras (this is a reformulation of part (1)). Show that there is a canonical isomorphism of A-algebras

   $$\varinjlim_{f \in I_S} A_f \cong S^{-1}A.$$

   Hint: use the universal property of the localization.

2.5. Let $S \subset A$ be a multiplicative subset. Let I be a directed set and let $\{A_i\}_{i \in I}$ be a direct system of A-algebras (so the transition maps $A_i \to A_j$ are A-linear ring homomorphisms, for $i \leq j$). We denote by $S^{-1}A_i$ the localization of A_i with respect to the image of S in A_i. Show that $\{S^{-1}A_i\}$ is naturally a direct system of $S^{-1}A$-algebras, and there is a canonical $S^{-1}A$-linear ring isomorphism

$$\varinjlim_{i \in I} S^{-1}A_i \cong S^{-1}(\varinjlim_{i \in I} A_i)$$

where the right side means the localization of the A-algebra $\varinjlim_{i \in I} A_i$ with respect to the image of S.

2.6. Show that $(S^{-1}A)[x] \cong S^{-1}(A[x])$ as $A[x]$-algebras. Here $S^{-1}(A[x])$ is the localization of $A[x]$ with respect to S, viewed as a multiplicative subset of $A[x]$ consisting of constant polynomials with values in S.

2.7. Let $A = \mathbb{Z}[[x]]$, the ring of formal power series with $\mathbb{Z}$-coefficients. Let $S = \mathbb{Z}\backslash\{0\}$, viewed as a multiplicative subset of A. Is $S^{-1}A$ the same as $\mathbb{Q}[[x]]$? Justify your answer.