Macdonald formula for curves with planar singularities

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Abstract. We generalize Macdonald's formula for the cohomology of Hilbert schemes of points on a curve from smooth curves to curves with planar singularities: we relate the cohomology of the Hilbert schemes to the cohomology of the compactified Jacobian of the curve. The new formula is a consequence of a stronger identity between certain perverse sheaves defined by a family of curves satisfying mild conditions. The proof makes essential use of Ngô's support theorem for compactified Jacobians and generalizes this theorem to the relative Hilbert scheme of such families. As a consequence, we give a cohomological interpretation of the numerator of the Hilbert-zeta function of curves with planar singularities.

1. Introduction

Let C be a smooth projective connected curve over an algebraically closed field k. Let $\mathrm{Sym}^n(C)$ be the n-th symmetric product of C. Macdonald's formula [21] says there is a canonical isomorphism between graded vector spaces,

$$(1.1) \quad \mathrm{H}^*(\mathrm{Sym}^n(C)) \cong \mathrm{Sym}^n(\mathrm{H}^*(C)) = \bigoplus_{i+j \le n, i, j \ge 0} \bigwedge^i (\mathrm{H}^1(C))[-i-2j](-j).$$

Here $[?]$ denotes the cohomological shift and $(?)$ denotes the Tate twist. This formula respects Hodge structures (when $k \subset \mathbb{C}$) and Galois actions (when C comes from a non-algebraically closed field via base change). If we define the cohomological zeta function of C to be the formal power series in one variable t with coefficients in graded vector spaces,

$$Z(t, C) := \frac{\bigoplus_{i=0}^{2g} (\bigwedge^i \mathrm{H}^1(C))[-i]t^i}{(1 - \mathbb{Q}_\ell t)(1 - \mathbb{Q}_\ell[-2](-1)t)},$$

then we may rewrite (1.1) for all n at once as an identity between formal power series,

$$\bigoplus_{n \ge 0} \mathrm{H}^*(\mathrm{Sym}^n(C))t^n \cong Z(t, C).$$

The purpose of this note is to generalize Macdonald's formula to projective integral curves C with planar singularities. In this case, we will work with the Hilbert schemes

$\mathrm{Hilb}^n(C)$ instead of symmetric powers. We can reinterpret $\bigwedge^i(\mathrm{H}^1(C))$ as the i-th cohomology of the Jacobian $\mathrm{Jac}(C)$ of C. In the case C is singular, $\mathrm{Jac}(C)$ will be replaced by the compactified Jacobian $\overline{\mathrm{Jac}}(C)$ classifying torsion-free, rank one coherent sheaves with a fixed degree on C.

The main result of this note is the following theorem, which was conjectured by L. Migliorini. While writing this note, we learned that L. Migliorini and V. Shende have an independent proof of the conjecture in the case $k = \mathbb{C}$, see [22].

1.1. Theorem. *Let C be an integral projective curve over k of arithmetic genus g_a with planar singularities. Assume either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > \max\{\mathrm{mult}_p(C); p \in C\}$ ($\mathrm{mult}_p(C)$ is the multiplicity of C at p). Then there exists a canonical increasing filtration $P_{\leq i}$ on $\mathrm{H}^*(\overline{\mathrm{Jac}}(C))$, normalized such that $\mathrm{Gr}_i^P \mathrm{H}^*(\overline{\mathrm{Jac}}(C)) = 0$ unless $0 \leq i \leq 2g_a$, such that for each non-negative integer n, there is an isomorphism between* graded *vector spaces (gradings are given by $*$, and are shifted by $[-2j]$ in the usual way),*

$$(1.2) \qquad \mathrm{H}^*(\mathrm{Hilb}^n(C)) \cong \bigoplus_{i+j \leq n, i,j \geq 0} \mathrm{Gr}_i^P(\mathrm{H}^*(\overline{\mathrm{Jac}}(C)))[-2j](-j).$$

The canonical filtration $P_{\leq i}$ on $\mathrm{H}^*(\overline{\mathrm{Jac}}(C))$ is defined by embedding the curve C into a suitable family of curves, and taking the perverse filtration on the direct image complex of the compactified Jacobian of the family. For this reason, we call $P_{\leq i}$ the perverse filtration. It turns out that this filtration is independent of the choice of such families (see Proposition 2.15 and §3.8).

If we define the cohomological zeta function of C with respect to the perverse filtration to be

$$Z_P(t, C) := \frac{\bigoplus_{i=0}^{2g_a} \mathrm{Gr}_i^P \mathrm{H}^*(\overline{\mathrm{Jac}}(C)) t^i}{(1 - \mathbb{Q}_\ell t)(1 - \mathbb{Q}_\ell[-2](-1)t)},$$

then we can restate the main theorem as an identity between formal power series in graded vector spaces,

$$\bigoplus_{n \geq 0} \mathrm{H}^*(\mathrm{Hilb}^n(C)) t^n \cong Z_P(t, C).$$

Qualitatively, this theorem says that the cohomological information of all the Hilbert schemes is already contained in the cohomology of the compactified Jacobian, equipped with the perverse filtration $P_{\leq i}$. Notice that, in this expression, the grading by number of points on the left-hand side is partially converted into the perverse filtration.

When the relevant cohomology groups carry weight filtrations (when $k \subset \mathbb{C}$) or Galois actions (when C comes from a non-algebraically closed field via base change), the isomorphism (1.2) respects these extra structures. When C is smooth, we recover (1.1) from (1.2).

1.2. Hilbert-zeta function. For a curve C defined over a finite field $\mathbb{F}_q$, one may consider its Hilbert-zeta function:

$$Z_{\mathrm{Hilb}}(t, C/\mathbb{F}_q) = \sum_{n \geq 0} \#\mathrm{Hilb}^n(C)(\mathbb{F}_q) t^n.$$

If C is an integral projective curve of arithmetic genus g_a over $\mathbb{F}_q$ with planar singularities, we have the following facts which generalize part of the Weil conjecture for smooth curves.

- $Z_{\mathrm{Hilb}}(t, C/\mathbb{F}_q)$ has the form $\frac{P(t)}{(1-t)(1-qt)}$ for some polynomial $P(t) \in \mathbb{Z}[t]$ of degree $2g_a$.
- $Z_{\mathrm{Hilb}}(t, C/\mathbb{F}_q)$ satisfies the functional equation

$$Z_{\mathrm{Hilb}}(t, C/\mathbb{F}_q) = (qt^2)^{(g_a - 1)} Z_{\mathrm{Hilb}}(q^{-1}t^{-1}, C/\mathbb{F}_q).$$

These two facts can be proved in an elementary way using the Riemann–Roch theorem for C. However, our main theorem gives a cohomological interpretation of the numerator $P(t)$ in the Hilbert-zeta function: the i-th coefficient of $P(t)$ is the trace of the geometric Frobenius Frob_q acting on $\mathrm{Gr}_i^P \mathrm{H}^*(\overline{\mathrm{Jac}}(C))$. In §3.9, we will give a functional equation for a two variable analog of $Z_{\mathrm{Hilb}}(t, C/\mathbb{F}_q)$.

1.3. Local version. We also have a local version of the above results, with C replaced by the completed local ring $\hat{\mathcal{O}}$ of a planar curve singularity. The local version is weaker than Theorem 1.1 in the sense that it is an identity between virtual Poincaré polynomials instead of the cohomology groups themselves. For more details, see §3.10 and Theorem 3.11.

In the local case, there is a conjectural relation between the virtual Poincaré polynomial of the punctual Hilbert scheme and the Khovanov–Rozansky homology of the associated link of the singular point, proposed by Oblomkov, Rasmussen and Shende [25, 26]. It would be interesting to see whether this relation can be better understood in terms of the perverse filtration on the cohomology of the compactified Jacobian.

1.4. Idea of the proof. Theorem 1.1 is proved by embedding C into a suitable family of curves, satisfying three conditions axiomatized as (A1)–(A3) in §2.1. For any such family of curves $\mathcal{C} \to \mathcal{B}$, we prove a global analog of the formula (1.2) in Theorem 2.13, which is an identity between perverse sheaves on the base of the family coming from the cohomology of the relative Hilbert schemes and the relative compactified Jacobian. The key step in the proof of the global formula is Proposition 2.12 saying that any simple perverse constituent of the direct image complex of $\mathrm{Hilb}^n(\mathcal{C}/\mathcal{B}) \to \mathcal{B}$ has full support $\mathcal{B}$. We use a cohomological correspondence argument for a descending induction on the number of points, reducing this support theorem to Ngô's support theorem for compactified Jacobians (Theorem 2.4). Therefore, our result is an application of Ngô's powerful technique to a more classical setting.

1.5. An example. Let C be the curve in $\mathbb{P}^2$ with affine equation $y^3 = x^4$. The only singularity of C is $(0, 0)$, and the smooth locus of C is isomorphic to the affine line $\mathbb{A}^1$. The arithmetic genus of C is 3. Let $\mathrm{Hilb}_{(0,0)}^i(C)$ be the Hilbert scheme of i points concentrated at $(0, 0)$. There is an action of $\mathbb{G}_m$ on C by $t \in \mathbb{G}_m : (x, y) \mapsto (t^3 x, t^4 y)$, which induces an action on $\mathrm{Hilb}_{(0,0)}^i(C)$. The Bialynicki–Birula decomposition with respect to this $\mathbb{G}_m$-action shows that $\mathrm{Hilb}_{(0,0)}^i(C)$ is paved by affine spaces whose dimensions are readily computable. One obtains an affine paving of $\mathrm{Hilb}^n(C)$ by observing that $\mathrm{Hilb}^n(C) = \bigcup_{i=0}^n \mathrm{Hilb}_{(0,0)}^i(C) \times \mathbb{A}^{n-i}$. This affine paving allows us to compute the two variable Hilbert–Poincaré series for C:

(1.3)
$$Z_{\mathrm{Hilb}}(s, t, C) := \sum_{n,i \geq 0} \dim \mathrm{H}^i(\mathrm{Hilb}^n(C)) s^i t^n = \frac{1 + s^2 t^2 + s^4 t^3 + s^4 t^4 + s^6 t^6}{(1 - t)(1 - s^2 t)}.$$

On the other hand, the compactified Jacobian $\overline{\mathrm{Jac}}(C)$ is the same as the local compactified Jacobian (or the affine Springer fiber) of the singularity at $(0, 0)$, which is studied by Lusztig–Smelt [20], Piontkowski [27] and others. It turns out that $\overline{\mathrm{Jac}}(C)$ also admits an affine paving

as $\mathbb{A}^3 \cup \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^0$, see [27, Table on p. 212, row $(p, q) = (3, 4)$] (note that the Betti numbers in [27] were mistakenly listed in the reverse order). Therefore the Poincaré polynomial of $\overline{\mathrm{Jac}}(C)$ is

$$\sum_{i=0}^{6} \dim \mathrm{H}^i(\overline{\mathrm{Jac}}(C)) s^i = 1 + s^2 + 2s^4 + s^6.$$

Comparing this with the numerator of (1.3), we see that the only difference between the perverse filtration on $\mathrm{H}^*(\overline{\mathrm{Jac}}(C))$ and the natural degree filtration is that the two-dimensional $\mathrm{H}^4(\overline{\mathrm{Jac}}(C))$ contributes one dimension to Gr_3^P and one dimension to Gr_4^P.

2. Macdonald formula for families of curves

2.1. Assumptions on the family of curves. In this section, k is any algebraically closed field. Let $\pi : \mathcal{C} \to \mathcal{B}$ be a locally projective flat family of curves over an irreducible base $\mathcal{B}$. Let g_a be the common arithmetic genus of the fibers of π. For each integer $n \geq 0$, let $f_n : \mathcal{H}_n = \mathrm{Hilb}^n(\mathcal{C}/\mathcal{B}) \to \mathcal{B}$ be the relative Hilbert scheme of n-points on the fibers of π. We assume:

(A1) Each geometric fiber $\mathcal{C}_b$ ($b \in \mathcal{B}$) is integral and has only planar singularities.

(A2) For each $0 \leq n \leq 2g_a - 1$, the total space $\mathcal{H}_n$ is smooth.

(A3) For every (not necessarily closed) point $b \in \mathcal{B}$, we have the δ-invariant $\delta(b)$ of the fiber $\mathcal{C}_b$. Then we have
$$\mathrm{codim}_{\mathcal{B}}(\overline{\{b\}}) \geq \delta(b).$$
Here $\overline{\{b\}}$ is the Zariski closure of b in $\mathcal{B}$.

Note that (A2) for $n = 0$ implies $\mathcal{B}$ is smooth. We denote $\dim \mathcal{B}$ by $d_{\mathcal{B}}$. Also, (A3) implies that the generic fiber of π is smooth, because the locus with $\delta(b) \geq 1$ has codimension at least one in $\mathcal{B}$.

2.2. Compactified Jacobians. In the following discussion, we assume:

(A4) The family π admits a section $s : \mathcal{B} \to \mathcal{C}^{\mathrm{sm}}$ where $\mathcal{C}^{\mathrm{sm}} \subset \mathcal{C}$ is the open subscheme with $\mathcal{C}_b^{\mathrm{sm}}$ being the smooth locus of $\mathcal{C}_b$ for each $b \in \mathcal{B}$.

With this assumption, one may define compactified Picard schemes $p_n : \overline{\mathrm{Pic}}_n \to \mathcal{B}$ of the family $\pi : \mathcal{C} \to \mathcal{B}$. More precisely, $\overline{\mathrm{Pic}}_n$ is the sheafification of the following presheaf: it sends every commutative k-algebra R to the set $\overline{\mathrm{Pic}}_n(R)$, the set of isomorphism classes of triples $(b, \mathcal{F}, \tau)$ where $b \in \mathcal{B}(R)$, $\mathcal{F}$ is a torsion-free, generically rank one coherent sheaf on $\mathcal{C}_b$ (a curve over $\mathrm{Spec}\,R$), flat over R, with

$$\chi(\mathcal{C}_{b'}, \mathcal{O}_{\mathcal{C}_{b'}}) - \chi(\mathcal{C}_{b'}, \mathcal{F}_{b'}) = n$$

for every geometric point $b' \in \mathrm{Spec}\,R$, and τ is an isomorphism of R-modules $\mathcal{F}_{s(b)} \xrightarrow{\sim} R$.

We can use the section s to identify the various components $\overline{\mathrm{Pic}}_n$. After this identification, we denote it by $\overline{\mathcal{J}}$, the relative compactified Jacobian.

2.3. Lemma. *Assume* (A1)–(A2) *and* (A4) *hold.*

(1) *The relative compactified Jacobian $\overline{\mathcal{J}}$ is smooth over k.*

(2) *Each geometric fiber $\overline{\mathcal{J}}_b$ of $\overline{\mathcal{J}}$ is irreducible of dimension g_a.*

Proof. By a result of Altman and Kleiman [2, Theorem 8.4 (v)], the morphism $\mathcal{H}_n \to \overline{\mathrm{Pic}}_n$ sending a subscheme of $\mathcal{C}$ to its ideal sheaf is a projective space bundle for $n \geq 2g_a - 1$. Since $\mathcal{H}_{2g_a-1}$ is smooth by (A2), so is $\overline{\mathrm{Pic}}_{2g_a-1}$, hence $\overline{\mathcal{J}}$. This proves part (1). Part (2) is the main result of [1]. $\qquad\square$

Let $p : \overline{\mathcal{J}} \to \mathcal{B}$ be the projection and set $L = \mathbf{R}p_* \mathbb{Q}_{\ell,\overline{\mathcal{J}}} \in D_c^b(\mathcal{B})$. Since $\overline{\mathcal{J}}$ is smooth and p is locally projective (see [2]), by the decomposition theorem [7, Théorème 6.2.5], we have (non-canonically)

$$(2.1) \qquad\qquad L \cong \bigoplus_i {}^p\mathrm{H}^i L[-i]$$

as objects in $D_c^b(\mathcal{B})$. Each perverse sheaf ${}^p\mathrm{H}^i L$ is semisimple.

2.4. Theorem (B. C. Ngô [23, Théorème 7.2.1]). *Assume* (A1)–(A4) *hold. For every $i \in \mathbb{Z}$ and every simple constituent M of ${}^p\mathrm{H}^i L$, the support of M is the whole of $\mathcal{B}$.*

Proof. We will apply the general result of Ngô [23, Théorème 7.2.1]. To this end, we need to check that the action of the relative Jacobian $\mathrm{Jac}(\mathcal{C}/\mathcal{B})$ over $\mathcal{B}$ on $\overline{\mathcal{J}}$ satisfies the assumptions in [23, §7.1]. First of all, $\overline{\mathcal{J}} \to \mathcal{B}$ is locally projective by Altman and Kleiman [2]. The condition [23, 7.1.2] follows from Lemma 2.3 (2). The condition [23, 7.1.3] is checked similarly as in [23, Corollaire 4.15.2], using the analog of the product formula [23, Proposition 4.15.1] in the setting of the $\mathrm{Jac}(C)$-action on $\overline{\mathrm{Jac}}(C)$. Details will be explained in §3.13 in preparation for the proof of the local Macdonald formula. The condition [23, 7.1.4] is checked in [23, §4.12] using the Weil pairing. The condition [23, 7.1.5] follows from (A3). Therefore [23, Théorème 7.2.1] is applicable.

Let Z be the support of M. The conclusion of [23, Théorème 7.2.1] is that M appears as a direct summand of $\mathbf{R}^{2g_a} p_! \mathbb{Q}_\ell$, at least when restricted to an étale neighborhood of the generic point of Z. But by Lemma 2.3 (2), $\mathbf{R}^{2g_a} p_! \mathbb{Q}_\ell = \mathbb{Q}_\ell$, which is an irreducible perverse sheaf up to shift, because $\mathcal{B}$ is smooth. Hence $Z = \mathcal{B}$. $\qquad\square$

2.5. What if there is no section? In this subsection, we discuss why the perverse sheaves ${}^p\mathrm{H}^i L$ and Theorem 2.4 still make sense even if we drop the assumption (A4).

Let $\varphi : \tilde{\mathcal{B}} \to \mathcal{B}$ be an étale surjective morphism, and $\tilde{s} : \tilde{\mathcal{B}} \to \mathcal{C}^{\mathrm{sm}}$ be a morphism such that $\pi \circ \tilde{s} = \varphi$. Such a pair $(\tilde{\mathcal{B}}, \tilde{s})$ always exists because $\mathcal{C}^{\mathrm{sm}} \to \mathcal{B}$ is smooth and surjective.

Let $\tilde{\mathcal{C}} = \tilde{\mathcal{B}} \times_{\mathcal{B}} \mathcal{C}$. Then $\tilde{\pi} : \tilde{\mathcal{C}} \to \tilde{\mathcal{B}}$ satisfies the assumptions (A1)–(A4). The above discussion on the relative compactified Jacobian makes sense for the family $\tilde{\pi}$. In particular, we have the direct image complex $\tilde{L} \in D_c^b(\tilde{\mathcal{B}})$ of the relative compactified Jacobian for $\tilde{\pi}$, and the perverse sheaves ${}^p\mathrm{H}^i \tilde{L}$ satisfy the conclusion of Theorem 2.4. The perverse sheaves ${}^p\mathrm{H}^i \tilde{L}$

carry obvious descent data with respect to the étale covering $\tilde{\mathcal{B}} \to \mathcal{B}$ (here we use the fact that the relative compactified Jacobian, once representable, is independent of the section of the family). By [7, 2.2.19], perverse sheaves satisfy étale descent, there exist (unique up to a unique isomorphism) perverse sheaves L^i on $\mathcal{B}$ such that $\varphi^* L^i \cong {}^p\mathrm{H}^i \tilde{L}$. Clearly, Theorem 2.4 holds for L^i as well. It is also easy to see that L^i is independent of the choice of $(\tilde{\mathcal{B}}, \tilde{s})$.

In the following we will use these perverse sheaves L^i when ${}^p\mathrm{H}^i L$ is not defined.

2.6. Lemma. *Under the assumptions* (A1)–(A2), *we have:*

(1) *The relative Hilbert scheme* $\mathcal{H}_n$ *is smooth and irreducible for all* $n \geq 0$.

(2) *The morphism* $f_n : \mathcal{H}_n \to \mathcal{B}$ *is flat, with all geometric fibers irreducible of dimension* n.

Proof. The statements in the lemma are local for the étale topology, so we may assume (A4) also holds, hence the compactified Jacobian $\overline{\mathcal{J}}$ exists. The argument and the statement of Lemma 2.3 (1) show that $\mathcal{H}_n$ is smooth over $\overline{\mathcal{J}}$ for $n \geq 2g_a - 1$, hence smooth. For $n \leq 2g_a - 1$, $\mathcal{H}_n$ is smooth by (A1). Hence $\mathcal{H}_n$ is smooth for all n.

The rest of the lemma will follow once we show that every geometric fiber $\mathrm{Hilb}^n(\mathcal{C}_b)$ of f_n is irreducible of dimension n for all n, which in turn follows from an argument of Rego in [28, p. 214, last paragraph]. □

2.7. The Hilbert–Chow map. For each $b \in \mathcal{B}$, there is a Zariski open neighborhood $\mathcal{B}'$ of b over which we can arrange an embedding $\mathcal{C}|_{\mathcal{B}'} \subset \mathbb{P}^N_{\mathcal{B}'}$. For a generic choice of linear projection, the induced map $\mathcal{C}_b \to \mathbb{P}^1$ is finite, and will remain so for $\mathcal{C}_{b'}$ for $b' \in \mathcal{B}''$ where $\mathcal{B}'' \subset \mathcal{B}'$ is another Zariski open neighborhood of b. Therefore we obtain a finite morphism $\mathcal{C}|_{\mathcal{B}''} \to \mathbb{P}^1 \times \mathcal{B}''$. This allows us, whenever the problem we consider is Zariski local for $\mathcal{B}$, to make the following assumption:

(A5) There is a smooth, connected, projective curve X over k and a finite morphism $\nu : \mathcal{C} \to X \times \mathcal{B}$ lifting π.

We now assume (A5) in the following discussion.

Consider the morphism

$$\nu_n : \mathcal{H}_n \to \mathrm{Sym}^n(\mathcal{C}/\mathcal{B}) \to \mathrm{Sym}^n(X) \times \mathcal{B}$$

where the first arrow is the Hilbert–Chow map relative to the base $\mathcal{B}$, and the second map is induced from the finite morphism $\nu : \mathcal{C} \to X \times \mathcal{B}$, which is proper. We understand $\mathrm{Sym}^0(X)$ as $\mathrm{Spec}\, k$.

We recall from [13, §6.2] the notion of a small map.

2.8. Definition. A proper surjective morphism $f : Y \to X$ between *irreducible* schemes over k is called *small* if for any $d \geq 1$, we have

$$\mathrm{codim}_X \{x \in X \mid \dim f^{-1}(x) \geq d\} \geq 2d + 1.$$

2.9. Lemma. *For each* $n \in \mathbb{Z}_{\geq 0}$, *the morphism* ν_n *is small.*

Proof. By Lemma 2.6 (1), $\mathcal{H}_n$ is irreducible. The morphism ν_n is clearly proper and surjective.

Fix an integer $d \geq 1$. Let $Z_d \subset \mathrm{Sym}^n(X) \times \mathcal{B}$ be the closed locus where the fibers of ν_n have dimension at least d. Let $\zeta = (D = m_1 x_1 + \cdots + m_r x_r, b)$ be a generic point of Z_d with values in some field F. Here $x_i \in X(F)$ are distinct points and $m_i > 0$. Consider the Hilbert–Chow map of $\mu_m : \mathrm{Hilb}^m(\mathcal{C}_b) \to \mathrm{Sym}^m(\mathcal{C}_b)$. Let $y \in \mathcal{C}_b$ be any closed point. Since $\mathcal{C}_b$ is locally planar, $\dim \mu_m^{-1}(my) \leq m - 1$ by a result of Iarrobino [17, Corollary 2]. On the other hand, $\mu_m^{-1}(my)$ classifies length m quotients of $\hat{\mathcal{O}}_{\mathcal{C}_b,y}$, hence $\mu_m^{-1}(my)$ is a subscheme of $\overline{\mathrm{Pic}}(\hat{\mathcal{O}}_{\mathcal{C}_b,y})$ to be defined in §3.10. Since $\overline{\mathrm{Pic}}(\hat{\mathcal{O}}_{\mathcal{C}_b,y})$ has dimension $\delta(\mathcal{C}_b; y)$ (the local δ-invariant of $\mathcal{C}_b$ at y), we have $\dim \mu_m^{-1}(my) \leq \delta(\mathcal{C}_b; y)$. Summarizing,

$$\dim \mu_m^{-1}(my) \leq \min\{m - 1, \delta(\mathcal{C}_b; y)\}, \quad \text{for all closed points } y \in \mathcal{C}_b, \ m \geq 1.$$

This implies that

$$(2.2) \qquad d \leq \dim \nu_n^{-1}(D, b) \leq \sum_{i=1}^r \min\{m_i - 1, \delta(\mathcal{C}_b; x_i)\}.$$

Here $\delta(\mathcal{C}_b; x) = \sum_{y \in \nu^{-1}(x,b)} \delta(\mathcal{C}_b; y)$ for $x \in X(F)$. In particular, $d \leq \delta(\mathcal{C}_b)$. Hence, by (A3),

$$(2.3) \qquad \mathrm{codim}_{\mathcal{B}}(\overline{\{b\}}) \geq \delta(\mathcal{C}_b) \geq d.$$

Let $S \subset X \otimes_k F$ be the finite subscheme consisting of those x with $\delta(\mathcal{C}_b; x) > 0$. The inequality (2.2) implies that at least $d + 1$ of the points in D (counted with multiplicities) are from S. This implies that

$$(2.4) \qquad \mathrm{codim}_{\mathrm{Sym}^n(X)}(\overline{\{D\}}) \geq d + 1.$$

Adding (2.3) and (2.4) together we get

$$\mathrm{codim}_{\mathrm{Sym}^n(X) \times \mathcal{B}}(\overline{\{\zeta\}}) \geq 2d + 1.$$

This being true for all generic points of Z_d, we conclude that

$$\mathrm{codim}_{\mathrm{Sym}^n(X) \times \mathcal{B}}(Z_d) \geq 2d + 1.$$

This verifies the smallness of ν_n. □

Let $E_n = \mathbf{R}\nu_{n,*} \mathbb{Q}_\ell \in D_c^b(\mathrm{Sym}^n(X) \times \mathcal{B})$. In particular, E_0 is the constant sheaf on $\mathcal{B}$. Also $E_1 = \nu_* \mathbb{Q}_\ell$ is a sheaf on $X \times \mathcal{B}$, and $E_1[d_{\mathcal{B}} + 1]$ is a perverse sheaf since ν is finite.

Let $U_n \subset \mathrm{Sym}^n(X)$ be the open subscheme consisting of multiplicity-free divisors. Let $\tilde{U}_n \subset X^n$ be the preimage of U_n, which is an S_n-torsor over U_n. The sheaf $E_1^{\boxtimes n}|_{\tilde{U}_n \times \mathcal{B}}$ on $\tilde{U}_n \times \mathcal{B}$ admits an obvious S_n-equivariant structure and hence descends to a sheaf $E_1^{(n)}$ on $U_n \times \mathcal{B}$.

2.10. Corollary. *The complex $E_n[d_{\mathcal{B}} + n]$ is a perverse sheaf on $\mathrm{Sym}^n(X) \times \mathcal{B}$, and we have a canonical isomorphism*

$$E_n[d_{\mathcal{B}} + n] \cong j_{n,!*}(E_1^{(n)}[d_{\mathcal{B}} + n]).$$

Here $j_n : U_n \times \mathcal{B} \hookrightarrow \mathrm{Sym}^n(X) \times \mathcal{B}$ is the open inclusion, and we are implicitly stating that $E_1^{(n)}[d_{\mathcal{B}} + n]$ is a perverse sheaf on $U_n \times \mathcal{B}$.

Proof. By Lemma 2.9, the morphism ν_n is small. Since $\mathcal{H}_n$ is smooth for all n by Lemma 2.6 (1), the complex $\mathbf{R}\nu_{n,*}\mathbb{Q}_\ell[d_\mathcal{B} + n] = E_n[d_\mathcal{B} + n]$ is perverse, and is the middle extension of its restriction to any open dense subset (in particular $U_n \times \mathcal{B}$) of $\mathrm{Sym}^n(X) \times \mathcal{B}$. Clearly $j_n^* E_n = E_1^{(n)}$, hence $E_n[d_\mathcal{B} + n]$ is the middle extension of the perverse sheaf $E_1^{(n)}[d_\mathcal{B} + n]$. $\square$

2.11. The shift operator.   Consider the following diagram:

(2.5)

$$
\begin{array}{ccc}
 & \mathcal{T}_n & \\
{}^{\overleftarrow{t}}\swarrow & & \searrow{}^{\overrightarrow{t}} \\
\mathcal{H}_n \times X & & \mathcal{H}_{n+1} \\
\downarrow{}^{\nu_n \times \mathrm{id}_X} & & \downarrow{}^{\nu_{n+1}} \\
\mathrm{Sym}^n(X) \times X \times \mathcal{B} \xrightarrow{\ \sigma_n\ } & & \mathrm{Sym}^{n+1}(X) \times \mathcal{B}
\end{array}
$$

For any k-algebra R, the set $\mathcal{T}_n(R)$ classifies the data $(b, x, \mathcal{I}' \subset \mathcal{I} \subset \mathcal{O}_{\mathcal{C}_b})$ where $(b, x) \in \mathcal{B}(R) \times X(R)$ such that $\mathcal{I}/\mathcal{I}'$ is an invertible R-module whose support is an R-point $y \in \mathcal{C}_b(R)$ over $x \in X(R)$. The morphism $\overleftarrow{t}$ (resp. $\overrightarrow{t}$) sends $(b, x, \mathcal{I}' \subset \mathcal{I})$ to $(\mathcal{I}, x)$ (resp. $\mathcal{I}'$). The morphism σ_n sends (D, x, b) to $(D + x, b)$.

The restriction of $\mathcal{T}_n$ to the generic point $\eta \in \mathcal{B}$ can be identified with

(2.6)

$$
\begin{array}{ccc}
 & \mathrm{Sym}^n(\mathcal{C}_\eta) \times \mathcal{C}_\eta & \\
{}^{\overleftarrow{t}}\swarrow & & \searrow{}^{\overrightarrow{t}} \\
\mathrm{Sym}^n(\mathcal{C}_\eta) \times X & & \mathrm{Sym}^{n+1}(\mathcal{C}_\eta)
\end{array}
$$

where $\overrightarrow{t}$ sends (D, y) to $D + y$. Let $\mathcal{T}_n'$ be the closure of the generic fiber $\mathcal{T}_{n,\eta}$ in $\mathcal{T}_n$. We have $\dim \mathcal{T}_n' = d_\mathcal{B} + n + 1$. Consider the fundamental class $[\mathcal{T}_n'] \in \mathrm{H}^{\mathrm{BM}}_{2(d_\mathcal{B}+n+1)}(\mathcal{T}_n)$ of $\mathcal{T}_n'$ in the Borel–Moore homology of $\mathcal{T}_n$.

Since $\mathcal{H}_{n+1}$ is smooth of dimension $d_\mathcal{B} + n + 1$, the fundamental class $[\mathcal{H}_{n+1}]$ identifies $\mathbb{D}_{\mathcal{T}_n}[-2(d_\mathcal{B}+n+1)](-d_\mathcal{B}-n-1)$ with $\overrightarrow{t}{}^! \mathbb{Q}_{\ell,\mathcal{H}_{n+1}}$, hence we may view $[\mathcal{T}_n']$ as an element in $\mathrm{H}^0(\mathcal{T}_n, \overrightarrow{t}{}^! \mathbb{Q}_{\ell,\mathcal{H}_{n+1}})$. By the formalism of cohomological correspondences ([15] or [31, Appendix A.1]), $[\mathcal{T}_n']$ can be viewed as a cohomological correspondence between the constant sheaves on $\mathcal{H}_n \times X$ and $\mathcal{H}_{n+1}$ with support on $\mathcal{T}_n$:

$$
[\mathcal{T}_n'] : \overleftarrow{t}{}^* \mathbb{Q}_{\ell,\mathcal{H}_n \times X} \to \overrightarrow{t}{}^! \mathbb{Q}_{\ell,\mathcal{H}_{n+1}}.
$$

It induces a map

$$
\tilde{T}_n = [\mathcal{T}_n']_\# : \sigma_{n,*}(E_n \boxtimes \mathbb{Q}_{\ell,X}) \to E_{n+1}
$$

between shifted perverse sheaves on $\mathrm{Sym}^{n+1}(X) \times \mathcal{B}$. Let

$$
K_n = \mathbf{R}f_{n,*}\mathbb{Q}_\ell \in D_c^b(\mathcal{B}).
$$

Taking the direct image of $\tilde{T}_n$ under $\mathbf{R}f_{n+1,*}$, we get

$$
T_n = \mathbf{R}f_{n+1,*}(\tilde{T}_n) : K_n \otimes \mathrm{H}^*(X) = \mathbf{R}f_{n+1,*}\sigma_{n,*}(E_n \boxtimes \mathbb{Q}_{\ell,X}) \to \mathbf{R}f_{n+1,*}E_{n+1} = K_{n+1}.
$$

2.12. Proposition. *Under* (A1)–(A3), *for each* $n \geq 0, i \in \mathbb{Z}$ *and every simple constituent* M *of* $^p\mathrm{H}^i K_n$, *the support of* M *is the whole of* $\mathcal{B}$.

Proof. The property of K_n to be proved is local for the étale topology of $\mathcal{B}$, hence we may assume (A4) and (A5) hold.

For $n \geq 2g_a - 1$, $h_n : \mathcal{H}_n \to \overline{\mathrm{Pic}}_n$ is a projective space bundle, the proposition is an easy consequence of Theorem 2.4. In fact, the relative ample line bundle $\mathcal{O}(1)$ along the fibers of h_n gives a decomposition

$$\mathbf{R}h_{n,*}K_n \cong \bigoplus_{j=0}^{n-g_a} \mathbb{Q}_\ell[-2j](-j).$$

Hence

$$^p\mathrm{H}^i K_n \cong \bigoplus_{j=0}^{n-g_a} {}^p\mathrm{H}^{i-2j} L(-j).$$

In particular, any simple constituent M of $^p\mathrm{H}^* K_n$ is also a simple constituent of $^p\mathrm{H}^* L$. By Theorem 2.4, the support of M is the whole of $\mathcal{B}$.

Now we apply backward induction to n. Assuming the statement is true for any simple constituent of $^p\mathrm{H}^* K_{n+1}$, we would like to deduce that the same is true for any simple constituent M of $^p\mathrm{H}^* K_n$. The idea is to show that M appears as a direct summand of $^p\mathrm{H}^* K_{n+1}$ via the map T_n.

The sheaf $E_1 = \nu_* \mathbb{Q}_{\ell,\mathcal{C}}$ on $X \times \mathcal{B}$ contains the constant sheaf as a direct summand. Fix a decomposition

$$E_1 = \mathbb{Q}_{\ell, X \times \mathcal{B}} \oplus V$$

where V is a sheaf on $X \times \mathcal{B}$. Then we can write

$$E_1^{(n)} = \bigoplus_{i=0}^{n} V_i^n$$

such that $V_i^n|_{\tilde{U}_n \times \mathcal{B}}$ is the sum of direct summands of $E_1^{\boxtimes n} = (\mathbb{Q}_\ell \oplus V)^{\boxtimes n}$ (under the binomial expansion) consisting of exactly i factors of V. Let

$$W_i^n = j_{n,!*} V_i^n.$$

Then by Corollary 2.10, we have

$$(2.7) \qquad\qquad E_n = \bigoplus_{i=0}^{n} W_i^n.$$

We would like to understand the effect of the map $\tilde{T}_n$ under the "binomial expansion" (2.7). Base changing the diagram (2.5) to the generic point $(x_1, \ldots, x_{n+1}, \eta) \in \tilde{U}_{n+1} \times \mathcal{B}$, using the diagram (2.6), we get

$$
\begin{array}{ccc}
 & \bigsqcup_{i=1}^{n+1} \Gamma(p_i) & \\
{}^{\overleftarrow{t}} \swarrow & & \searrow {}^{\overrightarrow{t}} \\
\bigsqcup_{i=1}^{n+1} \prod_{j \neq i} \nu^{-1}(x_j, \eta) \times \{x_i\} & & \prod_{j=1}^{n+1} \nu^{-1}(x_j, \eta)
\end{array}
$$

where $\Gamma(p_i)$ is the graph of the natural projection

$$p_i : \prod_{j=1}^{n+1} \nu^{-1}(x_j, \eta) \to \prod_{j \neq i} \nu^{-1}(x_j, \eta) \times \{x_i\}.$$

This implies that the fiber of $\tilde{T}_n$ at the point $(D = x_1 + x_2 + \cdots + x_{n+1}, \eta) \in U_{n+1} \times \mathcal{B}$ takes the form

$$\tilde{T}_n|_{(D,\eta)} = \bigoplus_{i=1}^{n+1} p_i^* : \bigoplus_{i=1}^{n+1} \bigotimes_{j \neq i} E_{1,x_j} \to \bigotimes_{j=1}^{n+1} E_{1,x_j}.$$

The pullback p_i^* is the identity on the factor E_{1,x_j} for $j \neq i$, and is the inclusion of the factor $\mathbb{Q}_\ell$ into E_{1,x_i} at the i-th factor. Using the expansion (2.7), we can rewrite $\tilde{T}_n$ at (D, η) as

$$\bigoplus_{j=1}^{n} \varphi_j : \bigoplus_{j=1}^{n} \bigoplus_{i=1}^{n+1} V_{j,D-x_i}^n \to \bigoplus_{j=1}^{n} V_{j,D}^{n+1} \subset \bigoplus_{j=1}^{n+1} V_{j,D}^{n+1}$$

where the map $\varphi_j : \bigoplus_{i=1}^{n+1} V_{j,D-x_i}^n \to V_{j,D}^{n+1}$ can be understood in the following way. In the following we omit the superscript of $V_{j,D}$ because it will be clear from the degree of D. By definition, we have

$$V_{j,D} = \bigoplus_{D' \subset D, \deg(D')=j} \bigotimes_{x' \in D'} V_{x'},$$

hence

$$\bigoplus_{i=1}^{n+1} V_{j,D-x_i} = \bigoplus_{D' \subset D, \deg(D')=j} \left(\bigoplus_{i, x_i \notin D'} \bigotimes_{x' \in D'} V_{x'} \right) = V_{j,D}^{\oplus n+1-j}.$$

The map φ_j can be identified with

$$\mathrm{id}^{\oplus n+1-j} : V_{j,D}^{\oplus n+1-j} \to V_{j,D}.$$

Both the source and the target of $\tilde{T}_n$ are middle extension perverse sheaves (up to the shift $[d_\mathcal{B} + n + 1]$) from any open dense subset of $\mathrm{Sym}^{n+1}(X) \times \mathcal{B}$ (the source being so because σ_n is finite). The above calculation on the generic point implies

$$\sigma_{n,*}(W_j^n \boxtimes \mathbb{Q}_{\ell,X}) \cong (W_j^{n+1})^{\oplus n+1-j},$$

and the map $\tilde{T}_n$ can be written as

$$\bigoplus_{j=1}^{n} \mathrm{id}^{\oplus n+1-j} : \bigoplus_{j=1}^{n} (W_j^{n+1})^{\oplus n+1-j} \to \bigoplus_{j=1}^{n} W_j^{n+1} \subset \bigoplus_{j=1}^{n+1} W_j^{n+1}.$$

In particular, every direct summand of $\sigma_{n,*}(E_n \boxtimes \mathbb{Q}_\ell)$ appears as a direct summand of E_{n+1}. Applying $\mathbf{R} f_{n+1,*}$, we conclude that all simple constituents of ${}^p\mathrm{H}^*(K_n \otimes \mathrm{H}^*(X))$ (which are the same as simple constituents of ${}^p\mathrm{H}^* K_n$) appear in ${}^p\mathrm{H}^* K_{n+1}$. This finishes the induction step.   □

2.13. Theorem. *Let* $\pi : \mathcal{C} \to \mathcal{B}$ *be a projective flat family of curves satisfying* (A1)–(A3) *in* §2.1. *Let* $f_n : \mathcal{H}_n \to \mathcal{B}$ *be the relative Hilbert scheme of n-points on the fibers of* π. *Let* L^i *be the perverse sheaves defined in* §2.5 *(which are the descent of the perverse cohomology sheaves of the family of compactified Jacobians). Then for any* $n \geq 0$ *and* $i \in \mathbb{Z}$, *there is a canonical isomorphism*

$$(2.8) \qquad {}^p\mathrm{H}^{i+d_{\mathcal{B}}}\mathbf{R}f_{n,*}\mathbb{Q}_\ell \cong \bigoplus_{\max\{i-n,0\}\leq j \leq i/2} L^{i+d_{\mathcal{B}}-2j}(-j).$$

Proof. By Proposition 2.12 and Theorem 2.4, both sides of (2.8) are middle extension perverse sheaves on $\mathcal{B}$. Therefore, it suffices to establish a canonical isomorphism in the form of (2.8) on the open dense subset $\mathcal{B}_0 \subset \mathcal{B}$ consisting of smooth fibers, or even at the generic point η of $\mathcal{B}$, i.e., a $\mathrm{Gal}(\overline{k(\eta)}/k(\eta))$-equivariant isomorphism

$$(2.9) \qquad \mathrm{H}^i(\mathrm{Sym}^n(\mathcal{C}_{\overline{\eta}})) \cong \bigoplus_{\max\{i-n,0\}\leq j \leq i/2} L_{\overline{\eta}}^{i+d_{\mathcal{B}}-2j}[-i-d_{\mathcal{B}}](-j)$$

where $\overline{\eta}$ is the geometric generic point above η. Since $L_{\overline{\eta}}^{i+d_{\mathcal{B}}}[-i-d_{\mathcal{B}}]$ is canonically $\mathrm{H}^i(\mathrm{Jac}(\mathcal{C}_{\overline{\eta}})) \cong \bigwedge^i \mathrm{H}^1(\mathcal{C}_{\overline{\eta}})$, (2.9) follows from the classical Macdonald formula (1.1) for $\mathcal{C}_{\overline{\eta}}$ (which is $\mathrm{Gal}(\overline{k(\eta)}/k(\eta))$-equivariant and canonical). This proves the theorem. $\qquad\square$

2.14. The perverse filtration. Suppose a family of curves $\mathcal{C} \to \mathcal{B}$ is such that $\mathcal{B}$ is irreducible, and $\overline{\mathcal{J}} = \overline{\mathrm{Jac}}(\mathcal{C}/\mathcal{B})$ is defined and smooth. For each geometric point $b \in \mathcal{B}$, we get a *perverse filtration* on the total cohomology $\mathrm{H}^*(\overline{\mathrm{Jac}}(\mathcal{C}_b)) = \mathrm{H}^*(\overline{\mathcal{J}}_b)$. This is the increasing filtration defined as

$$P_{\leq i}\mathrm{H}^*(\overline{\mathcal{J}}_b) := ({}^p\tau_{\leq i+d_{\mathcal{B}}}L)_b[-d_{\mathcal{B}}].$$

Note that ${}^p\tau_{\leq i+d_{\mathcal{B}}}L \to L$ is a direct summand by the decomposition (2.1), hence $P_{\leq i}\mathrm{H}^*(\overline{\mathcal{J}}_b)$ defined above is indeed a subspace of $\mathrm{H}^*(\overline{\mathcal{J}}_b)$.

The next result shows that the perverse filtration $P_{\leq i}$ is preserved under base change. This will be used in the proof of Theorem 1.1 to show that there is a canonical perverse filtration.

2.15. Proposition. *Let* $\mathcal{C}' \to \mathcal{B}'$ *be a family of curves satisfying* (A1)–(A4), *which is obtained from a family* $\mathcal{C} \to \mathcal{B}$ *via a base change* $\varphi : \mathcal{B}' \to \mathcal{B}$. *Assume* $\mathcal{B}$ *is irreducible,* $\overline{\mathcal{J}} = \overline{\mathrm{Jac}}(\mathcal{C}/\mathcal{B})$ *is defined and is smooth. Then for every geometric fiber* $b' \in \mathcal{B}'$ *with image* $b \in \mathcal{B}$, *the perverse filtration* $P_{\leq i}$ *on* $\mathrm{H}^*(\overline{\mathcal{J}}_b)$ *is the same as the perverse filtration* $P'_{\leq i'}$ *on* $\mathrm{H}^*(\overline{\mathcal{J}}_{b'})$ *(under the identification* $\overline{\mathcal{J}}_b = \overline{\mathcal{J}}_{b'}$).

Proof. Let $L \in D_c^b(\mathcal{B})$ and $L' \in D_c^b(\mathcal{B}')$ be the direct image complex of $\overline{\mathcal{J}}$ and $\overline{\mathcal{J}}'$. The (non-canonical) decomposition (2.1) applies to both L and L'. By proper base change, we have

$$\bigoplus_i \varphi^* {}^p\mathrm{H}^i L[-i] \cong \varphi^* L = L' \cong \bigoplus_i {}^p\mathrm{H}^i L'[-i].$$

To prove the proposition, it suffices to argue that $\varphi^* {}^p\mathrm{H}^{i+\dim\mathcal{B}}L \cong {}^p\mathrm{H}^{i+\dim\mathcal{B}'}L'$. Applying Proposition 2.12 to the family $\mathcal{C}' \to \mathcal{B}'$, every simple constituent of ${}^p\mathrm{H}^i L'$ has support equal to $\mathcal{B}'$, hence every simple constituent of ${}^p\mathrm{H}^j(\varphi^* {}^p\mathrm{H}^i L)$ also has support $\mathcal{B}'$.

Let $\mathcal{B}_0'$ (resp. $\mathcal{B}_0$) be the locus where the fiber curves are smooth, and let $\varphi_0 : \mathcal{B}_0' \to \mathcal{B}_0$ be the restriction of φ. Then ${}^p\mathrm{H}^{i+\dim \mathcal{B}} L|_{\mathcal{B}_0}$ and ${}^p\mathrm{H}^{i+\dim \mathcal{B}'} L'|_{\mathcal{B}_0'}$ are both lisse and

$$\varphi_0^*({}^p\mathrm{H}^{i+\dim \mathcal{B}} L|_{\mathcal{B}_0}) \cong {}^p\mathrm{H}^{i+\dim \mathcal{B}'} L'|_{\mathcal{B}_0'}.$$

By the support property of $\varphi^* {}^p\mathrm{H}^{i+\dim \mathcal{B}} L$ stated above, $\varphi^* {}^p\mathrm{H}^{i+\dim \mathcal{B}} L$ must also be the middle extension of its restriction to $\mathcal{B}_0'$. In particular, we conclude that

$$\varphi^* {}^p\mathrm{H}^{i+\dim \mathcal{B}} L \cong {}^p\mathrm{H}^{i+\dim \mathcal{B}'} L'.$$

This implies the proposition.   □

2.16. Perverse filtration vs. Lefschetz filtration. We first recall the definition of the determinant line bundle on the relative compactified Jacobian $\overline{\mathcal{J}}$. Let $\mathcal{F}^{\mathrm{univ}}$ be the universal object over $\overline{\mathcal{J}} \times_{\mathcal{B}} \mathcal{C}$. Let $\mathrm{pr}_{\overline{\mathcal{J}}} : \overline{\mathcal{J}} \times_{\mathcal{B}} \mathcal{C} \to \overline{\mathcal{J}}$ be the projection. The determinant line bundle is the line bundle

$$\mathcal{L}_{\det} := \det(\mathbf{R}\mathrm{pr}_{\overline{\mathcal{J}},*} \mathcal{F}^{\mathrm{univ}})$$

over $\overline{\mathcal{J}}$. The iterated cup product by $c_1(\mathcal{L}_{\det})$ induces a map:

(2.10) $$\cup c_1(\mathcal{L}_{\det})^{g_a-i} : {}^p\mathrm{H}^{d_{\mathcal{B}}+i} L \to {}^p\mathrm{H}^{d_{\mathcal{B}}+2g_a-i} L(g_a - i).$$

Let $\mathcal{B}_0 \subset \mathcal{B}$ be the locus where $\mathcal{C}_b$ is smooth. It is well known that $\mathcal{L}_{\det}$ is ample when restricted to $\overline{\mathcal{J}}_b$ for $b \in \mathcal{B}_0$. Therefore, by the relative hard Lefschetz theorem [7, Théorèmes 5.4.10, 6.2.10], the map $c_1(\mathcal{L}_{\det})^i$ is an isomorphism over $\mathcal{B}_0$, for $0 \leq i \leq g_a$. By Proposition 2.12, ${}^p\mathrm{H}^i L$ is the middle extension of ${}^p\mathrm{H}^i L|_{\mathcal{B}_0}$ for any i, hence (2.10) is an isomorphism over the whole $\mathcal{B}$.

For each geometric point $b \in \mathcal{B}$, we consider the Jacobson–Morozov filtration induced by the nilpotent action

$$\cup c_1(\mathcal{L}_{\det}) : \mathrm{H}^*(\overline{\mathcal{J}}_b) \to \mathrm{H}^*(\overline{\mathcal{J}}_b).$$

This is the unique increasing filtration $M_{\leq i}\mathrm{H}^*(\overline{\mathcal{J}}_b)$ such that $c_1(\mathcal{L}_{\det})M_{\leq i} \subset M_{\leq i-2}$ and that $c_1(\mathcal{L}_{\det})^i$ induces an isomorphism $\mathrm{Gr}_i^M \xrightarrow{\sim} \mathrm{Gr}_{-i}^M$ (see [8, Proposition 1.6.1]). We modify the filtration M by setting

$$F^{\geq i}\mathrm{H}^*(\overline{\mathcal{J}}_b) := M_{\leq g_a-i}\mathrm{H}^*(\overline{\mathcal{J}}_b).$$

Then $F^{\geq i}$ is a decreasing filtration on $\mathrm{H}^*(\overline{\mathcal{J}}_b)$, which we call the *Lefschetz filtration*.

The fact that (2.10) is an isomorphism suggests a stronger statement, which we formulate as a conjecture.

2.17. Conjecture. *Assume* (A1)–(A4) *hold for* $\mathcal{C} \to \mathcal{B}$. *Then for every geometric point* $b \in \mathcal{B}$, *the perverse filtration* $P_{\leq i}$ *and the Lefschetz filtration* $F^{\geq i}$ *on* $\mathrm{H}^*(\overline{\mathcal{J}}_b)$ *are opposite to each other.*

3. Applications

3.1. Spectral curves. In this subsection, we give an example of a family $\pi : \mathcal{C} \to \mathcal{B}$ satisfying the conditions in (A1)–(A3) coming from the Hitchin fibration.

Fix an integer $n \geq 1$. Let X be a smooth, projective and connected curve over k. Let n be a positive integer and let $\mathcal{L}$ be a line bundle over X. Let $\mathcal{A}$ be the affine space

$$\mathcal{A} = \bigoplus_{i=1}^{n} \mathrm{H}^0(X, \mathcal{L}^{\otimes i})$$

viewed as an affine scheme over k.

Let $\mathrm{Tot}_X(\mathcal{L}) = \underline{\mathrm{Spec}}_X(\bigoplus_{i \geq 0} \mathcal{L}^{\otimes -i} y^i)$ be the total space of the line bundle $\mathcal{L}$ (here y is a formal variable). We define a closed subscheme $\mathcal{Y} \subset \mathcal{A} \times \mathrm{Tot}_X(\mathcal{L})$ by the equation

$$y^n + a_1 y^{n-1} + \cdots + a_n = 0, \quad (a_1, \ldots, a_n) \in \mathcal{A}.$$

Let $\pi : \mathcal{Y} \to \mathcal{A}$ be the projection. This is the family of *spectral curves*. It appears in the study of Hitchin moduli space for the group GL_n (see Hitchin's original paper [16, §5.1]). In particular, $\mathcal{A}$ is the base of the Hitchin fibration.

Let $\mathcal{A}^{\mathrm{int}} \subset \mathcal{A}$ be the open locus where $\mathcal{Y}_a$ is integral (integrality is an open condition by [14, Théorème 12.2.1]). There is a stratification of $\mathcal{A}^{\mathrm{int}} = \bigsqcup_{\delta \geq 0} \mathcal{A}^{\mathrm{int}}_{\delta}$ by the δ-invariants of the spectral curves $\mathcal{Y}_a$. Recall the following codimension estimate.

3.2. Lemma. *The following assertions hold.*

(1) (Ngô [24, p. 4]) *If* $\mathrm{char}(k) = 0$, *then* $\mathrm{codim}_{\mathcal{A}^{\mathrm{int}}} \mathcal{A}^{\mathrm{int}}_{\delta} \geq \delta$ *for all* $\delta \geq 0$.

(2) (Ngô [23, Proposition 5.7.2], which is based on a result of Goresky, Kottwitz and MacPherson [12]) *If* $\mathrm{char}(k) > n$, *then for each fixed* $\delta_0 \geq 0$, *there is an integer* $N = N(\delta_0)$ *such that whenever* $\deg(\mathcal{L}) \geq N$ *and* $0 \leq \delta \leq \delta_0$, *we have*

$$(3.1) \qquad\qquad\qquad \mathrm{codim}_{\mathcal{A}^{\mathrm{int}}}(\mathcal{A}^{\mathrm{int}}_{\delta}) \geq \delta.$$

If $\mathrm{char}(k) = 0$, we take $\mathcal{B} = \mathcal{A}^{\mathrm{int}}$. If $\mathrm{char}(k) > n$, we fix $\delta_0 \geq 0$ and $\deg(\mathcal{L}) \geq N(\delta_0)$, and let $\mathcal{B} = \bigsqcup_{\delta \leq \delta_0} \mathcal{A}^{\mathrm{int}}_{\delta} \subset \mathcal{A}^{\mathrm{int}}$ be the open locus where the estimate (3.1) holds. We denote the restriction of π to $\mathcal{B}$ by the same symbol.

3.3. Proposition. *The family of curves* $\pi : \mathcal{Y} \to \mathcal{B}$ *satisfies* (A1)–(A3) *in* §2.1. *In particular, Theorem* 2.13 *applies to* π.

Proof. First of all, $\mathcal{Y}$ is closed in $\mathrm{Tot}_X(\mathcal{L}) \times \mathcal{B}$. Since we can compactify $\mathrm{Tot}_X(\mathcal{L})$ into a ruled surface over X (hence projective) by adding a divisor at infinity, π is a projective morphism. We check the conditions one by one.

(A1) Each fiber $\mathcal{Y}_a$ is integral because $\mathcal{B} \subset \mathcal{A}^{\mathrm{int}}$. Since $\mathcal{Y}_a \subset \mathrm{Tot}_X(\mathcal{L})$, it has planar singularities.

(A2) is proved by the second-named author in [32, Claim 1 in the proof of Proposition 3.2.6].

(A3) is guaranteed by the choice of $\mathcal{B}$ and Lemma 3.2.   □

3.4. Versal deformation of curves. Let C be an integral curve with planar singularities. We construct in this subsection a family of curves $\pi : \mathcal{C} \to \mathcal{B}$ with $C = \pi^{-1}(b_0)$ satisfying (A1)–(A3) in §2.1.

It follows from usual deformation-theoretic arguments (see [4, 30] for example) that C can be included in a family

$$
\begin{array}{ccc}
C & \hookrightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
b_0 & \hookrightarrow & \mathcal{B}
\end{array}
$$

that is versal at $b_0 \in \mathcal{B}$. More concretely, one can choose an embedding of C in $\mathbb{P}^N$ for which

$$
\mathrm{H}^1(C, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^N}(1)) = 0
$$

for all finite subschemes $Z \subset C$ with length$(Z) \leq 2g_a - 1$ where g_a is the arithmetic genus of C. By standard calculation, the above vanishing implies

$$
(3.2) \qquad \mathrm{H}^1(C, \mathcal{I}_Z \otimes N_{C/\mathbb{P}^N}) = 0
$$

for any such Z. Let $\mathrm{Hilb}^P(\mathbb{P}^N)$ be the Hilbert scheme of $\mathbb{P}^N$ with Hilbert polynomial P equal to that of C. Let $\pi : \mathcal{C} \to \mathrm{Hilb}^P(\mathbb{P}^N)$ be the universal curve.

3.5. Proposition. *Assume either* char$(k) = 0$ *or* char$(k) > \max\{\mathrm{mult}_p(C); p \in C\}$. *Under the above choice of the embedding* $C \hookrightarrow \mathbb{P}^N$, *there exists a Zariski neighborhood* $\mathcal{B} \subset \mathrm{Hilb}^P(\mathbb{P}^N)$ *of* $b_0 = [C]$ *over which the universal family* $\pi : \mathcal{C} \to \mathcal{B}$ *satisfies the conditions* (A1)–(A3) *in* §2.1.

Proof. (A1) The property of being locally planar is open: consider the relative cotangent sheaf $\Omega_{\mathcal{C}/\mathrm{Hilb}^P(\mathbb{P}^N)}$ and let $Z \subset \mathcal{C}$ be the locus where its stalk has dimension at least 3. Clearly Z is closed, hence $\pi(Z) \subset \mathrm{Hilb}^P(\mathbb{P}^N)$ is also closed. Then $\mathrm{Hilb}^P(\mathbb{P}^N) - \pi(Z)$ is precisely the locus where $\mathcal{C}_b$ is locally planar. Also being integral is an open condition [14, Théorème 12.2.1]. Therefore we can take a Zariski neighborhood of $[C]$ in $\mathrm{Hilb}^P(\mathbb{P}^N)$ over which the fibers are integral and locally planar.

(A2) When $k = \mathbb{C}$, this is proved by V. Shende in [29, Proposition 13]. In general, we proceed as follows. By (3.2), $\mathrm{H}^1(C, N_{C/\mathbb{P}^N}) = 0$, hence $[C]$ is in the smooth locus of $\mathrm{Hilb}^P(\mathbb{P}^N)$. Shrinking $\mathcal{B}$ if necessary, we may assume $\mathcal{B}$ is contained in the smooth locus of $\mathrm{Hilb}^P(\mathbb{P}^N)$.

We first need the following lemma:

3.6. Lemma. *Any finite subscheme* Z' *of length n of a locally planar curve* $C' \subset \mathbb{P}^N$ *lies in the smooth locus of* $\mathrm{Hilb}^n(\mathbb{P}^N)$.

Proof. Since Z' is planar, it lies in the closure of the locus of n distinct points on $\mathbb{P}^N$, so the local dimension at $[Z']$ is at least $n \cdot N$, and it suffices to bound the dimension of the tangent space. For this, we can assume that Z' is supported at a point and choose local coordinates so that $Z \subset S = \mathrm{Spec}(k[[x, y]]) \subset P = \mathrm{Spec}(k[[x, y, z_1, \ldots, z_{N-2}]])$. Since $I_{Z/P}$ is generated by the ideal $I_{Z/S}$ and $\{z_k\}$, we have a surjection

$$
I_{Z/S}/I_{Z/S}^2 \oplus \mathcal{O}_Z^{\oplus(N-2)} \to I_{Z/P}/I_{Z/P}^2 \to 0
$$

which leads to the inclusion

$$
0 \to \mathrm{Hom}(I_{Z/P}/I_{Z/P}^2, \mathcal{O}_Z) \to \mathrm{Hom}(I_{Z/S}/I_{Z/S}^2, \mathcal{O}_Z) \oplus \mathcal{O}_Z^{N-2}.
$$

The first term on the right side above is the tangent space of $\mathrm{Hilb}^n(\mathbb{A}^2)$ at Z, which has dimension $2n$; the second has length $n \cdot (N-2)$. Therefore the length of

$$T_Z \mathrm{Hilb}^n(\mathbb{P}^N) = \mathrm{Hom}(I_{Z/P}/I_{Z/P}^2, \mathcal{O}_Z)$$

is bounded above by $n \cdot N$. □

Now fix $0 \leq n \leq 2g_a - 1$. Let $\mathcal{I}_{\mathcal{C}}$ be the ideal sheaf of the universal curve $\mathcal{C} \subset \mathcal{B} \times \mathbb{P}^N$. Let $\mathcal{O}_{\mathcal{Z}}$ be the structure sheaf of the universal subscheme $\mathcal{Z} \subset \mathrm{Hilb}^n(\mathbb{P}^N) \times \mathbb{P}^N$. Let $\mathcal{E}$ be the complex $\mathbf{R}\mathrm{pr}_* \mathbf{R}\underline{\mathrm{Hom}}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_{\mathcal{Z}})$ where $\mathrm{pr} : \mathrm{Hilb}^n(\mathbb{P}^N) \times \mathcal{B} \times \mathbb{P}^N \to \mathrm{Hilb}^n(\mathbb{P}^N) \times \mathcal{B}$ is the projection. Over $(Z' \subset C') \in \mathrm{Hilb}^n(\mathcal{C}/\mathcal{B})$, we have $\mathrm{Ext}^{>0}_{\mathbb{P}^N}(\mathcal{I}_{C'}, \mathcal{O}_{Z'}) = 0$ for dimension reasons, hence $\mathcal{E}$ is concentrated in degree zero in a neighborhood of $\mathrm{Hilb}^n(\mathcal{C}/\mathcal{B})$ by semi-continuity, and is a vector bundle of rank $(N-1)n$ there. This vector bundle $\mathcal{E}$ has a canonical section s given by $\mathcal{I}_{\mathcal{C}} \hookrightarrow \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathcal{Z}}$. Now $\mathrm{Hilb}^n(\mathcal{C}/\mathcal{B})$ is cut off by the vanishing of s on the smooth locus of $\mathrm{Hilb}^n(\mathbb{P}^N) \times \mathcal{B}$, therefore the local dimension of $\mathrm{Hilb}^n(\mathcal{C}/\mathcal{B})$ is at least

$$\dim \mathrm{Hilb}^n(\mathbb{P}^N)^{\mathrm{sm}} + \dim \mathcal{B} - \mathrm{rank}\mathcal{E} = n + \dim \mathcal{B}.$$

On the other hand, the tangent space of $\mathrm{Hilb}^n(\mathcal{C}/\mathcal{B})$ at $(Z \subset C)$ is the kernel of the map

$$\mathrm{Hom}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) \oplus \mathrm{H}^0(C, N_{C/\mathbb{P}^N}) \to \mathrm{H}^0(Z, N_{C/\mathbb{P}^N}|_Z).$$

By (3.2), this map is surjective. We argue in the proof of Lemma 3.6 that

$$\dim \mathrm{Hom}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) = Nn.$$

Clearly $h^0(C, N_{C/\mathbb{P}^N}) = \dim \mathcal{B}$ and $h^0(Z, N_{C/\mathbb{P}^N}|_Z) = n \cdot \mathrm{rank}(N_{C/\mathbb{P}^N}) = n(N-1)$, therefore the dimension of the tangent space at $(Z \subset C)$ is $n + \dim \mathcal{B}$. This together with the lower bound above gives the smoothness of $\mathrm{Hilb}^n(\mathcal{C}/\mathcal{B})$ along the fiber $\mathrm{Hilb}^n(C)$. By openness of the smooth locus and properness of $\mathrm{Hilb}^n(\mathcal{C}/\mathcal{B}) \to \mathcal{B}$, we may shrink $\mathcal{B}$ further to ensure that $\mathrm{Hilb}^n(\mathcal{C}/\mathcal{B})$ is smooth.

Above we fixed an integer n and found a non-empty Zariski open subset of $\mathcal{B}(n) \subset \mathcal{B}$ over which $\mathrm{Hilb}^n(\mathcal{C}/\mathcal{B})$ is smooth. Now $\bigcap_{n=0}^{2g_a-1} \mathcal{B}(n)$ guarantees the smoothness condition (A2).

(A3) For every singularity $p \in C$, the deformation functor $\mathrm{Def}(\hat{\mathcal{O}}_{C,p})$ has an *algebraic* miniversal hull V_p by M. Artin's theorem [4, Theorem 3.3] and Elkik's theorem on isolated singularities [10]. More precisely, $V_p = \mathrm{Spec}\, R_p$ is of finite type over k equipped with a point $0_p \in V_p(k)$ and an R_p-flat family of algebras $\hat{\mathcal{O}}_{R_p}$ with an isomorphism

$$\hat{\mathcal{O}}_{R_p} \otimes_{R_p} k(0_p) \cong \hat{\mathcal{O}}_{C,p}.$$

Let $\hat{R}_p$ be the completion of R_p at 0_p. The canonical morphism $v_p : \mathrm{Spf}\hat{R}_p \to \mathrm{Def}(\hat{\mathcal{O}}_{C,p})$ is formally smooth and induces a bijection on the tangent spaces.

By the versality of V_p, there exists an étale neighborhood $\mathcal{B}'$ of $[C] \in \mathcal{B}$ and a pointed morphism

$$(3.3) \qquad (\mathcal{B}', [C]) \to \prod_{p \in C^{\mathrm{sing}}} (V_p, 0_p).$$

Since $\mathscr{B}$ is also versal, this morphism is smooth (see, for instance, [11, Section A] or [19, Appendix A]). Therefore the codimension estimate holds for $\mathscr{B}$ around $[C]$ with respect to the global δ-invariant if and only if the codimension estimate holds for each V_p around 0_p with respect to the local δ-invariant (here we use the fact that the locus $V_p^{\mathrm{sm}} \subset V_p$ parametrizing smooth deformations is not empty).

In characteristic zero, the codimension estimate for V_p is proven by Diaz and Harris [9, Theorem 4.15]. In arbitrary characteristic, we do not know a reference but it can be deduced from the family of spectral curves, see Lemma 3.7 below.   □

3.7. Lemma. *Let $\hat{\mathscr{O}}$ be the completed local ring of a planar curve singularity over k. Assume $\mathrm{char}(k)$ is either 0 or greater than the multiplicity of the singularity defined by $\hat{\mathscr{O}}$. Let V be an algebraic miniversal deformation of $\hat{\mathscr{O}}$, with the base point $0 \in V(k)$ corresponding to $\hat{\mathscr{O}}$. Then there is a Zariski open neighborhood V' of 0 such that*

$$\mathrm{codim}_{V'}(V'_\delta) \geq \delta$$

for any δ-constant stratum $V'_\delta \subset V'$.

Proof. By the Weierstrass preparation theorem, we may choose a non-unit $0 \neq t \in \hat{\mathscr{O}}$ such that $\hat{\mathscr{O}} \cong k[[t]][y]/(y^n + a_1^*(t)y^{n-1} + \cdots + a_n^*(t))$ for some $a_i^*(t) \in k[[t]]$ where n is the multiplicity of $\hat{\mathscr{O}}$. We may even assume $a_i^*(t) \in k[t]$ without changing the isomorphism type of $\hat{\mathscr{O}}$, by a result of Artin and Hironaka [3, Lemma 3.12].

Let $Y^* = \mathrm{Spec}\, k[t, y]/(f)$ where $f(t, y) = y^n + a_1^*(t)y^{n-1} + \cdots + a_n^*(t)$. This is an affine plane curve with isolated singularities at $\{p_1, \ldots, p_r\}$, and we may assume $\hat{\mathscr{O}}_{Y^*, p_1} \cong \hat{\mathscr{O}}$. We have the miniversal deformations $(V_i, 0_i)$ of the singularities p_i as in the proof of Proposition 3.5. Each V_i is smooth at 0_i because $\mathrm{Def}(\hat{\mathscr{O}}_{Y^*, p_i})$ is formally smooth [19, Théorème A.1.2 (3)].

Let $\mathscr{A}_N$ be the Hitchin base associated with the curve $\mathbb{P}^1$ and line bundle $\mathscr{O}(N)$. We trivialize $\mathscr{O}(N)$ over $\mathbb{A}_t^1 = \mathbb{P}^1 - \{\infty\}$, and identify $\mathscr{A}_N$ with an affine space with coordinates $\underline{a} = (a_i(t))_{1 \leq i \leq n}$ where $a_i(t) \in k[t]$ with $\deg(a_i) \leq Ni$. Then $\mathscr{A}_N$ parametrizes a family of affine spectral curves $\mathscr{Y} \to \mathscr{A}_N$ with

$$\mathscr{Y}_{\underline{a}} = \mathrm{Spec}\, k[t, y]/(y^n + a_1(t)y^{n-1} + a_2(t)y^{n-2} + \cdots + a_n(t)),$$

for $\underline{a} = (a_i(t)) \in \mathscr{A}_N$. We choose N large enough so that the original data $(a_i^*(t))$ gives a point $\underline{a}^* \in \mathscr{A}_N(k)$. By the versality of V_i, there is an étale neighborhood $\mathscr{A}_N'$ of $\underline{a}^* \in \mathscr{A}_N$ and a pointed morphism

$$\rho : (\mathscr{A}_N', \underline{a}^*) \to \prod_{i=1}^r (V_i, 0_i)$$

such that the family of affine spectral curves $\mathscr{Y} \to \mathscr{A}_N$ is, étale locally around the singularities of $Y^* = \mathscr{Y}_{\underline{a}^*}$, isomorphic to the pull-back of the disjoint union of the miniversal families over V_i.

We claim that ρ is smooth at $\underline{a}^*$ for large N. Since both $\mathscr{A}_N'$ and the V_i are smooth around the base points, we only need to show that the tangent map of ρ at $\underline{a}^*$ is surjective. We have a canonical isomorphism (see [5, Part 1, §4])

$$\bigoplus_{i=1}^r T_{0_i} V_i \cong k[t, y]/(f, \partial_y f, \partial_t f).$$

Identifying $T_{\underline{a}^*}\mathscr{A}'_N$ with $\mathscr{A}_N$ in the usual way, the tangent map $d\rho : T_{\underline{a}^*}\mathscr{A}'_N \to \bigoplus_i T_{0_i} V_i$ takes the form

$$(3.4) \qquad (a_i(t))_{1 \leq i \leq n} \mapsto \sum_{i=1}^n a_i(t) y^{n-i} \in k[t, y]/(f, \partial_y f, \partial_t f).$$

Since f has only isolated singularities, f does not have multiple factors. Since $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > n$, f and $\partial_y f$ are coprime as elements in $k(t)[y]$, hence the ideal $(f, \partial_y f) \subset k[t, y]$ contains some nonzero polynomial $g(t) \in k[t]$. Let $S = k[t]/(g(t))$, which is a finite-dimensional k-algebra. Then $\bigoplus_{i=1}^r T_{0_i} V_i$ is a quotient of $\sum_{i=0}^{n-1} S y^i$. For large N, the map $k[t]_{\deg \leq N} \to S$ is surjective, so is the tangent map (3.4). This proves ρ is smooth at $\underline{a}^*$ for large N.

By the remark made after (3.3) in the proof of Proposition 3.5, to prove the codimension estimate for V_1 around 0_1, we only need to show that the same codimension estimate holds for $\mathscr{A}_N$ around $\underline{a}^*$, for large enough N. To be precise, we would like to stratify $\mathscr{A}_N$ by the δ-invariants of the affine curves $\mathscr{Y}_{\underline{a}}$ (instead of the projective ones as considered in §3.1). We call these strata $\mathscr{A}_{N,\delta}$. For fixed δ_0, we would like to show that once N is large (depending on δ_0), we have

$$\mathrm{codim}_{\mathscr{A}_N}(\mathscr{A}_{N,\delta_0}) \geq \delta_0.$$

The proof for this is completely analogous to Ngô's argument in [23, Proposition 5.7.2], which works for spectral curves over any given curve, not necessarily complete. Our condition $\mathrm{char}(k) > n$ or $\mathrm{char}(k) = 0$ is also needed here because the argument in [23, Proposition 5.7.2] relies on the codimension calculation of Goresky, Kottwitz and MacPherson in [12], which was done under the assumption $\mathrm{char}(k) > n$ or $\mathrm{char}(k) = 0$. This completes the proof of the lemma. □

3.8. Proof of Theorem 1.1. Let C be an integral curve over k with planar singularities. Above we constructed a family $\pi : \mathcal{C} \to \mathscr{B}$ containing $C = \pi^{-1}(b_0)$ as a fiber, which satisfies (A1)–(A3) in §2.1. Making an étale base change of $\mathscr{B}$, we may further assume (A4) holds. Applying Theorem 2.13 to this family and taking the stalk of the relevant complexes at b_0, we obtain

$$\mathrm{H}^*(\mathrm{Hilb}^n(C)) \cong \bigoplus_{i+j \leq n, i, j \geq 0} ({}^p\mathrm{H}^{i+d_{\mathscr{B}}} L)_{b_0}[-d_{\mathscr{B}} - i - 2j](-j).$$

As discussed earlier in §2.14, the perverse filtration on L induces a filtration $P_{\leq i}$ on $\mathrm{H}^*(\overline{\mathrm{Jac}}(C)) = L_{b_0}$, and

$$({}^p\mathrm{H}^{i+d_{\mathscr{B}}} L)_{b_0}[-i - d_{\mathscr{B}}] = \mathrm{Gr}_i^P (\mathrm{H}^*(\overline{\mathrm{Jac}}(C))).$$

Now Theorem 1.1 is almost proved, except we need to show that the perverse filtration $P_{\leq i}$ on $\mathrm{H}^*(\overline{\mathrm{Jac}}(C))$ is independent of the choice of the deformation $\mathcal{C} \to \mathscr{B}$ satisfying (A1)–(A4). Let $\mathcal{C}_{\mathcal{V}} \to \mathcal{V}$ be a versal deformation of C, which satisfies (A1)–(A3) as proved in Proposition 3.5. Making an étale base change of $\mathcal{V}$ (which preserves versality), we may assume $\mathcal{C}_{\mathcal{V}} \to \mathcal{V}$ satisfies (A1)–(A4). By versality, there exists an étale neighborhood $\mathscr{B}'$ of $b_0 \in \mathscr{B}$ and a morphism $\varphi : \mathscr{B}' \to \mathcal{V}$ such that $\mathcal{C}' = \mathcal{C}|_{\mathscr{B}'}$ is obtained from $\mathcal{C}_{\mathcal{V}}$ via base change by φ. We thus have a diagram

$$\mathscr{B} \leftarrow \mathscr{B}' \xrightarrow{\varphi} \mathcal{V}.$$

Applying Proposition 2.15 to both arrows above, we conclude that $P_{\leq i}$ on $\mathrm{H}^*(\overline{\mathrm{Jac}}(C))$ defined using the perverse filtration of the family $\mathcal{B}$ is the same as the one defined using $\mathcal{V}$. Since $\mathcal{V}$ is fixed, $P_{\leq i}$ is independent of the choice of $\mathcal{B}$. This shows the canonicity of the perverse filtration $P_{\leq i}$, and finishes the proof of Theorem 1.1.

3.9. Functional equation. Let

$$Z_{\mathrm{Hilb}}(s, t, C) := \sum_{i, n \geq 0} \dim \mathrm{H}^i(\mathrm{Hilb}^n(C)) s^i t^n.$$

By Theorem 1.1, $Z_{\mathrm{Hilb}}(s, t, C)$ is a rational function of the form

$$Z_{\mathrm{Hilb}}(s, t, C) = \frac{P(s, t)}{(1 - t)(1 - s^2 t)}$$

where

$$P(s, t) = \sum_{i, j} \dim \mathrm{Gr}_i^P \mathrm{H}^j(\overline{\mathrm{Jac}}(C)) s^j t^i \in \mathbb{Z}[s, t]$$

is a polynomial of bidegree $(2g_a, 2g_a)$. Moreover, by the isomorphism (2.10), the cup product by $c_1(\mathcal{L}_{\mathrm{det}})$ gives an isomorphism

$$\cup c_1(\mathcal{L}_{\mathrm{det}})^i : \mathrm{Gr}_{g_a - i}^P \mathrm{H}^j(\overline{\mathrm{Jac}}(C)) \overset{\sim}{\to} \mathrm{Gr}_{g_a + i}^P \mathrm{H}^{j + 2i}(\overline{\mathrm{Jac}}(C))(i)$$

for any $i, j \geq 0$. This implies a symmetry on the polynomial $P(s, t)$, and hence the following functional equation for $Z_{\mathrm{Hilb}}(s, t, C)$:

$$Z_{\mathrm{Hilb}}(s, t, C) = (st)^{2g_a - 2} Z_{\mathrm{Hilb}}(s, s^{-2} t^{-1}, C).$$

3.10. A local Macdonald formula. Let $\hat{\mathcal{O}}$ be a complete local reduced k-algebra of dimension 1 with maximal ideal $\mathfrak{m}$ and residue field k. We say $\hat{\mathcal{O}}$ is *planar* if furthermore $\dim_k \mathfrak{m}/\mathfrak{m}^2 \leq 2$, i.e., $\hat{\mathcal{O}} \cong k[[x, y]]/(f)$ for some $0 \neq f \in k[[x, y]]$ without multiple factors. We would like to understand the cohomology of the Hilbert scheme $\mathrm{Hilb}^n(\hat{\mathcal{O}})$ of length n quotient algebras of $\hat{\mathcal{O}}$.

One can define a "compactified Picard scheme" for $\hat{\mathcal{O}}$ as follows. Let $\hat{K}$ be the ring of fractions of $\hat{\mathcal{O}}$, which is a finite product of $k((t))$. Let $\overline{\mathrm{Pic}}(\hat{\mathcal{O}})$ be the functor which associates to every noetherian k-algebra R the set of $R \hat{\otimes}_k \hat{\mathcal{O}}$-submodules $M \subset R \hat{\otimes}_k \hat{K}$ such that for some (equivalently, any) non-zero-divisor $t \in \mathfrak{m}$, there exists an integer $i \geq 0$ so that $R \hat{\otimes}_k t^i \hat{\mathcal{O}} \subset M \subset R \hat{\otimes}_k t^{-i} \hat{\mathcal{O}}$ and $(R \hat{\otimes}_k t^{-i} \hat{\mathcal{O}})/M$ (hence $M/(R \hat{\otimes}_k t^i \hat{\mathcal{O}})$) is a projective R-module. The functor $\overline{\mathrm{Pic}}(\hat{\mathcal{O}})$ is a disjoint union $\overline{\mathrm{Pic}}(\hat{\mathcal{O}}) = \bigsqcup_{n \in \mathbb{Z}} \overline{\mathrm{Pic}}^n(\hat{\mathcal{O}})$ according to the volume of M, defined as $\mathrm{rk}_R M/(R \hat{\otimes}_k t^i \hat{\mathcal{O}}) - \dim_k \hat{\mathcal{O}}/(t^i \hat{\mathcal{O}})$ for i large. Each $\overline{\mathrm{Pic}}^n(\hat{\mathcal{O}})$ is represented by an ind-scheme which is *not* of finite type. The reduced structure $\overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})$ is an infinite union of projective varieties, all of dimension $\delta(\hat{\mathcal{O}})$ (the local δ-invariant of $\hat{\mathcal{O}}$). The above results are consequences of the theory of affine Springer fibers (see [18], and also [23, §3]), because $\hat{\mathcal{O}}$ can be realized as the germ of a spectral curve.

Let $\mathrm{Pic}(\hat{\mathcal{O}}) = \hat{K}^\times/\hat{\mathcal{O}}^\times$, viewed as a group ind-scheme over k. Then $\mathrm{Pic}(\hat{\mathcal{O}})$ acts on $\overline{\mathrm{Pic}}(\hat{\mathcal{O}})$: an element $g \in \hat{K}^\times/\hat{\mathcal{O}}^\times$ sends $M \subset R \hat{\otimes}_k \hat{K}$ to Mg.

Let $\mathcal{O}_{\hat{K}}$ be the ring of integers of $\hat{K}$. Let $\mathrm{Jac}(\hat{\mathcal{O}}) = \mathcal{O}_{\hat{K}}^{\times}/\mathcal{O}^{\times}$, which is a connected affine algebraic group over k. We have an exact sequence of groups over k:

$$1 \to \mathrm{Jac}(\hat{\mathcal{O}}) \to \mathrm{Pic}^{\mathrm{red}}(\hat{\mathcal{O}}) \xrightarrow{\mathrm{val}} \Lambda \to 1$$

where $\Lambda = \hat{K}^{\times}/\mathcal{O}_{\hat{K}}^{\times}$ is a free abelian group of rank r (r is the number of analytic branches of $\hat{\mathcal{O}}$). Choosing a section of $\mathrm{val} : \mathrm{Pic}^{\mathrm{red}}(\hat{\mathcal{O}}) \to \Lambda$, we may identify Λ as a subgroup of $\mathrm{Pic}^{\mathrm{red}}(\hat{\mathcal{O}})$, hence Λ also acts on $\overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})$. It is proved by Kazhdan and Lusztig [18] that $\overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})/\Lambda$ is a projective variety. Two different sections of val can be transformed to each other by an element of $\mathrm{Jac}(\hat{\mathcal{O}})$, which is a connected group scheme over k acting also on $\overline{\mathrm{Pic}}(\hat{\mathcal{O}})$, therefore the cohomology $\mathrm{H}^*(\overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})/\Lambda)$ is independent of the choice of the section of val.

To state the local Macdonald formula, let us recall the notion of the virtual Poincaré polynomial. Let k be either $\overline{\mathbb{F}_p}$ or a subfield of $\mathbb{C}$. Let X be a scheme of finite type over k. In both cases, we have a weight filtration $W_{\leq i}\mathrm{H}^*(X)$ on the ℓ-adic cohomology of X: when $k = \overline{\mathbb{F}_p}$, we may assume $X = X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_p}$ for some X_0 over $\mathbb{F}_q$, then the weight filtration comes from the absolute values of the Frob_q-action on $\mathrm{H}^*(X)$ (we need to fix an embedding $\iota : \mathbb{Q}_{\ell} \hookrightarrow \mathbb{C}$); when $k \subset \mathbb{C}$, this comes from the weight filtration on the singular cohomology $\mathrm{H}^*(X^{\mathrm{an}}, \mathbb{Q})$ (X^{an} is the underlying analytic space of $X(\mathbb{C})$) and the comparison theorem between ℓ-adic and singular cohomology. Then we define

$$P^{\mathrm{vir}}(X, s) = \sum_{i,j} (-1)^j \dim \mathrm{Gr}_i^W \mathrm{H}^j(X) s^i.$$

Similarly, when $\mathrm{H}^*(X)$ carries a filtration $P_{\leq i}$ which is strictly compatible with the weight filtration, we can define the virtual Poincaré polynomial $P^{\mathrm{vir}}(\mathrm{Gr}_i^P \mathrm{H}^*(X), s)$.

3.11. Theorem. *Let $\hat{\mathcal{O}}$ be a planar complete local reduced k-algebra of dimension 1, with r analytic branches. Assume either $\mathrm{char}(k) = 0$ or $\mathrm{char}(k)$ is greater than the multiplicity of the singularity defined by $\hat{\mathcal{O}}$. Then there is a filtration $P_{\leq i}$ on $\mathrm{H}^*(\overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})/\Lambda)$, normalized such that $\mathrm{Gr}_i^P = 0$ unless $0 \leq i \leq 2\delta(\hat{\mathcal{O}})$, strictly compatible with the weight filtration, such that we have an identity in $\mathbb{Z}[[s, t]]$,*

$$\sum_n P^{\mathrm{vir}}(\mathrm{Hilb}^n(\hat{\mathcal{O}}), s) t^n = \frac{\sum_i P^{\mathrm{vir}}(\mathrm{Gr}_i^P \mathrm{H}^*(\overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})/\Lambda), s) t^i}{(1-t)^r}.$$

Before giving the proof, we record two product formulas.

3.12. Product formula for Hilbert schemes. Let C be an irreducible curve over k and $U = C - \{p_1, \ldots, p_m\}$ be a Zariski open subset. Then we have the following identity in $\mathrm{Var}_k[[t]]$ where Var_k is the Grothendieck group of varieties over k:

$$(3.5) \qquad \left(\sum_{n \geq 0} [\mathrm{Hilb}^n(U)] t^n \right) \prod_{i=1}^{m} \left(\sum_{n \geq 0} [\mathrm{Hilb}^n(\hat{\mathcal{O}}_{C,p_i})] t^n \right) = \sum_{n \geq 0} [\mathrm{Hilb}^n(C)] t^n.$$

3.13. Product formula for the compactified Jacobian.   The setup is the same as above. We have natural morphisms $\prod_i \operatorname{Pic}(\hat{\mathcal{O}}_{p_i}) \to \operatorname{Pic}(C)$ and $\prod_i \overline{\operatorname{Pic}}(\hat{\mathcal{O}}_{p_i}) \to \overline{\operatorname{Pic}}(C)$ given by gluing the local objects with the trivial line bundle on U. Therefore we get a canonical morphism

$$(3.6) \qquad \left(\prod_i \overline{\operatorname{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}}_{p_i}) \right) \overset{\prod_i \operatorname{Pic}(\hat{\mathcal{O}}_{p_i})}{\times} \operatorname{Pic}(C) \to \overline{\operatorname{Pic}}(C).$$

The product formula says that this is a homeomorphism. In the case p_i are unibranched singularities and C is rational, this is proved by Beauville [6, Proposition 3.7]. In general, the proof is analogous to Ngô's product formula [23, §4.15]. From (3.6), we see that the stabilizers of the $\operatorname{Pic}(C)$-action on $\overline{\operatorname{Pic}}(C)$ are affine, because these stabilizers are subgroups of the affine group $\prod_i \operatorname{Pic}(\hat{\mathcal{O}}_{p_i})$. This affineness is needed in the proof of Theorem 2.4.

Proof of Theorem 3.11.   Given $\hat{\mathcal{O}}$, there exists a rational projective curve C over k, together with a point $p \in C(k)$ such that

- C is nonsingular away from p;
- $\hat{\mathcal{O}}_{C,p} \cong \hat{\mathcal{O}}$.

  Let $\nu : \mathbb{P}^1 \to C$ be the normalization and let $\nu^{-1}(p) = \{p_1, \ldots, p_r\}$,

$$U = C - \{p\} = \mathbb{P}^1 - \{p_1, \ldots, p_r\}.$$

Applying the product formula to C we get

$$\left(\sum_{n \geq 0} [\operatorname{Hilb}^n(U)] t^n \right) \left(\sum_{n \geq 0} [\operatorname{Hilb}^n(\hat{\mathcal{O}})] t^n \right) = \sum_{n \geq 0} [\operatorname{Hilb}^n(C)] t^n.$$

Applying the product formula (3.5) to $\mathbb{P}^1$ we get

$$\frac{\sum_{n \geq 0} [\operatorname{Hilb}^n(U)] t^n}{(1-t)^r} = \left(\sum_{n \geq 0} [\operatorname{Hilb}^n(U)] t^n \right) \prod_{i=1}^{r} \left(\sum_{n \geq 0} [\operatorname{Hilb}^n(\hat{\mathcal{O}}_{\mathbb{P}^1, p_i})] t^n \right)$$

$$= \sum_{n \geq 0} [\operatorname{Hilb}^n(\mathbb{P}^1)] t^n = \sum_{n \geq 0} [\mathbb{P}^n] t^n = \frac{1}{(1-t)(1 - \mathbb{L}t)}$$

where $\mathbb{L}$ is the class of $\mathbb{A}^1$. Taking the quotient of the two identities, we get

$$\sum_{n \geq 0} [\operatorname{Hilb}^n(\hat{\mathcal{O}})] t^n = \frac{1 - \mathbb{L}t}{(1-t)^{r-1}} \sum_{n \geq 0} [\operatorname{Hilb}^n(C)] t^n.$$

Taking the virtual Poincaré polynomials of both sides, and applying Theorem 1.1 to C (which is applicable by the assumption on $\operatorname{char}(k)$), we get

$$(3.7) \qquad \sum_{n \geq 0} P^{\mathrm{vir}}(\operatorname{Hilb}^n(\hat{\mathcal{O}}), s) t^n = \frac{\sum_i P^{\mathrm{vir}}(\operatorname{Gr}_i^P \mathrm{H}^*(\overline{\operatorname{Jac}}(C)), s) t^i}{(1-t)^r}.$$

The local and global Picard fit into a commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_{\hat{K}}^{\times}/\hat{\mathcal{O}}^{\times} & \longrightarrow & \mathrm{Pic}(\hat{\mathcal{O}}) & \xrightarrow{\ \mathrm{val}\ } & \Lambda & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow{\scriptstyle\mathrm{sum}} & & \\
1 & \longrightarrow & \mathcal{O}_{\hat{K}}^{\times}/\hat{\mathcal{O}}^{\times} & \longrightarrow & \mathrm{Pic}(C) & \xrightarrow{\ \nu^{*}\ } & \mathrm{Pic}(\mathbb{P}^{1}) & \longrightarrow & 1
\end{array}
$$

Since $\mathrm{Pic}(\mathbb{P}^{1}) = \mathbb{Z}$, the map "sum" just means the sum map $\Lambda \cong \mathbb{Z}^{r} \to \mathbb{Z}$. Therefore, the product formula (3.6) gives homeomorphisms (after choosing a section of val)

$$
\overline{\mathrm{Pic}}(C) \cong \overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})/\ker(\mathrm{sum}) \quad \text{and} \quad \overline{\mathrm{Jac}}(C) \cong \overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})/\Lambda.
$$

Finally we define the filtration $P_{\leq i}$ on $\mathrm{H}^{*}(\overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})/\Lambda)$ to be the transport of the perverse filtration on $\mathrm{H}^{*}(\overline{\mathrm{Jac}}(C))$. It is easy to see that this filtration satisfies the requirements in the theorem. Therefore we get an isomorphism of bi-filtered graded vector spaces

$$
(\mathrm{H}^{*}(\overline{\mathrm{Jac}}(C)), P_{*}, W_{*}) \cong (\mathrm{H}^{*}(\overline{\mathrm{Pic}}^{\mathrm{red}}(\hat{\mathcal{O}})/\Lambda), P_{*}, W_{*}).
$$

This, together with (3.7), implies the desired formula. □

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