MATH 380A HOMEWORK 9

DUE ON NOV.8

Note: This homework consists of two parts. In the first part, please write down a complete proof for each theorem stated. You can consult any book you want, but you should write the proof according to your own understanding. The second part consists of exercises which you are supposed to work out independently. You will not lose credit if you choose not to do the problems marked "optional".

Notations:

- A denotes a commutative ring;
- $S \subset A$ denotes a multiplicative subset.

1. Theorems

Theorem 1 (Nakayama's lemma)**.** *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. Let M be a finitely generated A-module.*

(1) *Suppose $M = \mathfrak{m}M$ then $M = 0$.*
(2) *Let $x_1, \cdots, x_n \in M$ be such that their images in $M/\mathfrak{m}M$ form a k-basis. Then $x_1, \cdots, x_n$ generate M as an A-module.*

2. Exercises

2.1. The *annihilator* of an A-module M is defined to be the ideal $\mathrm{Ann}_A(M) = \{a \in A | am = 0, \forall m \in M\}$.

(1) Let M be a finitely generated A-module. Show that $S^{-1}(\mathrm{Ann}_A(M)) = \mathrm{Ann}_{S^{-1}A}(S^{-1}M)$ as ideals in $S^{-1}A$.
(2) (Optional) Give an example of A and M for which the above equality does not hold.

2.2. Let M be an A-module. The *support* of M is the collection of prime ideals $\mathfrak{p} \subset A$ such that $M_{\mathfrak{p}} \neq 0$. Let $\mathrm{Supp}(M)$ denote the support of M.

(1) If A is a domain and M is torsion-free and nonzero, show that $\mathrm{Supp}(M)$ is the set of all prime ideals.
(2) If $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{p} \in \mathrm{Supp}(M)$, show that $\mathfrak{q} \in \mathrm{Supp}(M)$.
(3) If $\mathrm{Supp}(M) = \varnothing$, show that $M = 0$.
(4) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Show that $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

2.3. Let M be a finitely generated A-module. Show that $\mathfrak{p} \in \mathrm{Supp}(M)$ if and only if $\mathfrak{p} \supset \mathrm{Ann}_A(M)$. (Hint: first consider when M is generated by one element).

2.4. Let A be a local ring with maximal ideal $\mathfrak{m}$ and $f : M \to N$ be an A-linear map between finitely generated A-modules. Let $\overline{f} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ be the induced map modulo $\mathfrak{m}$.

(1) If $\overline{f}$ is surjective then so is f.
(2) Suppose further that N is a free A-module and $\overline{f}$ is an isomorphism, show that f is also an isomorphism.

2.5. Let A be a DVR with maximal ideal $\mathfrak{m}$ and let $\pi \in \mathfrak{m} - \mathfrak{m}^2$ be a uniformizer. Let n be a positive integer and $B = A[x]/(x^n - \pi)$. Show that B is also a DVR.