MATH 380A HOMEWORK 8

DUE ON NOV.1

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notations:

- A denotes a commutative ring;
- $S \subset A$ denotes a multiplicative subset.
- For a non-nilpotent element $f \in A$, A_f denotes the localization of A with respect to the multiplicative set $\{f^n | n \geq 0\}$.

1. Theorems

Theorem 1. *Let $F : S^{-1}A - \underline{\mathrm{Mod}} \longrightarrow A - \underline{\mathrm{Mod}}$ be the forgetful functor. Show that the localization functor $S^{-1}(-) : A - \underline{\mathrm{Mod}} \longrightarrow S^{-1}A - \underline{\mathrm{Mod}}$ is a left adjoint of F.*

2. Exercises

2.1. Let I be a small category. Let $\{X_i\}_{i\in I}$ be an I-diagram in a category $\mathcal{C}$. Let $Y \in \mathcal{C}$ be any object.

(1) Let I^{op} be the opposite category to I. Show that the assignment $i \mapsto \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$ gives an I^{op}-diagram in the category of sets.

(2) Suppose the colimit $\varinjlim_{i\in I} X_i$ exists in $\mathcal{C}$. Show that there is a natural bijection of sets

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim_{i\in I} X_i, Y) \cong \varprojlim_{i\in I^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y).$$

2.2. Let $L : \mathcal{C} \to \mathcal{D}$ be a functor with a right adjoint $R : \mathcal{D} \to \mathcal{C}$. Let $\{X_i\}_{i\in I}$ be an I-diagram whose colimit $\varinjlim_{i\in I} X_i$ exists in $\mathcal{C}$. Explain how $\{L(X_i)\}_{i\in I}$ form an I-diagram in $\mathcal{D}$. Show that $L(\varinjlim_{i\in I} X_i)$ is the colimit of $\{L(X_i)\}_{i\in I}$ in $\mathcal{D}$.

Hint: use Exercise 2.1.

2.3. Let I be a small category and let $\{M_i\}_{i\in I}$ be an I-diagram in the category of A-modules.

(1) Show that $\{S^{-1}M_i\}_{i\in I}$ form an I-diagram in the category of $S^{-1}A$-modules.

(2) Show that there is a canonical isomorphism of $S^{-1}A$-modules

$$S^{-1}(\varinjlim_{i\in I} M_i) \cong \varinjlim_{i\in I} S^{-1}M_i.$$

2.4. Suppose A is a domain with field of fractions K. For any prime ideal $\mathfrak{p}$ of A we view the localization $A_{\mathfrak{p}}$ as a subring of K. Show that $A = \cap A_{\mathfrak{m}}$ where $\mathfrak{m}$ runs over all maximal ideals of A.

2.5. Let $f, g \in A$ be non-nilpotent elements such that $f - g$ is nilpotent. Show that there is a unique A-linear isomorphism of rings $A_f \cong A_g$.