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A $(-q)$-analogue of weight multiplicities

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Dedicated to Professor C. S. Seshadri on his 80th birthday.

Abstract. We prove a conjecture in [L11] stating that certain polynomials $P^{\sigma}_{y,w}(q)$ introduced in [LV11] for twisted involutions in an affine Weyl group give $(-q)$-analogues of weight multiplicities of the Langlands dual group $\check{G}$. We also prove that the signature of a naturally defined hermitian form on each irreducible representation of $\check{G}$ can be expressed in terms of these polynomials $P^{\sigma}_{y,w}(q)$.

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1. Statement of the main theorems

1.1 The P^{σ}-polynomials

Let W be a Coxeter group with simple reflections S. Let $\ell : W \to \mathbb{N}$ be the length function defined by the simple reflections S. In [KL79], for any two elements $y, w \in W$, a polynomial $P_{y,w}(q) \in \mathbb{Z}[q]$ is attached. Consider the Hecke algebra $\mathcal{H}$ over $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ (q is an indeterminate) with basis $\{T_w\}_{w \in W}$ and multiplication given by $T_w T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ and $(T_s + 1)(T_s - q) = 0$ for all $s \in S$. Then $\{\sum_{y \in W; y \leq w} P_{y,w}(q) T_y\}_{w \in W}$ is (up to a factor) the "new basis" of $\mathcal{H}$ introduced in [KL79].

In [LV11] (for W a Weyl group) and [L11] (in general), the authors work in the situation of a triple $(W, S, *)$ where (W, S) is as before and $*$ is

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an involution of (W, S). Let $I_* = \{w \in W | w^* = w^{-1}\}$ be the $*$-twisted involutions in W. From the data $(W, S, *)$, a refined version $P^{\sigma}_{y,w}(q) \in \mathbb{Z}[q]$ of $P_{y,w}(q)$ is defined for $y, w \in I_*$. They also introduced a free $\mathcal{A}$-module M with basis $\{a_w\}_{w \in I_*}$, which carries a natural module structure over the Hecke algebra $\mathcal{H}'$ with q replaced by q^2. Then $\{\sum_{y \leq w, y \in I_*} P^{\sigma}_{y,w}(q) a_y\}_{w \in I_*}$ is (up to a factor) the "new basis" of M introduced in [LV11, Theorem 0.3] and [L11, Theorem 0.4]. The algebraic definitions of $P^{\sigma}_{y,w}(q)$ will be reviewed in Section 2.1.

1.2 Affine Weyl group

For the rest of the note we consider the setting of [L11, Section 6]: (W, S) is the Coxeter group associated to an untwisted connected affine Dynkin diagram. Let $\Lambda \subset W$ be the subgroup of *translations*, i.e., those elements which have finite conjugacy classes. This is a free abelian normal subgroup of W of finite index. Let $\overline{W} = W/\Lambda$. We shall use additive notation for the group law in Λ. The conjugation action of $w \in \overline{W}$ on Λ is denoted by $\lambda \mapsto {}^{w}\lambda$.

Fix a hyperspecial vertex $s_0 \in S$ (i.e., a vertex in S with Dynkin label equal to 1). Then the finite Weyl group W_J generated by $J = S - \{s_0\}$ is a section of the natural projection $W \to \overline{W}$, and we henceforth identify W_J with $\overline{W}$. Let w_J be the longest element of W_J.

An element $\lambda \in \Lambda$ is *dominant* if $\ell(\lambda w_J) = \ell(\lambda) + \ell(w_J)$. Let Λ^+ denote the set of dominant translations. The set of double cosets $W_J \backslash W / W_J$ is in bijection with Λ^+: each W_J-double coset in W contains a unique $\lambda \in \Lambda^+$. For $\lambda \in \Lambda^+$, let $d_\lambda = \lambda w_J$ be the longest element in the double coset $W_J \lambda W_J$.

Let $*$ be the automorphism of W defined by

$$w^* := w_J w w_J, \text{ for } w \in W_J; \tag{1.1}$$
$$\lambda^* := -{}^{w_J}\lambda \text{ for } \lambda \in \Lambda.$$

This $*$ is an involution which stabilizes S and fixes s_0. In fact, if w_J acts by -1 on Λ, then $*$ is the identity; otherwise $*$ has order two. As shown in [L11, Proposition 8.2], every element d_λ belongs to I_*. Therefore we may consider the polynomials $P^{\sigma}_{d_\mu, d_\lambda}(q)$.

The following theorem is the main result of this note, which was conjectured by the first author in [L11, Conjecture 6.4].

Theorem 1.3. *Notation as above. Then for any $\lambda, \mu \in \Lambda^+$, we have*

$$P^{\sigma}_{d_\mu, d_\lambda}(q) = P_{d_\mu, d_\lambda}(-q).$$

The proof of the theorem will be given in Section 4, after some preparation regarding the geometric Satake equivalence in Section 3. In Section 7, we give a generalization of the above theorem to other involutions $\diamond$ of (W, S) which are closely related to $*$.

In [L11, Proposition 8.6], the first author proves special cases of this result by pure algebra.

It is proved in [L83, 6.1] that $P_{d_\mu,d_\lambda}(q)$ is a q-analogue of the μ-weight multiplicity in the irreducible representation V_λ of an algebraic group $\check{G}$ (see the discussion in Section 3.3). Therefore, we may interpret the above theorem as saying that $P^\sigma_{d_\mu,d_\lambda}(q)$ is a $(-q)$-analogue of weight multiplicities, hence the title of this note.

1.4 The Z^σ-functions

The polynomial $P_{y,w}(q)$ is the Poincaré polynomial of the local intersection cohomology of an affine Schubert variety indexed by w; the Poincaré polynomial of the global intersection cohomology of the same affine Schubert variety is given by

$$Z_w(q) = \sum_{y\in W; y\leq w} P_{y,w}(q)q^{\ell(y)} \in \mathbb{Z}[q]. \tag{1.2}$$

In Section 2.2, we will define certain rational functions $Z^\sigma_w(q)$ which are analogues of $Z_w(q)$ in the σ-twisted setting. We also set

$$\widetilde{Z}_{d_\lambda}(q) = Z_{d_\lambda}(q)Z_{w_J}(q)^{-1} \in \mathbb{Q}(q), \quad \widetilde{Z}^\sigma_{d_\lambda}(q) = Z^\sigma_{d_\lambda}(q)Z^\sigma_{w_J}(q)^{-1} \in \mathbb{Q}(q). \tag{1.3}$$

The function $\widetilde{Z}_{d_\lambda}(q)$ is in fact the Poincaré polynomial of the global intersection cohomology of an affine Schubert variety in the affine Grassmannian (see (5.11)), hence it belongs to $\mathbb{Z}[q]$. Our second main result is

Theorem 1.5. *For any $\lambda \in \Lambda^+$ we have $\widetilde{Z}^\sigma_{d_\lambda}(q) = \widetilde{Z}_{d_\lambda}(-q)$.*

We will present two proofs of the theorem, one geometric in Section 5 which is based on a cohomological interpretation of $Z^\sigma_w(q)$, and one algebraic in Section 6. Both proofs rely on Theorem 1.3.

1.6 Signatures

It is also observed in [L83] that $\widetilde{Z}_{d_\lambda}(q)$ is a q-analogue of the dimension of the irreducible representation V_λ of the group $\check{G}$. We will show in Section 6.6 that $\widetilde{Z}^\sigma_{d_\lambda}(q)$ is a q-analogue of the signature of V_λ under a naturally defined hermitian form introduced in [L97].

1.7 Gelfand's trick

It is interesting to notice the relation between the involution $*$ and "Gelfand's trick" in proving that the spherical Hecke algebra is commutative. In fact, for a split simply-connected almost simple group G over a local field F with Weyl group W_J, the double coset $G(\mathcal{O}_F)\backslash G(F)/G(\mathcal{O}_F)$ is in bijection with $W_J\backslash W/W_J$. The spherical Hecke algebra $\mathcal{H}^{\mathrm{sph}}$ consists of compactly supported bi-$G(\mathcal{O}_F)$-invariant functions on $G(F)$ with the algebra structure given by convolution. There is an involution $g \mapsto g^*$ of G which stabilizes a split maximal torus T and acts by $-w_J$ on $\mathbb{X}_*(T) = \Lambda$. The induced action on the affine Weyl group W is the same as the one given in Section 1.2. The anti-involution $\tau : g \mapsto (g^*)^{-1}$ induces an anti-involution on $\mathcal{H}^{\mathrm{sph}}$ while fixing each double coset $W_J\backslash W/W_J$, hence acting by identity on $\mathcal{H}^{\mathrm{sph}}$. This implies the commutativity of $\mathcal{H}^{\mathrm{sph}}$. Roughly speaking, the main theorem is a categorification of Gelfand's trick: it explains what τ does to the Satake category (categorification of $\mathcal{H}^{\mathrm{sph}}$) beyond the level of isomorphism classes of objects (on which it acts by identity).

1.8 Notation and conventions

By a tensor category, we mean a monoidal category with a commutativity constraint compatible with the associativity constraint.

For an algebraic torus T, let $\mathbb{X}_*(T)$ (resp. $\mathbb{X}^*(T)$) denote the group of cocharacters (resp. characters) of T. For a cocharacter $\lambda : \mathbb{G}_m \to T$, we use x^λ to mean the image of $x \in \mathbb{G}_m$ under λ; for a character $\alpha : T \to \mathbb{G}_m$, we use z^α to denote the image of $z \in T$ under α. Note that $(x^\lambda)^\alpha = x^{\langle \alpha,\lambda\rangle} \in \mathbb{G}_m$.

By an involution in a group, we mean an element of order at most two.

All algebraic varieties in this note are over $\mathbb{C}$; all complexes of sheaves are with $\mathbb{Q}$-coefficients.

For an algebraic variety X of dimension n, let $\mathrm{IH}^\bullet(X)$ denote its intersection cohomology groups with $\mathbb{Q}$-coefficients. We normalize it so that $\mathrm{IH}^i(X) = 0$ unless $0 \le i \le 2n$.

2. Algebraic definition of P^σ and Z^σ: recollections from [LV11] and [L11]

2.1 The P^σ-polynomials

We work with a general Coxeter group (W, S) with an involution $*$ as in the second paragraph of Section 1.1. Let M be the free $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$-module with basis $\{a_w\}_{w\in I_*}$. Let $\mathcal{H}'$ be the Hecke algebra of (W, S) with q replaced by q^2. Then [L11, Theorem 0.1] states that there is a unique $\mathcal{H}'$-module structure on M characterized by how T_s acts on a_w for each $s \in S$ and $w \in I_*$. Moreover, [L11, Theorem 0.2(a)] states that there is a unique involution $m \mapsto \overline{m}$ on M compatible with the bar involution on $\mathcal{H}'$ given by

$\bar{q} = q^{-1}$ and $\overline{T}_w = T_{w^{-1}}^{-1}$. Concretely, the bar involution on M is given by [L11, Theorem 0.2(b)]

$$\overline{a}_w = (-1)^{\ell(w)} T_{w^{-1}}^{-1} a_{w^{-1}}.$$

Next we extend scalars from $\mathcal{A}$ to $\underline{\mathcal{A}} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$. We denote $\underline{M} = \underline{\mathcal{A}} \otimes_{\mathcal{A}} M$. Then [L11, Theorem 0.4] states that there is a new $\underline{\mathcal{A}}$-basis $\{A_w\}_{w \in I_*}$ of $\underline{M}$ characterized by the property that $\overline{A}_w = A_w$ and that

$$A_w = q^{-\ell(w)/2} \sum_{y \leq w, y \in I_*} P^{\sigma}_{y,w}(q) a_y.$$

Here the polynomials $P^{\sigma}_{y,w}(q) \in \mathbb{Z}[q]$ are required to satisfy $P^{\sigma}_{w,w}(q) = 1$ and $\deg P^{\sigma}_{y,w} \leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$ for $y < w$. The uniqueness of A_w with these properties gives the definition of the polynomials $P^{\sigma}_{y,w}(q)$.

2.2 The Z^{σ}-functions

We want to define certain rational functions $Z^{\sigma}_w(q)$ which are analogues of $Z_w(q)$ in the σ-twisted setting.

In the untwisted setting, consider the $\mathcal{A}$-algebra homomorphism $\chi : \mathcal{H} \to \mathcal{A}$ given by $\chi(T_w) = q^{\ell(w)}$ for all $w \in W$. Then $Z_w(q)$ is the value of the new basis $\sum_{y \leq w} P_{y,w}(q) T_y$ under the homomorphism χ.

To define $Z^{\sigma}_w(q)$, we replace $\chi : \mathcal{H} \to \mathcal{A}$ by the following $\mathcal{A}$-linear map introduced in [L11, 5.7]

$$\begin{aligned} \zeta : M &\to \mathbb{Q}(q) \\ a_w &\mapsto q^{\ell(w)} \left(\frac{q-1}{q+1} \right)^{\phi(w)} \quad \text{for all } w \in I_* \end{aligned} \tag{2.1}$$

Here $\phi : I_* \to \mathbb{N}$ is defined in [L11, 4.5]. Concretely, for $w \in I_*$ with image $\overline{w} \in \overline{W}$, $\phi(w) = e(\overline{w}*) - e(*)$, where $e(*)$ (resp. $e(\overline{w}*)$) is the dimension of the (-1)-eigenspace of the involution $t \mapsto t^*$ (resp. $t \mapsto w(t^*)$) on $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

For $w \in I_*$ we let $Z^{\sigma}_w(q)$ be the image of the new basis of M under ζ:

$$\begin{aligned} Z^{\sigma}_w(q) &= \zeta \left(\sum_{y \in I_*; y \leq w} P^{\sigma}_{y,w}(q) a_y \right) \\ &= \sum_{y \in I_*; y \leq w} P^{\sigma}_{y,w}(q) q^{\ell(y)} \left(\frac{q-1}{q+1} \right)^{\phi(y)} \in \mathbb{Q}(q) \end{aligned} \tag{2.2}$$

3. Geometric definition of the P^σ-polynomials

3.1 Affine flag variety

In this section we give a geometric definition of the polynomials $P^\sigma_{x,y}(q)$. In fact, in the case of finite Weyl groups with $* = \mathrm{id}$, such a geometric definition is given in [LV11, Section 3] using the geometry of flag varieties. It is remarked in [LV11, Section 7.1-7.2] that such a geometric definition works for affine Weyl groups and general $*$, with the flag varieties replaced by affine flag varieties. This section is an elaboration of this remark.

Let G be the simply-connected almost simple group over $\mathbb{C}$ whose extended Dynkin diagram is the one we started with in Section 1.2, so that the usual Dynkin diagram of G is given by removing the vertex s_0. Fix a pinning for G; in particular, fix a maximal torus $T \subset G$, and a Borel B containing T. We may identify $(W_J, S - \{s_0\})$ with the Weyl group $N_G(T)/T$ together with the simple reflections determined by B. We may also identify Λ with the cocharacter lattice $\mathbb{X}_*(T)$, which is also the coroot lattice of G.

Let $G((t))$ be the loop group associated to G: it is the ind-scheme representing the functor $R \mapsto G(R((t)))$ for any $\mathbb{C}$-algebra R. Let $G[[t]] \subset G((t))$ be the subscheme representing the functor $R \mapsto G(R[[t]])$. The affine Weyl group W may be identified with the $\mathbb{C}$-points of $N_{G((t))}(T((t)))/T[[t]]$. For each $w \in W$, we choose a lifting $\dot{w}$ of it in $N_{G((t))}(T((t)))$. For example, if $\lambda \in \Lambda$, we may choose $\dot{\lambda}$ to be the point $t^\lambda \in T((t))$.

An Iwahori subgroup of $G((t))$ is one which is conjugate to $\mathbf{I} = \pi^{-1}(B) \subset G[[t]]$ where $\pi : G[[t]] \to G$ is the mod t reduction morphism. Let $\mathrm{Fl} = G((t))/\mathbf{I}$ be the affine flag variety of G classifying Iwahori subgroups of the loop group $G((t))$. This is a (locally finite) infinite union of projective varieties over $\mathbb{C}$ of increasing dimensions. The group scheme $\mathbf{I}$ acts on Fl from the left with orbits $\mathrm{Fl}_w = \mathbf{I}\dot{w}\mathbf{I}/\mathbf{I}$ indexed by $w \in W$. Each orbit Fl_w is isomorphic to an affine space of dimension $\ell(w)$ (with respect to the simple reflections S). Let $\mathrm{Fl}_{\le w}$ be the closure of Fl_w, which is the union of Fl_y for $y \le w$.

Consider the derived category $D_{\mathbf{I}}(\mathrm{Fl}) = \varinjlim_{w\in W} D_{\mathbf{I}}(\mathrm{Fl}_{\le w})$ of $\mathbf{I}$-equivariant $\mathbb{Q}$-complexes which are supported on the $\mathrm{Fl}_{\le w}$ for some $w \in W$. Note that for fixed w, the $\mathbf{I}$-action on $\mathrm{Fl}_{\le w}$ factors through a quotient group scheme $\mathbf{I}_w$ of finite type such that $\ker(\mathbf{I} \to \mathbf{I}_w)$ is pro-unipotent. We therefore understand $D_{\mathbf{I}}(\mathrm{Fl}_{\le w})$ as the category of $\mathbf{I}_w$-equivariant derived category of $\mathbb{Q}$-complexes on the projective variety $\mathrm{Fl}_{\le w}$ in the sense of [BL94].

3.2 Geometric interpretation of the P^σ-polynomials

Let $*$ denote the pinned automorphism of G such that $\lambda \mapsto ({}^{w_J}\lambda)^*$ acts by -1 on Λ. This involution induces an involution on the affine Weyl group (W, S)

which coincides with the $*$ defined in (1.1). The involution $*$ also induces an involution on $G((t))$ preserving the Iwahori $\mathbf{I}$, so that it induces an involution on Fl which we still denote by $*$.

Consider the anti-involution τ of $G((t))$ defined as

$$\tau(g) = (g^*)^{-1}.$$

We would like to define a functor:

$$\tau^* : D_{\mathbf{I}}(\mathrm{Fl}) \to D_{\mathbf{I}}(\mathrm{Fl})$$

given by pull-back along the map τ. We may identify each object of $D_{\mathbf{I}}(\mathrm{Fl})$ as a complex on $G((t))$ equivariant under the left and right translation by $\mathbf{I}$. Since each $\mathbf{I}$-double coset $\mathbf{I}\dot{w}\mathbf{I} \subset G((t))$ is sent to another double coset $\mathbf{I}(\dot{w}^*)^{-1}\mathbf{I}$, pull-back by τ preserves bi-$\mathbf{I}$-equivariance, and defines the functor τ^*.

For each object $\mathbf{K} \in D_{\mathbf{I}}(\mathrm{Fl})$ and $y \in W$, the restriction of $\mathbf{K}$ to Fl_y is a constant complex by $\mathbf{I}$-equivariance. We therefore have a vector space $\mathcal{H}^i_y\mathbf{K}$, which is canonically isomorphic to the i-th cohomology of the stalk of $\mathbf{S}_w$ at any point of Fl_y.

For each $w \in W$, one has the (shifted) intersection cohomology complex $\mathbf{S}_w \in D_{\mathbf{I}}(\mathrm{Fl})$ of $\mathrm{Fl}_{\leq w}$, which we normalize so that $\mathbf{S}_w|_{\mathrm{Fl}_w} \cong \mathbb{Q}$. If $w \in I_*$ (i.e., $(w^*)^{-1} = w$), we have a canonical isomorphism

$$\Phi_w : \tau^*\mathbf{S}_w \xrightarrow{\sim} \mathbf{S}_w \tag{3.1}$$

whose restriction to Fl_w is the identity map for the constant sheaf $\mathbb{Q}$. For each $y \in I_*$, $y \leq w$, the restriction of Φ_w induces an involution:

$$\mathcal{H}^i_y\Phi_w : \mathcal{H}^i_y\mathbf{S}_w = \tau^*\mathcal{H}^i_y(\tau^*\mathbf{S}_w) \to \mathcal{H}^i_y\mathbf{S}_w$$

where the first equality comes from the definition of τ^*. Then

$$P^{\sigma}_{y,w}(q) = \sum_{i\in\mathbb{Z}} \mathrm{tr}(\mathcal{H}^i_y\Phi_w, \mathcal{H}^i_y\mathbf{S}_w)q^{i/2}. \tag{3.2}$$

It is known that $\mathcal{H}^i_y\mathbf{S}_w = 0$ for odd i (see [KL80, Theorem 4.2] for the case W finite, [KL80, Theorem 5.5] for the case W affine; see also [G01, A.7] for the affine case), therefore $P^{\sigma}_{y,w} \in \mathbb{Z}[q]$.

3.3 Affine Grassmannian and the geometric Satake equivalence

Let $\mathrm{Gr} = G((t))/G[[t]]$ be the affine Grassmannian of G, which is also a locally finite union of projective varieties of increasing dimensions. The left translation by $G[[t]]$ on Gr has orbits indexed by W_J-orbits on Λ. For each dominant coweight $\lambda \in \Lambda^+$, there is a unique $G[[t]]$-orbit Gr_λ containing t^λ

(which also contains $t^{\lambda'}$ for any λ' in the same W_J-orbit of λ). The dimension of Gr_λ is $\langle 2\rho, \lambda\rangle$, where 2ρ is the sum of positive roots of G.

Let $\mathcal{S} = P_{G[[t]]}(\mathrm{Gr})$ be the category of $G[[t]]$-equivariant perverse sheaves on Gr which are supported on finitely many $G[[t]]$-orbits. This abelian category carries a convolution product $\odot : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ (see [L83, Corollary 8.7], see [G95, Proposition 2.2.1]), which is equipped with an obvious associativity constraint and a less obvious commutativity constraint (due to Drinfeld, see an exposition in [MV07, Section 5]) making $(\mathcal{S}, \odot)$ a tensor category (the convolution product is usually denoted by $*$ in literature, and we change it to $\odot$ to avoid confusion with the involution $*$). Let $\mathrm{Vec}^{\mathrm{gr}}$ be the category of finite dimensional graded $\mathbb{Q}$-vector spaces (the commutativity constraint is *not* adjusted by the Koszul sign convention, so $\mathrm{Vec}^{\mathrm{gr}} \cong \mathrm{Rep}(\mathbb{G}_m)$ as tensor categories). Consider the functor

$$\begin{aligned} \mathrm{H}^\bullet : \mathcal{S} &\to \mathrm{Vec}^{\mathrm{gr}} \\ \mathbf{K} &\mapsto \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(\mathrm{Gr}, \mathbf{K}). \end{aligned}$$

This functor carries a tensor structure (see [G95, Proposition 3.4.1] and [MV07, Proposition 6.3], note that the commutativity constraint of $\mathcal{S}$ is adjusted by a sign in [MV07, Paragraph after Remark 6.2] in order to make $\mathrm{H}^\bullet$ a tensor functor).

Composing $\mathrm{H}^\bullet$ with the forgetful functor $\mathrm{Vec}^{\mathrm{gr}} \to \mathrm{Vec}$ (the category of finite dimensional vector spaces), we get a fiber functor H of the tensor category $\mathcal{S}$, hence an algebraic group $\check{G} = \mathrm{Aut}^{\otimes}(\mathrm{H})$ over $\mathbb{Q}$. In [G95, Theorem 3.8.1] (with the corrected commutativity constraint by Drinfeld and based on results of [L83]), it is proved that $\check{G}$ is a connected split reductive group over $\mathbb{Q}$ whose root datum is dual to G. The proof in [MV07, Theorem 7.3] in fact equips $\check{G}$ with a maximal torus $\check{T}$ with a canonical identification $\mathbb{X}^*(\check{T}) = \Lambda = \mathbb{X}_*(T)$. In fact, letting Vec^Λ be the category of finite dimensional Λ-graded vector spaces, the functor $\mathrm{H}^\bullet$ factors as

$$\mathrm{H}^\bullet : \mathcal{S} \xrightarrow{\oplus_{\lambda\in\Lambda} F_\lambda} \mathrm{Vec}^\Lambda \xrightarrow{\langle 2\rho, -\rangle} \mathrm{Vec}^{\mathrm{gr}}$$

Here the first arrow is the sum of *weight functors* introduced in [MV07, Theorem 3.6]; the second functor turns a Λ-graded vector space $\oplus_\lambda V^\lambda$ into a $\mathbb{Z}$-graded one $V^i := \oplus_{\langle 2\rho,\lambda\rangle = i} V^\lambda$. Under the identification $\mathcal{S} \xrightarrow{\sim} \mathrm{Rep}(\check{G})$, the functor $\mathrm{H}^\bullet$ then factors as

$$\mathrm{Rep}(\check{G}) \to \mathrm{Rep}(\check{T}) \to \mathrm{Rep}(\mathbb{G}_m)$$

induced by the homomorphisms $2\rho : \mathbb{G}_m \to \check{T} \hookrightarrow \check{G}$.

3.4 Geometric interpretation of $P^{\sigma}_{d_\mu,d_\lambda}(q)$

For each $\lambda \in \Lambda^+$, let $\mathbf{C}_\lambda$ be the shifted intersection cohomology complex of the closure $\mathrm{Gr}_{\le\lambda}$ of Gr_λ, such that $\mathbf{C}_\lambda|_{\mathrm{Gr}_\lambda} = \mathbb{Q}$. The involution τ of $G((t))$ again induces a functor

$$\tau^* : \mathcal{S} \to \mathcal{S}. \tag{3.3}$$

One can similarly define the stalk $\mathcal{H}^i_\mu \mathbf{C}_\lambda$ for $\mu \le \lambda \in \Lambda^+$, which again vanishes for odd i. Each double coset $G[[t]]t^\lambda G[[t]]$ is sent to $G[[t]]t^{-\lambda^*}G[[t]]$. By the definition of $*$, we have $-\lambda^* = {}^{w_J}\lambda$, hence $G[[t]]t^{-\lambda^*}G[[t]] = G[[t]]t^\lambda G[[t]]$, i.e., each $G[[t]]$-double coset in $G((t))$ is stable under τ (this is equivalent to saying that the longest element in each W_J-double coset belongs to the set I_* of $*$-twisted involutions). This means one can fix an isomorphism

$$\Psi_\lambda : \tau^*\mathbf{C}_\lambda \xrightarrow{\sim} \mathbf{C}_\lambda \tag{3.4}$$

which is the identity when restricted to Gr_λ. This isomorphism similarly induces an involution:

$$\mathcal{H}^i_\mu \Psi_\lambda : \mathcal{H}^i_\mu \mathbf{C}_\lambda = \mathcal{H}^i_\mu(\tau^*\mathbf{C}_\lambda) \to \mathcal{H}^i_\mu \mathbf{C}_\lambda.$$

We have a projection map $\pi : \mathrm{Fl} \to \mathrm{Gr}$. For each $\lambda \in \Lambda^+$, the preimage $\pi^{-1}(\mathrm{Gr}_{\le\lambda}) = \mathrm{Fl}_{\le d_\lambda}$ (recall $d_\lambda \in W_J\lambda W_J$ is the longest element). Since $\mathrm{Fl}_{\le d_\lambda} \to \mathrm{Gr}_{\le\lambda}$ is smooth, we have an isomorphism $\phi_\lambda : \pi^*\mathbf{C}_\lambda \cong \mathbf{S}_{d_\lambda}$, which can be made canonical by requiring its restriction to Fl_{d_λ} to be the identity map on the constant sheaf. Moreover, the isomorphism ϕ_λ clearly intertwines Ψ_λ and Φ_{d_λ}. Using ϕ_λ, we get a commutative diagram

$$\begin{tikzcd}
\mathcal{H}^j_\mu \mathbf{C}_\lambda \arrow[r, "\mathcal{H}^j_\mu \phi_\lambda"] \arrow[d, "\mathcal{H}^j_\mu \Psi_\lambda"'] & \mathcal{H}^j_{d_\mu} \mathbf{S}_{d_\lambda} \arrow[d, "\mathcal{H}^j_\mu \Phi_\lambda"] \\
\mathcal{H}^j_\mu \mathbf{C}_\lambda \arrow[r, "\mathcal{H}^j_\mu \phi_\lambda"] & \mathcal{H}^j_{d_\mu} \mathbf{S}_{d_\lambda}
\end{tikzcd}$$

in which the horizontal arrows are isomorphisms. Therefore, from (3.2) we get

$$P^{\sigma}_{d_\mu,d_\lambda}(q) = \sum_{j\in\mathbb{Z}} \mathrm{tr}(\mathcal{H}^{2j}_\mu \Psi_\lambda, \mathcal{H}^{2j}_\mu \mathbf{C}_\lambda) q^j. \tag{3.5}$$

3.5 Loop group of a compact form

At certain points in the proof of the main theorem, it is convenient to take an alternative point of view of the affine Grassmannian Gr, namely the space of polynomial loops on the compact form of G. We remark that the switch of viewpoint is not necessary for the proof, but it makes the idea of the proof more transparent.

Let $K \subset G(\mathbb{C})$ be a compact real form which is stable under $*$ (for example, we may define K using the Cartan involution $\dot{w}_J *$, for any lifting of $\dot{w}_J$ of w_J to $N_G(T)$). Let $\Omega = \Omega_{\mathrm{pol}} K$ be the space of polynomial loops on K based at the identity element $1 \in K$ (see [PS86, §3.5]). By [PS86, Theorem 8.6.3], there is a homeomorphism

$$\imath : \Omega \overset{\widetilde{\imath}}{\hookrightarrow} G(\mathbb{C}((t))) \xrightarrow{p} \mathrm{Gr}(\mathbb{C}).$$

The stratification of Gr by $\{\mathrm{Gr}_\lambda\}_{\lambda \in \Lambda^+}$ gives a Whitney stratification of Ω. We denote the strata by Ω_λ with closure $\Omega_{\leq\lambda}$. Let $D^b(\Omega) = \varinjlim_\lambda D^b(\Omega_{\leq\lambda})$. Let $\mathcal{S}_K$ be the full subcategory of $D^b(\Omega)$ consisting of perverse sheaves which are locally constant along each strata Ω_λ.

Let $m_K : \Omega \times \Omega \to \Omega$ be the multiplication map. This is stratified in the sense that $m_K(\Omega_{\leq\lambda} \times \Omega_{\leq\mu}) = \Omega_{\leq\lambda+\mu}$ for $\lambda, \mu \in \Lambda^+$. Define

$$\begin{aligned} \odot_K : D^b(\Omega) \times D^b(\Omega) &\to D^b(\Omega) \\ (\mathbf{K}_1, \mathbf{K}_2) &\mapsto m_{K!}(\mathbf{K}_1 \boxtimes \mathbf{K}_2). \end{aligned}$$

Let

$$\mathrm{H}^\bullet : \mathcal{S}_K \to \mathrm{Vec}^{\mathrm{gr}}$$

be the functor of taking total cohomology.

The involution $\tau : k \mapsto (k^*)^{-1}$ on K induces an involution τ_K on Ω, which gives the pullback functor

$$\tau_K^* : D^b(\Omega) \to D^b(\Omega).$$

Lemma 3.6.

(1) *The functor $\odot_K$ has image in $\mathcal{S}_K$, and there is a natural associativity constraint making $(\mathcal{S}_K, \odot_K)$ a monoidal category;* $\mathrm{H}^\bullet : \mathcal{S}_K \to \mathrm{Vec}^{\mathrm{gr}}$ *is naturally a monoidal functor.*

(2) *The pull-back functor $\imath^*$ gives a monoidal equivalence $\imath^* : (\mathcal{S}, \odot) \to (\mathcal{S}_K, \odot_K)$.*

(3) *There is a natural isomorphism of monoidal functors* $\mathrm{H}^\bullet \circ \imath^* \cong \mathrm{H}^\bullet : \mathcal{S} \to \mathrm{Vec}^{\mathrm{gr}}$.

(4) *The functor τ_K^* sends $\mathcal{S}_K$ to $\mathcal{S}_K$; τ^* and τ_K^* are naturally intertwined under $\imath^*$.*

Proof. (1)(2) The functor $\imath^*$ identifies $\mathcal{S}_K$ with the category of perverse sheaves on Gr locally constant along the strata Gr_λ. By [MV07, Proposition A.1] the latter category is canonically equivalent to $\mathcal{S}$. To prove (1) and (2), it suffices to give ι^* a monoidal structure. Recall that the convolution product $\odot$ on $\mathcal{S}$ is defined as

$$\mathbf{K}_1 \odot \mathbf{K}_2 = m_!(\mathbf{K}_1 \boxdot \mathbf{K}_2)$$

Here $m : G((t)) \overset{G[[t]]}{\times} \mathrm{Gr} \to \mathrm{Gr}$ is the multiplication map, $\mathbf{K}_1 \boxdot \mathbf{K}_2$ is the perverse sheaf on $G((t)) \overset{G[[t]]}{\times} \mathrm{Gr}$ characterized by

$$p'^*\mathbf{K}_1 \boxdot \mathbf{K}_2 = p^*\mathbf{K}_1 \boxtimes \mathbf{K}_2 \text{ on } G((t)) \times \mathrm{Gr}, \tag{3.6}$$

where $p : G((t)) \to \mathrm{Gr}$, $p' : G((t)) \times \mathrm{Gr} \to G((t)) \overset{G[[t]]}{\times} \mathrm{Gr}$ are the projections. To give ι^* a tensor structure, we need to give a canonical isomorphism

$$m_{K!}(\iota^*\mathbf{K}_1 \boxtimes \iota^*\mathbf{K}_2) \cong \iota^* m_!(\mathbf{K}_1 \boxdot \mathbf{K}_2)$$

for any $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{S}$. Note that we have a commutative diagram

$$\begin{CD} \Omega \times \Omega @>{\iota_2}>> G((t)) \overset{G[[t]]}{\times} \mathrm{Gr} \\ @V{m_K}VV @VV{m}V \\ \Omega @>{\iota}>> \mathrm{Gr} \end{CD} \tag{3.7}$$

where ι_2 is given by the composition

$$\Omega \times \Omega \xrightarrow{\tilde{\iota} \times \iota} G((t)) \times \mathrm{Gr} \xrightarrow{p'} G((t)) \overset{G[[t]]}{\times} \mathrm{Gr}.$$

It is easy to see that ι_2 is also a homeomorphism, so (3.7) is a Cartesian diagram. Therefore, by proper base change, we have a canonical isomorphism

$$\begin{aligned} \iota^* m_!(\mathbf{K}_1 \boxdot \mathbf{K}_2) &\cong m_{K!}\iota_2^*(\mathbf{K}_1 \boxdot \mathbf{K}_2) \\ &= m_{K!}(\tilde{\iota} \times \iota)^* p'^*(\mathbf{K}_1 \boxdot \mathbf{K}_2) \overset{(3.6)}{=} m_{K!}(\tilde{\iota} \times \iota)^*(p^*\mathbf{K}_1 \boxtimes \mathbf{K}_2) \\ &= m_{K!}(\tilde{\iota}^* p^*\mathbf{K}_1 \boxtimes \iota^*\mathbf{K}_2) = m_{K!}(\iota^*\mathbf{K}_1 \boxtimes \iota^*\mathbf{K}_2) \end{aligned}$$

It is easy to check these isomorphisms are compatible with the associativity constraints.

(3) is obvious.

(4) For each $\mathbf{K} \in \mathcal{S}$, we need to give a functorial isomorphism

$$\iota^*\tau^*\mathbf{K} \xrightarrow{\sim} \tau_K^*\iota^*\mathbf{K}.$$

Recall ι factors as $\Omega \xrightarrow{\tilde{\iota}} G((t)) \xrightarrow{p} \mathrm{Gr}$ and $\tilde{\iota}\tau_K = \tau\tilde{\iota}$, where $\tau : g \mapsto (g^*)^{-1}$ is the anti-automorphism of $G((t))$. Therefore

$$\iota^*\tau^*\mathbf{K} = \tilde{\iota}^* p^*\tau^*\mathbf{K} = \tilde{\iota}^*\tau^* p^*\mathbf{K} = \tau_K^*\tilde{\iota}^* p^*\mathbf{K} = \tau_K^*\iota^*\mathbf{K}.$$

This gives the desired isomorphism. □

Using part (2) of Lemma 3.6, one can transfer the commutativity constraint of $(\mathcal{S}, \odot)$ to $(\mathcal{S}_K, \odot_K)$ making the latter a tensor category. Part (3) of Lemma 3.6 then gives the functor $\mathrm{H}^\bullet$ a tensor (in addition to monoidal) structure.

4. Proof of theorem 1.3

For a monoidal category $(\mathcal{C}, \otimes)$, we let $(\mathcal{C}, \otimes^\sigma)$ be the same category equipped with a new functor $\otimes^\sigma : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ given by $X \otimes^\sigma Y := Y \otimes X$. It is easy to check that $(\mathcal{C}, \otimes^\sigma)$ also carries a monoidal structure.

Lemma 4.1. *The functor $\tau^* : \mathcal{S} \to \mathcal{S}$ carries a natural structure of a monoidal functor*

$$\tau^* : (\mathcal{S}, \odot) \to (\mathcal{S}, \odot^\sigma).$$

Proof. Using Lemma 3.6(2) and (4), it suffices to construct the monoidal structure of τ_K^*. Let $\sigma : \Omega \times \Omega \to \Omega \times \Omega$ be the involution which interchanges two factors. Since τ_K is an anti-involution, we have a Cartesian diagram

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "\tau_K \times \tau_K"] \arrow[d, "m_K \circ \sigma"] & \Omega \times \Omega \arrow[d, "m_K"] \\ \Omega \arrow[r, "\tau_K"] & \Omega \end{tikzcd} \tag{4.1}$$

Therefore by proper base change, for any $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{S}_K$, we have a canonical isomorphism

$$\tau_K^* m_{K!}(\mathbf{K}_1 \boxtimes \mathbf{K}_2) \cong (m_K \circ \sigma)_!(\tau_K^* \mathbf{K}_1 \boxtimes \tau_K^* \mathbf{K}_2) = m_{K!}(\tau_K^* \mathbf{K}_2 \boxtimes \tau_K^* \mathbf{K}_2).$$

By the definition of $\odot_K$, we get a canonical isomorphism

$$\tau_K^*(\mathbf{K}_1 \odot_K \mathbf{K}_2) \xrightarrow{\sim} \tau_K^* \mathbf{K}_2 \odot_K \tau_K^* \mathbf{K}_1.$$

It is easy to check that these isomorphisms are compatible with the associativity constraint and the unit objects of $(\mathcal{S}_K, \odot_K)$ and $(\mathcal{S}_K, \odot_K^\sigma)$. This finishes the proof of the lemma. □

Let $\mathrm{H}^{\bullet,\sigma} : (\mathcal{S}, \odot^\sigma) \to (\mathrm{Vec}^{\mathrm{gr}}, \otimes)$ be the same functor as $\mathrm{H}^\bullet$, except that we change its monoidal structure to the one of $\mathrm{H}^\bullet$ composed with the commutativity constraint of $\otimes$ for $\mathrm{Vec}^{\mathrm{gr}}$, so that $\mathrm{H}^{\bullet,\sigma}$ is also a tensor functor.

Lemma 4.2. *There is a natural isomorphism $\gamma : \mathrm{H}^{\bullet,\sigma} \circ \tau^* \xrightarrow{\sim} \mathrm{H}^\bullet$, which preserves the monoidal structures of both functors.*

Proof. Using Lemma 3.6, it suffices to give a natural isomorphism γ_K : $\mathrm{H}^\bullet \circ \tau_K^* \xrightarrow{\sim} \mathrm{H}^\bullet$ between functors $\mathcal{S}_K \to \mathrm{Vec}^{\mathrm{gr}}$, which preserves the monoidal structures. Since τ_K is an automorphism of $\boldsymbol{\Omega}$, we have a canonical isomorphism $\mathrm{H}^\bullet(\boldsymbol{\Omega}, \tau_K^*\mathbf{K}) \xrightarrow{\sim} \mathrm{H}^\bullet(\boldsymbol{\Omega}, \mathbf{K})$, which gives the desired γ_K. It remains to check that γ preserves the monoidal structures. But this is also obvious from the natural monoidal structure of $\mathrm{H}^\bullet : \mathcal{S}_K \to \mathrm{Vec}^{\mathrm{gr}}$. □

Suppose we have two Tannakian categories $(\mathcal{C}, \otimes)$ and $(\mathcal{D}, \otimes)$ equipped with fiber functors $\omega_{\mathcal{C}}$ and $\omega_{\mathcal{D}}$ into Vec_k respectively (k is a field). Let $F : (\mathcal{C}, \otimes) \to (\mathcal{D}, \otimes)$ be a monoidal functor equipped with a monoidal isomorphism $\phi : \omega_{\mathcal{D}} \circ F \xrightarrow{\sim} \omega_{\mathcal{C}}$. Then ϕ induces a homomorphism of algebraic groups over k:

$$(F, \phi)^{\#} : \mathrm{Aut}^{\otimes}(\omega_{\mathcal{D}}) \to \mathrm{Aut}^{\otimes}(\omega_{\mathcal{C}})$$

$$(\omega_{\mathcal{D}} \xrightarrow{h} \omega_{\mathcal{D}}) \mapsto (\omega_{\mathcal{C}} \xrightarrow{\phi^{-1}} \omega_{\mathcal{D}} \circ F \xrightarrow{h \circ \mathrm{id}_F} \omega_{\mathcal{D}} \circ F \xrightarrow{\phi} \omega_{\mathcal{C}}).$$

Since the notion of tensor morphisms between tensor functors only use their structures as monoidal functors, the above definition makes sense even if F is only a monoidal functor. More generally, if $\omega_{\mathcal{C}}$ and $\omega_{\mathcal{D}}$ take values in another Tannakian category $\mathcal{V}$ equipped with a fiber functor $\omega : \mathcal{V} \to \mathrm{Vec}$, then F induces a homomorphism of algebraic groups $(F, \phi)^{\#} : \mathrm{Aut}^{\otimes}(\omega \circ \omega_{\mathcal{D}}) \to \mathrm{Aut}^{\otimes}(\omega \circ \omega_{\mathcal{C}})$ making the following diagram commutative

$$\begin{tikzcd} & \mathrm{Aut}^{\otimes}(\omega) \arrow[dl, "\omega_{\mathcal{D}}^{\#}"'] \arrow[dr, "\omega_{\mathcal{C}}^{\#}"] & \\ \mathrm{Aut}^{\otimes}(\omega \circ \omega_{\mathcal{D}}) \arrow[rr, "(F,\phi)^{\#}"] & & \mathrm{Aut}^{\otimes}(\omega \circ \omega_{\mathcal{C}}) \end{tikzcd}$$

We apply the above remarks to the situation

$$\begin{tikzcd} (\mathcal{S}, \odot) \arrow[rr, "\tau^*"] \arrow[dr, "\mathrm{H}^\bullet"'] & \arrow[d, phantom, "\overset{\gamma}{\Leftarrow}"] & (\mathcal{S}, \odot^{\sigma}) \arrow[dl, "\mathrm{H}^{\bullet,\sigma}"] \\ & \mathrm{Vec}^{\mathrm{gr}} & \end{tikzcd}$$

and get a commutative diagram of algebraic groups over $\mathbb{Q}$:

$$\begin{tikzcd} & \mathbb{G}_m \arrow[dl, "2\rho"'] \arrow[dr, "2\rho"] & \\ \check{G} \arrow[rr, "(\tau^*, \gamma)^{\#}"] & & \check{G} \end{tikzcd}$$

In other words, $(\tau^*, \gamma)^{\#}$ is an automorphism of $\check{G}$ fixing the elements in the torus $2\rho(\mathbb{G}_m)$ pointwise. Since τ^* does not change the isomorphism classes of irreducible objects in $\mathcal{S}$, this automorphism must be inner. Therefore $(\tau^*, \gamma)^{\#}$ determines an element $g \in \check{T}$ (note that $\check{G}$ is of adjoint form).

Using the commutative constraint of $(\mathcal{S}, \odot)$, the identity functor gives a monoidal equivalence

$$\mathrm{id}^{\sigma}_{\mathcal{S}} : (\mathcal{S}, \odot) \xrightarrow{\sim} (\mathcal{S}, \odot^{\sigma}).$$

There is a unique natural isomorphism of monoidal functors $\Theta : \tau^* \xrightarrow{\sim} \mathrm{id}^{\sigma}_{\mathcal{S}}$ making

$$\mathrm{id}_{\mathrm{H}^{\bullet,\sigma}} \circ \Theta = \gamma : \mathrm{H}^{\bullet,\sigma} \circ \tau^* \to \mathrm{H}^{\bullet,\sigma} \circ \mathrm{id}^{\sigma}_{\mathcal{S}} = \mathrm{H}^{\bullet}.$$

In fact, identifying $\mathcal{S}$ with $\mathrm{Rep}(\check{G})$, the functor τ^* sends $V \in \mathrm{Rep}(\check{G})$ (with the action $\alpha : \check{G} \to \mathrm{Aut}(V)$) to the same vector space V with the new action $\check{G} \xrightarrow{\mathrm{Ad}(g)} \check{G} \xrightarrow{\alpha} \mathrm{Aut}(V)$. Then the effect of the natural isomorphism Θ on V is given by $\alpha(g^{-1}) : V \to V$.

Lemma 4.3.

(1) *The element* $g \in \check{T}(\mathbb{Q})$ *is* $(-1)^{\rho}$*, the image of* -1 *under the cocharacter* $\rho : \mathbb{G}_m \to \check{T}$ *(note that* $\check{G}$ *is of adjoint type, so* ρ *is a cocharacter of* $\check{T}$*).*
(2) *The effect of the natural isomorphism* Θ *on the intersection complex* $\mathbf{C}_{\lambda}[\langle 2\rho, \lambda\rangle] \in \mathcal{S}$ *is* $(-1)^{\langle \rho, \lambda\rangle}\Psi_{\lambda}$*.*
(3) *The action of the involution* τ_K^* *on* $\mathrm{IH}^{2j}(\mathbf{\Omega}_{\leq\lambda})$ *is by* $(-1)^j$*.*

Proof. Let $\lambda \in \Lambda^+$. The action of g^{-1} on $\mathrm{H}^{\bullet}(\mathbf{\Omega}, \mathbf{C}_{\lambda})[\langle 2\rho, \lambda\rangle] = \mathrm{IH}^{\bullet}(\mathbf{\Omega}_{\leq\lambda})$ $[\langle 2\rho, \lambda\rangle] = V_{\lambda} \in \mathrm{Rep}(\check{G})$ is given by the composition

$$\begin{aligned} \mathrm{IH}^{\bullet}(\mathbf{\Omega}_{\leq\lambda}) \xrightarrow{\tau_K^*} \mathrm{IH}^{\bullet}(\mathbf{\Omega}_{\leq\lambda}) &= \mathrm{H}^{\bullet}(\mathbf{\Omega}, \tau_K^*\mathbf{C}_{\lambda}) \\ &\xrightarrow{\mathrm{H}^{\bullet}(\mathbf{\Omega}, \Theta_{\lambda})} \mathrm{H}^{\bullet}(\mathbf{\Omega}, \mathbf{C}_{\lambda}) = \mathrm{IH}^{\bullet}(\mathbf{\Omega}_{\leq\lambda}). \end{aligned}$$

where the first arrow is the pull-back along the anti-involution τ_K of $\mathbf{\Omega}_{\leq\lambda}$ and $\Theta_{\lambda} : \tau_K^*\mathbf{C}_{\lambda} \to \mathbf{C}_{\lambda}$ is induced from the effect of Θ on $\mathbf{C}_{\lambda}[\langle 2\rho, \lambda\rangle] \in \mathcal{S}$. Since the only automorphisms of $\mathbf{C}_{\lambda}$ are scalars, the isomorphisms Θ_{λ} and Ψ_{λ} must be related by $\Theta_{\lambda} = c_{\lambda}\Psi_{\lambda}$ for some constant $c_{\lambda} \in \mathbb{Q}^{\times}$: the restriction of Θ_{λ} on $\mathbf{\Omega}_{\lambda}$ is given by multiplication by c_{λ} on the constant sheaf.

The stratum $\mathbf{\Omega}_{\lambda}$ homotopy retracts to the K-orbit of t^{λ}, which is a partial flag variety $G/P_{\lambda} = K/P_{\lambda} \cap K$ (see [MV07, Top of page 100]). The action of τ_K on $\mathrm{Ad}(K)t^{\lambda} \cong K/P_{\lambda} \cap K$ is given by

$$kt^{\lambda}k^{-1} \mapsto (k^*t^{\lambda^*}k^{*,-1})^{-1} = k^*t^{-\lambda^*}k^{*,-1} = k^*\dot{w}_J t^{\lambda}\dot{w}_0^{-1}k^{*,-1}.$$

Therefore the induced action of τ_K^* on $K/P_{\lambda} \cap K$ is given by $k \mod P_{\lambda} \cap K \mapsto k^*w_J \mod P_{\lambda} \cap K$ (any lifting $\dot{w}_J \in N_{T\cap K}(K)$ normalizes $P_{\lambda} \cap K$, hence the right translation makes sense).

Let $j_\lambda : \Omega_\lambda \hookrightarrow \Omega_{\le\lambda}$ be the inclusion. We have a commutative diagram

$$\begin{tikzcd} \mathrm{IH}^i(\Omega_{\le\lambda}) \arrow[r, "j_\lambda^*"] \arrow[d, "c_\lambda^{-1}g^{-1}"] & \mathrm{H}^i(\Omega_\lambda) \arrow[r, "\sim"] \arrow[d, "\tau_K^*"] & \mathrm{H}^i(K/P_\lambda \cap K) \arrow[d, "w_J *"] \\ \mathrm{IH}^i(\Omega_{\le\lambda}) \arrow[r, "j_\lambda^*"] & \mathrm{H}^i(\Omega_\lambda) \arrow[r, "\sim"] & \mathrm{H}^i(K/P_\lambda \cap K) \end{tikzcd} \tag{4.2}$$

When $i \le 2$, the horizontal restriction maps are isomorphisms. In fact, from the stratification $\Omega_{\le\lambda}$ by the open Ω_λ and the closed complement $z : \Omega_{<\lambda} \hookrightarrow \Omega_{\le\lambda}$, we get an exact sequence

$$\mathrm{H}^i(\Omega_{<\lambda}, z^!\mathbf{C}_\lambda) \to \mathrm{IH}^i(\Omega_{\le\lambda}) \to \mathrm{H}^i(\Omega_\lambda) \to \mathrm{H}^i(\Omega_{<\lambda}, z^!\mathbf{C}_\lambda) \tag{4.3}$$

Since $\dim \Omega_{<\lambda} \le \langle 2\rho, \lambda\rangle - 2$ and $z^!\mathbf{C}_\lambda[\langle 2\rho, \lambda\rangle]$ lies in perverse degree ≥ 1, $z^!\mathbf{C}_\lambda$ lies in the usual cohomological degree ≥ 3. This implies $\mathrm{H}^i(\Omega_{<\lambda}, z^!\mathbf{C}_\lambda) = 0$ for $i \le 2$ hence the isomorphism follows from the exact sequence (4.3).

We claim that the action $\tau_K^* : k \mapsto k^* w_J$ on the partial flag variety $K/P_\lambda \cap K$ induces -1 on $\mathrm{H}^2(K/P_\lambda \cap K)$. In fact, $\mathrm{H}^2(K/P_\lambda \cap K, \mathbb{Q}) \hookrightarrow \mathrm{H}^2(K/T, \mathbb{Q}) \cong \mathbb{X}^*(T)_{\mathbb{Q}}$ by pull-back along the projection $K/T \cap K \to K/P_\lambda \cap K$, and this map is equivariant under the $(W \rtimes \mathrm{Out}(G))_\lambda$-actions (subscript λ means stabilizer of λ under the $W \rtimes \mathrm{Out}(G)$-action on $\Lambda = \mathbb{X}_*(T)$). Since $*w_J = w_J * \in W \rtimes \mathrm{Out}(G)$ acts on Λ by -1 by definition, the claim follows.

Since τ_K^* induces the identity action on $\mathrm{H}^0(K/P_\lambda \cap K)$, $c_\lambda^{-1}g^{-1}$ acts by identity on $\mathrm{IH}^0(\Omega_{\le\lambda})$ by diagram (4.2). Since τ_K^* acts by -1 on $\mathrm{H}^2(K/P_\lambda \cap K)$ by the above claim, $c_\lambda^{-1}g^{-1}$ acts on $\mathrm{IH}^2(\Omega_{\le\lambda})$ by multiplication by -1 by diagram (4.2).

Recall that the grading on $\mathrm{IH}^\bullet(\Omega_{\le\lambda})[\langle 2\rho, \lambda\rangle] \cong V_\lambda$ comes from the action of the cocharacter $2\rho : \mathbb{G}_m \to \check{T}$ on V_λ. Let $V_\lambda(\mu)$ be the weight space of weight μ under the $\check{T}$-action, we have

$$\mathrm{IH}^i(\Omega_{\le\lambda}) = \bigoplus_{\langle 2\rho, \mu\rangle = i} V_\lambda(\mu).$$

In particular,

$$\begin{aligned} \mathrm{IH}^0(\Omega_{\le\lambda}) &= V_\lambda({}^{w_J}\lambda); \\ \mathrm{IH}^2(\Omega_{\le\lambda}) &= \bigoplus V_\lambda({}^{w_J}\lambda + \alpha_i^\vee) \end{aligned}$$

where the sum is over the simple roots $\alpha_i^\vee$ of $\check{G}$. Therefore, the previous paragraph implies

$$\begin{aligned} c_\lambda^{-1} g^{-{}^{w_J}\lambda} &= 1; \\ c_\lambda^{-1} g^{-{}^{w_J}\lambda - \alpha_i^\vee} &= -1 \text{ for all simple roots } \alpha_i^\vee. \end{aligned} \tag{4.4}$$

Comparing these two equations we conclude that $g^{\alpha_i^\vee} = -1$ for all simple roots $\alpha_i^\vee$ of $\check{G}$. On the other hand, $(-1)^\rho$ also has this property. Since $\check{G}$ is adjoint, an element in $\check{T}$ is determined by its image under simple roots, therefore $g = (-1)^\rho$. This proves (2). Plugging this back into (4.4), we conclude that $c_\lambda = ((-1)^\rho)^{-w_J\lambda} = (-1)^{\langle\rho,-w_J\lambda\rangle} = (-1)^{\langle\rho,\lambda\rangle}$. This proves (1). Now (3) follows easily from (1) and (2). □

4.4 Completion of the proof

By (3.5), it suffices to show that $\mathcal{H}^{2j}_\mu\Psi_\lambda$ acts on $\mathcal{H}^{2j}_\mu\mathbf{C}_\lambda$ by $(-1)^j$.

We extend the partially ordered set $\{\mu \in \Lambda^+, \mu \leq \lambda\}$ into a totally ordered one, and denote the total ordering still by $\leq$. For any $\mu \leq \lambda$, let $\Omega_{[\mu,\lambda]} = \Omega_{\leq\lambda} - \Omega_{<\mu}$. Similarly $\Omega_{(\mu,\lambda]} = \Omega_{\leq\lambda} - \Omega_{\leq\mu}$. Then we have a long exact sequence

$$\cdots \to \mathrm{H}^i_c(\Omega_{(\mu,\lambda]}, \mathbf{C}_\lambda) \to \mathrm{H}^i_c(\Omega_{[\mu,\lambda]}, \mathbf{C}_\lambda) \to \mathrm{H}^i_c(\Omega_\mu, \mathbf{C}_\lambda|_{\Omega_\mu}) \to \cdots$$

Since $\mathbf{C}_\lambda|_{\Omega_\mu}$ is a complex of constant sheaves on Ω_μ with stalks $\mathcal{H}^*_\mu\mathbf{C}_\lambda$, the third term in the above long exact sequence is isomorphic to $\oplus_{a+b=i}\mathrm{H}^a_c(\Omega_\mu)\otimes\mathcal{H}^b_\mu\mathbf{C}_\lambda$. Since $\Omega_\mu = \mathrm{Gr}_\mu$ is an affine space bundle over a partial flag variety G/P_μ, we have that $\mathrm{H}^\bullet_c(\Omega_\mu) \cong \mathrm{H}^\bullet(G/P_\mu)[-\langle 2\rho,\mu\rangle + \dim G/P_\mu]$ which is concentrated in even degrees. We also know that $\mathcal{H}^b_\mu\mathbf{C}_\lambda$ vanishes for odd b. Therefore the third term in the above exact sequence vanishes for odd i. Using decreasing induction for μ (starting with λ), we conclude that each $\mathrm{H}^\bullet_c(\Omega_{[\mu,\lambda]}, \mathbf{C}_\lambda)$ is concentrated in even degrees, and the above long exact sequence becomes a short one for even i. This gives a canonical decreasing filtration

$$F^{\geq\mu}\mathrm{IH}^\bullet(\Omega_{\leq\lambda}) := \mathrm{H}^\bullet_c(\Omega_{[\mu,\lambda]}, \mathbf{C}_\lambda)$$

with associated graded pieces

$$\mathrm{gr}^\mu_F\mathrm{IH}^\bullet(\Omega_{\leq\lambda}) = \mathrm{H}^\bullet_c(\Omega_\mu)\otimes\mathcal{H}^\bullet_\mu\mathbf{C}_\lambda. \tag{4.5}$$

The action of τ^*_K preserves each $F^{\geq\mu}$, and the induced action on the associated graded pieces takes the form

$$\mathrm{gr}^\mu_F\tau^*_K = (\tau_K|_{\Omega_\mu})^*\otimes\mathcal{H}^\bullet_\mu\Psi_\lambda : \mathrm{H}^\bullet_c(\Omega_\mu)\otimes\mathcal{H}^\bullet_\mu\mathbf{C}_\lambda \to \mathrm{H}^\bullet_c(\Omega_\mu)\otimes\mathcal{H}^\bullet_\mu\mathbf{C}_\lambda. \tag{4.6}$$

By Lemma 4.3(3), the action of τ^*_K on the top-dimensional cohomology $\mathrm{H}^{2\langle 2\rho,\mu\rangle}_c(\Omega_\mu) \cong \mathrm{IH}^{2\langle 2\rho,\mu\rangle}(\Omega_{\leq\mu})$ is via multiplication by $(-1)^{\langle 2\rho,\mu\rangle} = 1$; the action of τ^*_K on $\mathrm{H}^{2\langle 2\rho,\mu\rangle}_c(\Omega_\mu)\otimes\mathcal{H}^{2j}_\mu\mathbf{C}_\lambda \subset \mathrm{gr}^\mu_F\mathrm{IH}^{2j+2\langle 2\rho,\lambda\rangle}(\Omega_{\leq\lambda})$, as a subquotient of $\mathrm{IH}^{2j+2\langle 2\rho,\lambda\rangle}(\Omega_{\leq\lambda})$, is via multiplication by $(-1)^{j+\langle 2\rho,\lambda\rangle} = (-1)^j$. Therefore, by (4.6), $\mathcal{H}^{2j}_\mu\Psi_\lambda$ acts on $\mathcal{H}^{2j}_\mu\mathbf{C}_\lambda$ via multiplication by $(-1)^j$. This finishes the proof of Theorem 1.3.

5. Geometric proof of theorem 1.5

The proof of Theorem 1.5 will become transparent once we give a cohomological interpretation of the Z^σ-polynomials.

5.1 Affine flag variety via a compact form

We already see that $\Omega = \Omega K \xrightarrow{\sim} \mathrm{Gr}(\mathbb{C})$ is a homeomorphism. We need an analogous statement for the affine flag variety. Let $T_c = K \cap T$ be a maximal torus in K. Then the inclusion $K \subset G(\mathbb{C})$ induces a homeomorphism $K/T_c \xrightarrow{\sim} (G/B)(\mathbb{C})$. The multiplication $(g, kT_c) \mapsto gk\mathbf{I}$ gives a continuous map $\iota_{\mathrm{Fl}} : \Omega \times K/T_c \to \mathrm{Fl}(\mathbb{C})$ making the following diagram commutative

$$\begin{tikzcd} \Omega \times K/T_c \arrow[r, "\iota_{\mathrm{Fl}}"] \arrow[d, "\mathrm{pr}_\Omega"] & \mathrm{Fl}(\mathbb{C}) \arrow[d, "\pi"] \\ \Omega \arrow[r, "\iota"] & \mathrm{Gr}(\mathbb{C}) \end{tikzcd} \tag{5.1}$$

It is easy to check that ι_{Fl} is bijective on points, hence a homeomorphism because it is a continuous map from a compact space to a Hausdorff one. Moreover, ι_{Fl} is T_c-equivariant, where T_c acts on $\Omega \times K/T_c$ diagonally by conjugation and left translation, and it acts on $\mathrm{Fl}(\mathbb{C})$ by left translation. Thus ι_{Fl} induces a homeomorphism

$$\widetilde{\iota}_{\mathrm{Fl}} : K \overset{T_c}{\times} \mathrm{Fl}(\mathbb{C}) \xrightarrow{\sim} K \overset{T_c}{\times} (\Omega \times K/T_c) \xrightarrow{\sim} K/T_c \times \Omega \times K/T_c$$

where the last arrow is $(k_1, g, k_2T_c) \mapsto (k_1T_c, k_1gk_1^{-1}, k_1k_2T_c)$. Let $\Xi = K/T_c \times \Omega \times K/T_c$, on which K acts diagonally via left translation on both factors K/T_c and via conjugation on Ω. Then $\widetilde{\iota}_{\mathrm{Fl}}$ is K-equivariant (with K acting on the first factor on $K \overset{T_c}{\times} \mathrm{Fl}$). We define Ξ_w (resp. $\Xi_{\le w}$) to be the image of $K \overset{T_c}{\times} \mathrm{Fl}_w$ (resp. $K \overset{T_c}{\times} \mathrm{Fl}_{\le w}$) under $\widetilde{\iota}_{\mathrm{Fl}}$. The isomorphism $\widetilde{\iota}_{\mathrm{Fl}}$ therefore induces an isomorphism

$$\mathrm{IH}^\bullet_{\mathbf{I}}(\mathrm{Fl}_{\le w}) \cong \mathrm{IH}^\bullet_{T_c}(\mathrm{Fl}_{\le w}) \xrightarrow{\sim} \mathrm{IH}^\bullet_K(\Xi_{\le w}). \tag{5.2}$$

Recall from (3.1) that we have an isomorphism $\Phi_w : \tau^*\mathbf{S}_w \xrightarrow{\sim} \mathbf{S}_w$ in the category $D_{\mathbf{I}}(\mathrm{Fl})$ for $w \in I_*$, where $\mathbf{S}_w$ is the shifted intersection cohomology sheaf of $\mathrm{Fl}_{\le w}$. This induces an involution on $\mathbf{I}$-equivariant cohomology

$$\tau^* = \mathrm{H}^\bullet_{\mathbf{I}}(\mathrm{Fl}, \Phi_w) : \mathrm{IH}^\bullet_{\mathbf{I}}(\mathrm{Fl}_{\le w}) \to \mathrm{IH}^\bullet_{\mathbf{I}}(\mathrm{Fl}_{\le w}). \tag{5.3}$$

On the other hand, Ξ also admits an involution $\widetilde{\tau} : (k_1T_c, g, k_2T_c) \mapsto (k_2^*T_c, (g^*)^{-1}, k_1^*T_c)$, which intertwines the original diagonal K-action and that action pre-composed with $*$. For $w \in I_*$, $\widetilde{\tau}$ induces an involution

$$\widetilde{\tau}^* : \mathrm{IH}^\bullet_K(\Xi_{\le w}) \to \mathrm{IH}^\bullet_K(\Xi_{\le w}). \tag{5.4}$$

The isomorphism (5.2) intertwines the involutions τ^* in (5.3) and $\widetilde{\tau}^*$ in (5.4). We prefer working with $\mathrm{IH}^\bullet_K(\Xi_{\le w})$ to working with $\mathrm{IH}^\bullet_{\mathbf{I}}(\mathrm{Fl}_{\le w})$ because the involution $\widetilde{\tau}$ can be seen on the nose for $\Xi_{\le w}$, without having to appeal to stacks such as $[\mathbf{I}\backslash\mathrm{Fl}]$.

Lemma 5.2. *Let r be the rank of G. Then*

$$\sum_{j\in\mathbb{Z}} \mathrm{tr}\left(\widetilde{\tau}^*, \mathrm{IH}^{2j}_K(\Xi_{\le w})\right) q^j = q^{\ell(w)}(1-q)^{e(*)-r}(1+q)^{-e(*)} Z^\sigma_w(q^{-1})$$

as elements in $\mathbb{Z}[[q]]$. Here $e(*)$ is the dimension of the (-1)-eigenspace of $* : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$.

Proof. Via the isomorphism $\widetilde{\iota}_{\mathrm{Fl}}$, we view $\mathbf{S}_w$ as the intersection complex of $\Xi_{\le w}$ which is the constant sheaf on Ξ_w. The stratification of Ξ by $\Xi_{\le w}$ gives a spectral sequence with the E_2-page consisting of $\mathrm{H}^\bullet_K(\Xi_y, i^!_y\mathbf{S}_w)$ abutting to $\mathrm{IH}^\bullet_K(\Xi_{\le w})$. Here $i_y : \Xi_y \hookrightarrow \Xi$ is the inclusion. Since $i^!_y\mathbf{S}_w$ is a sum of constant sheaves on Ξ_y concentrated on even degrees, and $\mathrm{H}^\bullet_K(\Xi_y) \cong \mathrm{H}^\bullet_T(\mathrm{pt})$ is also concentrated in even degrees, the spectral sequence necessarily degenerates at E_2. Therefore $\mathrm{IH}^\bullet_K(\Xi_{\le w})$ admits an increasing filtration indexed by $\{y \le w\}$ with $\mathrm{gr}_y\mathrm{IH}^\bullet_K(\Xi_{\le w}) = \mathrm{H}^\bullet_K(\Xi_y, i^!_y\mathbf{S}_w)$. The involution $\widetilde{\tau}^*$ on $\mathrm{IH}^\bullet_K(\Xi_{\le w})$ maps gr_y to $\mathrm{gr}_{(y^*)^{-1}}$, therefore its trace is the sum of traces on gr_y for $y \in I_*$, i.e.,

$$\sum_j \mathrm{tr}(\widetilde{\tau}^*, \mathrm{IH}^{2j}_K(\Xi_{\le w}))q^j = \sum_{y\le w, y\in I_*}\sum_{j\in\mathbb{Z}} \mathrm{tr}(\widetilde{\tau}^*, \mathrm{H}^{2j}_K(\Xi_y, i^!_y\mathbf{S}_w))q^j. \tag{5.5}$$

Verdier duality gives an isomorphism in $D_K(\Xi_y)$ commuting with the involutions induced by Φ_w

$$i^!_y\mathbf{S}_w \cong \bigoplus_k \mathcal{H}^{2\ell(w)-2\ell(y)-2k}\mathbf{S}_w[-2k]..$$

Hence

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \mathrm{tr}(\widetilde{\tau}^*, \mathrm{H}^{2j}_K(\Xi_y, i^!_y\mathbf{S}_w))q^j \qquad (5.6)\\
&= \sum_{k\in\mathbb{Z}} \mathrm{tr}(\mathcal{H}^{2\ell(w)-2\ell(y)-2k}_y\Phi_w, \mathcal{H}^{2\ell(w)-2\ell(y)-2k}_y\mathbf{S}_w)q^k\\
&\quad\times\sum_{j\in\mathbb{Z}} \mathrm{tr}(\widetilde{\tau}^*, \mathrm{H}^{2k}_K(\Xi_y))q^j\\
&= q^{\ell(w)-\ell(y)}P^\sigma_{y,w}(q^{-1})\sum_{j\in\mathbb{Z}} \mathrm{tr}(\widetilde{\tau}^*, \mathrm{H}^{2j}_K(\Xi_y))q^j.
\end{aligned}$$

We would like to calculate the $\widetilde{\tau}^*$-action on $\mathrm{H}^\bullet_K(\Xi_y)$. Write $y = \lambda\overline{y}$ for $\lambda \in \Lambda$ and $\overline{y} \in \overline{W}$. Note that $\overline{y}^* = \overline{y}^{-1}$ since $y \in I_*$. Under the isomorphism $\widetilde{\iota}_{\mathrm{Fl}} : K \overset{T_c}{\times} \mathrm{Fl}_y \xrightarrow{\sim} \Xi_y$, the base point $\dot{y}\mathbf{I} \in \mathrm{Fl}_y$ corresponds to $(T_c, t^\lambda, \overline{y}T_c) \in \Xi_y$. Consider the K-equivariant map

$$\begin{aligned} \gamma_y : K/T_c &\xrightarrow{\sim} K \cdot \dot{y} \subset \Xi_y \\ kT_c &\mapsto (kT_c, kt^\lambda k^{-1}, k\overline{y}T_c). \end{aligned}$$

This map in fact identifies K/T_c with the K-orbit $K \cdot \dot{y} \subset \Xi_y$. Since Fl_y is an affine space, the inclusion of the base point $\{\dot{y}\} \hookrightarrow \mathrm{Fl}_y$ induces an isomorphism $\mathrm{H}^\bullet_{T_c}(\mathrm{Fl}_y) \xrightarrow{\sim} \mathrm{H}^\bullet_{T_c}(\{\dot{y}\})$, therefore we get an isomorphism

$$\mathrm{H}^\bullet_K(\Xi_y) \cong \mathrm{H}^\bullet_{T_c}(\mathrm{Fl}_y) \xrightarrow{\sim} \mathrm{H}^\bullet_{T_c}(\{\dot{y}\}) \cong \mathrm{H}^\bullet_K(K \cdot \dot{y}) \xrightarrow{\gamma_y^*} \mathrm{H}^\bullet_K(K/T_c). \tag{5.7}$$

Direct calculation shows that $\widetilde{\tau}(\gamma_y(kT_c)) = \gamma_y(k^*\overline{y}^*T_c)$. Therefore the $\widetilde{\tau}^*$-action on $\mathrm{H}^\bullet_K(\Xi_y)$ can be identified with the involution on $\mathrm{H}^\bullet_K(K/T_c)$ induced from the automorphism $\epsilon : kT_c \mapsto k^*\overline{y}^*T_c$ of K/T_c. We may write ϵ as the composition of $* : K/T_c \to K/T_c$ with the *right* action of $\overline{y}^*$ on K/T_c. Therefore the involution ϵ^*on $\mathrm{H}^\bullet_K(K/T_c) \cong \mathrm{H}^\bullet_{T_c}(\mathrm{pt})$ is induced from the involution $t \mapsto \mathrm{Ad}(\overline{y}^*)^{-1}(t^*) = \mathrm{Ad}(\overline{y})(t^*)$ of T_c. This involution gives a decomposition $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_-$ of the Lie algebra $\mathfrak{t} \cong \Lambda_{\mathbb{C}}$ of T into $(+1)$ and (-1)-eigenspaces, with dimensions $r - e(\overline{y}*)$ and $e(\overline{y}*)$ respectively (see remarks following (2.1) for notations). Using (5.7) and the eigenspace decomposition $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_-$, we have

$$\begin{aligned} \mathrm{H}^\bullet_K(\Xi_y) \xrightarrow{\gamma_y^*} \mathrm{H}^\bullet_K(K/T_c) &\cong \mathrm{H}^\bullet_{T_c}(\mathrm{pt}) \cong \mathrm{Sym}(\mathfrak{t}^\vee[-2]) \\ &\cong \mathrm{Sym}(\mathfrak{t}_+^\vee[-2]) \otimes \mathrm{Sym}(\mathfrak{t}_-^\vee[-2]). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \mathrm{tr}(\widetilde{\tau}^*, \mathrm{H}^{2j}_K(\Xi_y))q^j &= \sum_{j\geq 0} \dim \mathrm{Sym}^j(\mathfrak{t}_+^\vee)q^j \sum_{k\geq 0} \dim \mathrm{Sym}^k(\mathfrak{t}_-^\vee)(-q)^k \\ &= (1-q)^{e(\overline{y}*)-r}(1+q)^{-e(\overline{y}*)}. \end{aligned}$$

Plugging this into (5.6) and then into (5.5), we get

$$\begin{aligned} &\sum_j \mathrm{tr}(\widetilde{\tau}^*, \mathrm{IH}^{2j}_K(\Xi_{\leq w}))q^j \\ &\quad = \sum_{y\leq w, y\in I_*} q^{\ell(w)-\ell(y)} P^\sigma_{y,w}(q^{-1})(1-q)^{e(\overline{y}*)-r}(1+q)^{-e(\overline{y}*)} \end{aligned}$$

$$= q^{\ell(w)}(1-q)^{-r} \sum_{y \le w, y \in I_*} P^\sigma_{y,w}(q^{-1}) q^{-\ell(y)} \left(\frac{q^{-1}-1}{q^{-1}+1}\right)^{e(\overline{y}*)}$$

$$= q^{\ell(w)}(1-q)^{-r} \left(\frac{q^{-1}-1}{q^{-1}+1}\right)^{e(*)} Z^\sigma_w(q^{-1})$$

$$= q^{\ell(w)}(1-q)^{e(*)-r}(1+q)^{-e(*)} Z^\sigma_w(q^{-1}). \qquad \square$$

In the situation of the affine Grassmannian, the isomorphism (3.4) induces an involution on the global sections $\tau_K^* : \mathrm{IH}^\bullet(\Omega_{\le\lambda}) \xrightarrow{\sim} \mathrm{IH}^\bullet(\Omega_{\le\lambda})$.

Lemma 5.3. *For $\lambda \in \Lambda^+$, we have*

$$\sum_{i \in \mathbb{Z}} \mathrm{tr}(\tau_K^*, \mathrm{IH}^{2i}(\Omega_{\le\lambda})) q^i = \widetilde{Z}^\sigma_{d_\lambda}(q). \tag{5.8}$$

Proof. Note that $\Xi_{\le d_\lambda} = K/T_c \times \Omega_{\le\lambda} \times K/T_c$. The map (5.2) induces an isomorphism on intersection cohomology commuting with the relevant involutions:

$$\mathrm{IH}^\bullet_{\mathbf{I}}(\mathrm{Fl}_{\le d_\lambda}) \cong \mathrm{IH}^\bullet_K(\Xi_{\le d_\lambda}) \cong \mathrm{IH}^\bullet_K(\Omega_{\le\lambda}) \otimes_{\mathrm{H}^\bullet_K(\mathrm{pt})} \mathrm{H}^\bullet_K(K/T_c \times K/T_c). \tag{5.9}$$

where the last equality comes from the degeneration of the Leray spectral sequence (for the projection $\Xi_{\le d_\lambda} \to \Omega_{\le\lambda}$) at E_2 since all the relevant cohomology groups are concentrated in even degrees. Note that in (5.9), the involution on $\mathrm{H}^\bullet_K(K/T_c \times K/T_c)$ is induced by $(k_1T_c, k_2T_c) \mapsto (k_2^*T_c, k_1^*T_c)$, and the involution on $\mathrm{H}^\bullet_K(\mathrm{pt})$ is induced by the involution $*$ of K.

Another spectral sequence argument shows that we have an isomorphism

$$\mathrm{IH}^\bullet_K(\Omega_{\le\lambda}) \cong \mathrm{H}^\bullet_K(\mathrm{pt}) \otimes \mathrm{IH}^\bullet(\Omega_{\le\lambda})$$

commuting with the obvious involutions (the one on $\mathrm{H}^\bullet_K(\mathrm{pt})$ is again induced by $*$, and the ones involving $\Omega_{\le\lambda}$ are given by τ_K^*). Combining this with (5.9) we get an isomorphism

$$\mathrm{IH}^\bullet_{\mathbf{I}}(\mathrm{Fl}_{\le d_\lambda}) \xrightarrow{\sim} \mathrm{IH}^\bullet(\Omega_{\le\lambda}) \otimes \mathrm{H}^\bullet_K(K/T_c \times K/T_c)$$

intertwining the involutions on both sides which we specified before. The special case $\lambda = 0$, $d_\lambda = w_J$ gives $\mathrm{IH}^\bullet_{\mathbf{I}}(\mathrm{Fl}_{\le w_J}) \cong \mathrm{H}^\bullet_K(K/T_c \times K/T_c)$. Therefore

$$\mathrm{IH}^\bullet_{\mathbf{I}}(\mathrm{Fl}_{\le d_\lambda}) \xrightarrow{\sim} \mathrm{IH}^\bullet(\Omega_{\le\lambda}) \otimes \mathrm{IH}^\bullet_{\mathbf{I}}(\mathrm{Fl}_{\le w_J})$$

commuting with the relevant involutions. Taking the Poincaré polynomials with respect to the traces of these involutions, and using Lemma 5.2, we get

$$q^{\ell(d_\lambda)}(1-q)^{e(*)-r}(1+q)^{-e(*)} Z^\sigma_{d_\lambda}(q^{-1})$$
$$= q^{\ell(w_J)}(1-q)^{e(*)-r}(1+q)^{-e(*)} Z^\sigma_{w_J}(q^{-1}) \sum_{j \in \mathbb{Z}} \mathrm{tr}(\tau_K^*, \mathrm{IH}^{2j}(\Omega_{\le\lambda})) q^j.$$

In view of the definition of $\widetilde{Z}^{\sigma}_{d_\lambda}(q)$ in (1.3), we get

$$\sum_{j\in\mathbb{Z}} \mathrm{tr}(\tau_K^*, \mathrm{IH}^{2j}(\Omega_{\le\lambda}))q^j = q^{\ell(d_\lambda)-\ell(w_J)}\widetilde{Z}^{\sigma}_{d_\lambda}(q^{-1}).$$

Let $Q_\lambda(q)$ denote the left side. Substituting q^{-1} for q in the above, we get

$$Q_\lambda(q^{-1}) = q^{-\ell(d_\lambda)+\ell(w_J)}\widetilde{Z}^{\sigma}_{d_\lambda}(q).$$

Poincaré duality for $\mathrm{IH}^\bullet(\Omega_{\le\lambda})$ (which has dimension $\langle 2\rho, \lambda\rangle$) implies $Q_\lambda(q) = q^{\langle 2\rho,\lambda\rangle}Q_\lambda(q^{-1})$. Therefore

$$Q_\lambda(q) = q^{\langle 2\rho,\lambda\rangle}Q_\lambda(q^{-1}) = q^{\langle 2\rho,\lambda\rangle+\ell(w_J)-\ell(d_\lambda)}\widetilde{Z}^{\sigma}_{d_\lambda}(q).$$

Since $\ell(d_\lambda) = \ell(w_J) + \langle 2\rho, \lambda\rangle$ (i.e, $\dim \mathrm{Fl}_{\le d_\lambda} = \dim G/B + \dim \mathrm{Gr}_{\le\lambda}$), the above equality implies (5.8). □

5.4 Completion of the proof

By Lemma 4.3(3), the involution τ_K^* acts on $\mathrm{IH}^{2j}(\Omega_{\le\lambda})$ via $(-1)^j$. Therefore, by Lemma 5.3, we have

$$\widetilde{Z}^{\sigma}_{d_\lambda}(q) = \sum_{j\in\mathbb{Z}}(-1)^j \dim \mathrm{IH}^{2j}(\Omega_{\le\lambda})q^j = \sum_{j\in\mathbb{Z}} \dim \mathrm{IH}^{2j}(\Omega_{\le\lambda})(-q)^j. \quad (5.10)$$

which is clearly in $\mathbb{Z}[q]$.

On the other hand, stratifying $\mathrm{Fl}_{\le w}$ into affine space Fl_y $(y \le w)$ and using the parity vanishing of the stalks $\mathcal{H}^\bullet_y\mathbf{S}_w$, we have $\sum_{j\in\mathbb{Z}} \dim \mathrm{IH}^{2j}(\mathrm{Fl}_{\le w})q^j = \sum_{y\le w} P_{y,w}(q)q^{\ell(y)}$. The argument is the same as the first part of Section 4.4. Comparing with the definition of $Z_w(q)$ in (1.2), we have

$$Z_w(q) = \sum_{j\in\mathbb{Z}} \dim \mathrm{IH}^{2j}(\mathrm{Fl}_{\le w})q^j.$$

Using the homeomorphism ι_{Fl} and the diagram (5.1), we have $\mathrm{IH}^\bullet(\mathrm{Fl}_{\le d_\lambda}) \cong \mathrm{IH}^\bullet(\Omega_{\le\lambda}) \otimes \mathrm{H}^\bullet(K/T_c) \cong \mathrm{IH}^\bullet(\Omega_\lambda) \otimes \mathrm{IH}^\bullet(\mathrm{Fl}_{\le w_J})$. Therefore $Z_{d_\lambda}(q)$ is the product of $Z_{w_J}(q)$ with the Poincaré polynomial of $\mathrm{IH}^\bullet(\Omega_{\le\lambda})$. By the definition of $\widetilde{Z}_{d_\lambda}(q)$ in (1.3), we have

$$\widetilde{Z}_{d_\lambda}(q) = Z_{d_\lambda}(q)Z_{w_J}(q)^{-1} = \sum_{j\in\mathbb{Z}} \dim \mathrm{IH}^{2j}(\Omega_{\le\lambda})q^j. \quad (5.11)$$

The theorem now follows by comparing (5.10) and (5.11).

6. Algebraic proof of theorem 1.5

Now we start the algebraic proof of Theorem 1.5. Using [L11, 3.6(f)] and Theorem 1.3 we see that

$$\begin{aligned}\widetilde{Z}^{\sigma}_{d_\lambda}(q) &= \sum_{\mu\in\Lambda^+;d_\mu\le d_\lambda} P^{\sigma}_{d_\mu,d_\lambda}(q)\zeta\left(\sum_{y\in W_J\mu W_J;y\in I_*} a_y\right) Z^{\sigma}_{w_J}(q)^{-1}\\ &= \sum_{\mu\in\Lambda^+;d_\mu\le d_\lambda} P_{d_\mu,d_\lambda}(-q)\zeta\left(\sum_{y\in W_J\mu W_J;y\in I_*} a_y\right) Z^{\sigma}_{w_J}(q)^{-1}.\end{aligned}$$

On the other hand

$$\widetilde{Z}_{d_\lambda}(-q) = \sum_{\mu\in\Lambda^+;d_\mu\le d_\lambda} P_{d_\mu,d_\lambda}(-q)\sum_{y\in W_J\mu W_J}(-q)^{\ell(y)}Z_{w_J}(-q)^{-1}.$$

Hence to prove Theorem 1.5 it is enough to show that for any double coset $W_J\mu W_J$ we have

$$\zeta\left(\sum_{y\in W_J\mu W_J\cap I_*} a_y\right) Z^{\sigma}_{w_J}(q)^{-1} = \sum_{y\in W_J\mu W_J}(-q)^{\ell(y)}Z_{w_J}(-q)^{-1}. \qquad (6.1)$$

We fix such a double coset $W_J\mu W_J$ for the rest of this section, where $\mu\in\Lambda^+\cap W_J\mu W_J$ is the unique dominant translation. Let $d=d_\mu$ (resp. b) be the element of maximal (resp. minimal) length in $W_J\mu W_J$.

We shall be interested also in some parabolic analogues of $Z_w(q)$, $Z^{\sigma}_w(q)$. For any $H\subsetneq S$ let W_H be the subgroup of W generated by H so that (W_H,H) is a finite Coxeter group; let w_H be the longest element of W_H. We also set $\mathbf{P}_H=\sum_{x\in W_H}q^{\ell(x)}\in\mathbb{N}[q]$ so that $Z_{w_H}(q)=\mathbf{P}_H(q)$. Recall that $J=S-\{s_0\}$, and our previous notation W_J, w_J is consistent with the new notation.

If in addition we are given an involution $\tau: W_H\to W_H$ leaving H stable, we set (as in [L11, 5.1]) $\mathbf{P}_{H,\tau}=\sum_{x\in W_H;\tau(x)=x}q^{\ell(x)}\in\mathbb{N}[q]$. By [L11, 5.9] we have $Z^{\sigma}_{w_J}(q)=\mathbf{P}_J(q^2)\mathbf{P}_{J,*}(q)^{-1}$ (we use also that $P^{\sigma}_{y,w_J}(q)=1$ for any $y\in W_J$, see [L11, 3.6(f)]).

Let $H=J\cap bJb^{-1}$. Let $\epsilon: W_{H^*}\to W_{H^*}$ be the involution $y\mapsto b^{-1}y^*b$ (H^* is the image of H under $*$). From [L11, 5.10] we have

$$\zeta\left(\sum_{y\in W_J\mu W_J\cap I_*} a_y\right) = \zeta(a_b)\mathbf{P}_J(q^2)\mathbf{P}_{H^*,\epsilon}(q)^{-1}.$$

Similarly,

$$\sum_{y\in W_J\mu W_J} q^{\ell(y)} = q^{\ell(b)}\mathbf{P}_J(q^2)\mathbf{P}_{H^*}(q)^{-1}.$$

We see that (6.1) is equivalent to the following statement:

$$\zeta(a_b)\mathbf{P}_{J,*}(q)\mathbf{P}_{H^*,\epsilon}(q)^{-1} = (-q)^{\ell(b)}\mathbf{P}_J(-q)\mathbf{P}_{H^*}(-q)^{-1}. \tag{6.2}$$

Lemma 6.1. *The involution ϵ on W_{H^*} is the same as $\mathrm{Ad}(w_{H^*})$; i.e., $b^{-1}y^*b = w_{H^*}yw_{H^*}$ for all $y \in W_{H^*}$.*

Proof. We shall denote the inverse of μ by μ^{-1} instead of $-\mu$ as before. We have $\mu w_J = d = w_J w_H b w_J$. Hence $\mu = w_J w_H b$. Now $W_J \times W_J$ acts transitively on $W_J \mu W_J$ by left and right multiplication and the isotropy group of μ is isomorphic to $W_\mu := \{w \in W_J; {}^w\mu = \mu\}$. Hence $|W_J\mu W_J| = |W_J|^2/|W_\mu|$. By [L11, 1.1] we have also $|W_J\mu W_J| = |W_J|^2/|W_H|$ hence $|W_\mu| = |W_H|$. We show that $W_\mu \subset W_{H^*}$. We have $W_H = W_J \cap bW_Jb^{-1}$; applying $*$ we deduce $W_{H^*} = W_J \cap b^{-1}W_Jb$. Hence it is enough to show that $W_\mu \subset b^{-1}W_Jb$. Since $\mu = w_Jw_Hb$ we have $\mu^{-1}W_J\mu = b^{-1}w_Hw_JW_Jw_Jw_Hb = b^{-1}W_Jb$. If $w \in W_\mu$ then $w\mu = \mu w$ hence $\mu w\mu^{-1} = w \in W_J$; thus $W_\mu \subset \mu^{-1}W_J\mu = b^{-1}W_Jb$. We have shown that $W_\mu \subset W_{H^*}$. Since the last two groups have the same order we see that $W_\mu = W_{H^*}$. Hence to prove (a) it is enough to show that for any $y \in W_\mu$ we have $b^{-1}y^*b = w_{H^*}yw_{H^*}$ that is (after applying $*$) $byb^{-1} = w_Hy^*w_H$. Since $b = w_Hw_J\mu$, it is enough to show that for $y \in W_\mu$ we have $w_J\mu y\mu^{-1}w_J = y^*$, or, using $\mu y = y\mu$, that $w_Jyw_J = y^*$. This follows from the definition of $*$ in Section 1.2. This proves the lemma. □

Lemma 6.2. *If $L \subsetneqq S$ and $\mathrm{Ad}(w_L)$ is the conjugation by w_L on W_L, then*

$$\mathbf{P}_{L,\mathrm{Ad}(w_L)}(q) = \mathbf{P}_L(-q)\left(\frac{1+q}{1-q}\right)^{n_L},$$

where n_L is the number of odd exponents of W_L.

Proof. Let $e_i (i \in X)$ be the exponents of W_L. We have $X = X' \sqcup X''$ where $X' = \{i \in X; e_i \text{ is odd}\}$, $X'' = \{i \in X; e_i \text{ is even}\}$. It is well known that

$$\mathbf{P}_L(q) = \prod_{i\in X}\frac{q^{e_i+1}-1}{q-1}.$$

It follows that

$$\mathbf{P}_L(-q) = \prod_{i\in X'}\frac{q^{e_i+1}-1}{-q-1}\prod_{i\in X''}\frac{q^{e_i+1}+1}{q+1} \tag{6.3}$$

We have

$$\mathbf{P}_{L,\mathrm{Ad}(w_L)}(q) = \prod_{i\in X'}\frac{q^{e_i+1}-1}{q-1}\prod_{i\in X''}\frac{q^{e_i+1}+1}{q+1}. \tag{6.4}$$

Here, the left hand side evaluated at a prime power q calculates $\#\mathcal{B}_{G'}(\mathbb{F}_q)$, where G' is a semisimple algebraic group defined over $\mathbb{F}_q$ with absolute Weyl group W_L, and $\mathcal{B}_{G'}$ is the variety of Borel subgroups of G'. More precisely, take any semisimple split group G'' over $\mathbb{F}_{q^2}$ with Weyl group W_L, and let G' be a form of G'' over $\mathbb{F}_q$ obtained by requiring that $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ acts on G'' via the opposition involution. This number $\#\mathcal{B}_{G'}(\mathbb{F}_q)$ can be computed from the known formula for the number of rational points of such a G' given in [S67, §11]. Now the lemma follows from (6.3) and (6.4). □

6.3. *Using Lemma 6.1 and 6.2 (applied to $L = H^*$) and the definition of ζ we see that the desired equality* (6.2) *is equivalent to*

$$q^{\ell(b)}\left(\frac{q-1}{q+1}\right)^{\phi(b)}\left(\frac{1+q}{1-q}\right)^{n_J-n_{H^*}} = (-q)^{\ell(b)},$$

that is, to the equality

$$\phi(b) = n_J - n_{H^*}. \tag{6.5}$$

Here we use that $\phi(w) \equiv \ell(w) \mod 2$ for any $w \in I_*$, see [L11, 4.5].

Now let R' be the reflection representation of W_{H^*}. For any linear map $A : R' \to R'$ we denote by $e'(A)$ the dimension of the (-1)*-eigenspace of A. For $z \in W_{H^*}$ satisfying $\epsilon(z) = z^{-1}$, define*

$$\phi'(z) = e'(z\epsilon) - e'(\epsilon). \tag{6.6}$$

Lemma 6.4. *For any $z \in W_{H^*}$ such that $\epsilon(z) = z^{-1}$, we have $\phi(bz) = \phi'(z) + \phi(b)$.*

Proof. We argue by induction on $\ell(z)$. If $z = 1$ the result is clear. Now assume that $z \neq 1$. We can find $s \in H^*$ such that $\ell(sz) < \ell(z)$. Assume first that $sz \neq z\epsilon(s)$. Then $\ell(sz\epsilon(s)) = \ell(z) - 2$ hence by the induction hypothesis we have $\phi(bsz\epsilon(s)) = \phi'(sz\epsilon(s)) + \phi(b)$. By definition, $\phi'(sz\epsilon(s)) = \phi'(z)$. We have $bsz\epsilon(s) = bsb^{-1}bz\epsilon(s) = \epsilon(s)^*bz\epsilon(s)$ and hence, by definition, $\phi(bsz\epsilon(s)) = \phi(\epsilon(s)^*bz\epsilon(s)) = \phi(bz)$. Thus $\phi(bz) = \phi'(z) + \phi(b)$. Next we assume that $sz = z\epsilon(s)$. Then $\ell(sz\epsilon(s)) = \ell(z) - 1$ hence by the induction hypothesis we have $\phi(bsz\epsilon(s)) = \phi'(sz\epsilon(s)) + \phi(b)$. By definition, $\phi'(sz\epsilon(s)) = \phi'(z) - 1$ and $\phi(bsz\epsilon(s)) = \phi(\epsilon(s)^*bz\epsilon(s)) = \phi(bz) - 1$. Thus $\phi(bz) = \phi'(z) + \phi(b)$. This completes the proof of the lemma. □

6.5 Completion of the proof

From Lemma 6.4 we deduce

$$\phi(bw_{H^*}) = \phi'(w_{H^*}) + \phi(b). \tag{6.7}$$

We have $d = cw_{H^*}c^{*-1}$ where $c = w_J w_H$ (see [L11, §1.2]). From the definition of ϕ we see that $\phi(d) = \phi(bw_{H^*})$ hence, using (6.7), we have

$$\phi(d) = \phi'(w_{H^*}) + \phi(b).$$

Hence (6.5) is equivalent to

$$\phi(d) - \phi'(w_{H^*}) = n_J - n_{H^*}. \tag{6.8}$$

For any linear map $A : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ (where $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$), recall $e(A)$ is the dimension of the (-1)-eigenspace of A. We claim that

$$\phi(d) = e(w_J). \tag{6.9}$$

In fact, if $w \in I_*$ with image $\overline{w} \in \overline{W}$, we have $\phi(w) = e(\overline{w}*) - e(*)$. Since the action of $*$ is given by $x \mapsto -w_J(x)$), we have $\phi(w) = e(-\overline{w}w_J) - e(-w_J)$. If $w = d$ then $d = tw_J$ (t is the dominant translation) hence $\overline{w} = w_J \in \overline{W} \cong W_J$ and $\phi(d) = e(-\mathrm{id}) - e(-w_J)$, which is equal to $e(w_J)$. This proves (6.9).

We also claim that

$$\phi'(w_{H^*}) = e'(w_{H^*}). \tag{6.10}$$

In fact, from the definition (6.6) we have $\phi'(w_{H^*}) = e'(w_{H^*}\epsilon) - e'(\epsilon)$. Note that both w_{H^*} and ϵ act naturally on R'; the action of ϵ is given by $x \mapsto -w_{H^*}x$ by Lemma 6.1. Thus we have $\phi'(w_{H^*}) = e'(-\mathrm{id}) - e'(-w_{H^*}) = e'(w_{H^*})$. This proves (6.10).

Using (6.9) and (6.10) we see that the desired equality (6.8) is equivalent to

$$e(w_J) - e'(w_{H^*}) = n_J - n_{H^*}. \tag{6.11}$$

Now for any finite Weyl group, the dimension of the (-1)-eigenspace of the longest element acting on the reflection representation is equal to the number of odd exponents of that Weyl group, as one easily verifies. It follows that $e(w_J) = n_J$, $e'(w_{H^*}) = n_{H^*}$. Thus (6.11) is proved. This completes the proof of Theorem 1.5.

6.6 Signature of a hermitian form

Let $\check{G}$ be the Langlands dual of G as before, with dual Cartan and Borel $\check{T} \subset \check{B}$. We identify the Weyl group of $\check{G}$ with W_J. Let $\lambda \in \Lambda^+$, viewed as a dominant weight of $\check{G}$, and let V_λ be the corresponding irreducible representation of $\check{G}$ with highest weight λ. In [L97] a hermitian form h_λ on V_λ is constructed in terms of a semisimple element $s \in \check{T}$ with $s^2 = 1$. Here we shall take $s = (-1)^\rho$. The hermitian form h_λ is invariant under a real form of $\check{G}$ which can be shown to be quasi-split (for our choice of s) and admits a compact Cartan subgroup. Moreover, by [L97, 2.9], the signature of h_λ is given by

$$\mathrm{Signature}(h_\lambda) = (-1)^{\langle\rho,\lambda\rangle}\mathrm{tr}((-1)^\rho, V_\lambda). \tag{6.12}$$

Recall the following results from [L83]. First, it is shown in [L83, 6.1] that the multiplicity of the weight μ in V_λ is equal to $P_{d_\mu,d_\lambda}(1)$. Second, we have the formula (see [L83, (8.10)] and its proof)

$$\widetilde{Z}_{d_\lambda}(q) = q^{\langle\rho,\lambda\rangle} \sum_{\mu\in\Lambda^+; d_\mu\le d_\lambda} P_{d_\mu,d_\lambda}(1) \sum_{\mu\in W_J\mu} q^{\langle\rho,\mu\rangle}. \tag{6.13}$$

Setting $q = 1$ we obtain that $\widetilde{Z}_{d_\lambda}(1) = \dim V_\lambda$. Setting $q = -1$ in (6.13), we obtain

$$\widetilde{Z}_{d_\lambda}(-1) = (-1)^{\langle\rho,\lambda\rangle}\mathrm{tr}((-1)^\rho, V_\lambda). \tag{6.14}$$

We may also obtain (6.14) from Lemma 4.3(3) and Lemma 5.3. Combining (6.14) with Theorem 1.5 and (6.12), we obtain

$$\mathrm{Signature}(h_\lambda) = \widetilde{Z}^\sigma_{d_\lambda}(1). \tag{6.15}$$

Thus, while $\widetilde{Z}_{d_\lambda}(q)$ is a q-analogue of the dimension of V_λ, $\widetilde{Z}^\sigma_{d_\lambda}(q) = \widetilde{Z}_{d_\lambda}(-q)$ is a q-analogue of the signature of the hermitian form h_λ on V_λ.

Remark 6.7. We expect that the hermitian form h_λ on V_λ is the complexification of the sum of the polarized Hodge structures $\mathrm{IH}^{2p}(\mathrm{Gr}_{\le\lambda})$ (which only has (p, p)-classes). By the Riemann-Hodge bilinear relation, this pairing is positive (resp. negative) definite on $\mathrm{IH}^{2p}(\mathrm{Gr}_{\le\lambda})$ when p is even (resp. odd). Therefore the signature on the total intersection cohomology $\mathrm{IH}^\bullet(\mathrm{Gr}_{\le\lambda})$ (which is also the signature of the Poincaré duality pairing) is also calculated by $\widetilde{Z}_{d_\lambda}(-1) = \widetilde{Z}^\sigma_{d_\lambda}(1)$.

7. Generalization

7.1 More involutions in affine Weyl groups

In Section 1.2, we fixed a hyperspecial vertex $s_0 \in S$ in the Dynkin diagram of (W, S). Let $A = \mathrm{Aut}(W, S)$. Then A has a subgroup

$$A_\Lambda := \{a \in \mathrm{Aut}(W, S) |\ \text{there exists } w \in W_J$$
$$\text{such that } a(\lambda) = {}^w\lambda \text{ for all } \lambda \in \Lambda\}.$$

One may identify A_Λ with the affine automorphisms fixing the standard alcove corresponding to S. It is easy to see that A_Λ is normal in A. Let $\overline{A} := A/A_\Lambda$. The stabilizer of s_0 under A is $A_J = \mathrm{Aut}(W_J, J)$, which projects isomorphically to $\bar{A}$.

We recall the extended affine Weyl group is the semi-direct product $\widetilde{W} = W \rtimes A_\Lambda$, and it fits into an exact sequence

$$1 \to \widetilde{\Lambda} \to \widetilde{W} \to \overline{W} \to 1$$

where $\widetilde{\Lambda}$ is a lattice containing Λ such that the projection $\widetilde{\Lambda} \hookrightarrow \widetilde{W} \to A_\Lambda$ induces an isomorphism $\widetilde{\Lambda}/\Lambda \cong A_\Lambda$.

Lemma 7.2. *Recall we have an involution $* \in A_J$ defined in* (1.1).

(1) *Every element in the coset $A_\Lambda * = * A_\Lambda \subset A$ is an involution.*
(2) *For any hyperspecial vertex $s_1 \in S$, there is a unique $a \in A_\Lambda *$ which sends s_0 to s_1.*

Proof.

(1) The group A_J acts on A_Λ by conjugation. This action can be seen explicitly as follows: $W_J \rtimes A_J$ acts on $\widetilde{\Lambda}$ by the reflection action stabilizing Λ. The action of A_J on the quotient $A_\Lambda = \widetilde{\Lambda}/\Lambda$ is then induced from this reflection action. In particular, the action of $* \in A_J$ on $\widetilde{\Lambda}$ is via $\lambda \mapsto -{}^{w_J}\lambda$, which is congruent to $-\lambda$ modulo Λ. Therefore $*$ acts on A_Λ by inversion, hence every element $a* \in A_\Lambda *$ satisfies $(a*)^2 = a(*a*) = aa^{-1} = 1$.
(2) It is well-known that A_Λ permutes the hyperspecial vertices simply transitively. Then for any $a \in A_\Lambda$, we have $(a*)(s_0) = a(s_0)$ which exhaust all hyperspecial vertices exactly once as a runs over A_Λ. $\square$ $\square$

Let s_1 be another hyperspecial vertex in S. Let $\diamond \in A_\Lambda *$ be the unique involution taking s_0 to s_1, hence taking J to $J^\diamond = S - \{s_1\}$. Let $I_\diamond = \{w \in W | w^\diamond = w^{-1}\}$ be the $\diamond$-twisted involutions in W. To avoid complicated subscripts, we denote $W_{J^\diamond}$ by $W_J^\diamond$ instead.

The following theorem generalizes Theorem 1.3.

Theorem 7.3.

(1) *Each double coset $W_J \backslash W / W_J^\diamond$ in W is stable under the anti-involution $w \mapsto (w^\diamond)^{-1}$. In particular, the longest element in each $(W_J, W_J^\diamond)$-double coset belongs to $I_\diamond$.*
(2) *For longest representatives d_1 and d_2 of $(W_J, W_J^\diamond)$-double cosets in W, we have*

$$P^{\sigma,\diamond}_{d_1,d_2}(q) = P_{d_1,d_2}(-q).$$

Here the polynomials $P^{\sigma,\diamond}_{y,w}(q)$ $(y, w \in I_\diamond)$ are the ones defined in [L11] in terms of $(W, S, \diamond)$.

7.4 Sketch of proof

We only indicate how to modify the proof of Theorem 1.3 to give the proof of this theorem.

The anti-involution $w \mapsto (w^*)^{-1}$ extends to an anti-involution on $\widetilde{W}$ by the same formula (1.1) (except that λ now is any element in $\widetilde{\Lambda}$). Again each double coset $W_J\backslash\widetilde{W}/W_J$ is stable under this anti-involution. Write $\diamond = a*$ for $a \in A_\Lambda$, then $W_J^\diamond = a(W_J)$. Multiplication by a on the right gives a bijection

$$W_J\backslash W/W_J^\diamond \leftrightarrow W_J\backslash W \cdot a/W_J \subset W_J\backslash\widetilde{W}/W_J.$$

This shows part (1) of Theorem 7.3.

In the situation of Section 3.1, G is a simply-connected group. Let G^{ad} be the adjoint form of G, with maximal torus $T^{\mathrm{ad}} = T/Z(G)$. Then we have a natural isomorphism $\widetilde{\Lambda} \cong \mathbb{X}_*(T^{\mathrm{ad}})$. The connected components of the affine Grassmannian $\mathrm{Gr}^{\mathrm{ad}}$ for G^{ad} are indexed by $\widetilde{\Lambda}/\Lambda$. The $G^{\mathrm{ad}}[[t]]$-orbits on $\mathrm{Gr}^{\mathrm{ad}}$ are indexed by $\widetilde{\Lambda}/W_J$, and the natural projection $\widetilde{\Lambda}/W_J \to \widetilde{\Lambda}/\Lambda$ indicates which orbit belongs to which connected component. Identifying A_Λ with $\widetilde{\Lambda}/\Lambda$, we denote the corresponding component of $\mathrm{Gr}^{\mathrm{ad}}$ by $\mathrm{Gr}^{\mathrm{ad}}_a$ ($a \in A_\Lambda$ such that $\diamond = a*$). We may similarly define the Satake category $\mathcal{S}^{\mathrm{ad}}$ for G^{ad} with simple objects $\mathbf{C}_\lambda[\langle 2\rho, \lambda\rangle]$, $\lambda \in \widetilde{\Lambda}^+$ (dominant coweights of G^{ad}). Via the fiber functor $\mathrm{H}^\bullet$, $\mathcal{S}^{\mathrm{ad}}$ is equivalent to $\mathrm{Rep}(\check{G}^{\mathrm{sc}})$, where $\check{G}^{\mathrm{sc}}$ is the simply-connected form of $\check{G}$. The same anti-involution τ^* defines a functor $(\mathcal{S}^{\mathrm{ad}}, \odot) \to (\mathcal{S}^{\mathrm{ad}}, \odot^\sigma)$, and there is an isomorphism $\Psi_\lambda : \tau^*\mathbf{C}_\lambda \xrightarrow{\sim} \mathbf{C}_\lambda$ normalized to be the identity on $\mathrm{Gr}^{\mathrm{ad}}_\lambda$, which induces an involution $\mathcal{H}^i_\mu\Psi_\lambda$ on the stalks $\mathcal{H}^i_\mu\mathbf{C}_\lambda$ for $\mu \le \lambda \in \widetilde{\Lambda}^+$. Note that in the partial ordering of $\widetilde{\Lambda}$, two elements are comparable only if they are congruent modulo Λ.

Let $\dot{a} \in N_{G^{\mathrm{ad}}((t))}(T^{\mathrm{ad}}((t)))$ be a lifting of $a \in A_\Lambda < \widetilde{W}$, then $\dot{a}G^{\mathrm{ad}}[[t]]\dot{a}^{-1}$ is a hyperspecial parahoric subgroup of $G^{\mathrm{ad}}((t))$ corresponding to the vertex $s_1 = \diamond(s_0)$. Let $\mathbf{P} \subset G((t))$ be the hyperspecial parahoric subgroup (containing $\mathbf{I}$) corresponding to s_1. Right multiplication by $\dot{a}$ induces an isomorphism

$$G((t))/\mathbf{P} \xrightarrow{\sim} G^{\mathrm{ad}}((t))/\dot{a}G^{\mathrm{ad}}[[t]]\dot{a}^{-1} \xrightarrow{\sim} \mathrm{Gr}^{\mathrm{ad}}_a \tag{7.1}$$

which is equivariant under the left actions by $G[[t]]$. The double coset $G[[t]]\backslash G((t))/\mathbf{P}$ is in bijection with $W_J\backslash W/W_J^\diamond$. As in (3.5), the coefficients of the polynomials $P^{\sigma,\diamond}_{d_1,d_2}(q)$ are expressible as the traces of an involution on the stalks of the intersection cohomology complexes on $G[[t]]$-orbits of $G((t))/\mathbf{P}$. Under the isomorphism (7.1), we have the following formula generalizing (3.5):

$$P^{\sigma,\diamond}_{d_1,d_2}(q) = \sum_{j\in\mathbb{Z}} \mathrm{tr}(\mathcal{H}^{2j}_\mu\Psi_\lambda, \mathcal{H}^{2j}_\mu\mathbf{C}_\lambda)q^j.$$

Here $\mu \leq \lambda \in \widetilde{\Lambda}^+$ have image equal to a in $\widetilde{\Lambda}/\Lambda$, and d_1 (resp. d_2) is the longest element in the double coset $W_J \mu a^{-1} W_J^\diamond$ (resp. $W_J \lambda a^{-1} W_J^\diamond$).

So in order to prove Theorem 7.3(2), it suffices to show that $\mathcal{H}_\mu^{2j}\Psi_\lambda$ acts on $\mathcal{H}_\mu^{2j}\mathbf{C}_\lambda$ via multiplication by $(-1)^j$ for any $\mu \leq \lambda \in \widetilde{\Lambda}^+$. The argument in Section 4 works up to Lemma 4.2. The pair (τ^*, γ) again determines the element $g = (-1)^\rho \in \check{T} < \mathrm{Aut}(\check{G}^{\mathrm{sc}})$. However, a monoidal isomorphism $\Theta : \tau^* \xrightarrow{\sim} \mathrm{id}^\sigma_{\mathcal{S}^{\mathrm{ad}}}$ is the same as the choice of an element $\widetilde{g} \in \check{T}^{\mathrm{sc}}$ lifting $(-1)^\rho$: the effect of Θ on $V \in \mathrm{Rep}(\check{G}^{\mathrm{sc}}) \cong \mathcal{S}^{\mathrm{ad}}$ is the action of $\widetilde{g}^{-1}$. Lemma 4.3(2) should say that the effect of Θ (or $\widetilde{g}^{-1}$) on $\mathbf{C}_\lambda[\langle 2\rho, \lambda\rangle]$ is $\widetilde{g}^{-{}^{w_J}\lambda}\tau_K^*$. In the rest of the argument, we use (4.6). The piece $\mathrm{H}_c^{2\langle 2\rho,\mu\rangle}(\Omega_\mu) \otimes \mathcal{H}_\mu^{2j}\mathbf{C}_\lambda$ appears in degree $2\langle 2\rho, \mu\rangle + 2j - \langle 2\rho, \lambda\rangle$ in $\mathrm{IH}^\bullet(\Omega_{\leq\lambda})[\langle 2\rho, \lambda\rangle] \cong V_\lambda$, hence it appears as a subquotient of $\oplus_\nu V_\lambda(\nu)$, where $\nu \in \widetilde{\widetilde{\Lambda}}$ has the same image as λ and μ in $\widetilde{\Lambda}/\Lambda$ and

$$\langle 2\rho, \nu\rangle = 2\langle 2\rho, \mu\rangle + 2j - \langle 2\rho, \lambda\rangle, \text{ or } j = \langle \rho, \nu + \lambda - 2\mu\rangle. \tag{7.2}$$

We write $\mathrm{H}_c^{2\langle 2\rho,\mu\rangle}(\Omega_\mu) \otimes \mathcal{H}_\mu^{2j}\mathbf{C}_\lambda = \oplus_\nu (\mathrm{H}_c^{2\langle 2\rho,\mu\rangle}(\Omega_\mu) \otimes \mathcal{H}_\mu^{j}\mathbf{C}_\lambda)_\nu$ according to the weight decomposition. Therefore $\widetilde{g}^{-1}$ or $\widetilde{g}^{-{}^{w_J}\lambda}\tau_K^*$ acts on $(\mathrm{H}_c^{2\langle 2\rho,\mu\rangle}(\Omega_\mu) \otimes \mathcal{H}_\mu^{2j}\mathbf{C}_\lambda)_\nu$ by $\widetilde{g}^{-\nu}$. Specializing to $\lambda = \mu = \nu$, $\widetilde{g}^{-{}^{w_J}\mu}\tau_K^*$ acts on $\mathrm{H}_c^{2\langle 2\rho,\mu\rangle}(\Omega_\mu) = \mathrm{IH}^{2\langle 2\rho,\mu\rangle}(\Omega_{\leq\mu})$ by $\widetilde{g}^{-\mu}$. Therefore, by (4.6), the action of $\mathcal{H}_\mu^j\Psi_\lambda$ on $\mathcal{H}_\mu^j\mathbf{C}_\lambda$ is given by

$$\widetilde{g}^{-\nu+{}^{w_J}\lambda} \cdot (\widetilde{g}^{-\mu+{}^{w_J}\mu})^{-1} = \widetilde{g}^{-\nu+\mu+{}^{w_J}(\lambda-\mu)}. \tag{7.3}$$

Since $-\nu + \mu \in \Lambda$, we have $\widetilde{g}^{-\nu+\mu} = g^{-\nu+\mu} = (-1)^{\langle\rho,-\nu+\mu\rangle}$. Since $\lambda - \mu \in \Lambda$, we also have $\widetilde{g}^{{}^{w_J}(\lambda-\mu)} = g^{{}^{w_J}(\lambda-\mu)} = (-1)^{\langle\rho,{}^{w_J}(\lambda-\mu)\rangle} = (-1)^{\langle-\rho,\lambda-\mu\rangle}$. Taking these two facts together we conclude that the expression (7.3) is equal to

$$(-1)^{\langle\rho,-\nu+\mu\rangle}(-1)^{\langle-\rho,\lambda-\mu\rangle} = (-1)^{\langle\rho,-\nu-\lambda+2\mu\rangle},$$

which is equal to $(-1)^j$ by (7.2). This finishes the proof of Theorem 7.3.

Remark 7.5. Theorem 1.5 can also be generalized to the setup in Section 7. Using this generalization, we may extend the discussion in Section 6.6 to the case where $\check{G}$ is simply connected. In this case, irreducible finite dimensional representations of $\check{G}$ still carry natural hermitian forms as in [L97], and their signatures can be expressed in terms analogous to (6.15). We omit the details.

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